Stat202C Project no. 2 (15 points)

Due date: May 12 Friday on CCLE

Exact sampling of the Ising/Potts model with coupled Markov Chains.

We consider Ising model in an n x n lattice (n=64, or you may try n= 128 if you have a fast computer) with 4-nearest neighbor. The state X is a binary image defined on the lattice and the variable X_s at each site or pixel s takes value in {0, 1}. The model is

$$\pi(X) = \frac{1}{Z}\exp\{\beta \sum_{<s,t>} 1(X_s = X_t)\}$$

We simulate two Markov Chains with the Gibbs sampler:

- MC1 starts with all sites being 1 (call it the white chain) and its state is denoted by X^1;
- MC2 starts with all sites being 0 (call it the black chain) and its state is denoted by X^2.

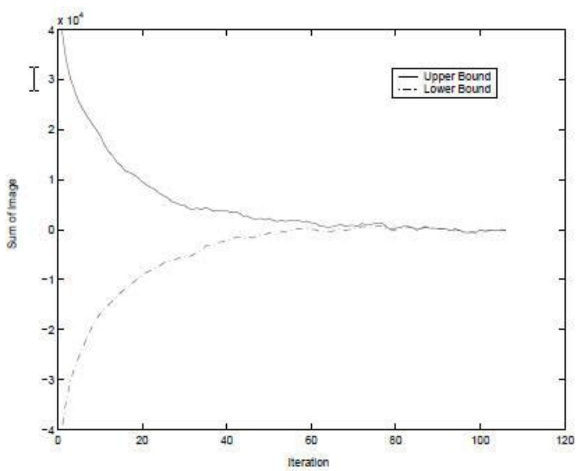
At each step, the Gibbs sampler picks up a site s in both images, and calculates the conditional probabilities, which only depends on its 4 nearest neighbor denoted by ∂s.

$$\pi(X_s^1 | X_{\partial s}^1) \quad \text{and} \quad \pi(X_s^2 | X_{\partial s}^2)$$

It updates the variables X_s^1 and X_s^2 according to the above two conditional probabilities, and <u>shares the same random number</u> r = rand[0,1]. The two Markov Chains are said to be "<u>*coupled*</u>".

1, Prove that $X_s^1 \geq X_s^2, \forall s$ in any time. That is, the white chain is always above the black chain.

2, When the two chains meet each other, i.e. $X_s^1 = X_s^2 \ \forall s$ after many sweeps, they are said to "coalesce". They will stay in the same state forever as they are driven by the same random number at each step. We denote the coalescence time (number of sweeps) by τ. The images after time τ are said to be <u>*exact samples*</u> from the Ising model.



Plot the two chain states (using their total sum $\sum_s X_s^1$ and $\sum_s X_s^2$) over the sweeps as it is shown above and <u>show the image</u> when the two chains coalesced.

Try values $\beta = 0.5,\ \ 0.65,\ \ 0.75,\ \ 0.83,\ \ 0.84,\ \ 0.85,\ \ 0.9,\ \ 1.0.$

3, Plot the curve of τ versus β (using the parameters above) to see a critical slowing-down around 0.84, which the physicists call "phase transition".