Risk Decomposition under FRTB: An Application of Euler's Theorem

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Abstract

This paper applies Euler's Theorem to portfolio standard deviation and the Fundamental Review of the Trading Book (FRTB) bucket-level capital requirements. Euler's Theorem provides a framework to decompose portfolio risk and capital requirements into marginal contributions, enhancing the understanding of risk allocation across individual assets and risk factors. We derive closed-form solutions for Marginal Contributions to Risk (MCR) and Contributions to Risk (CR) in both the portfolio and FRTB contexts, and we outline the implications of these decompositions for risk management and regulatory capital.

1 Introduction

The decomposition of risk in portfolios and capital requirements has become critical under recent regulatory frameworks such as the Fundamental Review of the Trading Book (FRTB). Euler's Theorem offers a natural framework to achieve this decomposition by leveraging the homogeneity properties of risk functions, such as the portfolio standard deviation. In this paper, we explore how Euler's Theorem can be applied to the standard deviation of portfolios and capital requirements in the FRTB framework. We provide explicit formulations for Marginal Contributions to Risk (MCR) and Contributions to Risk (CR) that can be readily implemented in risk management systems.

2 Euler's Theorem and Portfolio Standard Deviation

Euler's Theorem states that any homogeneous function of degree one can be expressed as the weighted sum of its partial derivatives with respect to each input. We begin by applying this theorem to the portfolio standard deviation, a key measure of risk in portfolio management.

2.1 Portfolio Standard Deviation

The standard deviation of a portfolio, denoted as $\sigma_p(x)$, is defined as:

$$\sigma_p(x) = \sqrt{x'\Sigma x} \tag{1}$$

where x is the vector of asset weights, Σ is the covariance matrix of asset returns, and $x'\Sigma x$ represents the portfolio variance.

2.2 Homogeneity of Degree One

To verify that the portfolio standard deviation is homogeneous of degree one, consider the scaling of the portfolio weights x by a constant c:

$$\sigma_p(c \cdot x) = \sqrt{(c \cdot x)'\Sigma(c \cdot x)} = c \cdot \sigma_p(x) \tag{2}$$

This confirms that $\sigma_p(x)$ is homogeneous of degree one.

2.3 Euler's Theorem Applied to Portfolio Risk

Euler's Theorem implies that for a homogeneous function of degree one, the function can be expressed as:

$$\sigma_p(x) = \sum_{i=1}^{N} x_i \frac{\partial \sigma_p(x)}{\partial x_i} \tag{3}$$

or, in matrix notation:

$$\sigma_p(x) = x' \nabla_x \sigma_p(x) \tag{4}$$

where $\nabla_x \sigma_p(x)$ is the gradient of $\sigma_p(x)$ with respect to the vector of asset weights x.

2.4 Risk Decomposition

The partial derivative of the portfolio standard deviation with respect to x_i is given by:

$$\frac{\partial \sigma_p(x)}{\partial x_i} = \frac{1}{\sigma_p(x)} \sum_{j=1}^{N} \Sigma_{ij} x_j \tag{5}$$

Thus, the portfolio standard deviation can be decomposed as:

$$\sigma_p(x) = \sum_{i=1}^{N} x_i \frac{1}{\sigma_p(x)} \sum_{j=1}^{N} \Sigma_{ij} x_j \tag{6}$$

This decomposition provides insight into the marginal contribution of each asset to the overall portfolio risk.

3 Euler's Theorem and FRTB Capital Requirements

FRTB introduces a bucket-level capital requirement, K_b, which, like portfolio standard deviation, can be decomposed using Euler's Theorem.

3.1 FRTB Bucket-Level Capital Requirement

The capital requirement for a bucket of risk factors under FRTB is given by:

$$K_b = \sqrt{s_b' \Sigma_b s_b} \tag{7}$$

where s_b is the vector of sensitivities to risk factors within bucket b, and Σ_b is the correlation matrix for bucket b.

