Mean-Field Approximation and Cumulant Expansion for a Generic Spin Hamiltonian

A. M. Rey, E. C. Chaparro*
Department of Physics, University of Colorado, Boulder

(Dated: 6th April 2021)

I. GENERIC SPIN HAMILTONIAN WITH DISSIPATION

Consider the following generic system hamiltonian of N written in the form:

$$\hat{H}_{\mathrm{S}} = \sum_{i} \overrightarrow{h_i} \cdot \overrightarrow{\hat{\sigma}_i} + \sum_{jk\beta\varepsilon} A_{jk} \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}, \tag{1}$$

$$\overrightarrow{\hat{\sigma}_i} = \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z\right), \tag{2}$$

$$\beta, \varepsilon \in \{x, y, z\}, \tag{3}$$

$$\hat{\sigma}_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

$$\hat{\sigma}_i^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{5}$$

$$\hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

here the set of $\overrightarrow{h_i}$ can be seen as a $N \times 3$ matrix and A is a matrix of size $N \times N$ which summarizes the interaction terms between the qubits(spins). In order to introduce a dissipation term we write the Linblandian operators of the system as:

$$\mathcal{L} \Rightarrow \sqrt{\gamma_+}\hat{\sigma}_i^+, \sqrt{\gamma_-}\hat{\sigma}_i^-, \sqrt{\gamma_z}\hat{\sigma}_i^z, \tag{7}$$

$$\hat{\sigma}_i^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$$\hat{\sigma}_i^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \tag{9}$$

$$\hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

$$\mathcal{L}\left[\hat{\rho}\right] = -\sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta} + \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\hat{\rho} - 2\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right), \tag{11}$$

$$\hat{A}_i^{\eta} = \hat{\sigma}_i^{\eta}, \eta \in \{+, -, z\}. \tag{12}$$

The final form of the master equation to consider is given by:

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta} + \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\hat{\rho} - 2\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right) \tag{13}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[H_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\left\{\rho, \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\right\} - 2\hat{A}_i^{\eta}\rho\hat{A}_i^{\dagger\eta}\right). \tag{14}$$

We introduce now a set of simplifications that can be useful for posterior computations:

$$\sum_{i,\eta}\gamma_\eta\hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \sum_i\sum_\eta\gamma_\eta\hat{A}_i^{\dagger\eta}\hat{A}_i^\eta, \tag{15}$$

$$\hat{\sigma}_i^{\dagger+}\hat{\sigma}_i^+ = \begin{pmatrix}0&0\\2&0\end{pmatrix}\begin{pmatrix}0&2\\0&0\end{pmatrix} \tag{16}$$

$$= \begin{pmatrix}0&0\\0&4\end{pmatrix} \tag{17}$$

$$= 4\frac{\hat{\mathbb{I}}-\hat{\sigma}_i^z}{2} \tag{18}$$

$$= 2\left(\hat{\mathbb{I}}-\hat{\sigma}_i^z\right), \tag{19}$$

$$\hat{\sigma}_i^{\dagger-}\hat{\sigma}_i^- = \begin{pmatrix}0&2\\0&0\end{pmatrix}\begin{pmatrix}0&0\\2&0\end{pmatrix} \tag{20}$$

$$= \begin{pmatrix}4&0\\0&0\end{pmatrix} \tag{21}$$

$$= 2\left(\hat{\mathbb{I}}+\hat{\sigma}_i^z\right), \tag{22}$$

$$\hat{\sigma}_i^{\dagger z}\hat{\sigma}_i^z = \hat{\mathbb{I}}, \tag{23}$$

$$\sum_\eta\gamma_\eta\hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \gamma_+2\left(\hat{\mathbb{I}}-\hat{\sigma}_i^z\right)+\gamma_-2\left(\hat{\mathbb{I}}+\hat{\sigma}_i^z\right)+\gamma_z\hat{\mathbb{I}} \tag{24}$$

$$= \left(\gamma_z+2\gamma_++2\gamma_-\right)\hat{\mathbb{I}}+2\left(\gamma_--\gamma_+\right)\hat{\sigma}_i^z. \tag{25}$$

Then the master equation (14) simplifies to:

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]-\sum_i\left(\left\{\hat{\rho},\left(\gamma_z+2\gamma_++2\gamma_-\right)\hat{\mathbb{I}}+2\left(\gamma_--\gamma_+\right)\hat{\sigma}_i^z\right\}\right)+\sum_{i,\eta}2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{26}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]-\sum_i\left(\left\{\hat{\rho},\left(\gamma_z+2\gamma_++2\gamma_-\right)\hat{\mathbb{I}}\right\}+\left\{\hat{\rho},2\left(\gamma_--\gamma_+\right)\hat{\sigma}_i^z\right\}\right)+\sum_{i,\eta}2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{27}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\hat{\rho}-2\left(\gamma_--\gamma_+\right)\sum_i\left\{\hat{\rho},\hat{\sigma}_i^z\right\}+\sum_{i,\eta}2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta}. \tag{28}$$

II. MEAN-FIELD APPROXIMATION

Using the evolution equation:

$$\partial_t\left\langle\hat{\sigma}_i^\alpha\left(t\right)\right\rangle = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\partial_t\hat{\rho}\right). \tag{29}$$

and the mean-field asymption on ρ of the form:

$$\hat{\rho} = \otimes_i\hat{\rho}_i. \tag{30}$$

then we can find that:

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \partial_t \hat{\rho}\right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left(-\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right] - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\rho} - 2\left(\gamma_- - \gamma_+\right)\sum_i \left\{\hat{\rho},\hat{\sigma}_i^z\right\} + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta \hat{\rho}\hat{A}_i^{\dagger\eta}\right)\right) \tag{31}$$

$$= -\frac{\mathrm{i}}{\hbar}\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle - 2\left(\gamma_- - \gamma_+\right)\sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right) \tag{32}$$

$$+ \sum_{j,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho}\hat{A}_j^{\dagger\eta}\right), \tag{33}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\rho}\hat{\sigma}_j^z + \hat{\sigma}_i^\alpha\hat{\sigma}_j^z\hat{\rho}\right) \tag{34}$$

$$= \mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\sigma}_i^\alpha\hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_j^z\hat{\rho}\right) \tag{35}$$

$$= \delta_{ij}\left(\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\sigma}_i^\alpha\hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_i^z\hat{\rho}\right)\right) + \left(1-\delta_{ij}\right)\left(\mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\sigma}_i^\alpha\hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_j^z\hat{\rho}\right)\right) \tag{36}$$

$$= \delta_{ij}\left(\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\sigma}_i^\alpha\hat{\rho}_i\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_i^z\hat{\rho}_i\right)\right) + 2(1-\delta_{ij})\mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\rho}_j\right)\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\rho}_i\right), \tag{37}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\sigma}_i^\alpha\hat{\rho}_i\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_i^z\hat{\rho}_i\right) = 2\delta_{z\alpha}, \tag{38}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right) = 2\delta_{ij}\delta_{z\alpha} + 2(1-\delta_{ij})\left\langle\hat{\sigma}_j^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle, \tag{39}$$

$$\sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right) = \sum_j\left(\delta_{ij}\delta_{z\alpha} + 2(1-\delta_{ij})\left\langle\hat{\sigma}_j^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle\right) \tag{40}$$

$$= \delta_{z\alpha} - 2\left\langle\hat{\sigma}_i^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle + 2\left\langle\hat{\sigma}_i^\alpha\right\rangle\sum_j\left\langle\hat{\sigma}_j^z\right\rangle \tag{41}$$

$$= \delta_{z\alpha} - 2\left\langle\hat{\sigma}_i^\alpha\right\rangle\left(\left\langle\hat{\sigma}_i^z\right\rangle - \sum_j\left\langle\hat{\sigma}_j^z\right\rangle\right), \tag{42}$$

$$2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\hat{A}_j^{\dagger\eta}\right) = 2\gamma_\eta\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\right) \tag{43}$$

$$= \left(1-\delta_{ij}\right)2\gamma_\eta\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\right) + \delta_{ij}2\gamma_\eta\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\right) \tag{44}$$

$$= \left(1-\delta_{ij}\right)2\gamma_\eta\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}_j\right)\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\rho}_i\right) + \delta_{ij}2\gamma_\eta\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_i^\eta\hat{\rho}_i\right), \tag{45}$$

$$\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}_j\right) = \delta_{+\eta}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} + \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{-\eta}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} - \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{\mathbb{I}}\hat{\rho}_j\right) \tag{46}$$

$$= 2\delta_{+\eta}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} + \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + 2\delta_{-\eta}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} - \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{\mathbb{I}}\hat{\rho}_j\right) \tag{47}$$

$$= 2\delta_{+\eta} + 2\delta_{-\eta} + \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle \tag{48}$$

$$= 2\delta_{+\eta} + 2\delta_{-\eta} + 2\delta_{z\eta} - \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle \tag{49}$$

$$= 2 - \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle, \tag{50}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_i^\eta\hat{\rho}_i\right) = \delta_{+\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^\alpha\hat{A}_i^+\hat{\rho}_i\right) + \delta_{-\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^\alpha\hat{A}_i^-\hat{\rho}_i\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^\alpha\hat{A}_i^z\hat{\rho}_i\right), \tag{51}$$

$$\left(\sigma_i^\pm\right)^2 = 0, \tag{52}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^\alpha\hat{A}_i^+\hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^+\hat{A}_i^+\hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^-\hat{A}_i^+\hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^z\hat{A}_i^+\hat{\rho}_i\right) \tag{53}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+}\hat{\sigma}_i^z\hat{A}_i^+\hat{\rho}_i\right) \tag{54}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right)\hat{\rho}_i\right) \tag{55}$$

