STATS 202C: Project 2
Exact sampling of the Ising/Potts Model with coupled Markov Chains

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Introduction

We consider the Ising model in an $n \times n$ lattice with 4 nearest neighbors. The state X is a binary image defined on the lattice X, each at site or pixel s takes value in $\{0, 1\}$. The model sis

$$\pi(X) = \frac{1}{Z} \exp\{\beta \sum_{\langle s,t \langle} \mathbb{1} \left(X_s = X_t\right)\} \tag{1}$$

We simulate 2 Markov Chains with the Gibbs sampler:

- MC1 starts with all sites being 1 (white chain) and its state is denoted by X^1
- MC2 starts with all sites being 0 (black chain) and its state is denoted by X^2

At each step, the Gibbs sampler picks up a site s in both images, and calculates the conditional probabilities, which only depends on its 4 nearest neighbors, denoted by ∂s

$$\pi\left(X_s^1 | X_{\partial s}^1\right) \qquad \pi\left(X_s^2 | X_{\partial s}^2\right)$$

It updates the variables X_s^1 and X_s^2 according to the above two conditional probabilities, and shares the same random number $r = \text{rand}[0, 1]$.

For this project, we are interested in the behavior of the two Markov chains and how this behavior varies with different values of β.

Probem 1

Prove that $X_s^1 \geq X_s^2$, for all s, in any time. That is, the white chain is always above the black chain.

Solution
We prove that $X_s^1 \geq X_s^2, \forall s$ at any time by induction on time.

Base Case: The inequality holds trivially, since at the initial step, all sites in the first Markov chain are 1, and all states in the second Markov chain are 0.

Inductive Hypothesis: Suppose that the inequality holds for all s, for time t. We will show that it also holds for $t+1$. Let s^* be the site chosen at time $t+1$. We consider two cases:

Case 1: s^* has never changed state. In this case, the inequality holds trivially, since s^* in the first Markov chain will be 1, and 0 in the second Markov chain.

Case 2: s^* has changed state. Let t_0 be the first time s^* changed state. At t_0, we know that the conditional probability for the second Markov chain is

$$\pi\left(X_{s^*}^2 | X_{\partial s^*}^2\right)$$

which must be less than the corresponding conditional probability associated with the first Markov chain because the conditional probabilities depend only the neighboring states of s^*. By the inductive hypothesis, we know that all neighboring states of s^* in the first Markov chain are greater than their corresponding states in the second Markov chain. Since $\pi\left(X_{s^*}^2 | X_{\partial s^*}^2\right) \leq \pi\left(X_{s^*}^1 | X_{\partial s^*}^1\right)$, then $X_{s^*}^2$ changes state if and only if $X_{s^*}^1$ (since they share the same random number $r \sim \text{Unif}[0,1]$). Applying this sequence of events forward to the time $t+1$, we see that this restriction holds during each update of s^*, so at $t+1$, it also follows that $X_{s^*}^2 \leq X_{s^*}^1$. We have thus shown that the inequality holds for $t+1$, and we conclude that $X_{s^*}^1 \geq X_{s^2}$ for all s at any time. $\square$

Problem 2

We plot the two chain states, using their total sum $\sum_s X_s^1$ and $\sum_s X_s^2$ over the sweeps, and we can see the image when the two chains coalesce. We observe the behavior of the Markov chains for the following values of β:

$$\beta = \{0.5, 0.65, 0.75, 0.83, 0.84, 0.85, 0.86, 0.88, 0.9, 1.0\}$$

The first four values of beta are shown below in Figure 1. The convergence begins to slow down for $\beta = 0.83$, but the Markov Chains still coalesce fairly quickly.

