Mean-Field Approximation and Cumulant Expansion for a Generic Spin Hamiltonian

A. M. Rey, E. C. Chaparro*
Department of Physics, University of Colorado, Boulder

(Dated: 6th April 2021)

I. GENERIC SPIN HAMILTONIAN WITH DISSIPATION

Consider the following generic system hamiltonian of N written in the form:

$$\hat{H}_{\mathrm{S}} = \sum_i \overrightarrow{h_i} \cdot \overrightarrow{\hat{\sigma}_i} + \sum_{jk\beta\varepsilon} A^{\beta\varepsilon}_{jk} \hat{\sigma}^{\beta}_j \hat{\sigma}^{\varepsilon}_k, \tag{1}$$

$$\overrightarrow{\hat{\sigma}_i} = \left(\hat{\sigma}^x_i, \hat{\sigma}^y_i, \hat{\sigma}^z_i\right), \tag{2}$$

$$\beta, \varepsilon \in \{x, y, z\}, \tag{3}$$

$$\hat{\sigma}^x_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

$$\hat{\sigma}^y_i = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{5}$$

$$\hat{\sigma}^z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

here the set of $\overrightarrow{h_i}$ can be seen as a $N \times 3$ matrix and A is a matrix of size $N \times N$ which summarizes the interaction terms between the qubits(spins). In order to introduce a dissipation term we write the Linblandian operators of the system as:

$$\mathcal{L} \Rightarrow \sqrt{\gamma_+}\hat{\sigma}^+_i, \sqrt{\gamma_-}\hat{\sigma}^-_i, \sqrt{\gamma_z}\hat{\sigma}^z_i, \tag{7}$$

$$\hat{\sigma}^+_i = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$$\hat{\sigma}^-_i = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \tag{9}$$

$$\hat{\sigma}^z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

$$\mathcal{L}\left[\hat{\rho}\right] = -\sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}^{\dagger\eta}_i \hat{A}^{\eta}_i + \hat{A}^{\dagger\eta}_i \hat{A}^{\eta}_i \hat{\rho} - 2\hat{A}^{\eta}_i \hat{\rho} \hat{A}^{\dagger\eta}_i\right), \tag{11}$$

$$\hat{A}^{\eta}_i = \hat{\sigma}^{\eta}_i, \eta \in \{+, -, z\}. \tag{12}$$

The final form of the master equation to consider is given by:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\hat{\rho}\hat{A}^{\dagger\eta}_i \hat{A}^{\eta}_i + \hat{A}^{\dagger\eta}_i \hat{A}^{\eta}_i \hat{\rho} - 2\hat{A}^{\eta}_i \hat{\rho} \hat{A}^{\dagger\eta}_i\right) \tag{13}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[H_{\mathrm{S}}, \hat{\rho}\right] - \sum_{i,\eta} \gamma_\eta \left(\left\{\rho, \hat{A}^{\dagger\eta}_i \hat{A}^{\eta}_i\right\} - 2\hat{A}^{\eta}_i \rho \hat{A}^{\dagger\eta}_i\right). \tag{14}$$

We introduce now a set of simplifications that can be useful for posterior computations:

$$\sum_{i,\eta}\gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \sum_i\sum_\eta \gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta, \tag{15}$$

$$\hat{\sigma}_i^{\dagger+}\hat{\sigma}_i^+ = \begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \tag{16}$$

$$= \begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix} \tag{17}$$

$$= 4\frac{\hat{\mathbb{I}} - \hat{\sigma}_i^z}{2} \tag{18}$$

$$= 2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right), \tag{19}$$

$$\hat{\sigma}_i^{\dagger-}\hat{\sigma}_i^- = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix} \tag{20}$$

$$= \begin{pmatrix}4 & 0\\ 0 & 0\end{pmatrix} \tag{21}$$

$$= 2\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right), \tag{22}$$

$$\hat{\sigma}_i^{\dagger z}\hat{\sigma}_i^z = \hat{\mathbb{I}}, \tag{23}$$

$$\sum_\eta \gamma_\eta \hat{A}_i^{\dagger\eta}\hat{A}_i^\eta = \gamma_+ 2\left(\hat{\mathbb{I}} - \hat{\sigma}_i^z\right) + \gamma_- 2\left(\hat{\mathbb{I}} + \hat{\sigma}_i^z\right) + \gamma_z\hat{\mathbb{I}} \tag{24}$$

$$= \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}} + 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z. \tag{25}$$

Then the master equation (14) simplifies to:

