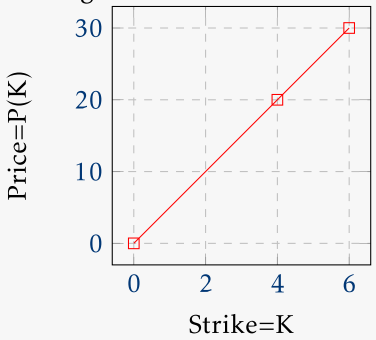
1 Introduction

- 2 put options , K=20,30 trading for \$4,\$6 - find arbitrage
  - 2x6-3x4=0, so try $2(20-S)^+ - 3(30-S)^+ > 0$
  - $\begin{cases} S \geq 30 \Rightarrow 0 \\ 20 \leq S \leq 30 \Rightarrow \text{-ve} \\ 0 \leq S \leq 20 \Rightarrow \text{-ve} \end{cases}$
  - 0 or -ve, so take opposite and done
  - graph (K,P(K)) not **strictly** convex $\Rightarrow$ arbitrage



- 2^{29} has 9 digits all different - find missing digit
  - key: 9 divides [n - Sum(digits n)]
  - **Proof**: $n = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0$
  - $S(n) = a_n + a_{n-1} + \ldots + a_1 + a_0$
  - $n - S(n) = a_n(10^n - 1) + a_{n-1}(10^{n-1} - 1) + \ldots + a_1(10^1 - 1) + a_0 - a_0$
  - $10^k - 1$ = all 9s = divisible by 9. so QED
  - Sum(digits n) $= \sum_1^9 i - x = 45 - x$
  - so $9|(2^{29} - 45 - x)$
  - but $2^{29} = 2^5 \times 2^{6\times4} = 2^5 \times 64^4 = 2^5 \times (63+1)^4 = 2^5 \times (k \times 63) + 1 = k \times 63 \times 2^5 + 2^5 = 9\times$ something for some k
  - so $9|(2^{29} - 45 - x)$ and $9|(2^{29} - 2^5)$ so $9|(2^{29} - 45 - x) - (2^{29} - 2^5)$
  - $9|(45 - x - 2^5) = (45 - x - 32) = (13 - x)$ so $x = 4$
- $\int_0^T W_t dt$ - what is distribution ? is it martingale ?
  - $X_T = \int_0^T W_t dt \Leftrightarrow dX_t = W_t dt + 0.dW_t =$ only drift so not martingale
  - $dX_t = W_t dt \Leftrightarrow X_T = \int_0^T W_t dt \Leftrightarrow X_T = [W_t t]_0^T - \int_0^T t dW_t = TW_T - \int_0^T t dW_t = T\int_0^T dW_t - \int_0^T t dW_t = \int_0^T (T-t) dW_t$
  - Recall stochastic integral $\int_0^T f(t) dW_t \sim N(0, \int_0^T |f(t)|^2 dt)$
  - $X_T = \int_0^T (T-t) dW_t \sim N(0, \int_0^T |T-t|^2) dt = N(0, \frac{T^3}{3})$
- Alice sends 20 ants towards Bob in straight line, Bob 50 towards Alice in straight line. Ants collide and go back. How many reach Bob? Alice ? How many collisions ?
  - imagine ants carry a flag and pass it on. so 50 reach Alice and 20 reach Bob.
  - num collisions: each of Bob's 50 flags must go though 20 collisions to reach Alice, each of Alice's 20 flags goes through 50 collisions to reach Bob: 2x 50 x 20 = 2000 collisions
- find all p such that this is a collrelation matrix $\Omega = \begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & p \\ -0.3 & p & 1 \end{bmatrix}$
  - Ω correlation matrix $\Leftrightarrow$ Ω symmetric positive definite
  - **Sylvester**, **Cholesky** , **definition**
  - **Sylvester** criterion: all the principal minor determinants must be >0 (principal minor: remove the same rows and same column indices form a matrix)
    1. remove 2,3 $\rightarrow \det(1) = 1$, remove 1,3 $\rightarrow \det(1) = 1$, remove 1,2 $\rightarrow \det(1) = 1$
    2. remove 3 $\rightarrow \det \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} = 0.64$
    3. remove 2 $\rightarrow \det \begin{bmatrix} 1 & -0.3 \\ -0.3 & 1 \end{bmatrix} = 0.91$
    4. remove 1 $\rightarrow \det \begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} = 1 - p^2$
    5. remove $\emptyset \rightarrow \det\Omega = \det \begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & p \\ -0.3 & p & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.6 & -0.3 & 1 & 0.6 \\ 0.6 & 1 & p & 0.6 & 1 \\ -0.3 & p & 1 & -0.3 & p \end{bmatrix} \rightarrow +$ down - up $\rightarrow 1 - 0.18p - 0.18p - 0.09 - p^2 - 0.36 = 0.55 - 0.36p - p^2$
  - $\begin{cases} 1 - p^2 > 0 \\ 0.55 - 0.36p - p^2 > 0 \end{cases}$
- how many draws N of U[0,1] such that $P(0.7 < U[0,1] < 0.72) \geq 0.95$ for one $U[0,1]$
  - P(none of the N rv in $[0.7, 0.72]$)=$0.98^N \rightarrow$ P(at least one in [0.7,0.72]) $= 1 - 0.98^N$ and solve $1 - 0.98^N \geq 0.95$
- prove pdf of N(0,1) is : $\int \text{pdf} = 1$
  - $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\frac{x^2}{2}) dx$ substitute $x^2/2 = t \rightarrow dx = \sqrt{(2)} dt \rightarrow$ need prove $I := \int_{-\infty}^{+\infty} \exp(-t^2) dt = \sqrt{\pi}$
  - $I^2 = \int_{-\infty}^{+\infty} \exp(-x^2) dx \int_{-\infty}^{+\infty} \exp(-y^2) dy = \int_{-\infty}^{+\infty} \exp(-x^2 + y^2) dx dy$
  - $x = r\cos\theta, y = r\sin\theta, dxdy = r dr d\theta$
  - $I^2 = \int_0^{+\infty} \int_0^{+2\pi} \exp(-r^2) r dr d\theta = 2\pi \int_0^{+\infty} \exp(-r^2) r dr = 2\pi [\frac{-1}{2} \exp(-r^2)]_0^{+\infty} = \pi \rightarrow I = \sqrt{\pi}$
- walk 1m south, 1m east, 1m north, then back at same point - where are you o Earth?
  - find latitute near south Pole such that round-the-world=1m. then any point 1m north of this works
  - also any latitude near south Pole such that round-the-world=$\frac{1}{k}$ works (go round the world k times but still end up same point)
- solve Oernstein-Uhlenbeck SDE - Vasicek for IR
  - $dr_t = \lambda(\theta - r_t) dt + \sigma dW_t \rightarrow dr + \lambda r_t dt = \lambda\theta dt + \sigma dW_t \overset{\times e^{\lambda t}}{\Rightarrow} e^{\lambda t}(dr + \lambda r_t dt) = e^{\lambda t}(\lambda\theta dt + \sigma dW_t) \rightarrow d(r_t e^{\lambda t}) = e^{\lambda t}(\lambda\theta dt + \sigma dW_t) \rightarrow [r_t e^{\lambda t}]_0^t = \int_0^t \lambda\theta e^{\lambda s} ds + \int_0^t \sigma e^{\lambda s} dW_s \rightarrow r_t e^{\lambda t} - r_0 = \theta(e^{\lambda t} - 1) + \sigma \int_0^t e^{\lambda s} dW_s \rightarrow r_t = e^{-\lambda t} r_0 + \theta(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(s-t)} dW_s$ and that is it.
  - note however $E(\int_0^t f(s) dW_s)$ = "sum of N(0,.)"=0 so $E(r_t) = e^{-\lambda t} r_0 + \theta(1 - e^{-\lambda t})$ which $\rightarrow \theta$ as $r \uparrow +\infty$ "mean reverting"

