Introduction to Robotics

Manipulation and Programming

Unit 2: Kinematics

DISPLACEMENT VECTORS

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Displacement Vectors

- In the previous lab, we discussed how coordinate frames rotate relative to each other. The goal was to find the orientation of the end effector of a robot (gripper, paint brush, robotic hand, vacuum suction cup, etc.) relative to the base of the robot.
- However, when a robotic arm moves around in the world, orientation is just half the puzzle. The end effector changes position as well. To account for this change in the position of the end effector, we use what is called a **displacement vector**.

Displacement Vectors

- Prismatic Joint, Cartesian Manipulator
- Transformation of Coordinate Frames

Rotational Frame/Matrix

SECTION 1

Relative Rotational Matrix

The projection of

$$
R_n^m = \begin{array}{c c}
 & \begin{array}{ccc} x_n & y_n & z_n \end{array} \\
\begin{array}{c} x_m \\ y_m \\ z_m \end{array} & \begin{bmatrix} v_{xx} & v_{yx} & v_{zx} \\ v_{xy} & v_{yy} & v_{zy} \\ v_{xz} & v_{zx} & v_{zz} \end{bmatrix}
\end{array}
$$

Rotation frame n with respective to rotation frame m.

Displacement Vector

SECTION 2

Displacement Vector

- A vector is a list of numbers. In robotics, we typically use three numbers (all organized in a single column), to represent displacement (i.e. change in position) of one frame relative to another frame in the x, y, and z directions.
- We'll use the following notation to represent the displacement of coordinate frame n relative to coordinate frame m.

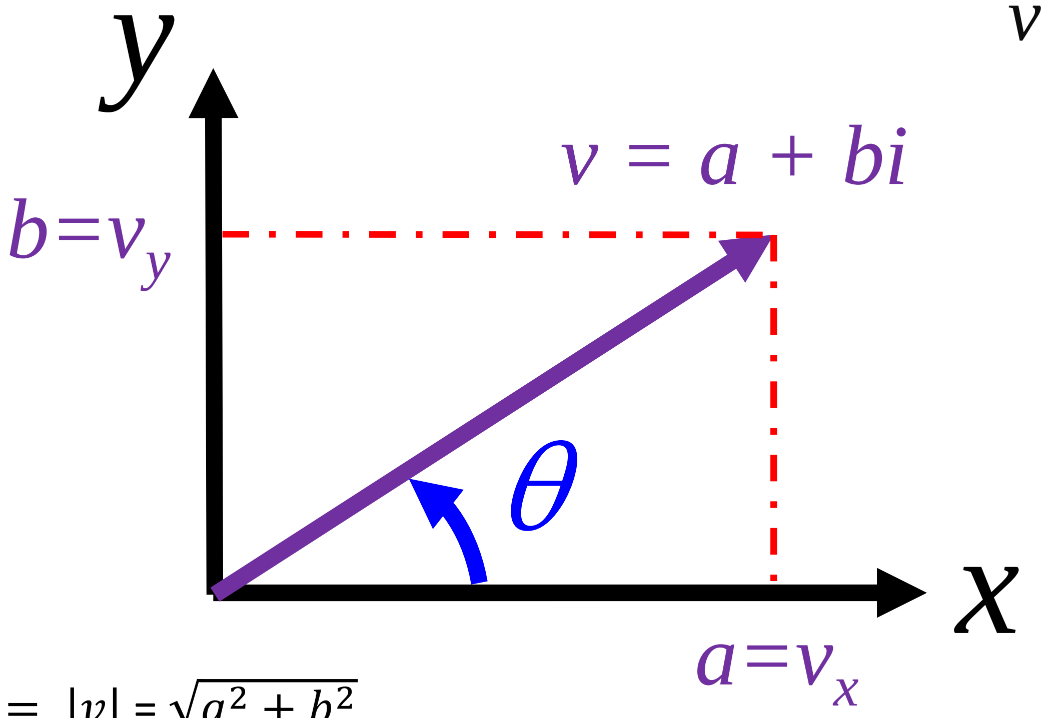
Displacement Vector

The Transition of

$$d_n^m = \begin{array}{cc} \begin{bmatrix} d_{xx} \\ d_{yy} \\ d_{zz} \end{bmatrix} & \begin{array}{c} x_n^m \\ y_n^m \\ z_n^m \end{array} \end{array}$$

Transitional displacement vector from frame n with respective to frame m.

Transition 2D



$r = |v| = \sqrt{a^2 + b^2}$
cos(θ) = a/r
sin(θ) = b/r

$$v = [v_x, v_y] = v_x \overrightarrow{e_x} + v_y \overrightarrow{e_y} =$$

$$= [v_x, v_y] \begin{bmatrix} \overrightarrow{e_x} \\ \overrightarrow{e_y} \end{bmatrix} = [v_x, v_y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [r\cos(\theta), r\sin(\theta)] = r\,[\cos(\theta), \sin(\theta)]$$

$$= r\cos(\theta) + i\,r\sin(\theta) = r\,(\cos(\theta) + i\sin(\theta))$$
$$= r\,e^{i\theta}$$