$$= 2\delta_{z\alpha}\left(1 - \left\langle\hat{\sigma}_i^z\right\rangle\right), \tag{56}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^\alpha\hat{A}_i^-\hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^+\hat{A}_i^-\hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^-\hat{A}_i^-\hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^z\hat{A}_i^-\hat{\rho}_i\right) \tag{57}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-}\hat{\sigma}_i^z\hat{A}_i^-\hat{\rho}_i\right) \tag{58}$$

$$= -2\delta_{z\alpha}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right)\hat{\rho}_i\right) \tag{59}$$

$$= -2\delta_{z\alpha}\left(1 + \left\langle\hat{\sigma}_i^z\right\rangle\right), \tag{60}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^\alpha\hat{A}_i^z\hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^+\hat{A}_i^z\hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^-\hat{A}_i^z\hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^z\hat{A}_i^z\hat{\rho}_i\right), \tag{61}$$

$$\hat{\sigma}_i^{\dagger z}\hat{\sigma}_i^\pm\hat{\sigma}_i^z = \hat{\sigma}_i^{\dagger z}\left(\hat{\sigma}_i^x \pm \mathrm{i}\hat{\sigma}_i^y\right)\hat{\sigma}_i^z \tag{62}$$

$$= \hat{\sigma}_i^z\left(\hat{\sigma}_i^x \pm \mathrm{i}\hat{\sigma}_i^y\right)\hat{\sigma}_i^z \tag{63}$$

$$= \hat{\sigma}_i^z\hat{\sigma}_i^x\hat{\sigma}_i^z \pm \mathrm{i}\hat{\sigma}_i^z\hat{\sigma}_i^y\hat{\sigma}_i^z \tag{64}$$

$$= \mathrm{i}\hat{\sigma}_i^y\hat{\sigma}_i^z \pm \mathrm{i}\hat{\sigma}_i^z\mathrm{i}\hat{\sigma}_i^x \tag{65}$$

$$
\begin{aligned}
&= \mathrm{i}^2 \hat{\sigma}_i^x \pm \mathrm{i}\mathrm{i}^2 \hat{\sigma}_i^y && (66)\\
&= -\hat{\sigma}_i^{\pm}, && (67)\\
\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^{\alpha} \hat{A}_i^{z} \hat{\rho}_i\right) &= \delta_{+\alpha} \mathrm{Tr}\left(-\hat{\sigma}_i^{+} \hat{\rho}_i\right) + \delta_{-\alpha} \mathrm{Tr}\left(-\hat{\sigma}_i^{-} \hat{\rho}_i\right) + \delta_{z\alpha} \mathrm{Tr}\left(\hat{\sigma}_i^{z} \hat{\rho}_i\right) && (68)\\
&= \delta_{z\alpha} \left\langle \hat{\sigma}_i^{z} \right\rangle - \delta_{+\alpha} \left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha} \left\langle \hat{\sigma}_i^{-} \right\rangle && (69)\\
\mathrm{Tr}\left(\hat{A}_i^{\dagger \eta} \hat{\sigma}_i^{\alpha} \hat{A}_i^{\eta} \hat{\rho}_i\right) &= \delta_{+\eta}(2\delta_{z\alpha}\left(1 - \left\langle \hat{\sigma}_i^{z} \right\rangle\right)) + \delta_{-\eta}(-2\delta_{z\alpha}\left(1 + \left\langle \hat{\sigma}_i^{z} \right\rangle\right)) + \delta_{z\eta}\big(\delta_{z\alpha} \left\langle \hat{\sigma}_i^{z} \right\rangle - \delta_{+\alpha}\left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha}\left\langle \hat{\sigma}_i^{-} \right\rangle\big) && (70)\\
&= 2\delta_{+\eta}\delta_{z\alpha}\left(1 - \left\langle \hat{\sigma}_i^{z} \right\rangle\right) - 2\delta_{-\eta}\delta_{z\alpha}\left(1 + \left\langle \hat{\sigma}_i^{z} \right\rangle\right) + \delta_{z\eta}\left(\delta_{z\alpha} \left\langle \hat{\sigma}_i^{z} \right\rangle - \delta_{+\alpha}\left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha}\left\langle \hat{\sigma}_i^{-} \right\rangle\right), && (71)\\
2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{A}_j^{\eta} \hat{\rho} \hat{A}_j^{\dagger\eta}\right) &= 2\gamma_\eta \left(\left(1 - \delta_{ij}\right) \mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{A}_j^{\eta} \hat{\rho}_j\right) \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{\rho}_i\right) + \delta_{ij} \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^{\alpha} \hat{A}_i^{\eta} \hat{\rho}_i\right)\right) && (72)\\
&= 2\gamma_\eta \delta_{ij}\big(2\delta_{+\eta}\delta_{z\alpha}(1 - \langle \hat{\sigma}_i^{z} \rangle) - 2\delta_{-\eta}\delta_{z\alpha}(1 + \langle \hat{\sigma}_i^{z} \rangle) + \delta_{z\eta}\big(\delta_{z\alpha}\langle \hat{\sigma}_i^{z} \rangle - \delta_{+\alpha}\left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha}\left\langle \hat{\sigma}_i^{-} \right\rangle\big)\big) && (73)\\
&\quad + 2\gamma_\eta \left(1 - \delta_{ij}\right)\left(2 - \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle && (74)\\
2\gamma_+ \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{A}_j^{+} \hat{\rho} \hat{A}_j^{\dagger +}\right) &= 2\gamma_+ \left(\left(1 - \delta_{ij}\right)\left(2 + 2\left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\left(2\delta_{z\alpha}\left(1 - \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right)\right) && (75)\\
&= 4\gamma_+ \left(\left(1 - \delta_{ij}\right)\left(1 + \left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\delta_{z\alpha}\left(1 - \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right), && (76)\\
2\gamma_- \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{A}_j^{-} \hat{\rho} \hat{A}_j^{\dagger -}\right) &= 2\gamma_- \left(\left(1 - \delta_{ij}\right)\left(2 - 2\left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\left(-2\delta_{z\alpha}\left(1 + \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right)\right) && (77)\\
&= 4\gamma_- \left(\left(1 - \delta_{ij}\right)\left(1 - \left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle - \delta_{ij}\delta_{z\alpha}\left(1 + \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right), && (78)\\
2\gamma_z \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{A}_j^{z} \hat{\rho} \hat{A}_j^{\dagger z}\right) &= 2\gamma_z \left(\left(1 - \delta_{ij}\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\left(\delta_{z\alpha}\left\langle \hat{\sigma}_i^{z} \right\rangle - \delta_{+\alpha}\left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha}\left\langle \hat{\sigma}_i^{-} \right\rangle\right)\right) && (79)\\
\sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{A}_j^{\eta} \hat{\rho} \hat{A}_j^{\dagger\eta}\right) &= 4\gamma_+ \left(\left(1 - \delta_{ij}\right)\left(1 + \left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\delta_{z\alpha}\left(1 - \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right) + 4\gamma_- \left(\left(1 - \delta_{ij}\right)\left(1 - \left\langle \hat{\sigma}_j^{z} \right\rangle\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle\right. && (80)\\
&\quad \left. - \delta_{ij}\delta_{z\alpha}\left(1 + \left\langle \hat{\sigma}_i^{z} \right\rangle\right)\right) + 2\gamma_z \left(\left(1 - \delta_{ij}\right) \left\langle \hat{\sigma}_i^{\alpha} \right\rangle + \delta_{ij}\left(\delta_{z\alpha}\left\langle \hat{\sigma}_i^{z} \right\rangle - \delta_{+\alpha}\left\langle \hat{\sigma}_i^{+} \right\rangle - \delta_{-\alpha}\left\langle \hat{\sigma}_i^{-} \right\rangle\right)\right), && (81)\\
\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right]\right) &= \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{H}_{\mathrm{S}} \hat{\rho} - \hat{\sigma}_i^{\alpha} \hat{\rho} \hat{H}_{\mathrm{S}}\right) && (82)\\
&= \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{H}_{\mathrm{S}} \hat{\rho}\right) - \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{\rho} \hat{H}_{\mathrm{S}}\right) && (83)\\
&= \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha} \hat{H}_{\mathrm{S}} \hat{\rho}\right) - \mathrm{Tr}\left(\hat{H}_{\mathrm{S}} \hat{\sigma}_i^{\alpha} \hat{\rho}\right) && (84)\\
&= \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{H}_{\mathrm{S}}\right] \hat{\rho}\right) && (85)\\
&= \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \sum_j \overrightarrow{h_j} \cdot \overrightarrow{\hat{\sigma}_j} + \sum_{jk\beta\varepsilon} A_{jk} \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}\right] \hat{\rho}\right) && (86)\\
&= \sum_j \overrightarrow{h_j} \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \overrightarrow{\hat{\sigma}_j}\right] \hat{\rho}\right) + \sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}\right] \hat{\rho}\right) && (87)\\
&= \sum_{j\beta} h_{j\beta} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\right] \hat{\rho}\right) + \sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}\right] \hat{\rho}\right), && (88)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\right] \hat{\rho}\right) &= \delta_{ij} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right] \hat{\rho}\right) + \cancelto{0}{\left(1 - \delta_{ij}\right) \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\right] \hat{\rho}\right)}, && (89)\\
&= \delta_{ij} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right] \hat{\rho}\right), && (90)\\
\sum_{j\beta} h_{j\beta} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\right] \hat{\rho}\right) &= \sum_{\beta} h_{i\beta} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right] \hat{\rho}\right), \beta \in \{x, y, z\}. && (91)
\end{aligned}
$$