2 Calculus

- i^i? $e^{i\frac{\pi}{2}i} = e^{-\frac{\pi}{2}}$
- $\pi^e >< e^\pi$? try $\ln(\pi^e) = e\ln(\pi)$ vs $ln(e^\pi) = \pi\ln(e)$ ie $\frac{\ln(\pi)}{\pi}$ vs $\frac{\ln(e)}{e}$ ie check $f(x) = \frac{\ln(x)}{x}$ ie $f'(x) = (\frac{\ln(x)}{x})' = \frac{1}{x}\frac{1}{x} + \ln(x)\frac{-1}{x^2} = \frac{1-\ln x}{x^2}$ ie $f'(x) < 0$ if $x > e$ and $f'(x) > 0$ if $x < e$ ie $f(x) \uparrow$ on $[0, e]$ and $f \downarrow$ on $[e, \ldots + \infty]$ and $e < \pi$ so $f(e) > f(\pi)$
- $\frac{e^x + e^y}{2} > e^{\frac{x+y}{2}}$? $e^{\frac{x+y}{2}} = \sqrt{e^{x+y}} = \sqrt{e^x}\sqrt{e^y}$ ie $a^2 + b^2 > 2ab$ ie $a^2 - 2ab + b^2 > 0$ which is true
- solve $x^6 = 64$ $2^6 = 64$ and $z^6 = 1 \Leftrightarrow z = e^{\frac{2ik\pi}{6}}, k \in 0\ldots5$ so $x = 2e^{\frac{2ik\pi}{6}}, k \in 0\ldots5$
- derivative of x^x ? $x^x = e^{x\ln(x)}$ so derivative $= g' \circ f.f'$ ie $e^{x\ln(x)}(\ln(x) + x\frac{1}{x}) = x^x(\ln(x) + 1)$
- compute $\sqrt{2 + \sqrt{2 + \sqrt{2\ldots}}}$? $l = \sqrt{2 + l}$ ie $l^2 = l + 2$ ie $(l-2)(l+1) = 0$ ie $l = 2$ since $l > 0$ assuming limit exists. but $x_{n+1} = \sqrt{x_n + 2}$ is increasing (same reason) and bounded abve, so l exists and l=2
- find $2 = x^{x^{x^{x\cdots}}}$ $2 = x^2$ so $x = \sqrt{2}$ if it exists.

