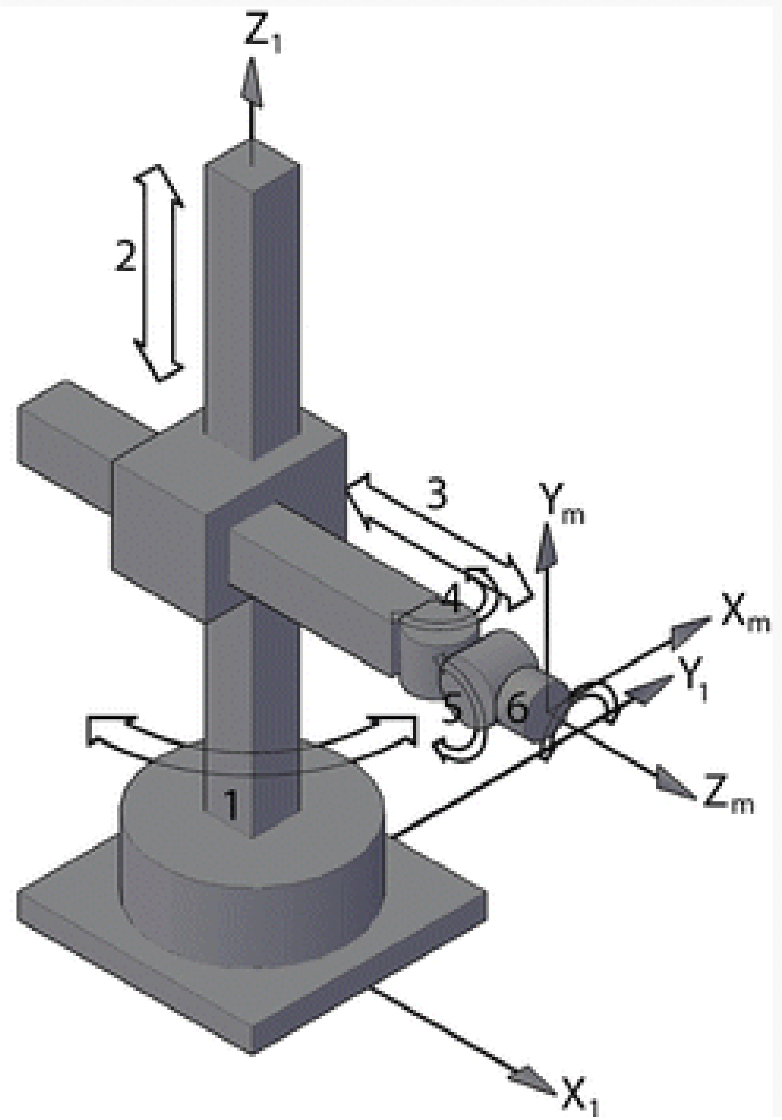
Introduction to Robotics

Manipulation and Programming

Unit 2: Kinematics

A CYLINDRICAL MANIPULATOR WITH SPHERICAL WRIST

DR. ERIC CHOU IEEE SENIOR MEMBER

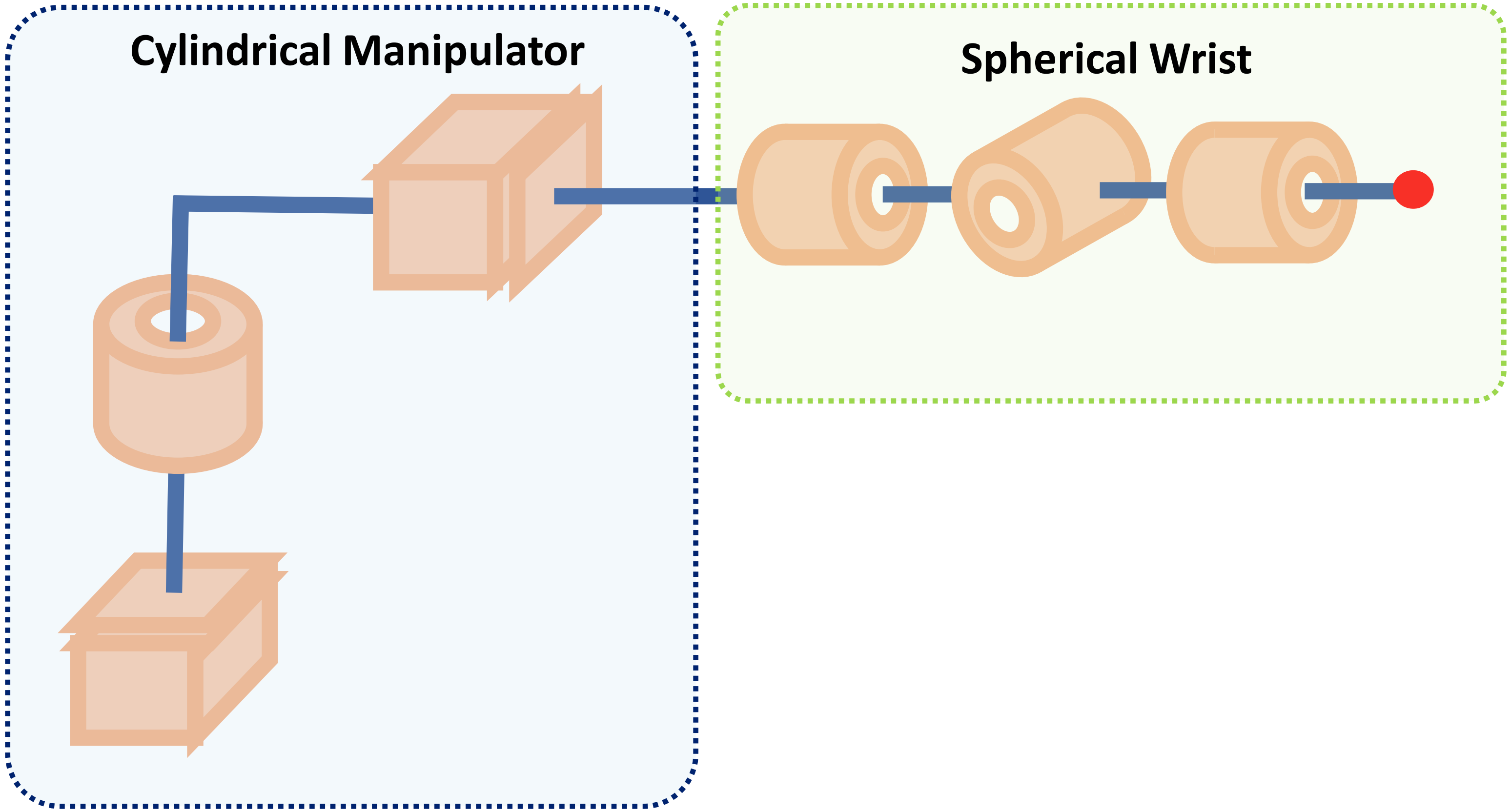


Objectives

- Modeling a manipulator for more than 3 DOF
- Use a PRPRRR (Cylindrical Arm, Spherical Wrist robot) as an example

Path-Planning for A Cylindrical Manipulator with Spherical Wrist

SECTION 1


Cylindrical Manipulator
Spherical Wrist

Assumption

- The **first three joints** are entirely responsible for **POSITIONING** the end-effector, and any additional joins are responsible for **ORIENTING** the end-effector.

