Continuum

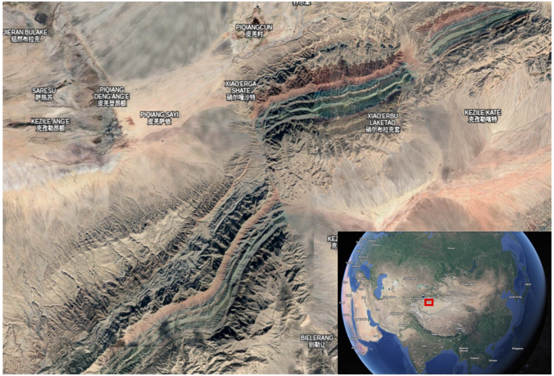
- Familiar with the classical Newtonian mechanics of *a system of particles* or *a rigid body*?
- A good starting point would be to see how continuum is different from those concepts.
  - Solids, liquids, gases. Is sand continuum?
  - A system of particles: particles separated by empty spaces.
  - Rigid bodies: Infinitely strong force prevents change in distance between particles.
  - Application of rigid body dynamics: `https://youtu.be/hI9HQfCAw64?si=mAis0X3lLHU-rWQK&t=104`

Continuum

- **Continuum**: Disregard the molecular or atomic structure of matter and picture it as being **without gaps or empty spaces.**
- Another central assumption: All the mathematical functions used to describe the material are **smoothly continuous** in the entire domain or in each of finite sub-domains, so that their derivatives exist.

Description of Motion

Typical objects of interest in geodynamics:



(`https://education.nationalgeographic.org/resource/fold-mountain/`) (from Google Earth)

Description of Motion

Other interesting phenomena that need continuum Kinematics:

- Wobbling liquid ball in space:
  https://www.youtube.com/watch?v=bKk_7NIKY3Y
- Lava flow on land 1/2:
  https://youtu.be/_hyE2NO7HnU?si=EMQybtO2215NWbF3&t=142
- Lava flow on land 2/2:
  https://youtu.be/_hyE2NO7HnU?si=Xcop12K827VK8gPg&t=224

Description of Motion

- When we are interested in how much a solid has deformed, it makes a lot of sense to play with some "relative" measurements. For instance, when we stretch an elastic bar, it makes a lot of sense to measure the amount of extension divided by the original length.
- ex) A 2 m bar extended to 2.002 m.
  0.002 m/2 m = 0.001 or 0.1 %. This dimensionless quantity is called ***strain***.
- On the other hand, the length change itself, 0.002 m, can be very large or very small depending on the original length of the bar, which can be very short (e.g., 0.002 m) or very long (e.g., $\tilde{2}$ km).
- But the meaning of 0.1 % strain is clearly understood for all cases.