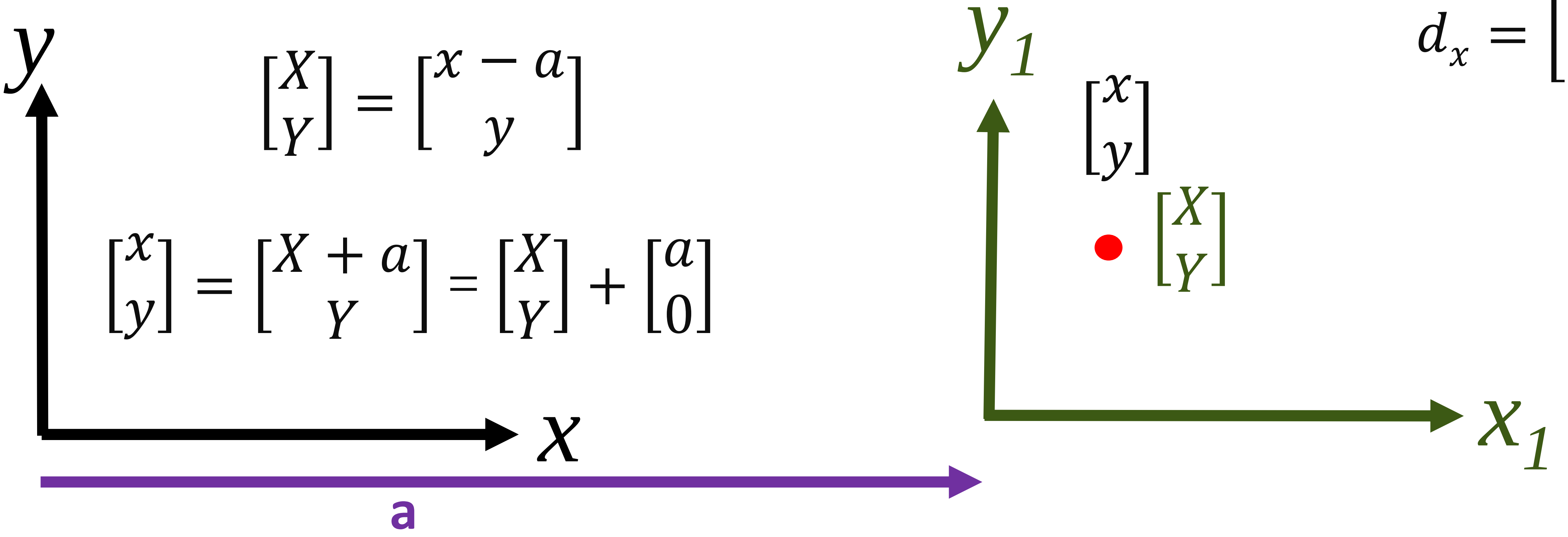
Transition

Vector displacement -a, or Frame displacement a

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x - a \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X + a \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$d_x = \begin{bmatrix} a \\ 0 \end{bmatrix}$$



Transition 3D

Vector displacement -a, or Frame displacement a
Object Rotational -θ, or Frame rotational θ

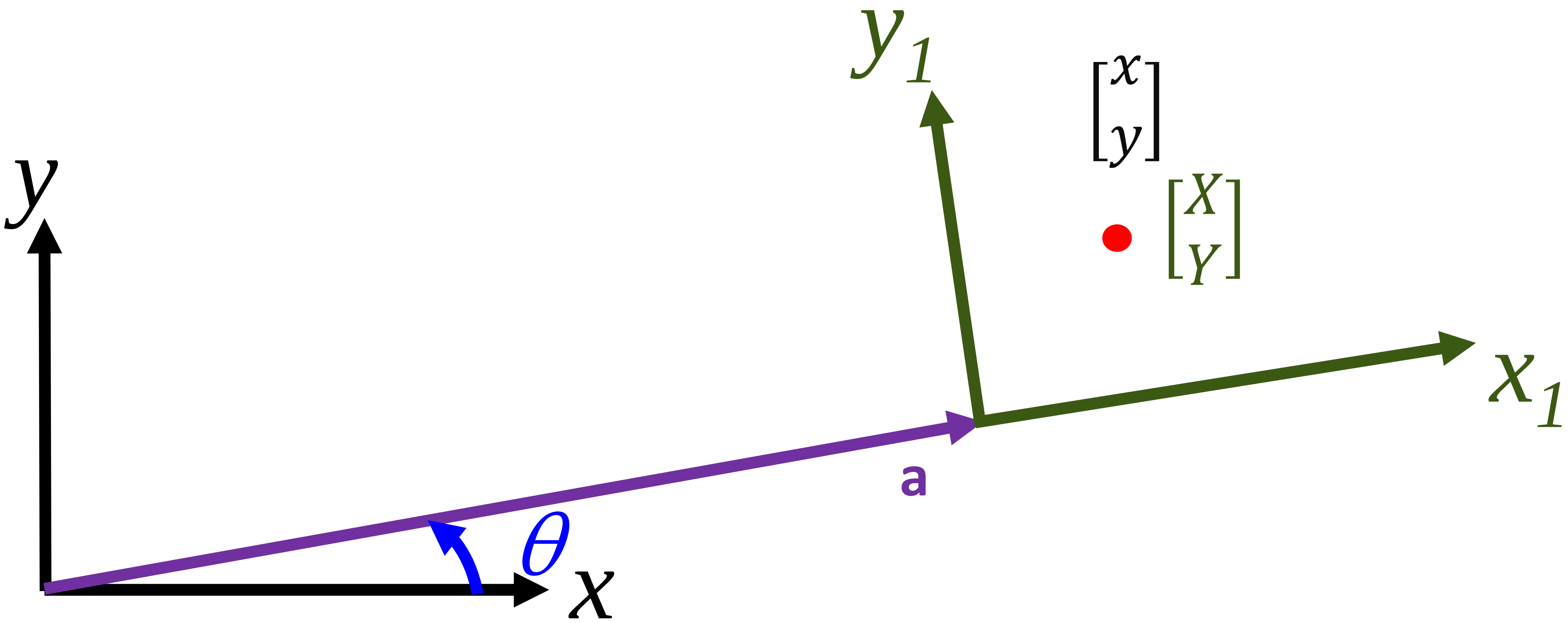
$$d_x = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad d_y = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \quad d_z = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

Combination of Rotation and Displacement

SECTION 3

Rotation and Transition

Vector displacement -a, or Frame displacement a



Rotation and Transition

Vector displacement -a, or Frame displacement a
Object Rotational -θ, or Frame rotational θ

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x - a \\ y \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (x-a)cos(\theta) - y\, sin(\theta) \\ (x-a)sin(\theta) + y\, cos(\theta) \end{bmatrix} = \begin{bmatrix} x\, cos(\theta) - y\, sin(\theta) \\ x\, sin(\theta) + y\, cos(\theta) \end{bmatrix} - \begin{bmatrix} a\cos(\theta) \\ a\sin(\theta) \end{bmatrix}$$

