Joshua Eckels - ME406 Controls

CLTF:

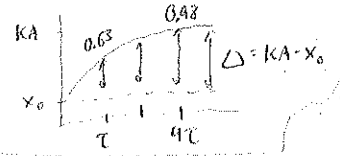
$$\frac{Y}{R} = \frac{CG}{1+CGH}$$

First-order:

A: step input

$$\tau\dot{x} + x = KA$$

$$x(t) = \begin{cases} KA + (x_0 - KA)e^{-t/\tau} \\ x_0 + \Delta(1 - e^{-t/\tau}) \end{cases}$$



Laplace

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

IVT: $f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

FVT: $f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\mathcal{L}\{u(t)\} = \frac{1}{s}$ $\quad \mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$ $\quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$ $\quad \mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$

$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$ $\quad " = \frac{s+a}{(s+a)^2+\omega^2}$

$$\frac{s+8}{(s+\sigma)^2+\omega^2} = \frac{A(\omega)}{(s+\sigma)^2+\omega^2} + \frac{B(s+\sigma)}{(s+\sigma)^2+\omega^2}$$

$s_{1,2} = -\sigma \pm i\omega$

Second order:

$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = Kf(t)$$

free response: $f(t)=0$

$s_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$

$s_{1,2} = -\zeta\omega_n \pm i\omega_d$

$x(t) = C_1e^{s_1t} + C_2e^{s_2t}$

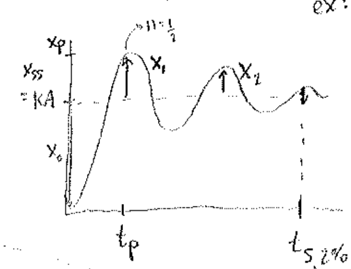
$e^{\pm i\omega_n t} = \cos(\omega_n t) \pm i\sin(\omega_n t)$

$\zeta=0$	$x(t) = A_1\cos\omega_n t + A_2\sin\omega_n t$
$0<\zeta<1$	$x(t) = e^{-\zeta\omega_n t}(A_1\cos\omega_d t + A_2\sin\omega_d t)$
$\zeta>1$	$x(t) = c_1e^{s_1t} + c_2e^{s_2t}$
$\zeta=1$	$x(t) = (C_1 + C_2t)e^{-\omega_n t}$

step response: $f(t) = A$

$s_{1,2} = -\zeta\omega_n \pm i\omega_d$

- $x(t) = KA\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right)$
- $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$



- $\%OS = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 100e^{-\frac{\zeta\omega_n}{\omega_d}\pi}$
- $\zeta = \sqrt{\frac{\ln^2(\%OS/100)}{\ln^2(\%OS/100)+\pi^2}}$
- $t_p = \frac{\pi}{\omega_d}$ $\quad$ • $t_s \approx \frac{4}{\zeta\omega_n}$
- $\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right)$

ex: $\left[\delta = \frac{1}{1/2}\ln\left(\frac{x_{ss}}{x_p - x_{ss}}\right)\right]$

$\left[\delta = \frac{1}{1}\ln\left(\frac{x_1}{x_2}\right)\right]$

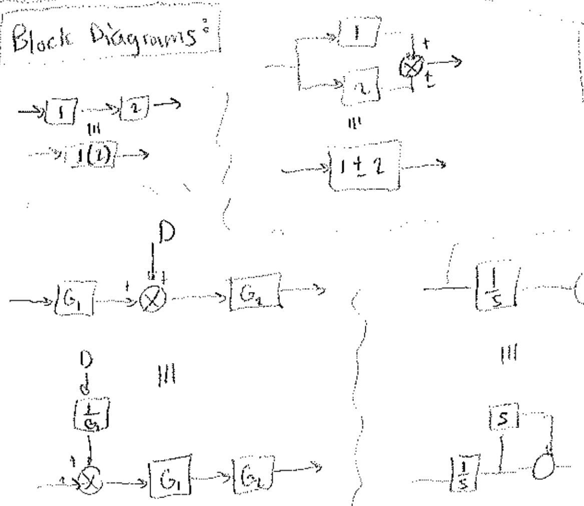
- $\zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}}$

Frequency response

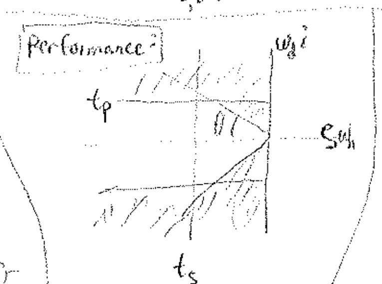
$$x_{ss} = A\,|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

$$M_{DB} = 20\log_{10}(|G(i\omega)|)$$

Block Diagrams:



Performance:



$t_p = \frac{\pi}{\omega_d}$ $\quad \cos\theta = \zeta$

$t_s = \frac{4}{\zeta\omega_n}$

Error:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+CG}R$$

$CG = L(s)$

$L(s) = \frac{K(s+z_1)\cdots(s+z_n)}{s^n(s+p_1)\cdots(s+p_n)}$ $\quad n = \text{type}$

Unit step: $e_{ss} = \lim_{s\to 0} s\frac{1/s}{1+L(s)}$

$= \lim_{s\to 0}\frac{1}{1+L(0)} \to K_p$

Unit ramp: $e_{ss} = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sL(s)}$

Unit parabolic: $e_{ss} = \frac{1}{K_a} = \frac{1}{\lim_{s\to 0} s^2L(s)}$

type	e_{ss} step	ramp	Para
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$1/K_v$	∞
2	0	0	$1/K_a$

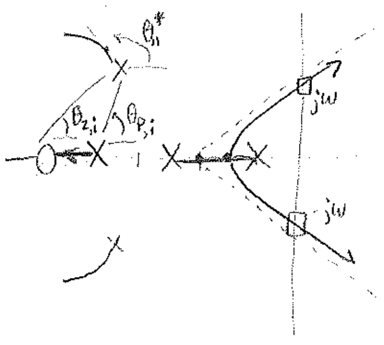
Root Locus:

CLTF: $\frac{KCG}{1+K(CGH)}$

OLTF: $L(s) = CGH$

Char. eqn: $1 + KL(s) = 0$

$\left|K\frac{N(s)}{D(s)}\right| = 1$ $\quad \angle KL = (2k+1)180°$



R_7: Departure angle complex open-loop pole

$$\theta_n^* = \Sigma\theta_{z_i} - \Sigma\theta_{p_i} \pm 180°$$

R_8: solve $1 + GH(j\omega) = 0$ (CE) for axis crossings

Rules: (open-loop)

R_1: n_p = # poles, n_z = # zeros

#branches = $\max(n_p, n_z)$

R_2: symmetric about real axis

R_3: pole $\times \to \bigcirc$ zero or $\times \to \infty \leftarrow \times$

R_4: branches on left of odd # poles, zeros

R_5: $n_p - n_z$ branches → asymptotes

$$\sigma_a = \frac{\Sigma(\text{poles}) - \Sigma(\text{zeros})}{n_p - n_z}$$

$$\alpha_K = \frac{(2k+1)180°}{n_p - n_z} \quad k = 0, \pm 1, \ldots$$

R_6: $\frac{dk}{ds} = 0$ → break in/departure

$\frac{d^2k}{ds^2}$: pos = rel min = break in

= neg = rel max = break away

Routh-Hurwitz:

CE: $a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	b_1	b_2	0
s^1	c_1	0	
s^0	b_2	0	

$b_1 = \frac{a_3a_2 - a_4a_1}{a_3}$

$b_2 = a_0$

$c_1 = \frac{b_1a_1 - a_3b_2}{b_1}$

*replace 0 with ε and find $\lim_{\varepsilon\to 0^+}$

sign changes = # unstable poles

PID

$$K_p + K_ds + \frac{K_i}{s}$$

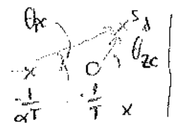
$$\frac{K_ds^2 + K_ps + K_i}{s} = \frac{K(s+z_1)(s+z_2)}{s}$$

$$\frac{Ks^2 + K(z_1+z_2)s + Kz_1z_2}{s}$$

Lead Compensator

$$C(s) = \frac{s + \frac{1}{T_1}}{s + \frac{1}{\alpha T_1}} \quad T>0, \quad 0<\alpha<1$$

$\rightarrow \otimes \rightarrow k \cdot C(s) \rightarrow G(s)$



$$\theta_d = \measuredangle C(s) = \theta_{zc} - \theta_{pc}$$

$\measuredangle C(s) > 0$ "lead"

Root Locus

Defect angle:

$$\theta_d + \angle L_{uc}(s_d) = (2k+1)180^\circ$$

s_d = desired pole

L_{uc} = uncompensated

1. choose $-\frac{1}{T} \rightarrow$ find θ_d, $\theta_d = \theta_{zc} - \theta_{pc} > 0$
2. Use θ_d to find $-\frac{1}{\alpha T}$
3. $K_{lead} = \left|\frac{1}{L_c(s_d)}\right|$, where $L_c(s) = K_{lead} \cdot C(s) \cdot L_{uc}(s)$