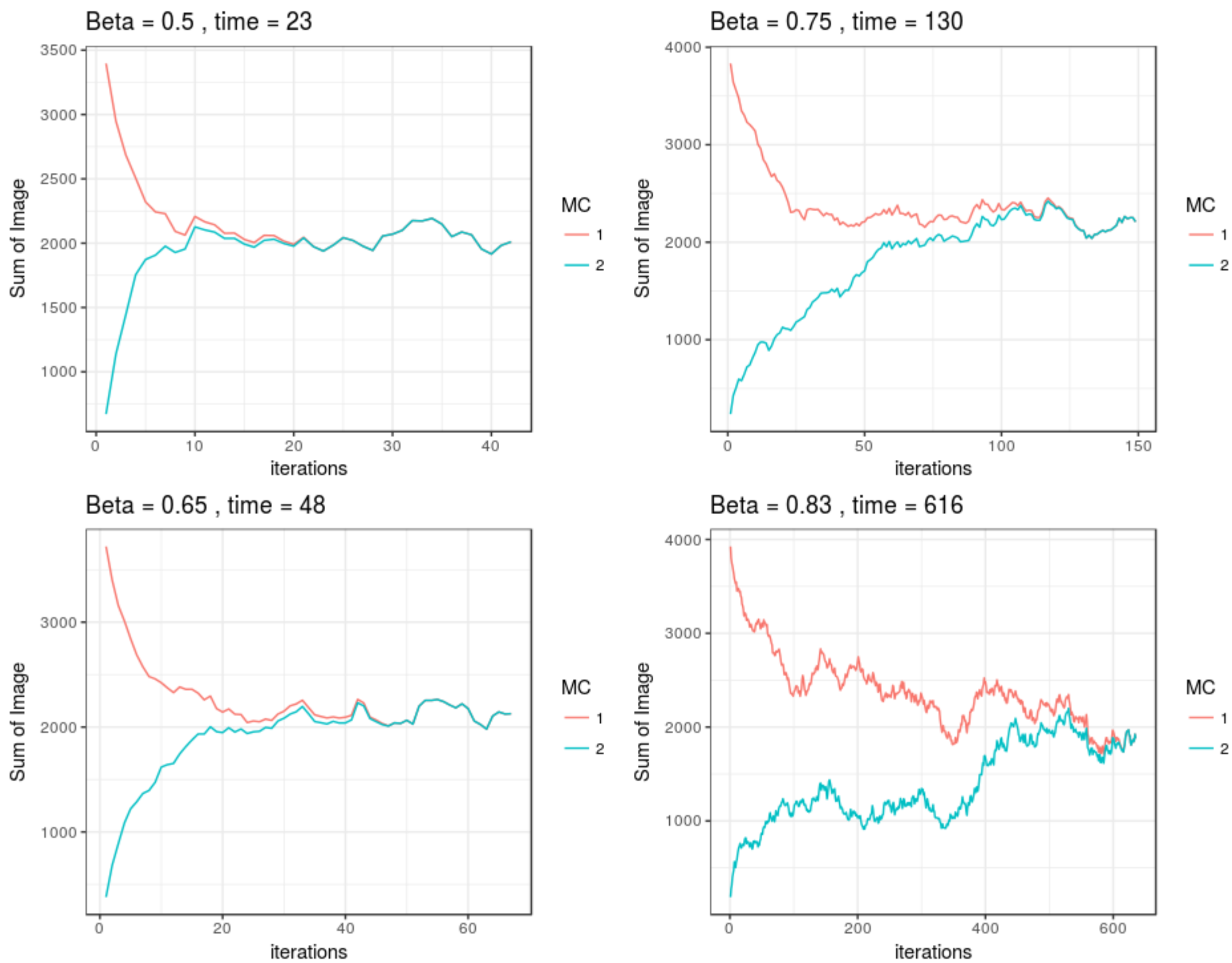

Figure 1: For these values of beta, we see very quick convergence.

However, as we increase the value of β to 0.84 and onward, the convergence slows down dramatically, as seen in Figure 2. In particular, for $\beta > 0.84$, we need upwards of several thousand iterations to reach convergence. We decrease our increments for β to investigate this "slowing-down" behavior.

