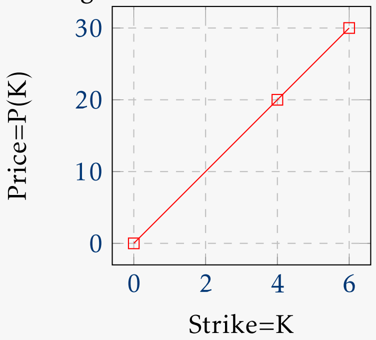
1 Introduction

- 2 put options , K=20,30 trading for \$4,\$6 - find arbitrage
  - 2x6-3x4=0, so try $2(20-S)^+ - 3(30-S)^+ > 0$
  - $\begin{cases} S \ge 30 \Rightarrow 0 \\ 20 \le S \le 30 \Rightarrow \text{-ve} \\ 0 \le S \le 20 \Rightarrow \text{-ve} \end{cases}$
  - 0 or -ve, so take opposite and done
  - graph (K,P(K)) not **strictly** convex ⇒ arbitrage



- 2^{29} has 9 digits all different - find missing digit
  - key: 9 divides [n - Sum(digits n)]
  - **Proof**: $n = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0$
  - $S(n) = a_n + a_{n-1} + \ldots + a_1 + a_0$
  - $n - S(n) = a_n(10^n - 1) + a_{n-1}(10^{n-1} - 1) + \ldots + a_1(10^1 - 1) + a_0 - a_0$
  - $10^k - 1$ = all 9s = divisible by 9. so QED
  - Sum(digits n) = $\sum_1^9 i - x = 45 - x$
  - so $9|(2^{29} - 45 - x)$
  - but $2^{29} = 2^5 \times 2^{6\times4} = 2^5 \times 64^4 = 2^5 \times (63+1)^4 = 2^5 \times (k \times 63) + 1 = k \times 63 \times 2^5 + 2^5 = 9\times$ something for some k
  - so $9|(2^{29} - 45 - x)$ and $9|(2^{29} - 2^5)$ so $9|(2^{29} - 45 - x) - (2^{29} - 2^5)$
  - $9|(45 - x - 2^5) = (45 - x - 32) = (13 - x)$ so $x = 4$
- $\int_0^T W_t dt$ - what is distribution ? is it martingale ?
  - $X_T = \int_0^T W_t dt \Leftrightarrow dX_t = W_t dt + 0.dW_t$ = only drift so not martingale
  - $dX_t = W_t dt \Leftrightarrow X_T = \int_0^T W_t dt \Leftrightarrow X_T = [W_t t]_0^T - \int_0^T t dW_t = TW_T - \int_0^T t dW_t = T\int_0^T dW_t - \int_0^T t dW_t = \int_0^T (T-t) dW_t$
  - Recall stochastic integral $\int_0^T f(t) dW_t \sim N(0, \int_0^T |f(t)|^2 dt)$
  - $X_T = \int_0^T (T-t) dW_t \sim N(0, \int_0^T |T-t|^2) dt = N(0, \frac{T^3}{3})$
- Alice sends 20 ants towards Bob in straight line, Bob 50 towards Alice in straight line. Ants collide and go back. How many reach Bob? Alice ? How many collisions ?
  - imagine ants carry a flag and pass it on. so 50 reach Alice and 20 reach Bob.
  - num collisions: each of Bob's 50 flags must go though 20 collisions to reach Alice, each of Alice's 20 flags goes through 50 collisions to reach Bob: 2x 50 x 20 = 2000 collisions
- find all p such that this is a collrelation matrix $\Omega = \begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & p \\ -0.3 & p & 1 \end{bmatrix}$
  - Ω correlation matrix ⇔ Ω symmetric positive definite
  - **Sylvester**, **Cholesky** , **definition**
  - **Sylvester** criterion: all the principal minor determinants must be >0 (principal minor: remove the same rows and same column indices form a matrix)
    1. remove 2,3 → $\det(1) = 1$, remove 1,3 → $\det(1) = 1$, remove 1,2 → $\det(1) = 1$
    2. remove 3 → $\det\begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} = 0.64$
    3. remove 2 → $\det\begin{bmatrix} 1 & -0.3 \\ -0.3 & 1 \end{bmatrix} = 0.91$
    4. remove 1 → $\det\begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} = 1 - p^2$
    5. remove $\emptyset$ → $\det\Omega = \det\begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & p \\ -0.3 & p & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.6 & -0.3 & 1 & 0.6 \\ 0.6 & 1 & p & 0.6 & 1 \\ -0.3 & p & 1 & -0.3 & p \end{bmatrix} \rightarrow$ + down - up → $1 - 0.18p - 0.18p - 0.09 - p^2 - 0.36 = 0.55 - 0.36p - p^2$
  - $\begin{cases} 1 - p^2 > 0 \\ 0.55 - 0.36p - p^2 > 0 \end{cases}$
- how many draws N of U[0,1] such that $P(0.7 < U[0,1] < 0.72) \ge 0.95$ for one $U[0,1]$
  - P(none of the N rv in $[0.7, 0.72]$)=0.98^N → P(at least one in $[0.7, 0.72]$) = $1 - 0.98^N$ and solve $1 - 0.98^N \ge 0.95$
- prove pdf of N(0,1) is : $\int$ pdf $= 1$
  - $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\frac{x^2}{2}) dx$ substitute $x^2/2 = t \to dx = \sqrt{(2)} dt$ → need prove $I := \int_{-\infty}^{+\infty} \exp(-t^2) dt = \sqrt{\pi}$
  - $I^2 = \int_{-\infty}^{+\infty} \exp(-x^2) dx \int_{-\infty}^{+\infty} \exp(-y^2) dy = \int_{-\infty}^{+\infty} \exp(-x^2 + y^2) dx dy$
  - $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$
  - $I^2 = \int_0^{+\infty} \int_0^{+2\pi} \exp(-r^2) r dr d\theta = 2\pi \int_0^{+\infty} \exp(-r^2) r dr = 2\pi [\frac{-1}{2}\exp(-r^2)]_0^{+\infty} = \pi \rightarrow I = \sqrt{\pi}$
- walk 1m south, 1m east, 1m north, then back at same point - where are you o Earth?
  - find latitute near south Pole such that round-the-world=1m. then any point 1m north of this works
  - also any latitude near south Pole such that round-the-world=$\frac{1}{k}$ works (go round the world k times but still end up same point)
- solve Oernstein-Uhlenbeck SDE - Vasicek for IR
  - $dr_t = \lambda(\theta - r_t) dt + \sigma dW_t \rightarrow dr + \lambda r_t dt = \lambda\theta dt + \sigma dW_t \overset{\times e^{\lambda t}}{\Rightarrow} e^{\lambda t}(dr + \lambda r_t dt) = e^{\lambda t}(\lambda\theta dt + \sigma dW_t) \rightarrow d(r_t e^{\lambda t}) = e^{\lambda t}(\lambda\theta dt + \sigma dW_t) \rightarrow [r_t e^{\lambda t}]_0^t = \int_0^t \lambda\theta e^{\lambda s} ds + \int_0^t \sigma e^{\lambda s} dW_s \rightarrow r_t e^{\lambda t} - r_0 = \theta(e^{\lambda t} - 1) + \sigma\int_0^t e^{\lambda s} dW_s \rightarrow r_t = e^{-\lambda t} r_0 + \theta(1 - e^{-\lambda t}) + \sigma\int_0^t e^{-\lambda(s-t)} dW_s$ and that is it.
  - note however $E(\int_0^t f(s) dW_s)$ = "sum of N(0,.)"=0 so $E(r_t) = e^{-\lambda t} r_0 + \theta(1 - e^{-\lambda t})$ which $\rightarrow \theta$ as $r \uparrow +\infty$ "mean reverting"