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_\mathrm{S}, \hat{\rho}\right] - \sum_i\left(\left\{\hat{\rho}, \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}} + 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z\right\}\right) + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{26}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_\mathrm{S}, \hat{\rho}\right] - \sum_i\left(\left\{\hat{\rho}, \left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\mathbb{I}}\right\} + \left\{\hat{\rho}, 2\left(\gamma_- - \gamma_+\right)\hat{\sigma}_i^z\right\}\right) + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta} \tag{27}$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_\mathrm{S}, \hat{\rho}\right] - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_-\right)\hat{\rho} - 2\left(\gamma_- - \gamma_+\right)\sum_i\left\{\hat{\rho}, \hat{\sigma}_i^z\right\} + \sum_{i,\eta} 2\gamma_\eta \hat{A}_i^\eta\hat{\rho}\hat{A}_i^{\dagger\eta}. \tag{28}$$

II. CUMULANT EXPANSION

We will extend our analysis to include the evolution of quantum correlations of second order, in particular we consider the quantum evolution equation of the form:

$$\partial_t\left\langle\hat{\sigma}_i^\alpha\hat{\sigma}_j^\beta\right\rangle = \mathrm{Tr}\left(\hat{\sigma}_i^\alpha\hat{\sigma}_j^\beta\partial_t\hat{\rho}\right). \tag{29}$$

with $i \neq j$.

Additionally, we truncate at 3rd order such that the following approximation is plausible by the cumulant expansion:

$$\left\langle\hat{\sigma}_a^\alpha\hat{\sigma}_b^\beta\hat{\sigma}_c^\gamma\right\rangle \approx \left\langle\hat{\sigma}_a^\alpha\right\rangle\left\langle\hat{\sigma}_b^\beta\hat{\sigma}_c^\gamma\right\rangle + \left\langle\hat{\sigma}_b^\beta\right\rangle\left\langle\hat{\sigma}_c^\gamma\hat{\sigma}_a^\alpha\right\rangle + \left\langle\hat{\sigma}_c^\gamma\right\rangle\left\langle\hat{\sigma}_a^\alpha\hat{\sigma}_b^\beta\right\rangle - 2\left\langle\hat{\sigma}_a^\alpha\right\rangle\left\langle\hat{\sigma}_b^\beta\right\rangle\left\langle\hat{\sigma}_c^\gamma\right\rangle. \tag{30}$$

in the case where a, b and c are distinct. With this in mind we will get the set of equations needed, at first for one-operator:

$$\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\partial_t\hat{\rho}\right)=\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left(-\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\hat{\rho}-2\left(\gamma_--\gamma_+\right)\sum_i\left\{\hat{\rho},\hat{\sigma}_i^z\right\}+\sum_{i,\eta}2\gamma_\eta\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right)\right) \tag{31}$$

$$=-\frac{\mathrm{i}}{\hbar}\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]\right)-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\langle\hat{\sigma}_i^{\alpha}\rangle-2\left(\gamma_--\gamma_+\right)\sum_j\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right) \tag{32}$$

$$+\sum_{j,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\eta}\hat{\rho}\hat{A}_j^{\dagger\eta}\right), \tag{33}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right)=\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\rho}\hat{\sigma}_j^z+\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^z\hat{\rho}\right) \tag{34}$$

$$=\mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\sigma}_i^{\alpha}\hat{\rho}\right)+\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^z\hat{\rho}\right) \tag{35}$$

$$=\delta_{ij}\left(\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\sigma}_i^{\alpha}\hat{\rho}\right)+\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\sigma}_i^z\hat{\rho}\right)\right)+\left(1-\delta_{ij}\right)\left(\mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\sigma}_i^{\alpha}\hat{\rho}\right)+\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^z\hat{\rho}\right)\right), \tag{36}$$

$$\sum_j\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left\{\hat{\rho},\hat{\sigma}_j^z\right\}\right)=\left(\mathrm{Tr}\left(\hat{\sigma}_i^z\hat{\sigma}_i^{\alpha}\hat{\rho}\right)+\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\sigma}_i^z\hat{\rho}\right)\right)+\sum_{j\neq i}\left(\mathrm{Tr}\left(\hat{\sigma}_j^z\hat{\sigma}_i^{\alpha}\hat{\rho}\right)+\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^z\hat{\rho}\right)\right) \tag{37}$$

$$=\langle\hat{\sigma}_i^{\alpha}\hat{\sigma}_i^z\rangle+\langle\hat{\sigma}_i^z\hat{\sigma}_i^{\alpha}\rangle+2\sum_{j\neq i}\langle\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^z\rangle, \tag{38}$$

$$2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\eta}\hat{\rho}\hat{A}_j^{\dagger\eta}\right)=2\gamma_\eta\left(\delta_{ij}\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_i^{\eta}\hat{\rho}\hat{A}_i^{\dagger\eta}\right)+\left(1-\delta_{ij}\right)\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\eta}\hat{\rho}\hat{A}_j^{\dagger\eta}\right)\right) \tag{39}$$

$$=2\gamma_\eta\left(\delta_{ij}\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{\alpha}\hat{A}_i^{\eta}\hat{\rho}\right)+\left(1-\delta_{ij}\right)\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right)\right), \tag{40}$$