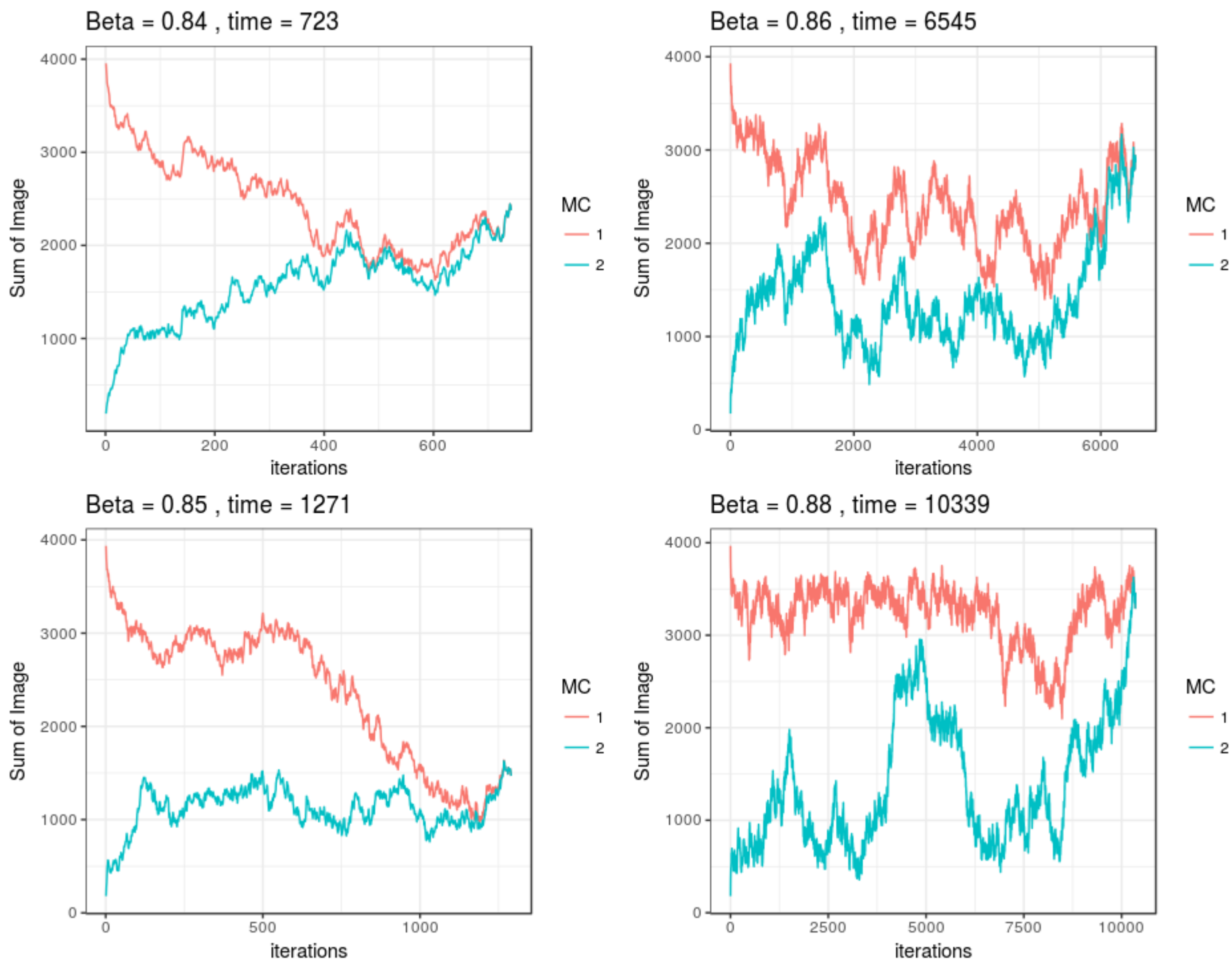

Figure 2: For these values of beta, we observe very slow convergence.

Finally, for $\beta = 0.9, 1.0$, we do not get convergence at all. We can see in Figure 3 that the movement in both of the Markov chains is minimal, and they show very little intention of coalescing.