2 Calculus

- i^i? $e^{i\frac{\pi}{2}i} = e^{-\frac{\pi}{2}}$
- $\pi^e >< e^\pi$? try $\ln(\pi^e) = e\ln(\pi)$ vs $ln(e^\pi) = \pi\ln(e)$ ie $\frac{\ln(\pi)}{\pi}$ vs $\frac{\ln(e)}{e}$ ie check $f(x) = \frac{\ln(x)}{x}$ ie $f'(x) = (\frac{\ln(x)}{x})' = \frac{1}{x}\frac{1}{x} + \ln(x)\frac{-1}{x^2} = \frac{1 - \ln x}{x^2}$ ie $f'(x) < 0$ if $x > e$ and $f'(x) > 0$ if $x < e$ ie $f(x) \uparrow$ on $[0, e]$ and $f \downarrow$ on $[e, \ldots + \infty]$ and $e < \pi$ so $f(e) > f(\pi)$
- $\frac{e^x + e^y}{2} > e^{\frac{x+y}{2}}$? $e^{\frac{x+y}{2}} = \sqrt{e^{x+y}} = \sqrt{e^x}\sqrt{e^y}$ ie $a^2 + b^2 > 2ab$ ie $a^2 - 2ab + b^2 > 0$ which is true
- solve $x^6 = 64$ $2^6 = 64$ and $z^6 = 1 \Leftrightarrow z = e^{\frac{2ik\pi}{6}}, k \in 0 \ldots 5$ so $x = 2e^{\frac{2ik\pi}{6}}, k \in 0 \ldots 5$
- derivative of x^x ? $x^x = e^{x\ln(x)}$ so derivative = $g' \circ f.f'$ ie $e^{x\ln(x)}(\ln(x) + x\frac{1}{x}) = x^x(\ln(x) + 1)$
- compute $\sqrt{2 + \sqrt{2 + \sqrt{2 \ldots}}}$? $l = \sqrt{2 + l}$ ie $l^2 = l + 2$ ie $(l - 2)(l + 1) = 0$ ie $l = 2$ since $l > 0$ assuming limit exists. but $x_{n+1} = \sqrt{x_n + 2}$ is increasing (same reason) and bounded abve, so l exists and l=2
- find $2 = x^{x^{x^{\cdots}}}$ $2 = x^2$ so $x = \sqrt{2}$ if it exists. prove $\sqrt{2} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$ let $x_0 = \sqrt{(2)}, x_{n+1} = x_n^{\sqrt{2}}$. Prove sequence is increasing, bounded from above by 2.therefore has a limit. Therefore that limit must = $\sqrt{2}$
- which one converges ? $\sum\frac{1}{k}, \sum\frac{1}{k^2}, \sum\frac{1}{k\ln(k)}$
  1. $\sum\frac{1}{k^2} \le \sum\frac{1}{k(k-1)} = \sum\frac{1}{k} - \frac{1}{k-1} \le 1 - 1/n < 2$ so $\uparrow$, bounded ie converges.
  2. $\sum\frac{1}{k} > \ln(n) + \frac{1}{n}$ because $\int_1^n \frac{1}{x} dx$ $\sum_1^{n-1}\int_k^{k+1}\frac{1}{x} dx < \sum_1^{n-1}\int_k^{k+1}\frac{1}{k} dx = \sum_1^{n-1}\frac{1}{k}[x]_k^{k+1} = \sum_1^{n-1}\frac{1}{k} = \sum_1^n \frac{1}{k} - \frac{1}{n}$
  3. $\sum\frac{1}{k\ln(k)}$: similalry compare $\int_1^n \frac{1}{x\ln(x)} dx$ and note $\int\frac{1}{x\ln(x)} = \ln(\ln(x))$
- compute $\int\frac{1}{1+x^2} dx$? substitute $x = \tan(z) = \frac{\sin z}{\cos z}$ so $dx = \frac{1}{\cos^2 z} dz [= \cos z\frac{1}{\cos z} + \sin z\frac{-1}{\cos^2 z}(-\sin z) = 1 + \frac{\sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}]$ so $\int\frac{1}{1+x^2} dx = \int\frac{1}{1+\tan^2 z}\frac{1}{\cos^2 z} dz = \int\frac{1}{\cos^2 z + \sin^2 z} dz = \int dz = z + C = \arctan x + C$
- compute $\int x\ln(x) dx$? $\int x\ln(x) dx = [\frac{x^2}{2}\ln x] - \int\frac{x^2}{2}\frac{1}{x}$
- compute $\int xe^x dx$? $\int xe^x dx = [xe^x] - \int e^x dx$
- compute $\int x^n \ln x\, dx$ $\int x^n \ln x\, dx = [\frac{x^{n+1}}{n+1}\ln x] - \int\frac{x^{n+1}}{n+1}\frac{1}{x} dx$
- compute $\int \ln^n x\, dx$? $\int \ln^n x dx = [x\ln^n x] - \int x n \ln^{n-1}(x)\frac{1}{x} dx = [x\ln^n x] - n\int \ln^{n-1}(x) dx$ so recursion $f_n(x) = [x\ln^n x] - nf_{n-1}(x)$
- solve $y'' - 4y' + 4y = 1$ general solution $y'' - 4y' + 4y = 0 \rightarrow z^2 - 4z + 4 = 0 \rightarrow (z-2)^2 = 0 \rightarrow y = C_1 e^{2x} + C_2 x e^{2x}$ + particular solution $y = \frac{1}{4}$
- solve $y' = y(1-y)$ $y' = y(1-y) \rightarrow \frac{y'}{y(1-y)} = 1 \rightarrow \int\frac{y'}{y(1-y)} dx = \int 1 dx \overset{y'=dy/dx}{\Rightarrow} \int\frac{dy}{y(1-y)} = \int 1 dx$ with $\int\frac{dy}{y(1-y)} = \int(\frac{1}{y} + \frac{1}{y-1}) dy$ so $\ln y -$

