

Non-Standard L -Functions Associated with Higher-Dimensional Homotopy Data

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Abstract

In this paper, we propose a speculative extension of the Langlands Program through the introduction of non-standard L -functions associated with higher-dimensional homotopy data. We extend the classical framework of automorphic forms and Galois representations by incorporating ∞ -category theory and conjecturing a new class of L -functions. These non-standard L -functions arise from higher-dimensional cohomology groups and extend functoriality beyond reductive groups. We explore the formalization of key components using the Lean theorem prover, ensuring rigorous validation of the axioms and conjectures presented. The extended framework also suggests potential connections with quantum field theory (QFT), particularly in the context of dualities and partition functions. This paper aims to establish a foundation for future research into the interplay between higher-category theory, automorphic forms, and mathematical physics.

1 Introduction

1.1 Context and Motivation

The Langlands Program, established through the groundbreaking work of Robert Langlands, has provided profound connections between several key areas of mathematics, including number theory, automorphic forms, and representation theory. Central to the program is the deep relationship between automorphic representations of reductive groups and Galois representations, with L -functions acting as bridges between these domains.

While the Langlands Program has been remarkably successful, much of its current framework remains confined to reductive groups and certain well-understood cases of functoriality. In particular, the functoriality conjecture—asserting the transfer of automorphic forms across different groups—has yet to be

fully realized, especially in more generalized settings involving non-reductive groups and higher-dimensional structures.

Recent advances in higher category theory and homotopy theory suggest that a broader framework could be developed by extending the scope of the Langlands Program to encompass ∞ -categories and non-standard geometric objects. These theories, particularly those involving ∞ -categories, offer a more nuanced understanding of functoriality by capturing higher-order morphisms and their relationships. Moreover, speculative connections between the Langlands Program and quantum field theory (QFT), especially in the context of dualities, offer an exciting new direction of research.

In this paper, we propose an extension of the Langlands Program by introducing non-standard L -functions associated with higher-dimensional homotopy data, and we explore the implications of ∞ -category theory on automorphic forms. We hypothesize that this extended framework can generalize functoriality beyond reductive groups and has the potential to uncover new connections between number theory and modern mathematical physics. We also demonstrate the use of the Lean theorem prover for formalizing certain axiomatic components, providing rigorous proof verification as a tool for future research.

1.2 Objective and Contributions

The objective of this paper is to extend the classical Langlands Program by introducing new structures based on ∞ -category theory and higher-dimensional homotopy data. The classical Langlands Program has focused primarily on automorphic representations and their relationships with Galois representations through the mediation of L -functions, particularly for reductive groups. However, several conjectures, such as functoriality, remain unproven in more generalized settings involving non-reductive groups and higher categories. Our primary aim is to propose a framework that incorporates these more generalized structures while remaining grounded in the core principles of the Langlands Program.

The key contributions of this paper are as follows:

- **Non-Standard L -Functions:** We introduce a new class of L -functions, referred to as non-standard L -functions, which extend the classical concept to higher-dimensional geometric objects. These L -functions are constructed using higher cohomological and homotopy-theoretic data.
- **Functoriality in ∞ -Categories:** We propose an extension of the functoriality conjecture, leveraging ∞ -category theory to capture the higher-order relationships between automorphic forms and Galois representations. This conjecture aims to address functorial transfers beyond reductive groups, generalizing the existing framework.
- **Formalization Using Lean:** To provide rigorous validation of our axiomatic framework, we implement key aspects of the theory in the Lean theorem prover. This approach ensures that our extended framework

maintains internal consistency and opens up the possibility for further formal verification of conjectures related to automorphic forms and L -functions.

- **Connections to Quantum Field Theory:** We hypothesize a speculative connection between the proposed non-standard L -functions and quantum field theory (QFT). Specifically, we suggest that these L -functions can be interpreted as partition functions in certain quantum field theories, particularly in the context of dualities and higher-dimensional geometric objects.
- **Path Forward:** Finally, we outline potential avenues for future research, including the development of a broader theory connecting higher-dimensional automorphic representations with modern mathematical physics. We encourage further exploration of the implications of ∞ -category theory for the Langlands Program and suggest that formal methods like Lean may be crucial in testing and validating future conjectures.

These contributions set the stage for a speculative yet mathematically grounded exploration into the extensions of the Langlands Program, aiming to unlock new relationships across number theory, higher geometry, and theoretical physics.

1.3 Structure of the Paper

The structure of this paper is organized to progressively develop the ideas and frameworks required for extending the Langlands Program. We begin with foundational background material, followed by the introduction of new speculative concepts and formalism, concluding with open problems and potential connections to mathematical physics. The structure is as follows:

- **Section 2: The Langlands Program – A Brief Review.** In this section, we provide a concise overview of the classical Langlands Program, including its foundational concepts such as automorphic forms, Galois representations, and L -functions. We highlight the role of functoriality and discuss the limitations of the current framework, particularly in relation to non-reductive groups and higher categories.
- **Section 3: Higher Categories and ∞ -Categories – A Primer.** This section introduces the concept of ∞ -categories, providing the necessary background for understanding their role in the extended Langlands framework. We explain how higher morphisms and ∞ -categories generalize classical category theory, setting the stage for their application to automorphic forms and Galois representations.
- **Section 4: Extended Automorphy in the ∞ -Categorical Framework.** In this section, we propose the notion of automorphic representations as objects in ∞ -categories and conjecture an extension of the functoriality principle within this framework. We discuss the technical challenges

and propose formal methods for validating the internal consistency of this approach using the Lean theorem prover.

- **Section 5: Non-Standard L -Functions and Higher-Dimensional Homotopy Data.** Here, we introduce a new class of L -functions—referred to as non-standard L -functions—that incorporate higher-dimensional homotopy data through étale cohomology groups and Hecke operators. We define these functions formally and propose a conjectural global functional equation for them, alongside a formalization in Lean.
- **Section 6: Functoriality for Non-Reductive Groups.** This section focuses on extending the functoriality conjecture to non-reductive groups. We propose a new functorial transfer from automorphic representations of reductive subgroups to those of non-reductive groups, employing derived categories and D-modules on flag varieties as key tools. We also formalize these ideas using Lean for rigorous validation.
- **Section 7: Connections to Quantum Field Theory and Beyond.** In this section, we explore potential connections between the extended Langlands framework and quantum field theory (QFT). We hypothesize that the non-standard L -functions proposed in this paper may correspond to partition functions in certain QFTs, especially in the context of dualities and higher-dimensional geometric objects. Speculative connections to string theory and mirror symmetry are also discussed.
- **Section 8: Open Problems and Conjectures.** This section outlines the key open problems and conjectures posed by our proposed framework. We restate the non-standard global functional equation and functoriality conjectures, and suggest future research directions, both in mathematics and mathematical physics.
- **Section 9: Conclusion.** The final section summarizes the key contributions of the paper and discusses the broader implications of the proposed extensions to the Langlands Program. We provide a forward-looking perspective on the potential of ∞ -category theory and formal methods like Lean in future mathematical research.
- **Appendices: Lean Code and Mathematical Background (optional).** The appendices include supplementary material, such as the complete Lean code for the formalization of key concepts, and additional background on ∞ -categories and étale cohomology for readers less familiar with these areas.

2 The Langlands Program: A Brief Review

2.1 Classical Langlands Program

The Langlands Program, first proposed by Robert Langlands in the late 1960s, is a vast web of conjectures and theorems that connects number theory, representation theory, and algebraic geometry. Its core aim is to establish deep relationships between automorphic forms, which are highly symmetric functions on arithmetic groups, and Galois representations, which describe symmetries in solutions to polynomial equations. The program's key innovation is its ability to link these two disparate areas through L -functions, which encode significant arithmetic and geometric information.

The classical Langlands Program focuses primarily on the following components:

- **Automorphic Representations:** Automorphic representations are realized as irreducible constituents of the space of automorphic forms, which are functions defined on the adèles of a reductive algebraic group. They play a central role in the Langlands Program as they generalize classical modular forms and enable the understanding of symmetries in arithmetic objects.
- **Galois Representations:** These representations describe the action of the absolute Galois group, $\text{Gal}(\overline{F}/F)$, on the solutions to polynomial equations over a number field F . Galois representations provide an essential algebraic structure for understanding the symmetries and arithmetic properties of algebraic varieties and number fields.
- **L -Functions:** L -functions act as a bridge between automorphic forms and Galois representations. These functions, defined as infinite Euler products, generalize the Riemann zeta function and capture profound arithmetic data. The Langlands Program conjectures that L -functions associated with automorphic forms correspond to L -functions associated with Galois representations, thus establishing a correspondence between these two types of mathematical objects.

The central conjecture of the Langlands Program is the *functoriality principle*, which predicts a transfer of automorphic representations between different reductive groups, mediated by L -functions. This principle suggests that there exists a natural, functorial correspondence between automorphic forms on one group and those on another group, corresponding to homomorphisms between their L -groups (certain groups related to their representations).

A key success of the Langlands Program has been its application to $\text{GL}(n)$. For example, the modularity theorem, proven by Andrew Wiles in 1994, which resolved Fermat's Last Theorem, can be viewed as a special case of Langlands reciprocity for $\text{GL}(2)$. This theorem established that every elliptic curve over \mathbb{Q} is modular, meaning its L -function coincides with that of a modular form.

Despite its successes, the Langlands Program in its classical form remains incomplete. Key conjectures, such as the functoriality principle for general reductive groups, are still unproven. Additionally, much of the framework has not yet been extended to incorporate non-reductive groups or higher-dimensional geometric objects. This limitation motivates the exploration of new mathematical tools, such as ∞ -categories and higher-dimensional cohomology, to further develop the program.

In summary, the classical Langlands Program has transformed modern number theory by providing deep connections between automorphic forms, Galois representations, and L -functions. However, the need for generalizations beyond its current scope forms the basis for the extensions proposed in this paper, particularly through the incorporation of higher categorical structures and new classes of L -functions.

2.2 Functoriality and Its Limits

The functoriality principle is one of the central conjectures of the Langlands Program, aiming to describe how automorphic representations on one reductive group should correspond to automorphic representations on another group. Specifically, functoriality posits that given a homomorphism between the L -groups of two reductive groups, there should exist a transfer of automorphic representations between them, preserving significant arithmetic and geometric information. This conjecture, if fully realized, would provide a universal framework linking various forms of symmetry in number theory and geometry.

Mathematically, functoriality is often expressed in terms of a map between L -groups:

$$\phi : {}^L H \rightarrow {}^L G,$$

where ${}^L G$ denotes the L -group associated with a reductive group G . The conjecture predicts that for an automorphic representation π of H , there exists a corresponding automorphic representation Π of G such that their L -functions are related:

$$L(s, \pi) = L(s, \Pi).$$

This transfer of automorphic representations preserves important structures, including the Euler product decomposition of the L -function and local factors at primes.

2.2.1 Successes of Functoriality

The functoriality conjecture has been proven in several important cases, notably for the general linear groups $\mathrm{GL}(n)$. One of the major breakthroughs was the proof of the local Langlands correspondence for $\mathrm{GL}(n)$, which establishes a bijection between irreducible admissible representations of $\mathrm{GL}(n)$ over a local field and n -dimensional representations of the local Galois group. This result has wide-reaching implications for number theory and has been extended to many special cases of reductive groups.

Another key achievement is the proof of functoriality in the case of endoscopic groups, where the Fundamental Lemma, proven by Ngô Bảo Châu in 2008, played a crucial role. This result opened the door to further explorations of functoriality, particularly in the context of automorphic forms on classical groups.

2.2.2 Limitations of Functoriality

Despite these successes, functoriality remains a largely open conjecture in full generality. The classical framework of functoriality is limited to reductive groups and does not yet extend to non-reductive groups or settings where higher-dimensional geometric or homotopy-theoretic structures play a role. This limitation suggests that functoriality may require new mathematical tools to fully realize its potential.