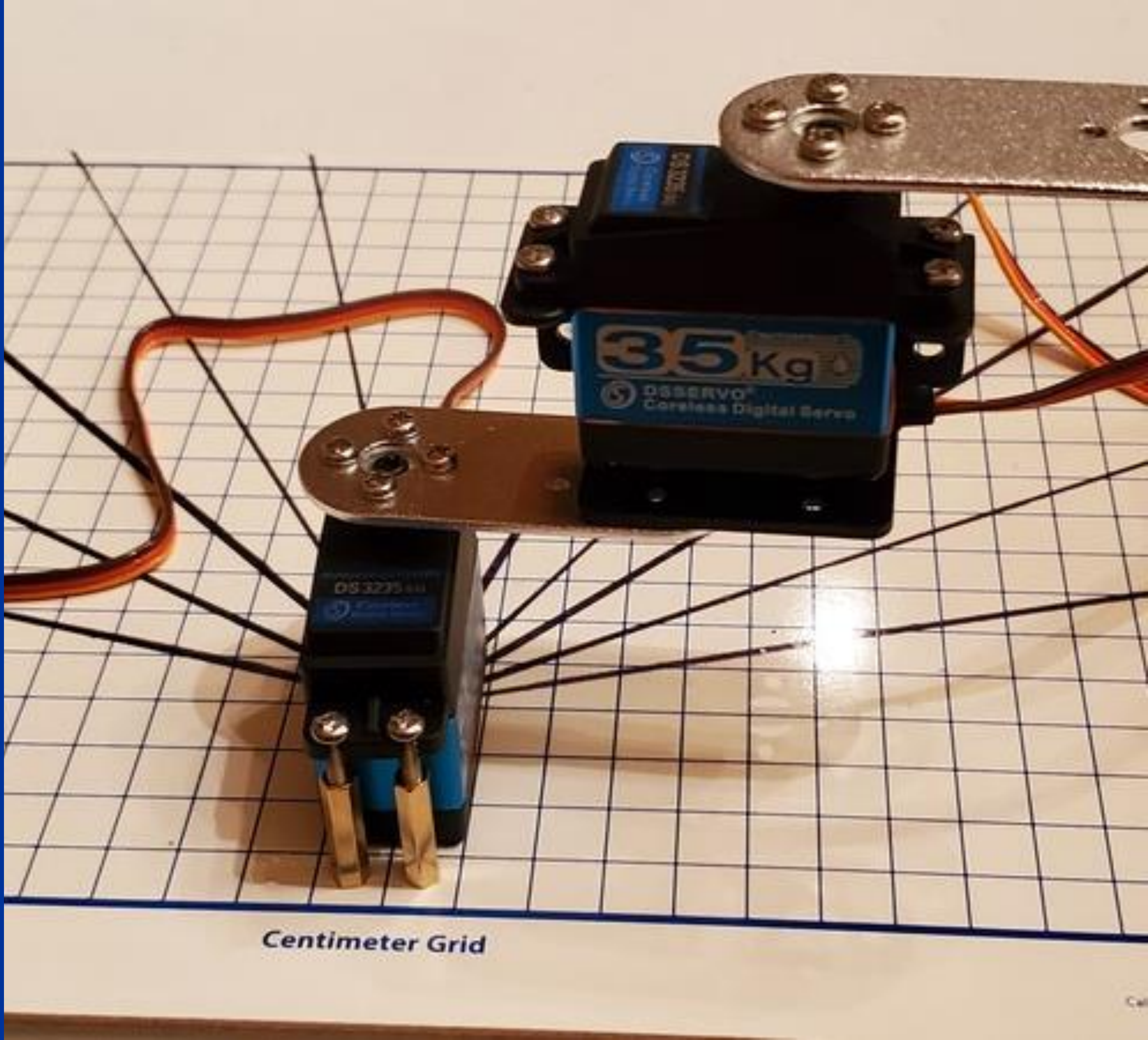
$$= \begin{bmatrix} x \\ y \end{bmatrix} R_{\theta} - a \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

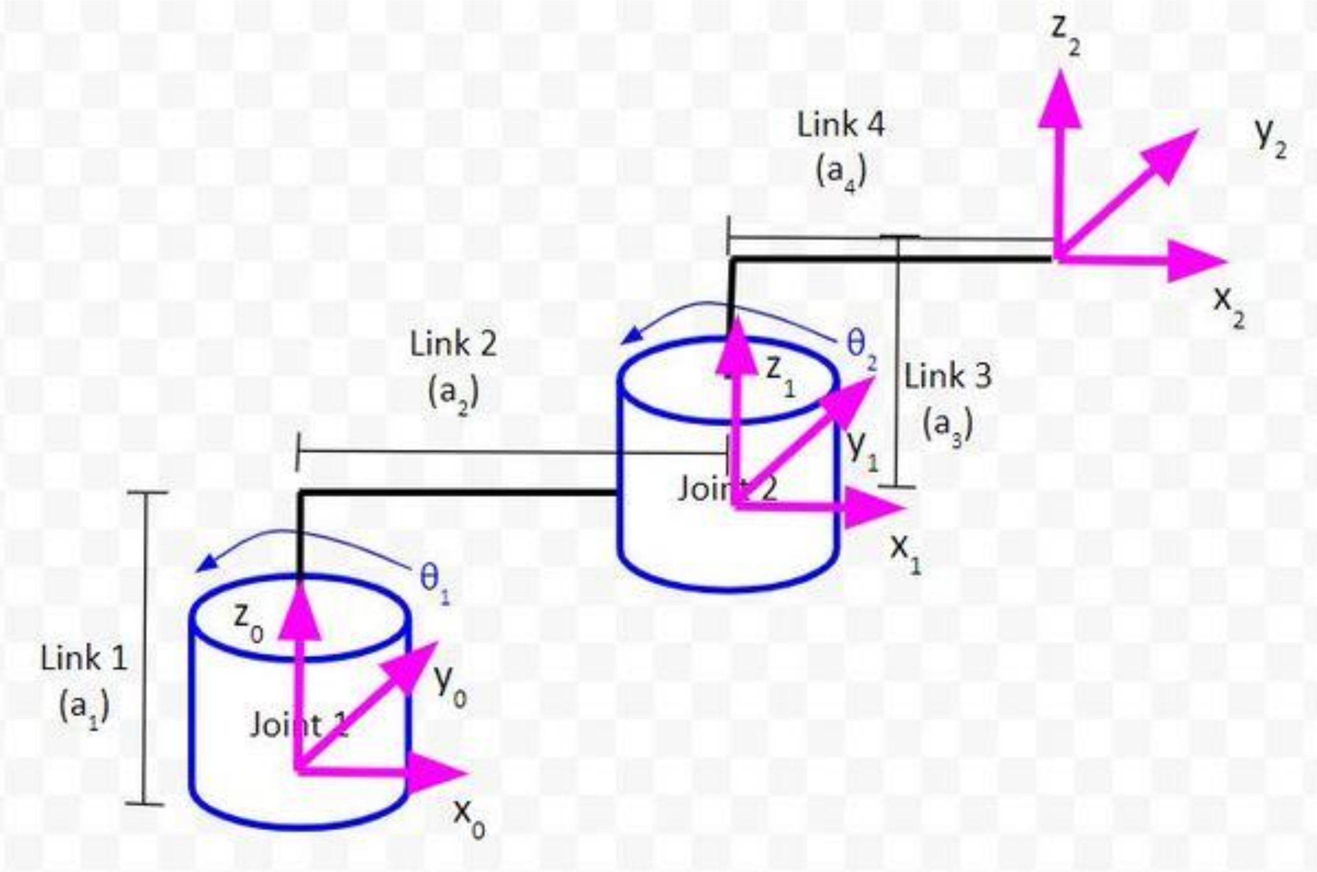
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R_{-\theta} + a \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Example - Two Degree of Freedom Robotic Arm