For the last part of this calculation given by $\sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}\right] \hat{\rho}\right)$ with $\beta, \varepsilon \in \{x, y, z\}$, we will be more detailed. If $i \neq j, j \neq k, k \neq i$ then by the factorizability hypothesis and the properties of the commutator we deduce that $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_k^{\varepsilon}\right] \hat{\rho}\right) = 0$, so for now we do not require the truncation at third order. Now if $i \neq j, j = k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_j^{\varepsilon}\right] \hat{\rho}\right) = 0$ by the factorization hyphotesis. If $i \neq j, i = k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta} \hat{\sigma}_i^{\varepsilon}\right] \hat{\rho}\right) = \left\langle [\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}] \right\rangle \left\langle \hat{\sigma}_j^{\beta} \right\rangle$ and if

$i=j, j\neq k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) = \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle\left\langle\hat{\sigma}_k^{\varepsilon}\right\rangle$, finally if $i=j=k$ then:

$$\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\hat{\sigma}_i^{\varepsilon}\right]\hat{\rho}\right) = \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \delta_{\beta\varepsilon} + \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\hat{\sigma}_i^{\gamma}\right]\hat{\rho}\right) \tag{92}$$
$$= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\gamma}\right]\hat{\rho}\right) \tag{93}$$
$$= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\gamma}\right]\right\rangle. \tag{94}$$

To have a suitable notation then we introduce our relevant sets where the expected values can be non-zero:

$$C_1 = \{(i,j,k)\,|i\neq j, i=k\}, \tag{95}$$
$$C_2 = \{(i,j,k)\,|i=j, j\neq k\}, \tag{96}$$
$$C_3 = \{(i,j,k)\,|i=j=k\}. \tag{97}$$

With this machinary we can write the expected value of the interaction hamiltonian as:

$$\sum_{jk\beta\varepsilon} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) = \sum_{\beta\varepsilon}\sum_{jk} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) \tag{98}$$
$$= \sum_{\beta\varepsilon}\left(\sum_{C_1} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) + \sum_{C_2} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) + \sum_{C_3} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)\right) \tag{99}$$
$$= \sum_{\beta\varepsilon}\left(\sum_{j\neq i} A_{ji}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle\left\langle\hat{\sigma}_j^{\beta}\right\rangle + \sum_{k\neq i} A_{ik}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle\left\langle\hat{\sigma}_k^{\varepsilon}\right\rangle + A_{ii}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\hat{\sigma}_i^{\varepsilon}\right]\hat{\rho}\right)\right) \tag{100}$$
$$= \sum_{\beta\varepsilon}\left(\sum_{j\neq i} A_{ji}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle\left\langle\hat{\sigma}_j^{\beta}\right\rangle + \sum_{j\neq i} A_{ij}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle\left\langle\hat{\sigma}_j^{\varepsilon}\right\rangle + A_{ii}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\hat{\sigma}_i^{\varepsilon}\right]\hat{\rho}\right)\right) \tag{101}$$
$$= \sum_{\beta\varepsilon}\left(\mathrm{i}A_{ii}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\gamma}\right]\right\rangle + \sum_{j\neq i}\left(A_{ji}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle\left\langle\hat{\sigma}_j^{\beta}\right\rangle + A_{ij}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle\left\langle\hat{\sigma}_j^{\varepsilon}\right\rangle\right)\right) \tag{102}$$
$$= \mathrm{i}A_{ii}\sum_{\beta\varepsilon}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\gamma}\right]\right\rangle + \sum_{j\neq i}\left(A_{ji}\sum_{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle\left\langle\hat{\sigma}_j^{\beta}\right\rangle + A_{ij}\sum_{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle\left\langle\hat{\sigma}_j^{\varepsilon}\right\rangle\right), \tag{103}$$
$$\hat{\Upsilon}_i = \sum_{\varepsilon}\hat{\sigma}_i^{\varepsilon} \tag{104}$$
$$= \begin{pmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & -1 \end{pmatrix} \tag{105}$$
$$= \hat{\sigma}_i^{z} + \frac{1-\mathrm{i}}{2}\hat{\sigma}_i^{+} + \frac{1+\mathrm{i}}{2}\hat{\sigma}_i^{-}, \tag{106}$$
$$\left\langle\hat{\Upsilon}_i\right\rangle = \left\langle\hat{\sigma}_i^{z}\right\rangle + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^{+}\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^{-}\right\rangle, \tag{107}$$
$$\sum_{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle\left\langle\hat{\sigma}_j^{\beta}\right\rangle = \left\langle\hat{\Upsilon}_j\right\rangle\sum_{\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle, \tag{108}$$
$$\sum_{\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle = \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}\right]\right\rangle \tag{109}$$
$$= \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right]\right\rangle, \tag{110}$$

$$\sum_{\beta\varepsilon} \langle[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\varepsilon}]\rangle \left\langle \hat{\sigma}_j^{\beta} \right\rangle = \left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right] \right\rangle \left\langle \hat{\Upsilon}_j \right\rangle, \tag{111}$$

$$\sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) = \mathrm{i}A_{ii}\sum_{\beta\varepsilon}\epsilon_{\beta\varepsilon\gamma}\langle[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\gamma}]\rangle + \left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right] \right\rangle \sum_{j\neq i}\left(A_{ji} + A_{ij}\right)\left\langle \hat{\Upsilon}_j \right\rangle, \tag{112}$$

$$\sum_{\beta\varepsilon}\epsilon_{\beta\varepsilon\gamma} = 0, \tag{113}$$

$$\sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) = \left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right] \right\rangle \sum_{j\neq i}\left(A_{ji} + A_{ij}\right)\left\langle \hat{\Upsilon}_j \right\rangle. \tag{114}$$

Summarizing we find that:

$$\begin{aligned}
\partial_t \langle\hat{\sigma}_i^{\alpha}(t)\rangle = &-\frac{\mathrm{i}}{\hbar}\left(\sum_{\beta} h_{i\beta}\left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\langle\hat{\sigma}_i^{\alpha}\rangle - 2\left(\gamma_- - \gamma_+\right) && (115)\\
&\times\left(\delta_{z\alpha} - 2\langle\hat{\sigma}_i^{\alpha}\rangle\left(\langle\hat{\sigma}_i^{z}\rangle - \sum_j\langle\hat{\sigma}_j^{z}\rangle\right)\right) + \sum_j\left(4\gamma_+\left((1-\delta_{ij})(1+\langle\hat{\sigma}_j^{z}\rangle)\langle\hat{\sigma}_i^{\alpha}\rangle + \delta_{ij}\delta_{z\alpha}\left(1-\langle\hat{\sigma}_i^{z}\rangle\right)\right) + 4\gamma_-\left((1-\delta_{ij})\right.\right. && (116)\\
&\times\left(1-\langle\hat{\sigma}_j^{z}\rangle\right)\langle\hat{\sigma}_i^{\alpha}\rangle - \delta_{ij}\delta_{z\alpha}\left(1+\langle\hat{\sigma}_i^{z}\rangle\right)\big) + 2\gamma_z\left((1-\delta_{ij})\langle\hat{\sigma}_i^{\alpha}\rangle + \delta_{ij}\left(\delta_{z\alpha}\langle\hat{\sigma}_i^{z}\rangle - \delta_{+\alpha}\left\langle\hat{\sigma}_i^{+}\right\rangle - \delta_{-\alpha}\left\langle\hat{\sigma}_i^{-}\right\rangle\right)\right)\big). && (117)
\end{aligned}$$

In particular we cannot assume symmetry due to the fact that the system hamiltonian is not the same for all the particles, in this case we have $2N$ differential equations (or $3N$ in case that we want to include the conjugate transpose part of σ^+ which is σ^-):