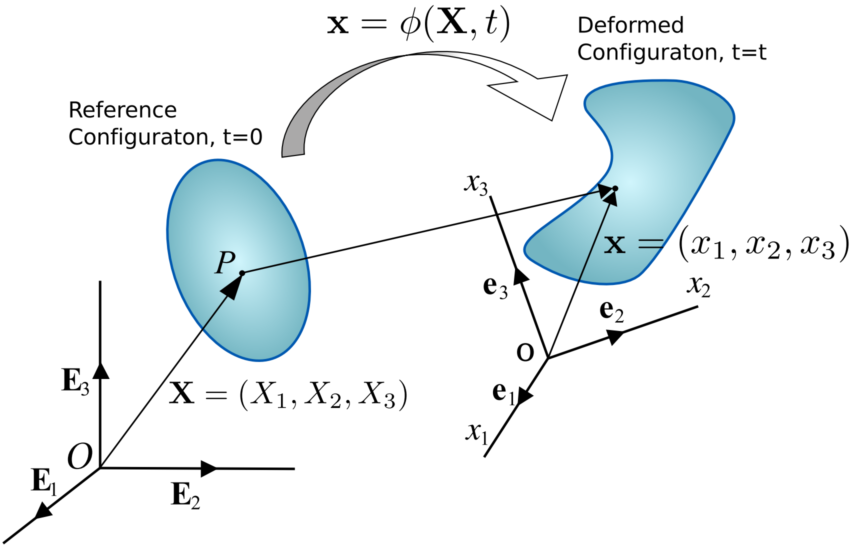
Description of Motion

- Measuring deformation of a coninuum is often complex, particularly for non-linear and/or history-dependent materials.
- So we will start with focusing on simple materials like ***linear*** elastic ones and their ***small deformation (strain)*** in most part of this course.
- It is not a serious limitation because many useful intuitions can be still acquired under this assumption.
- Even non-linear deformations look linear when observed for a short period of time.

Description of Motion

- We wish to describe the generic motion of a material body ($\mathcal{B}$), including translation and rigid body rotation as well as time dependent ones.
- To trace the motion of $\mathcal{B}$, we establish an absolutely fixed (inertial) frame of reference so that points in the Euclidean space ($\mathbf{R}^3$) can be identified by their position ($\mathbf{x}$) or their coordinates (x_i, i=1,2,3).
- The subsets of $\mathbf{R}^3$ occupied by $\mathcal{B}$ are called the ***configurations*** of the body. The *initially* known configuration is particularly called *reference configuration*.

Description of Motion



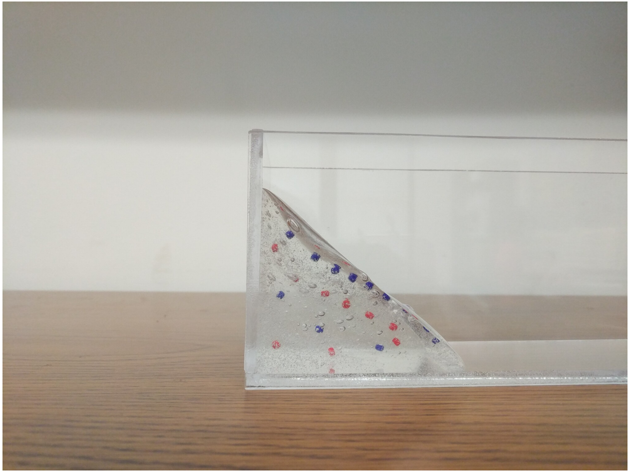
(from the continuum mechanics entry of Wikipedia)

Description of Motion

- It is conceptually important to distinguish the particles (P) of the body from their places in $\mathbf{R}^3$. The particles are physical entities - pieces of matter - whereas the places are merely positions in $\mathbf{R}^3$ in which particles may or may not be at any specific time.
- To identify particles, we label them in much the same way one labels discrete particles in classical dynamics. However, since $\mathcal{B}$ is an uncountable continuum of particles, we cannot use the integers to label them as in particle dynamics.

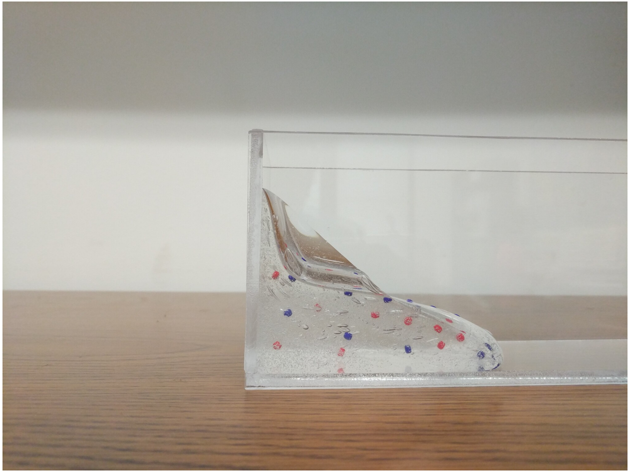
Description of Motion

Time-lapse images of slime flow: t = 0 (unitless)



