Proof of *Corollary*

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Corollary. *Let $L : V \to W$ be a linear transformation which is invertible. Then V is finite-dimensional if and only if W is finite dimensional, and if both are finite dimensional, then the dimension of V equals the dimension of W.*

Proof. Note first that, since $L : V \to W$ is invertible, $L^{-1} : W \to V$ is also invertible. Thus, the argument in the forward direction and the reverse direction may be applied equivalently.

Now, let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V, and let $\gamma = \{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)\}$ be the image of β under L. Fix any $\mathbf{w} \in W$. Since L is invertible, there is some $\mathbf{v} \in V$ such that $\mathbf{w} = L(\mathbf{v})$. This vector, of course, is a linear combination of vectors in β: that is, $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$. Since L is linear, it follows that $\mathbf{w} = \sum_{i=1}^{n} a_i L(\mathbf{v}_i)$. Since this is true for any $\mathbf{w} \in W$, γ is a generating set for W. We have shown that this means that γ contains a basis of W, which must have cardinality less than or equal to $n = \dim V$; thus W is finite, and $\dim W \leq \dim V$.

Applying the same argument to L^{-1}, we find that if W is finite-dimensional, then V is finite dimensional, proving the first half of the corollary. Moreover, we find that $\dim V \leq \dim W$. Of course, since $\dim W \leq \dim V$ and $\dim V \leq \dim W$, then V and W are of the same dimension. This proves the second half of the corollary, completing this proof. $\square$

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Additional credit for the final step of this proof—$\dim W \leq \dim V$ and $\dim V \leq \dim W$ implies $\dim V = \dim W$—goes to Hana Huber for their helpful contribution.