Mean Field Analysis and Cumulant Expansion on a Generic Spin Hamiltonian

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I. GENERIC SPIN HAMILTONIAN WITH DISSIPATION

Consider the following generic system hamiltonian of N written in the form:

$$\hat{H}_{\mathrm{S}} = \sum_{i} \overrightarrow{h_i} \cdot \overrightarrow{\hat{\sigma}_i} + \sum_{ij\alpha\beta} A_{ij} \hat{\sigma}_i^{\alpha} \hat{\sigma}_j^{\beta} \tag{1}$$

$$\overrightarrow{\hat{\sigma}_i} = \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z\right), \tag{2}$$

$$\alpha, \beta \in \{x, y, z\}, \tag{3}$$

$$\hat{\sigma}_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

$$\hat{\sigma}_i^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{5}$$

$$\hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

here the set of $\overrightarrow{h_i}$ can be seen as a $N \times 3$ matrix and A is a matrix of size $N \times N$ which summarizes the interaction terms between the qubits(spins). In order to introduce a dissipation term we write the Linblandian operators of the system as:

$$\mathcal{L} \Rightarrow \sqrt{\gamma_+}\hat{\sigma}_i^+, \sqrt{\gamma_-}\hat{\sigma}_i^-, \sqrt{\gamma_z}\hat{\sigma}_i^z, \tag{7}$$

$$\hat{\sigma}_i^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$$\hat{\sigma}_i^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \tag{9}$$

$$\hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

$$\mathcal{L}\left[\hat{\rho}\right] = -\sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta} + \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\hat{\rho} - 2\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right), \tag{11}$$

$$\hat{A}_i^{\eta} = \hat{\sigma}_i^{\eta}, \eta \in \{+, -, z\}. \tag{12}$$

The final form of the master equation to consider is given by:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta} + \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\hat{\rho} - 2\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right) \tag{13}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[H_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\left\{\rho, \hat{A}_i^{\dagger\eta}\hat{A}_i^{\eta}\right\} - 2\hat{A}_i^{\eta}\rho\hat{A}_i^{\dagger\eta}\right). \tag{14}$$

We introduce now a set of simplifications that can be useful for posterior computations:

$$\sum_{i,\eta}\gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \sum_i\sum_\eta \gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta, \tag{15}$$

$$\hat{\sigma}_i^{\dagger+}\hat{\sigma}_i^{+} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{17}$$

$$= 4\frac{\hat{\mathbb{I}} - \hat{\sigma}_i^z}{2} \tag{18}$$

$$= 2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right), \tag{19}$$

$$\hat{\sigma}_i^{\dagger-}\hat{\sigma}_i^{-} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \tag{21}$$

$$= 2\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right), \tag{22}$$

$$\hat{\sigma}_i^{\dagger z}\hat{\sigma}_i^{z} = \hat{\mathbb{I}}, \tag{23}$$

$$\sum_\eta \gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \gamma_+ 2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right) + \gamma_- 2\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right) + \gamma_z\hat{\mathbb{I}} \tag{24}$$

$$= \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}} + 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z. \tag{25}$$

Then the master equation (14) simplifies to:

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right] - \sum_i\left(\left\{\hat{\rho}, \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}} + 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z\right\}\right) + \sum_{i,\eta} 2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{26}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right] - \sum_i\left(\left\{\hat{\rho}, \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}}\right\} + \left\{\hat{\rho}, 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z\right\}\right) + \sum_{i,\eta} 2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{27}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right] - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\rho} - 2\left(\gamma_- - \gamma_+\right)\sum_i\left\{\hat{\rho},\hat{\sigma}_i^z\right\} + \sum_{i,\eta} 2\gamma_\eta\hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta}. \tag{28}$$

Using the evolution equation:

$$\partial_t\left\langle\hat{\sigma}_i^\alpha(t)\right\rangle = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\partial_t\hat{\rho}\right). \tag{29}$$

and the mean-field asymption on ρ of the form:

$$\hat{\rho} = \otimes_i\hat{\rho}_i. \tag{30}$$

we can find that:

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \partial_t \hat{\rho}\right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left(-\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right] - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\rho} - 2\left(\gamma_- - \gamma_+\right)\sum_i \left\{\hat{\rho}, \hat{\sigma}_i^z\right\} + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta \hat{\rho} \hat{A}_i^{\dagger\eta}\right)\right) \tag{31}$$

$$= -\frac{\mathrm{i}}{\hbar}\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right]\right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle - 2\left(\gamma_- - \gamma_+\right)\sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho}, \hat{\sigma}_j^z\right\}\right) \tag{32}$$

$$+ \sum_{j,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta}\right), \tag{33}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho}, \hat{\sigma}_j^z\right\}\right) = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\rho} \hat{\sigma}_j^z + \hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho}\right) \tag{34}$$

$$= \mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\sigma}_i^\alpha \hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho}\right) \tag{35}$$

$$= \delta_{ij}\left(\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\sigma}_i^\alpha \hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_i^z \hat{\rho}\right)\right) + \left(1 - \delta_{ij}\right)\left(\mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\sigma}_i^\alpha \hat{\rho}\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_j^z \hat{\rho}\right)\right) \tag{36}$$

$$= \delta_{ij}\left(\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\sigma}_i^\alpha \hat{\rho}_i\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_i^z \hat{\rho}_i\right)\right) + 2(1 - \delta_{ij})\mathrm{Tr}\left(\hat{\sigma}_j^z \hat{\rho}_j\right)\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\rho}_i\right), \tag{37}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\sigma}_i^\alpha \hat{\rho}_i\right) + \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\sigma}_i^z \hat{\rho}_i\right) = 2\delta_{z\alpha}, \tag{38}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho}, \hat{\sigma}_j^z\right\}\right) = 2\delta_{ij}\delta_{z\alpha} + 2(1 - \delta_{ij})\left\langle\hat{\sigma}_j^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle, \tag{39}$$

$$\sum_j \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left\{\hat{\rho}, \hat{\sigma}_j^z\right\}\right) = \sum_j\left(\delta_{ij}\delta_{z\alpha} + 2(1 - \delta_{ij})\left\langle\hat{\sigma}_j^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle\right) \tag{40}$$

$$= \delta_{z\alpha} - 2\left\langle\hat{\sigma}_i^z\right\rangle\left\langle\hat{\sigma}_i^\alpha\right\rangle + 2\left\langle\hat{\sigma}_i^\alpha\right\rangle\sum_j\left\langle\hat{\sigma}_j^z\right\rangle \tag{41}$$

$$= \delta_{z\alpha} - 2\left\langle\hat{\sigma}_i^\alpha\right\rangle\left(\left\langle\hat{\sigma}_i^z\right\rangle - \sum_j\left\langle\hat{\sigma}_j^z\right\rangle\right), \tag{42}$$

$$2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho} \hat{A}_j^{\dagger\eta}\right) = 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho}\right) \tag{43}$$

$$= \left(1 - \delta_{ij}\right) 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho}\right) + \delta_{ij} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_j^\eta \hat{\rho}\right) \tag{44}$$

$$= \left(1 - \delta_{ij}\right) 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho}_j\right)\mathrm{Tr}\left(\hat{\sigma}_i^\alpha \hat{\rho}_i\right) + \delta_{ij} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_i^\eta \hat{\rho}_i\right), \tag{45}$$

$$\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho}_j\right) = \delta_{+\eta}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} + \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{-\eta}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} - \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{\mathbb{I}}\hat{\rho}_j\right) \tag{46}$$

$$= 2\delta_{+\eta}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} + \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + 2\delta_{-\eta}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} - \hat{\sigma}_j^z\right)\hat{\rho}_j\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{\mathbb{I}}\hat{\rho}_j\right) \tag{47}$$

$$= 2\delta_{+\eta} + 2\delta_{-\eta} + \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle \tag{48}$$

$$= 2\delta_{+\eta} + 2\delta_{-\eta} + 2\delta_{z\eta} - \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle \tag{49}$$

$$= 2 - \delta_{z\eta} + 2\left(\delta_{+\eta} - \delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle, \tag{50}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^\alpha \hat{A}_i^\eta \hat{\rho}_i\right) = \delta_{+\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^\alpha \hat{A}_i^+ \hat{\rho}_i\right) + \delta_{-\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^\alpha \hat{A}_i^- \hat{\rho}_i\right) + \delta_{z\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^\alpha \hat{A}_i^z \hat{\rho}_i\right), \tag{51}$$

$$\left(\sigma_i^\pm\right)^2 = 0, \tag{52}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^\alpha \hat{A}_i^+ \hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^+ \hat{A}_i^+ \hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^- \hat{A}_i^+ \hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^z \hat{A}_i^+ \hat{\rho}_i\right) \tag{53}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^z \hat{A}_i^+ \hat{\rho}_i\right) \tag{54}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right)\hat{\rho}_i\right) \tag{55}$$