Modeling this Robot

Cylindrical Arm

Step 1: Draw a kinematic diagram of only the first 3 joints, and do inverse kinematics for the position.

Step 2: Do forward kinematics on the first three joints to get the rotation part, R0_3

Step 3: Find the inverse of the R0_3 matrix

Modeling this Robot

Spherical Wrist

Step 4: Do forward kinematics on the last three joints and pull out the rotation part, R3_6

Step 5: Specify what you want the rotation matrix R0_6 to be

Modeling this Robot

Path Planning

Step 6: Given a desired X, Y, and Z position, solve for the first three joints using the inverse kinematics equations from Step 1

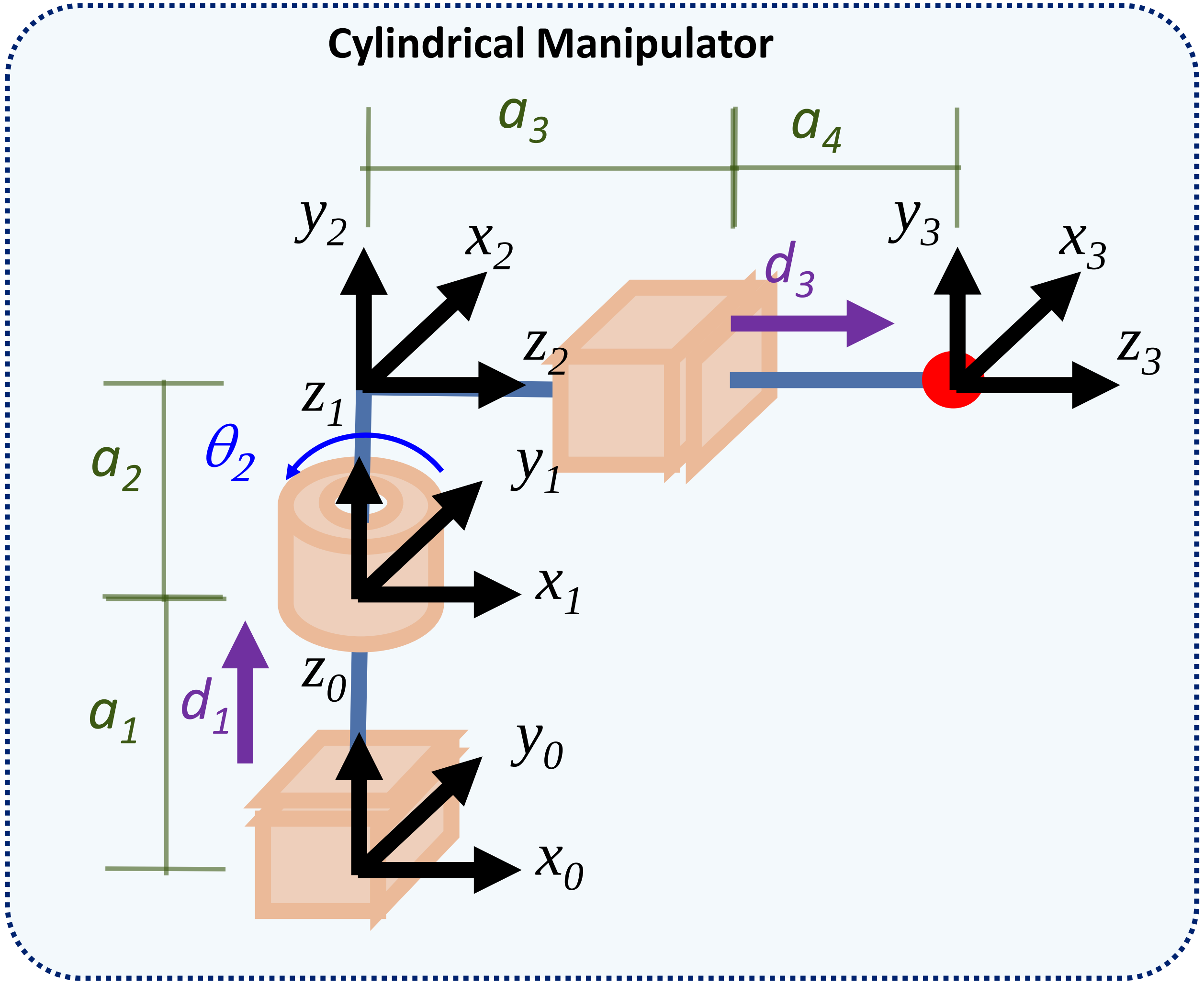
Step 7: Plug in those variables and use the rotation matrix to solve for the last three joints.