3.2 Marginal Contribution to Risk (MCR)

We define the Marginal Contribution to Risk (MCR) for each sensitivity s_i within the bucket as the partial derivative of K_b with respect to s_i:

$$\frac{\partial K_b}{\partial s_i} = \frac{1}{K_b} \sum_{j=1}^{N} \Sigma_{ij} s_j \tag{8}$$

In matrix notation:

$$\frac{\partial K_b}{\partial s} = \frac{1}{K_b} \Sigma_b s_b \tag{9}$$

3.3 Contribution to Risk (CR)

The Contribution to Risk (CR) for each sensitivity s_i is defined as the product of s_i and its MCR:

$$\mathrm{CR}_i = s_i \cdot \frac{1}{K_b} \sum_{j=1}^{N} \Sigma_{ij} s_j \tag{10}$$

In matrix form, the CR vector is:

$$\mathrm{CR} = s_b \circ \left(\frac{\Sigma_b s_b}{K_b} \right) \tag{11}$$

where $\circ$ denotes the Hadamard product (element-wise product).

4 Aggregating Risk across Buckets and Classes

The next step in FRTB is to aggregate the capital requirements across buckets and risk classes.

4.1 Class-Level Capital Requirement

The class-level capital requirement is given by:

$$K_{\text{class}} = \sqrt{K'\mathbf{\Gamma}K} \tag{12}$$

where K is the vector of bucket-level capital requirements, and $\mathbf{\Gamma}$ is the correlation matrix between buckets.

4.2 Marginal Contribution to Class-Level Risk

The MCR for each bucket-level capital requirement K_b is the partial derivative of K_{class} with respect to K_b:

$$\frac{\partial K_{\text{class}}}{\partial K_b} = \frac{1}{K_{\text{class}}} \sum_{c=1}^{B} \gamma_{bc} K_c \tag{13}$$

5 Applying Euler's Theorem to Portfolio Standard Deviation

Euler's theorem states that for a homogeneous function of degree one, the function can be decomposed into the weighted sum of its partial derivatives with respect to each of its inputs. In this case, we apply Euler's theorem to the formula for portfolio standard deviation $\sigma_p(x)$.

5.1 Step 1: Formula for Portfolio Standard Deviation

The standard deviation (volatility) of a portfolio, $\sigma_p(x)$, is given by:

$$\sigma_p(x) = \sqrt{x'\Sigma x}$$

Where:

- x is the vector of asset weights $(x_1, x_2, \ldots, x_N)$,
- Σ is the covariance matrix of asset returns,
- $x'\Sigma x$ is the portfolio variance.

5.2 Step 2: Homogeneity of Degree One

To confirm that the portfolio standard deviation is homogeneous of degree one, we check if scaling the weights x by a constant c scales $\sigma_p(x)$ by c as well:

$$\sigma_p(c \cdot x) = \sqrt{(c \cdot x)'\Sigma(c \cdot x)} = \sqrt{c^2 \cdot (x'\Sigma x)} = c \cdot \sqrt{x'\Sigma x} = c \cdot \sigma_p(x)$$

Thus, the portfolio standard deviation $\sigma_p(x)$ is homogeneous of degree one in the portfolio weights x.

5.3 Step 3: Euler's Theorem

Euler's theorem tells us that for a function homogeneous of degree one, it can be written as:

$$\sigma_p(x) = x_1 \frac{\partial \sigma_p(x)}{\partial x_1} + x_2 \frac{\partial \sigma_p(x)}{\partial x_2} + \cdots + x_N \frac{\partial \sigma_p(x)}{\partial x_N}$$

Or, in matrix notation:

$$\sigma_p(x) = x' \frac{\partial \sigma_p(x)}{\partial x}$$

5.4 Step 4: Partial Derivative of $\sigma_p(x)$

We now compute the partial derivative of the portfolio standard deviation with respect to each asset weight x_i. Starting with the formula:

$$\sigma_p(x) = \sqrt{x'\Sigma x}$$

The derivative of $\sigma_p(x)$ with respect to x_i is:

$$\frac{\partial \sigma_p(x)}{\partial x_i} = \frac{1}{2}(x'\Sigma x)^{-\frac{1}{2}} \cdot \frac{\partial (x'\Sigma x)}{\partial x_i}$$

Next, we compute the partial derivative of $x'\Sigma x$ with respect to x_i:

$$\frac{\partial (x'\Sigma x)}{\partial x_i} = 2 \sum_{j=1}^{N} \Sigma_{ij} x_j$$

