CS 5/7320
Artificial Intelligence

Constraint Satisfaction Problems
AIMA Chapter 6

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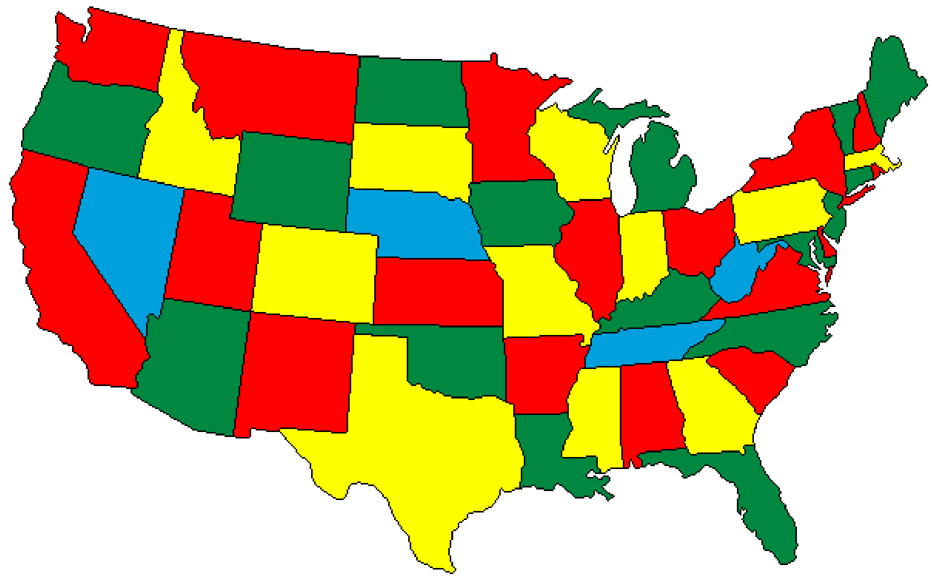
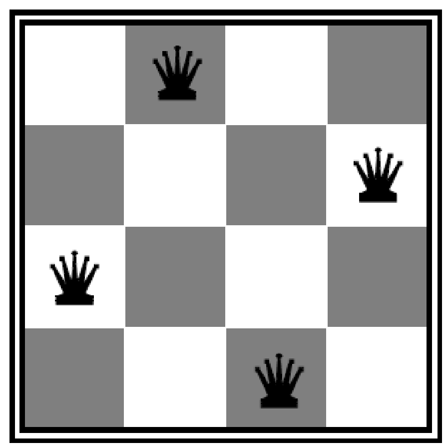
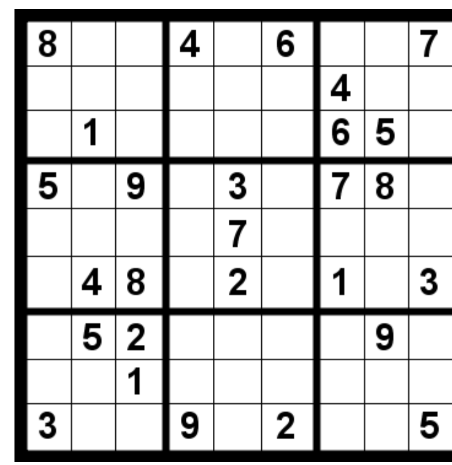
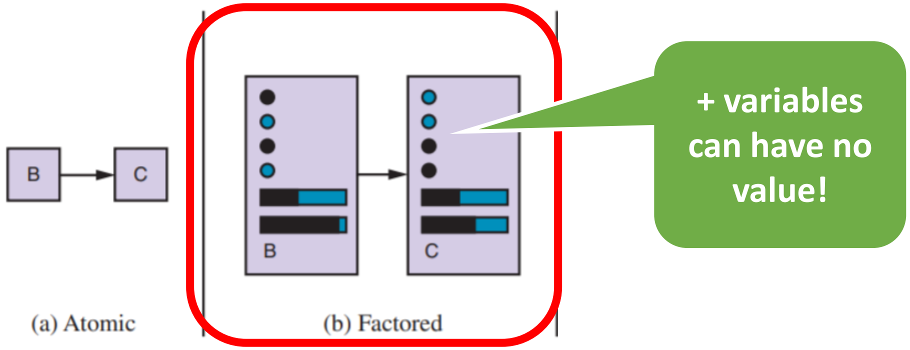
based on Slides by Svetlana Lazepnik
with figures from the AIMA textbook







