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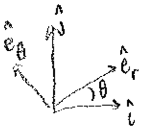
Theoretical Aerodynamics
2/5/20

- Div: $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$
- Curl: $\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$
- vorticity: $\xi = 2\omega = \nabla \times \vec{V}$
- streamline: $d\vec{s} \times \vec{V} = 0$ (tangent to velocity)



$\hookrightarrow w\,dy - v\,dz = 0$
$u\,dz - w\,dx = 0$
$v\,dx - u\,dy = 0 \longrightarrow \frac{dy}{dx} = \frac{v}{u}$

- $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$

stream: $\psi(x,y) = \psi(r,\theta) \rightarrow u = \frac{\partial \psi}{\partial y} \quad v = \frac{-\partial \psi}{\partial x} \quad V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad V_\theta = -\frac{\partial \psi}{\partial r}$

vel. potential: $\vec{V} = \nabla \phi \rightarrow \phi(x,y) \rightarrow u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad V_r = \frac{\partial \phi}{\partial r} \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

incompressible: $\nabla \cdot \vec{V} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{1}{r}\frac{\partial (r V_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$

irrotational: $\nabla \times \vec{V} = 0 \rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 = \frac{1}{r}\left[\frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right]$

Laplace: $\nabla \cdot \vec{V} = 0$
$\nabla \cdot (\nabla \phi) = 0$
$\nabla^2 \phi = 0$

$\rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

$\hookrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

$\rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

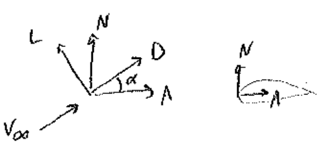
- $C_L = \frac{L}{q_\infty S} \quad C_D = \frac{D}{q_\infty S} \quad C_M = \frac{M}{q_\infty S c} \quad q_\infty = \frac{1}{2}\rho V_\infty^2$

unit span $S = c(1)$

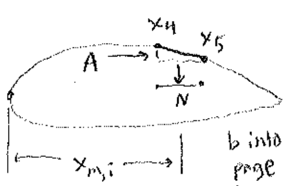
$c_l = \frac{L}{q_\infty c} \quad c_d = \frac{D}{q_\infty c} \quad c_m = \frac{M}{q_\infty c^2} \quad C_p = \frac{p - p_\infty}{q_\infty} \quad C_f = \frac{\tau}{q_\infty}$

- $C_n = \frac{1}{c}\left[\int_0^c (C_{p,l} - C_{p,u})\,dx + \int_0^c \left(C_{f,u}\frac{dy_u}{dx} + C_{f,l}\frac{dy_l}{dx}\right)dx\right]$
- $C_a = \frac{1}{c}\left[\int_0^c (C_{f,u} + C_{f,l})\,dx + \int_0^c \left(C_{p,u}\frac{dy_u}{dx} - C_{p,l}\frac{dy_l}{dx}\right)dx\right]$
- $C_{m,LE} = \frac{1}{c^2}\left[\int_0^c (C_{p,u} - C_{p,l})\,x\,dx - \int_0^c \left(C_{f,u}\frac{dy_u}{dx} + C_{f,l}\frac{dy_l}{dx}\right)x\,dx + \int_0^c \left(C_{f,u} + C_{p,u}\frac{dy_u}{dx}\right)y_u\,dx + \int_0^c \left(C_{f,l} - C_{p,l}\frac{dy_l}{dx}\right)y_l\,dx\right]$

- $C_l = C_n \cos\alpha - C_a \sin\alpha$
- $C_d = C_n \sin\alpha + C_a \cos\alpha$



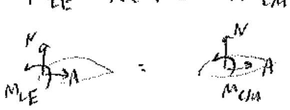
- $Re = \frac{\rho V c}{\mu} \quad M = \frac{V}{a} \quad a = \sqrt{\gamma R T} \quad p\forall = mRT \quad p = \rho R T \quad C_R = f(Re, M_\infty, \alpha, AR)$



$N_i = b(x_5 - x_4)\left(\frac{p_4 + p_5}{2}\right)$
$A_i = b(y_5 - y_4)\left(\frac{p_4 + p_5}{2}\right)$
$M_{LE,i} = N_i \cdot x_{m,i}$ *ignore A_i moments

$p_3 - p_{2y} = -\gamma(h_3 - h_{2y}) \qquad p + \frac{1}{2}\rho V^2 = \text{constant}$

$M_{LE} = N\left(\frac{c}{4}\right)(-1) + M_{CM} \qquad C_{m,x} = C_{m,c/4} + C_l\left(\frac{x}{c} - \frac{1}{4}\right)$