Thus, the partial derivative of $\sigma_p(x)$ with respect to x_i is:

$$\frac{\partial \sigma_p(x)}{\partial x_i} = \frac{1}{\sigma_p(x)} \sum_{j=1}^{N} \Sigma_{ij} x_j$$

5.5 Step 5: Applying Euler's Theorem

Using Euler's theorem, the total portfolio standard deviation can be expressed as:

$$\sigma_p(x) = \sum_{i=1}^{N} x_i \frac{1}{\sigma_p(x)} \sum_{j=1}^{N} \Sigma_{ij} x_j$$

This simplifies to:

$$\sigma_p(x) = \frac{1}{\sigma_p(x)} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \Sigma_{ij} x_j$$

Recognizing that $\sum_{i=1}^{N} \sum_{j=1}^{N} x_i \Sigma_{ij} x_j = x'\Sigma x = \sigma_p^2(x)$, we obtain:

$$\sigma_p(x) = \sigma_p(x)$$

This verifies that Euler's theorem holds for $\sigma_p(x)$.

5.6 Step 6: Risk Decomposition

We can now decompose the total portfolio standard deviation into the contributions from each asset:

$$\sigma_p(x) = \sum_{i=1}^{N} x_i \cdot \frac{\sum_{j=1}^{N} \Sigma_{ij} x_j}{\sigma_p(x)}$$

Here:

- $\frac{\sum_{j=1}^{N} \Sigma_{ij} x_j}{\sigma_p(x)}$ is the *marginal contribution to risk (MCR)* of asset i.
- $x_i \cdot \frac{\sum_{j=1}^{N} \Sigma_{ij} x_j}{\sigma_p(x)}$ is the *contribution to risk (CR)* of asset i, which represents the portion of the total portfolio risk attributable to asset i.

5.7 Step 7: Matrix Notation Restatement

We can restate the previous derivations more concisely using matrix notation. Consider the portfolio standard deviation formula $\sigma_p(x) = \sqrt{x'\Sigma x}$. To apply Euler's theorem in matrix form, we use the gradient of $\sigma_p(x)$ with respect to the vector of asset weights x, denoted as $\nabla_x \sigma_p(x)$.

The gradient of the quadratic form $x'\Sigma x$ with respect to x is given by:

$$\nabla_x (x'\Sigma x) = 2\Sigma x$$

Thus, the partial derivatives of $\sigma_p(x)$ with respect to each weight can be written as:

$$\nabla_x \sigma_p(x) = \frac{1}{2}(x'\Sigma x)^{-\frac{1}{2}} \cdot 2\Sigma x = \frac{\Sigma x}{\sigma_p(x)}$$

Now, using Euler's theorem, we express the portfolio standard deviation $\sigma_p(x)$ as a weighted sum of the partial derivatives:

$$\sigma_p(x) = x' \nabla_x \sigma_p(x)$$

Substituting the gradient $\nabla_x \sigma_p(x)$ into this expression, we get:

$$\sigma_p(x) = x' \frac{\Sigma x}{\sigma_p(x)}$$

Multiplying both sides by $\sigma_p(x)$, we recover the original expression for the portfolio variance:

$$\sigma_p^2(x) = x'\Sigma x$$

This confirms that Euler's theorem holds for the portfolio standard deviation in matrix form. Furthermore, the matrix notation restatement simplifies the expression for the contribution of each asset to portfolio risk, allowing for efficient computation in practical applications.

6 Applying Euler's Theorem to Portfolio Standard Deviation and FRTB

Euler's theorem states that for a homogeneous function of degree one, the function can be decomposed into the weighted sum of its partial derivatives with respect to each of its inputs. In this case, we apply Euler's theorem to the formula for portfolio standard deviation $\sigma_p(x)$ and the FRTB bucket-level capital requirement K_b.

6.1 Step 1: Formula for Portfolio Standard Deviation

The standard deviation (volatility) of a portfolio, $\sigma_p(x)$, is given by:

$$\sigma_p(x) = \sqrt{x'\Sigma x}$$

Where:

- x is the vector of asset weights $(x_1, x_2, \ldots, x_N)$,
- Σ is the covariance matrix of asset returns,
- $x'\Sigma x$ is the portfolio variance.

