VARIATIONAL APPROXIMATION FOR "FINE-MAPPING" MODEL

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First, we define the "fine-mapping" model:

$$\begin{aligned} p(y \mid \mathbf{X}, \beta, \sigma^2) &= N(y \mid \mathbf{X}\beta_1 + \cdots + \mathbf{X}\beta_K, \sigma^2 I) \\ p(\beta_{jk} \mid \gamma_k = j) &= N(0, \sigma^2 \sigma_a^2) \\ p(\beta_{jk} \mid \gamma_k \neq j) &= \delta_0 \\ p(\gamma_k = j) &= \pi_j. \end{aligned}$$

Next, we define the variational approximation:

$$q(\beta, \gamma; \phi) = \prod_{k=1}^{K} q(\beta_k, \gamma_k; \phi_k)$$

where the individual factors are of the form

$$q(\beta_k, \gamma_k; \phi_k) = \prod_{j=1}^{p} \left\{ \alpha_{jk} N(\beta_{jk} \mid \mu_{jk}, s_{jk}^2) \right\}^{\delta(\gamma_k = j)} \delta_0(\beta_{jk})^{\delta(\gamma_k \neq j)}$$

Under this approximation, the coefficients ("effects") for the kth mixture component are independent of the coefficients for the other mixture components, conditioned on the hyperparameters.

The variational lower bound

$$F(\theta, \phi) \equiv \iint q(\beta, \gamma; \phi) \log \left\{ p(y, \beta, \gamma \mid \mathbf{X}, \theta) / q(\beta, \gamma; \phi) \right\} d\beta \, d\gamma$$

is derived to be

$$\begin{aligned} F(\theta, \phi) = &- \frac{n}{2} \log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} + \frac{1}{2\sigma^2} \sum_{k=1}^{K} \|\mathbf{X}r_k\|^2 \\ &- \frac{1}{2\sigma^2} \sum_{k=1}^{K} \sum_{j=1}^{p} (\mathbf{X}^T\mathbf{X})_{jj} \alpha_{jk} (\mu_{jk}^2 + s_{jk}^2) + \sum_{k=1}^{K} \sum_{j=1}^{p} \alpha_{jk} \log\left(\frac{\pi_j}{\alpha_{jk}}\right) \\ &+ \sum_{k=1}^{K} \sum_{j=1}^{p} \frac{\alpha_{jk}}{2} \left[1 + \log\left(\frac{s_{jk}^2}{\sigma^2 \sigma_a^2}\right) - \frac{\mu_{jk}^2 + s_{jk}^2}{\sigma^2 \sigma_a^2} \right], \end{aligned}$$

in which r_k is a vector with entries $r_{jk} = \alpha_{jk}\mu_{jk}$, and r is a vector with entries $r_j = \sum_{k=1}^{K} \alpha_{jk}\mu_{jk}$.

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