$$\sum_{j,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\eta}\hat{\rho}\hat{A}_j^{\dagger\eta}\right)=\sum_\eta2\gamma_\eta\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{\alpha}\hat{A}_i^{\eta}\hat{\rho}\right)+\sum_{j\neq i,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right), \tag{41}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\left[\hat{H}_{\mathrm{S}},\hat{\rho}\right]\right)=\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{H}_{\mathrm{S}}\hat{\rho}-\hat{\sigma}_i^{\alpha}\hat{\rho}\hat{H}_{\mathrm{S}}\right) \tag{42}$$

$$=\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{H}_{\mathrm{S}}\hat{\rho}\right)-\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{\rho}\hat{H}_{\mathrm{S}}\right) \tag{43}$$

$$=\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{H}_{\mathrm{S}}\hat{\rho}\right)-\mathrm{Tr}\left(\hat{H}_{\mathrm{S}}\hat{\sigma}_i^{\alpha}\hat{\rho}\right) \tag{44}$$

$$=\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{H}_{\mathrm{S}}\right]\hat{\rho}\right) \tag{45}$$

$$=\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\sum_j\overrightarrow{h_j}\cdot\overrightarrow{\hat{\sigma}_j}+\sum_{jk\beta\varepsilon}A_{jk}\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) \tag{46}$$

$$=\sum_j\overrightarrow{h_j}\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\overrightarrow{\hat{\sigma}_j}\right]\hat{\rho}\right)+\sum_{jk\beta\varepsilon}A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) \tag{47}$$

$$=\sum_{j\beta}h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\right]\hat{\rho}\right)+\sum_{jk\beta\varepsilon}A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right), \tag{48}$$

$$h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\right]\hat{\rho}\right)=\delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\rho}\right)+\left(1-\delta_{ij}\right)h_{j\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\right]\hat{\rho}\right) \tag{49}$$

$$=\delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\rho}\right), \tag{50}$$

$$\sum_{j\beta}\overrightarrow{h_j}\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\overrightarrow{\hat{\sigma}_j}\right]\hat{\rho}\right)=\sum_{j\beta}\delta_{ij}h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\rho}\right) \tag{51}$$

$$=\sum_\beta h_{i\beta}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\rho}\right) \tag{52}$$

$$=\overrightarrow{h}_i\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\left(\hat{\sigma}_i^x,\hat{\sigma}_i^y,\hat{\sigma}_i^z\right)\right]\hat{\rho}\right). \tag{53}$$

Again we come back to the term $\sum_{jk\beta\varepsilon}A_{jk}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)$ with $\beta,\varepsilon\in\{x,y,z\}$. If $i\neq j,j\neq k,k\neq i$ then $\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]=0$ so $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)=0$. Now if $i\neq j,j=k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_j^{\varepsilon}\right]\hat{\rho}\right)=0$ because $\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_j^{\varepsilon}\right]=0$ due to the fact that the operators belong to different Hilbert spaces. If $i\neq j,i=k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_i^{\varepsilon}\right]\hat{\rho}\right)=$

$\mathrm{Tr}\left(\left(\hat{\sigma}_j^{\beta}\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]+\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\right]\hat{\sigma}_i^{\varepsilon}\right)\hat{\rho}\right)=\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]\hat{\sigma}_j^{\beta}\hat{\rho}\right)=\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]\hat{\sigma}_j^{\beta}\right\rangle$ and if $i=j, j\neq k$ then $\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)=\mathrm{Tr}\left(\left(\hat{\sigma}_i^{\beta}\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_k^{\varepsilon}\right]+\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_k^{\varepsilon}\right)\hat{\rho}\right)=\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_k^{\varepsilon}\hat{\rho}\right)=\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_k^{\varepsilon}\right\rangle$, finally if $i=j=k$ then:

$$\begin{aligned}
\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\hat{\sigma}_i^{\varepsilon}\right]\hat{\rho}\right) &= \mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\delta_{\beta\varepsilon}+\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\hat{\sigma}_i^{\gamma}\right]\hat{\rho}\right) && (54)\\
&= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\hat{\rho}\right) && (55)\\
&= \mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle. && (56)
\end{aligned}$$