Constraint satisfaction problems (CSPs)



Definition:

- **State** is defined by a set of **variables** X_i (= factored state description)
  - Each variable can have a **value** from **domain** D_i or be **unassigned** (partial solution).
- **Constraints** are a set of rules specifying allowable combinations of values for subsets of variables (e.g., $X_1 \neq X_7$ or $X_2 > X_9 + 3$)
- **Solution**: a state that is a
  a) **Consistent assignment**: satisfies all constraints
  b) **Complete assignment:** assigns value to each variable

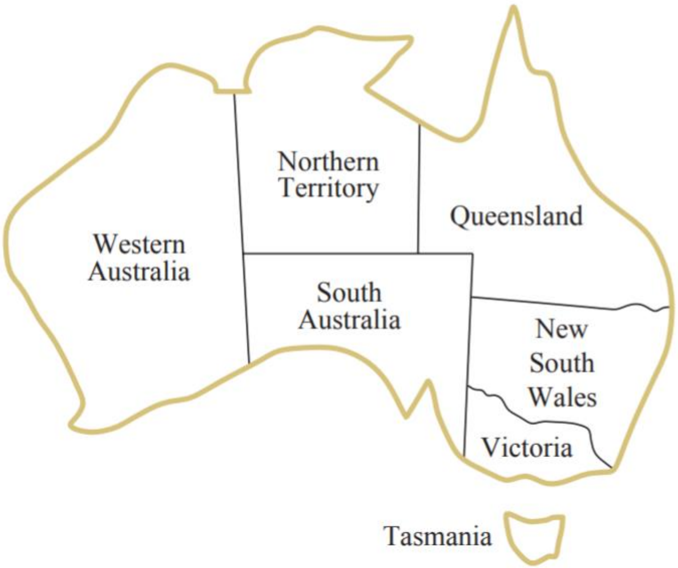
This makes the problem different from the "generic" tree search formulation where we have:

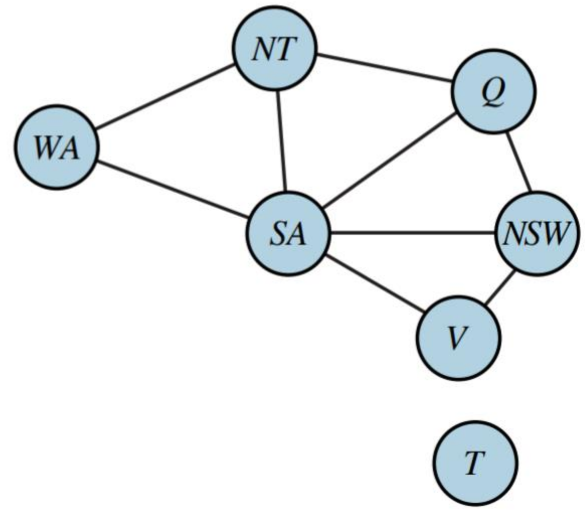
- Atomic states
- States are always compete assignments.
- Constrains are implicit in the transition function.

General-purpose algorithms for CSP with more power than standard search algorithms exit.