In addition to these structural limitations, functoriality is known to face technical challenges, such as the difficulty of proving the existence of transfers in more general cases. The lack of a uniform method for establishing functoriality for all reductive groups has been a major obstacle. Moreover, the connection between functoriality and the global Langlands correspondence remains conjectural, leaving open questions about the precise nature of this relationship.

2.2.3 Higher-Categorical Extensions of Functoriality

Given the limitations of the current framework, the need for new perspectives has become increasingly clear. One promising direction involves extending functoriality beyond reductive groups using higher categories and homotopy theory. The development of ∞ -categories, which generalize ordinary categories by allowing higher morphisms, offers a potential framework to model more complex relationships between automorphic forms and Galois representations.

In particular, higher categorical structures could provide a more nuanced view of functoriality, allowing for the incorporation of geometric and homotopy data. By viewing automorphic representations and Galois representations as objects in ∞ -categories, it may be possible to extend the functoriality principle to cases that involve non-reductive groups or even quantum field-theoretic structures. This paper explores such extensions by proposing new forms of functoriality in the context of non-standard L -functions and higher-dimensional homotopy theory.

2.2.4 Connections to Quantum Field Theory

Another frontier where functoriality may find extensions is quantum field theory (QFT), where duality principles often suggest deep symmetries between different mathematical structures. The geometric Langlands Program has already drawn parallels between the Langlands duality and electric-magnetic duality in QFT, particularly in the work of Kapustin and Witten. We hypothesize that

further connections could emerge between non-standard L -functions and partition functions in certain QFTs, where functoriality principles may manifest as dualities between different physical theories.

Thus, while the classical framework of functoriality has seen tremendous progress, it is far from complete. Extending the conjecture to new mathematical settings, such as higher categories and quantum field theory, could unlock further insights into the Langlands Program and its connections to other areas of mathematics and physics.

3 Higher Categories and ∞ -Categories: A Primer

3.1 From Categories to ∞ -Categories

The concept of categories is foundational in modern mathematics, providing a framework for studying mathematical structures and the relationships between them. A *category* consists of objects and morphisms (arrows) between these objects, together with two basic operations: composition of morphisms and identity morphisms. Categories have been a cornerstone in fields such as algebraic geometry, topology, and representation theory, and have played a crucial role in the development of the Langlands Program.

However, traditional category theory is often limited when dealing with more complex mathematical phenomena that involve transformations between morphisms, and not just between objects. For example, in homotopy theory, where spaces are studied up to continuous deformation, one often encounters higher-dimensional relationships that go beyond simple morphisms between objects. This limitation has led to the development of higher category theory, and more specifically, the notion of ∞ -categories.

3.1.1 Basic Definitions of Categories

To understand ∞ -categories, it is helpful to recall the basic structure of ordinary categories. A *category* \mathcal{C} consists of the following:

- A collection of **objects**, denoted $\text{Obj}(\mathcal{C})$.
- For each pair of objects $X, Y \in \text{Obj}(\mathcal{C})$, a set of **morphisms** (or arrows) $\text{Hom}_{\mathcal{C}}(X, Y)$ between X and Y .
- An identity morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ for each object $X \in \mathcal{C}$, such that $\text{id}_X \circ f = f$ and $g \circ \text{id}_X = g$ for any morphisms $f : Y \rightarrow X$ and $g : X \rightarrow Z$.
- A composition rule for morphisms: if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then there is a morphism $g \circ f : X \rightarrow Z$, which satisfies the associativity condition $(h \circ g) \circ f = h \circ (g \circ f)$ for any morphism $h : Z \rightarrow W$.

In classical category theory, only objects and morphisms are considered, which allows for a relatively simple and structured analysis of mathematical concepts.

However, many modern problems require more intricate structures, where morphisms themselves are subject to transformations, and these transformations need to be tracked systematically.

3.1.2 ∞ -Categories: A Generalization

∞ -categories generalize the concept of categories by introducing higher-dimensional morphisms between morphisms themselves. Instead of merely having objects and arrows, an ∞ -category allows for:

- **0-morphisms**, which correspond to the objects in the category.
- **1-morphisms**, which are the arrows between objects, just like in an ordinary category.
- **2-morphisms**, which are transformations between 1-morphisms. For example, if $f, g : X \rightarrow Y$ are two morphisms, a 2-morphism would be a way to transform f into g .
- **3-morphisms**, which are transformations between 2-morphisms, and so on.

This hierarchy of morphisms continues indefinitely, creating an intricate web of relationships between objects and their transformations. ∞ -categories allow us to track not only how objects relate to one another but also how these relationships evolve at higher levels. This flexibility is particularly important in fields like homotopy theory, where paths between points (1-morphisms) can themselves have higher-dimensional homotopies between them (2-morphisms and higher).

3.1.3 Formal Definition of ∞ -Categories

There are several models for ∞ -categories, each suited to different applications. The two most common approaches are:

- **Quasi-Categories:** Introduced by Joyal, quasi-categories are a generalization of simplicial sets, a powerful combinatorial model for topological spaces. Quasi-categories satisfy a weakened version of the Kan condition, which allows for higher-dimensional homotopies between morphisms.
- **Complete Segal Spaces:** Introduced by Rezk, this model views an ∞ -category as a simplicial space (a collection of spaces with face and degeneracy maps) that satisfies the Segal condition. Complete Segal spaces allow for more topological intuition and are closely connected to the study of moduli spaces in algebraic geometry and topological field theories.

In both models, an ∞ -category captures the idea that there are infinitely many levels of morphisms between objects, with each level encoding deeper relationships between these objects.

3.1.4 Applications of ∞ -Categories in the Langlands Program

The notion of ∞ -categories has far-reaching implications for the Langlands Program. By viewing automorphic representations and Galois representations as objects in an ∞ -category, we open the door to capturing higher-order relationships that were previously inaccessible using classical categories. In particular, functoriality—the idea that automorphic forms can be transferred between different groups—could be realized in a more general setting using ∞ -functors (maps between ∞ -categories) that account for higher-dimensional transformations.

Moreover, ∞ -categories allow us to incorporate homotopy-theoretic data into the Langlands Program. This is crucial for the study of non-standard L -functions, which rely on higher-dimensional cohomology groups and their interactions with automorphic forms. The rich structure of ∞ -categories provides a natural framework for formalizing these relationships and extending the scope of the Langlands Program to more generalized settings.

In summary, ∞ -categories offer a powerful tool for expanding the mathematical structures used in the Langlands Program. By incorporating higher morphisms and capturing more intricate relationships between objects, they provide a flexible and robust framework for addressing the limitations of the classical program, particularly in the context of functoriality and non-standard L -functions.

3.2 Applications in Automorphic Forms

The introduction of ∞ -categories offers new avenues for understanding automorphic forms and their deep connections to number theory, representation theory, and algebraic geometry. Automorphic forms are central objects in the Langlands Program, and higher-category theory provides a powerful framework to explore new relationships between these forms and other mathematical structures, such as Galois representations, via functoriality.

In this section, we explore how ∞ -categories can be applied to automorphic forms, focusing on higher-dimensional generalizations of automorphic representations, the role of ∞ -functors, and the conjectural functoriality in this new context. This higher categorical perspective opens the door to generalizing the Langlands Program beyond the classical setting, potentially unlocking new insights into non-reductive groups, quantum field theory, and higher cohomological data.

3.2.1 Automorphic Representations as Objects in ∞ -Categories

Traditionally, automorphic representations have been viewed as irreducible components of spaces of automorphic forms, which are highly symmetric functions on arithmetic groups. In the classical Langlands Program, these representations are studied in the context of reductive groups, and their connections to Galois representations are mediated through L -functions.

In the higher-categorical setting, we propose to treat automorphic representations as objects in an ∞ -category, denoted $\mathcal{A}^{(\infty)}(G(F))$, where G is a reductive group defined over a number field F . The key idea is that higher-dimensional morphisms between automorphic representations capture more intricate symmetries and relationships than what is possible in classical categories.

Formally, an ∞ -category $\mathcal{A}^{(\infty)}(G(F))$ consists of:

- **0-morphisms:** These correspond to the automorphic representations of $G(F)$.
- **1-morphisms:** These are natural transformations between automorphic representations, capturing symmetries or functorial relationships at the level of the representations.
- **2-morphisms and higher:** These capture homotopy-theoretic relationships between the transformations themselves, allowing for a deeper understanding of the connections between different automorphic forms.

This higher-categorical perspective on automorphic forms allows us to explore functoriality at multiple levels of structure, including transformations between the automorphic forms themselves.

3.2.2 Functoriality in ∞ -Categories

One of the most promising aspects of using ∞ -categories is the potential to extend the functoriality principle beyond its classical form. Functoriality, as conjectured in the Langlands Program, describes the relationship between automorphic representations of different reductive groups. By lifting this to an ∞ -categorical setting, we can define ∞ -functors between categories of automorphic representations, potentially generalizing the classical notion of functoriality.

Given two reductive groups G and H , we define a conjectural ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

which should induce a transfer of automorphic representations from G to H . This functorial relationship not only respects the structure of automorphic forms but also captures higher-dimensional morphisms between them, including potential homotopy data.

The advantage of working with ∞ -categories is that ∞ -functors can be flexible enough to encode richer structures, such as:

- **Homotopy-theoretic transformations:** These higher-dimensional transformations may correspond to additional symmetries or relationships between automorphic forms that are not captured by classical functors.
- **Derived functoriality:** ∞ -functors can be designed to respect derived categories, providing a more nuanced picture of the cohomological structures underlying automorphic forms.

This ∞ -categorical generalization of functoriality could help overcome some of the limitations of the classical Langlands Program, especially in cases involving non-reductive groups or other complex algebraic structures.

3.2.3 Geometric Realizations and Homotopy-Theoretic Data

A key feature of ∞ -categories is their natural connection to geometry and homotopy theory. In particular, the study of automorphic forms in the context of ∞ -categories opens the possibility of using geometric objects and their cohomological properties to gain a deeper understanding of the Langlands Program.

One example is the use of higher-dimensional cohomology groups, such as étale cohomology or derived categories of sheaves on moduli spaces, to study automorphic representations. These cohomological structures can be organized into ∞ -categories, where automorphic forms correspond to geometric objects, and the higher-dimensional morphisms between these forms encode geometric and homotopy-theoretic information.

For instance, if X is a smooth projective variety over a number field F , then the étale cohomology groups $H_{\text{et}}^n(X, \mathbb{Q}_\ell)$ can be viewed as objects in an ∞ -category of sheaves on X . The automorphic representations of a reductive group G could correspond to these cohomological objects, with higher morphisms representing homotopies between cohomology classes. This perspective provides a natural setting for defining non-standard L -functions, as explored later in the paper.

3.2.4 Connections to Derived Categories and D-Modules

Another promising application of ∞ -categories in the context of automorphic forms is through their connection to derived categories and D-modules. Derived categories are fundamental objects in modern algebraic geometry, capturing complex interactions between cohomological data. By viewing automorphic forms as objects in derived categories, and studying their relationships using D-modules on flag varieties or other moduli spaces, we can potentially gain a deeper understanding of functoriality and other key aspects of the Langlands Program.

For example, automorphic representations could correspond to D-modules on flag varieties associated with a reductive group G , with ∞ -functors capturing the derived equivalences between these D-modules. This approach could be especially useful for understanding the transfer of automorphic forms in cases involving non-reductive groups or more exotic geometric settings.

3.2.5 Potential Generalizations to Non-Reductive Groups

The flexibility of ∞ -categories makes them an ideal tool for exploring functoriality beyond the classical setting of reductive groups. Non-reductive groups, which fall outside the traditional scope of the Langlands Program, pose significant challenges for classical functoriality. However, by organizing automorphic

forms and representations in an ∞ -categorical framework, we may be able to extend functoriality to these more general cases.

In particular, higher morphisms between automorphic forms in ∞ -categories may capture symmetries and transformations that are inaccessible through classical methods. This opens up the possibility of developing new conjectures for functoriality in non-reductive settings, potentially linking the Langlands Program with broader areas of mathematics, including non-abelian Hodge theory, derived algebraic geometry, and mathematical physics.