$\ln(1-y) = ln(\frac{y}{1-y}) = x + C$ so $\frac{y}{1-y} = e^{x+C}$

- derive BS PDE set $\Pi = V - \frac{\partial V}{\partial S}S$ so $d\Pi = dV - \frac{\partial V}{\partial S}dS$ and $\frac{dS}{S} = \mu dt + \sigma dW$ and $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$ and $(dS)^2 = \sigma^2 S^2 dt$ so $dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial V}{\partial S}dS$ so $d\Pi$
  $= (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt \overset{\text{risk free growth}}{=}$
  $= r\Pi dt$
  $= r(V - \frac{\partial V}{\partial S}S)dt$ so $(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)$
  $= r(V - \frac{\partial V}{\partial S}S)$
  rearrange $\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + r\frac{\partial V}{\partial S}S - rV = 0$

3 Linear algebra

- cov Σ_X & corr Ω_X matrices are +ve sem.-def
  - $\Sigma(j,k) = cov(X_j, X_k) = cov(X_k, X_j) = \Sigma_X(k,j)$ and $\Omega(j,k) = corr(X_j, X_k) = corr(X_k, X_j) = \Omega(k,j)$. Also $(\star) var(\sum_1^n c_i X_i) = C^T \Sigma_X C$ with C= $\begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix}$ and var(.)>= 0 so Σ_X is positive, and symmetric (see above), so +ve semi-definite
  - Proof of $(\star)$: let $Y = \sum_1^n c_i X_i$ then $Y - E[Y] = \sum_1^n c_i(X_i - \mu_i)$ and Var(Y)
    $= E[Y - E(Y)]^2$
    $= E[\sum_1^n c_i(X_i - \mu_i)^2]$
    $= E[\sum_{1\le j,k \le n} c_j c_k (X_j - \mu_j)(X_k - \mu_k)]$
    $= \sum_{1\le j,k \le n} c_j c_k E[(X_j - \mu_j)(X_k - \mu_k)]$
    $= \sum_{1\le j,k \le n} c_j c_k cov(X_j, X_k)$
    $= \sum_{1\le j,k \le n} c_j c_k \Sigma_X(j,k)$
    $= C^T \Sigma_X C$
  - For correlation: $\Sigma_X = D_{\sigma_X}\Omega_X D_{\sigma_X}$ where $D_{\sigma_X} = Diag(\sigma_i), \sigma_i^2 = var(X_i)\ , i \in 1..n$ and $w^T\Sigma_X w = w^T D_{\sigma_X}\Omega_X D_{\sigma_X} w = v^T D_{\sigma_X}^{-1} D_{\sigma_X}\Omega_X D_{\sigma_X} D_{\sigma_X}^{-1} v = v^T \Omega_X v \ge 0$ so correlation Ω_X positive semi-definite.
- Find correlation Ω for covariance matrix $\Sigma = \begin{bmatrix} 1 & 0.36 & -1.144 \\ 0.36 & 4 & 0.8 \\ -1.44 & 0.8 & 9 \end{bmatrix}$
  $\Sigma_X = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}\Omega_X\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$
  $\Omega_X = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3} \end{bmatrix}\Sigma_X\begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3} \end{bmatrix}$ with $\sigma_1 = \sqrt{\Sigma(1,1)} = 1, \sigma_2 = \sqrt{\Sigma(2,2)} = 2, \sigma_3 = \sqrt{\Sigma(3,3)} = 3$
- Find allowable p for $\Omega = \begin{bmatrix} 1 & p & \dots & p \\ p & 1 & \ddots & p \\ \vdots & \ddots & \ddots & p \\ p & p & p & 1 \\ p & p & p & 1 \end{bmatrix}$
  - Find eigenvalues of Ω then state that all e.v must be +ve to find the condition on p
  - $\Omega = (1-p)I + pM$ with I = identity. M=all 1s. But ev of M are easy to find : $\mathbf{Mv} = \lambda\mathbf{v} \to v_1 + v_2 + \ldots + v_n = \lambda v_1 = \lambda v_2 = \ldots = \lambda v_n$ so $\begin{cases} \lambda = 0 \\ v_1 = v_2 = v_n \to v_1 + v_2 + v_n = nv_1 = \lambda v_1 \end{cases}$ so $\lambda = 0, n$ and for $\Omega v = (1-p)Iv + pMv = (1-p)v + p\lambda v = (1 - p + p\lambda)v$ so $\begin{cases} \lambda = 0 \to 1 - p \ge 0 \\ \lambda = n \to 1 - p + pn \ge 0 \end{cases}$ so $\frac{1}{1-n} \le p \le 1$
- prove nxn matrix has n eigenvalues
  - $Av = \lambda v \Leftrightarrow (\lambda I - A)v = 0 \Leftrightarrow (\lambda I - A)$ singular $\Leftrightarrow det(\lambda I - A) = 0 \Leftrightarrow P_A(\lambda) = 0$ so n roots, so n eigenvalues but some may be the same
  - An eigenvalue of multiplicity m has at least 1 eigenvector and at most m linearly independent eigenvectors.
- find $X^2 = A$ and $YY^T = A$ for $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$
  - A symmetric ! $\Leftrightarrow A = O^T\Lambda O$ with $\Lambda = Diag(evals)$ and O orthogonal and made up of the evectors ie
    $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, O = (v_1 v_2)$,
    $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \|v_1\| = 1, \|v_2\| = 1$
  - $X = O^T\Lambda^{\frac{1}{2}}O$, $O^TO = I$ as O=orthogonal
  - A symmetric +ve definite $\Leftrightarrow$ Cholesky $\Leftrightarrow A = U^TU$ with $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & 0 \end{bmatrix}$ so $Y = U^T$
  - find the evals :
    $det(\lambda I - A) = det\begin{bmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{bmatrix} = 0$
  - find the evecs with $\|.\| = 1$
  - find the Cholesky U by solving linear equation $\begin{bmatrix} u_{11} & 0 \\ u_{12} & u_{22} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = A$
- if $\lambda_1 = 2, \lambda_2 = -3, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$
  find $Av, v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
  - find $c_1, c_2 : v = c_1v_1 + c_2v_2$ (solve equation)
  - $Av = c_1Av_1 + c_2Av_2 = c_1\lambda_1v_1 + c_2\lambda_2v_2$
- show $trace(AB) = trace(BA)$ for $A, B = nxn$
  - $(\star)P_{AB}(\lambda) = P_{BA}(\lambda) = det(\lambda I - AB) = det(\lambda I - BA)$
  - $P_M(\lambda) = det(\lambda I - M) = \lambda^n - trace(M) + \ldots + (-1)^n det(M)$
  - so $trace(AB) = trace(BA)$
  - Proof $\star$:
    $det(\lambda I - AB)$
    $= det(\lambda IB^{-1}B - B^{-1}BAB)$
    $= det(B^{-1})det(\lambda IB - BAB)$
    $= det(B^{-1})det(\lambda I - BA)det(B)$
    $= det(B^{-1})det(B)det(\lambda I - BA)$
    $= det(B^{-1}B)det(\lambda I - BA)$
    $= det(\lambda I - BA)$
  - for singular B, use $B - \epsilon I$ which is non-singular apart from finite $\epsilon = evalue$ so let $\epsilon \to 0$
- solve $AB - BA - I = 0$ for $A, B = nxn$
  - no solution since $trace(AB - BA) = trace(AB) - trace(BA) = 0$
- show A, B prob. Mat $\Rightarrow AB$ prob. Mat
  - A prob. Mat := sum of rows ==1 $\Leftrightarrow A\mathbb{1} = \mathbb{1}$ with $\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
  - $(AB)\mathbb{1} = A(B\mathbb{1}) = A\mathbb{1} = \mathbb{1}$
- all $\rho : \Omega := \begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{bmatrix}$ = corr matrix
  - Sylvester, Cholesky, Definition
  - **Sylvester**: all the principal minors must be >0 (princicipal minors: remove same row and same column, compute determinant) – see above
  - **Cholesky**: the determinant of the 2x2 matrix M in the first step of the Cholesky algorithm must be ≥ 0 :
    $M = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} 0.6 \\ -0.3 \end{bmatrix}\begin{bmatrix} 0.6 & -0.3 \end{bmatrix}$
    $= \begin{bmatrix} 0.64 & \rho + 0.18 \\ \rho + 0.18 & 0.91 \end{bmatrix}$
    $\det M = \det\begin{bmatrix} 0.64 & \rho + 0.18 \\ \rho + 0.18 & 0.91 \end{bmatrix} \ge 0$
  - **Definition** $x^t\Omega x \ge 0 \quad \forall x$ ie $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ge 0$ ie $x_1^2 + x_2^2 + x_3^2 + 1.2x_1x_2 - 0.6x_1x_3 + 2\rho x_1x_3 \ge 0$ then complete squares $x_1^2 + x_2^2 + x_3^2 + 1.2x_1x_2 - 0.6x_1x_3 + 2\rho x_1x_3 = x_1^2 + 1.2x_1x_2 - 0.6x_1x_3 + x_2^2 + x_3^2 + 2\rho x_1x_3 = x_1^2 + 2x_1(0.6x_2 - 0.3x_3) + x_2^2 + x_3^2 + 2\rho x_1x_3 = (x_1 + 0.6x_2 - 0.3x_3)^2 - (0.6x_2 - 0.3x_3)^2 + +x_2^2 + x_3^2 + 2\rho x_1x_3 = (x_1 + 0.6x_2 - 0.3x_3)^2 + 0.64x_2^2 + 2x_2x_3(\rho + 0.18) + 0.91x_3^2$ and complete last sqaure again same method