Example: Map Coloring (Graph coloring)

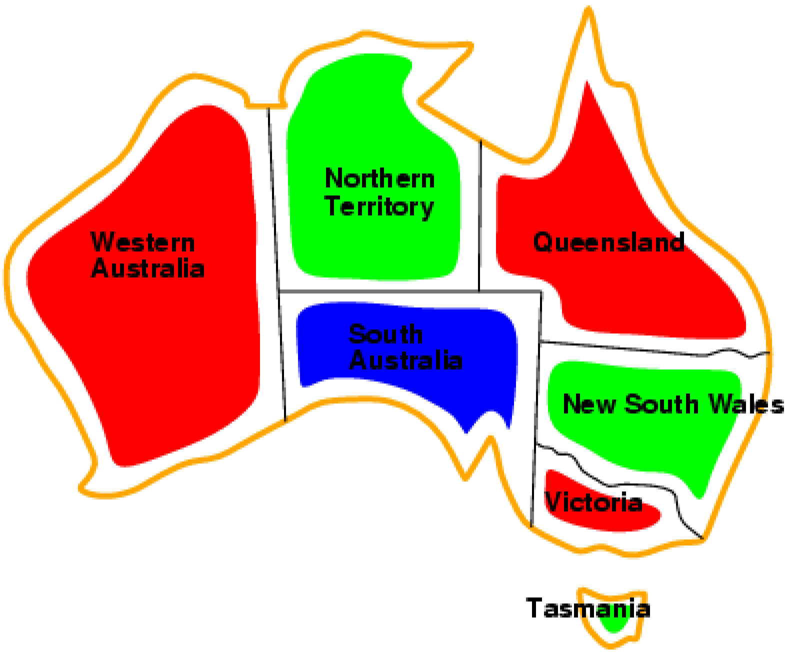
Problem → Constraint graph





- **Variables representing state:** WA, NT, Q, NSW, V, SA, T
- **Variable Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g.,
  WA ≠ NT ⇔ (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

Example: Map Coloring

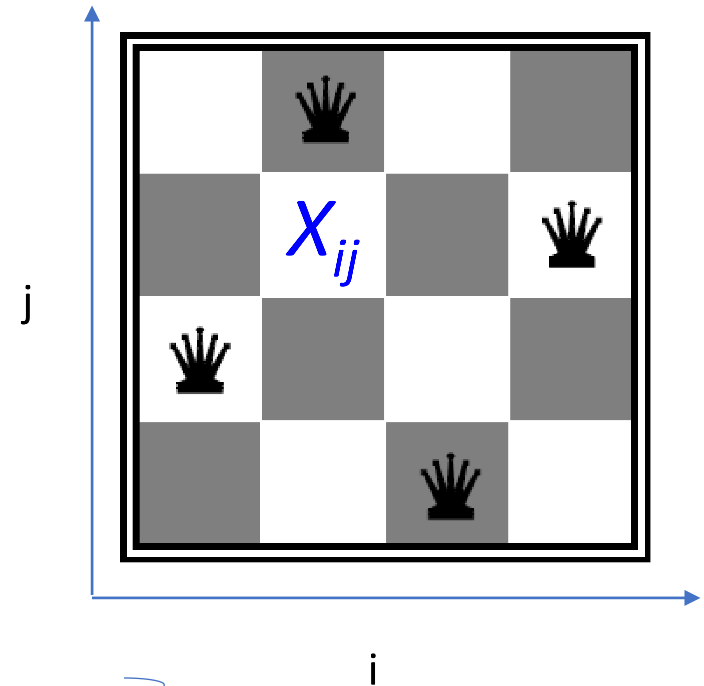


Solutions are ***complete*** and **consistent** assignments, e.g.,

WA = red, NT = green, Q = red, NSW = green,
V = red, SA = blue, T = green

Example: N-Queens

- **Variables:** X_{ij} for $i, j \in \{1, 2, \ldots, N\}$
- **Domains:** $\{0, 1\}$ # Queen: no/yes



- **Constraints:**

$\Sigma_{i,j}\, X_{ij} = N$

$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be in same col.

$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be in same row.