$$= 2\delta_{z\alpha}\left(1 - \left\langle\hat{\sigma}_i^z\right\rangle\right), \tag{56}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^\alpha \hat{A}_i^- \hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^+ \hat{A}_i^- \hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^- \hat{A}_i^- \hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^z \hat{A}_i^- \hat{\rho}_i\right) \tag{57}$$

$$= \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^z \hat{A}_i^- \hat{\rho}_i\right) \tag{58}$$

$$= -2\delta_{z\alpha}\mathrm{Tr}\left(\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right)\hat{\rho}_i\right) \tag{59}$$

$$= -2\delta_{z\alpha}\left(1 + \left\langle\hat{\sigma}_i^z\right\rangle\right), \tag{60}$$

$$\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^\alpha \hat{A}_i^z \hat{\rho}_i\right) = \delta_{+\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^+ \hat{A}_i^z \hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^- \hat{A}_i^z \hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^z \hat{A}_i^z \hat{\rho}_i\right), \tag{61}$$

$$\hat{\sigma}_i^{\dagger z} \hat{\sigma}_i^\pm \hat{\sigma}_i^z = \hat{\sigma}_i^{\dagger z}\left(\hat{\sigma}_i^x \pm \mathrm{i}\hat{\sigma}_i^y\right)\hat{\sigma}_i^z \tag{62}$$

$$= \hat{\sigma}_i^z\left(\hat{\sigma}_i^x \pm \mathrm{i}\hat{\sigma}_i^y\right)\hat{\sigma}_i^z \tag{63}$$

$$= \hat{\sigma}_i^z \hat{\sigma}_i^x \hat{\sigma}_i^z \pm \mathrm{i}\hat{\sigma}_i^z \hat{\sigma}_i^y \hat{\sigma}_i^z \tag{64}$$

$$= \mathrm{i}\hat{\sigma}_i^y \hat{\sigma}_i^z \pm \mathrm{i}\hat{\sigma}_i^z \mathrm{i}\hat{\sigma}_i^x \tag{65}$$

