COMP4418 Assignment 3

Question 1

a) The uncovered set

A	{1: b, d}	{2: g, e, f, c}	√
B	{1: c, d}	{2: g, e, f, a}	√
C	{1: a, d}	{2: g, e, f, b}	√
D	{1: e, f, g}	{2: a, b, c}	√
E	{1: f, a}	{2: b, g, d}	×
F	{1: b, g}	{2: e, d, c}	×
G	{1: c, e}	{2: a, d, f}	×

Hence, the uncovered set is {a, b, c, d}

b) The top cycle

An alternative is in the top cycle if and only if it can reach every other alternative by a path in the tournament and any member of the uncovered set is a member of the top cycle.

Hence, the top cycle set should be {a, b, c, d, e, f, g}.

c) The set of Copeland winners

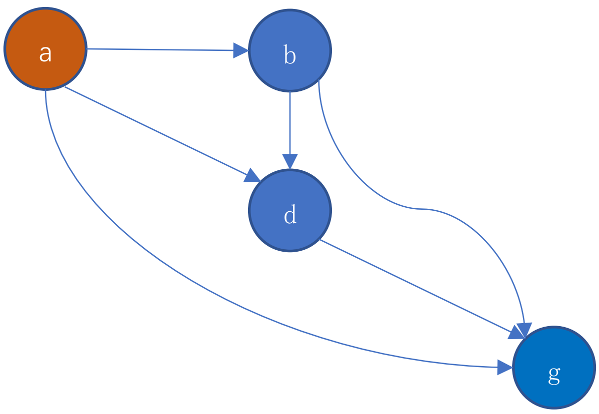
A	(a,b) (a,d) (a,f) (a,g)	Score = 4
B	(b,c) (b,d) (b,e) (b,g)	Score = 4
C	(c,a) (c,d) (c,e) (c,f)	Score = 4
D	(d,e) (d,f) (d,g)	Score = 3
E	(e,a) (e,f)	Score = 2
F	(f,b) (f,g)	Score = 2
G	(g,c) (g,e)	Score = 2

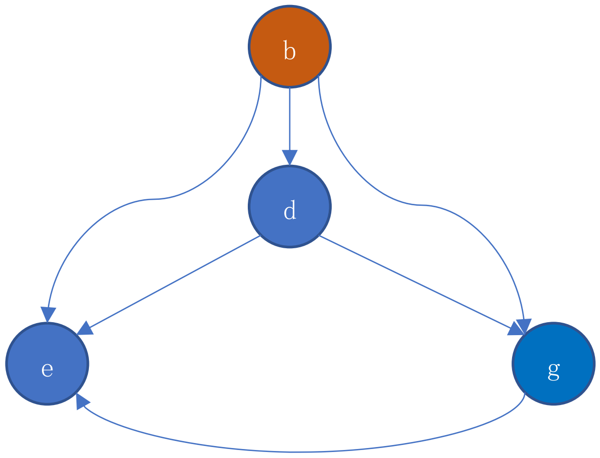
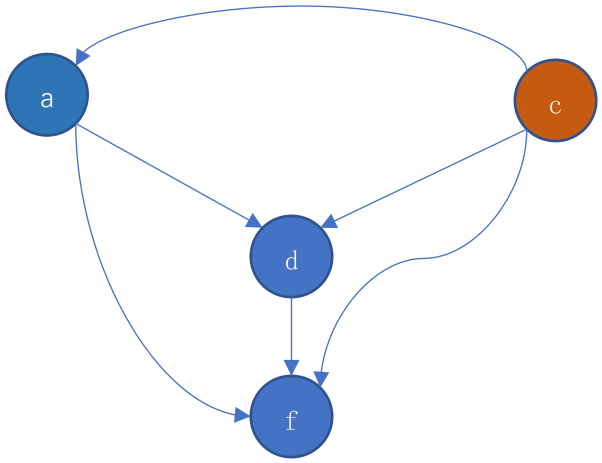
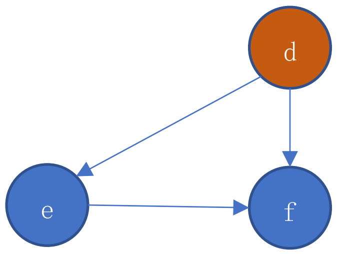
Hence, the top score of all the vertex is 3, so the set of Copeland winners is {a, b, c}