4 Financial instruments

- 3 put options (strike=K,Price(K))= (40,10),(50,20),(70,30). Is there arbitrageif so build it
  - graph $K \mapsto Price(K)$ must be **strictly convex** else arbitrage
  - line through (40,10) and (70,30) : $C : K \mapsto \frac{70-K}{30}10 + \frac{K-40}{30}30 = 10 + \frac{x-40}{70-40}(30-10) = 10 + \frac{30-10}{70-40}(x - 40)$
  - on that line $P(50) = 20 > C(50) = \frac{70-50}{30}10 + \frac{50-40}{30}30 = 50/3$ so not convex (convex:curve must be below line) so arbitrage exists
  - construct arbitrage : find porfolio in the puts so that 1. V(0) < 0, 2. $V(T) > 0\ \forall T$ with $V(T) =$
  - try $2x10 + 30 - 3x20 = 2P_{40} + P_{50} - 3P_{70} = 2(40-K)^+ + (50-K)^+ - 3(70-K)^+$
- Price of a stock now P=50. In 3m, either P=47 or P=52 with prob 50-50. How much for ATM put ? Assume no dividends and Interest rate = 0
  - real world probability irrelevant
  - standard solutions follows:
  - $P(0) = p_{up}P_{up} + p_{down}P_{down}$
    $= pP_{up} + (1-p)P_{down}$
    $= puP(0) + (1-p)dP(0)$
  - $P(0) == puP(0) + (1-p)dP(0)$
  - $1 = pu + (1-p)d$ with $u = \frac{P_{up}}{P(0)} = \frac{52}{50}, d = \frac{47}{50}$
  - $1 = p(u-d) + d \Rightarrow p = \frac{1-d}{u-d} = \ldots = 0.6$
  - $O(0) = pO_{up} + (1-p)O_{down}$
  - (here) O=ATM put , ATM means strike=Price now = 50
  - $O(0) = p0 + (1-p)3 = 0.6 \times 0 + 0.4 \times 3 = 1.2$
  - alternative solution follows:
  - Set up portfolio $\Pi = +1$ Option $+ (-1)\Delta_{Option}$ Stock with $\Delta_{Option} = \frac{O_{up} - O_{down}}{S_{up} - S_{down}} =$ (here) $\frac{0-3}{52-47} = 0.6$
  - (here) $\Pi = +1$ Option $+ (-1)(-0.6)$ Stock