Summarizing we have:

$$\begin{aligned}
\sum_{jk\beta\varepsilon}A_{jk}^{\beta\varepsilon}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) &= \sum_{\beta\varepsilon}\sum_{jk}A_{jk}^{\beta\varepsilon}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right) && (57)\\
&= \sum_{\beta\varepsilon}\left(\sum_{C_1}A_{jk}^{\beta\varepsilon}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)+\sum_{C_2}A_{jk}^{\beta\varepsilon}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)+\sum_{C_3}A_{jk}^{\beta\varepsilon}\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_j^{\beta}\hat{\sigma}_k^{\varepsilon}\right]\hat{\rho}\right)\right) && (58)\\
&= \sum_{\beta\varepsilon}\left(\sum_{j\neq i}A_{ji}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]\hat{\sigma}_j^{\beta}\right\rangle+\sum_{k\neq i}A_{ik}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_k^{\varepsilon}\right\rangle+A_{ii}^{\beta\varepsilon}\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle\right) && (59)\\
&= \sum_{\beta\varepsilon}\left(\sum_{j\neq i}A_{ji}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]\hat{\sigma}_j^{\beta}\right\rangle+\sum_{j\neq i}A_{ij}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_j^{\varepsilon}\right\rangle+A_{ii}^{\beta\varepsilon}\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle\right) && (60)\\
&= \sum_{\beta\varepsilon}\left(\sum_{j\neq i}\left(A_{ji}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\varepsilon}\right]\hat{\sigma}_j^{\beta}\right\rangle+A_{ij}^{\beta\varepsilon}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_j^{\varepsilon}\right\rangle\right)+A_{ii}^{\beta\varepsilon}\mathrm{i}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle\right) && (61)\\
&= \sum_{j\neq i,\beta\varepsilon}\left(A_{ji}^{\varepsilon\beta}+A_{ij}^{\beta\varepsilon}\right)\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_j^{\varepsilon}\right\rangle+\mathrm{i}\sum_{\beta\varepsilon}A_{ii}^{\beta\varepsilon}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle. && (62)
\end{aligned}$$

Joining the set of expressions obtained previously we find that:

$$\begin{aligned}
\partial_t\left\langle\hat{\sigma}_i^{\alpha}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}&\left(\overrightarrow{h}_i\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^{\alpha},\left(\hat{\sigma}_i^{x},\hat{\sigma}_i^{y},\hat{\sigma}_i^{z}\right)\right]\hat{\rho}\right)+\sum_{j\neq i,\beta\varepsilon}\left(A_{ji}^{\varepsilon\beta}+A_{ij}^{\beta\varepsilon}\right)\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\beta}\right]\hat{\sigma}_j^{\varepsilon}\right\rangle+\mathrm{i}\sum_{\beta\varepsilon}A_{ii}^{\beta\varepsilon}\epsilon_{\beta\varepsilon\gamma}\left\langle\left[\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{\gamma}\right]\right\rangle\right) && (63)\\
&-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\left\langle\hat{\sigma}_i^{\alpha}\right\rangle-2\left(\gamma_--\gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^{\alpha},\hat{\sigma}_i^{z}\right\}\right\rangle+2\sum_{j\neq i}\left\langle\hat{\sigma}_i^{\alpha}\hat{\sigma}_j^{z}\right\rangle\right)+\sum_{\eta}2\gamma_{\eta}\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^{\alpha}\hat{A}_i^{\eta}\hat{\rho}\right) && (64)\\
&+\sum_{j\neq i,\eta}2\gamma_{\eta}\mathrm{Tr}\left(\hat{\sigma}_i^{\alpha}\hat{A}_j^{\dagger\eta}\hat{A}_j^{\eta}\hat{\rho}\right). && (65)
\end{aligned}$$

Explicitly we can find the equations associated with $\partial_t\left\langle\hat{\sigma}_i^{+}(t)\right\rangle$ and $\partial_t\left\langle\hat{\sigma}_i^{z}(t)\right\rangle$:

$$\partial_t\left\langle\hat{\sigma}_i^+(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^+,\left(\hat{\sigma}_i^x,\hat{\sigma}_i^y,\hat{\sigma}_i^z\right)\right]\hat{\rho}\right)+\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\left[\hat{\sigma}_i^+,\hat{\Upsilon}_i\right]\hat{\Upsilon}_j\right\rangle\right)-2N(\gamma_z+2\gamma_++2\gamma_-)\left\langle\hat{\sigma}_i^+\right\rangle \quad (66)$$

$$-2\left(\gamma_--\gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^+,\hat{\sigma}_i^z\right\}\right\rangle+2\sum_{j\neq i}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)+\sum_\eta 2\gamma_\eta\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^+\hat{A}_i^\eta\hat{\rho}\right)+\sum_{j\neq i,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^+\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}\right), \quad (67)$$

$$= -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\hat{\sigma}_i^+\hat{\Upsilon}_j\right\rangle\right)-2N(\gamma_z+2\gamma_++2\gamma_-)\left\langle\hat{\sigma}_i^+\right\rangle \quad (68)$$

$$-2\left(\gamma_--\gamma_+\right)\left(\left\langle\left\{\hat{\sigma}_i^+,\hat{\sigma}_i^z\right\}\right\rangle+2\sum_{j\neq i}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)+\sum_\eta 2\gamma_\eta\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^+\hat{A}_i^\eta\hat{\rho}\right)+\sum_{j\neq i,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^+\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}\right), \quad (69)$$