  prove $\sqrt{2} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}\cdots}}$ let $x_0 = \sqrt{(2)}, x_{n+1} = x_n^{\sqrt{2}}$. Prove sequence is increasing, bounded from above by 2.therefore has a limit. Therefore that limit must $= \sqrt{2}$
- which one converges ? $\sum\frac{1}{k}, \sum\frac{1}{k^2}, \sum\frac{1}{k\ln(k)}$
  1. $\sum\frac{1}{k^2} \leq \sum\frac{1}{k(k-1)} = \sum\frac{1}{k} - \frac{1}{k-1} \leq 1 - 1/n < 2$ so $\uparrow$, bounded ie converges.
  2. $\sum\frac{1}{k} > \ln(n) + \frac{1}{n}$ because $\int_1^n \frac{1}{x} dx = \sum_1^{n-1} \int_k^{k+1} \frac{1}{x} dx < \sum_1^{n-1} \int_k^{k+1} \frac{1}{k} dx = \sum_1^{n-1} \frac{1}{k}[x]_k^{k+1} = \sum_1^{n-1} \frac{1}{k} = \sum_1^n \frac{1}{k} - \frac{1}{n}$
  3. $\sum\frac{1}{k\ln(k)}$: similalry compare $\int_1^n \frac{1}{x\ln(x)} dx$ and note $\int \frac{1}{x\ln(x)} = \ln(\ln(x))$
- compute $\int \frac{1}{1+x^2} dx$? substitute $x = \tan(z) = \frac{\sin z}{\cos z}$ so $dx = \frac{1}{\cos^2 z} dz [= \cos z \frac{1}{\cos z} + \sin z \frac{-1}{\cos^2 z}(-\sin z) = 1 + \frac{\sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}]$ so $\int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2 z} \frac{1}{\cos^2 z} dz = \int \frac{1}{\cos^2 z + \sin^2 z} dz = \int dz = z + C = \arctan x + C$
- compute $\int x\ln(x) dx$? $\int x\ln(x) dx = [\frac{x^2}{2}\ln x] - \int \frac{x^2}{2}\frac{1}{x}$
- compute $\int xe^x dx$? $\int xe^x dx = [xe^x] - \int e^x dx$
- compute $\int x^n \ln x \, dx$ $\int x^n \ln x \, dx = [\frac{x^{n+1}}{n+1}\ln x] - \int \frac{x^{n+1}}{n+1}\frac{1}{x} dx$
- compute $\int \ln^n x \, dx$? $\int \ln^n x dx = [x\ln^n x] - \int x n \ln^{n-1}(x)\frac{1}{x} dx = [x\ln^n x] - n\int \ln^{n-1}(x) dx$ so recursion $f_n(x) = [x\ln^n x] - nf_{n-1}(x)$
- solve $y'' - 4y' + 4y = 1$ general solution $y'' - 4y' + 4y = 0 \rightarrow z^2 - 4z + 4 = 0 \rightarrow (z-2)^2 = 0 \rightarrow y = C_1 e^{2x} + C_2 x e^{2x}$ + particular solution $y = \frac{1}{4}$
- solve $y' = y(1-y)$ $y' = y(1-y) \rightarrow \frac{y'}{y(1-y)} = 1 \rightarrow \int \frac{y'}{y(1-y)} dx = \int 1 dx \overset{y'=dy/dx}{\Rightarrow} \int \frac{dy}{y(1-y)} = \int 1 dx$ with $\int \frac{dy}{y(1-y)} = \int (\frac{1}{y} + \frac{1}{y-1}) dy$ so $\ln y -$

$\ln(1-y) = ln(\frac{y}{1-y}) = x + C$ so $\frac{y}{1-y} = e^{x+C}$

- derive BS PDE set $\Pi = V - \frac{\partial V}{\partial S}S$ so $d\Pi = dV - \frac{\partial V}{\partial S}dS$ and $\frac{dS}{S} = \mu dt + \sigma dW$ and $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$ and $(dS)^2 = \sigma^2 S^2 dt$ so $dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial V}{\partial S}dS$ so $d\Pi$
  $= (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt \overset{\text{risk free growth}}{=}$
  $= r\Pi dt$
  $= r(V - \frac{\partial V}{\partial S}S)dt$ so $(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)$
  $= r(V - \frac{\partial V}{\partial S}S)$
  rearrange $\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + r\frac{\partial V}{\partial S}S - rV = 0$