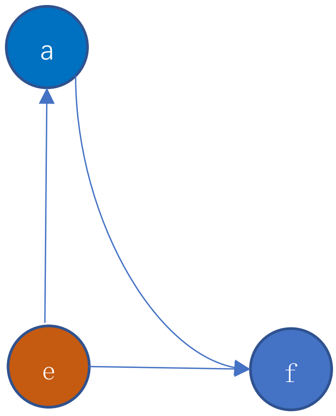
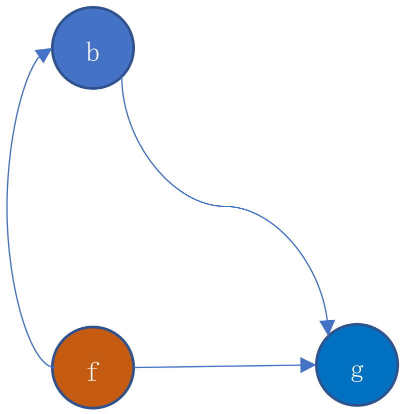
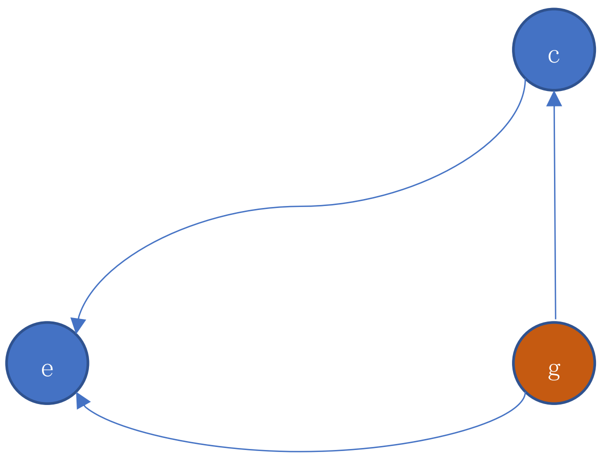
d) The set of Banks winners

Any member of the Banks set is a member of the uncovered set.

Under the Banks rule, an alternative x is a Banks winner if it is a top element in a maximal acyclic subgraph of the tournament.

A	

B	
C	
D	

E	
F	
G	

Hence, because the {a, b, c} are the root of the largest acyclic subgraph of the origin graph, the bank winners set should be {a, b, c}

e) The set of Condorcet winners

Condorcet winner: an alternative that is pairwise preferred by a majority of voters over every other alternative.

Hence the Condorcet winner shouldn't have any parent vertex in the graph. There's no vertex in the graph has no parent, so the Condorcet winner is {}.

Question 2

a) Th Prove or disprove that the Condorcet winner always has the maximum Borda score among all the alternatives.

Disprove.

If someone in {a1… am} is the Condorcet winner, and there are voters set {1…n}.

If the number of voters is odd, so the voters can be seen as {1…2n+1}, so the Condorcet winner must win at least n+1 times in all voters. So the Borda score of the winner should be more than (m-1)*(n+1). Then we assume that there are a person who have a higher score but not the Condorcet winner, the highest mark he can get is

(m-2)*(n+1)+(m-1)*n, we calculate the difference of the min of winner and the max of the competitor: (m-1)*(n+1) – ((m-2)*(n+1)+(m-1)*n) = n(2-m)+1. If it is a competition, m must larger than 2, so (2-m) always <0, so if n(2-m)< -1 the other competitor may got a higher score than the Condorcet winner.

If the number of voters is even, so the voters can be seen as {1…2n}, so the Condorcet winner must win at least n+1 times in all voters. So the Borda score of the winner should be more than (m-1)*(n+1). Then we assume that there are a person who have a higher score but not the Condorcet winner, the highest mark he can get is

(m-2)*(n+1)+(m-1)*(n-1) we calculate the difference of the min of winner and the max of the competitor: (m-1)*(n+1) – ((m-2)*(n+1)+(m-1)*(n-1)) = n(2-m)+m. If it is a competition, m must larger than 2, so (2-m) always <0, so if n(2-m)< -m the other competitor may got a higher score than the Condorcet winner.

So the state is not always true.

b) Prove or disprove that the Condorcet winner has at least half of the Borda score of the Borda winner.

Prove.

Case 1 even voters: Under the model as voters: {1…2n}, alternatives: {1…m}

The max score of Borda winner can be: B=(m-1)2n = 2mn-2n

The least score of a Condorcet winner is C=(m-1)(n+1)=mn+m-n+1

So B-2C = 2mn-2n – 2mn-2n+2n-2=2n-2, n>=1 so the C must be at least half of the Borda score.