- $\Pi(T) = \begin{cases} S(T)=52 \Rightarrow 0+0.6\times 52=31.2 \\ S(T)=47\Rightarrow 3+0.6\times 47 = 31.2\end{cases}$
- $\Pi(0)$ = discounted $\Pi(T) \overset{IR=0}{=} \Pi(T) = 31.2 = O(0)+0.6\times S(0) = O(0)+0.6\times 50 \Rightarrow O(0)=31.2-0.6\times 50 = 31.2-30=1.2$
- What is risk neutral pricing ?
- $V(0)=\mathbb{E}\left[e^{-rT}V(S(T))\right]$
- with $S(T)=S(0)e^{(r-q-\frac{\sigma^2}{2})T+\sigma\sqrt{T}N(0,1)}$
- not OK for path-dependent
- How to derive BS ?
- 12 possible ways actually ...
- risk neutral pricing
- BS PDE. .
  - Payoff = boundary conditions
  - Transform to heat equation
- binomial tree, with calibration:
  - drift = risk free rate
  - terminal dist = lognormal when num time steps $\to\infty$
- Approximate formula for ATM put ?
- $\text{Put}_{ATM} \approx 0.4 S_0\sigma\sqrt{T}$ when total variance $=\sigma^2T$ small
- if the price of a stock doubles, how does call option change ?
- depends if call option ITM, ATM, OTM
- ITM:
  - $C-P=S-K$ $\looparrowleft$ Put-call parity
  - $C+Ke^{-rT}=S\Rightarrow C=S-Ke^{-rT}$ so $C\times 2$
- ATM: call option $\to$ ITM, $C\uparrow$ $(\times 10)$
- OTM: call option $\to$ ATM, $C\uparrow\uparrow\uparrow$ $(\times 10^n)$
- what are the possible values of Delta of an option ?
- Call option:
  - long Call: [0 (OTM) ... 1 (ITM)]
- Put option:
  - $C+P=S$ $\looparrowleft$ Put-Call parity
  - $\Delta_C+\Delta_P=1$
- what is Delta of +1 ATM Call ? what is Delta of +1 ATM Put ?
- Delta of +1 ATM Call = 0.5
- Delta of +1 ATM Put = −0.5
- $\Delta_{Call}=N(d_1)$ with $d_1=\frac{\ln\frac{S}{K}+(r-d+\frac{\sigma^2}{2})}{\sigma\sqrt{T}}$
  - ATM $\to K=S$, assume r=q=0 $\to d_1=\frac{\sigma\sqrt{T}}{2}$
  - $\Delta_{Call}=N(d_1)=N(0.5\sigma\sqrt{T})$
  - $0.5\sigma\sqrt{T}$ normaly small eg for $\sigma=0.5, T=1\Rightarrow 0.5\times 0.5\sqrt{1}=0.25$
  - $N(x)$ about 0 is $N(x)=0.5+\frac{x}{\sqrt{2\pi}}$
  - so $\Delta_{Call}=N(d1)\approx 0.5$ since $d_1=0.5\sigma\sqrt{T}$ small
- $\Delta_{Put}=-N(-d_1)$
  - so $\Delta_{Put}=-N(-d_1)\approx -0.5$
- $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{\frac{-z^2}{2}}dz = 0.5+\frac{1}{\sqrt{2\pi}}\int_0^x e^{\frac{-z^2}{2}}dz = 0.5+\frac{1}{\sqrt{2\pi}}\int_0^x[1+...]dz\approx 0.5+\frac{1}{\sqrt{2\pi}}x$
- what is Put-Call parity ?
- Call - Put = Forward
- $(S-K)^+-(K-S)^+=S-K$
- $C(0)-P(0)=[S(0)e^{rT}-K]e^{-rT}$
- $C(0)-P(0)=S(0)-Ke^{-rT}$
- $C(t)-P(t)=S(t)-Ke^{-r(T-t)}$
- Show that **the time value of an option** is highest ATM
- time value of call option := call option value - intrinsic value := $C(t)-(S_t-K)^+$
- at any time t, for which value of S is this highest ? $S=K$?
- fix t, define: $f(S)=C(S)-(S-K)^+$
- $\begin{cases}S\le K\Rightarrow f(S)=C(S)\\ S>K\Rightarrow f(S)=C(S)-(S-K)\end{cases}$
- $\begin{cases}S\le K\Rightarrow \text{call option so}\uparrow\text{with S}\\ S>K\Rightarrow f'(S)=\Delta_C-1<0\end{cases}$
- so max at $S=K$
- What is implied volatility ? volatility smile ? volatility skew ?
- implied volatility = the σ : $BS(..,\sigma)$ = observed price. Unique σ because $\sigma\uparrow\Rightarrow BS(..,\sigma)\uparrow$
- Volatility smile $\smile$: $\sigma_{ITM}>\sigma_{ATM}<\sigma_{OTM}$ [currency options]
- Volatility skew $\frown$: $\sigma_{ITM}<\sigma_{ATM}>\sigma_{OTM}$ [index options, equity options,com. options]
- What is the Gamma of an option ? Why is it better to have small Gamma ? Why is Gamma of plain vanilla options positive ?
- Gamma $\Gamma=\frac{\partial^2 V}{\partial S^2}$
- small Gamma $\Rightarrow$ Delta does not change quickly $\Rightarrow$ easier to keep Delta-neutral
- Delta of Call/Put $\uparrow$ with $\uparrow$ S , hence Gamma +ve
- Gamma $\approxeq 0$ for OTM / deep ITM, Gamma max ATM
