VARIATIONAL APPROXIMATION FOR "FINE-MAPPING" MODEL

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First, we define the "fine-mapping" model:

$$\begin{aligned} p(y \,|\, \mathbf{X}, \beta, \sigma^2) &= N(y \,|\, \mathbf{X}\beta_1 + \cdots + \mathbf{X}\beta_K, \sigma^2 I) \\ p(\beta_{jk} \,|\, \gamma_k = j) &= N(0, \sigma^2 \sigma_a^2) \\ p(\beta_{jk} \,|\, \gamma_k \neq j) &= \delta_0 \\ p(\gamma_k = j) &= \pi_j. \end{aligned}$$

Next, we define the variational approximation:

$$q(\beta, \gamma; \phi) = \prod_{k=1}^{K} q(\beta_k, \gamma_k; \phi_k)$$

where the individual factors are of the form

$$q(\beta_k, \gamma_k; \phi_k) = \prod_{j=1}^{p} \left\{ \alpha_{jk} N(\beta_{jk} \,|\, \mu_{jk}, s_{jk}^2) \right\}^{\delta(\gamma_k = j)} \delta_0(\beta_{jk})^{\delta(\gamma_k \neq j)}$$

Under this approximation, the coefficients ("effects") for the kth mixture component are independent of the coefficients for the other mixture components, conditioned on the hyperparameters.

The variational lower bound

$$F(\theta, \phi) \equiv \iint q(\beta, \gamma; \phi) \log \left\{ p(y, \beta, \gamma \,|\, \mathbf{X}, \theta) / q(\beta, \gamma; \phi) \right\} d\beta \, d\gamma$$

is derived to be

$$\begin{aligned} F(\theta, \phi) = &-\frac{n}{2} \log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} + \frac{1}{2\sigma^2} \sum_{k=1}^{K} \|\mathbf{X} r_k\|^2 \\ &- \frac{1}{2\sigma^2} \sum_{k=1}^{K} \sum_{j=1}^{p} (\mathbf{X}^T\mathbf{X})_{jj} \alpha_{jk} (\mu_{jk}^2 + s_{jk}^2) + \sum_{k=1}^{K} \sum_{j=1}^{p} \alpha_{jk} \log\left(\frac{\pi_j}{\alpha_{jk}}\right) \\ &+ \sum_{k=1}^{K} \sum_{j=1}^{p} \frac{\alpha_{jk}}{2} \left[1 + \log\left(\frac{s_{jk}^2}{\sigma^2 \sigma_a^2}\right) - \frac{\mu_{jk}^2 + s_{jk}^2}{\sigma^2 \sigma_a^2} \right], \end{aligned}$$

in which r_k is a vector with entries $r_{jk} = \alpha_{jk}\mu_{jk}$, and r is a vector with entries $r_j = \sum_{k=1}^{K} \alpha_{jk}\mu_{jk}$.

The coordinate descent updates for optimizing the variational lower bound can be obtained by taking partial derivatives of the Kullback-Leibler divergence, setting the partial derivatives to zero, and solving for the parameters α_{jk}, μ_{jk} and s_{jk}^2. This

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yields coordinate updates

$$
\begin{aligned}
s_{jk}^2 &= \frac{\sigma^2}{(\mathbf{X}^T\mathbf{X})_{jj} + 1/\sigma_a^2)} \\
\mu_{jk} &= \frac{s_{jk}^2}{\sigma^2}\left((\mathbf{X}^T y)_j - \sum_{j'=1}^{p}(\mathbf{X}^T\mathbf{X})_{jj'} \sum_{k' \neq k} \alpha_{j'k'}\mu_{j'k'}\right) \\
\alpha_{jk} &\propto \pi_j \times \frac{s_{jk}}{\sigma\sigma_a} \times e^{\mathrm{SSR}_{jk}/2},
\end{aligned}
$$

where $\mathrm{SSR}_{jk} = \mu_{jk}^2 / s_{jk}^2$.