Introduction to Robotics

Manipulation and Programming

Unit 2: Kinematics

INVERSE JACOBIAN MATRIX

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Objectives

- Jacobian Matrix and Inverse Jacobian Matrix
- How to find the joint variable function?
- Pseudo Inverse Matrix

Jacobian Matrix

SECTION 1

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$I = A \; A^{-1}$$

$$I = A^{-1} A$$

$$I = J \; J^{-1}$$

Jacobian Matrix

$$J^{-1}J\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \vdots\\ \dot{q}_n\end{bmatrix}=\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \vdots\\ \dot{q}_n\end{bmatrix}=J^{-1}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix}$$

$$I = A\ A^{-1}$$

$$I = A^{-1}A$$

$$I = J\ J^{-1}$$

How to find the Joint Variables

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}$$

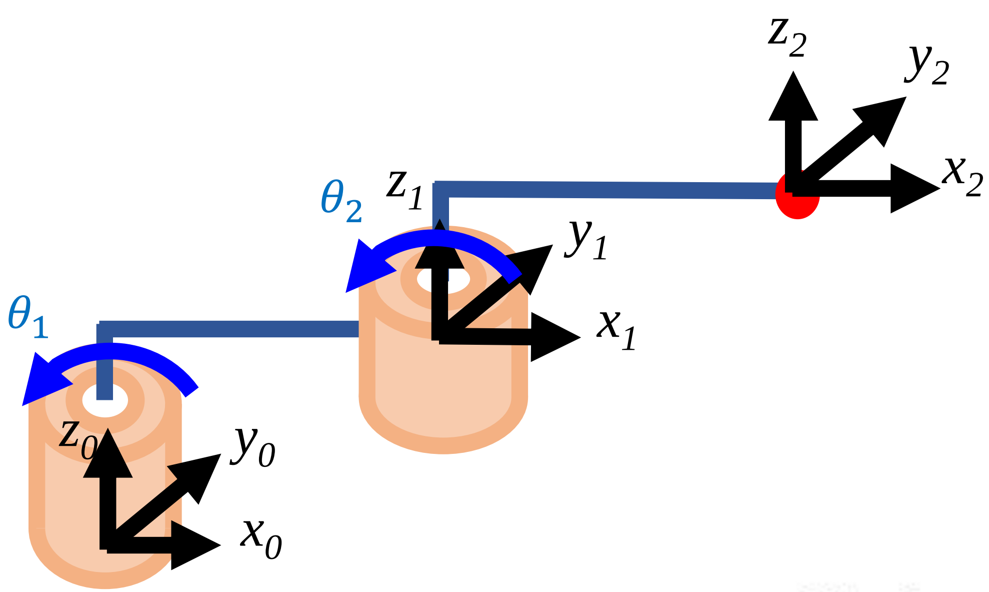
$$\Delta(\mathrm{q_i(t)}\text{-}\mathrm{q_i(0)}) = \dot{q}_i t$$