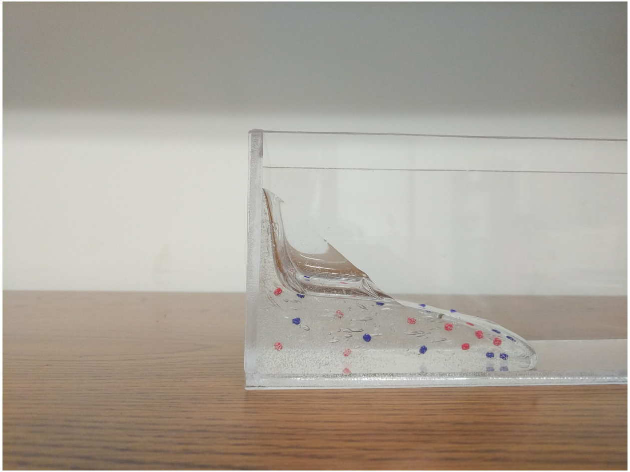
Description of Motion

Time-lapse images of slime flow: t = 8 (unitless)



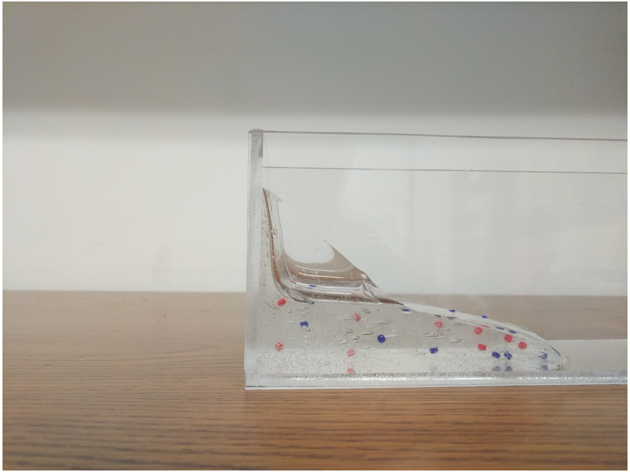
Description of Motion

Time-lapse images of slime flow: t = 16 (unitless)



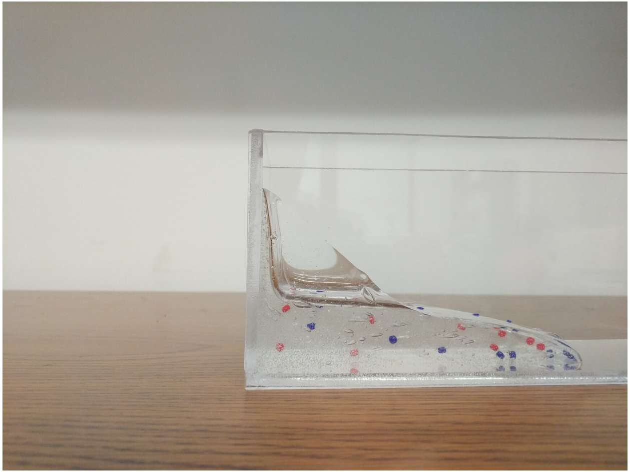
Description of Motion

Time-lapse images of slime flow: t = 24 (unitless)



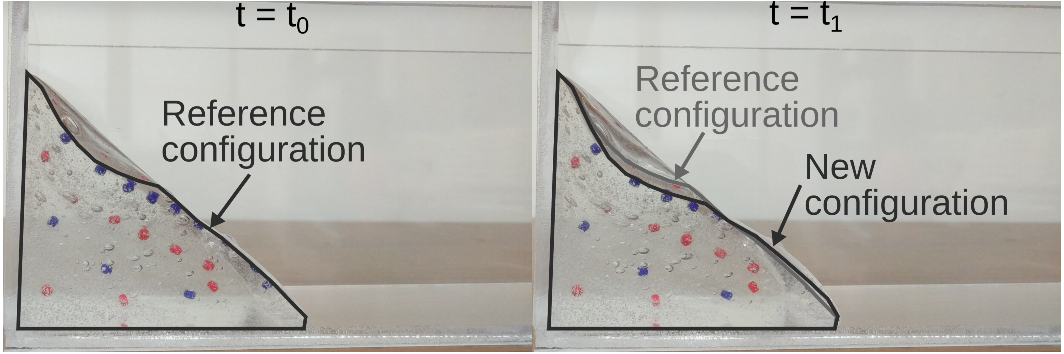
Description of Motion

Time-lapse images of slime flow: t = 32 (unitless)