In summary, ∞ -categories provide a robust and flexible framework for studying automorphic forms and extending the Langlands Program. By capturing higher-dimensional relationships and encoding homotopy-theoretic data, they allow for a deeper exploration of functoriality, cohomological structures, and geometric realizations of automorphic representations. These tools offer promising new directions for future research in number theory, algebraic geometry, and related fields.

3.3 Extensions to Galois Representations

The study of Galois representations is central to the Langlands Program, providing a deep connection between number theory and the symmetries of algebraic equations. In the classical Langlands Program, Galois representations are finite-dimensional representations of the absolute Galois group of a number field, $\text{Gal}(\overline{F}/F)$, where F is a number field. These representations encode arithmetic information and play a crucial role in the correspondence with automorphic forms. The introduction of ∞ -categories provides an opportunity to extend our understanding of Galois representations, allowing us to explore higher-dimensional symmetries and transformations that go beyond the classical framework.

In this section, we propose an extension of the classical Galois representations by viewing them as objects in an ∞ -category, and we explore how this new perspective can deepen our understanding of the Langlands correspondence, especially in the context of higher-dimensional cohomological structures.

3.3.1 Galois Representations as Objects in an ∞ -Category

Traditionally, Galois representations are finite-dimensional representations of the absolute Galois group, $\text{Gal}(\overline{F}/F)$, over fields such as \mathbb{Q}_ℓ . These representations describe how the Galois group acts on the solutions to polynomial equations, capturing symmetries in number fields and arithmetic varieties.

In the ∞ -categorical framework, we propose to generalize Galois representations by viewing them as objects in an ∞ -category, denoted $\mathcal{G}^{(\infty)}(F)$. This allows us to capture more nuanced symmetries and relationships between Galois representations, particularly when considering higher-dimensional cohomological data.

In this framework, the Galois representations are generalized as follows:

- **0-morphisms:** These correspond to classical Galois representations of $\text{Gal}(\overline{F}/F)$.
- **1-morphisms:** These are natural transformations between Galois representations, representing homotopies or symmetries between different representations.
- **2-morphisms and higher:** These encode higher-order transformations between the 1-morphisms, capturing more complex symmetries that are invisible in the classical framework.

This higher-categorical structure allows us to explore more general Galois representations that are not constrained by finite-dimensionality or traditional algebraic constraints. It also opens the possibility of studying new types of symmetries and homotopies between Galois representations, which may have applications in the study of non-standard L -functions and their cohomological properties.

3.3.2 Higher Galois Cohomology and Homotopy-Theoretic Data

A key motivation for considering Galois representations in the context of ∞ -categories is the desire to incorporate homotopy-theoretic data and higher-dimensional cohomology. Classical Galois representations are typically studied in the context of Galois cohomology, which measures the failure of certain algebraic objects (such as field extensions or algebraic varieties) to be split by the action of the Galois group. Galois cohomology plays a central role in many important conjectures and results in arithmetic geometry, such as the study of torsors and Brauer groups.

In the ∞ -categorical setting, we propose to extend the classical Galois cohomology groups $H^n(\text{Gal}(\overline{F}/F), M)$, where M is a Galois module, to higher cohomology groups that capture homotopy-theoretic data. These extended cohomology groups, denoted $H_\infty^n(\text{Gal}(\overline{F}/F), M)$, encode not only the standard cohomological obstructions but also higher-order homotopies between these obstructions.

For instance, étale cohomology groups $H_{\text{et}}^n(X, \mathbb{Q}_\ell)$ of a smooth projective variety X over a number field F can be viewed as objects in an ∞ -category, where the higher morphisms correspond to homotopies between cohomology classes. In this sense, Galois representations can be understood as acting on these cohomology groups, with the higher morphisms capturing deeper relationships between different cohomological data.

3.3.3 Functoriality in the ∞ -Categorical Framework

One of the most significant potential applications of extending Galois representations to ∞ -categories is in the context of the functoriality conjecture. In the classical Langlands Program, functoriality describes a relationship between automorphic representations and Galois representations, mediated by L -functions.

By generalizing Galois representations to an ∞ -categorical setting, we propose that functoriality can also be extended to capture higher-dimensional transformations between automorphic forms and Galois representations.

Given an ∞ -category $\mathcal{A}^{(\infty)}(G(F))$ of automorphic representations and an ∞ -category $\mathcal{G}^{(\infty)}(F)$ of Galois representations, we propose the existence of an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{G}^{(\infty)}(F),$$

which generalizes the classical functoriality principle by encoding higher-order morphisms between automorphic forms and Galois representations. This ∞ -functor could be viewed as a conjectural extension of the classical Langlands correspondence, capturing new types of symmetries and relationships that are inaccessible in the classical framework.

Moreover, by studying the relationships between higher-dimensional Galois cohomology and automorphic forms, we can explore the potential for defining new classes of L -functions that incorporate higher-order homotopy data. These non-standard L -functions, introduced later in this paper, could provide deeper insights into the arithmetic properties of Galois representations and automorphic forms.

3.3.4 Connections to Derived Categories and Algebraic Geometry

In algebraic geometry, Galois representations are often studied in connection with derived categories of sheaves on algebraic varieties. The introduction of ∞ -categories allows us to extend this perspective, viewing Galois representations as acting on objects in derived categories, such as D-modules on flag varieties or perverse sheaves on moduli spaces.

For example, let X be a smooth projective variety over a number field F , and consider the derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves on X . Galois representations can be understood as acting on objects in this derived category, and the ∞ -categorical structure allows us to track not only the action of the Galois group on the sheaves themselves but also the higher-order transformations between these actions.

This higher-categorical perspective has significant implications for the study of functoriality, as it provides a more flexible framework for understanding the transfer of automorphic forms and Galois representations between different groups. It also opens the possibility of extending the Langlands Program to non-reductive groups, where the traditional tools of representation theory are less effective.

3.3.5 Towards a Higher-Categorical Langlands Correspondence

In summary, the extension of Galois representations to ∞ -categories provides a powerful new framework for studying automorphic forms, L -functions, and arithmetic geometry. By viewing Galois representations as objects in an ∞ -category, we can capture higher-dimensional symmetries and homotopies that are inaccessible in the classical setting. This new perspective opens the door

to extending functoriality, defining new classes of L -functions, and exploring deeper connections between automorphic forms, cohomology, and Galois representations.

This higher-categorical Langlands correspondence has the potential to address some of the limitations of the classical Langlands Program, particularly in cases involving non-reductive groups or higher-dimensional geometric objects. As we explore in the following sections, this new framework also has promising connections to quantum field theory, where higher-dimensional symmetries play a fundamental role in understanding dualities and partition functions.

4 Extended Automorphy in the ∞ -Categorical Framework

4.1 Definition of Extended Automorphy

In the classical Langlands Program, automorphic forms are associated with representations of adelic groups, specifically reductive algebraic groups. The correspondence between these automorphic forms and Galois representations is mediated through L -functions and encapsulates deep arithmetic properties. In this section, we extend the notion of automorphy to the realm of ∞ -categories, where automorphic representations are generalized to higher categorical structures, allowing us to capture more complex interactions and higher-dimensional data.

We propose a definition of *extended automorphy*, where automorphic representations are viewed as objects in an ∞ -category, and the relationships between them, mediated by ∞ -morphisms, encode deeper symmetries and transformations. This perspective allows us to explore automorphy in settings that go beyond reductive groups and classical number-theoretic contexts, opening up new possibilities for the Langlands correspondence in higher dimensions.

4.1.1 Higher Automorphic Representations

Let G be a reductive algebraic group over a number field F , and let $\mathcal{A}(G(F))$ denote the category of automorphic representations of $G(F)$ in the classical setting. In the ∞ -categorical framework, we generalize this category to an ∞ -category $\mathcal{A}^{(\infty)}(G(F))$, where the objects are higher automorphic representations, and the morphisms correspond to transformations between these representations.

In this extended framework:

- Objects of $\mathcal{A}^{(\infty)}(G(F))$ are higher-dimensional analogs of classical automorphic representations.
- 1-morphisms are natural transformations between these representations, capturing symmetries at the level of automorphic forms.

- Higher morphisms (2-morphisms, 3-morphisms, etc.) represent more complex interactions between the automorphic forms, incorporating homotopy-theoretic data and derived structures.

These higher automorphic representations extend the classical Langlands Program by allowing us to model more intricate relationships between automorphic forms and the arithmetic objects they encode.

4.1.2 Extended Functoriality

The extension of automorphic forms to ∞ -categories naturally leads to a generalization of the classical notion of functoriality. We define *extended functoriality* as the transfer of automorphic representations between ∞ -categories associated with different groups. Specifically, given two reductive algebraic groups G and H , we propose the existence of an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

which generalizes the classical functorial transfer between automorphic representations by encoding higher-dimensional symmetries and transformations.

This extended functoriality conjecture suggests that the Langlands correspondence can be realized in a more generalized setting, where automorphic forms and Galois representations are objects in ∞ -categories. This provides a flexible and powerful framework for exploring new types of arithmetic relationships, including those that involve non-reductive groups or quantum field-theoretic structures.

4.1.3 Homotopy-Theoretic Automorphy

Incorporating homotopy-theoretic data into automorphic forms opens new possibilities for understanding automorphy in a geometric context. By associating automorphic representations with higher cohomology groups, such as those arising in étale cohomology, we can explore how these representations act on more sophisticated geometric objects. The use of ∞ -categories allows us to capture not just the classical automorphic forms but also the transformations between them in a homotopy-theoretic sense.

For example, we can consider automorphic forms as acting on higher-dimensional cohomology groups, where the transformations between these forms correspond to homotopies between cohomology classes. This perspective enriches the classical framework by incorporating higher-dimensional structures and allowing us to explore new types of automorphy, such as those involving derived categories or D-modules on moduli spaces.

4.1.4 Extended Automorphy for Non-Reductive Groups

One of the primary motivations for extending automorphy to ∞ -categories is the potential to explore functoriality in the context of non-reductive groups.

The classical Langlands Program focuses primarily on reductive groups, but many important arithmetic and geometric structures are associated with non-reductive groups. By defining automorphic forms and representations in an ∞ -categorical setting, we gain the flexibility to study these structures in a more generalized context.

In particular, extended automorphy allows us to explore new types of symmetries and relationships between non-reductive groups and their associated automorphic forms. These symmetries are captured by higher morphisms in the ∞ -category, allowing us to model interactions that are not accessible through classical methods.

4.1.5 Implications for the Langlands Correspondence

The introduction of extended automorphy has significant implications for the Langlands correspondence. By generalizing automorphic representations and their relationships through ∞ -categories, we open up the possibility of extending the Langlands correspondence to new settings, including non-reductive groups, higher-dimensional cohomology, and quantum field theory. This extended correspondence provides a more flexible framework for exploring the deep connections between automorphic forms, Galois representations, and L -functions.

In the following sections, we build on this foundation by introducing non-standard L -functions and exploring their connections to extended automorphy. We also explore the role of homotopy-theoretic data in defining these new classes of L -functions and their implications for functoriality and the Langlands Program as a whole.

4.2 Extended Functoriality

Extended functoriality is a generalization of the classical Langlands functoriality conjecture, which predicts a transfer of automorphic representations between different groups, mediated by L -functions. In the ∞ -categorical setting, functoriality is enhanced to include higher-dimensional relationships and transformations between automorphic forms, allowing for more complex interactions and symmetries that are inaccessible in the classical framework.

In this section, we introduce the formal definition of extended functoriality and explore its implications for both reductive and non-reductive groups, as well as its potential applications in quantum field theory and higher-dimensional cohomology.

4.2.1 Definition of Extended Functoriality

Let G and H be two reductive algebraic groups over a number field F . In the classical Langlands Program, functoriality is realized as a transfer of automorphic representations from G to H , guided by a homomorphism between their L -groups:

$$\phi : {}^L G \rightarrow {}^L H.$$

The goal of extended functoriality is to generalize this transfer to an ∞ -categorical setting. Specifically, we propose the existence of an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

where $\mathcal{A}^{(\infty)}(G(F))$ and $\mathcal{A}^{(\infty)}(H(F))$ are the ∞ -categories of automorphic representations associated with G and H , respectively. This ∞ -functor captures not only the transfer of automorphic representations but also the higher morphisms that exist between these representations, thus encoding more intricate relationships and symmetries.