$$\left\langle\left\{\hat{\sigma}_i^+,\hat{\sigma}_i^z\right\}\right\rangle = \begin{pmatrix}0&2\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&2\\0&0\end{pmatrix} \quad (70)$$

$$= \begin{pmatrix}0&-2\\0&0\end{pmatrix}+\begin{pmatrix}0&2\\0&0\end{pmatrix} \quad (71)$$

$$= \begin{pmatrix}0&0\\0&0\end{pmatrix}, \quad (72)$$

$$\partial_t\left\langle\hat{\sigma}_i^+(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left\langle\hat{\sigma}_i^+\hat{\Upsilon}_j\right\rangle\right)-2N(\gamma_z+2\gamma_++2\gamma_-)\left\langle\hat{\sigma}_i^+\right\rangle \quad (73)$$

$$-4\left(\gamma_--\gamma_+\right)\sum_{j\neq i}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle+\sum_\eta 2\gamma_\eta\mathrm{Tr}\left(\hat{A}_i^{\dagger\eta}\hat{\sigma}_i^+\hat{A}_i^\eta\hat{\rho}\right)+\sum_{j\neq i,\eta}2\gamma_\eta\mathrm{Tr}\left(\hat{\sigma}_i^+\hat{A}_j^{\dagger\eta}\hat{A}_j^\eta\hat{\rho}\right), \quad (74)$$

$$\left\langle\hat{\sigma}_i^+\hat{\Upsilon}_j\right\rangle = \left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle+\frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle+\frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle, \quad (75)$$

$$\overrightarrow{v_1} = \left(\frac{1-\mathrm{i}}{2}, \frac{1+\mathrm{i}}{2}, 1\right), \tag{76}$$

$$\left\langle \hat{\sigma}_i^+ \hat{\Upsilon}_j \right\rangle = \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right), \tag{77}$$

$$\partial_t \left\langle \hat{\sigma}_i^+ (t) \right\rangle = -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{78}$$

$$- 2N (\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4(\gamma_- - \gamma_+) \sum_{j \neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle + \sum_{\eta} 2\gamma_\eta \operatorname{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^+ \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j \neq i, \eta} 2\gamma_\eta \tag{79}$$

$$\times \operatorname{Tr}\left(\hat{\sigma}_i^+ \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right) \tag{80}$$

$$\hat{A}_i^{\dagger\pm} \hat{\sigma}_i^+ \hat{A}_i^\pm = 0_{2\times 2}, \tag{81}$$

$$\hat{A}_i^{\dagger z} \hat{\sigma}_i^+ \hat{A}_i^z = -\hat{\sigma}_i^+, \tag{82}$$

$$\partial_t \left\langle \hat{\sigma}_i^+ (t) \right\rangle = -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{83}$$

$$- 2N (\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4(\gamma_- - \gamma_+) \sum_{j \neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle - 2\gamma_z \left\langle \hat{\sigma}_i^+ \right\rangle + \sum_{j \neq i, \eta} 2\gamma_\eta \operatorname{Tr}\left(\hat{\sigma}_i^+ \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right), \tag{84}$$

$$\hat{A}_j^{\dagger+} \hat{A}_j^+ = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{85}$$

$$= 2\left(\mathbb{I} - \hat{\sigma}_j^z \right), \tag{86}$$

$$\hat{A}_j^{\dagger-} \hat{A}_j^- = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \tag{87}$$

$$= 2\left(\mathbb{I} + \hat{\sigma}_j^z \right), \tag{88}$$

$$\hat{A}_j^{\dagger z} \hat{A}_j^z = \mathbb{I}, \tag{89}$$

$$\partial_t \left\langle \hat{\sigma}_i^+ (t) \right\rangle = -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{90}$$

$$- 2N (\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4(\gamma_- - \gamma_+) \sum_{j \neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle - 2\gamma_z \left\langle \hat{\sigma}_i^+ \right\rangle + \sum_{j \neq i} \left(2\gamma_+ \operatorname{Tr}\left(\hat{\sigma}_i^+ 2\left(\mathbb{I} - \hat{\sigma}_j^z \right) \hat{\rho} \right) \right. \tag{91}$$

$$\left. + 2\gamma_- \operatorname{Tr}\left(\hat{\sigma}_i^+ 2\left(\mathbb{I} + \hat{\sigma}_j^z \right) \hat{\rho} \right) + 2\gamma_z \operatorname{Tr}\left(\hat{\sigma}_i^+ \mathbb{I} \hat{\rho} \right) \right) \tag{92}$$

$$= -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{93}$$

$$- 2N (\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4(\gamma_- - \gamma_+) \sum_{j \neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle - 2\gamma_z \left\langle \hat{\sigma}_i^+ \right\rangle + \sum_{j \neq i} \left(4\gamma_+ \operatorname{Tr}\left(\hat{\sigma}_i^+ \left(\mathbb{I} - \hat{\sigma}_j^z \right) \hat{\rho} \right) \right. \tag{94}$$

