VARIATIONAL APPROXIMATION FOR NON-ZERO-CENTERED PRIOR

PETER CARBONETTO*

Here we consider a small extension to the original model from the *Bayesian Analysis* paper in which the "slab" in the spike-and-slab prior on the regression coefficients is normal with mean $\sigma\beta_0$ and standard deviation $\sigma\sigma_a$. The original model is of course recovered as a special case when $\beta_0 = 0$.

With this prior, the variational lower bound to the marginal log-likelihood has the following analytical expression:

$$
\begin{aligned}
F(\theta;\phi) = &-\frac{n}{2}\log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{j=1}^{p}(\mathbf{X}^T\mathbf{X})_{jj}\mathrm{Var}[\beta_j] \\
&- \sum_{j=1}^{p}\alpha_j \log\left(\frac{\alpha_j}{\pi_j}\right) - \sum_{j=1}^{p}(1-\alpha_j)\log\left(\frac{1-\alpha_j}{1-\pi_j}\right) \\
&+ \sum_{j=1}^{p}\frac{\alpha_j}{2}\left[1 + \log\left(\frac{s_j^2}{\sigma^2\sigma_a^2}\right) - \frac{s_j^2 + (\mu_j - \sigma\beta_0)^2}{\sigma^2\sigma_a^2}\right]. \qquad (1)
\end{aligned}
$$

Setting $\beta_0 = 0$ recovers the variational lower bound described in the *Bayesian Analysis* paper.

Taking partial derivatives of the free parameters α_j, μ_j and s_j^2 with respect to this lower bound, setting the partial derivatives to zero, and solving for these parameters yields the following co-ordinate ascent updates:

$$
s_j^2 = \frac{\sigma^2}{(\mathbf{X}^T\mathbf{X})_{jj} + 1/\sigma_a^2} \qquad (2)
$$

$$
\mu_j = \frac{s_j^2}{\sigma^2}\left(\sigma\beta_0/\sigma_a^2 + (\mathbf{X}^T y)_j - \sum_{k \neq j}(\mathbf{X}^T\mathbf{X})_{jk}\alpha_k\mu_k\right) \qquad (3)
$$

$$
\frac{\alpha_j}{1-\alpha_j} = \frac{\pi_j}{1-\pi_j} \times \frac{s_j}{\sigma_a\sigma} \times \exp\left\{\frac{1}{2}\left(\frac{\mu_j^2}{s_j^2} - \frac{\beta_0^2}{\sigma_a^2}\right)\right\}. \qquad (4)
$$

Taking partial derivatives of the hyperparameters σ^2, σ_a^2 and β_0 yields the following approximate M-step updates:

$$
\sigma^2 = \qquad (5)
$$

$$
\sigma_a^2 = \qquad (6)
$$

$$
\beta_0 = \frac{n_{\beta_0}\mu_{\beta_0} + \sum_{j=1}^{p}\alpha_j\mu_j}{\sigma(n_{\beta_0} + \sum_{j=1}^{p}\mu_j)}. \qquad (7)
$$

*Research Computing Center and the Department of Human Genetics, University of Chicago, Chicago, IL, 60637