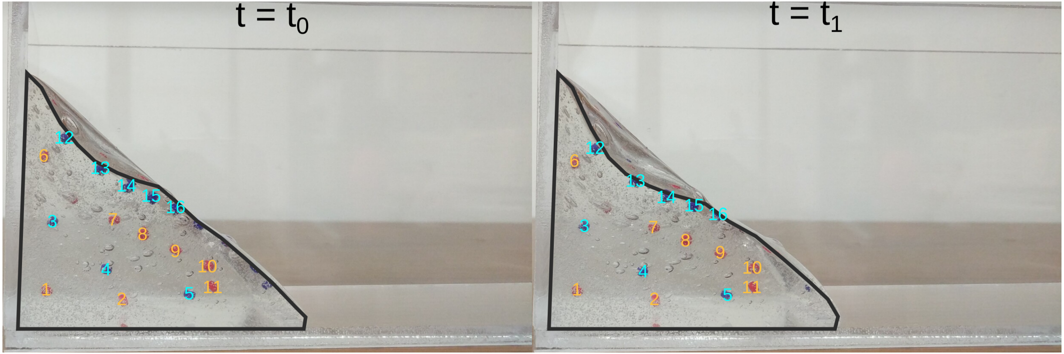
Description of Motion

Reference and current configurations



Description of Motion

Integer labels are not **dense** enough although sufficient for many applications (e.g., numerical approximation).



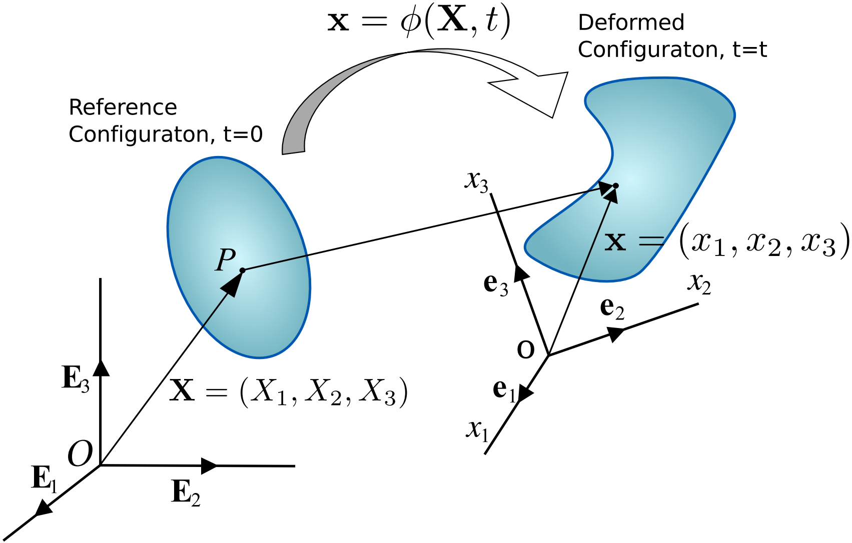
Description of Motion

- The problem is resolved by placing each particle in $\mathcal{B}$ in correspondence with an ordered triple $\mathbf{X} = (X_1, X_2, X_3)$ of *real numbers*. Mathematically, this "correspondence" is a *homeomorphism* from $\mathcal{B}$ into $\mathbf{R}^3$ and we make no distinction between $\mathcal{B}$ and the set of particle labels.
- The numbers X_i associated with particle $\mathbf{X} \in \mathcal{B}$ are called the ***material coordinates*** of $\mathbf{X}$.