$(X_{ij}, X_{i+k,\, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be diagonal

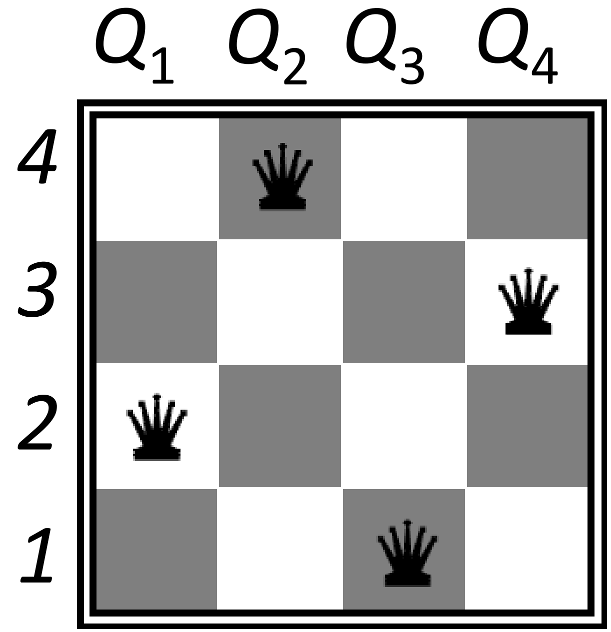
$(X_{ij}, X_{i+k,\, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be diagonal

for $i, j, k \in \{1, 2, \ldots, N\}$

N-Queens: Alternative formulation

- **Variables:** $Q_1, Q_2, \ldots, Q_N$
- **Domains:** $\{1, 2, \ldots\ , N\}$ # row for each col.
- **Constraints:**

  $\forall$ *i*, *j* non-threatening (Q_i , Q_j)



Example:
Q1 = 2, Q2 = 4, Q3 = 1, Q4 = 3

Example: Cryptarithmetic Puzzle

- **Variables:** T, W, O, F, U, R
  X_1, X_2
- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**
  Alldiff(T, W, O, F, U, R)
  $O + O = R + 10 * X_1$
  $W + W + X_1 = U + 10 * X_2$
  $T + T + X_2 = O + 10 * F$
  $T \neq 0, F \neq 0$

Given Puzzle:
Find values for the letters.
Each letter stands for a different digit.

$$\begin{array}{cccc} & X_2 & X_1 & \\ & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

Example: Sudoku

- **Variables:** X_{ij}
- **Domains:** {1, 2, …, 9}
- **Constraints:**

  Alldiff(X_{ij} in the same *unit)*

  Alldiff(X_{ij} in the same *row)*

  Alldiff(X_{ij} in the same *column*)

					8			4
	8	4		1	6			
			5			1		
1		3	8			9		
6		8		X_{ij}		4		3
		2			9	5		1
		7			2			
			7	8		2	6	
2			3					

Some Popular Types of CSPs

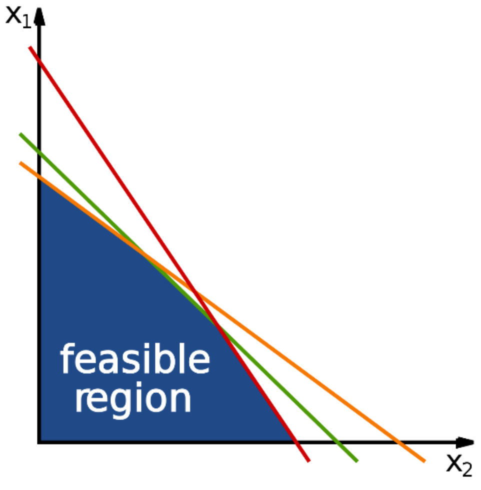
- **Boolean Satisfiability Problem (SAT)**
  Find variable assignments that makes a Boolean expression (often expressed in conjunctive normal form) evaluate as true.

$$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge \neg x_1 = \text{True}$$

- **Integer Programming**
  Variables are restricted to integers. Find a feasible solution that satisfies all constraints. The traveling salesman problem can be expressed as an integer program.