$$\left. + 4\gamma_- \operatorname{Tr}\left(\hat{\sigma}_i^+ \left(\mathbb{I} + \hat{\sigma}_j^z \right) \hat{\rho} \right) + 2\gamma_z \operatorname{Tr}\left(\hat{\sigma}_i^+ \hat{\rho} \right) \right) \tag{95}$$

$$= -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{96}$$

$$- 2N (\gamma_z + 2\gamma_+ + 2\gamma_-) \left\langle \hat{\sigma}_i^+ \right\rangle - 4(\gamma_- - \gamma_+) \sum_{j \neq i} \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle - 2\gamma_z \left\langle \hat{\sigma}_i^+ \right\rangle + \sum_{j \neq i} \left(4\gamma_+ \left(\left\langle \hat{\sigma}_i^+ \right\rangle - \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right. \tag{97}$$

$$\left. + 4\gamma_- \left(\left\langle \hat{\sigma}_i^+ \right\rangle + \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) + 2\gamma_z \left\langle \hat{\sigma}_i^+ \right\rangle \right) \tag{98}$$

$$= -\frac{\mathrm{i}}{\hbar} \left(2\overrightarrow{h}_i \cdot \left(\left\langle \hat{\sigma}_i^z \right\rangle, \mathrm{i} \left\langle \hat{\sigma}_i^z \right\rangle, -\left\langle \hat{\sigma}_i^+ \right\rangle \right) - 4(1+\mathrm{i}) \sum_{j \neq i} (A_{ji} + A_{ij}) \overrightarrow{v_1} \cdot \left(\left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^- \right\rangle, \left\langle \hat{\sigma}_i^+ \hat{\sigma}_j^z \right\rangle \right) \right) \tag{99}$$

$$
\begin{aligned}
&-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\left\langle\hat{\sigma}_i^+\right\rangle-4\left(\gamma_--\gamma_+\right)\sum_{j\neq i}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle-2\gamma_z\left\langle\hat{\sigma}_i^+\right\rangle+\left(N-1\right)\left(4\gamma_++4\gamma_-+2\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle && (100)\\
&-4\sum_{j\neq i}\left(\gamma_+-\gamma_-\right)\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle && (101)\\
=&-\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v_1}\cdot\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)\right) && (102)\\
&-2N\left(\gamma_z+2\gamma_++2\gamma_-\right)\left\langle\hat{\sigma}_i^+\right\rangle-2\gamma_z\left\langle\hat{\sigma}_i^+\right\rangle+2\left(N-1\right)\left(2\gamma_++2\gamma_-+\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle && (103)\\
=&-\frac{\mathrm{i}}{\hbar}\left(2\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-4\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v_1}\cdot\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)\right) && (104)\\
&-2\gamma_z\left\langle\hat{\sigma}_i^+\right\rangle-2\left(2\gamma_++2\gamma_-+\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle && (105)\\
=&-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-2\left(1+\mathrm{i}\right)\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v_1}\cdot\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)\right) && (106)\\
&-4\left(\gamma_++\gamma_-+\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle. && (107)
\end{aligned}
$$

In mean-field approximation $\left\langle\hat{a}_i\hat{b}_j\right\rangle\approx\left\langle\hat{a}_i\right\rangle\left\langle\hat{b}_j\right\rangle$ with $i\neq j$ this is just:

$$
\begin{aligned}
\partial_t\left\langle\hat{\sigma}_i^+\left(t\right)\right\rangle=&-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-2\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\overrightarrow{v_1}\cdot\left(\left\langle\hat{\sigma}_j^+\right\rangle,\left\langle\hat{\sigma}_j^-\right\rangle,\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) && (108)\\
&-4\left(\gamma_++\gamma_-+\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle && (109)\\
=&-\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle,\mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle,-\left\langle\hat{\sigma}_i^+\right\rangle\right)-2\left(1+\mathrm{i}\right)\left\langle\hat{\sigma}_i^+\right\rangle\sum_{j\neq i}\left(A_{ji}+A_{ij}\right)\left(\mathrm{Re}\left(\left\langle\hat{\sigma}_j^+\right\rangle\right)+\mathrm{Im}\left(\left\langle\hat{\sigma}_j^+\right\rangle\right)+\left\langle\hat{\sigma}_j^z\right\rangle\right)\right) && (110)\\
&-4\left(\gamma_++\gamma_-+\gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle. && (111)
\end{aligned}
$$