3 Linear algebra

- cov Σ_X & corr Ω_X matrices are +ve sem.-def
  - $\Sigma(j,k) = cov(X_j, X_k) = cov(X_k, X_j) = \Sigma_X(k,j)$ and $\Omega(j,k) = corr(X_j, X_k) = corr(X_k, X_j) = \Omega(k,j)$. Also $(\star) var(\sum_1^n c_i X_i) = C^T \Sigma_X C$ with C= $\begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix}$ and var(.)>= 0 so Σ_X is positive, and symmetric (see above), so +ve semi-definite
  - Proof of $(\star)$: let $Y = \sum_1^n c_i X_i$ then $Y - E[Y] = \sum_1^n c_i(X_i - \mu_i)$ and Var(Y)
    $= E[Y - E(Y)]^2$
    $= E[\sum_1^n c_i(X_i - \mu_i)^2]$
    $= E[\sum_{1 \le j,k \le n} c_j c_k (X_j - \mu_j)(X_k - \mu_k)]$
    $= \sum_{1 \le j,k \le n} c_j c_k E[(X_j - \mu_j)(X_k - \mu_k)]$
    $= \sum_{1 \le j,k \le n} c_j c_k cov(X_j, X_k)$
    $= \sum_{1 \le j,k \le n} c_j c_k \Sigma_X(j,k)$
    $= C^T \Sigma_X C$
  - For correlation: $\Sigma_X = D_{\sigma_X} \Omega_X D_{\sigma_X}$ where $D_{\sigma_X} = Diag(\sigma_i), \sigma_i^2 = var(X_i), i \in 1..n$ and $w^T \Sigma_X w = w^T D_{\sigma_X} \Omega_X D_{\sigma_X} w = v^T D_{\sigma_X}^{-1} D_{\sigma_X} \Omega_X D_{\sigma_X} D_{\sigma_X}^{-1} v = v^T \Omega_X v \ge 0$ so correlation Ω_X positive semi-definite.
- Find correlation Ω for covariance matrix
  $\Sigma = \begin{bmatrix} 1 & 0.36 & -1.144 \\ 0.36 & 4 & 0.8 \\ -1.44 & 0.8 & 9 \end{bmatrix}$
  $\Sigma_X = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \Omega_X \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$
  $\Omega_X = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3} \end{bmatrix} \Sigma_X \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3} \end{bmatrix}$ with
  $\sigma_1 = \sqrt{\Sigma(1,1)} = 1, \sigma_2 = \sqrt{\Sigma(2,2)} = 2, \sigma_3 = \sqrt{\Sigma(3,3)} = 3$
- Find allowable p for $\Omega = \begin{bmatrix} 1 & p & \dots & p \\ p & 1 & \ddots & p \\ \vdots & \ddots & \ddots & p \\ p & p & p & 1 \\ p & p & p & 1 \end{bmatrix}$
  - Find eigenvalues of Ω then state that all e.v must be +ve to find the condition on p
  - $\Omega = (1-p)I + pM$ with I = identity. M=all 1s. But ev of M are easy to find : $\mathbf{Mv} = \lambda \mathbf{v} \to v_1 + v_2 + \ldots + v_n = \lambda v_1 = \lambda v_2 = \ldots = \lambda v_n$ so $\begin{cases} \lambda = 0 \\ v_1 = v_2 = v_n \to v_1 + v_2 + v_n = nv_1 = \lambda v_1 \end{cases}$ so $\lambda = 0, n$ and for $\Omega v = (1-p)Iv + pMv = (1-p)v + p\lambda v = (1 - p + p\lambda)v$ so $\begin{cases} \lambda = 0 \to 1 - p \ge 0 \\ \lambda = n \to 1 - p + pn \ge 0 \end{cases}$ so $\frac{1}{1-n} \le p \le 1$
- prove nxn matrix has n eigenvalues
  - $Av = \lambda v \Leftrightarrow (\lambda I - A)v = 0 \Leftrightarrow (\lambda I - A)$ singular $\Leftrightarrow det(\lambda I - A) = 0 \Leftrightarrow P_A(\lambda) = 0$ so n roots, so n eigenvalues but some may be the same
  - An eigenvalue of multiplicity m has at least 1 eigenvector and at most m linearly independent eigenvectors.
- find $X^2 = A$ and $YY^T = A$ for $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$
  - A symmetric ! $\Leftrightarrow A = O^T \Lambda O$ with $\Lambda = Diag(evals)$ and O orthogonal and made up of the evectors ie $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, O = (v_1 v_2)$,
    $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \|v_1\| = 1, \|v_2\| = 1$
  - $X = O^T \Lambda^{\frac{1}{2}} O, O^T O = I$ as O=orthogonal
  - A symmetric +ve definite $\Leftrightarrow$ Cholesky $\Leftrightarrow A = U^T U$ with $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & 0 \end{bmatrix}$ so $Y = U^T$
  - find the evals :
    $det(\lambda I - A) = det\begin{bmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{bmatrix} = 0$
  - find the evecs with $\|.\| = 1$
  - find the Cholesky U by solving linear equation $\begin{bmatrix} u_{11} & 0 \\ u_{12} & u_{22} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = A$
- if $\lambda_1 = 2, \lambda_2 = -3, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$
  find $Av, v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
  - find $c_1, c_2 : v = c_1 v_1 + c_2 v_2$ (solve equation)
  - $Av = c_1 Av_1 + c_2 Av_2 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$
- show $trace(AB) = trace(BA)$ for $A, B = nxn$
  - $(\star) P_{AB}(\lambda) = P_{BA}(\lambda) = det(\lambda I - AB) = det(\lambda I - BA)$
  - $P_M(\lambda) = det(\lambda I - M) = \lambda^n - trace(M) + \ldots + (-1)^n det(M)$
  - so $trace(AB) = trace(BA)$
  - Proof $\star$:
    $det(\lambda I - AB)$
    $= det(\lambda IB^{-1}B - B^{-1}BAB)$
    $= det(B^{-1})det(\lambda IB - BAB)$
    $= det(B^{-1})det(\lambda I - BA)det(B)$
    $= det(B^{-1})det(B)det(\lambda I - BA)$
    $= det(B^{-1}B)det(\lambda I - BA)$
    $= det(\lambda I - BA)$
  - for singular B, use $B - \epsilon I$ which is non-singular apart from finite $\epsilon = evalue$ so let $\epsilon \to 0$
- solve $AB - BA - I = 0$ for $A, B = nxn$
  - no solution since $trace(AB - BA) = trace(AB) - trace(BA) = 0$
- show A, B prob. Mat $\Rightarrow AB$ prob. Mat
  - A prob. Mat := sum of rows ==1 $\Leftrightarrow A\mathbb{1} = \mathbb{1}$ with $\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
  - $(AB)\mathbb{1} = A(B\mathbb{1}) = A\mathbb{1} = \mathbb{1}$
- all $\rho : \Omega := \begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{bmatrix}$ = corr matrix
  - Sylvester, Cholesky, Definition
  - **Sylvester**: all the principal minors must be >0 (pricincipal minors: remove same row and same column, compute determinant) – see above
  - **Cholesky**: the determinant of the 2x2 matrix M in the first step of the Cholesky algorithm must be ≥ 0 :
    $M = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} 0.6 \\ -0.3 \end{bmatrix}[0.6 \quad -0.3]$
    $= \begin{bmatrix} 0.64 & \rho + 0.18 \\ \rho + 0.18 & 0.91 \end{bmatrix}$
    $\det M = \det\begin{bmatrix} 0.64 & \rho + 0.18 \\ \rho + 0.18 & 0.91 \end{bmatrix} \ge 0$
  - **Definition** $x^t \Omega x \ge 0 \quad \forall x$ ie $[x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ge 0$ ie $x_1^2 + x_2^2 + x_3^2 + 1.2x_1x_2 - 0.6x_1x_3 + 2\rho x_1x_3 \ge 0$ then complete squares $x_1^2 + x_2^2 + x_3^2 + 1.2x_1x_2 - 0.6x_1x_3 + 2\rho x_1x_3 = x_1^2 + 1.2x_1x_2 - 0.6x_1x_3 + x_2^2 + x_3^2 + 2\rho x_1x_3 = x_1^2 + 2x_1(0.6x_2 - 0.3x_3) + x_2^2 + x_3^2 + 2\rho x_1x_3 = (x_1 + 0.6x_2 - 0.3x_3)^2 - (0.6x_2 - 0.3x_3)^2 + + x_2^2 + x_3^2 + 2\rho x_1x_3 = (x_1 + 0.6x_2 - 0.3x_3)^2 + 0.64x_2^2 + 2x_2x_3(\rho + 0.18) + 0.91x_3^2$ and complete last sqaure again same method