Uniform: $\phi = V_\infty x = V_\infty r\cos\theta$
$\psi = V_\infty y = V_\infty r\sin\theta$

Circular Cylinder: $R^2 = \frac{K}{2\pi V_\infty}$
$C_p = 1 - \frac{V^2}{V_\infty^2}$
$C_p = 1 - 4\sin^2\theta$

Source: $\phi = \frac{\Lambda}{2\pi}\ln r \quad \psi = \frac{\Lambda}{2\pi}\theta$

Doublet: $\phi = \frac{K}{2\pi r}\cos\theta \quad \psi = \frac{-K}{2\pi r}\sin\theta$

Vortex: $\phi = \frac{-\Gamma}{2\pi}\theta \quad \psi = \frac{\Gamma}{2\pi}\ln r$

$V_\theta = \frac{-\Gamma}{2\pi r} \quad V_r = 0$

Lifting cylinder:

$\psi = V_\infty r\sin\theta - \frac{K}{2\pi r}\sin\theta + \frac{\Gamma}{2\pi}\ln r \rightarrow \psi_{stag} = \frac{\Gamma}{2\pi}\ln R$

$V_\theta = -2V_\infty\sin\theta - \frac{\Gamma}{2\pi R} \rightarrow C_p = 1 - 4\left[\sin\theta + \frac{\Gamma}{4\pi V_\infty R}\right]^2$

$\rightarrow C_n = \frac{\Gamma}{V_\infty R} \rightarrow \frac{N}{b} = \boxed{N' = \rho V_\infty \Gamma}$ Kutta-Joukowski

$\Gamma = -\oint \vec{V}\cdot d\vec{s} \qquad \oint \vec{V}\cdot d\vec{s} = \iint_S (\nabla\times\vec{V})\cdot d\vec{S}$ Stokes

$\Gamma = \frac{1}{c}\int_{x=0}^{c} \gamma(x)\,dx$

$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$

$C_l = \frac{2\Gamma}{V_\infty c}$

$V_r = V_\infty\cos\theta\left[1 - \frac{R^2}{r^2}\right] \qquad V_\theta = -V_\infty\sin\theta\left[1 + \frac{R^2}{r^2}\right] - \frac{\Gamma}{2\pi r}$

$u = V_r\cos\theta - V_\theta\sin\theta \qquad v = V_r\sin\theta + V_\theta\cos\theta$
$V_r = u\cos\theta + v\sin\theta \qquad V_\theta = -u\sin\theta + v\cos\theta$

$\frac{du}{dt} \approx \frac{u(t+\Delta t) - u(t)}{\Delta t}$ finite-diff

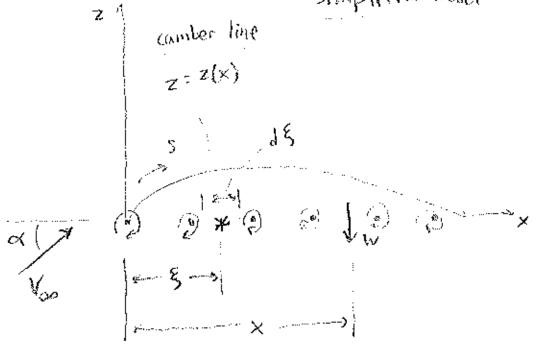
$\frac{d^2 u}{dy^2} \approx \frac{u(y+\Delta y) + u(y-\Delta y) - 2u(y)}{\Delta y^2}$

$\frac{d^2 u}{dy^2} = 0$
$\int \frac{d}{dy}\left(\frac{du}{dy}\right)dy = \int 0\,dy$
$\frac{du}{dy} = C_1 \rightarrow u = C_1 y + C_2$

$(y=0, u=0) \rightarrow C_2 = 0$
$(y=d, u=V_p) \rightarrow C_1 = \frac{V_p}{d}$

TAT

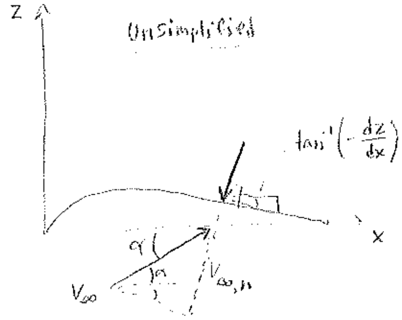
Simplified model



- at any point ξ, vortex strength $d\Gamma = \gamma(\xi)\,d\xi$
- $\perp$ velocity at $x \rightarrow dw = \dfrac{-\gamma(\xi)d\xi}{2\pi(x-\xi)}$
- $w(x) = \displaystyle\int_0^c \frac{-\gamma(\xi)}{2\pi(x-\xi)}\,d\xi$
- $V_{\infty,n} + V_{vortex} = 0$ ($\perp$ velocity $= 0$)