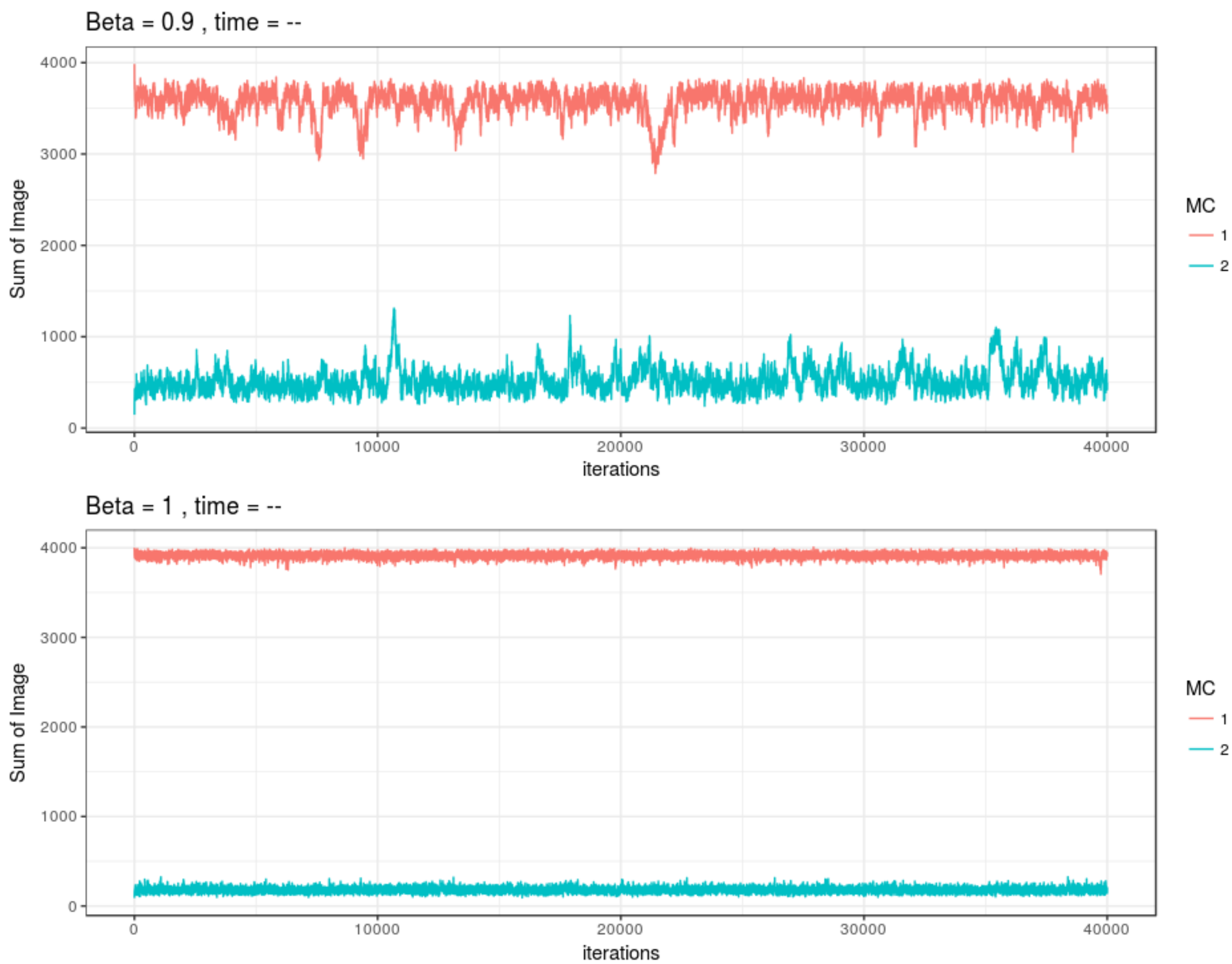

Figure 3: For these values of beta, we do not observe convergence.

Problem 3

Next, we plot the curve of τ, the coalescence times, against varying values of β. Note in Figure 4 the slowing down around $\beta = 0.84$, called the phase transition. Shortly after this point, we fail to get convergence for values of $\beta > 0.88$ in a timely manner.

