

VARIATIONAL APPROXIMATION FOR NON-ZERO-CENTERED PRIOR

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Here we consider a small extension to the original model from the *Bayesian Analysis* paper in which the “slab” in the spike-and-slab prior on the regression coefficients is normal with mean $\sigma\beta_0$ and standard deviation $\sigma\sigma_a$. The original model is of course recovered as a special case when $\beta_0 = 0$.

With this prior, the variational lower bound to the marginal log-likelihood has the following analytical expression:

$$\begin{aligned} F(\theta; \phi) = & -\frac{n}{2} \log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} - \frac{1}{2\sigma^2} \sum_{j=1}^p (\mathbf{X}^T \mathbf{X})_{jj} \text{Var}[\beta_j] \\ & - \sum_{j=1}^p \alpha_j \log\left(\frac{\alpha_j}{\pi_j}\right) - \sum_{j=1}^p (1 - \alpha_j) \log\left(\frac{1 - \alpha_j}{1 - \pi_j}\right) \\ & + \sum_{j=1}^p \frac{\alpha_j}{2} \left[1 + \log\left(\frac{s_j^2}{\sigma^2 \sigma_a^2}\right) - \frac{s_j^2 + (\mu_j - \sigma\beta_0)^2}{\sigma^2 \sigma_a^2} \right]. \end{aligned} \quad (1)$$

Setting $\beta_0 = 0$ recovers the variational lower bound described in the *Bayesian Analysis* paper.

Taking partial derivatives of the free parameters α_j , μ_j and s_j^2 with respect to this lower bound, setting the partial derivatives to zero, and solving for these parameters yields the following co-ordinate ascent updates:

$$s_j^2 = \frac{\sigma^2}{(\mathbf{X}^T \mathbf{X})_{jj} + 1/\sigma_a^2} \quad (2)$$

$$\mu_j = \frac{s_j^2}{\sigma^2} \left(\sigma\beta_0/\sigma_a^2 + (\mathbf{X}^T y)_j - \sum_{k \neq j} (\mathbf{X}^T \mathbf{X})_{jk} \alpha_k \mu_k \right) \quad (3)$$

$$\frac{\alpha_j}{1 - \alpha_j} = \frac{\pi_j}{1 - \pi_j} \times \frac{s_j}{\sigma_a \sigma} \times \exp \left\{ \frac{1}{2} \left(\frac{\mu_j^2}{s_j^2} - \frac{\beta_0^2}{\sigma_a^2} \right) \right\}. \quad (4)$$

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