$$\begin{aligned}
\partial_t\left\langle\hat{\sigma}_i^{+}(t)\right\rangle = &-\frac{\mathrm{i}}{\hbar}\left(\sum_{\beta} h_{i\beta}\left\langle\left[\hat{\sigma}_i^{+}, \hat{\sigma}_i^{\beta}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{+}, \hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N(\gamma_z + 2\gamma_+ + 2\gamma_-)\left\langle\hat{\sigma}_i^{+}\right\rangle - 2\left(\gamma_- - \gamma_+\right) && (118)\\
&\times\left(\delta_{z+} - 2\left\langle\hat{\sigma}_i^{+}\right\rangle\left(\langle\hat{\sigma}_i^{z}\rangle - \sum_j\langle\hat{\sigma}_j^{z}\rangle\right)\right) + \sum_j\left(4\gamma_+\left((1-\delta_{ij})\left(1+\langle\hat{\sigma}_j^{z}\rangle\right)\left\langle\hat{\sigma}_i^{+}\right\rangle + \delta_{ij}\delta_{z+}\left(1-\langle\hat{\sigma}_i^{z}\rangle\right)\right)\right. && (119)\\
&+ 4\gamma_-(1-\delta_{ij})\left(1-\langle\hat{\sigma}_j^{z}\rangle\right)\left\langle\hat{\sigma}_i^{+}\right\rangle - \delta_{ij}\delta_{z+}\left(1+\langle\hat{\sigma}_i^{z}\rangle\right) + 2\gamma_z\left(\delta_{ij}\left(\delta_{z+}\langle\hat{\sigma}_i^{z}\rangle - \delta_{++}\left\langle\hat{\sigma}_i^{+}\right\rangle - \delta_{-+}\delta_{-+}\left\langle\hat{\sigma}_i^{-}\right\rangle\right)\right. && (120)\\
&+ (1-\delta_{ij})\left\langle\hat{\sigma}_i^{+}\right\rangle\big)\big) && (121)\\
= &-\frac{\mathrm{i}}{\hbar}\left(\sum_{\beta} h_{i\beta}\left\langle\left[\hat{\sigma}_i^{+}, \hat{\sigma}_i^{\beta}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{+}, \hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^{+}\right\rangle - 2\left(\gamma_-\right. && (122)\\
&\left.-\gamma_+\right)\left(-2\left\langle\hat{\sigma}_i^{+}\right\rangle\left(\langle\hat{\sigma}_i^{z}\rangle - \sum_j\langle\hat{\sigma}_j^{z}\rangle\right)\right) + \sum_j\left(4\gamma_+\left(1-\delta_{ij}\right)\left(1+\langle\hat{\sigma}_j^{z}\rangle\right)\left\langle\hat{\sigma}_i^{+}\right\rangle + 4\gamma_-\left(1-\delta_{ij}\right)\left(1-\langle\hat{\sigma}_j^{z}\rangle\right)\right. && (123)\\
&\times\left\langle\hat{\sigma}_i^{+}\right\rangle + 2\gamma_z\left((1-\delta_{ij})\left\langle\hat{\sigma}_i^{+}\right\rangle - \delta_{ij}\left\langle\hat{\sigma}_i^{+}\right\rangle\right)\big), && (124)\\
= &-\frac{\mathrm{i}}{\hbar}\left(\sum_{\beta} h_{i\beta}\left\langle\left[\hat{\sigma}_i^{+}, \hat{\sigma}_i^{\beta}\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^{+}, \hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N(\gamma_z + 2\gamma_+ + 2\gamma_-)\left\langle\hat{\sigma}_i^{+}\right\rangle - 4\left(\gamma_-\right. && (125)\\
&\left.-\gamma_+\right)\left\langle\hat{\sigma}_i^{+}\right\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^{z}\rangle + \left\langle\hat{\sigma}_i^{+}\right\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\langle\hat{\sigma}_j^{z}\rangle\right) + 4\gamma_-\left(1-\langle\hat{\sigma}_j^{z}\rangle\right) + 2\gamma_z\right) - 2\gamma_z\left\langle\hat{\sigma}_i^{+}\right\rangle && (126)
\end{aligned}$$

$$\begin{aligned}
\left[\hat{\sigma}_i^{+}, \hat{\sigma}_i^{x}\right] &= \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} && (127)\\
&= \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix} && (128)\\
&= 2\hat{\sigma}_i^{z}, && (129)
\end{aligned}$$

$$[\hat{\sigma}_i^+, \hat{\sigma}_i^y] = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} - \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \tag{130}$$

$$= \begin{pmatrix}2\mathrm{i} & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 2\mathrm{i}\end{pmatrix} \tag{131}$$

$$= 2\mathrm{i}\hat{\sigma}_i^z, \tag{132}$$

$$[\hat{\sigma}_i^+, \hat{\sigma}_i^z] = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \tag{133}$$

$$= \begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \tag{134}$$

$$= -2\hat{\sigma}_i^+, \tag{135}$$

$$\left\langle \left[\hat{\sigma}_i^+, \hat{\Upsilon}_i\right]\right\rangle = \left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^x + \hat{\sigma}_i^y + \hat{\sigma}_i^z\right]\right\rangle \tag{136}$$

$$= \left\langle \left[\hat{\sigma}_i^+, 2\hat{\sigma}_i^z + 2\mathrm{i}\hat{\sigma}_i^z - 2\hat{\sigma}_i^+\right]\right\rangle \tag{137}$$

$$= 2\left(1+\mathrm{i}\right)\left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^z\right]\right\rangle \tag{138}$$

$$= -4\left(1+\mathrm{i}\right)\left\langle \hat{\sigma}_i^+\right\rangle, \tag{139}$$

$$\sum_\beta h_{i\beta}\left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^\beta\right]\right\rangle = h_{ix}\left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^x\right]\right\rangle + h_{iy}\left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^y\right]\right\rangle + h_{iz}\left\langle \left[\hat{\sigma}_i^+, \hat{\sigma}_i^z\right]\right\rangle \tag{140}$$

$$= h_{ix}\left\langle 2\hat{\sigma}_i^z\right\rangle + h_{iy}\left\langle 2\mathrm{i}\hat{\sigma}_i^z\right\rangle + h_{iz}\left\langle -2\hat{\sigma}_i^+\right\rangle \tag{141}$$

$$= 2\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right), \tag{142}$$

$$\partial_t\left\langle\hat{\sigma}_i^+(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - 4\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left(\left\langle\hat{\sigma}_j^z\right\rangle + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_j^+\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) \tag{143}$$

$$-2N(\gamma_z+2\gamma_++2\gamma_-)\left\langle\hat{\sigma}_i^+\right\rangle - 4\left(\gamma_--\gamma_+\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle + \left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right) + 4\gamma_-\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\right. \tag{144}$$

$$\left.+2\gamma_z\right) - 2\gamma_z\left\langle\hat{\sigma}_i^+\right\rangle \tag{145}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - 4\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left(\left\langle\hat{\sigma}_j^z\right\rangle + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_j^+\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) \tag{146}$$

$$-\left\langle\hat{\sigma}_i^+\right\rangle(2N\left(\gamma_z+2\gamma_++2\gamma_-\right)+2\gamma_z) - 4\left(\gamma_--\gamma_+\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle + \left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right) + 4\gamma_-\right. \tag{147}$$

$$\left.\times\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right) + 2\gamma_z\right) \tag{148}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - 4\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left(\left\langle\hat{\sigma}_j^z\right\rangle + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_j^+\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) \tag{149}$$

$$-\left\langle\hat{\sigma}_i^+\right\rangle(2N(\gamma_z+2\gamma_++2\gamma_-)+2\gamma_z) + \left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right) + 4\gamma_-\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right) + 2\gamma_z - 4\left(\gamma_--\gamma_+\right)\right. \tag{150}$$

$$\left.\times\left\langle\hat{\sigma}_j^z\right\rangle\right), \tag{151}$$

$$= -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - (1+\mathrm{i})\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left(2\left\langle\hat{\sigma}_j^z\right\rangle + (1-\mathrm{i})\left\langle\hat{\sigma}_j^+\right\rangle + (1+\mathrm{i})\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) \tag{152}$$

$$-\left\langle\hat{\sigma}_i^+\right\rangle(2N(\gamma_z+2\gamma_++2\gamma_-)+2\gamma_z) + \left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right) + 4\gamma_-\left(1-\left\langle\hat{\sigma}_j^z\right\rangle - 4\left(\gamma_--\gamma_+\right)\left\langle\hat{\sigma}_j^z\right\rangle\right)\right. \tag{153}$$

$$\left.+2\gamma_z\right), \tag{154}$$

$$\overrightarrow{v} = (1-\mathrm{i}, 1+\mathrm{i}, 2), \tag{155}$$

$$\partial_t\left\langle\hat{\sigma}_i^+(t)\right\rangle = -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - \left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle, \left\langle\hat{\sigma}_j^-\right\rangle, \left\langle\hat{\sigma}_j^z\right\rangle\right)\right) - \left\langle\hat{\sigma}_i^+\right\rangle \tag{156}$$

$$\times\left(4N\left(\gamma_++\gamma_-\right) + 2\left(N+1\right)\gamma_z\right) + 2\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(2\gamma_+\left(1+2\left\langle\hat{\sigma}_j^z\right\rangle\right) + 2\gamma_-\left(1-2\left\langle\hat{\sigma}_j^z\right\rangle\right) + \gamma_z\right) \tag{157}$$

$$= -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - \left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle, \left\langle\hat{\sigma}_j^-\right\rangle, \left\langle\hat{\sigma}_j^z\right\rangle\right)\right) \tag{158}$$

$$-\left\langle\hat{\sigma}_i^+\right\rangle\left(4N\left(\gamma_++\gamma_-\right)+2\left(N+1\right)\gamma_z\right)+2\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(2\gamma_++2\gamma_-+\gamma_z\right)+2\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left\langle\hat{\sigma}_j^z\right\rangle-4\gamma_-\left\langle\hat{\sigma}_j^z\right\rangle\right) \tag{159}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)+2\left\langle\hat{\sigma}_i^+\right\rangle\left(N-1\right) \tag{160}$$

$$\times\left(2\gamma_++2\gamma_-+\gamma_z\right)-\left\langle\hat{\sigma}_i^+\right\rangle\left(4N\left(\gamma_++\gamma_-\right)+2\left(N+1\right)\gamma_z\right)+2\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(4\gamma_+\left\langle\hat{\sigma}_j^z\right\rangle-4\gamma_-\left\langle\hat{\sigma}_j^z\right\rangle\right) \tag{161}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)+8\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle \tag{162}$$

$$\times\left(\gamma_+-\gamma_-\right)+\left\langle\hat{\sigma}_i^+\right\rangle\left(2\left(N-1\right)\left(2\gamma_++2\gamma_-+\gamma_z\right)-\left(4N\left(\gamma_++\gamma_-\right)+2\left(N+1\right)\gamma_z\right)\right) \tag{163}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)+8\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle \tag{164}$$

$$\times\left(\gamma_+-\gamma_-\right)+\left\langle\hat{\sigma}_i^+\right\rangle\left(4\left(N-1\right)\left(\gamma_++\gamma_-\right)+2\left(N-1\right)\gamma_z-4N\left(\gamma_++\gamma_-\right)-2\left(N+1\right)\gamma_z\right) \tag{165}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)+8\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle \tag{166}$$

$$\times\left(\gamma_+-\gamma_-\right)+\left\langle\hat{\sigma}_i^+\right\rangle\left(-4\left(\gamma_++\gamma_-\right)-4\gamma_z\right) \tag{167}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)+8\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle \tag{168}$$

$$\times\left(\gamma_+-\gamma_-\right)-4\left\langle\hat{\sigma}_i^+\right\rangle\left(\gamma_++\gamma_-+\gamma_z\right). \tag{169}$$

