Inhomogeneous Phase Transition

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DERIVATION OF MASTER EQUATION

In the Thompson QED Cavity and in the presence of a driving field, the atom-light evolution is given by the following master equation:

$$\dot{\rho} = -i\Big[-\Delta \hat{a}^\dagger \hat{a} + \Omega_p(\hat{a}+\hat{a}^\dagger) + \sum_i g_i\big(\hat{a}^\dagger s_-^i + \hat{a} s_+^i\big), \rho\Big] + \kappa\Big(\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}\Big) \tag{1}$$

Adiabatic elimination proceeds heuristically by computing the equations of motion for $\hat{a}$:

$$\dot{\hat{a}} = i\Delta\hat{a} - \frac{\kappa}{2}\hat{a} - i\Omega_p - i\sum_i g_i s_-^i, \tag{2}$$

setting $\dot{\hat{a}} = 0$ and replacing the result into the Liouvillian. Then, we obtain effective Hamiltonian and jump operators:

$$\begin{aligned} H_{\rm eff} &= \frac{\Delta}{\Delta^2+(\kappa/2)^2}\sum_{i,j} g_i g_j s_+^i s_-^j + \frac{\Omega_p}{\sqrt{\Delta^2+(\kappa/2)^2}}\sum_i g_i(e^{i\phi}s_-^i + e^{-i\phi}s_+^i) \\ L &= \sqrt{\frac{\kappa}{\Delta^2+(\kappa/2)^2}}\sum_i g_i s_-^i, \end{aligned} \tag{3}$$

where $\phi = \arctan(\kappa/2\Delta)$. The system can then be put in the form

$$\begin{aligned} H_{\rm eff} &= \chi \tilde{J}^+\tilde{J}^- + \frac{\Omega_R}{2}\big(\tilde{J}^- e^{i\phi} + \tilde{J}^+ e^{-i\phi}\big) \\ L &= \Gamma \tilde{J}^-, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \chi &= \frac{\bar{g}^2\Delta}{\Delta^2+(\kappa/2)^2} \\ \Gamma &= \frac{\bar{g}^2\kappa}{\Delta^2+(\kappa/2)^2} \\ \Omega_R &= \frac{\bar{g}\Omega_p}{\sqrt{\Delta^2+(\kappa/2)^2}} \\ \tilde{J}_{x,y,z} &= \sum_i \frac{g_i}{\bar{g}} s^i_{x,y,z} \equiv \sum_i \eta_i s^i_{x,y,z} \end{aligned} \tag{5}$$

and $\bar{g}$ is some coupling, which could be some average of the g_i or the maximum of the g_i. The steady state solution is then very easy to write

$$\rho_{ss} = \mathcal{C}\left(\frac{1}{\tilde{J}^- + N\alpha}\right)\left(\frac{1}{\tilde{J}^+ + N\alpha}\right), \tag{6}$$

where $\mathcal{C}$ is a normalization factor and $\alpha = \Omega_R/N\sqrt{\chi^2+(\Gamma/2)^2}$. We can always make α positive by a spin rotation about z so we will only consider positive α.

ANALYSIS OF STEADY STATE

Eq. (6) is still in an unwieldy form, so we re-express it as the two dimensional integral of two exponentials

$$\rho_{ss} = \mathcal{C}\int_0^\infty\int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} e^{-x\tilde{J}^-} e^{-y\tilde{J}^+}. \tag{7}$$

This form is particularly convenient because the integrand factorizes into operators acting on different spin degrees of freedom:

$$\rho_{ss} = \mathcal{C}\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} \prod_i e^{-x\eta_i s_-^i} e^{-y\eta_i s_+^i}, \tag{8}$$

which makes the evaluation of expectation values much easier. Therefore, if we want to calculate $\langle O\rangle$, we use the standard recipe

$$\langle \hat{O}\rangle = \frac{\mathrm{Tr}(\rho_{ss}\hat{O})}{\mathrm{Tr}(\rho_{ss})} \tag{9}$$

and evaluate the trace by using the overcomplete set of coherent states for each spin. Thus, for example

