Sideband ground-state cooling

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Consider the Heisenberg-Langevin equation:

$$\mathrm{d}\hat{O} = \frac{\mathrm{i}}{\hbar}\left[H, \hat{O}\right] + \mathcal{L}^{*}\left[\hat{O}\right], \tag{1}$$

$$H = \omega_{\mathrm{m}} b^{\dagger} b + \Omega_{\mathrm{mw}} S_x - \Delta_{\mathrm{mw}} S_z + 2g\left(b + b^{\dagger}\right) S_z, \tag{2}$$

$$\mathcal{L}\left[\rho\right] = \frac{\gamma_m}{2}\left(n_{\mathrm{th}} + 1\right)\mathcal{D}_b\left[\rho\right] + \frac{\gamma_m}{2} n_{\mathrm{th}} \mathcal{D}_{b^{\dagger}}\left[\rho\right] + \Gamma \mathcal{D}_{\sigma_-}\left[\rho\right] + \frac{\tilde{\Gamma}}{4}\mathcal{D}_{\sigma_z}\left[\rho\right]. \tag{3}$$

Summarizing the results obtained in the unitary evolution, where we transform the operators to phase space variables $\sigma \to s$:

$$\mathrm{d}s_i^x\big|_{\mathrm{unit}} = \left(\Delta_{\mathrm{mw}} - 4g\alpha_x\right) s_i^y \mathrm{d}t, \tag{4}$$

$$\mathrm{d}s_i^y\big|_{\mathrm{unit}} = \left(4g\alpha_x - \Delta_{\mathrm{mw}}\right) s_i^x \mathrm{d}t - \Omega_{\mathrm{mw}} s_i^z \mathrm{d}t, \tag{5}$$

$$\mathrm{d}s_i^z\big|_{\mathrm{unit}} = \Omega_{\mathrm{mw}} s_i^y \mathrm{d}t, \tag{6}$$

$$\mathrm{d}\alpha\big|_{\mathrm{unit}} = -\mathrm{i}\left(\omega_{\mathrm{m}}\alpha + 2g S_z\right)\mathrm{d}t. \tag{7}$$

The loss terms with their respective stochastic noise terms are:

$$\mathrm{d}s_i^x\big|_{\mathrm{loss}} = -\left(\frac{\Gamma}{2} + \tilde{\Gamma}\right) s_i^x \mathrm{d}t - \sqrt{2\tilde{\Gamma}} s_i^y \mathrm{d}W_{i,\phi} - \sqrt{\Gamma}\frac{\left(s_i^y + 1\right)\mathrm{d}W_{i,\mathrm{dec}}^1 + \left(s_i^y - 1\right)\mathrm{d}W_{i,\mathrm{dec}}^2}{2}, \tag{8}$$

$$\mathrm{d}s_i^y\big|_{\mathrm{loss}} = -\left(\frac{\Gamma}{2} + \tilde{\Gamma}\right) s_i^y \mathrm{d}t + \sqrt{2\tilde{\Gamma}} s_i^x \mathrm{d}W_{i,\phi} + \sqrt{\Gamma}\frac{\left(s_i^x + 1\right)\mathrm{d}W_{i,\mathrm{dec}}^1 + \left(s_i^x - 1\right)\mathrm{d}W_{i,\mathrm{dec}}^2}{2}, \tag{9}$$

$$\mathrm{d}s_i^z\big|_{\mathrm{loss}} = -\Gamma\left(s_i^z + 1\right)\mathrm{d}t + \sqrt{\frac{\Gamma}{2}}\left(s_i^z + 1\right)\left(\mathrm{d}W_{i,\mathrm{dec}}^1 - \mathrm{d}W_{i,\mathrm{dec}}^2\right), \tag{10}$$

$$\mathrm{d}\alpha\big|_{\mathrm{loss}} = -\frac{\gamma_{\mathrm{m}}}{2}\alpha \mathrm{d}t + \sqrt{\frac{\gamma_m}{4}}\left(\mathrm{d}W_1 + \mathrm{i}\mathrm{d}W_2\right). \tag{11}$$

