STATS 202C: Project 2
Exact sampling of the Ising/Potts Model with coupled Markov Chains

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Introduction

We consider the Ising model in an $n \times n$ lattice with 4 nearest neighbors. The state X is a binary image defined on the lattice X, each at site or pixel s takes value in $\{0, 1\}$. The model sis

$$\pi(X) = \frac{1}{Z} \exp\{\beta \sum_{\langle s,t\langle} 1\left(X_s = X_t\right)\} \tag{1}$$

We simulate 2 Markov Chains with the Gibbs sampler:

- MC1 starts with all sites being 1 (white chain) and its state is denoted by X^1
- MC2 starts with all sites being 0 (black chain) and its state is denoted by X^2

At each step, the Gibbs sampler picks up a site s in both images, and calculates the conditional probabilities, which only depends on its 4 nearest neighbors, denoted by ∂s

$$\pi\left(X_s^1 | X_{\partial s}^1\right) \qquad \pi\left(X_s^2 | X_{\partial s}^2\right)$$

It updates the variables X_s^1 and X_s^2 according to the above two conditional probabilities, and shares the same random number $r = \text{rand}[0, 1]$.

Probem 1

Prove that $X_s^1 \geq X_s^2$, for all s, in any time. That is, the white chain is always above the black chain.

Solution

Problem 2

We plot the two chain states, using their total sum $\sum_s X_s^1$ and $\sum_s X_s^2$ over the sweeps, and we can see the image when the two chains coalesce. We use the following values for β:

$$\beta = \{0.5, 0.65, 0.75, 0.83, 0.84, 0.85, 0.9, 1.0\}$$

Problem 3

We plot the curve of τ versus β to see a critical slowing-down around 0.84.