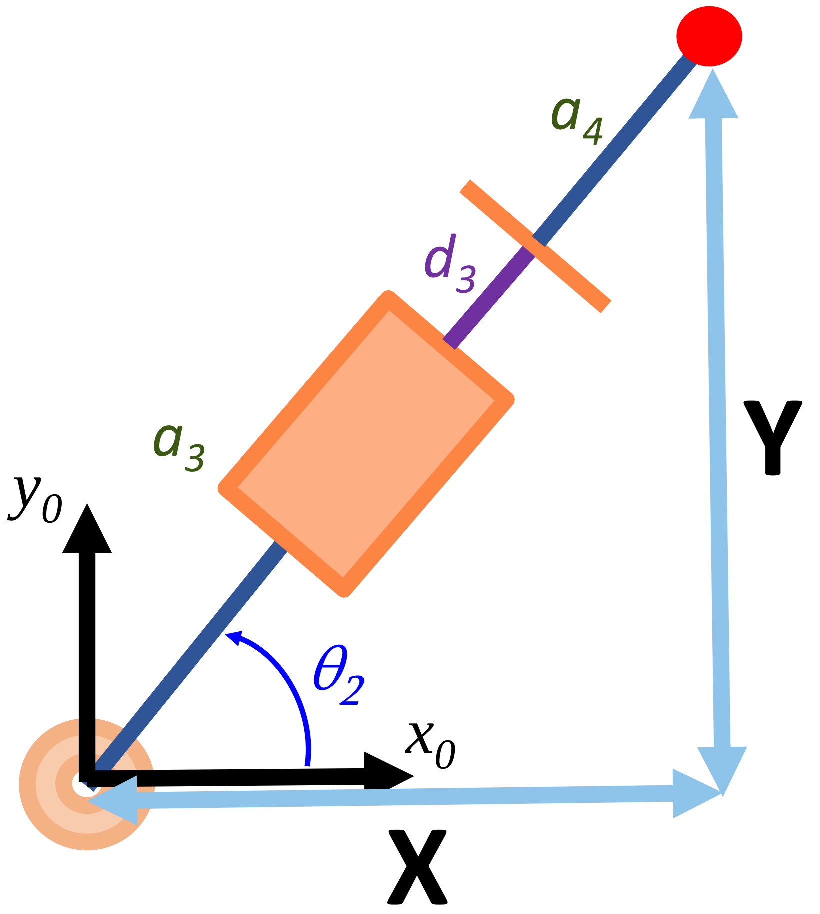
Modelling Cylindrical Arm

SECTION 2

Step 1

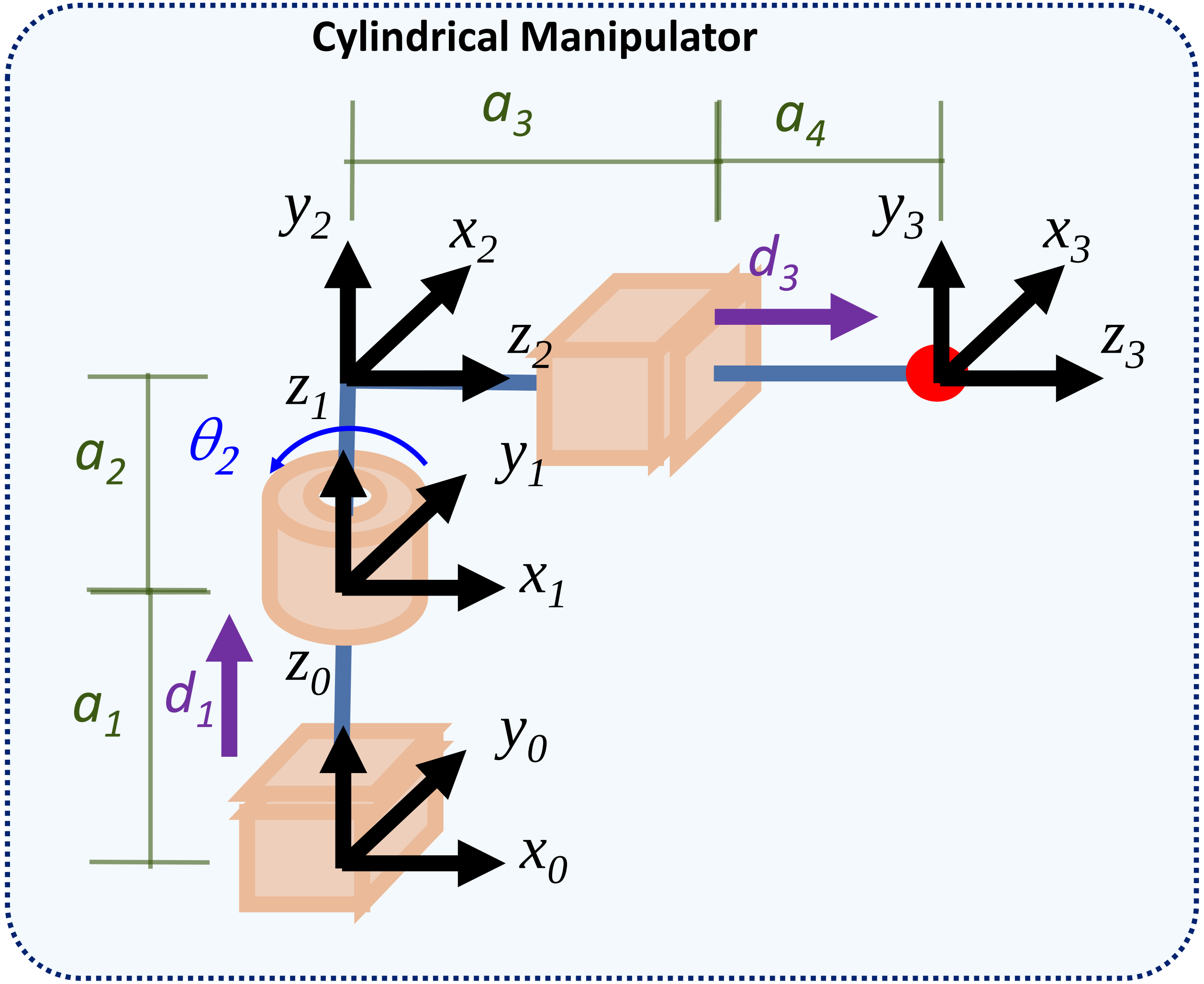
Draw a kinematic diagram of only the first 3 joints, and do inverse kinematics for the position.