The extended functoriality conjecture predicts that this ∞ -functor respects the structure of automorphic forms at every level, including higher-order symmetries and homotopies. In particular, the functor $\Phi^{(\infty)}$ should map automorphic representations to their corresponding higher-dimensional counterparts, preserving key arithmetic and geometric properties.

4.2.2 Functoriality for Non-Reductive Groups

A major motivation for extending functoriality is to apply it to non-reductive groups, which are beyond the scope of the classical Langlands Program. Non-reductive groups, such as solvable or nilpotent groups, often arise in various contexts within arithmetic geometry and mathematical physics, but their automorphic forms and representations are less understood due to the lack of a functoriality framework.

By introducing extended functoriality, we aim to develop a more flexible approach that can handle non-reductive groups. The ∞ -categorical framework allows us to generalize the classical functorial transfer to non-reductive settings by utilizing higher-dimensional morphisms. In this case, the conjecture becomes:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

where G may be non-reductive. The extended functoriality conjecture for non-reductive groups suggests that automorphic forms on non-reductive groups can still be transferred to reductive groups, or between non-reductive groups themselves, through higher-dimensional relationships.

This approach has potential applications in areas where non-reductive groups play a prominent role, such as in the study of unipotent representations or certain quantum field theories, where symmetries are captured by non-reductive groups.

4.2.3 Higher-Dimensional Symmetries and Cohomology

One of the key features of extended functoriality is its ability to capture higher-dimensional symmetries and cohomological structures. In the classical setting, functoriality is primarily concerned with the transfer of automorphic forms and their associated L -functions, but the higher-categorical approach allows us to explore more intricate relationships involving derived categories and cohomology groups.

For example, automorphic representations in ∞ -categories can be linked to higher-dimensional cohomology groups, such as étale cohomology or perverse sheaves on moduli spaces. The extended functoriality conjecture predicts that these cohomological structures are preserved under the transfer between groups. In particular, the ∞ -functor $\Phi^{(\infty)}$ should map cohomological objects in $\mathcal{A}^{(\infty)}(G(F))$ to their corresponding cohomological objects in $\mathcal{A}^{(\infty)}(H(F))$.

This higher-dimensional functoriality is crucial for understanding the role of non-standard L -functions, which involve higher cohomological data and more complex geometric structures. By capturing these symmetries, extended functoriality provides a deeper insight into the connections between automorphic forms, Galois representations, and their associated cohomology.

4.2.4 Connections to Quantum Field Theory

The extended functoriality conjecture also has potential connections to quantum field theory (QFT), particularly in the context of dualities between different physical theories. In QFT, dualities often suggest deep symmetries between seemingly unrelated theories, and these symmetries can often be captured by higher-categorical structures. For example, the geometric Langlands Program has been linked to electric-magnetic duality in gauge theory, providing a mathematical framework for understanding these dualities.

Extended functoriality may provide a way to model dualities in QFT by treating automorphic forms and their symmetries as objects in ∞ -categories. The higher-dimensional relationships encoded by ∞ -functors could correspond to transformations between different quantum field theories, providing a new perspective on the connections between the Langlands Program and physics.

4.2.5 Towards a Generalized Functoriality Principle

In summary, extended functoriality provides a powerful generalization of the classical Langlands functoriality principle, allowing us to explore new relationships between automorphic forms, Galois representations, and higher-dimensional symmetries. By viewing these objects as part of an ∞ -categorical framework, we can capture more intricate interactions and extend the Langlands correspondence to new settings, including non-reductive groups and quantum field theory.

The potential applications of extended functoriality are wide-ranging, from number theory and arithmetic geometry to mathematical physics. As we develop the tools necessary to formalize these ideas, we anticipate that extended functoriality will provide new insights into the deep connections between automorphic forms, L -functions, and higher-dimensional cohomology. In the next section, we will explore how this extended framework applies to the definition and study of non-standard L -functions.

4.3 Non-Standard L -Functions

Non-standard L -functions are speculative extensions of the classical automorphic L -functions, incorporating higher-dimensional cohomological and homotopy-theoretic data. These generalized L -functions, defined within an ∞ -categorical framework, provide a natural extension of the classical Langlands Program and open the door to new arithmetic and geometric phenomena that were previously inaccessible.

In this section, we introduce the definition of non-standard L -functions and explore their potential connections to higher-dimensional cohomology, automorphic representations, and quantum field theory. We also discuss the role of extended automorphy and extended functoriality in defining and understanding these new L -functions.

4.3.1 Classical L -Functions

In the classical Langlands Program, automorphic L -functions are associated with automorphic representations and Galois representations. These L -functions, which are meromorphic functions of a complex variable s , encode deep arithmetic information about automorphic forms and number fields. A classical automorphic L -function has the following Euler product form:

$$L(s, \pi) = \prod_p L_p(s, \pi_p),$$

where $L_p(s, \pi_p)$ are the local factors associated with the automorphic representation π at the prime p . The functional equation satisfied by these L -functions is one of the central conjectures of the Langlands Program.

The motivation for extending L -functions comes from the desire to incorporate additional geometric and homotopy-theoretic data, as well as to generalize the classical correspondence to higher dimensions.

4.3.2 Definition of Non-Standard L -Functions

We propose a definition of non-standard L -functions in the context of ∞ -categories. Let $\pi \in \mathcal{A}^{(\infty)}(G(F))$ be an automorphic representation in the ∞ -category associated with a reductive group G . The non-standard L -function associated with π , denoted $L^{(\infty)}(s, \pi)$, is defined as:

$$L^{(\infty)}(s, \pi) = \prod_{n=1}^{\infty} \frac{1}{\det(1 - q^{-s} T_n | H_{\text{et}}^n(X, \mathbb{Q}_l))},$$

where $H_{\text{et}}^n(X, \mathbb{Q}_l)$ represents the n -th étale cohomology group of a smooth projective variety X associated with the automorphic representation π , and T_n are Hecke operators acting on these cohomology groups.

This definition extends the classical L -function by incorporating higher-dimensional cohomology and Hecke operators. The introduction of these cohomological objects allows us to model more complex arithmetic and geometric relationships, which are not captured by classical L -functions.

4.3.3 Cohomological and Homotopy-Theoretic Data

The key innovation in non-standard L -functions is the use of higher-dimensional cohomology and homotopy-theoretic data. In the classical setting, L -functions are constructed from representations of the absolute Galois group and their action on cohomological objects such as étale cohomology groups. Non-standard L -functions extend this construction by incorporating higher-order cohomology groups, capturing more subtle geometric and arithmetic information.

For example, the étale cohomology groups $H_{\text{ét}}^n(X, \mathbb{Q}_l)$ that appear in the definition of $L^{(\infty)}(s, \pi)$ encode higher-dimensional information about the arithmetic and geometry of the variety X . The Hecke operators T_n acting on these cohomology groups capture symmetries and transformations between cohomological classes, and the determinant in the definition of the L -function reflects the action of these operators on the cohomology.

The use of homotopy-theoretic data in this context is also significant. By viewing automorphic forms and their cohomological representations in an ∞ -categorical framework, we can incorporate homotopies between cohomology classes, which capture additional symmetries and relationships. These higher morphisms play a crucial role in understanding the structure of non-standard L -functions.

4.3.4 Functional Equations for Non-Standard L -Functions

One of the central conjectures in the study of classical L -functions is the functional equation, which relates the value of the L -function at s to its value at $1 - s$. In the case of non-standard L -functions, we propose a generalized functional equation of the form:

$$L^{(\infty)}(s, \pi) = \epsilon(s, \pi) L^{(\infty)}(1 - s, \pi^\vee),$$

where $\epsilon(s, \pi)$ is an epsilon factor that depends on the homotopy-theoretic and cohomological data associated with the automorphic representation π , and π^\vee is the contragredient representation of π in the ∞ -category.

This functional equation suggests that non-standard L -functions possess a deep symmetry, similar to that of classical L -functions, but with additional structure provided by higher cohomology and homotopy data. The epsilon factor, which plays a crucial role in the functional equation, encodes the corrections necessary to account for the extended symmetries in the ∞ -categorical framework.

4.3.5 Applications to Arithmetic Geometry and Quantum Field Theory

The study of non-standard L -functions has implications for both arithmetic geometry and quantum field theory. In arithmetic geometry, the use of higher cohomology groups in the definition of these L -functions allows for a more detailed understanding of the arithmetic properties of varieties over number fields.

The homotopy-theoretic perspective also provides new tools for understanding the symmetries of these varieties and their associated automorphic forms.

In quantum field theory, non-standard L -functions may play a role in the study of partition functions and dualities. The geometric Langlands Program has already revealed deep connections between L -functions and quantum field theory, particularly in the context of electric-magnetic duality. By incorporating higher-dimensional cohomology and homotopy data, non-standard L -functions may provide a new perspective on these dualities and their mathematical underpinnings.

Moreover, the extended functoriality conjecture introduced in the previous section plays a crucial role in understanding how non-standard L -functions behave under the transfer of automorphic forms between different groups. The higher morphisms in the ∞ -category provide the flexibility to capture the more intricate relationships between L -functions and automorphic representations, allowing for a deeper exploration of their arithmetic and geometric properties.

4.3.6 Future Directions

The study of non-standard L -functions is still in its speculative stages, and many open questions remain. One of the key challenges is to rigorously define the higher cohomology groups and homotopy-theoretic structures that appear in the ∞ -categorical framework. Developing the necessary mathematical tools to formalize these ideas will be essential for proving the conjectures introduced in this paper.

Another direction for future research is the exploration of specific examples of non-standard L -functions in arithmetic geometry and quantum field theory. By studying concrete cases, we can gain insight into the behavior of these L -functions and their implications for both number theory and physics. In particular, the connections between non-standard L -functions and quantum field theory dualities provide a promising avenue for further investigation.

In summary, non-standard L -functions provide a natural extension of the classical Langlands Program, incorporating higher-dimensional cohomology and homotopy-theoretic data. These L -functions open new avenues for research in arithmetic geometry and quantum field theory, offering a deeper understanding of automorphic forms and their associated symmetries.

5 Non-Standard L -Functions and Higher-Dimensional Homotopy Data

5.1 Classical L -Functions and Their Functional Equations

Classical L -functions arise in the Langlands Program as key objects connecting automorphic forms, Galois representations, and number theory. These functions are typically defined by an Euler product, reflecting local data at each prime,

and they satisfy global functional equations that relate their values at s to their values at $1 - s$.

For an automorphic representation π of a reductive group $G(F)$, where F is a number field, the associated L -function $L(s, \pi)$ can be expressed as an Euler product:

$$L(s, \pi) = \prod_v L_v(s, \pi_v),$$

where v ranges over all places of F , and $L_v(s, \pi_v)$ are the local factors at each place. Each local factor $L_v(s, \pi_v)$ encodes information about the local Galois representation or automorphic form at the place v . At unramified places, the local factors take the simple form:

$$L_v(s, \pi_v) = \frac{1}{\det(1 - q_v^{-s} T_v)},$$

where q_v is the cardinality of the residue field at v , and T_v is a Frobenius element acting on the local representation.

One of the most important properties of classical L -functions is the global functional equation, which relates the L -function evaluated at s with its value at $1 - s$. For many automorphic L -functions, this equation takes the form:

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi^\vee),$$

where π^\vee is the contragredient (or dual) representation of π , and $\epsilon(s, \pi)$ is a global root number, or epsilon factor, encoding arithmetic and geometric information about π .

5.2 Defining Non-Standard L -Functions

To generalize classical L -functions, we propose a new class of *non-standard* L -functions that incorporate higher-dimensional cohomology and homotopy-theoretic data. These non-standard L -functions are motivated by the extended Langlands Program, which seeks to extend the classical correspondences between automorphic forms and Galois representations to include higher categories, higher-dimensional geometric objects, and their associated cohomological data.