$$\begin{aligned}
\langle s_z^j\rangle &= \frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} \langle \theta_j,\phi_j|\, (e^{-x\eta_j s_-^j})(e^{-y\eta_j s_+^j}) s_z^j\, |\theta_j,\phi_j\rangle \prod_{i\neq j}\int_{\Omega_i} \langle \theta_i,\phi_i|\, (e^{-x\eta_i s_-^i})(e^{-y\eta_i s_+^i})\, |\theta_i,\phi_i\rangle}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} \prod_i \int_{\Omega_i} \langle \theta_i,\phi_i|\, (e^{-x\eta_i s_-^i})(e^{-y\eta_i s_+^i})\, |\theta_i,\phi_i\rangle} \\
\langle s_+^j\rangle &= \frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} \langle \theta_j,\phi_j|\, (e^{-x\eta_j s_-^j})(e^{-y\eta_j s_+^j}) s_+^j\, |\theta_j,\phi_j\rangle \prod_{i\neq j}\int_{\Omega_i} \langle \theta_i,\phi_i|\, (e^{-x\eta_i s_-^i})(e^{-y\eta_i s_+^i})\, |\theta_i,\phi_i\rangle}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y} \prod_i \int_{\Omega_i} \langle \theta_i,\phi_i|\, (e^{-x\eta_i s_-^i})(e^{-y\eta_i s_+^i})\, |\theta_i,\phi_i\rangle}
\end{aligned} \tag{10}$$

For this purpose it is useful to calculate

$$\begin{aligned}
\int_\Omega \langle \theta,\phi|\, (e^{-x\eta s_-})(e^{-y\eta s_+})\, |\theta,\phi\rangle &= \int_\Omega \langle \theta,\phi|\, (1-x\eta s_-)(1-y\eta s_+)\, |\theta,\phi\rangle \\
&= \int d\phi\, d\theta\, \sin(\theta)\Big[1 - \sin(\theta/2)\cos(\theta/2)\eta(xe^{-i\phi} + ye^{i\phi}) + \eta^2 xy \sin(\theta/2)^2\Big] \\
&= 4\pi\Big(1 + \frac{\eta^2 xy}{2}\Big)
\end{aligned} \tag{11}$$

In the same way

$$\begin{aligned}
\int_\Omega \langle \theta,\phi|\, (e^{-x\eta s_-})(e^{-y\eta s_+}) s_z\, |\theta,\phi\rangle &= -4\pi \frac{\eta^2 xy}{4} \\
\int_\Omega \langle \theta,\phi|\, (e^{-x\eta s_-})(e^{-y\eta s_+}) s_+\, |\theta,\phi\rangle &= -4\pi \frac{\eta x}{2}
\end{aligned} \tag{12}$$

Thus

$$\begin{aligned}
\langle s_z^j\rangle &= -\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y}\Big(\frac{\eta_j^2 xy}{4}\Big)\prod_{i\neq j}\Big(1+\frac{\eta_i^2 xy}{2}\Big)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y}\prod_i\Big(1+\frac{\eta_i^2 xy}{2}\Big)} \\
\langle s_+^j\rangle &= -\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y}\Big(\frac{\eta_j x}{2}\Big)\prod_{i\neq j}\Big(1+\frac{\eta_i^2 xy}{2}\Big)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha x - N\alpha y}\prod_i\Big(1+\frac{\eta_i^2 xy}{2}\Big)}
\end{aligned} \tag{13}$$

The latter form is particularly suited for numerical computation. For analytics, we use a different representation

$$\begin{aligned}
\langle s_z^j\rangle &= -\frac{1}{2}\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-NS(x,y)}\Big(\frac{\eta_j^2 xy}{2+\eta_j^2 xy}\Big)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-NS(x,y)}} \\
\langle s_+^j\rangle &= -\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-NS(x,y)}\Big(\frac{\eta_j x}{2+\eta_j^2 xy}\Big)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-NS(x,y)}},
\end{aligned} \tag{14}$$

where

$$S(x,y) = \alpha(x+y) - \int d\eta\, \rho(\eta) \log\left(1 + \frac{\eta^2 xy}{2}\right) \tag{15}$$

and

$$\rho(\eta) = \frac{1}{N}\sum_i \delta(\eta - \eta_i) \tag{16}$$

is the distribution function of couplings. The behaviour of S is what determines the integral in the large N limit. At small x and y the function is linear and increasing and it can be seen that this happens at very large x and y too. Depending on α at intermediate x and y the integral may have a saddle point. If this saddle point exists and is such that $S < 0$, then it dominates the integral when $N \to \infty$. If it is such that $S > 0$, even if it exists, the $x, y \approx 0$ region dominates the integral. This can be seen to hold when α is large enough, since then the linear growth of the first term in S is more than enought to compensate the growth of the logarithmic term (assuming $\rho(\eta)$ is well behaved). In this regime