6.2 Step 2: Formula for FRTB Bucket-Level Capital Requirement K_b

Similarly, in the context of FRTB Sensitivities-Based Method, the capital requirement within a risk bucket is given by:

$$K_b = \sqrt{s_b'\Sigma_b s_b}$$

Where:

- s_b is the vector of sensitivities to risk factors within bucket b,
- Σ_b is the correlation matrix for bucket b,
- K_b is the total capital requirement for bucket b.

6.3 Step 3: Marginal Contribution to Risk (MCR) in FRTB

To compute the Marginal Contribution to Risk (MCR) for each sensitivity s_i within the bucket, we differentiate K_b with respect to s_i. The MCR measures how much each individual risk sensitivity contributes to the overall capital requirement.

The derivative of K_b with respect to s_i is:

$$\frac{\partial K_b}{\partial s_i} = \frac{1}{2} \cdot \frac{1}{K_b} \cdot \frac{\partial \left(s_b' \Sigma_b s_b \right)}{\partial s_i}$$

Since $\frac{\partial}{\partial s_i} \left(s_b' \Sigma_b s_b \right) = 2 \sum_{j=1}^{N} \Sigma_{ij} s_j$, we obtain:

$$\frac{\partial K_b}{\partial s_i} = \frac{1}{K_b} \sum_{j=1}^{N} \Sigma_{ij} s_j$$

In matrix notation, this is compactly expressed as:

$$\frac{\partial K_b}{\partial s} = \frac{1}{K_b} \Sigma_b s_b$$

This represents the vector of marginal contributions to risk (MCR) for all sensitivities in bucket b.

6.4 Step 4: Contribution to Risk (CR) in FRTB

The Contribution to Risk (CR) for each sensitivity s_i is given by the product of the sensitivity s_i and its Marginal Contribution to Risk (MCR):

$$\mathrm{CR}_i = s_i \cdot \frac{1}{K_b} \sum_{j=1}^{N} \Sigma_{ij} s_j$$

In matrix notation, the CR vector is written as:

$$\mathrm{CR} = s_b \circ \left(\frac{\Sigma_b s_b}{K_b} \right)$$

Where $\circ$ denotes the element-wise (Hadamard) product. This gives the portion of the total capital requirement K_b attributable to each sensitivity s_i.

6.5 Step 5: Euler's Theorem Restated for FRTB

Euler's theorem tells us that for a homogeneous function of degree one, the function can be written as the sum of its inputs weighted by their partial derivatives. For the bucket-level capital requirement K_b, we can express this as:

$$K_b = s_b' \frac{\partial K_b}{\partial s}$$

Substituting $\frac{\partial K_b}{\partial s} = \frac{\Sigma_b s_b}{K_b}$, we obtain:

$$K_b = s_b' \frac{\Sigma_b s_b}{K_b}$$

Multiplying both sides by K_b, we recover the original formula for K_b:

$$K_b^2 = s_b' \Sigma_b s_b$$

This confirms that Euler's theorem holds for the bucket-level capital requirement in FRTB, and the total capital requirement can be decomposed into the sum of marginal contributions from each sensitivity.

6.6 Step 6: Comparison with Portfolio Standard Deviation

The FRTB capital requirement formula $K_b = \sqrt{s_b' \Sigma_b s_b}$ is analogous to the portfolio risk formula $\sigma_p = \sqrt{x' \Sigma x}$, where the vector s_b represents sensitivities in FRTB, and x represents asset weights in a portfolio. Both involve a quadratic form of the inputs (sensitivities or asset weights) and a correlation (or covariance) matrix.

6.7 Step 7: Class-Level Capital Requirement K_{class} and Marginal Risk Contribution

After aggregating risk sensitivities within buckets, the next step is to aggregate across buckets to compute the class-level capital requirement K_{class}. This aggregation is done using a correlation matrix $\mathbf{\Gamma}$ for buckets within a risk class.