In general we have that the dynamics of the phase-space variables is given by:

$$\mathrm{d}s_i^x = \mathrm{d}s_i^x\big|_{\mathrm{unit}} + \mathrm{d}s_i^x\big|_{\mathrm{loss}}, \tag{12}$$

$$\mathrm{d}s_i^y = \mathrm{d}s_i^y\big|_{\mathrm{unit}} + \mathrm{d}s_i^y\big|_{\mathrm{loss}}, \tag{13}$$

$$\mathrm{d}s_i^z = \mathrm{d}s_i^z\big|_{\mathrm{unit}} + \mathrm{d}s_i^z\big|_{\mathrm{loss}}, \tag{14}$$

$$\mathrm{d}\alpha = \mathrm{d}\alpha\big|_{\mathrm{unit}} + \mathrm{d}\alpha\big|_{\mathrm{loss}}. \tag{15}$$

We want to use real variables, so our system rewritten with $\alpha = \alpha_x + \mathrm{i}\alpha_y$ is:

$$\mathrm{d}\alpha_x|_{\mathrm{unit}} = \omega_m \alpha_y \mathrm{d}t, \tag{16}$$
$$\mathrm{d}\alpha_y|_{\mathrm{unit}} = -\left(\omega_m \alpha_x + g\Sigma_z\right)\mathrm{d}t, \tag{17}$$
$$\mathrm{d}\alpha_x|_{\mathrm{loss}} = -\frac{\gamma_{\mathrm{m}}}{2}\alpha_x \mathrm{d}t + \sqrt{\frac{\gamma_m}{4}}\mathrm{d}W_1, \tag{18}$$
$$\mathrm{d}\alpha_y|_{\mathrm{loss}} = -\frac{\gamma_{\mathrm{m}}}{2}\alpha_y \mathrm{d}t + \sqrt{\frac{\gamma_m}{4}}\mathrm{d}W_2. \tag{19}$$

The set of stochastic differential equations in Ito's form is:

$$\mathrm{d}s_i^x = \left(\Delta_{\mathrm{mw}} - 4g\alpha_x\right) s_i^y \mathrm{d}t - \left(\frac{\Gamma}{2} + \tilde{\Gamma}\right) s_i^x \mathrm{d}t - \sqrt{2\tilde{\Gamma}} s_i^y \mathrm{d}W_{i,\phi} \tag{20}$$
$$- \sqrt{\Gamma}\frac{\left(s_i^y + 1\right)\mathrm{d}W_{i,\mathrm{dec}}^1 + \left(s_i^y - 1\right)\mathrm{d}W_{i,\mathrm{dec}}^2}{2}, \tag{21}$$
$$\mathrm{d}s_i^y = -\left(\Delta_{\mathrm{mw}} - 4g\alpha_x\right) s_i^x \mathrm{d}t - \Omega_{\mathrm{mw}} s_i^z \mathrm{d}t - \left(\frac{\Gamma}{2} + \tilde{\Gamma}\right) s_i^y \mathrm{d}t + \sqrt{2\tilde{\Gamma}} s_i^x \mathrm{d}W_{i,\phi} \tag{22}$$
$$+ \sqrt{\Gamma}\frac{\left(s_i^x + 1\right)\mathrm{d}W_{i,\mathrm{dec}}^1 + \left(s_i^x - 1\right)\mathrm{d}W_{i,\mathrm{dec}}^2}{2}, \tag{23}$$
$$\mathrm{d}s_i^z = \Omega_{\mathrm{mw}} s_i^y \mathrm{d}t - \Gamma\left(s_i^z + 1\right)\mathrm{d}t + \sqrt{\frac{\Gamma}{2}}\left(s_i^z + 1\right)\left(\mathrm{d}W_{i,\mathrm{dec}}^1 - \mathrm{d}W_{i,\mathrm{dec}}^2\right), \tag{24}$$
$$\mathrm{d}\alpha_x = \omega_m \alpha_y \mathrm{d}t - \frac{\gamma_{\mathrm{m}}}{2}\alpha_x \mathrm{d}t + \sqrt{\frac{1}{2}\gamma_m\left(n_{\mathrm{th}} + 1\right)}\mathrm{d}W_1, \tag{25}$$
$$\Sigma_z \equiv \sum_i \sigma_i^z, \tag{26}$$
$$\mathrm{d}\alpha_y = -\left(\omega_m \alpha_x + g\Sigma_z\right)\mathrm{d}t - \frac{\gamma_{\mathrm{m}}}{2}\alpha_y \mathrm{d}t + \sqrt{\frac{1}{2}\gamma_m\left(n_{\mathrm{th}} + 1\right)}\mathrm{d}W_2. \tag{27}$$

