

VARIATIONAL APPROXIMATION FOR “FINE-MAPPING” MODEL

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First, we define the “fine-mapping” model:

$$\begin{aligned} p(y | \mathbf{X}, \beta, \sigma^2) &= N(y | \mathbf{X}\beta_1 + \cdots + \mathbf{X}\beta_K, \sigma^2 I) \\ p(\beta_{jk} | \gamma_k = j) &= N(0, \sigma^2 \sigma_a^2) \\ p(\beta_{jk} | \gamma_k \neq j) &= \delta_0 \\ p(\gamma_k = j) &= \pi_j. \end{aligned}$$

Next, we define the variational approximation:

$$q(\beta, \gamma; \phi) = \prod_{k=1}^K q(\beta_k, \gamma_k; \phi_k)$$

where the individual factors are of the form

$$q(\beta_k, \gamma_k; \phi_k) = \prod_{j=1}^p \left\{ \alpha_{jk} N(\beta_{jk} | \mu_{jk}, s_{jk}^2) \right\}^{\delta(\gamma_k=j)} \delta_0(\beta_{jk})^{\delta(\gamma_k \neq j)}$$

Under this approximation, the coefficients (“effects”) for the k th mixture component are independent of the coefficients for the other mixture components, conditioned on the hyperparameters.

The variational lower bound

$$F(\theta, \phi) \equiv \iint q(\beta, \gamma; \phi) \log \{ p(y, \beta, \gamma | \mathbf{X}, \theta) / q(\beta, \gamma; \phi) \} d\beta d\gamma$$

is derived to be

$$\begin{aligned} F(\theta, \phi) &= -\frac{n}{2} \log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} + \frac{1}{2\sigma^2} \sum_{k=1}^K \|\mathbf{X}r_k\|^2 \\ &\quad - \frac{1}{2\sigma^2} \sum_{k=1}^K \sum_{j=1}^p (\mathbf{X}^T \mathbf{X})_{jj} \alpha_{jk} (\mu_{jk}^2 + s_{jk}^2) + \sum_{k=1}^K \sum_{j=1}^p \alpha_{jk} \log \left(\frac{\pi_j}{\alpha_{jk}} \right) \\ &\quad + \sum_{k=1}^K \sum_{j=1}^p \frac{\alpha_{jk}}{2} \left[1 + \log \left(\frac{s_{jk}^2}{\sigma^2 \sigma_a^2} \right) - \frac{\mu_{jk}^2 + s_{jk}^2}{\sigma^2 \sigma_a^2} \right], \end{aligned}$$

in which r_k is a vector with entries $r_{jk} = \alpha_{jk} \mu_{jk}$, and r is a vector with entries $r_j = \sum_{k=1}^K \alpha_{jk} \mu_{jk}$.

The coordinate descent updates for optimizing the variational lower bound can be obtained by taking partial derivatives of the Kullback-Leibler divergence, setting the partial derivatives to zero, and solving for the parameters α_{jk} , μ_{jk} and s_{jk}^2 . This

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yields coordinate updates

$$\begin{aligned}
 s_{jk}^2 &= \frac{\sigma^2}{(\mathbf{X}^T \mathbf{X})_{jj} + 1/\sigma_a^2} \\
 \mu_{jk} &= \frac{s_{jk}^2}{\sigma^2} \left((\mathbf{X}^T y)_j - \sum_{j'=1}^p (\mathbf{X}^T \mathbf{X})_{jj'} \sum_{k' \neq k} \alpha_{j'k'} \mu_{j'k'} \right) \\
 \alpha_{jk} &=
 \end{aligned}$$