Our first pair of equations are:

$$\partial_t\left\langle\hat{\sigma}_i^+\left(t\right)\right\rangle=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-(1+\mathrm{i})\langle\hat{\sigma}_i^+\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right)-4\left\langle\hat{\sigma}_i^+\right\rangle \tag{170}$$

$$\times\left(\gamma_++\gamma_-+\gamma_z\right)+8\left\langle\hat{\sigma}_i^+\right\rangle\left(\gamma_+-\gamma_-\right)\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle, \tag{171}$$

$$\partial_t\left\langle\hat{\sigma}_i^-\left(t\right)\right\rangle=\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}^*\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,-\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^-\right\rangle\right)-\left(1-\mathrm{i}\right)\left\langle\hat{\sigma}_i^-\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)^*\overrightarrow{v}^*\cdot\left(\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) \tag{172}$$

$$-4\left\langle\hat{\sigma}_i^-\right\rangle\left(\gamma_++\gamma_-+\gamma_z\right)^*+8\left\langle\hat{\sigma}_i^-\right\rangle\left(\gamma_+-\gamma_-\right)^*\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle \tag{173}$$

$$=-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}^*\cdot\left(-\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,\left\langle\hat{\sigma}_i^-\right\rangle\right)+\left(1-\mathrm{i}\right)\left\langle\hat{\sigma}_i^-\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)^*\overrightarrow{v}^*\cdot\left(\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) \tag{174}$$

$$-4\left\langle\hat{\sigma}_i^-\right\rangle\left(\gamma_++\gamma_-+\gamma_z\right)+8\left\langle\hat{\sigma}_i^-\right\rangle\left(\gamma_+-\gamma_-\right)\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle. \tag{175}$$

The latter was found using $\left(\hat{\sigma}_i^+\left(t\right)\right)^\dagger=\hat{\sigma}_i^-\left(t\right)$. The equation for $\partial_t\left\langle\hat{\sigma}_i^z\left(t\right)\right\rangle$ is:

$$\partial_t \left\langle \hat{\sigma}_i^z\left(t\right)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\sum_\beta h_{i\beta}\left\langle\left[\hat{\sigma}_i^z,\hat{\sigma}_i^\beta\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^z,\hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N(\gamma_z+2\gamma_+ +2\gamma_-)\langle\hat{\sigma}_i^z\rangle - 2\left(\gamma_- - \gamma_+\right) \tag{176}$$

$$\times\left(\delta_{zz} - 2\left\langle\hat{\sigma}_i^z\right\rangle\left(\left\langle\hat{\sigma}_i^z\right\rangle - \sum_j\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) + \sum_j\left(4\gamma_+\left(\left(1-\delta_{ij}\right)\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\delta_{zz}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 4\gamma_- \right. \tag{177}$$

$$\times\left(\left(1-\delta_{ij}\right)\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{ij}\delta_{zz}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 2\gamma_z\left(\left(1-\delta_{ij}\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\left(\delta_{zz}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+z}\left\langle\hat{\sigma}_i^+\right\rangle - \delta_{-z}\left\langle\hat{\sigma}_i^-\right\rangle\right)\right)\Big) \tag{178}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\sum_\beta h_{i\beta}\left\langle\left[\hat{\sigma}_i^z,\hat{\sigma}_i^\beta\right]\right\rangle + \left\langle\left[\hat{\sigma}_i^z,\hat{\Upsilon}_i\right]\right\rangle\sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z+2\gamma_+ +2\gamma_-\right)\langle\hat{\sigma}_i^z\rangle - 2\left(\gamma_- - \gamma_+\right) \tag{179}$$

$$\times\left(1-2\left\langle\hat{\sigma}_i^z\right\rangle\left(\left\langle\hat{\sigma}_i^z\right\rangle - \sum_j\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) + \sum_j\left(4\gamma_+\left(\left(1-\delta_{ij}\right)\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 4\gamma_-\left(\left(1-\delta_{ij}\right)\right.\right. \tag{180}$$

$$\times\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{ij}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\Big) + 2\gamma_z\left(\left(1-\delta_{ij}\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\left\langle\hat{\sigma}_i^z\right\rangle\right)\Big), \tag{181}$$

$$\sum_\beta h_{i\beta}\left\langle\left[\hat{\sigma}_i^z,\hat{\sigma}_i^\beta\right]\right\rangle = h_{ix}\left\langle[\hat{\sigma}_i^z,\hat{\sigma}_i^x]\right\rangle + h_{iy}\left\langle[\hat{\sigma}_i^z,\hat{\sigma}_i^y]\right\rangle + h_{iz}\left\langle[\hat{\sigma}_i^z,\hat{\sigma}_i^z]\right\rangle \tag{182}$$

$$= h_{ix}\left\langle\mathrm{i}\hat{\sigma}_i^y\right\rangle + h_{iy}\left\langle-\mathrm{i}\hat{\sigma}_i^x\right\rangle + h_{iz}\cdot 0 \tag{183}$$

$$= \mathrm{i}\overrightarrow{h_i}\cdot\left(\left\langle\hat{\sigma}_i^y\right\rangle, -\left\langle\hat{\sigma}_i^x\right\rangle, 0\right), \tag{184}$$

$$\hat{\sigma}_i^x = \frac{\hat{\sigma}_i^+ + \hat{\sigma}_i^-}{2}, \tag{185}$$

$$\hat{\sigma}_i^y = \mathrm{i}\frac{\hat{\sigma}_i^- - \hat{\sigma}_i^+}{2}, \tag{186}$$

$$\sum_\beta h_{i\beta}\left\langle\left[\hat{\sigma}_i^z,\hat{\sigma}_i^\beta\right]\right\rangle = \mathrm{i}\overrightarrow{h_i}\cdot\left(\left\langle\mathrm{i}\frac{\hat{\sigma}_i^- - \hat{\sigma}_i^+}{2}\right\rangle, -\left\langle\frac{\hat{\sigma}_i^+ + \hat{\sigma}_i^-}{2}\right\rangle, 0\right) \tag{187}$$

$$= \overrightarrow{h_i}\cdot\left(\frac{\left\langle\hat{\sigma}_i^+\right\rangle - \left\langle\hat{\sigma}_i^-\right\rangle}{2}, -\mathrm{i}\frac{\left\langle\hat{\sigma}_i^+\right\rangle + \left\langle\hat{\sigma}_i^-\right\rangle}{2}, 0\right), \tag{188}$$

$$\left\langle\left[\hat{\sigma}_i^z,\hat{\Upsilon}_i\right]\right\rangle = \left\langle[\hat{\sigma}_i^z,\hat{\sigma}_i^x + \hat{\sigma}_i^y + \hat{\sigma}_i^z]\right\rangle \tag{189}$$

$$= \left\langle[\hat{\sigma}_i^z,\hat{\sigma}_i^x] + [\hat{\sigma}_i^z,\hat{\sigma}_i^y] + [\hat{\sigma}_i^z,\hat{\sigma}_i^z]\right\rangle \tag{190}$$

$$= \left\langle\mathrm{i}\hat{\sigma}_i^y - \mathrm{i}\hat{\sigma}_i^x\right\rangle \tag{191}$$

$$= \mathrm{i}\left\langle\hat{\sigma}_i^y - \hat{\sigma}_i^x\right\rangle \tag{192}$$

$$= \mathrm{i}\left\langle\mathrm{i}\frac{\hat{\sigma}_i^- - \hat{\sigma}_i^+}{2} - \frac{\hat{\sigma}_i^+ + \hat{\sigma}_i^-}{2}\right\rangle \tag{193}$$

$$= \left\langle-\frac{\hat{\sigma}_i^- - \hat{\sigma}_i^+}{2} - \mathrm{i}\frac{\hat{\sigma}_i^+ + \hat{\sigma}_i^-}{2}\right\rangle \tag{194}$$

$$= -\frac{1}{2}\left(\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^-\right\rangle - \left(1-\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\right) \tag{195}$$

$$= \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\right\rangle - \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\right\rangle, \tag{196}$$

$$\partial_t\left\langle\hat{\sigma}_i^z\left(t\right)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\frac{\left\langle\hat{\sigma}_i^+\right\rangle - \left\langle\hat{\sigma}_i^-\right\rangle}{2}, -\mathrm{i}\frac{\left\langle\hat{\sigma}_i^+\right\rangle + \left\langle\hat{\sigma}_i^-\right\rangle}{2}, 0\right) + \left(\frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\right\rangle - \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\right\rangle\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left(\left\langle\hat{\sigma}_j^z\right\rangle\right.\right. \tag{197}$$

$$\left.\left. + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_j^+\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) - 2N\left(\gamma_z+2\gamma_+ +2\gamma_-\right)\left\langle\hat{\sigma}_i^z\right\rangle - 2\left(\gamma_- - \gamma_+\right)\left(1+2\left\langle\hat{\sigma}_i^z\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle\right) \tag{198}$$

$$+ \sum_j\left(4\gamma_+\left(\left(1-\delta_{ij}\right)\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 4\gamma_-\left(\left(1-\delta_{ij}\right)\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{ij}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\right)\right. \tag{199}$$