SECTION 4

Example 1: Two Degree of Freedom Robotic Arm





Kinematic Diagram

Rotation

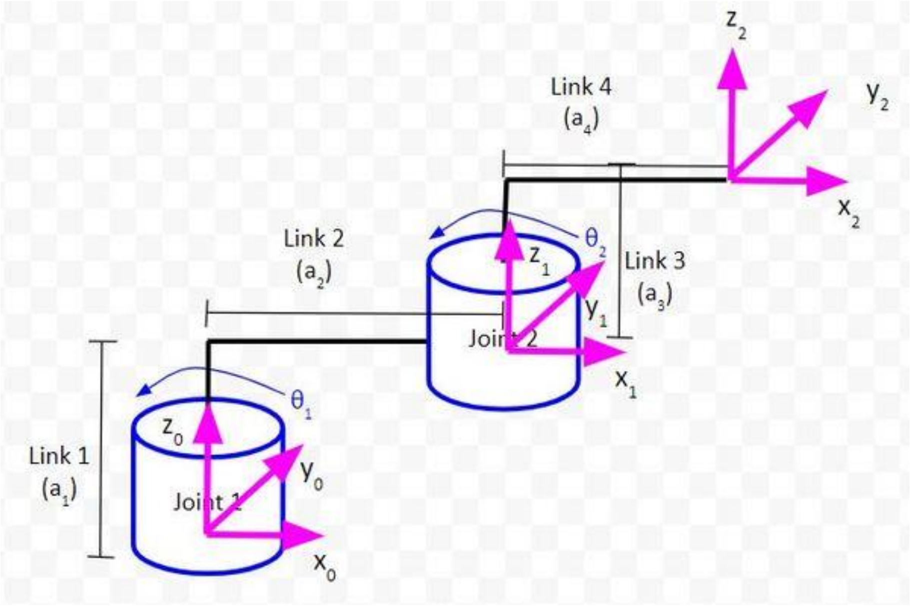
$$R_1^0 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

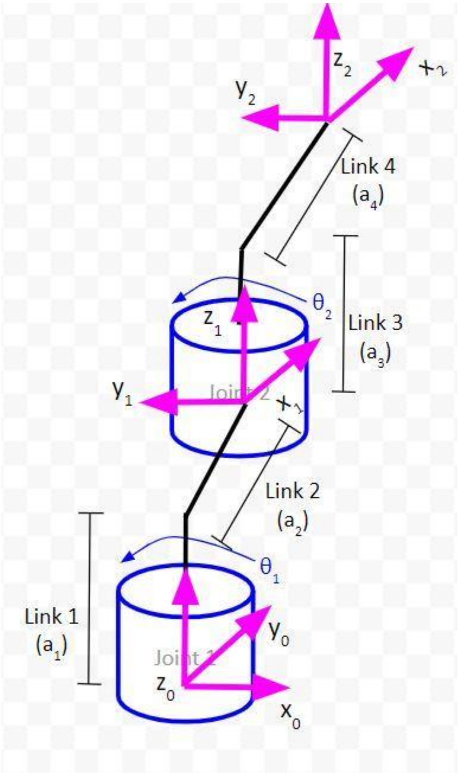
$$R_2^1 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2^0 = R_1^0 R_2^1$$

Displacement Vector

Let's determine the displacement vector from frame 0 to frame 1. We need to find the displacement in the x_0, y_0, and z_0 direction.