Description of Motion

- For convenience, it is customary to choose the material coordinates of $\mathbf{X}$ to exactly coincide with the ***spatial coordinates***, $\mathbf{x}$, when $\mathcal{B}$ occupies its reference configuration.
- A ***motion*** of $\mathcal{B}$ is a time-dependent family of configurations, written $\mathbf{x} = \phi(\mathbf{X}, t)$. Of course, $\mathbf{X} = \phi(\mathbf{X}, 0)$.
- To prevent weird, non-realistic behaviors, we also require configurations (i.e., the mapping ϕ) to be ***sufficiently smooth*** (to be able to take derivatives), ***invertible*** (to prevent self-penetration, for instance), and ***orientation preserving*** (to prevent a mapping to a mirror image).

Description of Motion



(from the continuum mechanics entry of Wikipedia)

Description of Motion

- ***Material velocity*** of a point $\mathbf{X}$ is defined by

$$\mathbf{V}(\mathbf{X}, t) = (\partial/\partial t)\phi(\mathbf{X}, t)$$

- Velocity viewed as a function of $(\mathbf{x}, t)$, denoted $\mathbf{v}(\mathbf{x}, t)$, is called ***spatial velocity***.

$$\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$$

- ***Material acceleration*** of a motion $\phi(\mathbf{X}, t)$ is defined by

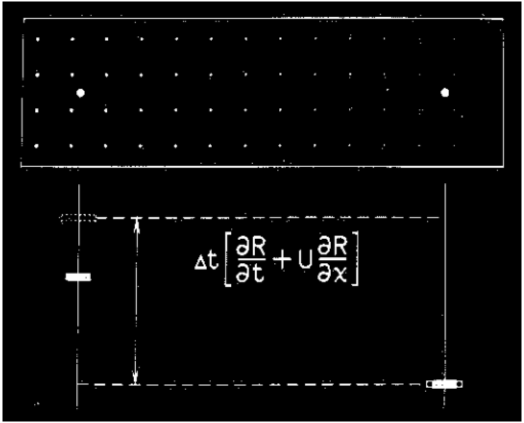
$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \phi}{\partial t^2}(\mathbf{X}, t) = \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t)$$

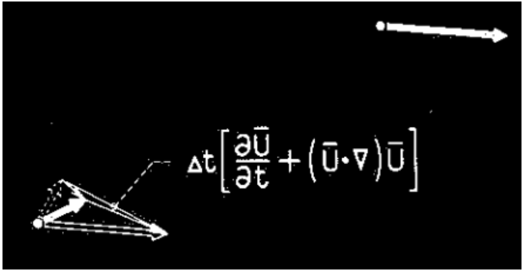
- The relationship between material and spatial acceleration is given by the chain rule,

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

- Watch `https://www.youtube.com/watch?v=mdN8OOkx2ko` to consolidate your underdtangin.

Description of Motion

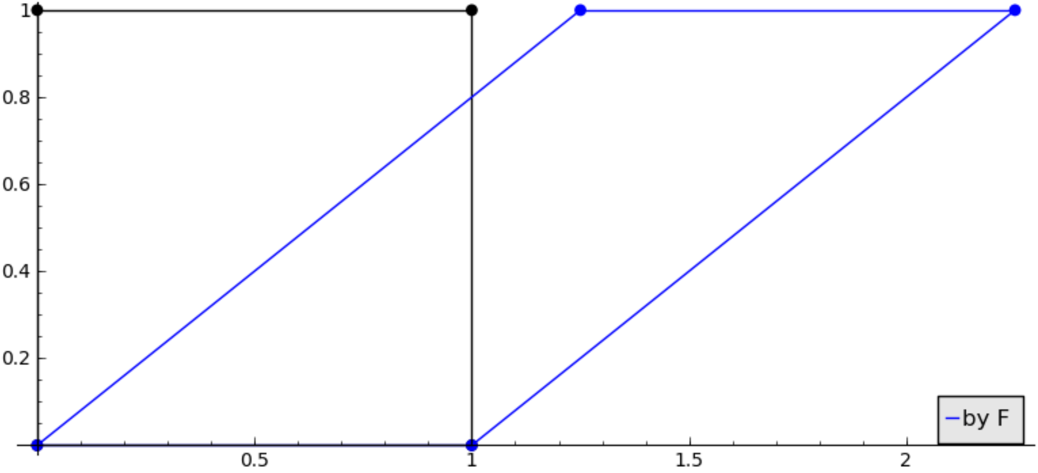




Description of Motion

Example:

- Find $\phi(\mathbf{X}, t)$, $\mathbf{V}$ and $\mathbf{v}$ in this simple shear motion? Assume that the top nodes are moving at a constant speed c to the right and the bottom nodes are fixed.



Description of Motion

- In general, if $Q(\mathbf{X}, t)$ is a material quantity–a given function of $(\mathbf{X}, t)$– and $q(\mathbf{x}, t) = Q(\mathbf{X}, t)$ is the same quantity expressed as a function of $(\mathbf{x}, t)$, then the chain rule gives
$$\frac{\partial Q}{\partial t} = \frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla)q.$$
- The right-hand side is called the ***material derivative*** of q and is denoted $Dq/Dt = \dot{q}$.
- Thus Dq/Dt is the derivative of q with respect to t, holding $\mathbf{X}$ fixed, while $\partial q/\partial t$ is the derivative of q with respect to t holding $\mathbf{x}$ fixed. In particular

$$\dot{\mathbf{v}} = D\mathbf{v}/Dt = \partial \mathbf{V}/\partial t = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

Description of Motion

- ***Deformation gradient***: The 3×3 matrix of partial derivatives of ϕ with respect to $\mathbf{X}$, denoted $\mathbf{F}$ and given as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad \text{or} \quad F_{ij} = \frac{\partial x_i}{\partial X_j}.$$

- Some trivial cases:
  If $\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix;
  if $\mathbf{x} = \mathbf{X} + ct\mathbf{E}_1$ (translation along X_1-axis with speed c), $\mathbf{F} = \mathbf{I}$. Consistent with the intuition that a simple translation is not a "deformation" of the usual sense.

- Exercises
  - What are ϕ and $\mathbf{F}$ of a "pure shear" deformation?
  - What are ϕ and $\mathbf{F}$ of a "simple shear" deformation?

- Think about why this first order partial derivative is sufficient for describing ANY general deformations.

Description of Motion

- ***Polar decomposition***: From linear algebra, we know $\mathbf{F}$ can be uniquely decomposed as

$$\mathbf{F} = \mathbf{RU}(\mathbf{X}) = \boldsymbol{v}(\mathbf{x})\mathbf{R},$$

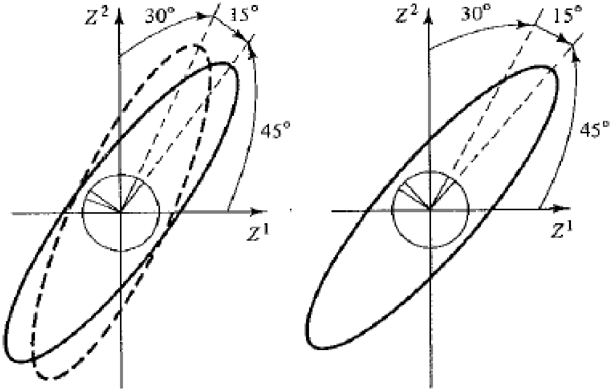
  where $\mathbf{R}$ is a proper (i.e., det($\mathbf{R}$) = +1) *orthogonal matrix* (i.e., $\mathbf{RR}^T$=$\mathbf{I}$) called the *rotation*, and $\mathbf{U}$ and $\boldsymbol{v}$ are positive-definite and symmetric and called right and left *stretch tensors*.
  - Note that $\mathbf{U}$ is associated with material coordinates while $\boldsymbol{v}$ with spatial coordinates.
  - Note that symbols, $\boldsymbol{U}$ and $\boldsymbol{v}$ are also used for denoting material displacement (see below) and spatial velocity.
- $\boldsymbol{v}(\mathbf{x}) = \mathbf{RUR}^T$
  - $\mathbf{R}^T : \mathbf{x} \longrightarrow \mathbf{X}$
  - $\mathbf{U} : \mathbf{X} \longrightarrow \mathbf{X}$
  - $\mathbf{R} : \mathbf{X} \longrightarrow \mathbf{x}$