CE: $1 + K_{lead} L_c(s_d) = 0$

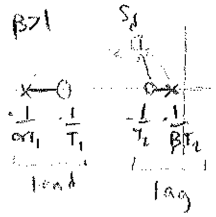
Lag Compensator

$\beta > 1$

$\theta_{c_{lag}} \approx 0$

$$e_{ss} = \lim_{s\to 0} s E(s)$$

$$C_{lag}(s) = \frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}$$



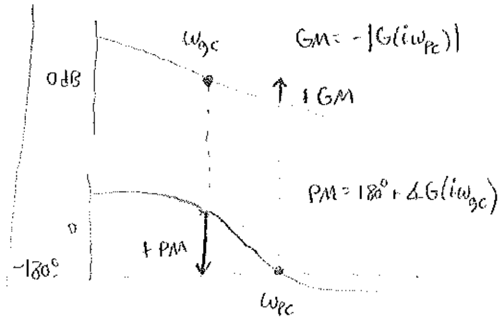
$$L_c(0) = \underbrace{K_{lead}\left(\frac{1/T_1}{1/\alpha T_1}\right)}_{lead}\underbrace{\left(\frac{1/T_2}{1/\beta T_2}\right)}_{lag}\underbrace{G(0)}_{plant}$$

1. Choose $\frac{1}{T_2} \approx 0.1$
2. Solve for β from e_{ss}
3. Place $\frac{1}{T_2}$ and $\frac{1}{\beta T_2}$

$$L_c(s) = K_{lead}\left[\frac{s+\frac{1}{T_1}}{s+\frac{1}{\alpha T_1}}\right]\left[\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right]G(s)$$

Gain and Phase Margin

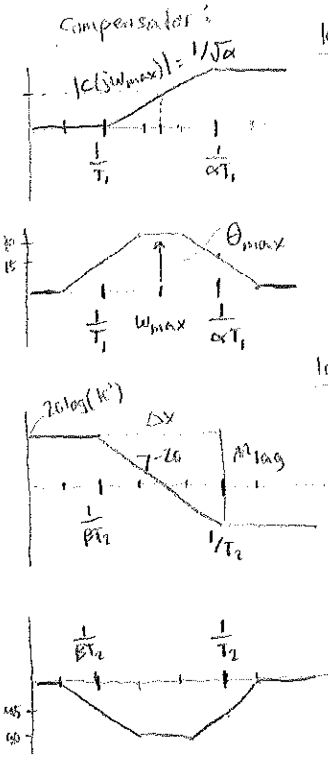
unstable when $\Phi = -180^\circ$ AND $|G(i\omega)| = 0\,dB$

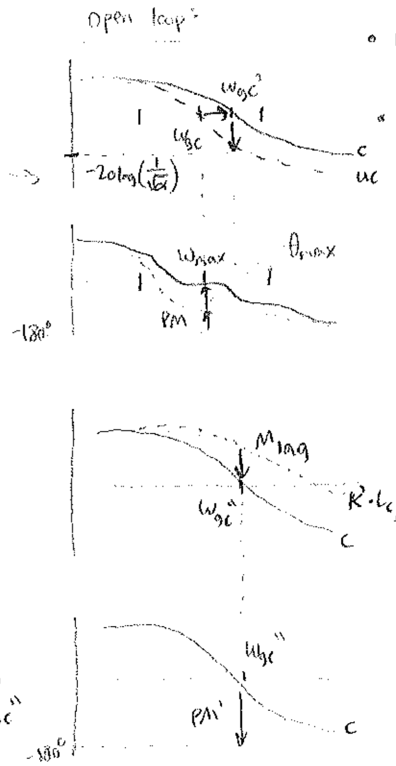


Lead Compensator

Bode

- $$PM_d = \left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right] \approx 100\zeta \quad (\zeta \le 0.6)$$
- $\measuredangle C_{lead}(\omega_{max}) = \theta_{max} = \theta_{lead} = PM_d - PM_{uc} + \Delta$, $\Delta \approx 12^\circ$





- $$\omega_{max} = \frac{1}{T_1\sqrt{\alpha}}$$
- $$\alpha = \frac{1 - \sin(\theta_{lead})}{1 + \sin(\theta_{lead})}$$
- ω_{gc}' where $|L_{uc}(i\omega)|_{dB} = -20\log\left(\frac{1}{\sqrt{\alpha}}\right)$
- $$T_1 = \frac{1}{\omega_{gc}'\sqrt{\alpha}}$$ $\rightarrow$ zero: $\frac{1}{T_1}$ $\rightarrow$ pole: $\frac{1}{\alpha T_1}$
- $$C_{lead}(s) = \frac{1}{\alpha}\,\frac{s+\frac{1}{T_1}}{s+\frac{1}{\alpha T_1}}$$