Let π be an automorphic representation of a reductive group $G(F)$, as in the classical setting. We define the non-standard L -function $L^{(\infty)}(s, \pi)$ as follows:

$$L^{(\infty)}(s, \pi) = \prod_{n=0}^{\infty} \frac{1}{\det(1 - q^{-s} T_n | H_{\text{et}}^n(X, \mathbb{Q}_l))},$$

where:

- $H_{\text{et}}^n(X, \mathbb{Q}_l)$ is the n -th étale cohomology group of a smooth projective variety X associated with the automorphic representation π ,
- T_n are Hecke operators acting on the cohomology group $H_{\text{et}}^n(X, \mathbb{Q}_l)$,

- q is the cardinality of the residue field at a given prime of F , and
- s is a complex variable.

This definition generalizes the classical case by replacing the local Galois representations with higher-dimensional cohomological data. The higher cohomology groups $H_{\text{et}}^n(X, \mathbb{Q}_l)$ capture more refined geometric and arithmetic information than classical L -functions. Additionally, the introduction of higher-dimensional Hecke operators T_n reflects the influence of more complex symmetries and geometric structures on the automorphic representation.

These non-standard L -functions not only extend the classical framework but also incorporate homotopy-theoretic corrections, as the higher-dimensional data allows for a deeper connection between automorphic forms, Galois representations, and the geometry of higher categorical structures.

5.3 Conjectural Global Functional Equation

We propose a conjectural global functional equation for the non-standard L -function, extending the classical equation to incorporate the higher-dimensional cohomology and homotopy-theoretic data. This functional equation takes the following form:

$$L^{(\infty)}(s, \pi) = \epsilon(s, \pi) L^{(\infty)}(1 - s, \pi^\vee),$$

where π^\vee is the contragredient representation of π , and $\epsilon(s, \pi)$ is an epsilon factor that now encodes homotopy-theoretic corrections.

In this setting, the epsilon factor $\epsilon(s, \pi)$ not only accounts for local arithmetic data but also includes contributions from the higher-dimensional cohomology groups and the actions of the Hecke operators on those cohomology groups. These corrections reflect the influence of higher homotopies and symmetries on the automorphic representation.

The conjectural functional equation suggests a deep symmetry between the automorphic representation π and its dual π^\vee , generalized to an ∞ -categorical framework. Just as in the classical case, the equation hints at profound relationships between arithmetic, geometry, and representation theory, but now extended to higher dimensions and more complex geometric structures.

5.4 Lean Formalization

The formalization of non-standard L -functions and their related structures in theorem-proving software such as Lean provides a rigorous method for validating the internal consistency of the definitions and conjectures presented in this framework. By encoding the key elements of the non-standard L -functions, such as étale cohomology groups and Hecke operators, we can ensure that the higher-dimensional constructions are mathematically sound and adhere to the rules of category theory and homotopy theory.

In Lean, we can define basic structures such as categories, functors, and L -functions. For example, a simplified version of the non-standard L -function can be represented as follows:

```

1  -- Define the basic structure for an L-function in Lean
2  structure LFunction :=
3    (s : ) -- Variable for L-function
4    (value : → ) -- L-function as a real-valued function
5
6  -- Define a Hecke operator acting on cohomology groups
7  structure HeckeOperator :=
8    (T : → ) -- Operator acting on higher-dimensional data
9
10 -- Define étale cohomology (simplified version)
11 structure EtaleCohomology :=
12   (n : ) -- Degree of the cohomology group
13   (group : Type) -- The cohomology group itself
14
15 -- Define a non-standard L-function (simplified version)
16 def NonStandardLFunction (H : → EtaleCohomology) (T : HeckeOperator) :
17   LFunction :=
18   { s := 0,
19     value := s, n in (finset.range 100), 1 / (1 - real.exp (-s * T.T n
20       )) }

```

This Lean formalization provides a basic foundation for non-standard L -functions, Hecke operators, and étale cohomology. Further extensions of this code can formalize the higher-dimensional aspects of these constructions, allowing for rigorous validation of the conjectures and structures introduced in the extended Langlands framework.

Formalizing these ideas in Lean not only increases the rigor of the mathematical framework but also opens the possibility for computer-assisted proofs and the verification of complex conjectures in higher-dimensional geometry and number theory.

6 Implications for Quantum Field Theory

6.1 Connections Between the Langlands Program and Quantum Field Theory

The relationship between the Langlands Program and quantum field theory (QFT) has become a topic of increasing interest over the past few decades. The geometric Langlands Program, in particular, has revealed deep connections between dualities in QFT and the mathematical structures underlying the Langlands correspondence. These connections have opened up new avenues for research, where higher-dimensional and homotopy-theoretic data play a crucial role in understanding physical symmetries and dualities.

In this section, we explore how the ∞ -categorical framework and the extension of automorphic representations, functoriality, and L -functions can provide new insights into quantum field theory. We focus on how the structures developed in the extended Langlands Program can be applied to specific quantum field theories, particularly those that exhibit electric-magnetic duality and other types of symmetry.

6.1.1 Electric-Magnetic Duality and the Geometric Langlands Program

One of the most well-known connections between the Langlands Program and quantum field theory is through electric-magnetic duality, a phenomenon in gauge theory where the roles of electric and magnetic fields can be exchanged. This duality can be understood mathematically through the geometric Langlands Program, where it corresponds to the duality between certain categories of sheaves on moduli spaces of bundles on algebraic curves.

In the geometric Langlands Program, automorphic forms are replaced by sheaves or D-modules on moduli spaces, and the Langlands dual group plays a central role in understanding these dualities. For example, Kapustin and Witten [5] demonstrated that the geometric Langlands correspondence can be understood as a manifestation of electric-magnetic duality in a specific class of quantum field theories known as S -duality. In these theories, duality relates the physical observables of the theory to the automorphic representations in the geometric Langlands Program.

The extension of the Langlands Program to ∞ -categories, as proposed in this paper, provides a natural framework to study more complex dualities that arise in quantum field theory. By incorporating higher-dimensional morphisms and homotopy-theoretic data, the ∞ -categorical approach offers a more flexible way of capturing the symmetries of these dualities and understanding their deeper mathematical structure.

6.1.2 Higher-Dimensional Automorphic Forms and Field Theories

The use of ∞ -categories in the study of automorphic representations naturally leads to applications in higher-dimensional quantum field theories. In traditional quantum field theory, fields are defined on space-time manifolds, and the symmetries of the theory are described by group actions on these fields. Higher-dimensional field theories, such as string theory or M-theory, involve more intricate structures where the symmetries are encoded not just in groups but in higher categories.

In the ∞ -categorical framework, automorphic representations can be viewed as objects that capture the symmetries of fields in these higher-dimensional theories. The higher morphisms in the ∞ -categories correspond to transformations between field configurations that cannot be captured by classical group actions alone. These morphisms are particularly useful for understanding non-perturbative effects in field theory, where classical methods are insufficient.

Moreover, the extended functoriality conjecture introduced in previous sections suggests that higher-dimensional automorphic forms can be transferred between different field theories through ∞ -functors. This could provide a new mathematical framework for studying dualities in higher-dimensional field theories, where the transfer of automorphic data between different groups corresponds to physical symmetries between different theories.

6.1.3 Non-Standard L -Functions and Partition Functions in QFT

Non-standard L -functions, introduced in Section 4.3, have potential applications in quantum field theory through their connection to partition functions. In quantum field theory, the partition function is a central object that encodes the statistical properties of the theory. It is typically computed as an integral over all possible field configurations, weighted by the action of the theory. Partition functions are often closely related to L -functions, particularly in theories with symmetries described by automorphic forms.

In the extended Langlands framework, non-standard L -functions incorporate higher-dimensional cohomology and homotopy-theoretic data, making them well-suited for describing partition functions in higher-dimensional field theories. These L -functions can encode more subtle symmetries and topological features of the theory that are not captured by classical L -functions.

For example, the higher cohomological data that appears in non-standard L -functions may correspond to topological defects or solitons in a field theory, which play a significant role in the non-perturbative dynamics of the theory. By studying the relationship between non-standard L -functions and partition functions, we can gain new insights into the structure of quantum field theories and their symmetries.

6.1.4 Quantum Field Theory and Higher Cohomology

Another promising application of the extended Langlands framework is the study of higher cohomology in quantum field theory. In traditional field theory, cohomology is used to study the topological features of space-time manifolds and field configurations. Higher cohomology groups, such as those used in the definition of non-standard L -functions, provide a natural way to capture the more intricate topological and homotopy-theoretic structures that arise in higher-dimensional theories.

For instance, in string theory, the study of higher cohomology groups is essential for understanding the topology of target spaces and the moduli spaces of field configurations. By applying the extended Langlands framework to these higher-dimensional cohomology groups, we can develop a more refined understanding of the symmetries and dualities in quantum field theory. In particular, the use of ∞ -categories allows us to track not just the field configurations themselves but also the transformations between them, providing a deeper understanding of the field theory's structure.

6.1.5 Future Directions and Open Questions

The connection between the Langlands Program and quantum field theory is a rapidly evolving area of research, and the extension of the Langlands framework to ∞ -categories opens up new possibilities for exploration. Some open questions for future research include:

- **Dualities in Higher-Dimensional Theories:** How can the extended

functoriality conjecture be applied to study dualities in higher-dimensional quantum field theories, such as string theory or M-theory? Can ∞ -categories provide a unifying framework for understanding these dualities?

- **Non-Standard L -Functions and Quantum Symmetries:** What role do non-standard L -functions play in the study of partition functions and quantum symmetries? Can these L -functions be used to describe new types of topological or geometric defects in quantum field theory?
- **Higher Cohomology in Field Theories:** How can higher cohomology groups be used to study the topology of space-time manifolds and field configurations in quantum field theory? Can the tools of ∞ -categories and extended automorphy provide new insights into these structures?

In summary, the extension of the Langlands Program to ∞ -categories has significant implications for the study of quantum field theory. By incorporating higher-dimensional automorphic forms, non-standard L -functions, and extended functoriality, we can develop new mathematical tools to study dualities, partition functions, and the symmetries of higher-dimensional field theories.

6.2 Higher-Dimensional Symmetries in Quantum Field Theory

Quantum field theory (QFT) has long been a fertile ground for exploring symmetries, particularly those that go beyond classical group structures. Symmetries play a critical role in organizing the behavior of quantum fields, and dualities between seemingly different quantum field theories often hint at deeper, more intricate symmetries. These symmetries can be better understood through the lens of higher-dimensional categories and homotopy-theoretic structures, which naturally arise in the extended Langlands framework.

In this section, we examine how the ∞ -categorical perspective on automorphic forms, functoriality, and L -functions can enhance our understanding of higher-dimensional symmetries in QFT. We also explore the implications of these symmetries for the study of dualities, topological defects, and non-perturbative effects in higher-dimensional field theories.

6.2.1 Classical Symmetries in Quantum Field Theory

Classical quantum field theories, particularly those based on gauge symmetries, are often described using Lie groups and their representations. For example, in Yang-Mills theory, the gauge group G (a Lie group) acts on the quantum fields, and the observables of the theory are invariant under this action. This structure allows physicists to organize the states of the theory and predict the behavior of particles, forces, and interactions.

However, in higher-dimensional and more sophisticated quantum field theories, such as those encountered in string theory or M-theory, the symmetries

are often more intricate and require more refined mathematical tools. In particular, these symmetries may not be fully captured by classical group theory but instead involve higher categorical structures that account for transformations between fields, interactions between symmetries, and the topological nature of the field configurations.

6.2.2 Higher Symmetries and ∞ -Categories

The ∞ -categorical framework introduced in the extended Langlands Program is ideally suited to describe higher-dimensional symmetries. In this framework, automorphic representations and their associated symmetries are not just objects in a classical category but are instead objects in an ∞ -category, where higher morphisms represent transformations between symmetries at different levels.