$$\begin{aligned}
\langle s_z^j \rangle &= -\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha(x+y)} \left(\frac{\eta_j^2 xy}{4}\right)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha(x+y)}} = -\frac{\eta_j^2}{4N^2\alpha^2} \\
\langle s_+^j \rangle &= -\frac{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha(x+y)} \left(\frac{\eta_j x}{2}\right)}{\int_0^\infty \int_0^\infty dx\, dy\, e^{-N\alpha(x+y)}} = -\frac{\eta_j}{2N\alpha}
\end{aligned} \tag{17}$$

From this we can calculate the collective observables:

$$\begin{aligned}
\langle J_z \rangle &= -\frac{\langle \eta^2 \rangle}{4N\alpha^2} \\
\langle J_+ \rangle &= -\frac{\langle \eta \rangle}{2\alpha} \\
\langle \tilde{J}_+ \rangle &= -\frac{\langle \eta^2 \rangle}{2\alpha}
\end{aligned} \tag{18}$$

On the other hand, for small enough α, the logarithm term will dominate for intermediate x and y and there will be a saddle point, which is determined by

$$\begin{aligned}
\partial_x S &= \alpha - \int d\eta\, \rho(\eta) \frac{\eta^2 y_s}{2 + \eta^2 x_s y_s} = 0 \\
\partial_y S &= \alpha - \int d\eta\, \rho(\eta) \frac{\eta^2 x_s}{2 + \eta^2 x_s y_s} = 0
\end{aligned} \tag{19}$$

They imply $x_s = y_s$ and $x_s \equiv x_s(\alpha)$ is a function of α defined implicitly by:

$$\alpha = x_s \int d\eta\, \frac{\rho(\eta)\eta^2}{2 + \eta^2 x_s^2} = \frac{1}{x_s}\left(1 - 2\int d\eta\, \frac{\rho(\eta)}{2 + \eta^2 x_s^2}\right) \tag{20}$$

From this, the single particle observables are

$$\begin{aligned}
\langle s_z^j \rangle &= -\frac{1}{2}\left(\frac{\eta_j^2 x_s^2}{2 + \eta_j^2 x_s^2}\right) \\
\langle s_+^j \rangle &= -\frac{\eta_j x_s}{2 + \eta_j^2 x_s^2}
\end{aligned} \tag{21}$$

and the colllective ones are

$$\begin{aligned}
\langle J_z\rangle &= -\frac{N}{2}\int d\eta\,\rho(\eta)\frac{\eta^2 x_s^2}{2+\eta^2 x_s^2} = -\frac{N\alpha\, x_s(\alpha)}{2} \\
\langle J_+\rangle &= -N\int d\eta\,\frac{x_s\eta\rho(\eta)}{2+\eta^2 x_s^2} \\
\langle \tilde{J}_+\rangle &= -N\int d\eta\,\frac{x_s\eta^2\rho(\eta)}{2+\eta^2 x_s^2} = -N\alpha
\end{aligned} \tag{22}$$

where we have emphasized that x_s is a function of α and used Eq. (20) to simplify the formulas a bit. Note that in this case the collective observables do behave collectively, that is , they scale as N. Remarkably, $\langle \tilde{J}_+\rangle$ is independent of the distribution of couplings $\rho(\eta)$. The critical α where the change in behaviour occurs is given implicitly by

$$S(x_s,x_s) = 2\alpha_c\, x_s(\alpha) - \int d\eta\,\rho(\eta)\log\left(1+\frac{\eta^2\, x_s(\alpha)^2}{2}\right) = 0 \tag{23}$$

Sinusoidal distribution of couplings

For this example, let's consider the couplings to be such that

$$\rho(\eta)\,d\eta = \frac{d\phi}{2\pi} \tag{24}$$

where $\eta = \cos(\phi)$. Then Eq. (20) becomes

$$\alpha x_s = 1 - \frac{1}{\sqrt{1+x_s^2/2}} \tag{25}$$

which has a closed form but cumbersome solution. For all practical purposes, we can use its expansion about $\alpha = 0$:

$$x_s = \frac{1}{\alpha} - \sqrt{2} - 3\sqrt{2}\alpha - 12\alpha^3 - \frac{55}{\sqrt{2}}\alpha^4 - 136\alpha^5 + ... \tag{26}$$