Surprisingly $\tilde{n}_{\mathrm{th}}$disappeared, this is because the cavity noise operators $\hat{b}$ and $\hat{b}^\dagger$ have the same decay rate γ_m associated.

Parameters

$$\Delta_{\mathrm{mw}} = c_\Delta \omega_{\mathrm{m}}, \tag{28}$$
$$g = 4.536 \times 10^{-3}\omega_{\mathrm{m}}, \tag{29}$$
$$\Gamma = 2.578 \times 10^{-2}\omega_{\mathrm{m}}, \tag{30}$$
$$\tilde{\Gamma} = 0.015\omega_{\mathrm{m}}, \tag{31}$$
$$\gamma_{\mathrm{m}} = 5 \times 10^{-6}\omega_{\mathrm{m}}, \tag{32}$$
$$\omega_{\mathrm{m}} = \omega_{\mathrm{m}}, \tag{33}$$
$$\Omega_{\mathrm{mw}} = c_\Omega \omega_{\mathrm{m}}, \tag{34}$$
$$n_{\mathrm{th}} = 6. \tag{35}$$

Using a time scale in ω_{m} then we can rewrite the parameters in the new scaled time as:

$$\Delta'_{\mathrm{mw}} = c_{\Delta}, \tag{36}$$
$$g' = 4.536 \times 10^{-3}, \tag{37}$$
$$\Gamma' = 2.578 \times 10^{-2}, \tag{38}$$
$$\tilde{\Gamma}' = 0.015, \tag{39}$$
$$\gamma'_{\mathrm{m}} = 5 \times 10^{-6}, \tag{40}$$
$$\omega'_{\mathrm{m}} = 1, \tag{41}$$
$$\Omega'_{\mathrm{mw}} = c_{\Omega}, \tag{42}$$
$$n_{\mathrm{th}} = 6. \tag{43}$$

where the time has the form $t' = \omega_{\mathrm{m}} t$

The equations are:

$$\mathrm{d}s_i^x = \left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) s_i^y \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) s_i^x \mathrm{d}t' - \sqrt{2\tilde{\Gamma}'} s_i^y \mathrm{d}W'_{i,\phi} \tag{44}$$
$$- \sqrt{\Gamma'} \frac{\left(s_i^y + 1\right) \mathrm{d}W^{1\prime}_{i,\mathrm{dec}} + \left(s_i^y - 1\right) \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}}{2}, \tag{45}$$
$$\mathrm{d}s_i^y = -\left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) s_i^x \mathrm{d}t' - \Omega'_{\mathrm{mw}} s_i^z \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) s_i^y \mathrm{d}t' + \sqrt{2\tilde{\Gamma}'} s_i^x \mathrm{d}W'_{i,\phi} \tag{46}$$
$$+ \sqrt{\Gamma'} \frac{\left(s_i^x + 1\right) \mathrm{d}W^{1\prime}_{i,\mathrm{dec}} + \left(s_i^x - 1\right) \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}}{2}, \tag{47}$$
$$\mathrm{d}s_i^z = \Omega'_{\mathrm{mw}} s_i^y \mathrm{d}t' - \Gamma' \left(s_i^z + 1\right) \mathrm{d}t' + \sqrt{\frac{\Gamma'}{2}} \left(s_i^z + 1\right) \left(\mathrm{d}W^{1\prime}_{i,\mathrm{dec}} - \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}\right), \tag{48}$$
$$\mathrm{d}\alpha_x = \omega'_m \alpha_y \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_x \mathrm{d}t' + \sqrt{\frac{1}{2}\gamma'_m \left(n_{\mathrm{th}} + 1\right)} \mathrm{d}W'_1, \tag{49}$$
$$\mathrm{d}\alpha_y = -\left(\omega'_m \alpha_x + g'\Sigma_z\right) \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_y \mathrm{d}t' + \sqrt{\frac{1}{2}\gamma'_m \left(n_{\mathrm{th}} + 1\right)} \mathrm{d}W'_2. \tag{50}$$