$$\left. +2\gamma_z\left(\left(1-\delta_{ij}\right)\left\langle\hat{\sigma}_i^z\right\rangle + \delta_{ij}\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) \tag{200}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\frac{\left\langle\hat{\sigma}_i^+\right\rangle - \left\langle\hat{\sigma}_i^-\right\rangle}{2}, -\mathrm{i}\frac{\left\langle\hat{\sigma}_i^+\right\rangle + \left\langle\hat{\sigma}_i^-\right\rangle}{2}, 0\right) + \left(\frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\right\rangle - \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\right\rangle\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left(\left\langle\hat{\sigma}_j^z\right\rangle\right.\right. \tag{201}$$

$$\left.\left. + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_j^+\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_j^-\right\rangle\right)\right) - 2N\left(\gamma_z+2\gamma_+ +2\gamma_-\right)\left\langle\hat{\sigma}_i^z\right\rangle - 2\left(\gamma_- - \gamma_+\right)\left(1+2\left\langle\hat{\sigma}_i^z\right\rangle\sum_{j\neq i}\left\langle\hat{\sigma}_j^z\right\rangle\right) \tag{202}$$

$$+\sum_j \delta_{ij}\left(4\gamma_+(1-\langle\hat{\sigma}_i^z\rangle)-4\gamma_-\left(1+\langle\hat{\sigma}_i^z\rangle\right)+2\gamma_z\langle\hat{\sigma}_i^z\rangle\right)+\sum_j(1-\delta_{ij})\Big(4\gamma_+\left(1+\langle\hat{\sigma}_j^z\rangle\right)\langle\hat{\sigma}_i^z\rangle+4\gamma_-\left(1-\langle\hat{\sigma}_j^z\rangle\right)\langle\hat{\sigma}_i^z\rangle+2\gamma_z\langle\hat{\sigma}_i^z\rangle\Big) \tag{203}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{204}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)\Bigg(1+2\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle\Bigg)+(4\gamma_+(1-\langle\hat{\sigma}_i^z\rangle)-4\gamma_-(1+\langle\hat{\sigma}_i^z\rangle)+2\gamma_z\langle\hat{\sigma}_i^z\rangle) \tag{205}$$

$$+\sum_{j\neq i}\left(4\gamma_+\left(1+\langle\hat{\sigma}_j^z\rangle\right)\langle\hat{\sigma}_i^z\rangle+4\gamma_-\left(1-\langle\hat{\sigma}_j^z\rangle\right)\langle\hat{\sigma}_i^z\rangle+2\gamma_z\langle\hat{\sigma}_i^z\rangle\right) \tag{206}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{207}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)-4(\gamma_--\gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_+\langle\hat{\sigma}_i^z\rangle-4\gamma_--4\gamma_-\langle\hat{\sigma}_i^z\rangle \tag{208}$$

$$+2\gamma_z\langle\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\left(4\gamma_+\left(1+\langle\hat{\sigma}_j^z\rangle\right)+4\gamma_-\left(1-\langle\hat{\sigma}_j^z\rangle\right)+2\gamma_z\right) \tag{209}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{210}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)-4(\gamma_--\gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_+\langle\hat{\sigma}_i^z\rangle-4\gamma_--4\gamma_-\langle\hat{\sigma}_i^z\rangle \tag{211}$$

$$+2\gamma_z\langle\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\left(4\gamma_++4\gamma_+\langle\hat{\sigma}_j^z\rangle+4\gamma_--4\gamma_-\langle\hat{\sigma}_j^z\rangle+2\gamma_z\right) \tag{212}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{213}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)-4(\gamma_--\gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_+\langle\hat{\sigma}_i^z\rangle-4\gamma_--4\gamma_-\langle\hat{\sigma}_i^z\rangle \tag{214}$$

$$+2\gamma_z\langle\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}(4\gamma_++4\gamma_-+2\gamma_z)+4\langle\hat{\sigma}_i^z\rangle(\gamma_+-\gamma_-)\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle \tag{215}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{216}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)-4(\gamma_--\gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_+\langle\hat{\sigma}_i^z\rangle-4\gamma_--4\gamma_-\langle\hat{\sigma}_i^z\rangle \tag{217}$$

$$+2\gamma_z\langle\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\rangle(N-1)(4\gamma_++4\gamma_-+2\gamma_z)+4\langle\hat{\sigma}_i^z\rangle(\gamma_+-\gamma_-)\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle \tag{218}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{219}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)-2N(\gamma_z+2\gamma_++2\gamma_-)\langle\hat{\sigma}_i^z\rangle-2(\gamma_--\gamma_+)-8(\gamma_--\gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_+\langle\hat{\sigma}_i^z\rangle-4\gamma_--4\gamma_-\langle\hat{\sigma}_i^z\rangle \tag{220}$$

$$+2\gamma_z\langle\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\rangle(N-1)(4\gamma_++4\gamma_-+2\gamma_z) \tag{221}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{222}$$

$$\times\langle\hat{\sigma}_j^-\rangle\Big)\Big)+\langle\hat{\sigma}_i^z\rangle(-2N(\gamma_z+2\gamma_++2\gamma_-)-4\gamma_+-4\gamma_-+2\gamma_z+(N-1)(4\gamma_++4\gamma_-+2\gamma_z))-2(\gamma_--\gamma_+)-8(\gamma_--\gamma_+) \tag{223}$$

$$\times\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle+4\gamma_+-4\gamma_- \tag{224}$$

$$=-\frac{\mathrm{i}}{\hbar}\Bigg(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle-\langle\hat{\sigma}_i^-\rangle}{2},-\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle+\langle\hat{\sigma}_i^-\rangle}{2},0\right)+\left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle-\frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle+\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle+\frac{1+\mathrm{i}}{2} \tag{225}$$

$$\times \langle\hat{\sigma}_j^-\rangle)) + \langle\hat{\sigma}_i^z\rangle(-4\gamma_+ - 4\gamma_- + 2\gamma_z - (4\gamma_+ + 4\gamma_- + 2\gamma_z)) - 8(\gamma_- - \gamma_+)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle + 6(\gamma_+ - \gamma_-) \tag{226}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle - \langle\hat{\sigma}_i^-\rangle}{2}, -\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle + \langle\hat{\sigma}_i^-\rangle}{2}, 0\right) + \left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle - \frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\sum_{j\neq i}(A_{ji}+A_{ij})\Big(\langle\hat{\sigma}_j^z\rangle + \frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle + \frac{1+\mathrm{i}}{2}\right. \tag{227}$$

$$\times \langle\hat{\sigma}_j^-\rangle)) - 8\langle\hat{\sigma}_i^z\rangle(\gamma_+ + \gamma_-) + 8(\gamma_+ - \gamma_-)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle + 6(\gamma_+ - \gamma_-). \tag{228}$$

Summarizing, our mean-field equations are given by:

$$\overrightarrow{v}_+ = (1-\mathrm{i}, 1+\mathrm{i}, 2), \tag{229}$$

$$\partial_t\langle\hat{\sigma}_i^+\rangle = -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\langle\hat{\sigma}_i^z\rangle, \mathrm{i}\langle\hat{\sigma}_i^z\rangle, -\langle\hat{\sigma}_i^+\rangle\right) - (1+\mathrm{i})\langle\hat{\sigma}_i^+\rangle\,\overrightarrow{v}_+\cdot\sum_{j\neq i}(A_{ji}+A_{ij})\left(\langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^-\rangle, \langle\hat{\sigma}_j^z\rangle\right)\right) - 4\langle\hat{\sigma}_i^+\rangle \tag{230}$$

$$\times(\gamma_+ + \gamma_- + \gamma_z) + 8\langle\hat{\sigma}_i^+\rangle(\gamma_+ - \gamma_-)\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle, \tag{231}$$

$$\partial_t\langle\hat{\sigma}_i^-\rangle = -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}^*\cdot\left(-\langle\hat{\sigma}_i^z\rangle, \mathrm{i}\langle\hat{\sigma}_i^z\rangle, \langle\hat{\sigma}_i^-\rangle\right) + (1-\mathrm{i})\langle\hat{\sigma}_i^-\rangle\,\overrightarrow{v}_+^*\cdot\sum_{j\neq i}(A_{ji}+A_{ij})^*\left(\langle\hat{\sigma}_j^-\rangle, \langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^z\rangle\right)\right) - 4\langle\hat{\sigma}_i^-\rangle \tag{232}$$

$$\times(\gamma_+ + \gamma_- + \gamma_z) + 8\langle\hat{\sigma}_i^-\rangle(\gamma_+ - \gamma_-)\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle, \tag{233}$$

$$\overrightarrow{v}_z = \left(\frac{1-\mathrm{i}}{2}, \frac{1+\mathrm{i}}{2}, 1\right), \tag{234}$$

$$\partial_t\langle\hat{\sigma}_i^z\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h_i}\cdot\left(\frac{\langle\hat{\sigma}_i^+\rangle - \langle\hat{\sigma}_i^-\rangle}{2}, -\mathrm{i}\frac{\langle\hat{\sigma}_i^+\rangle + \langle\hat{\sigma}_i^-\rangle}{2}, 0\right) + \left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle - \frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^-\rangle\right)\overrightarrow{v}_z\cdot\sum_{j\neq i}(A_{ji}+A_{ij})\right. \tag{235}$$

$$\times\left(\langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^-\rangle, \langle\hat{\sigma}_j^z\rangle\right)\Big) - 8\langle\hat{\sigma}_i^z\rangle(\gamma_+ + \gamma_-) + 8(\gamma_+ - \gamma_-)\langle\hat{\sigma}_i^z\rangle\sum_{j\neq i}\langle\hat{\sigma}_j^z\rangle + 6(\gamma_+ - \gamma_-). \tag{236}$$