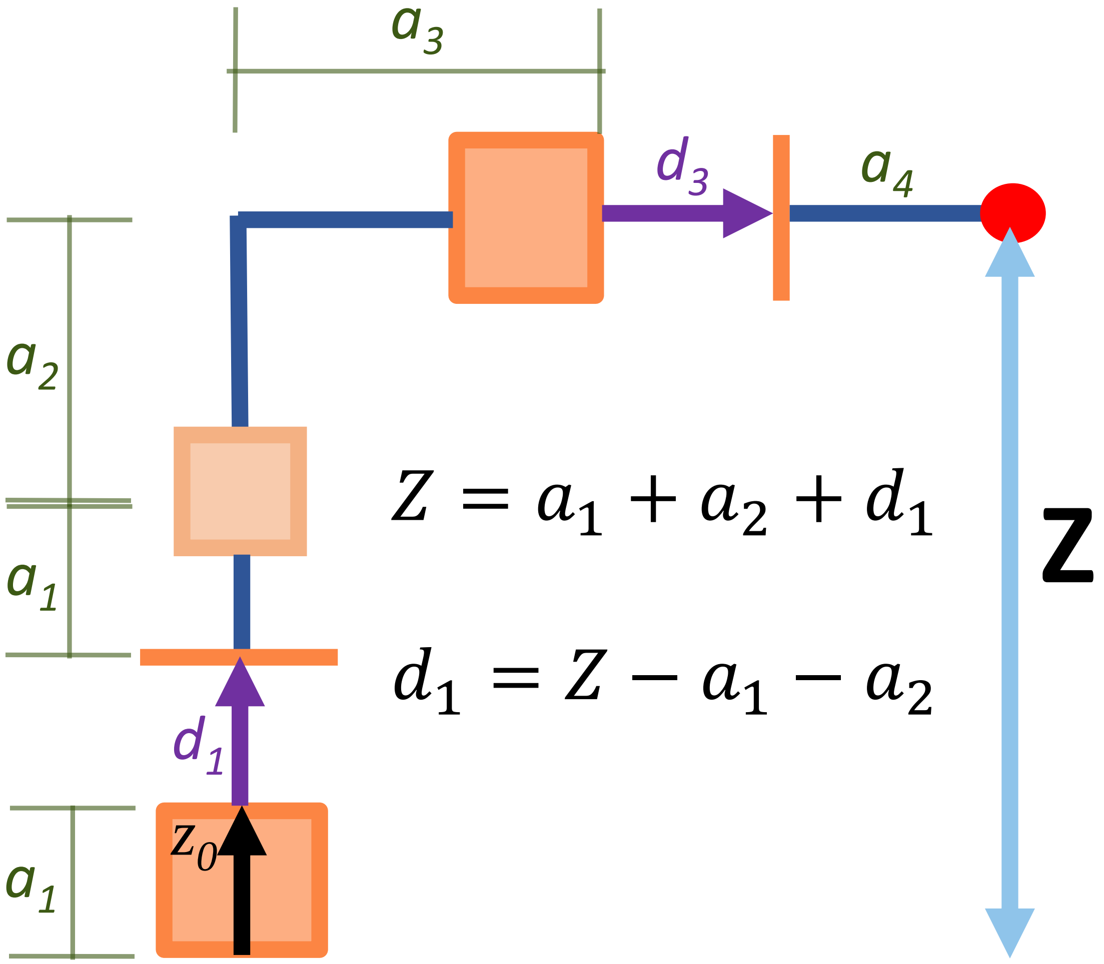




$$\theta_2 = tan^{-1}\left(\frac{Y}{X}\right)$$

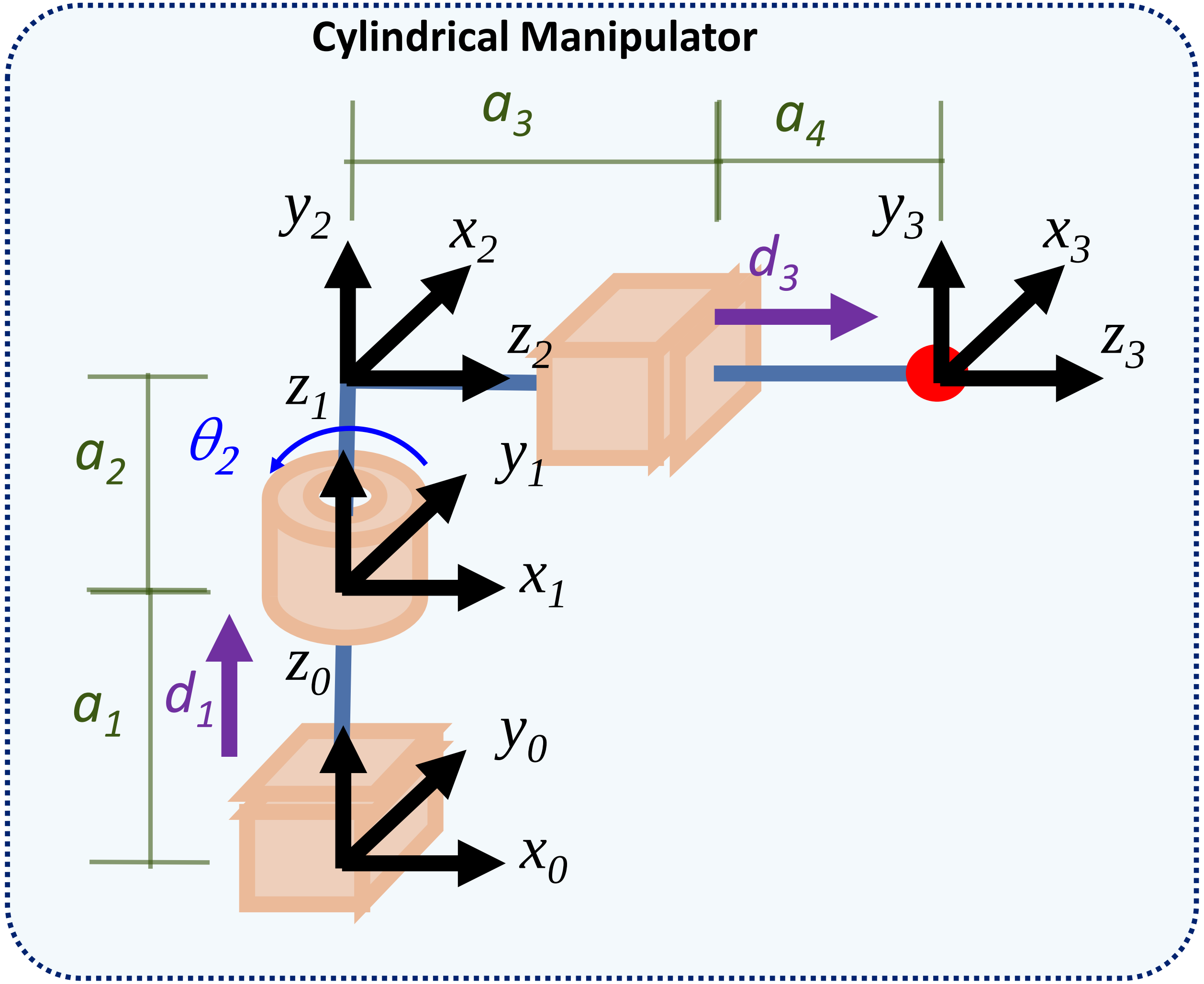
$$d_3 = \sqrt{X^2 + Y^2} - a_3 - a_4$$





$$Z = a_1 + a_2 + d_1$$

$$d_1 = Z - a_1 - a_2$$



	θ	α	r	d
1	0	0	0	a_1+d_1
2	$90+\theta_2$	90	0	a_2
3	0	0	0	$a_3+a_4+d_3$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 2