Displacement Vector

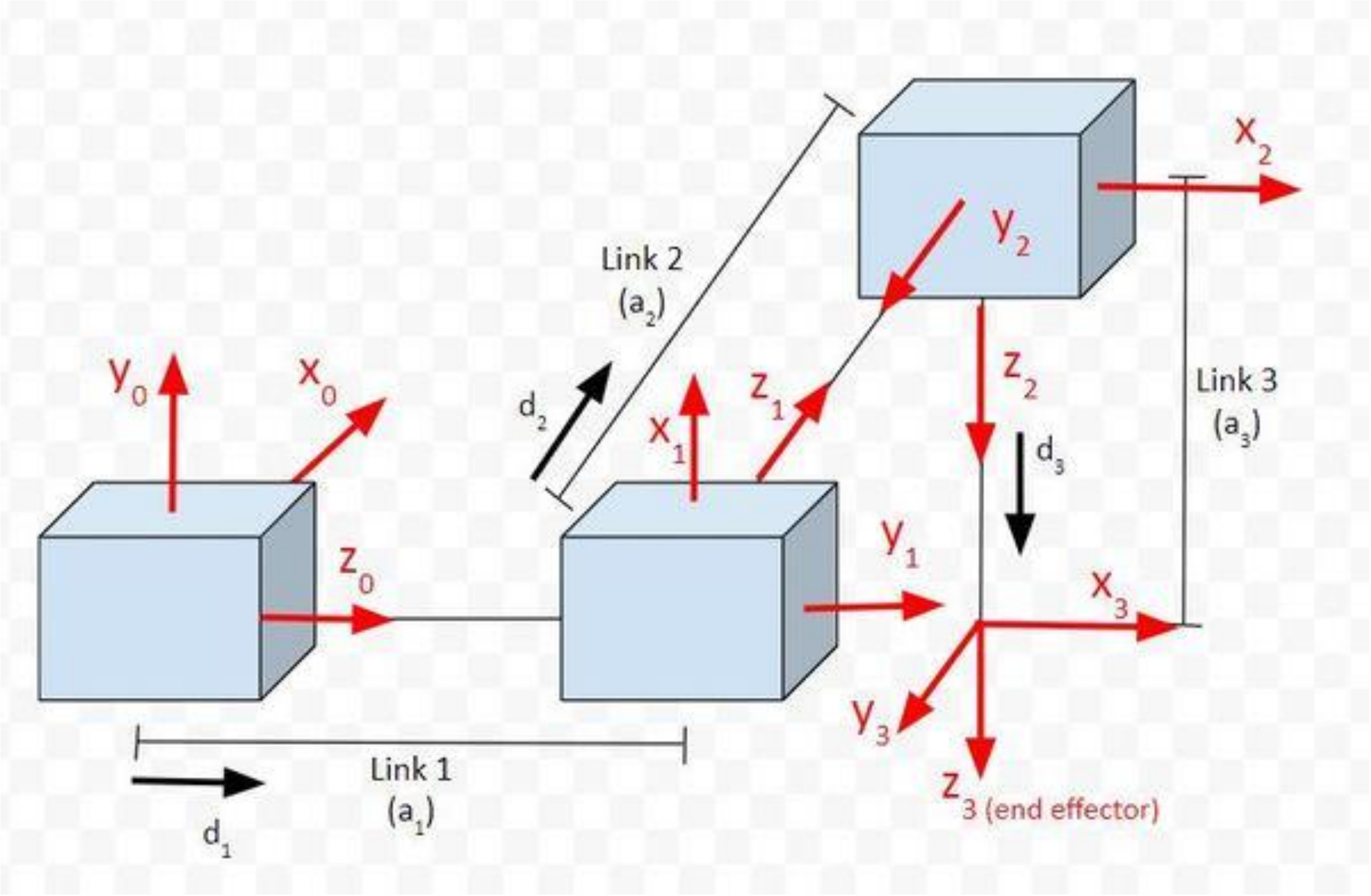
$$d_1^0 = \begin{bmatrix} a_2 \cos(\theta_1) \\ a_2 \sin(\theta_1) \\ a_1 \end{bmatrix} \qquad d_2^1 = \begin{bmatrix} a_4 \cos(\theta_2) \\ a_4 \sin(\theta_2) \\ a_3 \end{bmatrix}$$

