VARIATIONAL APPROXIMATION FOR NON-ZERO-CENTERED PRIOR

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Here we consider a small extension to the original model from the *Bayesian Analysis* paper in which the "slab" in the spike-and-slab prior on the regression coefficients is normal with mean $\sigma\beta_0$ and standard deviation $\sigma\sigma_a$. The original model is of course recovered as a special case when $\beta_0 = 0$.

With this prior, the variational lower bound has the following analytical expression:

$$\begin{aligned} F(\theta;\phi) = & -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\|y - \mathbf{X}r\|^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{j=1}^{p}(\mathbf{X}^T\mathbf{X})_{jj}\mathrm{Var}[\beta_j] \\ & - \sum_{j=1}^{p}\alpha_j \log\left(\frac{\alpha_j}{\pi}\right) - \sum_{j=1}^{p}(1-\alpha_j)\log\left(\frac{1-\alpha_j}{1-\pi}\right) \\ & + \sum_{j=1}^{p}\frac{\alpha_j}{2}\left[1 + \log\left(\frac{s_j^2}{\sigma_a^2\sigma^2}\right) - \frac{s_j^2 + \mu_j^2}{\sigma_a^2\sigma^2}\right], \end{aligned} \tag{1}$$

Taking partial derivatives of the free parameters α_j, μ_j and s_j^2 with respect to this lower bound, setting the partial derivatives to zero, and solving for these parameters yields the following co-ordinate ascent updates:

$$s_j^2 = \frac{\sigma^2}{(\mathbf{X}^T\mathbf{X})_{jj} + 1/\sigma_\beta^2} \tag{2}$$

$$\mu_j = \frac{s_j^2}{\sigma^2}\left((\mathbf{X}^T y)_j - \sum_{k\neq j}(\mathbf{X}^T\mathbf{X})_{jk}\alpha_k\mu_k\right) \tag{3}$$

$$\frac{\alpha_j}{1-\alpha_j} = \frac{\pi}{1-\pi} \times \frac{s_j}{\sigma_\beta\sigma} \times e^{\mu_j^2/(2s_j^2)}. \tag{4}$$

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