4 Financial instruments

- 3 put options (strike=K,Price(K))=(40,10),(50,20),(70,30). Is there arbitrageif so build it
  - graph $K \mapsto Price(K)$ must be **strictly convex** else arbitrage
  - line through (40,10) and (70,30) : $C : K \mapsto \frac{70-K}{30}10 + \frac{K-40}{30}30 = 10 + \frac{x-40}{70-40}(30-10) = 10 + \frac{30-10}{70-40}(x-40)$
  - on that line $P(50) = 20 > C(50) = \frac{70-50}{30}10 + \frac{50-40}{30}30 = 50/3$ so not convex (convex:curve must be below line) so arbitrage exists
  - construct arbitrage : find porfolio in the puts so that 1. V(0) < 0, 2. $V(T) > 0 \ \forall T$ with $V(T) =$
  - try $2x10 + 30 - 3x20 = 2P_{40} + P_{50} - 3P_{70} = 2(40-K)^+ + (50-K)^+ - 3(70-K)^+$
- Price of a stock now P=50. In 3m, either P=47 or P=52 with prob 50-50. How much for ATM put ? Assume no dividends and Interest rate = 0
  - real world probability irrelevant
  - standard solutions follows:
  - $P(0) = p_{\text{up}} P_{\text{up}} + p_{\text{down}} P_{\text{down}}$
    $= pP_{\text{up}} + (1-p)P_{\text{down}}$
    $= puP(0) + (1-p)dP(0)$
  - $P(0) == puP(0) + (1-p)dP(0)$
  - $1 = pu + (1-p)d$ with $u = \frac{P_{\text{up}}}{P(0)} = \frac{52}{50}, d = \frac{47}{50}$
  - $1 = p(u-d) + d \Rightarrow p = \frac{1-d}{u-d} = \ldots = 0.6$
  - $O(0) = pO_{\text{up}} + (1-p)O_{\text{down}}$
  - (here) O=ATM put , ATM means strike=Price now = 50
  - $O(0) = p0 + (1-p)3 = 0.6 \times 0 + 0.4 \times 3 = 1.2$
  - alternative solution follows:
  - Set up portfolio $\Pi = +1$ Option $+ (-1)\Delta_{\text{Option}}$ Stock with $\Delta_{\text{Option}} = \frac{O_{\text{up}} - O_{\text{down}}}{S_{\text{up}} - S_{\text{down}}} =$ (here) $\frac{0-3}{52-47} = 0.6$
  - (here) $\Pi = +1$ Option $+ (-1)(-0.6)$ Stock

- $\Pi(T) = \begin{cases} S(T) = 52 \Rightarrow 0 + 0.6 \times 52 = 31.2 \\ S(T) = 47 \Rightarrow 3 + 0.6 \times 47 = 31.2 \end{cases}$
- $\Pi(0)$ = discounted $\Pi(T) \overset{IR=0}{=} \Pi(T) = 31.2 = O(0) + 0.6 \times S(0) = O(0) + 0.6 \times 50 \Rightarrow O(0) = 31.2 - 0.6 \times 50 = 31.2 - 30 = 1.2$