Example - Cartesian Robot

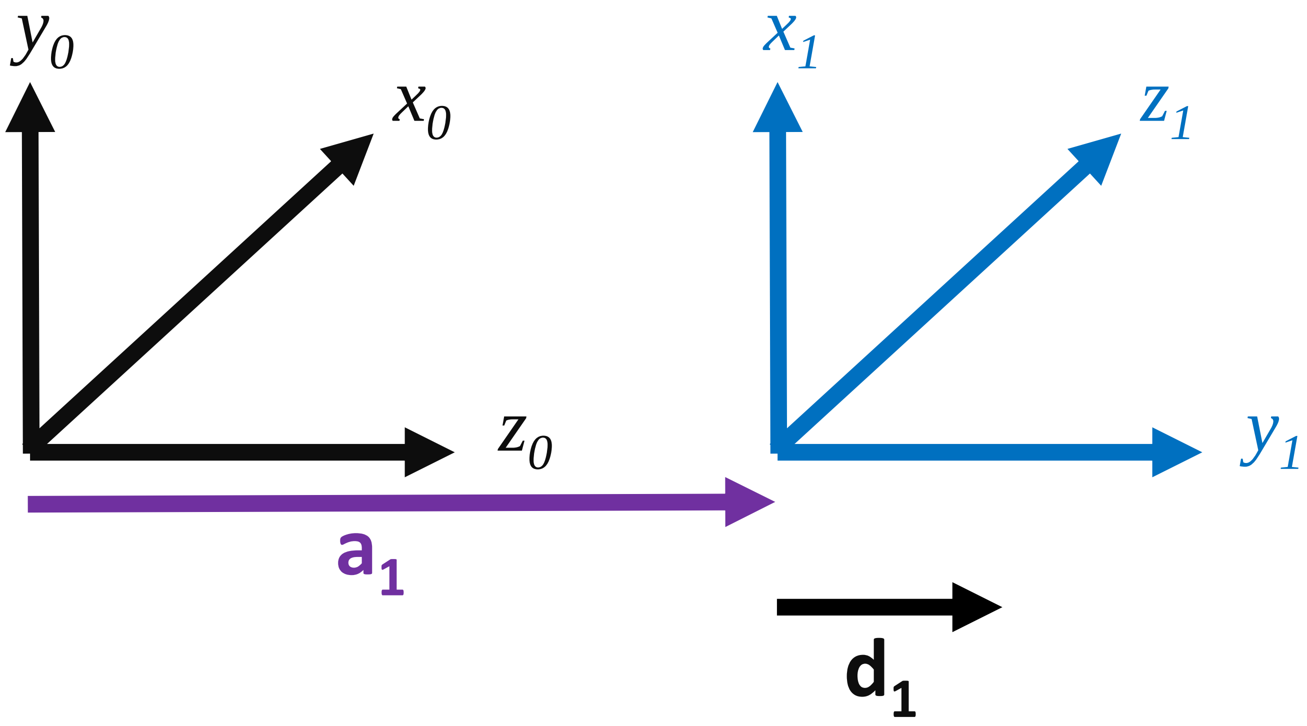
SECTION 4

Cartesian Robot

Cartesian Robot



d_1^0



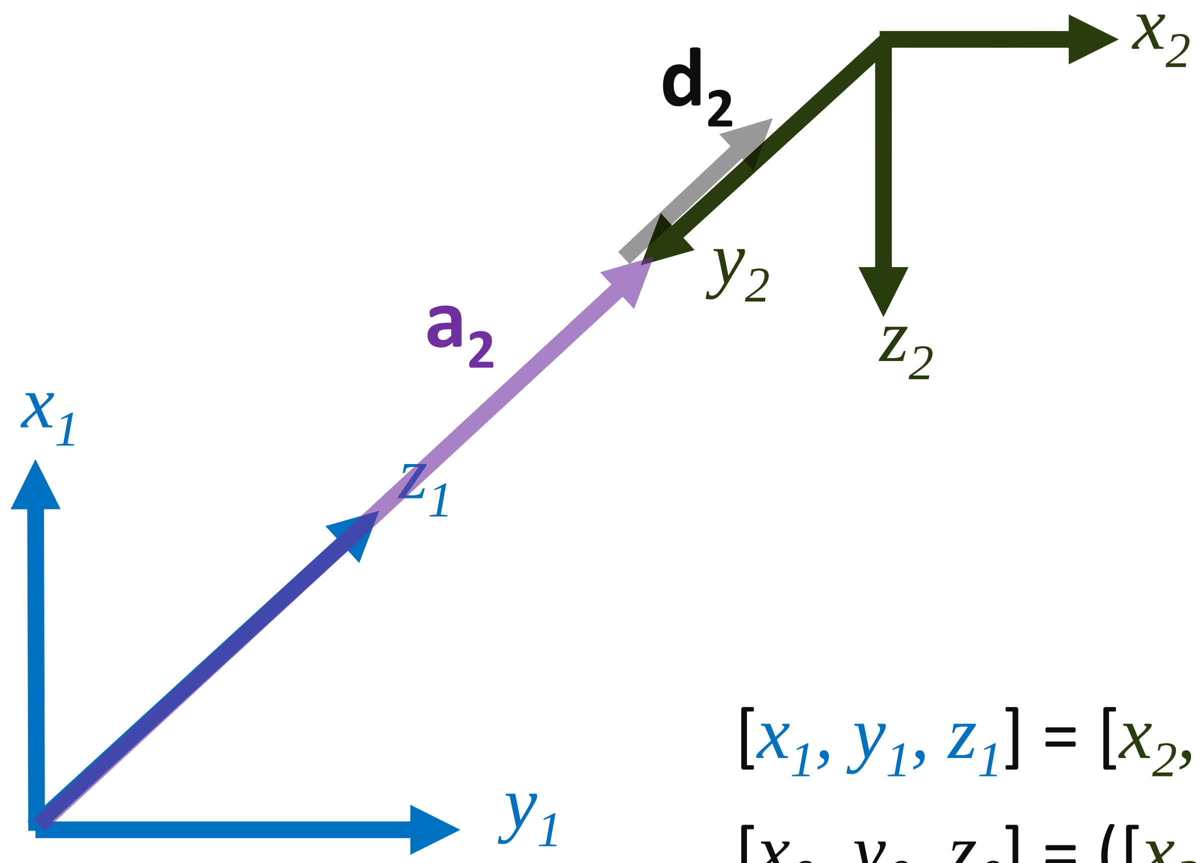
$$[x_0, y_0, z_0] = [x_1, y_1, z_1]\ R_1$$

$$= [x_1, y_1, z_1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$d_1^0 = \begin{bmatrix} 0 \\ 0 \\ a_1 + d_1 \end{bmatrix}$$

$$[x_0, y_0, z_0] = [x_1, y_1, z_1]\ R_1 + d_1^0$$

d_2^1



$$[x_1, y_1, z_1] = [x_2, y_2, z_2]\, R_2$$

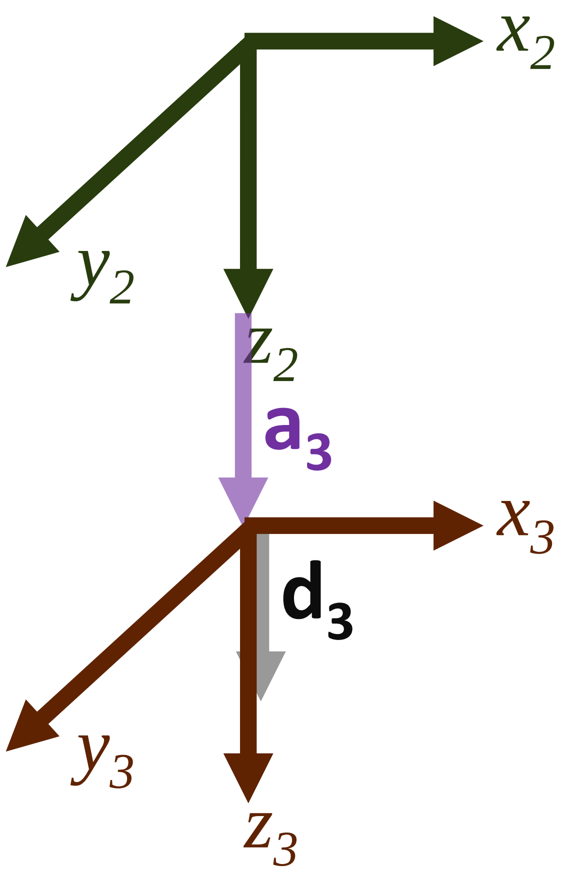
$$= [x_2, y_2, z_2] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$d_2^1 = \begin{bmatrix} 0 \\ 0 \\ a_2 + d_2 \end{bmatrix}$$

$$[x_1, y_1, z_1] = [x_2, y_2, z_2]\, R_2 + d_2^1$$

$$[x_0, y_0, z_0] = ([x_2, y_2, z_2]\, R_2 + d_2^1)\, R_1 + d_1^0$$

d_3^2



$$[x_2, y_2, z_2] = [x_3, y_3, z_3]\, R_3$$

$$= [x_3, y_3, z_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_3^2 = \begin{bmatrix} 0 \\ 0 \\ a_3 + d3 \end{bmatrix}$$

$$[x_2, y_2, z_2] = [x_3, y_3, z_3]\, R_3 + d_3^2 = [x_3, y_3, z_3] + d_3^2$$

$$[x_0, y_0, z_0] = ((([x_3, y_3, z_3]\, R_3 + d_3^2\,)R_2 + d_2^1\,)\, R_1 + d_1^0$$