The class-level capital requirement is defined as:

$$K_{\text{class}} = \sqrt{K' \mathbf{\Gamma} K}$$

Where:

- K is the vector of bucket-level capital requirements within the class,
- $\mathbf{\Gamma}$ is the correlation matrix between the buckets.

6.7.1 Marginal Contribution to Risk (MCR) for Class-Level Capital Requirement

The Marginal Contribution to Risk (MCR) for each bucket-level capital requirement K_b is the partial derivative of K_{class} with respect to K_b:

$$\frac{\partial K_{\text{class}}}{\partial K_b} = \frac{1}{K_{\text{class}}} \sum_{c=1}^{B} \gamma_{bc} K_c$$

In matrix notation, this can be restated as:

$$\frac{\partial K_{\text{class}}}{\partial K} = \frac{1}{K_{\text{class}}}\mathbf{\Gamma}K$$

This expression gives us the vector of marginal risk contributions for all buckets in the class.

6.7.2 Contribution to Risk (CR) for Class-Level Capital Requirement

The Contribution to Risk (CR) for each bucket K_b is computed by multiplying the bucket-level capital requirement K_b by its corresponding MCR:

$$\text{CR}_b = K_b \cdot \frac{1}{K_{\text{class}}} \sum_{c=1}^{B} \gamma_{bc} K_c$$

In matrix form, this becomes:

$$\text{CR} = K \circ \left(\frac{\mathbf{\Gamma}K}{K_{\text{class}}} \right)$$

Where:

- CR is the vector of contributions to risk for each bucket,
- K is the vector of bucket-level capital requirements,
- $\circ$ denotes the Hadamard product (element-wise product).

6.8 Step 8: Total Capital Requirement K_{total} and Marginal Risk Contribution

The total capital requirement K_{total} is the aggregation of the class-level capital requirements across all risk classes. In some cases, it may involve selecting the maximum capital requirement across delta, vega, and curvature risks:

$$K_{\text{total}} = \max\left(K_{\text{delta}}, K_{\text{vega}}, K_{\text{curvature}}\right)$$

Alternatively, if aggregation follows a correlation structure similar to that for class-level requirements, the total capital requirement is computed as:

$$K_{\text{total}} = \sqrt{K'\mathbf{P}K}$$

Where:

- K is the vector of class-level capital requirements,
- $\mathbf{P}$ is the correlation matrix between the risk classes.

6.8.1 Marginal Contribution to Risk (MCR) for Total Capital Requirement

The Marginal Contribution to Risk (MCR) for each class K_k is the partial derivative of K_{total} with respect to K_k:

$$\frac{\partial K_{\text{total}}}{\partial K_k} = \frac{1}{K_{\text{total}}} \sum_{l=1}^{N} \rho_{kl} K_l$$

In matrix notation, this can be written as:

$$\frac{\partial K_{\text{total}}}{\partial K} = \frac{1}{K_{\text{total}}} \mathbf{P} K$$

6.8.2 Contribution to Risk (CR) for Total Capital Requirement

The Contribution to Risk (CR) for each class K_k is the product of the class-level capital K_k and its MCR:

$$\text{CR}_k = K_k \cdot \frac{1}{K_{\text{total}}} \sum_{l=1}^{N} \rho_{kl} K_l$$

In matrix form, this becomes:

$$\text{CR} = K \circ \left(\frac{\mathbf{P} K}{K_{\text{total}}} \right)$$

Where:

- CR is the vector of contributions to risk for each risk class,
- K is the vector of class-level capital requirements,
- $\mathbf{P}K$ represents the weighted sum of correlations across classes,
- $\circ$ denotes the Hadamard product.

6.9 Conclusion: Risk Decomposition using Hadamard Product

Using the Hadamard product, we are able to efficiently compute the Marginal Contribution to Risk (MCR) and Contribution to Risk (CR) at both the class and total levels. These matrix formulations allow for systematic risk decomposition at each aggregation step, while ensuring consistency with Euler's theorem.

7 Notes on Practical Implementation and Interpretation

A comparison may be drawn between FRTB and the Lun-class ekranoplan (a Soviet ground-effect vehicle). The Ekranoplan had a maximum flying altitude of 4 to 10 meters, while requiring eight engines and 28,600 lbf (127.4kN) of thrust force.