- When is Call = Put ?
- $C_0-P_0=(S_0e^{rT}-K)e^{-rT}=S_0-Ke^{-rT}=0\Leftrightarrow K=S_0e^{rT}$
- so $C_0=P_0\Rightarrow$ K = forward value of asset
- What is 2-year volatility of an asset with 30 % 6m volatility ?
- $\sigma(t)=\sigma_{1Y}\sqrt{T}$
- $\sigma(6m)=\sigma_{1Y}\sqrt{0.5}$
- $\sigma(2y)=\sigma_{1Y}\sqrt{2}=\sigma(6m)\frac{\sqrt{2}}{\sqrt{0.5}}$
- Value fix/float swap ?
- PV = fix - float (say)
- fix pv = sum of future flows
- float = right after payment, sumof all flows = 1 (notional)
- so float = pv of next payment + pv of (1) at next payment date
- example: 6m swap, N=10M, K=3%, next payment in 1m, then 7m, 13m, 19m, with last reset saying next flt payment = 125k
  - fix cpn $=Nr\tau=10M\times 0.03\times 0.5=150k$ with τ = year fraction
  - fix pv = $150k\times df(\frac{1}{12})+150k\times df(\frac{7}{12})+150k\times df(\frac{13}{12})+(10M+150k)\times df(\frac{19}{12})$
  - float pv = $(10M+125k)\times df(\frac{1}{12})$
  - example df for semi-annual LIBOR = $df(t)=[(1+L\times 0.5)\times(1+L\times 0.5)]^{-t}$ $\looparrowleft$ start with 1, keep on re-investing. Divide 1/(what you get) to get the df = price of zero coupon delivering 1 at T
- Price change of a 10y ZC bond if yield increases by 10bp ?
- Duration: $\Delta B=\frac{\partial B}{\partial y}dy=-\overbrace{\left(-\frac{1}{B}\frac{\partial B}{\partial y}\right)}^{Duration}B\,dy:=-DB\,dy$
- so $\frac{\Delta B}{B}=-D\,dy$
- (here)
  - 10y ZC $\Rightarrow$ D=10
  - $dy=10bps=10\times 1e-4=1e-3$
  - $\frac{\Delta B}{B}=-D\,dy=-10\times 1e-3=-0.01$
  - price $\downarrow$ 10%
- A 5y ZC bond with Duration D=3.5y has P=102. What is P if yield $\downarrow$ 50bp ?
- $\frac{\Delta B}{B}=-D\,dy$
- (here)
  - $D=3.5$
  - $dy=-50bp=-5\times 1e-3$
  - $B=102$
  - $\Delta B=-3.5\times 102\times(-5\times 1e-3)\approxeq +1.785$
  - $B_{new}\approxeq B+1.785=102+1.785=103.785$
- What is a forward contract ?
- (long position) agrees to buy an asset at a specific price at specific time in future
- (short position) agrees to sell an asset at a specific price at specific time in future
- that specific price is called the forward price
- at T=0 the value of the forward contact is 0 $\looparrowleft$ obv price of the forward contract $\neq$ forward price !! (completely unrelated quantities)
- Forward Price= $F=S_0e^{(r-q)T}$ = price of the asset in the future
- Forward price of Treasury FUTURES contract vs Forward price of a Commodity FUTURES contract ?
- For Treasury (ie bond) FUTURES contract , price now includes the bond flows between now and T
- But you won't receive these when you get the bond, so for you, price of bond must exclude them
- ie Forward price = F_{bond} $=\left\{PV_{now}(\text{Bond})-PV_{now}(\text{Coupons})\right\}e^{rT}$
- For Com future, situation is reversed: when you get the Com, you will not have incurred storage costs etc
- ie Forward price = F_{com} $=\left\{PV_{now}(\text{Com})+PV_{now}(\text{Storage})\right\}e^{rT}$
- Difference between Future and Forward ?
- daily settlement for futures, not so for forward
- Standardised maturities, contracts for futures, not so for forwards
- Exchange involved for futures (reduced CPTY risk), not so for forwards (OTC)
- Range of delivery dates for (COM) futures, not so for forwards
- Forwards can be cash-settled, even for COMs
- 10-day VAR @ 99% of a portfolio with 5-day VAR @ 95% = USD 100M ?
- $VAR(N,C)=\sigma_{V,1Y}P(Z\le C)\sqrt{\frac{N}{252}}V(0)$ with
  - N = time horizon eg N=10-days
  - P = P for Normal distribution
  - C = confidence level = eg 95 % = 0.95
  - $\sigma_{V,1Y}$ = annualised std dev of the portfolio PNL
  - $V(0)$ = PV now of portfolio
- eliminate the $\sigma_{V,1Y}V(0)$ by division ...