Extending to the conjugate $\langle\hat{\sigma}_j^+\rangle^* = \langle\hat{\sigma}_i^-\rangle$ and reducing to real and imaginary parts we have:

$$\langle\hat{\sigma}_i^+\rangle - \langle\hat{\sigma}_i^+\rangle^* = 2\mathrm{i}\mathrm{Im}\left(\langle\hat{\sigma}_i^+\rangle\right) \tag{237}$$

$$\langle\hat{\sigma}_i^+\rangle + \langle\hat{\sigma}_i^+\rangle^* = 2\mathrm{Re}\left(\langle\hat{\sigma}_i^+\rangle\right) \tag{238}$$

$$\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle - \frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle^* = \frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle - \left(\frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_i^+\rangle\right)^* \tag{239}$$

$$= \mathrm{i}\mathrm{Im}\left((1-\mathrm{i})\langle\hat{\sigma}_i^+\rangle\right) \tag{240}$$

$$= \mathrm{i}\left(\mathrm{Im}\left(\langle\hat{\sigma}_i^+\rangle\right) - \mathrm{Re}\left(\langle\hat{\sigma}_i^+\rangle\right)\right), \tag{241}$$

$$\overrightarrow{v}_+\cdot\left(\langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^+\rangle^*, \langle\hat{\sigma}_j^z\rangle\right) = (1-\mathrm{i})\langle\hat{\sigma}_j^+\rangle + (1+\mathrm{i})\langle\hat{\sigma}_j^+\rangle^* + 2\langle\hat{\sigma}_j^z\rangle \tag{242}$$

$$= 2\left(\mathrm{Re}\left(\langle\hat{\sigma}_j^+\rangle\right) + \mathrm{Im}\left(\langle\hat{\sigma}_j^+\rangle\right) + \langle\hat{\sigma}_j^z\rangle\right), \tag{243}$$

$$\overrightarrow{v}_z\cdot\left(\langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^+\rangle^*, \langle\hat{\sigma}_j^z\rangle\right) = \left(\frac{1-\mathrm{i}}{2}, \frac{1+\mathrm{i}}{2}, 1\right)\cdot\left(\langle\hat{\sigma}_j^+\rangle, \langle\hat{\sigma}_j^+\rangle^*, \langle\hat{\sigma}_j^z\rangle\right) \tag{244}$$

$$= \frac{1-\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle + \frac{1+\mathrm{i}}{2}\langle\hat{\sigma}_j^+\rangle^* + \langle\hat{\sigma}_j^z\rangle \tag{245}$$

$$= \mathrm{Re}\left(\langle\hat{\sigma}_j^+\rangle\right) + \mathrm{Im}\left(\langle\hat{\sigma}_j^+\rangle\right) + \langle\hat{\sigma}_j^z\rangle. \tag{246}$$

In this scheme we arrive to a $2N$ system of differential equations:

$$\partial_t \left\langle \hat{\sigma}_i^+ \right\rangle = -\frac{2\mathrm{i}}{\hbar} \left(\overrightarrow{h_i} \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 2\left(1+\mathrm{i}\right) \left\langle \hat{\sigma}_i^+ \right\rangle \sum_{j\neq i} \left(A_{ji} + A_{ij}\right) \left(\mathrm{Re}\left(\left\langle \hat{\sigma}_j^+ \right\rangle \right) + \mathrm{Im}\left(\left\langle \hat{\sigma}_j^+ \right\rangle \right) + \left\langle \hat{\sigma}_j^z \right\rangle \right) \right) \tag{247}$$

$$- 4 \left\langle \hat{\sigma}_i^+ \right\rangle \left(\gamma_+ + \gamma_- + \gamma_z\right) + 8 \left\langle \hat{\sigma}_i^+ \right\rangle \left(\gamma_+ - \gamma_-\right) \sum_{j\neq i} \left\langle \hat{\sigma}_j^z \right\rangle, \tag{248}$$

$$\partial_t \left\langle \hat{\sigma}_i^z \right\rangle = \frac{1}{\hbar} \left(\overrightarrow{h_i} \cdot \left(\mathrm{Im}\left(\left\langle \hat{\sigma}_i^+ \right\rangle \right), -\mathrm{Re}\left(\left\langle \hat{\sigma}_i^+ \right\rangle \right), 0 \right) + \left(\mathrm{Im}\left(\left\langle \hat{\sigma}_i^+ \right\rangle \right) - \mathrm{Re}\left(\left\langle \hat{\sigma}_i^+ \right\rangle \right) \right) \sum_{j\neq i} \left(A_{ji} + A_{ij}\right) \left(\mathrm{Re}\left(\left\langle \hat{\sigma}_j^+ \right\rangle \right) + \mathrm{Im}\left(\left\langle \hat{\sigma}_j^+ \right\rangle \right) \right. \right. \tag{249}$$

$$\left. \left. + \left\langle \hat{\sigma}_j^z \right\rangle \right) \right) - 8 \left\langle \hat{\sigma}_i^z \right\rangle \left(\gamma_+ + \gamma_-\right) + 8 \left(\gamma_+ - \gamma_-\right) \left\langle \hat{\sigma}_i^z \right\rangle \sum_{j\neq i} \left\langle \hat{\sigma}_j^z \right\rangle + 6 \left(\gamma_+ - \gamma_-\right). \tag{250}$$

III. CUMULANT EXPANSION

We will extend our analysis to include the evolution of quantum correlations of second order, in particular we consider the quantum evolution equation of the form:

$$\partial_t \left\langle \hat{\sigma}_i^\alpha \hat{\sigma}_j^\beta \right\rangle = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^\beta \partial_t \hat{\rho} \right). \tag{251}$$

with $i \neq j$.
Additionally, we truncate at 3rd order such that the following approximation is plausible by the cumulant expansion:

$$\left\langle \hat{\sigma}_a^\alpha \hat{\sigma}_b^\beta \hat{\sigma}_c^\gamma \right\rangle \approx \left\langle \hat{\sigma}_a^\alpha \right\rangle \left\langle \hat{\sigma}_b^\beta \hat{\sigma}_c^\gamma \right\rangle + \left\langle \hat{\sigma}_b^\beta \right\rangle \left\langle \hat{\sigma}_c^\gamma \hat{\sigma}_a^\alpha \right\rangle + \left\langle \hat{\sigma}_c^\gamma \right\rangle \left\langle \hat{\sigma}_a^\alpha \hat{\sigma}_b^\beta \right\rangle - 2 \left\langle \hat{\sigma}_a^\alpha \right\rangle \left\langle \hat{\sigma}_b^\beta \right\rangle \left\langle \hat{\sigma}_c^\gamma \right\rangle. \tag{252}$$

in the case where a, b and c are distinct. With this in mind we will get the set of equations needed, at first for one-operator:

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \partial_t \hat{\rho} \right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left(-\frac{\mathrm{i}}{\hbar} \left[\hat{H}_{\mathrm{S}}, \hat{\rho} \right] - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right) \hat{\rho} - 2\left(\gamma_- - \gamma_+\right) \sum_i \left\{ \hat{\rho}, \hat{\sigma}_i^z \right\} + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta \hat{\rho} \hat{A}_i^{\dagger\eta} \right) \right) \tag{253}$$

$$= -\frac{\mathrm{i}}{\hbar} \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left[\hat{H}_{\mathrm{S}}, \hat{\rho} \right] \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right) \left\langle \hat{\sigma}_i^\alpha \right\rangle - 2\left(\gamma_- - \gamma_+\right) \sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left\{ \hat{\rho}, \hat{\sigma}_j^z \right\} \right) \tag{254}$$

$$+ \sum_{j,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta} \right), \tag{255}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left\{ \hat{\rho}, \hat{\sigma}_j^z \right\} \right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\rho} \hat{\sigma}_j^z + \hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho} \right) \tag{256}$$

$$= \mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\sigma}_i^\alpha \hat{\rho} \right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho} \right) \tag{257}$$

$$= \delta_{ij} \left(\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\sigma}_i^\alpha \hat{\rho} \right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_i^z \hat{\rho} \right) \right) + \left(1 - \delta_{ij}\right) \left(\mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\sigma}_i^\alpha \hat{\rho} \right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho} \right) \right), \tag{258}$$

$$\sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left\{ \hat{\rho}, \hat{\sigma}_j^z \right\} \right) = \left(\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\sigma}_i^\alpha \hat{\rho} \right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_i^z \hat{\rho} \right) \right) + \sum_{j\neq i} \left(\mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\sigma}_i^\alpha \hat{\rho} \right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho} \right) \right) \tag{259}$$

$$= \left\langle \hat{\sigma}_i^\alpha \hat{\sigma}_i^z \right\rangle + \left\langle \hat{\sigma}_i^z \hat{\sigma}_i^\alpha \right\rangle + 2 \sum_{j\neq i} \left\langle \hat{\sigma}_i^\alpha \hat{\sigma}_j^z \right\rangle, \tag{260}$$

$$2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta} \right) = 2\gamma_\eta \left(\delta_{ij} \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_i^\eta \hat{\rho} \hat{A}_i^{\dagger\eta} \right) + \left(1 - \delta_{ij}\right) \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta} \right) \right) \tag{261}$$

$$= 2\gamma_\eta \left(\delta_{ij} \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_i^\eta \hat{\rho} \right) + \left(1 - \delta_{ij}\right) \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right) \right), \tag{262}$$

$$\sum_{j,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta} \right) = \sum_\eta 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right), \tag{263}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \left[\hat{H}_{\mathrm{S}}, \hat{\rho} \right] \right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{H}_{\mathrm{S}} \hat{\rho} - \hat{\sigma}_i^\alpha \hat{\rho} \hat{H}_{\mathrm{S}} \right) \tag{264}$$

$$= \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{H}_\mathrm{S}\hat{\rho}\right) - \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\rho}\hat{H}_\mathrm{S}\right) \tag{265}$$