For example, in a gauge theory, the automorphic forms corresponding to the gauge fields can be viewed as objects in an ∞ -category. The higher morphisms between these automorphic forms capture transformations between field configurations, including homotopies between gauge connections. These higher morphisms are particularly relevant in the study of non-perturbative phenomena, where classical symmetries are insufficient to describe the full structure of the theory.

One important consequence of this perspective is that higher symmetries allow for the existence of topological objects, such as solitons, vortices, and monopoles, which arise as stable, non-trivial configurations of the fields. These topological objects are associated with higher cohomology groups, and their behavior can be captured by the higher morphisms in the ∞ -category of automorphic representations.

6.2.3 Dualities and Higher Symmetries in Field Theories

Dualities between different quantum field theories are among the most profound manifestations of higher symmetries in modern physics. A duality is a relationship between two seemingly different theories, where the physical observables of one theory correspond to those of another. These dualities often suggest that the two theories are different manifestations of the same underlying structure.

In many cases, dualities in quantum field theory, such as electric-magnetic duality or S -duality, can be understood through higher categorical symmetries. For instance, the geometric Langlands Program interprets electric-magnetic duality in gauge theory as a correspondence between certain categories of sheaves on moduli spaces of bundles. The extension of the Langlands Program to ∞ -categories provides a framework for understanding even more complex dualities, where the symmetries of the theory are encoded in higher morphisms between automorphic forms.

In this context, extended functoriality provides a way to transfer automorphic data between different quantum field theories. For example, given two dual field theories with gauge groups G and H , the extended functoriality conjecture

predicts the existence of an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

where $\mathcal{A}^{(\infty)}(G(F))$ and $\mathcal{A}^{(\infty)}(H(F))$ are the ∞ -categories of automorphic representations associated with the gauge groups G and H , respectively. This functor captures the higher symmetries underlying the duality, providing a mathematical explanation for how the two theories are related.

6.2.4 Topological Defects and Non-Perturbative Effects

One of the key features of higher-dimensional symmetries in quantum field theory is the presence of topological defects, such as monopoles, solitons, and instantons. These defects are stable, non-trivial solutions to the field equations that are characterized by their topological properties. They play a significant role in the non-perturbative dynamics of the theory and are often responsible for phenomena such as confinement and symmetry breaking.

In the extended Langlands framework, topological defects can be understood as higher-dimensional objects in the ∞ -category of automorphic representations. The cohomology groups that appear in the definition of non-standard L -functions are naturally associated with these topological objects, capturing their geometric and topological properties. The higher morphisms between automorphic forms represent transformations between different topological configurations, providing a powerful tool for studying the non-perturbative dynamics of the theory.

For instance, monopoles in gauge theory can be viewed as cohomology classes in a higher-dimensional cohomology group, and their interactions can be described by higher morphisms in the ∞ -category. The extended functoriality conjecture suggests that these topological objects can be transferred between different quantum field theories, providing a deeper understanding of their role in dualities and symmetry breaking.

6.2.5 Applications to String Theory and M-Theory

String theory and M-theory are two areas of quantum field theory where higher-dimensional symmetries are especially prominent. In string theory, the fields are not just point particles but one-dimensional strings, and the symmetries of the theory involve transformations of these extended objects. M-theory generalizes this idea to higher-dimensional membranes, or "branes," leading to even more complex symmetries.

The extended Langlands framework, with its emphasis on ∞ -categories and higher morphisms, provides a natural setting for studying the symmetries of string theory and M-theory. The automorphic forms in this context correspond to the various field configurations of the theory, and the higher morphisms between them capture the transformations between different brane configurations. This approach provides a new perspective on the dualities in string theory, such

as T-duality and mirror symmetry, where the field configurations of one theory correspond to those of another.

Moreover, the non-standard L -functions introduced in the extended Langlands framework may provide a way to understand partition functions and other key observables in string theory and M-theory. The higher-dimensional cohomological data that appears in these L -functions is closely related to the topological features of the target space in string theory, providing new insights into the geometry and physics of these theories.

6.2.6 Future Research Directions

The study of higher-dimensional symmetries in quantum field theory through the lens of the extended Langlands Program opens up many exciting research directions. Some of the key questions that remain include:

- **Higher-Dimensional Automorphic Representations:** How can we explicitly construct higher-dimensional automorphic representations for specific quantum field theories, particularly in the context of string theory and M-theory?
- **Topological Defects and Dualities:** How do topological defects, such as monopoles and solitons, behave under dualities in higher-dimensional field theories? Can the extended functoriality conjecture be used to track these objects across different theories?
- **Non-Perturbative Effects and L -Functions:** How can non-standard L -functions be used to capture non-perturbative effects in quantum field theory, such as instantons or topological defects? Can these L -functions be computed explicitly in specific models of quantum field theory?

In summary, the extension of the Langlands Program to higher-dimensional symmetries provides a powerful new framework for studying quantum field theory. By incorporating ∞ -categories, non-standard L -functions, and extended functoriality, we can explore dualities, topological defects, and non-perturbative effects in a new light, offering deeper insights into the symmetries and structure of quantum field theories.

6.3 Non-Perturbative Phenomena and Topological Structures

Non-perturbative phenomena play a critical role in modern quantum field theory (QFT) and string theory, as they reveal physical processes that are not accessible through standard perturbative methods. These phenomena are often tied to topological structures such as solitons, instantons, monopoles, and other topological defects, which arise as stable, non-trivial field configurations. The study of these phenomena requires new mathematical tools that go beyond

classical perturbative expansions, and the extended Langlands Program, particularly through its use of ∞ -categories and higher-dimensional symmetries, offers a promising framework for addressing these challenges.

In this section, we explore the connections between non-perturbative phenomena, topological structures, and the extended Langlands framework. Specifically, we focus on how non-standard L -functions and higher cohomological data provide a natural language for describing these phenomena and the role that ∞ -categories play in capturing the complex transformations between topological objects in QFT and string theory.

6.3.1 Non-Perturbative Effects in Quantum Field Theory

In perturbative quantum field theory, physical quantities such as scattering amplitudes and correlation functions are computed by expanding the action in terms of a small coupling constant, with each term in the expansion representing a different order of interaction. However, many important physical effects cannot be captured by such an expansion. For instance, phenomena like confinement in quantum chromodynamics (QCD), the existence of solitons, or instanton transitions in gauge theory are inherently non-perturbative and require alternative techniques for their study.

Non-perturbative phenomena often arise from the topological structure of the field configurations. For example, instantons in gauge theory are solutions to the Euclidean field equations that correspond to tunneling events between different vacua of the theory. These solutions are classified by topological invariants, such as the instanton number, which are not apparent in perturbative expansions. Solitons, monopoles, and other topological defects similarly correspond to non-trivial field configurations that are stable due to their topological properties.

In the extended Langlands framework, these non-perturbative effects can be studied using higher-dimensional cohomology and homotopy-theoretic data. The ∞ -categorical perspective allows us to capture the transformations between different topological configurations of the fields and to describe how these configurations contribute to the non-perturbative dynamics of the theory.

6.3.2 Topological Defects and Extended Automorphy

Topological defects, such as monopoles, solitons, and instantons, are stable, localized configurations of fields that are characterized by their non-trivial topological properties. These defects play a significant role in non-perturbative dynamics, as they represent configurations that cannot be deformed into trivial solutions through continuous transformations. In many cases, topological defects correspond to solutions of the classical field equations that minimize the action, but they do so in a non-trivial topological sector.

In the extended Langlands Program, topological defects can be understood as higher-dimensional objects in the ∞ -category of automorphic representations. These representations, which are generalized in the extended framework

to include cohomological and homotopy-theoretic data, capture the non-trivial topological structure of the defects and their transformations. The higher morphisms between automorphic representations correspond to transitions between different topological sectors, such as instanton transitions or soliton interactions.

Non-standard L -functions, which incorporate higher cohomology groups in their definition, provide a natural tool for describing the contribution of topological defects to the physical observables of the theory. For instance, the instanton number in a gauge theory can be viewed as a cohomological invariant that appears in the non-standard L -function associated with the theory. These L -functions encode the contribution of the topological defects to quantities such as partition functions and correlation functions, offering a new way to understand their role in non-perturbative dynamics.

6.3.3 Instantons and Solitons in Gauge Theory

Instantons and solitons are two of the most prominent examples of topological defects that contribute to non-perturbative phenomena in gauge theory. Instantons are finite-action solutions to the Euclidean field equations that correspond to tunneling events between different vacua of the theory. These solutions are classified by an integer-valued topological invariant known as the instanton number, which measures the winding of the gauge field configurations.

Solitons, on the other hand, are stable, localized solutions to the field equations that exist in real-time (as opposed to Euclidean time). These solutions are typically characterized by topological invariants, such as the winding number or magnetic charge, which prevent the soliton from decaying into trivial vacuum configurations. In many cases, solitons correspond to classical field configurations that minimize the energy of the system within a given topological sector.

The study of instantons and solitons requires tools that can capture their topological nature and their interactions with other field configurations. In the extended Langlands framework, instantons and solitons can be described as cohomological classes in the ∞ -category of automorphic representations. The higher morphisms in this category capture the transitions between different topological configurations, such as instanton transitions between vacua or soliton-soliton interactions.

6.3.4 Non-Standard L -Functions and Non-Perturbative Physics

Non-standard L -functions provide a powerful tool for understanding non-perturbative phenomena in quantum field theory. These L -functions, introduced in Section 4.3, incorporate higher-dimensional cohomological data and homotopy-theoretic structures, making them well-suited for describing the contribution of topological defects to the physical observables of the theory.

For example, the partition function of a quantum field theory, which encodes the statistical properties of the system, often has contributions from instantons, solitons, and other topological defects. In the extended Langlands framework, the non-standard L -function associated with the theory captures these contribu-

tions through its dependence on higher cohomology groups. These cohomology groups are naturally associated with the topological invariants of the defects, such as the instanton number or the soliton winding number.

The functional equation of the non-standard L -function, which relates its value at s to its value at $1 - s$, suggests a deep symmetry between different topological sectors of the theory. This symmetry may be interpreted as a manifestation of dualities in the quantum field theory, such as electric-magnetic duality or S -duality, where the roles of different topological defects are exchanged.

6.3.5 Applications to String Theory and M-Theory

In string theory and M-theory, non-perturbative effects are even more pronounced, as the fundamental objects in the theory—strings and branes—are extended objects that naturally give rise to higher-dimensional topological defects. These defects, such as D-branes and M-branes, play a crucial role in the non-perturbative dynamics of the theory, and their interactions are governed by higher-dimensional symmetries and cohomological structures.

The extended Langlands framework, with its emphasis on ∞ -categories and higher cohomology, provides a natural setting for studying the non-perturbative effects in string theory and M-theory. The automorphic representations in this framework correspond to the different brane configurations, and the higher morphisms between these representations capture the transitions between different topological sectors. Non-standard L -functions provide a way to encode the contributions of these topological objects to partition functions and other physical observables, offering new insights into the non-perturbative dynamics of string theory and M-theory.

6.3.6 Future Directions

The study of non-perturbative phenomena and topological structures in quantum field theory and string theory through the extended Langlands framework opens up many exciting directions for future research. Some key questions that remain include:

- **Cohomology of Topological Defects:** How can the higher cohomology groups that appear in non-standard L -functions be explicitly computed for specific topological defects, such as instantons or solitons, in gauge theory?
- **Dualities and Non-Perturbative Effects:** What role do dualities, such as electric-magnetic duality or S -duality, play in the non-perturbative dynamics of quantum field theory? Can these dualities be understood in terms of the functional equations of non-standard L -functions?
- **Applications to String Theory and M-Theory:** How can the extended Langlands framework be applied to the study of D-branes, M-branes, and other extended objects in string theory and M-theory? Can non-standard L -functions provide new insights into the non-perturbative dynamics of these theories?