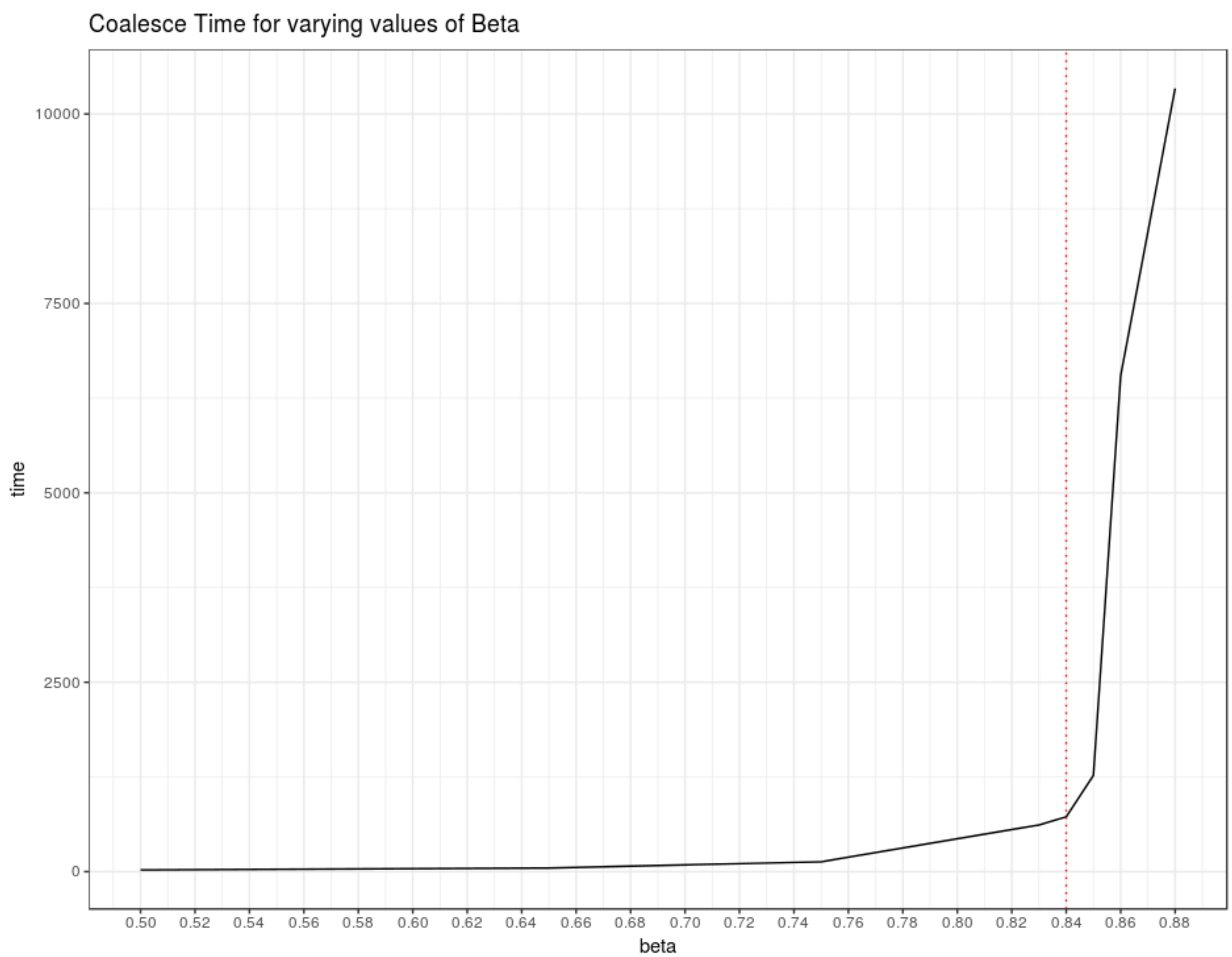

Figure 4: Note the critical slowing down around $\beta = 0.84$.