- What is risk neutral pricing ?
  - $V(0) = \mathbb{E}\left[e^{-rT} V(S(T))\right]$
  - with $S(T) = S(0)e^{(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}N(0,1)}$
  - not OK for path-dependent
- How to derive BS ?
  - 12 possible ways actually ...
  - risk neutral pricing
  - BS PDE. .
    - Payoff = boundary conditions
    - Transform to heat equation
  - binomial tree, with calibration:
    - drift = risk free rate
    - terminal dist = lognormal when num time steps $\to \infty$
- Approximate formula for ATM put ?
  - $\text{Put}_{ATM} \approx 0.4 S_0 \sigma \sqrt{T}$ when total variance $= \sigma^2 T$ small
- if the price of a stock doubles, how does call option change ?
  - depends if call option ITM, ATM, OTM
  - ITM:
    - $C - P = S - K \hookleftarrow$ Put-call parity
    - $C + Ke^{-rT} = S \Rightarrow C = S - Ke^{-rT}$ so $C \times 2$
  - ATM: call option $\to$ ITM, $C \uparrow (\times 10)$
  - OTM: call option $\to$ ATM, $C \uparrow\uparrow\uparrow (\times 10^n)$
- what are the possible values of Delta of an option ?
  - Call option:
    - long Call: [0 (OTM) ... 1 (ITM)]
  - Put option:
    - $C + P = S \hookleftarrow$ Put-Call parity
    - $\Delta_C + \Delta_P = 1$
- what is Delta of +1 ATM Call ? what is Delta of +1 ATM Put ?
  - Delta of +1 ATM Call = 0.5
  - Delta of +1 ATM Put = -0.5
  - $\Delta_{Call} = N(d_1)$ with $d_1 = \frac{\ln\frac{S}{K} + (r - d + \frac{\sigma^2}{2})}{\sigma\sqrt{T}}$
    - ATM $\to K = S$, assume r=q=0 $\to d_1 = \frac{\sigma\sqrt{T}}{2}$
    - $\Delta_{Call} = N(d_1) = N(0.5\sigma\sqrt{T})$
    - $0.5\sigma\sqrt{T}$ normaly small eg for $\sigma = 0.5, T = 1 \Rightarrow 0.5 \times 0.5\sqrt{1} = 0.25$
    - $N(x)$ about 0 is $N(x) = 0.5 + \frac{x}{\sqrt{2\pi}}$
    - so $\Delta_{Call} = N(d1) \approx 0.5$ since $d_1 = 0.5\sigma\sqrt{T}$ small
  - $\Delta_{Put} = -N(-d_1)$
    - so $\Delta_{Put} = -N(-d_1) \approx -0.5$
  - $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-z^2}{2}} dz = 0.5 + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{\frac{-z^2}{2}} dz = 0.5 + \frac{1}{\sqrt{2\pi}} \int_0^x [1 + ...] dz \approx 0.5 + \frac{1}{\sqrt{2\pi}} x$
- what is Put-Call parity ?
  - Call - Put = Forward
  - $(S-K)^+ - (K-S)^+ = S - K$
  - $C(0) - P(0) = [S(0)e^{rT} - K]e^{-rT}$
  - $C(0) - P(0) = S(0) - Ke^{-rT}$
  - $C(t) - P(t) = S(t) - Ke^{-r(T-t)}$
- Show that **the time value of an option** is highest ATM
  - time value of call option := call option value - intrinsic value := $C(t) - (S_t - K)^+$
  - at any time t, for which value of S is this highest ? $S = K$?
  - fix t, define: $f(S) = C(S) - (S-K)^+$
  - $\begin{cases} S \le K \Rightarrow f(S) = C(S) \\ S > K \Rightarrow f(S) = C(S) - (S-K) \end{cases}$
  - $\begin{cases} S \le K \Rightarrow \text{call option so} \uparrow \text{with S} \\ S > K \Rightarrow f'(S) = \Delta_C - 1 < 0 \end{cases}$
  - so max at $S = K$
- What is implied volatility ? volatility smile ? volatility skew ?
  - implied volatility = the σ : $BS(..,\sigma)$ = observed price. Unique σ because $\sigma \uparrow \Rightarrow BS(..,\sigma) \uparrow$
  - Volatility smile $\smile$: $\sigma_{ITM} > \sigma_{ATM} < \sigma_{OTM}$ [currency options]
  - Volatility skew $\frown$: $\sigma_{ITM} < \sigma_{ATM} > \sigma_{OTM}$ [index options, equity options,com. options]
- What is the Gamma of an option ? Why is it better to have small Gamma ? Why is Gamma of plain vanilla options positive ?
  - Gamma $\Gamma = \frac{\partial^2 V}{\partial S^2}$
  - small Gamma $\Rightarrow$ Delta does not change quickly $\Rightarrow$ easier to keep Delta-neutral
  - Delta of Call/Put $\uparrow$ with $\uparrow$ S , hence Gamma +ve
  - Gamma $\approxeq 0$ for OTM / deep ITM, Gamma max ATM
- When is Call = Put ?
  - $C_0 - P_0 = (S_0 e^{rT} - K)e^{-rT} = S_0 - Ke^{-rT} = 0 \Leftrightarrow K = S_0 e^{rT}$