In summary, non-perturbative phenomena and topological structures are central to the study of quantum field theory and string theory, and the extended Langlands Program provides a powerful new framework for exploring these phenomena. By incorporating higher-dimensional symmetries, cohomology, and non-standard L -functions, we can gain a deeper understanding of the role that topological defects play in the non-perturbative dynamics of these theories.

7 Functoriality in the Langlands Program

7.1 Overview of Functoriality in the Langlands Program

Functoriality is one of the central themes in the Langlands Program, representing deep conjectures that propose relationships between automorphic forms and Galois representations, mediated by maps between different reductive groups. In broad terms, functoriality conjectures predict that, given a homomorphism between two reductive groups, there should exist a corresponding transfer of automorphic representations from one group to the other, as well as a transfer of the associated L -functions.

The Langlands functoriality conjectures can be described as follows: let G and H be two reductive groups over a number field F , and suppose there exists a homomorphism:

$$\phi : {}^L G \rightarrow {}^L H,$$

where ${}^L G$ and ${}^L H$ are the respective Langlands dual groups of G and H . The functoriality conjecture states that for each automorphic representation π of $G(\mathbb{A}_F)$, there should exist an automorphic representation Π of $H(\mathbb{A}_F)$ such that the L -functions associated with π and Π are related by the map ϕ . More precisely, the local factors of the L -functions at each place v should satisfy:

$$L_v(s, \Pi) = L_v(s, \pi, \phi).$$

This conjecture suggests that the automorphic representations of different groups are not independent but are connected by intricate functorial relations, governed by homomorphisms between their Langlands dual groups. In this way, functoriality is a powerful tool that enables the transfer of automorphic and arithmetic data between groups, providing a unified framework for understanding a wide variety of phenomena in number theory, representation theory, and algebraic geometry.

7.1.1 Historical Background and Key Results

The origins of the functoriality conjectures can be traced back to Robert Langlands' original formulation of the Langlands Program in the 1960s. Since then, functoriality has played a prominent role in shaping modern research in automorphic forms and number theory. The conjectures are deeply connected to the

principle of reciprocity, which seeks to generalize classical results, such as the quadratic reciprocity law, to higher dimensions and more general fields.

One of the most significant achievements in functoriality is the proof of the modularity theorem (formerly known as the Taniyama-Shimura-Weil conjecture), which states that every elliptic curve over \mathbb{Q} is modular. This result, proven by Andrew Wiles in the 1990s, established a functorial relationship between elliptic curves and modular forms, providing a striking realization of Langlands' vision.

In addition to the modularity theorem, functoriality has been proven in various other cases, such as the Langlands correspondence for $\mathrm{GL}(2)$ and $\mathrm{GL}(n)$, and the base change for $\mathrm{GL}(2)$. However, the general functoriality conjecture remains open and is one of the most important unsolved problems in modern mathematics.

7.1.2 Automorphic Transfer and L -Functions

The concept of automorphic transfer, central to the functoriality conjecture, asserts that an automorphic representation of one group can be transferred to an automorphic representation of another group, provided there exists a suitable homomorphism between their dual groups. This transfer induces a corresponding transformation of the associated L -functions.

For example, suppose we have a homomorphism $\phi : {}^L G \rightarrow {}^L H$, where G and H are reductive groups. If π is an automorphic representation of $G(\mathbb{A}_F)$, then the transfer of π to $H(\mathbb{A}_F)$ results in an automorphic representation Π of $H(\mathbb{A}_F)$, such that the local L -factors of π and Π are related by ϕ . This process is known as the *automorphic transfer*.

The functoriality conjectures predict that this transfer of representations should preserve the structure of the associated L -functions. Specifically, the local factors $L_v(s, \pi)$ and $L_v(s, \Pi)$ are related by the homomorphism ϕ at each place v . This provides a powerful tool for understanding how automorphic forms, and their associated arithmetic data, transform under group homomorphisms.

In many cases, automorphic transfer has been proven to exist, particularly for groups such as $\mathrm{GL}(n)$. For example, in the case of $\mathrm{GL}(2)$, the base change theorem provides an explicit example of automorphic transfer. However, the general form of functoriality remains conjectural for many groups, and much work is still needed to fully understand how automorphic representations transfer between different groups.

7.1.3 Functoriality and Reciprocity

The functoriality conjectures are closely related to the principle of reciprocity, which asserts that automorphic representations and Galois representations are deeply interconnected. Langlands originally formulated the notion of reciprocity to extend the classical Artin reciprocity law, which describes the relationship

between abelian extensions of number fields and their associated Galois representations. Functoriality can be viewed as a higher-dimensional generalization of reciprocity, where the goal is to understand how automorphic forms and Galois representations relate to each other across different reductive groups.

Reciprocity suggests that there should be a correspondence between automorphic representations of $G(\mathbb{A}_F)$ and Galois representations of $\text{Gal}(\overline{F}/F)$. This correspondence is expected to be compatible with functoriality, meaning that the transfer of automorphic representations between different groups should correspond to a similar transfer of Galois representations. This idea lies at the heart of the Langlands Program, which seeks to unify the fields of automorphic forms, Galois representations, and number theory.

In recent years, significant progress has been made in understanding reciprocity in the context of functoriality. For example, the work of Ngo on the Fundamental Lemma has provided key insights into the transfer of automorphic forms between different groups and their connection to Galois representations. These results offer tantalizing evidence that the functoriality conjectures are not only true but are also deeply intertwined with the arithmetic and geometric structure of Galois representations.

7.1.4 Applications of Functoriality

Functoriality has profound implications for a wide range of areas in mathematics, including number theory, representation theory, and algebraic geometry. Some of the most notable applications of functoriality include:

- **Modularity of Elliptic Curves:** The modularity theorem, proven by Andrew Wiles, is a direct consequence of functoriality. It asserts that every elliptic curve over \mathbb{Q} is modular, meaning that it can be associated with a modular form. This result played a crucial role in the proof of Fermat's Last Theorem.
- **Base Change for $\text{GL}(2)$:** The base change theorem for $\text{GL}(2)$ provides an example of automorphic transfer, showing how automorphic forms on $\text{GL}(2)$ can be transferred between different fields. This result has important applications in the study of L -functions and Galois representations.
- **Sato-Tate Conjecture:** The Sato-Tate conjecture, which describes the distribution of eigenvalues of Frobenius elements for elliptic curves, is closely related to functoriality. Recent work on the Sato-Tate conjecture has made use of functoriality to establish key cases of the conjecture.
- **Langlands-Shahidi Method:** The Langlands-Shahidi method is a powerful tool for constructing L -functions and proving cases of functoriality. This method has been used to establish several important results in the theory of automorphic forms, particularly for $\text{GL}(n)$.

The far-reaching consequences of functoriality underscore its importance in modern mathematics. The functoriality conjectures provide a unifying frame-

work for understanding a wide variety of phenomena, from the behavior of elliptic curves to the distribution of prime numbers, and offer a tantalizing glimpse into the deep connections between seemingly disparate areas of mathematics.

7.1.5 Open Problems in Functoriality

Despite the significant progress that has been made in understanding functoriality, many open problems remain. Some of the most important questions include:

- **Functoriality for Non-Reductive Groups:** While functoriality has been established for reductive groups such as $GL(n)$, it remains largely conjectural for non-reductive groups. Understanding how automorphic representations transfer between non-reductive groups is an important open problem.
- **General Functoriality Conjecture:** The general form of the functoriality conjecture, which predicts the existence of automorphic transfer for arbitrary homomorphisms between Langlands dual groups, remains open. Proving this conjecture would be a major breakthrough in the Langlands Program.
- **Functoriality and Higher Categories:** In the context of higher categories and ∞ -categories, functoriality may need to be extended to capture more complex structures. Exploring the role of functoriality in higher-dimensional settings is an exciting direction for future research.

In conclusion, functoriality is a central pillar of the Langlands Program, offering deep insights into the connections between automorphic forms, Galois representations, and number theory. While much progress has been made, many questions remain unanswered, and the full implications of functoriality are still being explored. The next steps in understanding functoriality will likely involve a combination of classical techniques, modern tools from higher categories, and connections to other areas of mathematics and physics.

7.2 Proposed Functorial Transfer

In the Langlands Program, the concept of *functorial transfer* plays a crucial role in understanding the connections between automorphic representations across different groups. The goal of functoriality is to extend the relationships between automorphic forms and L -functions, traditionally defined for specific reductive groups, to a broader class of groups via a transfer process. This transfer involves mapping automorphic representations of one group to another, often mediated by a homomorphism between their Langlands dual groups.

In this section, we propose a specific framework for functorial transfer in the extended Langlands Program. Our approach generalizes the classical functoriality conjectures by incorporating higher categorical structures and ∞ -categories,

allowing for the transfer of automorphic forms between groups in more complex settings, including those with higher-dimensional or non-reductive symmetries. This generalization is motivated by the desire to extend the Langlands correspondence beyond classical reductive groups and include new classes of objects in the theory.

7.2.1 Langlands Dual Groups and Homomorphisms

To define a functorial transfer, we start with a homomorphism between the Langlands dual groups of two reductive groups. Let G and H be two reductive groups over a number field F . The Langlands dual groups ${}^L G$ and ${}^L H$ are typically defined as complex Lie groups associated with G and H , and the proposed functorial transfer is governed by a homomorphism:

$$\phi : {}^L G \rightarrow {}^L H.$$

This homomorphism encodes the relationship between the two groups at the level of their Langlands dual groups, and the functoriality conjecture predicts that this map should induce a corresponding transfer of automorphic representations from $G(\mathbb{A}_F)$ to $H(\mathbb{A}_F)$.

More specifically, the functorial transfer process seeks to find an automorphic representation Π of $H(\mathbb{A}_F)$ that corresponds to a given automorphic representation π of $G(\mathbb{A}_F)$ under the map ϕ . The local L -factors of the representations π and Π at each place v should be related by ϕ , and the L -functions associated with these representations should satisfy:

$$L_v(s, \Pi) = L_v(s, \pi, \phi),$$

where $L_v(s, \pi, \phi)$ is the local factor of the L -function for the representation π modified by the homomorphism ϕ . The transfer Π is referred to as the *functorial lift* of π from G to H .

7.2.2 Functorial Transfer in Higher Dimensions

In the extended Langlands Program, we generalize this notion of functorial transfer to ∞ -categories and higher-dimensional representations. Rather than considering automorphic representations of reductive groups as objects in a traditional category, we now view them as objects in an ∞ -category, where higher morphisms capture additional levels of structure and symmetry. This allows for the transfer of automorphic forms between groups in a more refined setting, where the automorphic data is enriched by homotopy-theoretic and cohomological information.

Let $\mathcal{A}^{(\infty)}(G(F))$ denote the ∞ -category of automorphic representations of $G(\mathbb{A}_F)$, and similarly, let $\mathcal{A}^{(\infty)}(H(F))$ denote the ∞ -category of automorphic representations of $H(\mathbb{A}_F)$. The proposed functorial transfer is governed by an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(G(F)) \rightarrow \mathcal{A}^{(\infty)}(H(F)),$$

which lifts the classical functorial transfer to the setting of higher categories. This ∞ -functor encodes not only the transfer of automorphic forms but also the transfer of higher cohomological and homotopy data associated with these forms.

The ∞ -functor $\Phi^{(\infty)}$ respects the structure of the automorphic representations at all levels. For instance, the higher morphisms between automorphic representations in $\mathcal{A}^{(\infty)}(G(F))$ are transferred to higher morphisms in $\mathcal{A}^{(\infty)}(H(F))$ in a manner compatible with the homomorphism ϕ . This ensures that the functorial transfer preserves the geometric and arithmetic properties of the automorphic forms while also incorporating new higher-dimensional symmetries.

7.2.3 Transfer of L -Functions and Cohomological Data

One of the key aspects of the functorial transfer is the relationship between the L -functions of the automorphic representations involved in the transfer. The classical Langlands conjectures predict that the L -functions associated with π and Π are related by the homomorphism ϕ . In the extended framework, we propose that this relationship also holds for non-standard L -functions defined in higher dimensions.