5 C++

- declare an array
  - `T foo [3]`
  - `T bar [] = {1,2,3}`
  - `int T* baz = new int[3]`
- declare an array of pointers

- `T* foo [3]`
- `T* bar [] = {&a,&b,&c}`
- `int T** baz = new int* [3]`

• const pointer to object, const pointer to const object etc
- Remember the clockwise and backward rule
- `T* bar [10]; //` array of 10 pointers to T
- `char const *chptr;` // pointer to a const char == pointer to a read-only char
- `char * const chptr;` // const pointer to a char == read-only pointer to a char
- `char const * const chptr;` // const pointer to a const char == read-only pointer to a read-only char

• template functions

```
template typename <T>
T temp_sum (T a, T b) return {a+b;}
```

// accepts 2 Ts, return a+b , which is also a T

• find max subarray of +ve numbers– check Kadane algorithm

```
max_ending += a[i];
max_ending_hr = max(0,max_ending_hr)
max_so_far = max(0,max_ending_here)
```

// keep track of largest so far

• Reverse a linked list

```
max_ending += a[i];
max_ending_hr = max(0,max_ending_hr)
max_so_far = max(0,max_ending_here)
```

// keep track of largest so far

6 MC simulation

• How to estimate π ? what is the variance of your method ?
- Acceptance-Rejection on πR^2 in an $2R\times 2R$ square with $R=1$
  - * $\frac{\pi 1^2}{2\times 2} = \frac{\pi}{4} = \frac{A}{N}$
  - * N = num points tried altogether
  - * A = num accepted points
- $X_i = \mathbb{1}_{D(0,R=1)}(U_i)$ with $U_i \overset{\text{uniform}}{=} [-1,1]\times[-1,1]$
- $\mathbb{E}[X_i] = \mathbb{E}\left[\mathbb{1}_{D(0,R=1)}(U_i)\right]$

$= \iint_{D(0,R=1)} \frac{1}{4}\,dx\,dy = \frac{\pi}{4}$
- for variance note $X_i^2 \equiv X_i$ since indicator variable !
- so $var\left(\frac{\sum X_i}{N}\right)$

$= \frac{1}{N^2}\sum_1^N var(X_i)$

$= \frac{1}{N^2}\sum_1^N \left(\mathbb{E}\left[X_i^2\right] - \mathbb{E}\left[X_i\right]^2\right)$

$= \frac{N}{N^2}\left(\frac{\pi}{4} - (\frac{\pi}{4})^2\right)$

• How to generate N(.,.) rv ?
- Acceptance rejection
- Box-Mueller
- Inverse Transform