$$
\begin{aligned}
&= \mathrm{i}^2\hat{\sigma}_i^x \pm \mathrm{i}\mathrm{i}^2\hat{\sigma}_i^y && (66)\\
&= -\hat{\sigma}_i^{\pm}, && (67)\\
\mathrm{Tr}\left(\hat{A}_i^{\dagger z}\hat{\sigma}_i^\alpha\hat{A}_i^z\hat{\rho}_i\right) &= \delta_{+\alpha}\mathrm{Tr}\left(-\hat{\sigma}_i^+\hat{\rho}_i\right) + \delta_{-\alpha}\mathrm{Tr}\left(-\hat{\sigma}_i^-\hat{\rho}_i\right) + \delta_{z\alpha}\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\rho}_i\right) && (68)\\
&= \delta_{z\alpha}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+\alpha}\left\langle\hat{\sigma}_i^+\right\rangle - \delta_{-\alpha}\left\langle\hat{\sigma}_i^-\right\rangle && (69)\\
\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_i^\eta\hat{\rho}_i\right) &= \delta_{+\eta}(2\delta_{z\alpha}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)) + \delta_{-\eta}(-2\delta_{z\alpha}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)) + \delta_{z\eta}\big(\delta_{z\alpha}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+\alpha}\big\langle\hat{\sigma}_i^+\big\rangle - \delta_{-\alpha}\big\langle\hat{\sigma}_i^-\big\rangle\big) && (70)\\
&= 2\delta_{+\eta}\delta_{z\alpha}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right) - 2\delta_{-\eta}\delta_{z\alpha}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right) + \delta_{z\eta}\left(\delta_{z\alpha}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+\alpha}\left\langle\hat{\sigma}_i^+\right\rangle - \delta_{-\alpha}\left\langle\hat{\sigma}_i^-\right\rangle\right), && (71)\\
2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\hat{A}_j^{\dagger\eta}\right) &= 2\gamma_\eta\left(\left(1-\delta_{ij}\right)\mathrm{Tr}\left(\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}_j\right)\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\rho}_i\right) + \delta_{ij}\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^\alpha\hat{A}_i^\eta\hat{\rho}_i\right)\right) && (72)\\
&= 2\gamma_\eta\delta_{ij}\big(2\delta_{+\eta}\delta_{z\alpha}(1-\langle\hat{\sigma}_i^z\rangle) - 2\delta_{-\eta}\delta_{z\alpha}(1+\langle\hat{\sigma}_i^z\rangle) + \delta_{z\eta}\big(\delta_{z\alpha}\langle\hat{\sigma}_i^z\rangle - \delta_{+\alpha}\big\langle\hat{\sigma}_i^+\big\rangle - \delta_{-\alpha}\big\langle\hat{\sigma}_i^-\big\rangle\big)\big) && (73)\\
&\quad + 2\gamma_\eta\left(1-\delta_{ij}\right)\left(2-\delta_{z\eta}+2\left(\delta_{+\eta}-\delta_{-\eta}\right)\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle && (74)\\
2\gamma_+\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^+\hat{\rho}\hat{A}_j^{\dagger +}\right) &= 2\gamma_+\left(\left(1-\delta_{ij}\right)\left(2+2\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\left(2\delta_{z\alpha}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right)\right) && (75)\\
&= 4\gamma_+\left(\left(1-\delta_{ij}\right)\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\delta_{z\alpha}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right), && (76)\\
2\gamma_-\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^-\hat{\rho}\hat{A}_j^{\dagger -}\right) &= 2\gamma_-\left(\left(1-\delta_{ij}\right)\left(2-2\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\left(-2\delta_{z\alpha}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\right)\right) && (77)\\
&= 4\gamma_-\left(\left(1-\delta_{ij}\right)\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle - \delta_{ij}\delta_{z\alpha}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\right), && (78)\\
2\gamma_z\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^z\hat{\rho}\hat{A}_j^{\dagger z}\right) &= 2\gamma_z\left(\left(1-\delta_{ij}\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\left(\delta_{z\alpha}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+\alpha}\left\langle\hat{\sigma}_i^+\right\rangle - \delta_{-\alpha}\left\langle\hat{\sigma}_i^-\right\rangle\right)\right) && (79)\\
\sum_\eta 2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{A}_j^\eta\hat{\rho}\hat{A}_j^{\dagger\eta}\right) &= 4\gamma_+\left(\left(1-\delta_{ij}\right)\left(1+\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\delta_{z\alpha}\left(1-\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 4\gamma_-\left(\left(1-\delta_{ij}\right)\left(1-\left\langle\hat{\sigma}_j^z\right\rangle\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle\right. && (80)\\
&\quad \left.-\delta_{ij}\delta_{z\alpha}\left(1+\left\langle\hat{\sigma}_i^z\right\rangle\right)\right) + 2\gamma_z\left(\left(1-\delta_{ij}\right)\left\langle\hat{\sigma}_i^\alpha\right\rangle + \delta_{ij}\left(\delta_{z\alpha}\left\langle\hat{\sigma}_i^z\right\rangle - \delta_{+\alpha}\left\langle\hat{\sigma}_i^+\right\rangle - \delta_{-\alpha}\left\langle\hat{\sigma}_i^-\right\rangle\right)\right), && (81)\\
\mathrm{Tr}\left(\hat{\sigma}_i^\alpha\left[\hat{H}_\mathrm{S},\hat{\rho}\right]\right) &= \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{H}_\mathrm{S}\hat{\rho} - \hat{\sigma}_i^\alpha\hat{\rho}\hat{H}_\mathrm{S}\right) && (82)\\
&= \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{H}_\mathrm{S}\hat{\rho}\right) - \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\rho}\hat{H}_\mathrm{S}\right) && (83)\\
&= \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{H}_\mathrm{S}\hat{\rho}\right) - \mathrm{Tr}\left(\hat{H}_\mathrm{S}\hat{\sigma}_i^\alpha\hat{\rho}\right) && (84)\\
&= \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{H}_\mathrm{S}\right]\hat{\rho}\right) && (85)\\
&= \mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\sum_j\overrightarrow{h_j}\cdot\overrightarrow{\hat{\sigma}_j} + \sum_{jk\beta\varepsilon}A_{jk}\hat{\sigma}_j^\beta\hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) && (86)\\
&= \sum_j\overrightarrow{h_j}\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\overrightarrow{\hat{\sigma}_j}\right]\hat{\rho}\right) + \sum_{jk\beta\varepsilon}A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_j^\beta\hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right) && (87)\\
&= \sum_{j\beta}h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_j^\beta\right]\hat{\rho}\right) + \sum_{jk\beta\varepsilon}A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_j^\beta\hat{\sigma}_k^\varepsilon\right]\hat{\rho}\right), && (88)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_j^\beta\right]\hat{\rho}\right) &= \delta_{ij}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_i^\beta\right]\hat{\rho}\right) + \cancelto{0}{\left(1-\delta_{ij}\right)\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_j^\beta\right]\hat{\rho}\right)}, && (89)\\
&= \delta_{ij}\mathrm{Tr}\left(\left[\hat{\sigma}_i^\alpha,\hat{\sigma}_i^\beta\right]\hat{\rho}\right). && (90)
\end{aligned}
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II. BIBLIOGRAPHY

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