Now $\partial_t\left\langle\hat{\sigma}_i^z\left(t\right)\right\rangle$ is given by:

$$\partial_t \langle \hat{\sigma}_i^z(t) \rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{112}$$

$$- 2\left(\gamma_- - \gamma_+ \right) \left(\langle \{ \hat{\sigma}_i^z, \hat{\sigma}_i^z \} \rangle + 2\sum_{j\neq i} \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle \right) + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^z \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right) \tag{113}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{114}$$

$$+ 4\left(\gamma_+ - \gamma_- \right) \sum_{j\neq i} \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^z \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right) \tag{115}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{116}$$

$$+ 4\left(\gamma_+ - \gamma_- \right) \sum_{j\neq i} \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle + \sum_{\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{A}_i^{\dagger\eta} \hat{\sigma}_i^z \hat{A}_i^\eta \hat{\rho} \right) + \sum_{j\neq i,\eta} 2\gamma_\eta \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger\eta} \hat{A}_j^\eta \hat{\rho} \right), \tag{117}$$

$$\hat{A}_i^{\dagger+} \hat{\sigma}_i^z \hat{A}_i^+ = 2\left(\mathbb{I} - \hat{\sigma}_i^z \right), \tag{118}$$

$$\hat{A}_i^{\dagger-} \hat{\sigma}_i^z \hat{A}_i^- = -2\left(\mathbb{I} + \hat{\sigma}_i^z \right), \tag{119}$$

$$\hat{A}_i^{\dagger z} \hat{\sigma}_i^z \hat{A}_i^z = \hat{\sigma}_i^z, \tag{120}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger+} \hat{A}_j^+ \hat{\rho} \right) = \mathrm{Tr}\left(\hat{\sigma}_i^z 2\left(\mathbb{I} - \hat{\sigma}_j^z \right) \hat{\rho} \right) \tag{121}$$

$$= 2\left(\langle \hat{\sigma}_i^z \rangle - \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle \right), \tag{122}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger-} \hat{A}_j^- \hat{\rho} \right) = \mathrm{Tr}\left(\hat{\sigma}_i^z 2\left(\mathbb{I} + \hat{\sigma}_j^z \right) \hat{\rho} \right) \tag{123}$$

$$= 2\left(\langle \hat{\sigma}_i^z \rangle + \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle \right), \tag{124}$$

$$\mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger z} \hat{A}_j^z \hat{\rho} \right) = \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\rho} \right) \tag{125}$$

$$= \langle \hat{\sigma}_i^z \rangle, \tag{126}$$

$$\partial_t \langle \hat{\sigma}_i^z(t) \rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{127}$$

$$+ 4\left(\gamma_+ - \gamma_- \right) \sum_{j\neq i} \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle + 2\gamma_+ \mathrm{Tr}\left(\hat{A}_i^{\dagger+} \hat{\sigma}_i^z \hat{A}_i^+ \hat{\rho} \right) + 2\gamma_- \mathrm{Tr}\left(\hat{A}_i^{\dagger-} \hat{\sigma}_i^z \hat{A}_i^- \hat{\rho} \right) + 2\gamma_z \mathrm{Tr}\left(\hat{A}_i^{\dagger z} \hat{\sigma}_i^z \hat{A}_i^z \hat{\rho} \right) \tag{128}$$

$$+ \sum_{j\neq i} 2\left(\gamma_+ \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger+} \hat{A}_j^+ \hat{\rho} \right) + \gamma_- \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{A}_j^{\dagger-} \hat{A}_j^- \hat{\rho} \right) + \gamma_z \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\rho} \right) \right) \tag{129}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{130}$$

$$+ 4\left(\gamma_+ - \gamma_- \right) \sum_{j\neq i} \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle + 2\gamma_+ \mathrm{Tr}\left(2\left(\mathbb{I} - \hat{\sigma}_i^z \right) \hat{\rho} \right) + 2\gamma_- \mathrm{Tr}\left(-2\left(\mathbb{I} + \hat{\sigma}_i^z \right) \hat{\rho} \right) + 2\gamma_z \mathrm{Tr}\left(\hat{\sigma}_i^z \hat{\rho} \right) \tag{131}$$

$$+ \sum_{j\neq i} 2\left(\gamma_+ 2\left(\langle \hat{\sigma}_i^z \rangle - \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle \right) + \gamma_- 2\left(\langle \hat{\sigma}_i^z \rangle + \left\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \right\rangle \right) + \gamma_z \langle \hat{\sigma}_i^z \rangle \right) \tag{132}$$

$$= -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i \cdot \mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z \right) \right] \hat{\rho} \right) + \sum_{j\neq i} \left(A_{ji} + A_{ij} \right) \left\langle \left[\hat{\sigma}_i^z, \hat{\Upsilon}_i \right] \hat{\Upsilon}_j \right\rangle \right) - 2N\left(\gamma_z + 2\gamma_+ + 2\gamma_- \right) \langle \hat{\sigma}_i^z \rangle \tag{133}$$