Do forward kinematics on the first three joints to get the rotation part, R0_3

	θ	α	r	d
1	0	0	0	a_1+d_1
2	$90+\theta_2$	90	0	a_2
3	0	0	0	$a_3+a_4+d_3$

$$H_n^{n-1} = \begin{bmatrix} C(\theta_n) & -S(\theta_n)C(\alpha_n) & S(\theta_n)S(\alpha_n) & r_n C(\theta_n) \\ S(\theta_n) & C(\theta_n)C(\alpha_n) & -C(\theta_n)S(\alpha_n) & r_n S(\theta_n) \\ 0 & S(\alpha_n) & C(\alpha_n) & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_3^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 & 0 \\ C\theta_2 & 0 & S\theta_2 & 0 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 & 0 \\ C\theta_2 & 0 & S\theta_2 & 0 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 & 0 \\ C\theta_2 & 0 & S\theta_2 & 0 \\ 0 & 1 & 0 & a_1+a_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_3+a_4+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 & 0 \\ C\theta_2 & 0 & S\theta_2 & 0 \\ 0 & 1 & 0 & a_1+a_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_3+a_4+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 & (a_3+a_4+d_3)C\theta_2 \\ C\theta_2 & 0 & S\theta_2 & (a_3+a_4+d_3)S\theta_2 \\ 0 & 1 & 0 & a_1+a_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P – R – P Manipulator

	Prismatic	Revolute
Linear	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix} \quad R_3^0 = \begin{bmatrix} -S\theta_2 & 0 & C\theta_2 \\ C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_1 \times (\Omega_3 - \Omega_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \mathrm{r}C\theta_2 \\ \mathrm{r}S\theta_2 \\ a_2 \end{bmatrix}$$

$$r = (a_3 + a_4 + d_3)$$

$$Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad Z_2 = \begin{bmatrix} C\theta_2 \\ S\theta_2 \\ 0 \end{bmatrix} \quad Z_3 = \begin{bmatrix} C\theta_2 \\ S\theta_2 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \mathrm{r}C\theta_2 & \mathrm{r}S\theta_2 & a_2 \end{vmatrix} = \begin{bmatrix} -\mathrm{r}S\theta_2 \\ \mathrm{r}C\theta_2 \\ 0 \end{bmatrix}$$

$$\Omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Omega_1 = \begin{bmatrix} 0 \\ 0 \\ a_1 + d_1 \end{bmatrix} \quad \Omega_2 = \begin{bmatrix} 0 \\ 0 \\ a_1 + a_2 + d_1 \end{bmatrix} \quad \Omega_3 = \begin{bmatrix} (a_3 + a_4 + d_3)C\theta_2 \\ (a_3 + a_4 + d_3)S\theta_2 \\ a_1 + a_2 + d_1 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -\mathrm{r}S\theta_2 & C\theta_2 \\ 0 & \mathrm{r}C\theta_2 & S\theta_2 \\ 1 & 0 & 0 \\ 0 & C\theta_2 & 0 \\ 0 & S\theta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{r}S\theta_2 & C\theta_2 \\ 0 & \mathrm{r}C\theta_2 & S\theta_2 \\ 1 & 0 & 0 \\ 0 & C\theta_2 & 0 \\ 0 & S\theta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$r = (a_3 + a_4 + d_3)$$

Step 3

Find the inverse of the R0_3 matrix

Note: Can be done by our inverse matrix solver

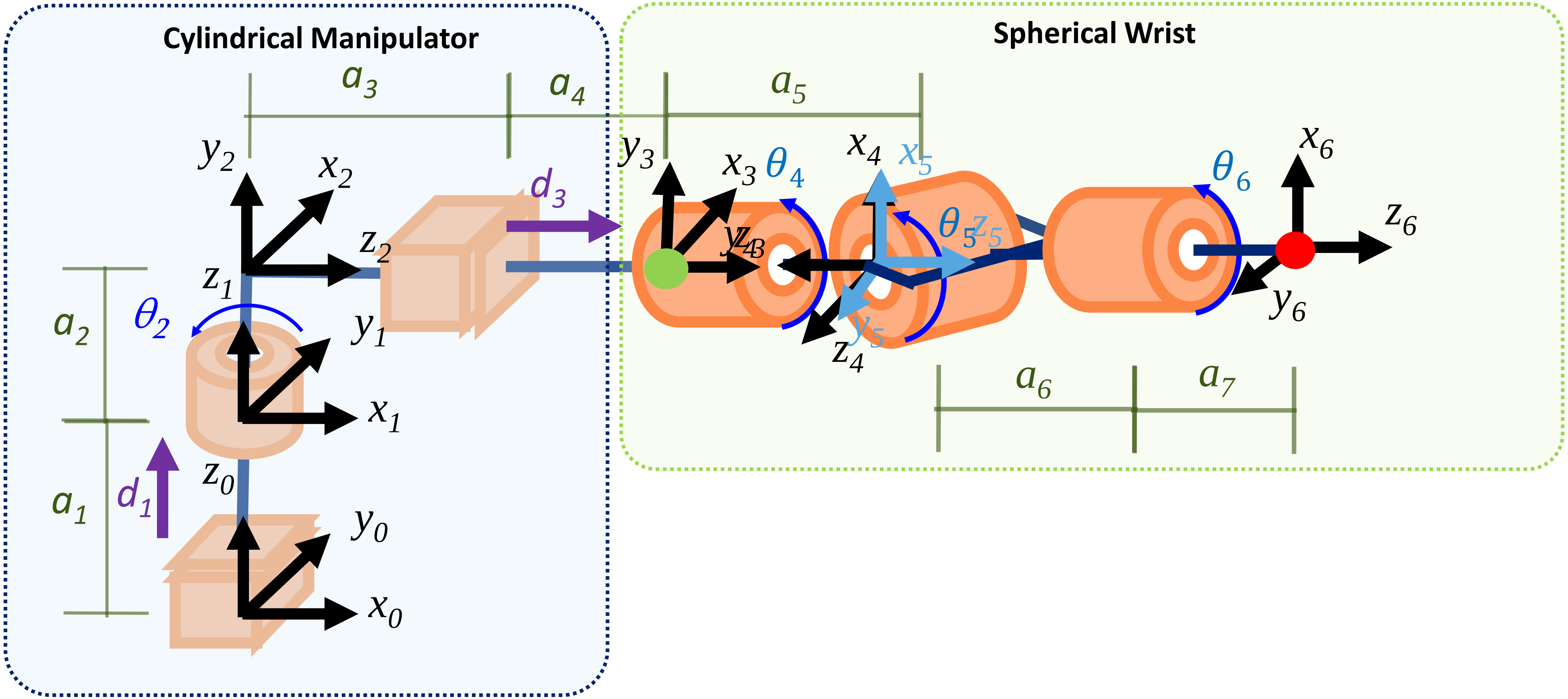
Modelling Wrist

(Please also refer to B8A)