The equations are equivalent and $\omega'_{\mathrm{m}} = 1$. No new dynamics added up to a change in the scale factor. Without noise:

$$\mathrm{d}s_i^x = \left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) s_i^y \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) s_i^x \mathrm{d}t' \tag{51}$$

$$\mathrm{d}s_i^y = -\left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) s_i^x \mathrm{d}t' - \Omega'_{\mathrm{mw}} s_i^z \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) s_i^y \mathrm{d}t' \tag{52}$$

$$\mathrm{d}s_i^z = \Omega'_{\mathrm{mw}} s_i^y \mathrm{d}t' - \Gamma' \left(s_i^z + 1\right) \mathrm{d}t' \tag{53}$$

$$\mathrm{d}\alpha_x = \omega'_m \alpha_y \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_x \mathrm{d}t' \tag{54}$$

$$\mathrm{d}\alpha_y = -\left(\omega'_m \alpha_x + g'\Sigma_z\right) \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_y \mathrm{d}t' \tag{55}$$

Using collective operators:

$$\mathrm{d}\Sigma_x = \left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) \Sigma_y \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) \Sigma_x \mathrm{d}t', \tag{56}$$

$$\mathrm{d}\Sigma_y = -\left(\Delta'_{\mathrm{mw}} - 4g'\alpha_x\right) \Sigma_x \mathrm{d}t' - \Omega'_{\mathrm{mw}} \Sigma_z \mathrm{d}t' - \left(\frac{\Gamma'}{2} + \tilde{\Gamma}'\right) \Sigma_y \mathrm{d}t', \tag{57}$$

$$\mathrm{d}\Sigma_z = \Omega'_{\mathrm{mw}} \Sigma_y \mathrm{d}t' - \Gamma' \left(\Sigma_z + N\right) \mathrm{d}t', \tag{58}$$

$$\mathrm{d}\alpha_x = \omega'_m \alpha_y \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_x \mathrm{d}t', \tag{59}$$

$$\mathrm{d}\alpha_y = -\left(\omega'_m \alpha_x + g'\Sigma_z\right) \mathrm{d}t' - \frac{\gamma'_{\mathrm{m}}}{2} \alpha_y \mathrm{d}t'. \tag{60}$$

The noise terms are:

$$\mathrm{d}s_i^x|_{\mathrm{noise}} = -\sqrt{2\tilde{\Gamma}'} s_i^y \mathrm{d}W'_{i,\phi} - \sqrt{\Gamma'} \frac{\left(s_i^y + 1\right) \mathrm{d}W^{1\prime}_{i,\mathrm{dec}} + \left(s_i^y - 1\right) \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}}{2}, \tag{61}$$

$$\mathrm{d}s_i^y|_{\mathrm{noise}} = \sqrt{2\tilde{\Gamma}'} s_i^x \mathrm{d}W'_{i,\phi} + \sqrt{\Gamma'} \frac{\left(s_i^x + 1\right) \mathrm{d}W^{1\prime}_{i,\mathrm{dec}} + \left(s_i^x - 1\right) \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}}{2}, \tag{62}$$

$$\mathrm{d}s_i^z|_{\mathrm{noise}} = \sqrt{\frac{\Gamma'}{2}} \left(s_i^z + 1\right) \left(\mathrm{d}W^{1\prime}_{i,\mathrm{dec}} - \mathrm{d}W^{2\prime}_{i,\mathrm{dec}}\right), \tag{63}$$

$$\mathrm{d}\alpha_x|_{\mathrm{noise}} = \sqrt{\frac{\gamma'_m}{2} \left(n_{\mathrm{th}} + 1\right)} \mathrm{d}W'_1, \tag{64}$$

$$\mathrm{d}\alpha_y|_{\mathrm{noise}} = \sqrt{\frac{\gamma'_m}{2} \left(n_{\mathrm{th}} + 1\right)} \mathrm{d}W'_2. \tag{65}$$