$V_\infty \sin\alpha - \dfrac{\Gamma_x}{2\pi(x-\xi)} = 0$

Unsimplified



$V_{\infty,n} = V_\infty \sin\left[\alpha - \tan^{-1}\left(\frac{dz}{dx}\right)\right]$

$V_{\infty,n} \approx V_\infty\left[\alpha - \frac{dz}{dx}\right]$

$V_\infty\left[\alpha - \frac{dz}{dx}\right] + \displaystyle\int_{\xi=0}^{c} \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$

$\rightarrow \gamma(\theta) = 2V_\infty\left[A_0\left(\frac{1+\cos\theta}{\sin\theta}\right) + \displaystyle\sum_{n=1}^{\infty} A_n \sin(n\theta)\right]$

$\boxed{\frac{\xi}{c} = \frac{1}{2}(1-\cos\theta)}$

$\boxed{A_0 = \alpha - \frac{1}{\pi}\int_{\theta=0}^{\pi} \frac{dz}{dx}\,d\theta}$ $\boxed{A_n = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos(n\theta)\,d\theta}$

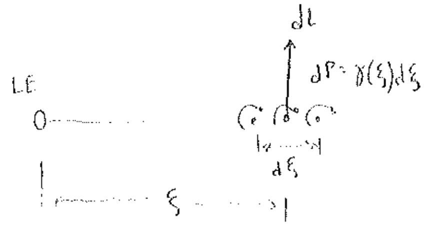
$C_\ell = 2\pi(\alpha - \alpha_{L=0})$

$\Gamma = \displaystyle\int_{\theta=0}^{\pi} \gamma(\theta)\,d\theta = \pi c V_\infty\left[A_0 + \frac{A_1}{2}\right] \rightarrow L = \rho V_\infty \Gamma$

$\rightarrow C_\ell = \pi(2A_0 + A_1)$

$\frac{dC_\ell}{d\alpha} = 2\pi$

$\boxed{\alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}(\cos\theta - 1)\,d\theta}$



$dL = \rho V_\infty\, d\Gamma b$ $dM_{LE} = -\xi\, dL$

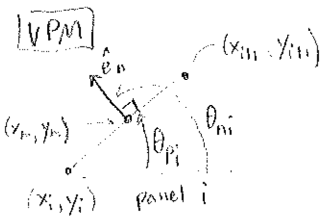
$C_{M,LE} = \frac{M_{LE}}{q_\infty c^2 b} = -\frac{2}{V_\infty c^2}\displaystyle\int_{\xi=0}^{c} \xi\gamma(\xi)\,d\xi$

$C_{M,LE} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)$

$\boxed{C_{M,c/4} = \frac{\pi}{4}(A_2 - A_1)}$

$\frac{dC_{m,ac}}{d\alpha} = 0 \rightarrow x_{ac} = \frac{c}{4}$

VPM

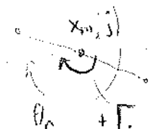


$\theta_{p_i} = \tan^{-1}\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)$

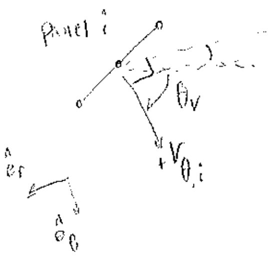
$\theta_{n_i} = \theta_{p_i} + \frac{\pi}{2}$

$\theta_r = \tan^{-1}\left(\frac{y_{m,i} - y_{m,j}}{x_{m,i} - x_{m,j}}\right)$

r_{ij}



$\theta_V = \theta_r + \frac{\pi}{2}$



$\hat{e}_\theta = \cos\theta_V\,\hat{i} + \sin\theta_V\,\hat{j}$

$\hat{e}_n = \cos\theta_n\,\hat{i} + \sin\theta_n\,\hat{j}$

$\vec{V}_\infty = V_\infty[\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}]$