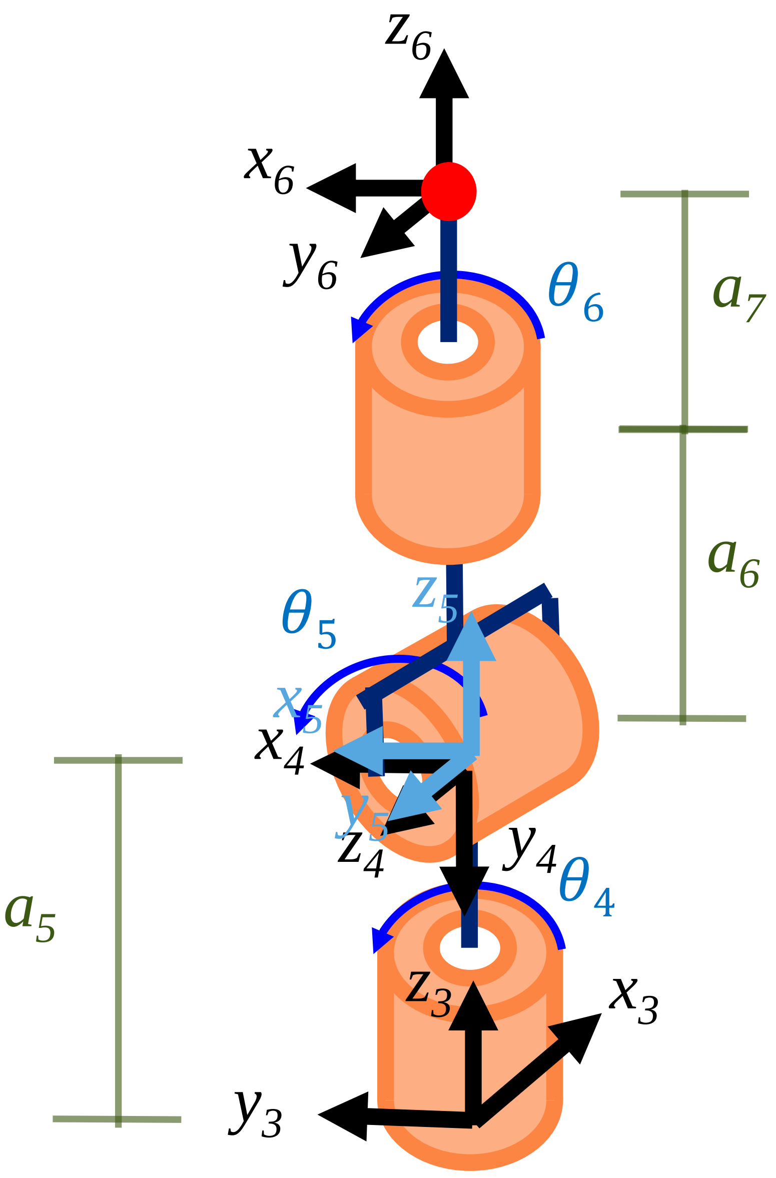
SECTION 3

Step 4

Do forward kinematics on the last three joints and pull out the rotation part, R3_6



Cylindrical Manipulator
Spherical Wrist
a_1
a_2
a_3
a_4
a_5
a_6
a_7
d_1
d_3
θ_2
θ_4
θ_5
θ_6
x_0
y_0
z_0
x_1
y_1
z_1
x_2
y_2
z_2
x_3
y_3
z_3
x_4
y_4
z_4
x_5
y_5
z_5
x_6
y_6
z_6



z_6
x_6
y_6
θ_6
a_7
a_6
z_5
θ_5
x_5
x_4
y_5
z_4
y_4
θ_4
a_5
z_3
x_3
y_3

Configuration $Q(q_1, q_2, \ldots, q_n)$

- Different configuration leads to different coordination system.
- Different coordination system may lead to different matrix
- R_6^3 can be derived from R_6^0 and R_3^0, because you can provide the end-effector's position (X, Y, Z) and find the R_6^0, then find R_6^3 by numerical result.
- Or, you may use Denavit-Hartenberg to find the symbolic result.