  - so $C_0 = P_0 \Rightarrow$ K = forward value of asset
- What is 2-year volatility of an asset with 30 % 6m volatility ?
  - $\sigma(t) = \sigma_{1Y}\sqrt{T}$
  - $\sigma(6m) = \sigma_{1Y}\sqrt{0.5}$
  - $\sigma(2y) = \sigma_{1Y}\sqrt{2} = \sigma(6m)\frac{\sqrt{2}}{\sqrt{0.5}}$
- Value fix/float swap ?
  - PV = fix - float (say)
  - fix pv = sum of future flows
  - float = right after payment, sumof all flows = 1 (notional)
  - so float = pv of next payment + pv of (1) at next payment date
  - example: 6m swap, N=10M, K=3%, next payment in 1m, then 7m, 13m, 19m, with last reset saying next flt payment = 125k
    - fix cpn $= N r\tau = 10M \times 0.03 \times 0.5 = 150k$ with τ = year fraction
    - fix pv $= 150k \times df(\frac{1}{12}) + 150k \times df(\frac{7}{12}) + 150k \times df(\frac{13}{12}) + (10M + 150k) \times df(\frac{19}{12})$
    - float pv $= (10M + 125k) \times df(\frac{1}{12})$
    - example df for semi-annual LIBOR = $df(t) = [(1 + L \times 0.5) \times (1 + L \times 0.5)]^{-t} \hookleftarrow$ start with 1, keep on re-investing. Divide 1/(what you get) to get the df = price of zero coupon delivering 1 at T
- Price change of a 10y ZC bond if yield increases by 10bp ?
  - Duration: $\Delta B = \frac{\partial B}{dy} dy = -\overbrace{\left(-\frac{1}{B}\frac{\partial B}{dy}\right)}^{Duration} B\, dy := -DB\, dy$
  - so $\frac{\Delta B}{B} = -D\, dy$
  - (here)
    - 10y ZC $\Rightarrow$ D=10
    - $dy = 10bps = 10 \times 1e-4 = 1e-3$
    - $\frac{\Delta B}{B} = -D\, dy = -10 \times 1e-3 = -0.01$
    - price $\downarrow$ 10%
- A 5y ZC bond with Duration D=3.5y has P=102. What is P if yield $\downarrow$ 50bp ?
  - $\frac{\Delta B}{B} = -D\, dy$
  - (here)
    - $D = 3.5$
    - $dy = -50bp = -5 \times 1e-3$
    - $B = 102$
    - $\Delta B = -3.5 \times 102 \times (-5 \times 1e-3) \approxeq +1.785$
    - $B_{new} \approxeq B + 1.785 = 102 + 1.785 = 103.785$
- What is a forward contract ?
  - (long position) agrees to buy an asset at a specific price at specific time in future
  - (short position) agrees to sell an asset at a specific price at specific time in future
  - that specific price is called the forward price
  - at T=0 the value of the forward contact is 0 $\hookleftarrow$ obv price of the forward contract $\neq$ forward price !! (completely unrelated quantities)
  - Forward Price= $F = S_0 e^{(r-q)T}$ = price of the asset in the future
- Forward price of Treasury FUTURES contract vs Forward price of a Commodity FUTURES contract ?
  - For Treasury (ie bond) FUTURES contract , price now includes the bond flows between now and T
  - But you won't receive these when you get the bond, so for you, price of bond must exclude them
  - ie Forward price = F_{bond} $= \left\{PV_{now}(\text{Bond}) - PV_{now}(\text{Coupons})\right\} e^{rT}$
  - For Com future, situation is reversed: when you get the Com, you will not have incurred storage costs etc
  - ie Forward price = F_{com} $= \left\{PV_{now}(\text{Com}) + PV_{now}(\text{Storage})\right\} e^{rT}$
- Difference between Future and Forward ?
  - daily settlement for futures, not so for forward
  - Standardised maturities, contracts for futures, not so for forwards
  - Exchange involved for futures (reduced CPTY risk), not so for forwards (OTC)
  - Range of delivery dates for (COM) futures, not so for forwards
  - Forwards can be cash-settled, even for COMs
- 10-day VAR @ 99% of a portfolio with 5-day VAR @ 95% = USD 100M ?
  - $VAR(N,C) = \sigma_{V,1Y} P(Z \le C) \sqrt{\frac{N}{252}} V(0)$ with
    - N = time horizon eg N=10-days
    - P = P for Normal distribution
    - C = confidence level = eg 95 % = 0.95
    - $\sigma_{V,1Y}$ = annualised std dev of the portfolio PNL
    - $V(0)$ = PV now of portfolio
  - eliminate the $\sigma_{V,1Y} V(0)$ by division ...