$$= \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{H}_\mathrm{S}\hat{\rho}\right) - \mathrm{Tr}\left(\hat{H}_\mathrm{S}\hat{\sigma}_i^\alpha \hat{\rho}\right) \tag{266}$$

$$= \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{H}_\mathrm{S}\right]\hat{\rho}\right) \tag{267}$$

$$= \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \sum_j \overrightarrow{h_j}\cdot\overrightarrow{\hat{\sigma}_j} + \sum_{jk\beta\varepsilon} A_{jk}\hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) \tag{268}$$

$$= \sum_j \overrightarrow{h_j}\cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \overrightarrow{\hat{\sigma}_j}\right]\hat{\rho}\right) + \sum_{jk\beta\varepsilon} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) \tag{269}$$

$$= \sum_{j\beta} h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta\right]\hat{\rho}\right) + \sum_{jk\beta\varepsilon} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right), \tag{270}$$

$$h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta\right]\hat{\rho}\right) = \delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\rho}\right) + (1-\delta_{ij})\,h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta\right]\hat{\rho}\right) \tag{271}$$

$$= \delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\rho}\right), \tag{272}$$

$$\sum_{j\beta} \overrightarrow{h_j}\cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \overrightarrow{\hat{\sigma}_j}\right]\hat{\rho}\right) = \sum_{j\beta}\delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\rho}\right) \tag{273}$$

$$= \sum_{\beta} h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\rho}\right) \tag{274}$$

$$= \overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, (\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z)\right]\hat{\rho}\right). \tag{275}$$

Again we come back to the term $\sum_{jk\beta\varepsilon} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right)$ with $\beta,\varepsilon \in \{x,y,z\}$. If $i\neq j, j\neq k, k\neq i$ then $\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right] = 0$ so $\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) = 0$. Now if $i\neq j, j=k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_j^\varepsilon\right]\hat{\rho}\right) = 0$ because $\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_j^\varepsilon\right] = 0$ due to the fact that the operators belong to different Hilbert spaces. If $i\neq j, i=k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_i^\varepsilon\right]\hat{\rho}\right) = \mathrm{Tr}\left(\left(\hat{\sigma}_j^\beta\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\varepsilon\right] + \left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta\right]\hat{\sigma}_i^\varepsilon\right)\hat{\rho}\right) = \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\varepsilon\right]\hat{\sigma}_j^\beta\hat{\rho}\right) = \left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\varepsilon\right]\hat{\sigma}_j^\beta\right\rangle$ and if $i=j, j\neq k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) = \mathrm{Tr}\left(\left(\hat{\sigma}_i^\beta\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_k^\varepsilon\right] + \left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_k^\varepsilon\right)\hat{\rho}\right) = \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_k^\varepsilon\hat{\rho}\right) = \left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_k^\varepsilon\right\rangle$, finally if $i=j=k$ then:

$$\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta \hat{\sigma}_i^\varepsilon\right]\hat{\rho}\right) = \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \delta_{\beta\varepsilon} + \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\hat{\sigma}_i^\gamma\right]\hat{\rho}\right) \tag{276}$$

$$= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\gamma\right]\hat{\rho}\right) \tag{277}$$

$$= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\gamma\right]\right\rangle. \tag{278}$$

Summarizing we have:

$$\sum_{jk\beta\varepsilon} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) = \sum_{\beta\varepsilon}\sum_{jk} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) \tag{279}$$

$$= \sum_{\beta\varepsilon}\left(\sum_{C_1} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) + \sum_{C_2} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) + \sum_{C_3} A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta \hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right)\right) \tag{280}$$

$$= \sum_{\beta\varepsilon}\left(\sum_{j\neq i} A_{ji}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\varepsilon\right]\hat{\sigma}_j^\beta\right\rangle + \sum_{k\neq i} A_{ik}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_k^\varepsilon\right\rangle + A_{ii}\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\gamma\right]\right\rangle\right) \tag{281}$$

$$= \sum_{\beta\varepsilon}\left(\sum_{j\neq i} A_{ji}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\varepsilon\right]\hat{\sigma}_j^\beta\right\rangle + \sum_{j\neq i} A_{ij}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_j^\varepsilon\right\rangle + A_{ii}\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\gamma\right]\right\rangle\right) \tag{282}$$

$$= \sum_{\beta\varepsilon}\sum_{j\neq i}\left(A_{ji}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_j^\varepsilon\right\rangle + A_{ij}\left\langle\left[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta\right]\hat{\sigma}_j^\varepsilon\right\rangle\right) \tag{283}$$

$$= \sum_{j\neq i} \left(A_{ji}+A_{ij}\right) \sum_{\beta\varepsilon} \left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{\beta}\right] \hat{\sigma}_j^{\varepsilon} \right\rangle, \tag{284}$$

$$\hat{\Upsilon}_i = \sum_{\varepsilon} \hat{\sigma}_i^{\varepsilon}, \tag{285}$$

$$\sum_{jk\beta\varepsilon} A_{jk} \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) = \sum_{j\neq i} \left(A_{ji}+A_{ij}\right) \left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right] \hat{\Upsilon}_j \right\rangle, \tag{286}$$

Joining the set of expressions obtained previously we find that:

$$\partial_t \left\langle \hat{\sigma}_i^{\alpha}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha}, \left(\hat{\sigma}_i^{x}, \hat{\sigma}_i^{y}, \hat{\sigma}_i^{z}\right)\right]\hat{\rho}\right) + \sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle \left[\hat{\sigma}_i^{\alpha}, \hat{\Upsilon}_i\right]\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^{\alpha}\right\rangle \tag{287}$$

$$-2\left(\gamma_- - \gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^{\alpha}, \hat{\sigma}_i^{z}\right\}\right\rangle + 2\sum_{j\neq i}\left\langle\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^{z}\right\rangle\right) + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{\alpha}\hat{A}_i^{\eta}\hat{\rho}\right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right). \tag{288}$$

Explicitly we can find the equations associated with $\partial_t \left\langle \hat{\sigma}_i^{+}(t)\right\rangle$ and $\partial_t \left\langle \hat{\sigma}_i^{z}(t)\right\rangle$:

$$\partial_t \left\langle \hat{\sigma}_i^{+}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^{+}, \left(\hat{\sigma}_i^{x}, \hat{\sigma}_i^{y}, \hat{\sigma}_i^{z}\right)\right]\hat{\rho}\right) + \sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle \left[\hat{\sigma}_i^{+}, \hat{\Upsilon}_i\right]\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^{+}\right\rangle \tag{289}$$

$$-2\left(\gamma_- - \gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^{+}, \hat{\sigma}_i^{z}\right\}\right\rangle + 2\sum_{j\neq i}\left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{z}\right\rangle\right) + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{+}\hat{A}_i^{\eta}\hat{\rho}\right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{+}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right), \tag{290}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i \cdot \left(\left\langle\hat{\sigma}_i^{z}\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^{z}\right\rangle, -\left\langle\hat{\sigma}_i^{+}\right\rangle\right) - 4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\hat{\sigma}_i^{+}\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^{+}\right\rangle \tag{291}$$

$$-2\left(\gamma_- - \gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^{+}, \hat{\sigma}_i^{z}\right\}\right\rangle + 2\sum_{j\neq i}\left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{z}\right\rangle\right) + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{+}\hat{A}_i^{\eta}\hat{\rho}\right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{+}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right), \tag{292}$$

$$\left\langle\left\{\hat{\sigma}_i^{+}, \hat{\sigma}_i^{z}\right\}\right\rangle = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{293}$$

$$= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{294}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{295}$$

$$\partial_t \left\langle \hat{\sigma}_i^{+}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i \cdot \left(\left\langle\hat{\sigma}_i^{z}\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^{z}\right\rangle, -\left\langle\hat{\sigma}_i^{+}\right\rangle\right) - 4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\hat{\sigma}_i^{+}\hat{\Upsilon}_j\right\rangle\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^{+}\right\rangle \tag{296}$$

$$-4\left(\gamma_- - \gamma_+\right)\sum_{j\neq i}\left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{z}\right\rangle + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{+}\hat{A}_i^{\eta}\hat{\rho}\right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^{+}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right), \tag{297}$$

$$\left\langle\hat{\sigma}_i^{+}\hat{\Upsilon}_j\right\rangle = \left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{z}\right\rangle + \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{+}\right\rangle + \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^{+}\hat{\sigma}_j^{-}\right\rangle, \tag{298}$$

$$\overrightarrow{v}_1 = \left(\frac{1-\mathrm{i}}{2}, \frac{1+\mathrm{i}}{2}, 1\right), \tag{299}$$

$$\left\langle \hat{\sigma}_i^+ \hat{\Upsilon}_j \right\rangle = \overrightarrow{v}_1 \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right), \tag{300}$$

$$\partial_t \left\langle \hat{\sigma}_i^+ (t) \right\rangle = -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4\,(1+\mathrm{i}) \sum_{j\neq i} (A_{ji} + A_{ij})\, \overrightarrow{v}_1 \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{301}$$

$$- 2N\,(\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4\,(\gamma_- - \gamma_+) \sum_{j\neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle + \sum_{\eta} 2\,\gamma_\eta \,\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^+ \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j\neq i,\eta} 2\,\gamma_\eta \tag{302}$$

$$\times\, \mathrm{Tr}\left(\hat{\sigma}_i^+ \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right) \tag{303}$$

IV. BIBLIOGRAPHY

[1] Tucker, K, Barberena, D, Lewis-Swan, RJ, Thompson, JK, Restrepo, JG, Rey, AM 2020 Phys. Rev. A 2469-9926

---

* edch5956@colorado.edu