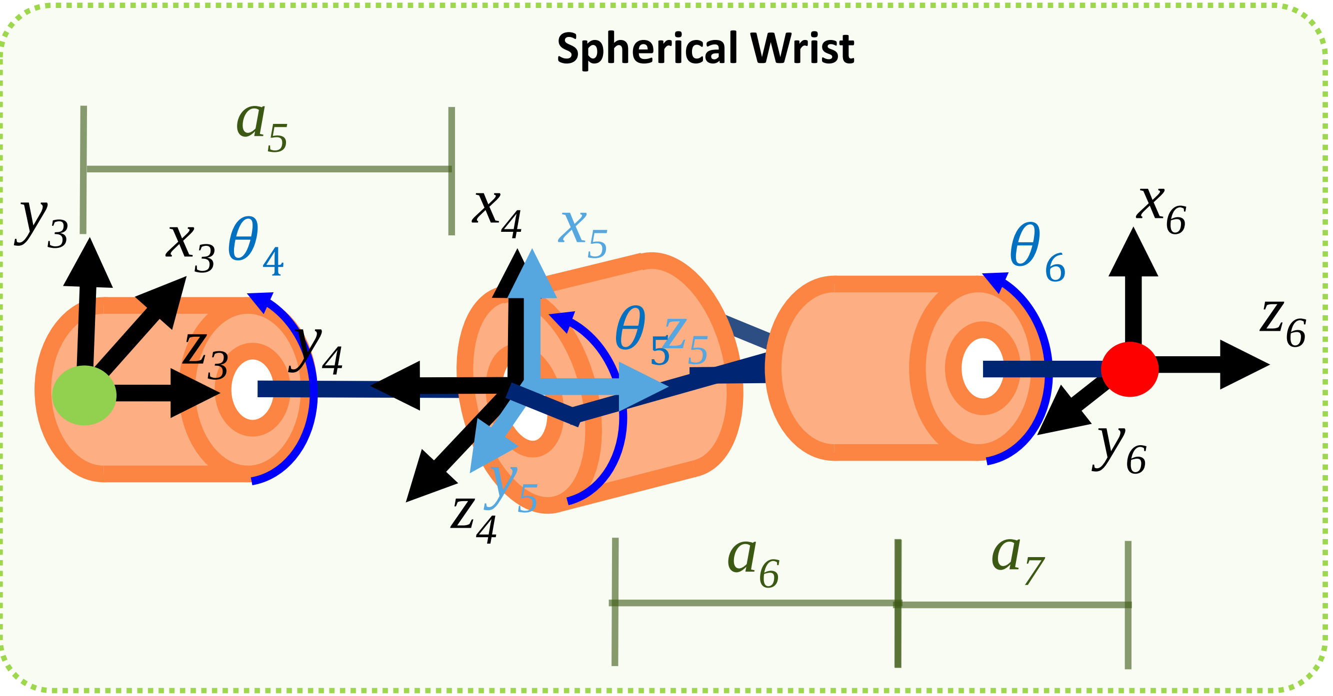
Combination of Arm and Wrist

- The rotation matrix can be obtained by picking up the top 3 X 3 sub matrix

$$R_6^0 = R_3^0 R_6^3$$

$${R_3^0}^{-1} R_6^0 = {R_3^0}^{-1} R_3^0 R_6^3$$

$$R_6^3 = {R_3^0}^{-1} R_6^0$$



	θ	α	r	d
4	$90+\theta_4$	-90	0	a_5
5	θ_5	90	0	0
6	θ_6	0	0	a_6+a_7

$$H_4^3 = \begin{bmatrix} -S\theta_4 & 0 & -C\theta_4 & 0 \\ C\theta_4 & 0 & -S\theta_4 & 0 \\ 0 & -1 & 0 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_5^4 = \begin{bmatrix} C\theta_5 & 0 & S\theta_5 & 0 \\ S\theta_5 & 0 & -C\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C(\theta_n+90) = -S\theta_n$$
$$S(\theta_n+90) = C\theta_n$$

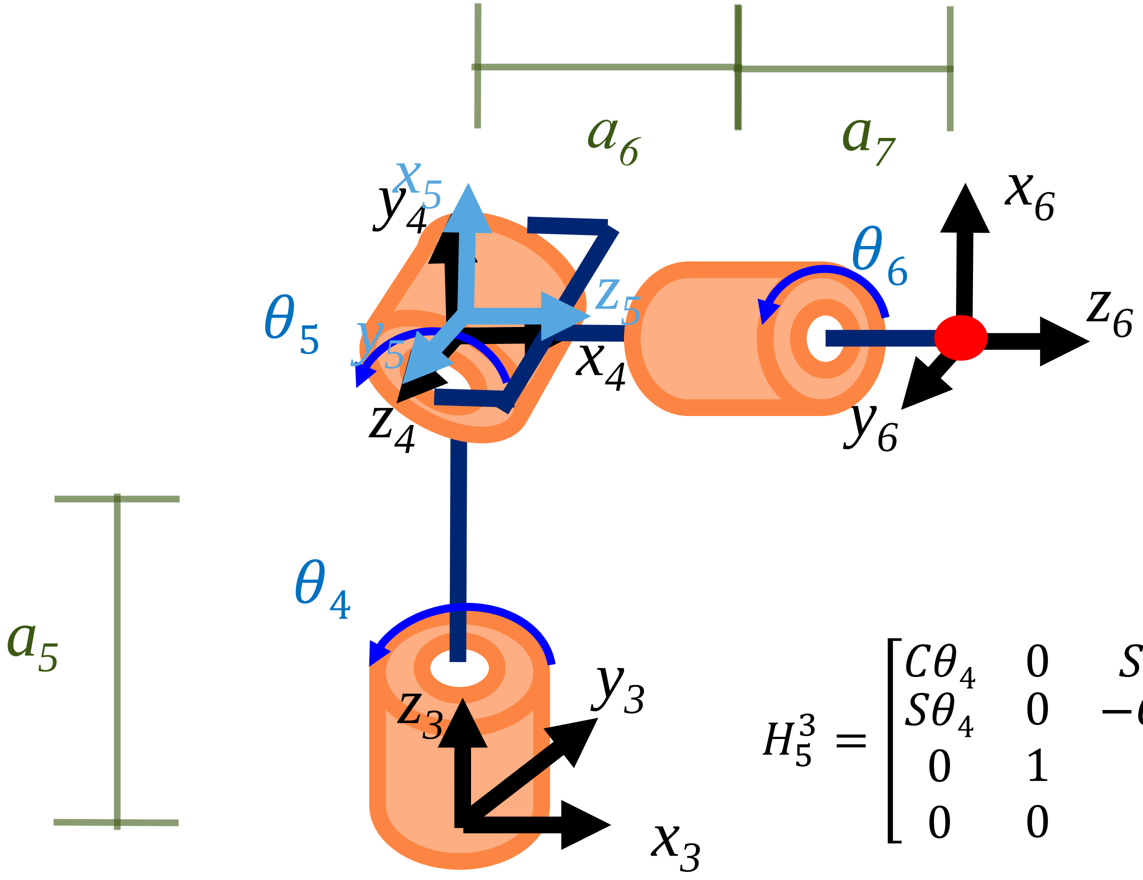
$$H_5^3 = \begin{bmatrix} -S\theta_4 & 0 & -C\theta_4 & 0 \\ C\theta_4 & 0 & -S\theta_4 & 0 \\ 0 & -1 & 0 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_5 & 0 & S\theta_5 & 0 \\ S\theta_5 & 0 & -C\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -S\theta_4 C\theta_5 & -C\theta_4 & -S\theta_4 S\theta_5 & 0 \\ C\theta_4 C\theta_5 & -S\theta_4 & C\theta_4 S\theta_5 & 0 \\ -S\theta_5 & 0 & C\theta_5 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_n^{n-1} = \begin{bmatrix} C\theta_n & -S\theta_n C\alpha_n & S\theta_n S\alpha_n & r_n C\theta_n \\ S\theta_n & C\theta_n C\alpha_n & -C\theta_n S\alpha_n & r_n S\theta_n \\ 0 & S\alpha_n & C\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^5 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & a_6+a_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^3 = \begin{bmatrix} -S\theta_4 C\theta_5 & -C\theta_4 & -S\theta_4 S\theta_5 & 0 \\ C\theta_4 C\theta_5 & -S\theta_4 & C\theta_4 S\theta_5 & 0 \\ -S\theta_5 & 0 & C\theta_5 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & a_6+a_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S\theta_4 C\theta_5 C\theta_6 - C\theta_4 S\theta_6 & S\theta_4 C\theta_5 S\theta_6 - C\theta_4 C\theta_6 & -S\theta_4 S\theta_5 & -(a_6+a_7) S\theta_4 S\theta_5 \\ C\theta_4 C\theta_5 C\theta_6 - S\theta_4 S\theta_6 & -C\theta_4 C\theta_5 S\theta_6 - S\theta_4 C\theta_6 & C\theta_4 S\theta_5 & (a_6+a_7) C\theta_4 S\theta_5 \\ -S\theta_5 C\theta_6 & S\theta_5 S\theta_6 + C\theta_5 & C\theta_5 & (a_6+a_7) C\theta_5 + a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