* solve for K if given e_{ss}, otherwise K=1

Lag compensator

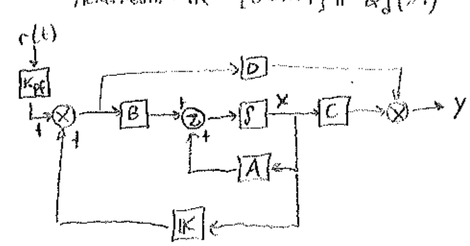
- $PM' = PM_d + \Delta$, $\Delta \approx 5^\circ$
- $e_{ss} \rightarrow$ solve for K'
- Find ω_{gc}'' and M_{lag}
- $$\frac{1}{T_2} = \frac{\omega_{gc}''}{10}$$
- $$\frac{1}{\beta T_2} = 10^{\log_{10}\left(\frac{1}{T_2}\right) - \frac{M_{lag}}{20}}$$
- $$\beta = \frac{1/T_2}{1/\beta T_2}$$ $\rightarrow$ zero: $\frac{1}{T_2}$ $\rightarrow$ pole: $\frac{1}{\beta T_2}$

$$C_{lag}(s) = \frac{K'}{\beta}\,\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}$$

State-Space

$$\dot{\vec{x}} = A\vec{x} + Bu(t)$$
$$y = C\vec{x} + Du(t)$$

- for each nth order EOM, define n lowest derivatives as state variables
- Solve for highest derivative
- write as matrix
- Controllability: $\mathbb{P} = [B \,\vdots\, AB \,\vdots\, \cdots A^{n-1}B]$, $\det(P) \neq 0$
- $\alpha_{sys}(s) = \alpha_d(s)$; $\alpha_d(s) = (s-p_1)(s-p_2)\ldots$, $\alpha_d(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$
- Ackermann: $\mathbb{K} = [0\,0 \ldots 1]\,\mathbb{P}^{-1}\alpha_d(\mathbb{A})$



State → TF

$$\vec{X} = (s\mathbb{I} - A)^{-1}\mathbb{B}\mathbb{U}$$

$$TF = \frac{Y}{U} = \mathbb{C}(s\mathbb{I} - A)^{-1}\mathbb{B} + \mathbb{D}$$

$$\mathbb{M}^{-1} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

TF → state-space

- control canonical form:

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n\end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & 0 & 0 \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n\end{bmatrix} + \begin{bmatrix}1\\0\\0\\ \vdots \\ 0\end{bmatrix}u(t)$$

$$y = [b_1\ b_2 \cdots b_{n-1}\ b_n]\begin{bmatrix}x_1\\x_2\\ \vdots \\ x_{n-1} \\ x_n\end{bmatrix} + 0\cdot u(t)$$

state space $\xrightarrow{unique}$ TF; TF $\xrightarrow{not\ unique}$ state space

Free response

$\dot{x} = Ax \mid y = Cx$

$(A - s\mathbb{I})\vec{v} = \vec{0}$

OL CE: $\det(s\mathbb{I} - A) = 0$

→ eigenvalues of A = poles (TF)

→ solution: $\vec{x}(t) = A_1\vec{v}_1 e^{s_1 t} + A_2\vec{v}_2 e^{s_2 t} + \ldots + A_n\vec{v}_n e^{s_n t}$

IC: A_i, eigenvalue s_k, eigenvector $\vec{v}_k$

Full state feedback: $u(t) = -\mathbb{K}\vec{x}$

$\dot{\vec{x}} = (A - BK)\vec{x} = A_{cl}\vec{x} \mid y = (C - DK)x = C_{cl}\vec{x}$

$A_{cl}\vec{x} = \lambda\vec{x} \rightarrow \det(A_{cl} - \lambda I) = 0$

→ CL CE: $\det(s\mathbb{I} - A + \mathbb{B}\mathbb{K}) = 0$

Pre-filter gain

$u(t) = -\mathbb{K}x + K_{pf}r(t)$

$$\begin{bmatrix}\vec{N}_x \\ N_u\end{bmatrix} = \begin{bmatrix}\mathbb{A} & B \\ C & D\end{bmatrix}^{-1}\begin{bmatrix}\vec{0} \\ 1\end{bmatrix}$$

$$K_{pf} = N_u + \mathbb{K}\vec{N}_x$$

Time delay

$G_2 = G_1 e^{-sT}$, T = time delay

$-\omega_{gc}T_{max} = PM$

* time delay reduces PM and GM by decreasing ϕ linearly