NP-complete

- **Linear Programming**
  Variables are continuous and constraints are linear (in)equalities.
  Find a feasible solution using, e.g., the simplex algorithm.



Real-world CSPs

- Assignment problems

  e.g., who teaches what class for a fixed schedule. Teacher cannot be in two classes at the same time!
- Timetable problems

  e.g., which class is offered when and where? No two classes in the same room at the same problem.
- Scheduling in transportation and production (e.g., order of production steps).
- Many problems can naturally also be formulated as CSPs.

- More examples of CSPs: http://www.csplib.org/

CSP as a Standard Search Formulation

State:

- Values assigned so far

Initial state:

- The empty assignment { } (all variables are unassigned)

Successor function:

- Choose an unassigned variable and assign it a value that does not violate any constraints
- Fail if no legal assignment is found

Goal state:

- Any complete and consistent assignment.

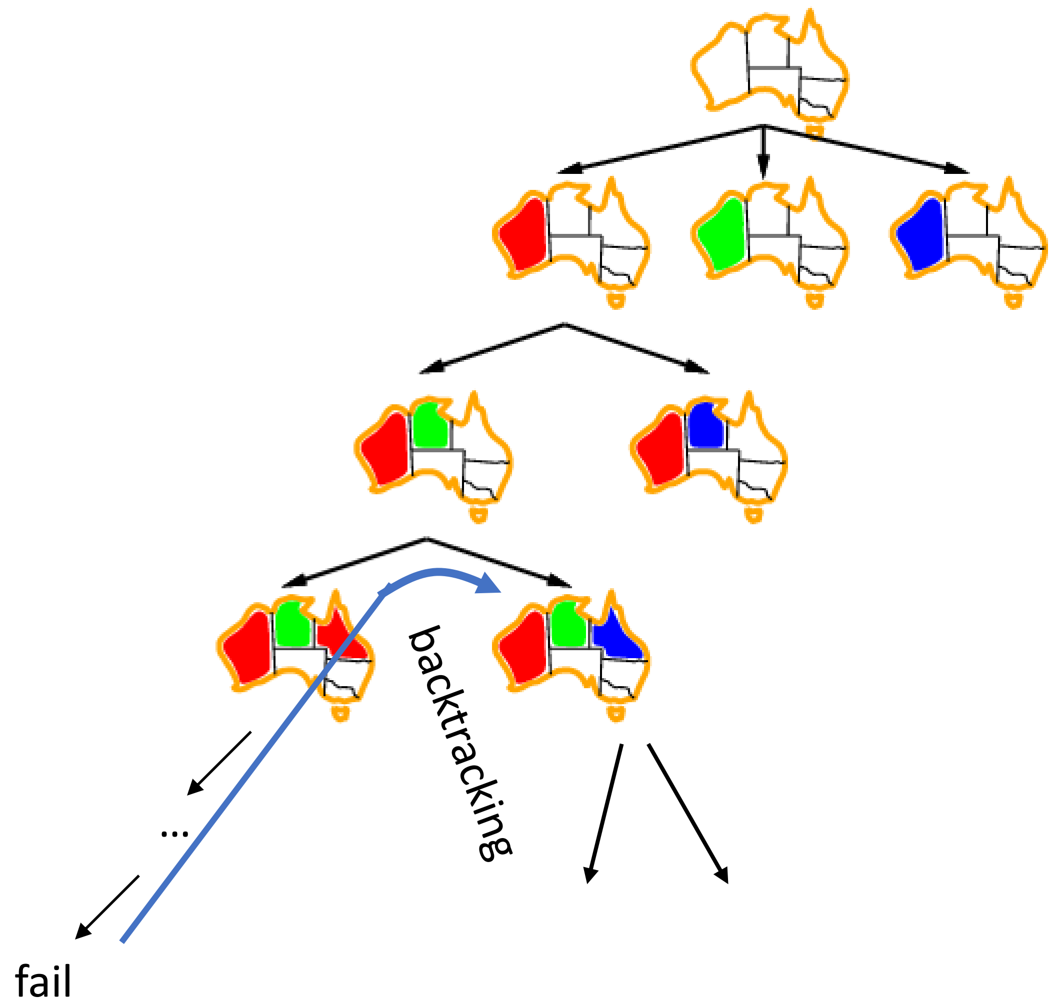
Backtracking search

- In CSP's, variable assignments are **commutative**
  For example,
  [WA = red then NT = green] is the same as
  [NT = green then WA = red]. → Order is not important

- We can build a search tree that assigns the value to one variable per level.
  - Tree depth ***n*** (number of variables)
  - Number of leaves: d^n (d is the number of values per variable)

- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

Example: Backtracking search (DFS)



Backtracking search algorithm

```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Call: `Recursive-Backtracking({}, csp)`

Improving backtracking efficiency:

- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?

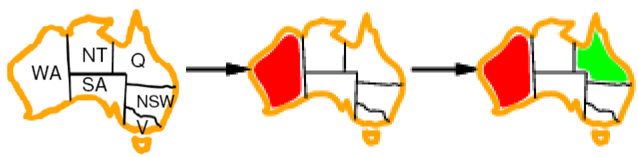
Which variable should be assigned next?
In which order should its values be tried?