To determine α_c, we also need Eq. (23)

$$2\alpha_c x_s - 2\log\left(\frac{1+\sqrt{1+x_s^2/2}}{2}\right) = 0 \tag{27}$$

which numerically gives

$$\alpha_c = 0.174 \tag{28}$$

Thus

$$\begin{aligned}
\langle J_z\rangle &= \begin{cases} -\frac{N}{2}\left(1-\sqrt{2}\alpha - 3\sqrt{2}\alpha^2 - 12\alpha^4 + ...\right) & \text{if } \alpha < 0.174 \\ -\frac{1}{8N\alpha^2} & \text{if } \alpha > 0.174 \end{cases} \\
\langle \tilde{J}_+\rangle &= \begin{cases} -N\alpha & \text{if } \alpha < 0.174 \\ -\frac{1}{4\alpha} & \text{if } \alpha > 0.174 \end{cases}
\end{aligned} \tag{29}$$

while $\langle J_+\rangle = 0$. This can be checked against numerical simulations, which are shown in Fig. 1. Second order correlators are also important. We can calculate standard squeezing, defined by

$$\xi^2 = \min_{\mathbf{n}_\perp}\left(N\frac{\langle(\mathbf{J}\cdot\mathbf{n}_\perp)^2\rangle}{|\,\langle\mathbf{J}\rangle\,|^2}\right), \tag{30}$$

where $\mathbf{J} = (J_x, J_y, J_z)$ are collective spin observables. As shown in Fig. 2(a), there's no metrological gain because of the fact that the spin length is reduced ($J_x \approx J_y \approx 0$) whereas the size of the variance is kept approximately

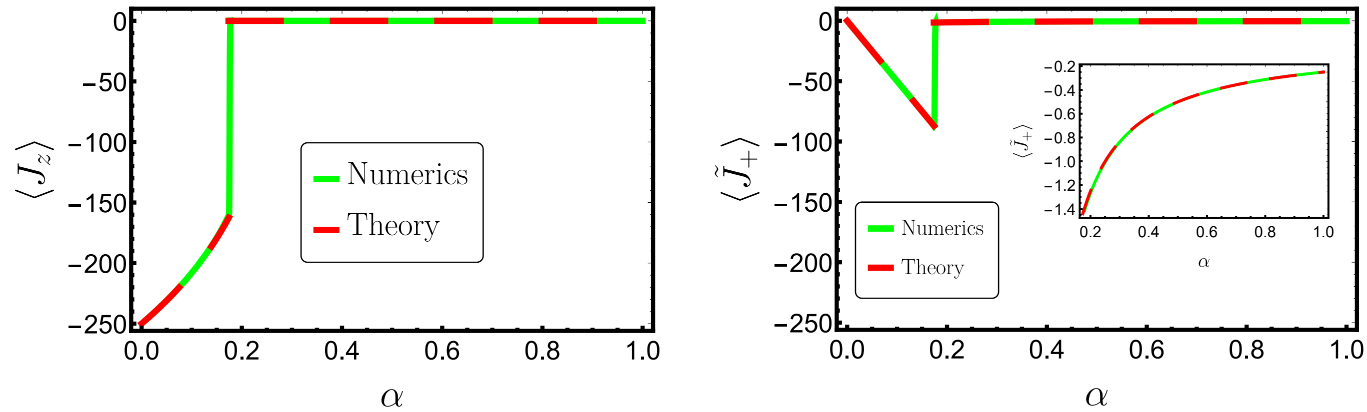

FIG. 1. Steady state inversion (left) and $\langle \tilde{J}_+ \rangle$ (right) as a function of α for $N = 500$. We can see that the simple formulas of Eq. (29) capture correctly the behaviour of the observables and the nature and location of the transition. Inset in right figure shows a zoom in of the region $\alpha > 0.2$, where $\langle \tilde{J}_+ \rangle$ is of order 1 rather than order N.