$$= [x_3, y_3, z_3]\, R_3R_2R_1 + d_3^2\, R_2R_1 + d_2^1 R_1 + d_1^0$$

Displacement Terms

$$d_3^2 \,\mathrm{R}_2\mathrm{R}_1 + d_2^1\mathrm{R}_1 + d_1^0$$

Rotational Term

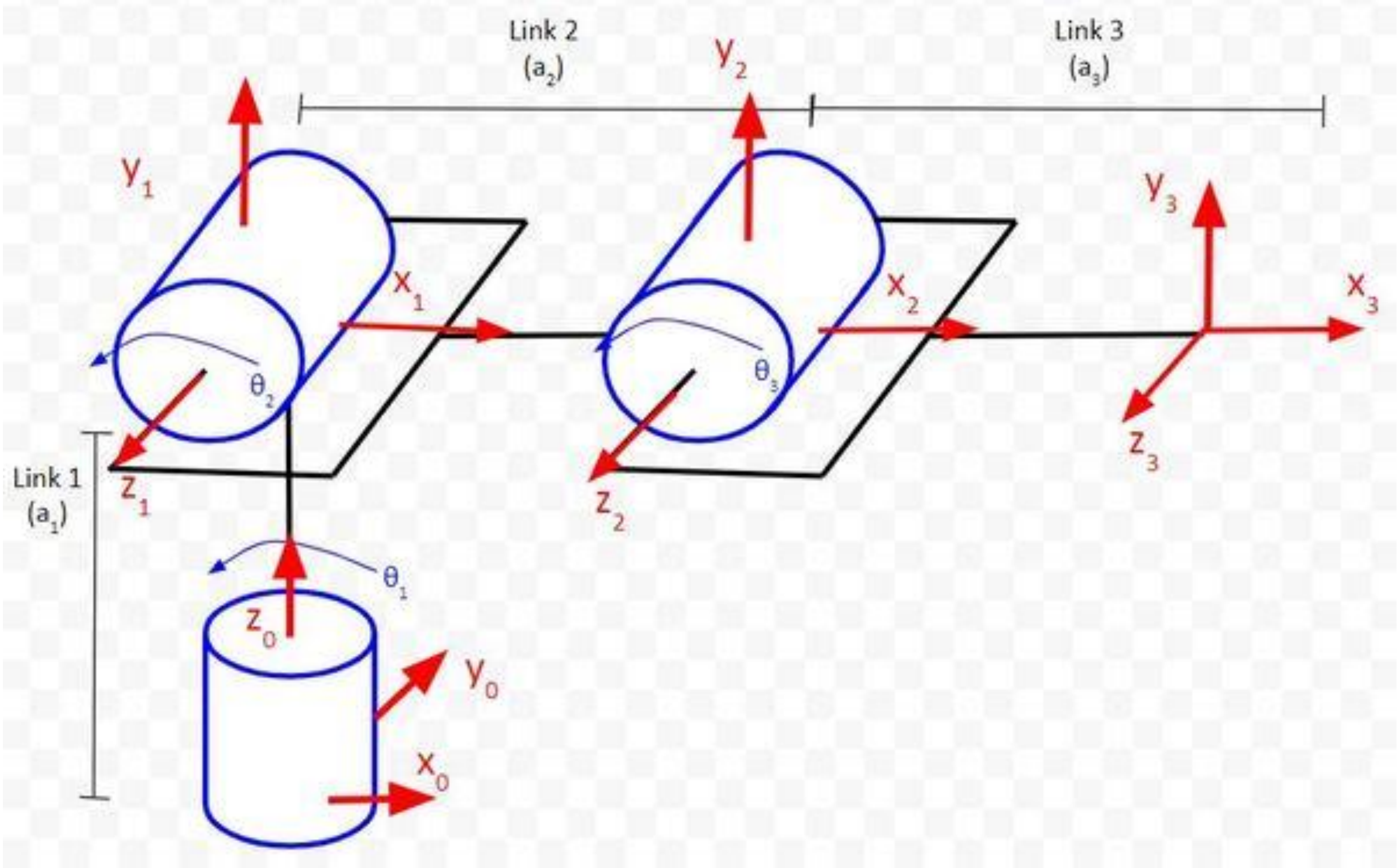
$$[x_3, y_3, z_3]\ \mathrm{R}_3\mathrm{R}_2\mathrm{R}_1$$

Example –Articulated Manipulator

SECTION 5

Articulated Robot

Articulated Robot



Steps

Step 1: Formulate all the rotational matrix.

R_1^0, R_2^1, R_3^2

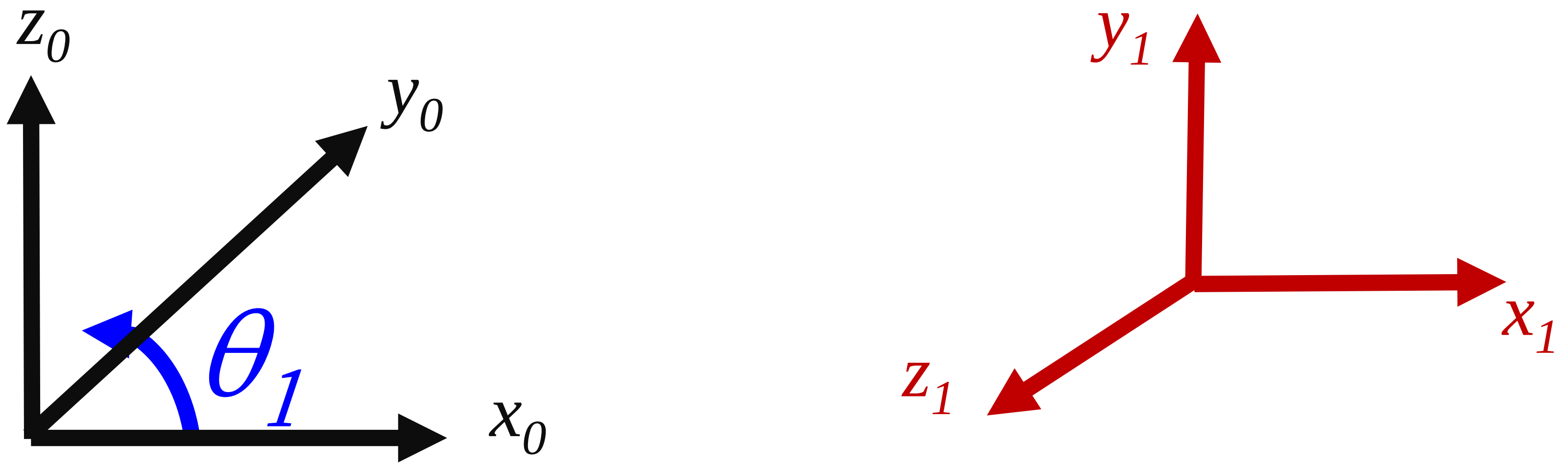
Step 2: Formulate all the Displacement Vectors.