$V_n = \vec{V}_\theta \cdot \hat{e}_n$

$V_n = \frac{-\Gamma}{2\pi r_{ij}}\left[\hat{e}_\theta \cdot \hat{e}_n\right]$

$V_{\infty,n} = V_\infty\left[\hat{e}_\infty \cdot \hat{e}_n\right]$

$[A]\{\Gamma\} = \{b\}$

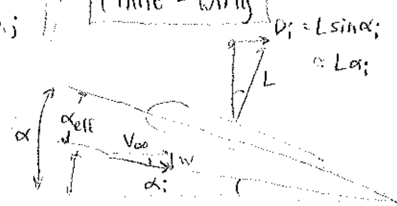
$\boxed{V_{\infty,n} + V_n = 0}$

$\begin{bmatrix} V_n \rightarrow \text{panel 1} \\ \rightarrow \text{panel 2} \\ \vdots \\ \rightarrow \text{panel } n \\ 0 \; 0 \ldots 1 \; 1 \ldots 0 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_n \end{bmatrix} = \begin{bmatrix} -V_{\infty,n,1} \\ \vdots \\ 0 \end{bmatrix}$

$\boxed{C_\ell = \frac{2\sum \Gamma_i}{cV_\infty}}$

Kutta: $\Gamma_{mid} + \Gamma_{mid+1} = 0$ $\displaystyle\sum_{\text{first}}^{\text{last}} \rightarrow$ mid, mid+1

Finite-wing

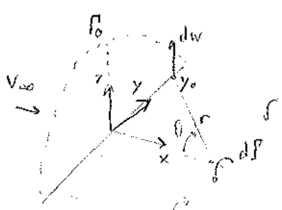


$\alpha_i = \frac{w}{V_\infty}$ $\alpha_{eff} + \alpha_i = \alpha$

$C_L = a_0(\alpha_{eff} - \alpha_{L=0})$

$C_L = a(\alpha - \alpha_{L=0})$

$a = a_0\left[1 + \frac{w/V_\infty}{C_L/a}\right]$



$d\Gamma = \left(\frac{d\Gamma}{dy}\right)dy$ $dw = \frac{-\left(\frac{d\Gamma}{dy}\right)dy}{4\pi(y_0 - y)}$

elliptical

$\Gamma(y) = \Gamma_0\left[1 - \left(\frac{2y}{b}\right)^2\right]^{1/2}$ $w(y_0) = \displaystyle\int_{-b/2}^{b/2} dw$

$L = \rho V_\infty \int \Gamma(y)\,dy = \rho V_\infty \Gamma_0 \frac{\pi b}{4}$ $w(y_0) = -\frac{\Gamma_0}{2b}$

$\boxed{a = \frac{a_0}{1 + \frac{a_0}{\pi e AR}}}$ $e = \frac{1}{1+\delta}$

$\delta \sim f\left(AR, \frac{c_t}{c_r}\right)$

$\boxed{C_{D_i} = \frac{C_L^2}{\pi e AR}}$ $e = 1$ for elliptical wing

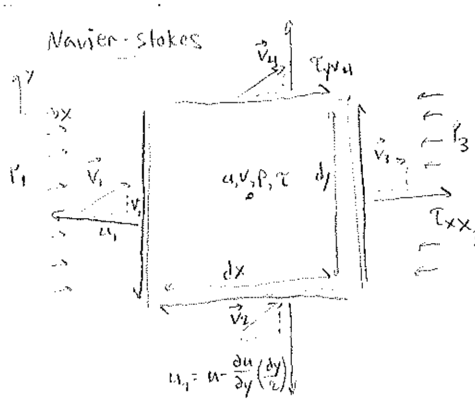
$C_L = \frac{\Gamma_0 \pi b}{2 S V_\infty}$

$D = q_\infty S(C_{D,0} + C_{D,i})$

$D = qSC_{D,0} + \frac{W^2}{\pi AR\, q S}$

parasitic induced

Navier-Stokes



pressure: $(P_1 - P_2)\,dy\,dz$

mass flow: $(\rho u_1 dy dz)u_1 + (\rho v_2 dx dz)u_2 - (\rho u_3 dy dz)u_3 - (\rho v_4 dx dz)u_4$

friction: $\tau_{xx_3} dy dz - \tau_{xx_1} dy dz + \tau_{yx_4} dx dz - \tau_{yx_2} dx dz$

$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$ $\tau_{yx} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$

color x:

$\rho\frac{\partial u}{\partial t} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]}_{\text{viscous}} \underbrace{- \rho u\frac{\partial u}{\partial x} - \rho v\frac{\partial u}{\partial y}}_{mV_x}$ $\rightarrow$ assume: incompressible, inviscous, x-uniform

$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \longrightarrow \frac{\partial u}{\partial x}^{\,0} + \frac{\partial v}{\partial y} = 0$, $v = 0$ everywhere

$u(t+\Delta t) = u(t) + \left[\frac{\Delta t\,\mu}{\rho\,\Delta y^2}\right]\left[u(y+\Delta y) + u(y-\Delta y) - 2u(y)\right]$

CFL #