Description of Motion

- In general $\mathbf{U} \neq \boldsymbol{v}$.
- $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\boldsymbol{v} = \sqrt{\mathbf{F}\mathbf{F}^T}$. Furthermore, we call $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ the *right Cauchy-Green tensor* and $\mathbf{b} = \mathbf{F}\mathbf{F}^T = \boldsymbol{v}^2$ is the *left Cauchy-Green tensor*.
- Since $\mathbf{U}$ and $\boldsymbol{v}$ are *similar*, their eigenvalues are equal;
- since $\mathbf{U}$ and $\boldsymbol{v}$ are positive definite and symmetric, their eigenvalues are real and positive.
- These eigenvalues are called the ***principal stretches***.
- The deviation of principal stretches from unity measures the amount of *strain* in a deformation. Analogy can be found in the earlier simplistic example: 2.002 m/ 2 m = 1.001. Here, 0.001 is the “deviation from the unity” and represents the actual deformation.

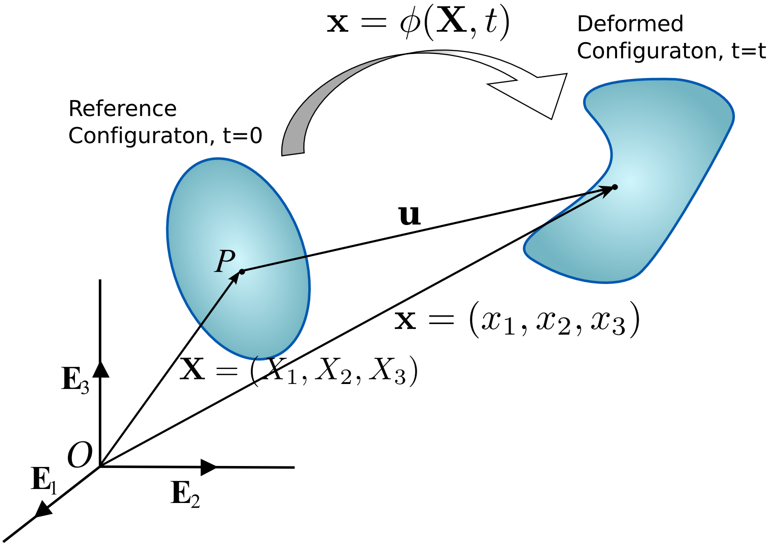
Description of Motion

- The meaning of the polar decomposition is that a deformation is locally given to first order by a rotation followed by a stretching by amounts corresponding to eigenvalues along three principal directions or vice versa.



(Fig. 1.3.1 in *Mathematical Foundations of Elasticity* (Marsden and Hughes, Dover, 1994))

Description of Motion



(modified from the continuum mechanics entry of Wikipedia)

Description of Motion

- ***Displacement*** is denoted $\mathbf{U}(\mathbf{X})$ and defined as

$$\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$$

  - $\mathbf{U}(\mathbf{X}, t) = \mathbf{U}(\phi^{-1}(\mathbf{x}, t), t) = \mathbf{u}(\mathbf{x}, t)$.
  - Like the material and spatial velocity, material and spatial displacements represent the same vector field (i.e., functions returning the same numerical values for given $\mathbf{x}$ and $\mathbf{X}$ that are related by $\mathbf{x} = \phi(\mathbf{X})$.