$$\mathrm{q_i(t)} = \mathrm{q_i(0)} + \dot{q}_i t$$

Simplified SCARA Manipulator

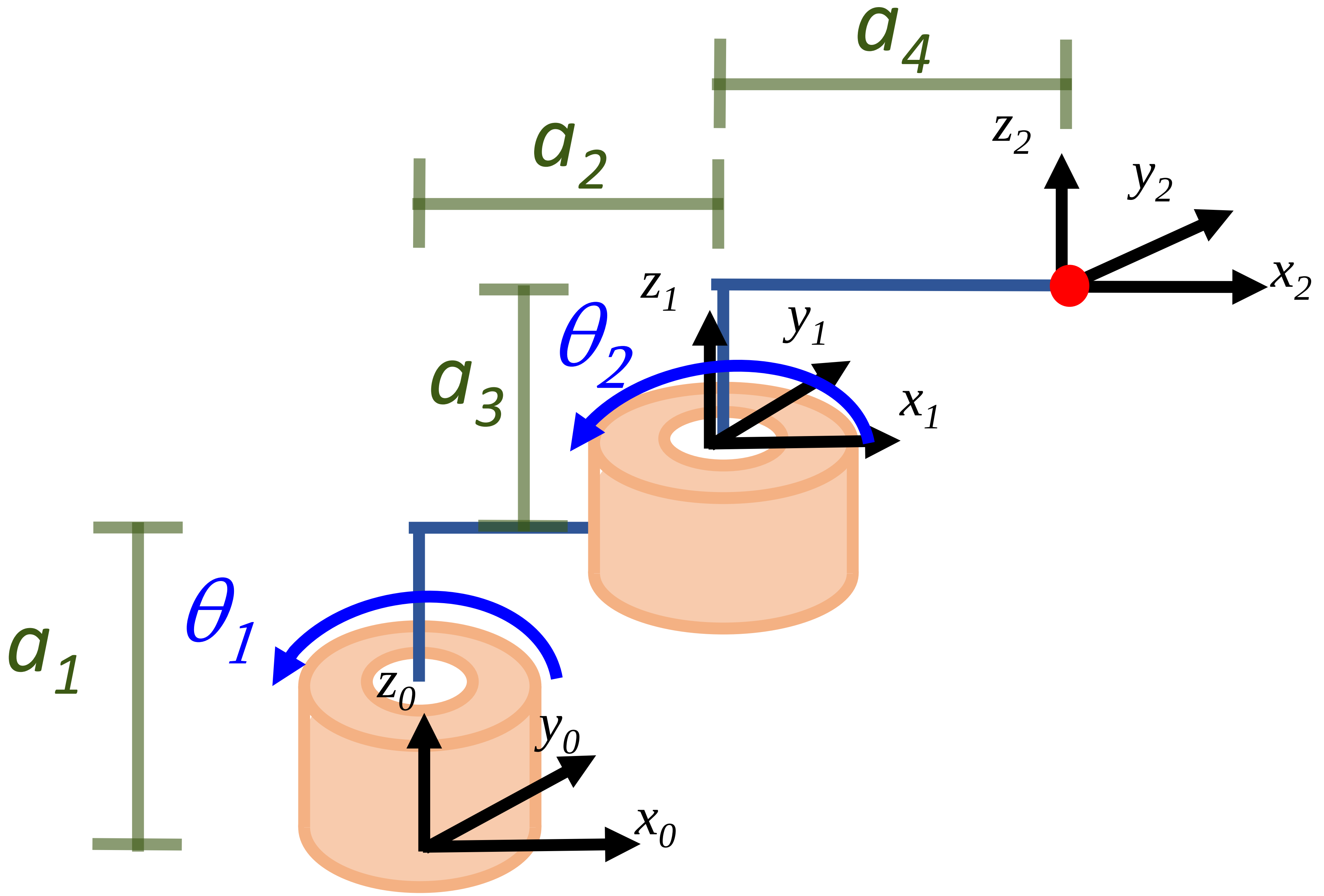
SECTION 3

Simplified Rotational Matrix and Jacobian Matrix



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$



a4
a2
z2
y2
x2
z1
y1
x1
θ2
a3
θ1
a1
z0
y0
x0

$$R = R_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_1^0 = \begin{bmatrix} a_2 \cos(\theta_1) \\ a_2 \sin(\theta_1) \\ a_1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_2 \cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_2 \sin(\theta_1) \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = R_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_2^1 = \begin{bmatrix} a_4 \cos(\theta_2) \\ a_4 \sin(\theta_2) \\ a_3 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_4 \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_4 \sin(\theta_2) \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 \, H_2^1$$

$$H_2^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_2\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_2\sin(\theta_1) \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_4\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_4\sin(\theta_2) \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_2c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_2s\theta_1 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_4c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_4s\theta_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1c\theta_2 - s\theta_1s\theta_2 & -c\theta_1s\theta_2 - s\theta_1c\theta_2 & 0 & a_4c\theta_1c\theta_2 - a_4s\theta_1s\theta_2 + a_2c\theta_1 \\ s\theta_1c\theta_2 + c\theta_1s\theta_2 & -s\theta_1s\theta_2 + c\theta_1c\theta_2 & 0 & a_4s\theta_1c\theta_2 + a_4c\theta_1s\theta_2 + a_2s\theta_1 \\ 0 & 0 & 1 & a_1 + a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_2\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_2\sin(\theta_1) \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & -c\theta_1 s\theta_2 - s\theta_1 c\theta_2 & 0 & a_4 c\theta_1 c\theta_2 - a_4 s\theta_1 s\theta_2 + a_2 c\theta_1 \\ s\theta_1 c\theta_2 + c\theta_1 s\theta_2 & -s\theta_1 s\theta_2 + c\theta_1 c\theta_2 & 0 & a_4 s\theta_1 c\theta_2 + a_4 c\theta_1 s\theta_2 + a_2 s\theta_1 \\ 0 & 0 & 1 & a_1 + a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Omega_1 = \begin{bmatrix} a_2\, c\theta_1 \\ a_2\, s\theta_1 \\ a_1 \end{bmatrix} = d_1 = \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Omega_2 = \begin{bmatrix} a_4 c\theta_1 c\theta_2 - a_4 s\theta_1 s\theta_2 + a_2 c\theta_1 \\ a_4 s\theta_1 c\theta_2 + a_4 c\theta_1 s\theta_2 + a_2 s\theta_1 \\ a_1 + a_3 \end{bmatrix} = d_2 = \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

	Prismatic	Revolute
Linear	$R_{i-1}^{0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^{0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n - d_{i-1})$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^{0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$Z_0 \times (d_2 - d_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ d_{21} & d_{22} & d_{23} \end{vmatrix} = -d_{22}i + d_{21}j + 0k = \begin{bmatrix} -d_{22} \\ d_{21} \\ 0 \end{bmatrix}$$

$$Z_1 \times (d_2 - d_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} d_{21} - d_{11} \\ d_{22} - d_{12} \\ d_{23} - d_{13} \end{bmatrix} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ d_{21} - d_{11} & d_{22} - d_{12} & d_{23} - d_{13} \end{vmatrix} = (d_{12} - d_{22})i + (d_{22} - d_{12})j + 0k$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\omega_z = \dot{\theta}_1 + \dot{\theta}_2$$