• How to generate GBM stock path ?
- generate path for $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
- discretise $[0..T]$ into m steps of size $\delta t = \frac{T}{m}$
- let $t_j = j\delta t$
- $S_{t_{j+1}} - S_{t_j} = \int_{t_j}^{t_{j+1}} \mu S_t dt + \int_{t_j}^{t_{j+1}} \sigma S_t dW_t dt$
- Method 1: Simple method (but can end up with negative paths)
  - * $\int_{t_j}^{t_{j+1}} \mu S_t dt \approx \mu S_{t_j}(t_{j+1} - t_j) = \mu S_{t_j}\delta t$
  - * $\int_{t_j}^{t_{j+1}} \sigma S_t dW_t dt$
    $\approx \sigma S_{t_j}(W_{t_{j+1}} - W_{t_j})$
    $\approx \sigma S_{t_j}\sqrt{t_{j+1} - t_j}N(0,1)$
    $= \sigma S_{t_j}\sqrt{\delta t}N(0,1)$
  - * $S_{t_{j+1}} - S_{t_j} = \mu S_{t_j}(t_{j+1} - t_j) + \sigma S_{t_j}\sqrt{\delta t}N(0,1)$
  - * $S_{t_{j+1}} = S_{t_j}\left\{1 + \mu\delta t + \sigma\sqrt{\delta t}N(0,1)\right\}$
- Method 2: with Ito - No negative paths
  - * $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
  - * $d(\ln(S_t)) \overset{\text{Ito}}{=} (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$
  - * $\ln(S_{t_{j+1}}) - \ln(S_{t_j})$
    $= \ln\left(\frac{S_{t_{j+1}}}{S_{t_j}}\right)$
    $= (\mu - \frac{\sigma^2}{2})(t_{j+1} - t_j) + \sigma(W_{t_{j+1}} - W_{t_j})$
    $= (\mu - \frac{\sigma^2}{2})\delta t + \sigma\sqrt{\delta t}N(0,1)$
  - * $S_{t_{j+1}} = S_{t_j}\exp\left\{(\mu - \frac{\sigma^2}{2})\delta t + \sigma\sqrt{\delta t}N(0,1)\right\}$

• How to generate an rv N(0,1) from 12 rv $u_1,\ldots,u_1$ 2 with u_1=Uniform $[0,1]$?
- Answer: $X = \sum u_i - 6 \approx N(0,1)$
- Recall Central limit $Z \sim \frac{\frac{1}{N}\sum_1^N X_i - \mathbb{E}[X]}{\frac{\sigma(X)}{\sqrt{N}}}$
- $E(U) = \int_0^1 u\,du = \frac{1}{2}$
- $E(U^2) = \int_0^1 u^2 du = \frac{1}{3}$
- $\sigma^2(U) = E(U^2) = E(U)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$
- $Z \sim \frac{\frac{1}{12}\sum_1^{12} U_i - \frac{1}{2}}{\frac{\sqrt{\frac{1}{12}}}{\sqrt{12}}} = \sum U_i - 6$
- note: inefficient: requires 12 u_i for just 1 N(0,1). Box-Muller uses 2 for 1.

• Convergence rate of MC method ?
- m time steps, n simulations
- then convergence $\sim O\left\{\max(\frac{T}{m}, \frac{1}{\sqrt{n}})\right\}$

• What does reducing variance for MC ? how to do it ?
- it means: reduce the constant factor of the O(.)
- Methods:
  - * antithetic variables
  - * control variates
  - * moment matching

• How to generate correlated N(.,.) variables with correlation ρ ?
- given 2 uncorr N_1, N_2
- $X_1 = N_1$
- $X_2 = \rho N_1 + \sqrt{1-\rho^2}N_2$
- $corr(X_1, X_2) = corr(N_1, \rho N_1 + \sqrt{1-\rho^2}N_2) = corr(N_1, \rho N_1) + corr(N_1, \sqrt{1-\rho^2}N_2) = \rho corr(N_1, N_1) + \sqrt{1-\rho^2}corr(N_1, N_2) = \rho$

• What is Newton's method? How does it converge ?
- Newton=recursive method for finding x : $f(x) = 0$
- $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \looparrowleft f(x_{k+1}) - f(x_k) = (x_{k+1} - x_k)f'(x_k)$ with $f(x_{k+1}) = 0$
- Convergence rate : quadratic
- Proof
  - * let $f(x^*) = 0$
  - * Taylor: $f(x^*) = f(x_k) + (x^* - x_k)f'(x_k) + \frac{1}{2}(x^* - x_k)^2 f''(c_k)$
  - * $x_{k+1} - x_k = -\frac{f(x_k)}{f'(x_k)}$
  - * $x_{k+1} - x^*$
    $= x_k - x^* - \frac{f(x_k)}{f'(x_k)}$
    $= x_k - x^* - \frac{f(x_k) - f(x^*)}{f'(x_k)}$
    $= \frac{f(x^*) - f(x_k) + f'(x_k)(x^* - x_k)}{f'(x_k)}$
  - * substitute Taylor etc

• Finite difference method for Trinomial Tree?
- Forward Euler == Trinomial Tree

7 Probability

• What is exponential distribution ? what are its mean and variance ?
- pdf $= f(x) = \alpha e^{-\alpha x}, x \geq 0$
- mean $= E(x) = \int_0^\infty x f(x)dx = \int_0^\infty x\alpha e^{-\alpha x}dx$
- var $= E(x^2) - E(x)^2$
- by parts : $\int_0^\infty x e^{-\alpha x}dx = [x\frac{e^{-\alpha x}}{-\alpha}]_0^\infty - \int_0^\infty \frac{e^{-\alpha x}}{-\alpha}dx = 0 + \int_0^\infty \frac{e^{-\alpha x}}{\alpha}dx = \frac{1}{\alpha}[\frac{e^{-\alpha x}}{-\alpha}]_0^\infty = \frac{1}{\alpha^2}$
- by parts : $\int_0^\infty x^2 e^{-\alpha x}dx = [x^2\frac{e^{-\alpha x}}{-\alpha}]_0^\infty - \int_0^\infty 2x\frac{e^{-\alpha x}}{-\alpha}dx = 0 + \frac{2}{\alpha}\int_0^\infty x e^{-\alpha x}dx = \frac{2}{\alpha}\frac{1}{\alpha^2} = \frac{2}{\alpha^3}$