- Since $\mathbf{x} = \mathbf{U} + \mathbf{X}$,

$$\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{U}}{\partial \mathbf{X}}.$$

- $\mathbf{C}$, the right Cauchy-Green tensor, is defined as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{U}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T \frac{\partial \mathbf{U}}{\partial \mathbf{X}}$$

Description of Motion

- Note that the rotational part (**R**) is not involved according to this definition. So, **C** is all about stretches.
- Green's (material or Lagrangian) strain tensor ("deviation from the unity"):

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

- Note that

$$\begin{aligned}(ds)^2 = dx_i dx_i = F_{ij} dX_j F_{ik} dX_k &= dX_j F^T_{ji} F_{ik} dX_k \\ &= d\mathbf{X} \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X},\end{aligned}$$

and

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{I} \cdot d\mathbf{X}.$$

Description of Motion

- From the previous result, we realize that Green's strain tensor quantifies the change in the square of the length of the material vector $d\mathbf{X}$.

$$ds^2 - dS^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X} \quad \text{or} \quad dx_i dx_i - dX_i dX_i = 2\, dX_i\, E_{ij}\, dX_j.$$

- With further linearization, i.e., dropping the quadratic term under the assumption of infinitely small displacements, we get the familiar form of the **(small or infinitesimal) strain tensor** (ε):

$$\varepsilon = \frac{1}{2}\left[\frac{\partial \mathbf{U}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T\right] \text{ or } \varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$$

- Also note that the following decomposition is always possible:

$$\frac{\partial \mathbf{U}}{\partial \mathbf{X}} = \frac{1}{2}\left[\frac{\partial \mathbf{U}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T\right] + \frac{1}{2}\left[\frac{\partial \mathbf{U}}{\partial \mathbf{X}} - \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T\right] = \varepsilon + \boldsymbol{\omega}$$

The second term represents "(rigid body) rotation".

Description of Motion

- Strain and rotation, only when combined together, describe the entire motion. Then, why do we care so much about strain and only occasionally about rotation?
  - The answer is that only strain is related to stress. More on this point later.
- ***Principal strains***, eigenvalues of a small strain tensor, have the same meaning with principal streches.
- The trace of strain (ε_{ii}) is called ***dilatation*** and often denoted e.
- ***Invariants*** of a strain tensor are all often used in various contexts. Dilatation is, for instance, the first invariant.

Description of Motion

Invariants: Three coefficients of the characteristic equation of a rank-2 tensor (**T**).

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - \mathrm{I}_{\mathbf{T}}\lambda^2 + \mathrm{II}_{\mathbf{T}}\lambda - \mathrm{III}_{\mathbf{T}} = 0,$$

$$\begin{aligned} \mathrm{I}_{\mathbf{T}} &= T_{11} + T_{22} + T_{33}, \\ \mathrm{II}_{\mathbf{T}} &= T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33} \\ &\quad - T_{12}T_{21} - T_{13}T_{31} - T_{23}T_{32} \\ &= \frac{1}{2}(\mathbf{T}:\mathbf{T} - \mathrm{I}_{\mathbf{T}}^2), \\ \mathrm{III}_{\mathbf{T}} &= \det \mathbf{T}. \end{aligned}$$

Exercises

- What are **E** and ε of a pure shear deformation?
- What are **E** and ε of a simple shear deformation?
- Find **F**, **E** and ε for the following motion:

$$x(\mathbf{X},t) = X(1+at)\cos\frac{\pi t}{2} - Y(1+bt)\sin\frac{\pi t}{2}$$
$$y(\mathbf{X},t) = X(1+at)\sin\frac{\pi t}{2} + Y(1+bt)\cos\frac{\pi t}{2}$$

- Find **F**, **E** and ε for the following motion:

$$x(\mathbf{X},t) = X^2(1+at)\cos\frac{\pi t}{2} - Y^2(1+bt)\sin\frac{\pi t}{2}$$
$$y(\mathbf{X},t) = X^2(1+at)\sin\frac{\pi t}{2} + Y^2(1+bt)\cos\frac{\pi t}{2}$$