5 C++

- declare an array
  - `T foo [3]`
  - `T bar [] = {1,2,3}`
  - `int T* baz = new int[3]`
- declare an array of pointers

- `T* foo [3]`
- `T* bar [] = {&a,&b,&c}`
- `int T** baz = new int* [3]`

- const pointer to object, const pointer to const object etc
  - Remember the clockwise and backward rule
  - `T* bar [10]; //` array of 10 pointers to T
  - `char const *chptr;` // pointer to a const char == pointer to a read-only char
  - `char * const chptr;` // const pointer to a char == read-only pointer to a char
  - `char const * const chptr;` // const pointer to a const char == read-only pointer to a read-only char
- template functions

```
template typename <T>
T temp_sum (T a, T b) return {a+b;}
```

// accepts 2 Ts, return a+b , which is also a T

- find max subarray of +ve numbers– check Kadane algorithm

```
max_ending += a[i];
max_ending_hr = max(0,max_ending_hr)
max_so_far = max(0,max_ending_here)
```

// keep track of largest so far

- Reverse a linked list

```
max_ending += a[i];
max_ending_hr = max(0,max_ending_hr)
max_so_far = max(0,max_ending_here)
```

// keep track of largest so far

6 MC simulation

- How to estimate π ? what is the variance of your method ?
  - Acceptance-Rejection on πR^2 in an $2R \times 2R$ square with $R = 1$
    * $\frac{\pi 1^2}{2 \times 2} = \frac{\pi}{4} = \frac{A}{N}$
    * N = num points tried altogether
    * A = num accepted points
  - $X_i = \mathbb{1}_{D(0,R=1)}(U_i)$ with $U_i \overset{\text{uniform}}{=} [-1,1] \times [-1,1]$
  - $\mathbb{E}[X_i] = \mathbb{E}\left[\mathbb{1}_{D(0,R=1)}(U_i)\right]$

    $= \iint_{D(0,R=1)} \frac{1}{4}\, dx\, dy = \frac{\pi}{4}$
  - for variance note $X_i^2 \equiv X_i$ since indicator variable !
  - so $var\left(\frac{\sum X_i}{N}\right)$

    $= \frac{1}{N^2} \sum_1^N var(X_i)$

    $= \frac{1}{N^2} \sum_1^N \left(\mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2\right)$

    $= \frac{N}{N^2} \left(\frac{\pi}{4} - (\frac{\pi}{4})^2\right)$