	θ	α	r	d
4	θ_4	90	0	a_5
5	$90+\theta_5$	90	0	0
6	θ_6	0	0	a_6+a_7

$$H_4^3 = \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & 0 \\ S\theta_4 & 0 & -C\theta_4 & 0 \\ 0 & 1 & 0 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_5^4 = \begin{bmatrix} -S\theta_5 & 0 & C\theta_5 & 0 \\ C\theta_5 & 0 & S\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_5^3 = \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & 0 \\ S\theta_4 & 0 & -C\theta_4 & 0 \\ 0 & 1 & 0 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -S\theta_5 & 0 & C\theta_5 & 0 \\ C\theta_5 & 0 & S\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -C\theta_4 S\theta_5 & S\theta_4 & C\theta_4 C\theta_5 & 0 \\ -S\theta_4 S\theta_5 & -C\theta_4 & S\theta_4 C\theta_5 & 0 \\ C\theta_5 & 0 & S\theta_5 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^5 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & a_6+a_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_n^{n-1} = \begin{bmatrix} C\theta_n & -S\theta_n C\alpha_n & S\theta_n S\alpha_n & r_n C\theta_n \\ S\theta_n & C\theta_n C\alpha_n & -C\theta_n S\alpha_n & r_n S\theta_n \\ 0 & S\alpha_n & C\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^3 = \begin{bmatrix} -C\theta_4 S\theta_5 & S\theta_4 & C\theta_4 C\theta_5 & 0 \\ -S\theta_4 S\theta_5 & -C\theta_4 & S\theta_4 C\theta_5 & 0 \\ C\theta_5 & 0 & S\theta_5 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & a_6+a_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -C\theta_4 S\theta_5 C\theta_6 + S\theta_4 S\theta_6 & C\theta_4 S\theta_5 S\theta_6 + S\theta_4 C\theta_6 & C\theta_4 C\theta_5 & (a_6+a_7) C\theta_4 C\theta_5 \\ -S\theta_4 S\theta_5 C\theta_6 - C\theta_4 S\theta_6 & S\theta_4 S\theta_5 S\theta_6 - C\theta_4 C\theta_6 & S\theta_4 C\theta_5 & (a_6+a_7) S\theta_4 C\theta_5 \\ C\theta_5 C\theta_6 & -C\theta_5 S\theta_6 & S\theta_5 & (a_6+a_7) S\theta_5 + a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note:

1. A little bit complicated.   The wrist resting condition may matter for the sine or cosine function. This example is chosen just for the best demonstration purpose of the matrix.   May not be the best for joint control.

Combination of Arm and Wrist

(Please also refer to B8A)

SECTION 4

PRPRRR Manipulator

	Prismatic	Revolute
Linear	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$Z_0 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad Z_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad Z_2 = \begin{bmatrix}C\theta_2\\S\theta_2\\0\end{bmatrix} \quad Z_3 = \begin{bmatrix}C\theta_2\\S\theta_2\\0\end{bmatrix}$$

$$\Omega_0 = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Omega_1 = \begin{bmatrix}0\\0\\a_1+d_1\end{bmatrix} \quad \Omega_2 = \begin{bmatrix}0\\0\\a_1+a_2+d_1\end{bmatrix} \quad \Omega_3 = \begin{bmatrix}(a_3+a_4+d_3)C\theta_2\\(a_3+a_4+d_3)S\theta_2\\a_1+a_2+d_1\end{bmatrix}$$

$$H_3^0 = \begin{bmatrix}-S\theta_2 & 0 & C\theta_2 & (a_3+a_4+d_3)C\theta_2\\ C\theta_2 & 0 & S\theta_2 & (a_3+a_4+d_3)S\theta_2\\ 0 & 1 & 0 & a_1+a_2+d_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_4^3 = \begin{bmatrix}-S\theta_4 & 0 & -C\theta_4 & 0\\ C\theta_4 & 0 & -S\theta_4 & 0\\ 0 & -1 & 0 & a_5\\ 0 & 0 & 0 & 1\end{bmatrix} \quad H_5^3 = \begin{bmatrix}-S\theta_4C\theta_5 & -C\theta_4 & -S\theta_4S\theta_5 & 0\\ C\theta_4C\theta_5 & -S\theta_4 & C\theta_4S\theta_5 & 0\\ -S\theta_5 & 0 & C\theta_5 & a_5\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_4^0 = H_3^0 H_4^3$$

$$H_5^0 = H_3^0 H_5^3$$

$$H_6^3 = \begin{bmatrix}-C\theta_4S\theta_5C\theta_6 + S\theta_4S\theta_6 & C\theta_4S\theta_5S\theta_6 + S\theta_4C\theta_6 & C\theta_4C\theta_5 & (a_6+a_7)C\theta_4C\theta_5\\ -S\theta_4S\theta_5C\theta_6 - C\theta_4S\theta_6 & S\theta_4S\theta_5S\theta_6 - C\theta_4C\theta_6 & S\theta_4C\theta_5 & (a_6+a_7)S\theta_4C\theta_5\\ C\theta_5C\theta_6 & -C\theta_5S\theta_6 & S\theta_5 & (a_6+a_7)S\theta_5 + a_5\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_6^0 = H_3^0 H_6^3$$

R_3^0, R_6^3, R_6^0 can all be extracted

Step 4

Do forward kinematics on the last three joints and pull out the rotation part, R3_6

Step 5

Specify what you want the rotation matrix R0_6 to be

Step 6

Given a desired X, Y, and Z position, solve for the first three joints using the inverse kinematics equations from Step 1

Step 7

Plug in those variables and use the rotation matrix to solve for the last three joints.

Summary

SECTION 5

Summary

- Up to this point, we are able to find the Jacobian Matrix for this PRPRRR manipulator.
- We can also solve the inverse Jacobian matrix to provide the path-planning formula.
- In the next Lab, we will discuss the issue in Python in more details.