- **Most constrained variable:**
  - Keep track of remaining legal values for unassigned variables (using constraints)
  - Choose the variable with the fewest legal values left
  - A.k.a. **minimum remaining values** (MRV) heuristic

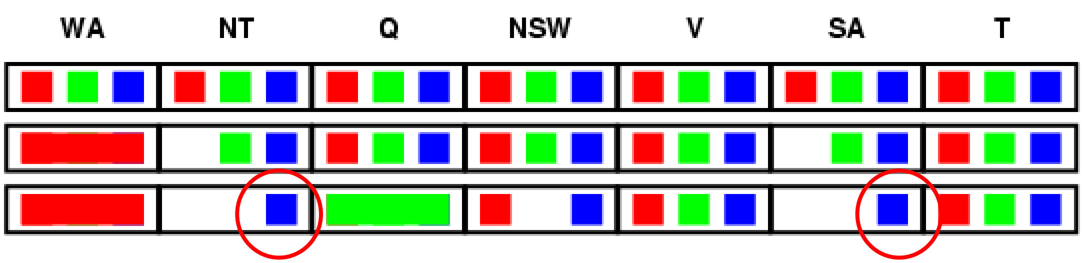
- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

Early detection of failure – Forward checking
Node consistency

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values (i.e., minimum remaining values = 0)



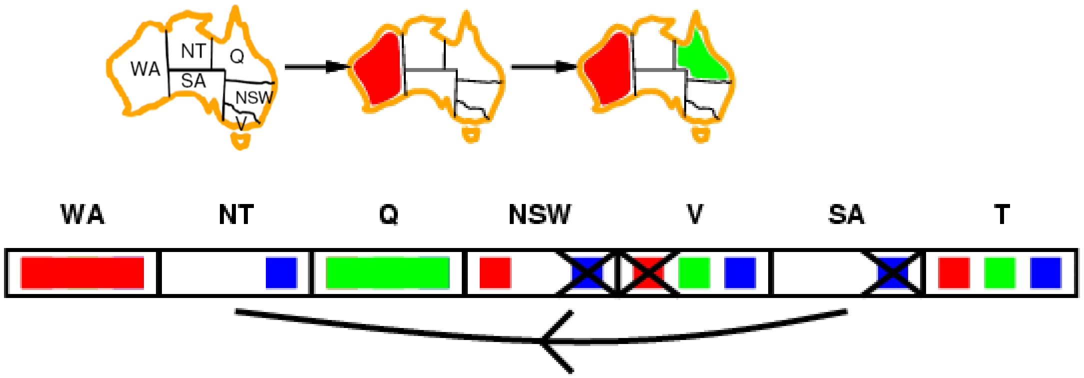
Stop and backtrack



- NT and SA cannot both be blue! This violates the constraint.

Early detection of failure – Forward checking Arc consistency

- *X* is arc consistent wrt *Y* iff for every value of *X* there is some allowed value of *Y.*
- Make *X* arc consistent wrt *Y* by throwing out any values of *X* for which there is no allowed value of *Y*.



1. NWS cannot be blue because SA has to be blue.
2. V cannot be red because NSW has to be red.
3. SA cannot be blue because NT is blue.
4. Fail and backtrack

- Arc consistency detects failure earlier than node consistency
- There are more consistency checks (path consistency, K-consistency)

Backtracking search with inference

function RECURSIVE-BACKTRACKING(*assignment*, *csp*)
    if *assignment* is complete **then return** *assignment*
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*], *assignment*, *csp*)
    for each *value* **in** ORDER-DOMAIN-VALUES(*var*, *assignment*, *csp*)
        if *value* is consistent with *assignment* given CONSTRAINTS[*csp*]
            add {*var* = *value*} to *assignment*
            result ← RECURSIVE-BACKTRACKING(*assignment*, *csp*)
            if *result* ≠ *failure* **then return** *result*
            remove {*var* = *value*} from *assignment*
    return *failure*

Call: `Recursive-Backtracking({}, csp)`