We focus our attention on:

$$\hat{\Upsilon}_j = \hat{\sigma}_j^z + \frac{1-\mathrm{i}}{2}\hat{\sigma}_j^+ + \frac{1+\mathrm{i}}{2}\hat{\sigma}_j^-, \tag{165}$$

$$\left\langle\left(\left(\frac{1+\mathrm{i}}{2}\right)\hat{\sigma}_i^- - \left(\frac{1-\mathrm{i}}{2}\right)\hat{\sigma}_i^+\right)\hat{\Upsilon}_j\right\rangle = \left(\frac{1+\mathrm{i}}{2}\right)\left\langle\hat{\sigma}_i^-\hat{\Upsilon}_j\right\rangle - \left(\frac{1-\mathrm{i}}{2}\right)\left\langle\hat{\sigma}_i^+\hat{\Upsilon}_j\right\rangle \tag{166}$$

$$= \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\left(\hat{\sigma}_j^z + \frac{1-\mathrm{i}}{2}\hat{\sigma}_j^+ + \frac{1+\mathrm{i}}{2}\hat{\sigma}_j^-\right)\right\rangle - \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\left(\hat{\sigma}_j^z + \frac{1-\mathrm{i}}{2}\hat{\sigma}_j^+ + \frac{1+\mathrm{i}}{2}\hat{\sigma}_j^-\right)\right\rangle \tag{167}$$

$$= \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle + \frac{1}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle + \frac{\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^-\right\rangle - \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle + \frac{\mathrm{i}}{2}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle - \frac{1}{2}\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle \tag{168}$$

$$= \frac{1+\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle + \frac{1}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle + \frac{\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^-\right\rangle - \frac{1-\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle^* + \frac{\mathrm{i}}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^-\right\rangle^* - \frac{1}{2}\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle^* \tag{169}$$

$$= \mathrm{iIm}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle\right) + \mathrm{iRe}\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle\right) + \mathrm{iIm}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right) + \mathrm{iRe}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right) \tag{170}$$

$$= \mathrm{i}\left(\mathrm{Im}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle\right) + \mathrm{Re}\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle\right) + \mathrm{Im}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right) + \mathrm{Re}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right)\right). \tag{171}$$

So we obtained that:

$$\tag{172}$$

$$\partial_t\left\langle\hat{\sigma}_i^z(t)\right\rangle = \frac{1}{\hbar}\Bigg(\overrightarrow{h}_i\cdot\left(\mathrm{Im}\left(\left\langle\hat{\sigma}_i^+\right\rangle\right), -\mathrm{Re}\left(\left\langle\hat{\sigma}_i^+\right\rangle\right), 0\right) + \sum_{j\neq i}(A_{ji}+A_{ij})\big(\mathrm{Im}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^+\right\rangle\right) + \mathrm{Re}\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle\right) + \mathrm{Im}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right) \tag{173}$$

$$+\mathrm{Re}\left(\left\langle\hat{\sigma}_i^-\hat{\sigma}_j^z\right\rangle\right)\big)\Bigg) + 4\left(\gamma_+ - \gamma_-\right) - 8\left\langle\hat{\sigma}_i^z\right\rangle\left(\gamma_+ + \gamma_-\right), \tag{174}$$

$$\partial_t\left\langle\hat{\sigma}_i^+(t)\right\rangle = -\frac{2\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\left(\left\langle\hat{\sigma}_i^z\right\rangle, \mathrm{i}\left\langle\hat{\sigma}_i^z\right\rangle, -\left\langle\hat{\sigma}_i^+\right\rangle\right) - 2(1+\mathrm{i})\sum_{j\neq i}(A_{ji}+A_{ij})\,\overrightarrow{v_1}\cdot\left(\left\langle\hat{\sigma}_i^+\hat{\sigma}_j^+\right\rangle, \left\langle\hat{\sigma}_i^+\hat{\sigma}_j^-\right\rangle, \left\langle\hat{\sigma}_i^+\hat{\sigma}_j^z\right\rangle\right)\right) \tag{175}$$

$$-4\left(\gamma_+ + \gamma_- + \gamma_z\right)\left\langle\hat{\sigma}_i^+\right\rangle. \tag{176}$$

In the mean-field approximation:

$$\partial_t\left\langle\hat{\sigma}_i^z(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left(\overrightarrow{h}_i\cdot\mathrm{Tr}\left(\left[\hat{\sigma}_i^z, \left(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z\right)\right]\hat{\rho}\right) + \sum_{j\neq i}(A_{ji}+A_{ij})\left\langle\left[\hat{\sigma}_i^z, \hat{\Upsilon}_i\right]\hat{\Upsilon}_j\right\rangle\right) + 4\left(\gamma_+ - \gamma_-\right) - 8\left\langle\hat{\sigma}_i^z\right\rangle \tag{177}$$

$$\tag{178}$$

$$\tag{179}$$

III. BIBLIOGRAPHY

[1] Tucker, K, Barberena, D, Lewis-Swan, RJ, Thompson, JK, Restrepo, JG, Rey, AM 2020 Phys. Rev. A 2469-9926

* edch5956@colorado.edu