Case 2 odd voters: Under the model as voters: {1…2n+1}, alternatives: {1…m}

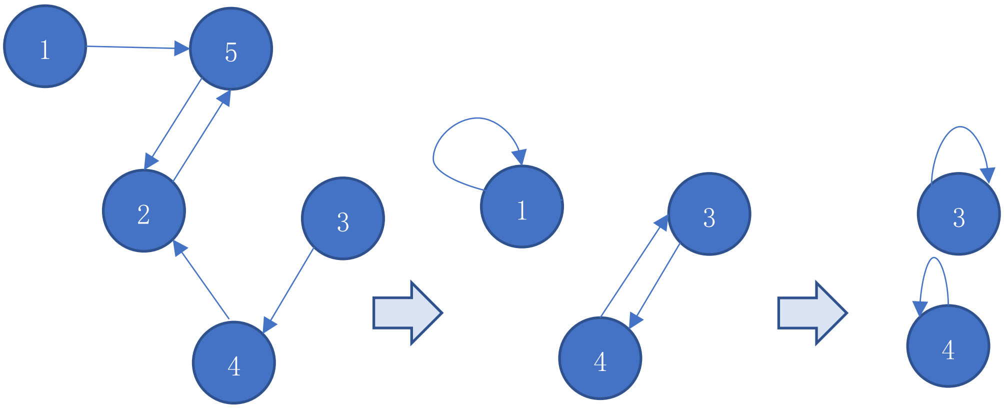
The max score of Borda winner can be: B=(m-1)(2n+1) = 2mn-2n+m-1

The least score of a Condorcet winner is C=(m-1)(n+1)=mn+m-n+1

So B-2C = 2mn-2n +m-1– 2mn-2n+2n-2=2n+m-3, m,n>=1 so the C must be at least half of the Borda winner.

Question 3

a) Find the outcome of the TTC (top trading cycles) algorithm.



The outcome is

1	2	3	4	5
O1	O5	O4	O3	O2

b) Can agent 4 misport her preference to get a better a match? Prove or disprove that the outcome is individually rational.
No. 4 cannot get a better match. Because at first 2 and 5 have got a circle, so they must be exchanged first, and then the loop begin from 1 to 5 again, 1 have got the next preference which is himself. Then the 4 can only got its 4th choice, no matter what the sequence of 4 it will all circle with 3. And the TTC is a strategy proof approach, which means the misport of anyone won't affect the final result.

Question 4

a) Find the outcome matching of the student proposing deferred acceptance algorithm and explain how you found the matching.

	result	Actions
1	{a: 4},{b: 2},{c:},{d: 5},{e: 1}	3 and 4 all apply to a, a reject 3
2	{a: 4},{b: 3},{c:},{d: 5},{e: 1 }	3 then apply b, b reject 2
3	{a: 2},{b: 3},{c:},{d: 5},{e: 1 }	2 then apply a, a reject 4
4	{a: 2},{b: 3},{c:},{d: 5},{e: 1 }	4 then apply b, b reject 4
5	{a: 2},{b: 3},{c: 4},{d: 5},{e: 1 }	4 then apply c

So the result of student proposing is

1	2	3	4	5
e	a	b	c	d

b) Prove or disprove that the resultant matching is Pareto optimal for the students.
Although lectures told us the DA algorithm is the pareto optimal in all envyfreeness allocations of the students. But If we allocate the students as {{1, e}, {2, b}, {3, a}, {4, c}, {5, d}}, we can get a better allocation than DA of all the students. So the DA method is not Pareto optimal for the students.

Question 5

The general principle is:

1. The sum of the values of the two is as close as possible to each allocation
2. Always allocate high value items first

Algorithm:

Step1. Sort the value table of all agents from high to low as an input.

Step2. The value owned by all agents is initialized to 0 (indicated by a list V, every agent gets a v value as 0)

Loop:

    If multiple agents have the same v value:

        Choose the agent that has the lowest utility for the next item

    Else:

        Find the agent with the smallest v value

    Select the highest value from the items to be allocate to him.

    Re-statistical v-value: the agent's v plus the current value of the agent.

The complexity of the algorithm is O(m*n), with m as the number of items and n as the number of agents, which should be able to find the EF1 solutions of all the cases.