d_1^0, d_2^1, d_3^2

Step 3:Combine the overall vectors and matrix.

$$[x_0, y_0, z_0] = ((([x_3, y_3, z_3]\ R_3^2 + d_3^2)\ R_2^1 + d_2^1)\ R_1^0 + d_1^0$$
$$= [x_3, y_3, z_3]\ R_3^2\ R_2^1\ R_1^0 + d_3^2 R_2^1 R_1^0 + d_2^1 R_1^0 + d_1^0$$

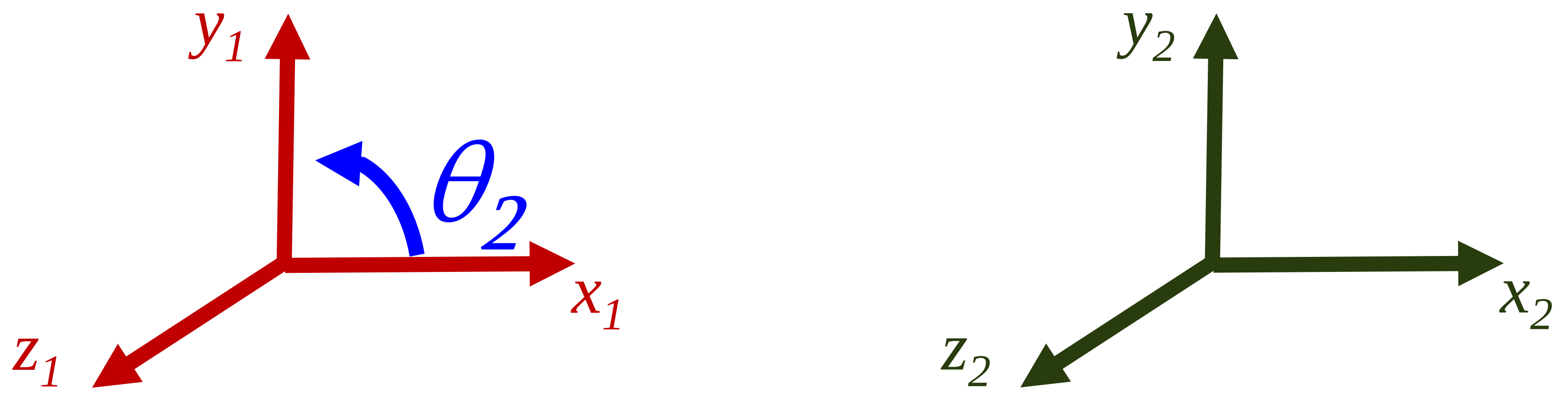
R_1^0



$$[x, y, z] = [x_1, y_1, z_1]\, R_1^0$$

$$= [x_1, y_1, z_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

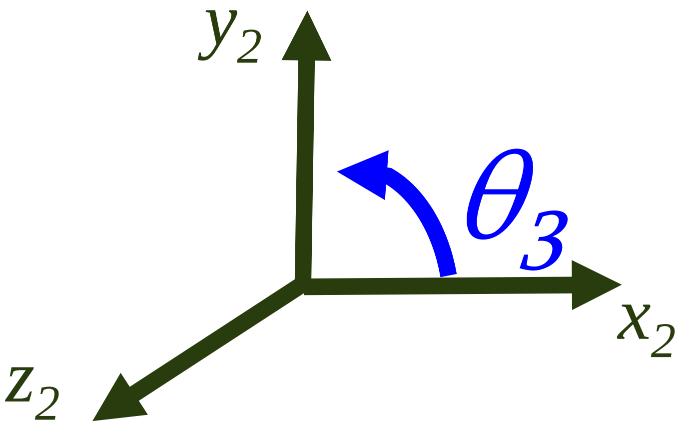
R_2^1

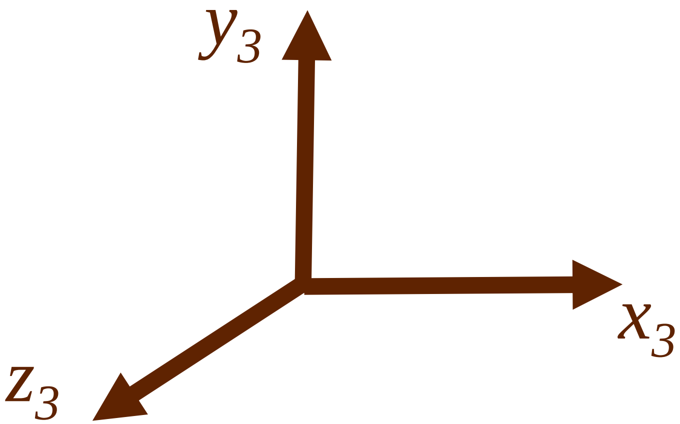


$$[x_1, y_1, z_1] = [x_2, y_2, z_2]\, R_2^1$$

$$= [x_2, y_2, z_2] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_3^2





$$[x_2, y_2, z_2] = [x_3, y_3, z_3]\, R_3^2$$

$$= [x_3, y_3, z_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Displacement Vector

$$d_1^0 = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \qquad d_2^1 = \begin{bmatrix} a_2 \cos(\theta_2) \\ a_2 \sin(\theta_2) \\ 0 \end{bmatrix} \qquad d_3^2 = \begin{bmatrix} a_3 \cos(\theta_3) \\ a_3 \sin(\theta_3) \\ 0 \end{bmatrix}$$

Modeling and Simulation

Step 4: Put the formula into Mathematical Model

$$[x_0, y_0, z_0] = (([x_3, y_3, z_3]\, R_3^2 + d_3^2\,)\, R_2^1 + d_2^1\,)\, R_1^0 + d_1^0$$
$$= [x_3, y_3, z_3]\, R_3^2\, R_2^1\, R_1^0 + d_3^2 R_2^1 R_1^0 + d_2^1 R_1^0 + d_1^0$$