```
If (inference(csp, var, assignment) == failure)
      return failure
```

Check consistency here (called "inference") and backtrack if we know that the branch will lead to failure.

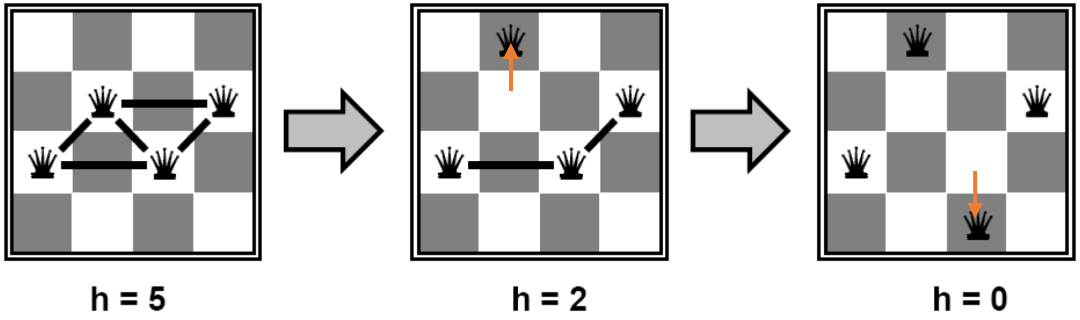
Local search for CSPs

CSP algorithms allow **incomplete states**, but only if **they satisfy all constraints**.

Local Search (e.g., Hill-climbing and simulated annealing) works only with **"complete" states**, i.e., all variables assigned, but we can **allow states with unsatisfied constraints**.

Attempt to improve states by the **min-conflicts** heuristic:

1. Select a conflicted variable and
2. Choose a new value that produces violates the fewest constraints (local improvement step)
3. Repeat till all constraints are met.



Local search is often very effective for CSPs.

Summary

- CSPs are a special type of search problem:
  - States are **structured** and defined by a set of variables and values assignments
  - Variables can be unassigned
  - Goal test defined by
    - **Consistency** with constraints
    - **Completeness** of assignment

- **Backtracking search** = depth-first search where a successor state is generated by a consistent value assignment to a single unassigned variable
  - Starts with {} and only considers consistent assignments.
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure

- Local search can be used to search the space of all complete assignments for consistent assignments = **min-conflicts heuristic.**