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -a_4 s\theta_1 c\theta_2 - a_4 c\theta_1 s\theta_2 - a_2 s\theta_1 & -a_4 s\theta_1 c\theta_2 - a_4 c\theta_1 s\theta_2 \\ a_4 c\theta_1 c\theta_2 - a_4 s\theta_1 s\theta_2 + a_2 c\theta_1 & a_4 c\theta_1 c\theta_2 - a_4 s\theta_1 s\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

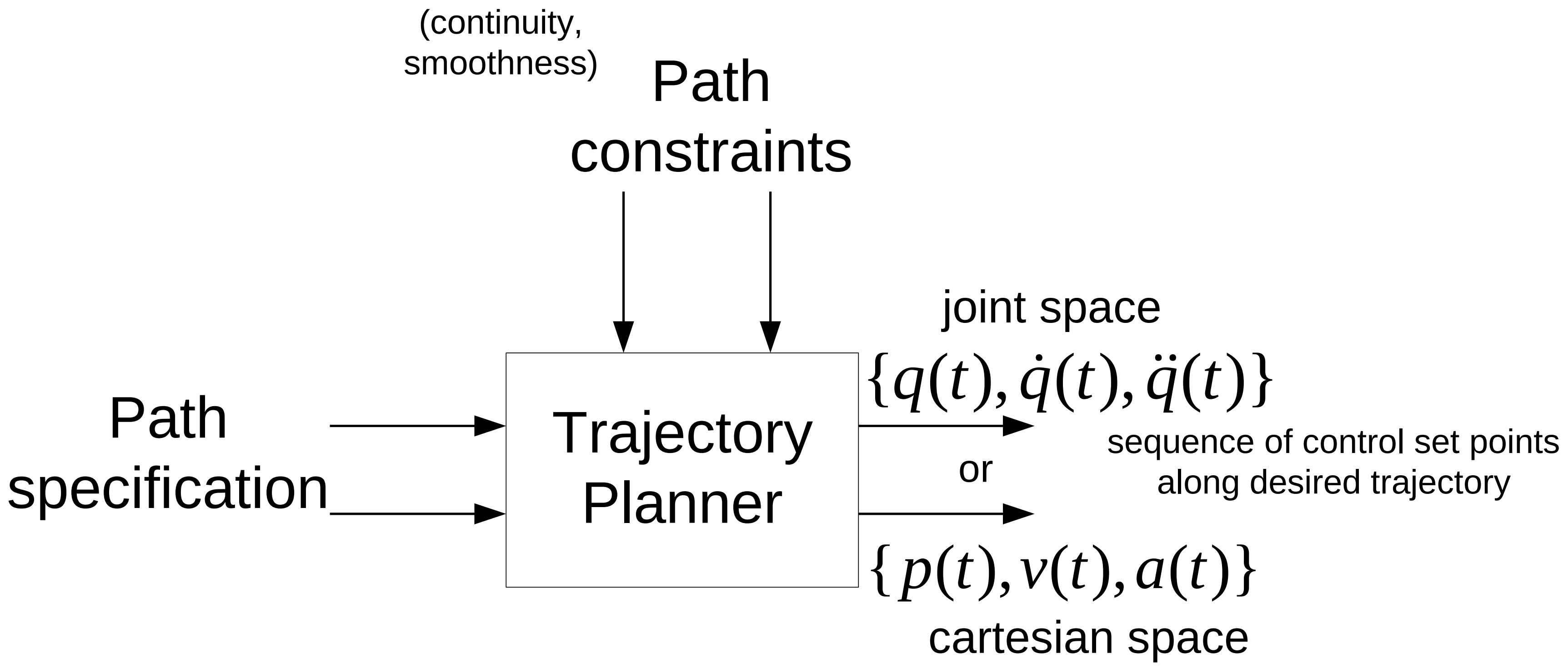
$$J^{-1} = \frac{\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}}{\begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}}$$

$$\dot{\theta}_1 = {J^{-1}}_{11}\, \dot{x} + {J^{-1}}_{12}\, \dot{y}$$
$$\dot{\theta}_2 = {J^{-1}}_{21}\, \dot{x} + {J^{-1}}_{22}\, \dot{y}$$

Python Program

SECTION 3

Trajectory Planning



Trajectory Planning

- **Problem statement**
  - Turn a specified Cartesian-space trajectory of Pe into appropriate joint position reference values
- **Input**
  - Cartesian space path
  - Path constraints including velocity and acceleration limits and singularity analysis.
- **Output**
  - a series of joint position/velocity reference values to send to the controller