constant. However, Eq. (30) is based on the fact that the experimentally accesible observables are $\mathbf{J}$. In the case of the inhomogeneous system, the accesible observables are $\tilde{\mathbf{J}}_p = (\tilde{J}_x, \tilde{J}_y)$, and J_z. So a better definiton of squeezing is motivated by calculating the minimal angular rotation that can be discerned using these variables. Since there is no way to transform them among each other through a unitary process except under a rotation around the z axis, it only makes sense to consider the angular resolution of rotations in the xy plane:

$$(\delta\theta)^2 = \frac{\langle (\tilde{\mathbf{J}}_p \cdot \mathbf{n}_\perp)^2 \rangle}{|\langle \tilde{\mathbf{J}}_p \rangle|^2}, \tag{31}$$

and then compare it to the angular resolution obtained by optimizing over all possible uncorrelated states. The optimal resolution is given by

$$(\delta\theta)^2_{\rm opt} = \frac{\sum_i \eta_i^2}{\left(\sum_i |\eta_i|\right)^2} \tag{32}$$

In the case of the sinusoidal distribution of couplings, this number is $(\delta\theta)^2_{\rm opt} \approx 1.23/N$. Thus, the relevant squeezing parameter is

$$\tilde{\xi}^2 = \frac{N}{1.23} \frac{\langle (\tilde{\mathbf{J}} \cdot \mathbf{n}_\perp)^2 \rangle}{|\langle \tilde{\mathbf{J}} \rangle|^2}, \tag{33}$$

which is shown in Fig. 2(b). In the best case scenario, we would be able to do all rotations and mix $\tilde{J}_x$, $\tilde{J}_y$ and $\tilde{J}_z$. Then we can use the full $\tilde{\mathbf{J}} = (\tilde{J}_x, \tilde{J}_y, \tilde{J}_z)$ instead of just $\tilde{\mathbf{J}}_p$:

$$\tilde{\xi}_b^2 = \frac{N}{1.23} \min_{\mathbf{n}_\perp} \left(\frac{\langle (\tilde{\mathbf{J}} \cdot \mathbf{n}_\perp)^2 \rangle}{|\langle \tilde{\mathbf{J}} \rangle|^2} \right), \tag{34}$$

In our case, however, because of the alternating signs of the couplings, $\langle \tilde{J}_z \rangle = O(1)$ instead of $O(N)$, so $\tilde{\xi}_b^2 \approx \tilde{\xi}^2$.

$$\delta S_y \tag{35}$$

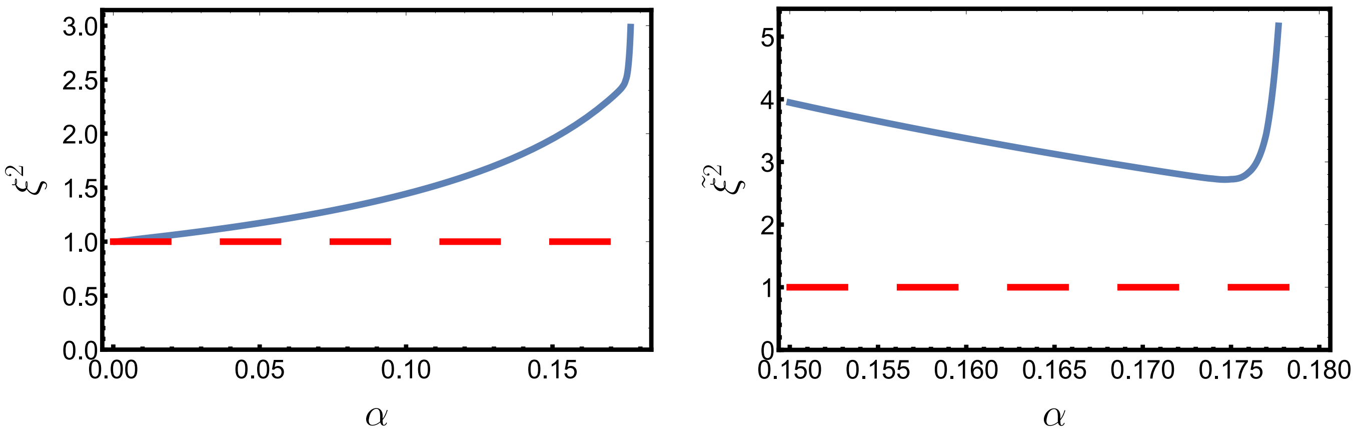

FIG. 2. ξ^2 (left) and $\tilde{\xi}^2$ (right) as a function of α for $N = 200$. None of them go below 1, which would indicate a gain over all uncorrelated states. The usual squeezing parameter ξ^2 is affected by a reduction in spin length whereas $\tilde{\xi}^2$ is affected by the lack of any $\langle \tilde{J}_z \rangle$.