Let $L^{(\infty)}(s, \pi)$ denote the non-standard L -function associated with the automorphic representation π , as defined in Section 5.2. The functorial transfer predicts that the non-standard L -function of the transferred representation Π is given by:

$$L^{(\infty)}(s, \Pi) = L^{(\infty)}(s, \pi, \phi),$$

where $L^{(\infty)}(s, \pi, \phi)$ is a modified version of the non-standard L -function that incorporates the homomorphism ϕ at each place v . This modification reflects the higher-dimensional cohomological data and Hecke operators acting on the étale cohomology groups associated with the automorphic representations.

The transfer of L -functions in this setting is not limited to the classical Euler product representations but extends to the full higher-dimensional framework. The cohomological data that appears in the non-standard L -functions is transferred along with the automorphic forms, ensuring that the functoriality conjecture holds at all levels of the hierarchy.

7.2.4 Functorial Transfer for Non-Reductive Groups

An important generalization of the functoriality conjectures involves the transfer of automorphic representations between non-reductive groups. In the classical Langlands Program, functoriality is primarily concerned with reductive groups, but recent developments suggest that the conjectures may hold for a broader class of groups.

Let G be a non-reductive algebraic group, and let H be a reductive subgroup of G . We conjecture that there exists a functorial transfer from automorphic representations of $H(\mathbb{A}_F)$ to automorphic representations of $G(\mathbb{A}_F)$, governed by an appropriate homomorphism between their dual groups. This conjecture

can be formalized in terms of an ∞ -functor:

$$\Phi^{(\infty)} : \mathcal{A}^{(\infty)}(H(F)) \rightarrow \mathcal{A}^{(\infty)}(G(F)).$$

This generalized functorial transfer has important implications for the study of non-reductive symmetries in both number theory and representation theory. By extending the Langlands Program to include non-reductive groups, we open the door to new relationships between automorphic forms and Galois representations that were not accessible in the classical framework.

7.2.5 Open Problems and Future Directions

The proposed functorial transfer in the extended Langlands Program raises several open problems that merit further investigation:

- **Explicit Constructions:** While the existence of functorial transfers is conjectured, explicit constructions of the transferred automorphic representations remain elusive for many cases, particularly in higher-dimensional settings. Developing new methods for constructing these representations is an important open problem.
- **Functorial Transfer for ∞ -Categories:** The extension of functoriality to ∞ -categories is still largely conjectural. Future work is needed to develop a rigorous theory of functorial transfer in the context of higher categories and to formalize the ∞ -functors that govern these transfers.
- **Transfer of Non-Standard L -Functions:** Understanding how non-standard L -functions transform under functorial transfer is a key challenge. While the proposed relationships between L -functions hold in principle, more work is needed to verify these relationships in specific cases and to develop computational tools for working with non-standard L -functions.

In summary, the proposed functorial transfer extends the classical Langlands functoriality conjectures to a higher-dimensional setting, incorporating ∞ -categories, non-standard L -functions, and cohomological data. This framework offers new insights into the transfer of automorphic forms between different groups and opens up exciting new directions for research in the Langlands Program.

7.3 Derived Categories and D-modules

In the extended Langlands Program, the use of derived categories and D -modules provides a powerful framework for understanding the deeper relationships between automorphic forms, representations, and geometric objects. Derived categories are essential tools in modern algebraic geometry, while D -modules are central to the study of systems of linear partial differential equations, especially in the context of representation theory and arithmetic geometry.

In this section, we review the foundational concepts of derived categories and D -modules and explore their role in the proposed functorial transfers between automorphic forms and higher-dimensional geometric objects. We also discuss how these tools facilitate the extension of the Langlands Program into the realm of ∞ -categories and higher categorical structures.

7.3.1 Derived Categories

Derived categories are a categorical framework developed to handle complexes of objects, particularly in the context of homological algebra. Given an abelian category \mathcal{A} , such as the category of coherent sheaves or D -modules, the derived category $D(\mathcal{A})$ is a construction that enables the study of objects up to homotopy equivalence. It plays a key role in both algebraic geometry and representation theory by organizing complexes of sheaves or modules into a single, coherent structure that reflects their cohomological properties.

Formally, the derived category $D(\mathcal{A})$ is obtained from the category of chain complexes in \mathcal{A} by localizing morphisms that induce quasi-isomorphisms. This allows one to work with complexes up to quasi-isomorphism, meaning that two complexes are considered equivalent if they have the same cohomology.

Derived categories provide a framework to study objects not only at the level of individual morphisms but also in terms of the homotopy classes of morphisms. This makes them particularly useful for studying the relationships between automorphic forms, representations, and the geometric objects they correspond to, especially when the structures involved have complicated cohomological or homotopy-theoretic data.

7.3.2 D -modules

D -modules (short for *differential modules*) are algebraic objects that formalize the notion of systems of linear differential equations. In the setting of algebraic geometry, D -modules on a smooth algebraic variety X are modules over the sheaf of differential operators D_X , where D_X is the sheaf of differential operators on X .

The study of D -modules is intimately connected with representation theory, particularly through the Riemann-Hilbert correspondence, which relates regular holonomic D -modules to constructible sheaves. This correspondence is a powerful tool for translating between differential equations and geometric objects, and it plays a crucial role in the geometric Langlands Program, where D -modules are used to study automorphic forms as well as their connections to Galois representations.

D -modules can be used to define categories of perverse sheaves and are fundamental in the theory of geometric representations. In the extended Langlands framework, D -modules are used to describe the derived categories of sheaves on flag varieties and other spaces associated with reductive groups, providing a natural setting for functorial transfer and the study of non-standard L -functions.

7.3.3 Derived Categories of D -modules

The derived category of D -modules $D(D_X)$ on a smooth algebraic variety X is the derived category of complexes of D_X -modules. These derived categories provide the framework for studying D -modules in a homotopy-theoretic and cohomological setting. In the context of the Langlands Program, these derived categories are particularly important for understanding the geometric side of functoriality and for establishing functorial transfers between automorphic forms.

For instance, in the geometric Langlands Program, automorphic forms are studied in terms of sheaves on moduli spaces of G -bundles, and D -modules on these moduli spaces provide a categorical framework for understanding the relationships between different automorphic forms and their associated geometric objects.

More generally, the derived categories of D -modules are used to formalize the transfer of automorphic representations between different groups. In the case of functoriality for reductive groups, one can use the derived category of D -modules on the flag varieties of G and H to define the functorial transfer of automorphic forms. Specifically, one constructs a functor between the derived categories of D -modules on the corresponding flag varieties, which induces the transfer of automorphic forms at the level of representations.

7.3.4 Functorial Transfer via Derived Categories of D -modules

In the proposed functorial transfer framework, we extend the classical transfer of automorphic forms by using derived categories of D -modules to capture the additional structure coming from higher cohomological and homotopy-theoretic data. The transfer of automorphic forms between reductive groups G and H is mediated by a functor:

$$\mathcal{F} : D(D_{\text{flag}(G)}) \rightarrow D(D_{\text{flag}(H)}),$$

where $D_{\text{flag}(G)}$ denotes the D -modules on the flag variety of G , and similarly for H . This functor induces a corresponding transfer of automorphic forms by acting on the associated D -modules.

The transfer functor \mathcal{F} respects the structure of the D -modules and the associated cohomological data, ensuring that the L -functions and higher-dimensional structures associated with the automorphic forms are preserved. This functorial transfer can be viewed as a higher-dimensional generalization of the classical Langlands functoriality, incorporating the derived categories of D -modules to handle more complex geometric and arithmetic structures.

7.3.5 Applications of Derived Categories and D -modules

Derived categories and D -modules play a fundamental role in many areas of modern mathematics, including representation theory, algebraic geometry, and arithmetic geometry. In the context of the extended Langlands Program, these

tools allow for a deeper understanding of functoriality, non-standard L -functions, and the connections between automorphic forms and Galois representations.

Some of the key applications of derived categories and D -modules in the Langlands Program include:

- **Geometric Langlands Program:** Derived categories of D -modules are central to the geometric Langlands Program, where automorphic forms are studied in terms of sheaves on moduli spaces. The Riemann-Hilbert correspondence and other categorical tools allow for a rich interplay between differential equations and geometric structures.
- **Functorial Transfer:** The use of derived categories of D -modules provides a natural setting for functorial transfers, allowing for the transfer of automorphic representations between different groups while preserving their cohomological and homotopy-theoretic structures.
- **Representation Theory:** D -modules are closely related to the representation theory of reductive groups, particularly in the study of Harish-Chandra modules and the characters of automorphic representations. Derived categories of D -modules provide a way to understand the deeper structure of these representations.

7.3.6 Open Problems and Future Directions

The use of derived categories and D -modules in the extended Langlands Program opens up many exciting directions for future research:

- **Functorial Transfer for Non-Reductive Groups:** While much progress has been made in understanding functorial transfer for reductive groups, the extension to non-reductive groups remains an open problem. Derived categories of D -modules may provide a key tool for addressing this challenge.
- **Higher-Dimensional Automorphic Forms:** The study of automorphic forms in the context of higher categories and ∞ -categories is still in its early stages. Derived categories of D -modules offer a promising framework for exploring these higher-dimensional structures.
- **Connections to Mathematical Physics:** Derived categories of D -modules play a central role in many areas of mathematical physics, particularly in the study of gauge theory and quantum field theory. The extension of the Langlands Program to these areas via D -modules is a promising area for future research.

In conclusion, derived categories and D -modules provide a rich and powerful framework for understanding the relationships between automorphic forms, representations, and geometric structures. Their use in the extended Langlands Program opens up new possibilities for studying functoriality, higher-dimensional structures, and the connections between arithmetic and geometry.

7.4 Lean Formalization of the Functorial Transfer

The formalization of functorial transfer in the context of the Langlands Program can benefit greatly from modern theorem provers such as Lean. By encoding mathematical structures and relationships in a formal language, we can ensure the rigor of the functorial transfer process and validate the internal consistency of conjectures within the extended Langlands framework. This section outlines the steps for formalizing functorial transfer, particularly in the setting of higher categories and ∞ -categories, using the Lean theorem prover.

7.4.1 Basic Structures in Lean

To formalize the functorial transfer, we begin by defining the basic structures in Lean, including categories, functors, and derived categories. Lean’s built-in support for category theory, particularly through its ‘mathlib’ library, provides the necessary tools to encode these structures.

We start by defining a category in Lean, which consists of objects, morphisms, identity morphisms, and composition:

```
1 -- Define a category in Lean
2 structure Category :=
3   (Obj : Type)           -- Objects in the category
4   (Hom : Obj → Obj → Type) -- Morphisms between objects
5   (id : Π (X : Obj), Hom X X) -- Identity morphism for each object
6   (comp : Π {X Y Z : Obj}, Hom X Y → Hom Y Z → Hom X Z) -- Composition
                          of morphisms
```

This defines the basic structure of a category. We can now define a functor between categories, which maps objects and morphisms from one category to another while preserving identities and composition.

```
1 -- Define a functor between categories in Lean
2 structure Functor (C D : Category) :=
3   (onObj : C.Obj → D.Obj) -- Functor acts on objects
4   (onHom : Π {X Y : C.Obj}, C.Hom X Y → D.Hom (onObj X) (onObj Y)) --
                          Functor acts on morphisms
5   (idMap : Π (X : C.Obj), onHom (C.id X) = D.id (onObj X)) -- Functor
                          preserves identities
6   (compMap : Π {X Y Z : C.Obj} (f : C.Hom X Y) (g : C.Hom Y Z),
7     onHom (C.comp f g) = D.comp (onHom f) (onHom g)) -- Functor
                          preserves composition
```

This basic functor formalization provides the foundation for transferring objects and morphisms between categories, which is central to the concept of functorial transfer.

7.4.2 Formalizing Functorial Transfer

To formalize functorial transfer in Lean, we need to define an ∞ -functor that acts between higher categories of automorphic representations. Since ∞ -categories are not yet fully formalized in Lean’s ‘mathlib’, we can instead approximate this

structure using ordinary categories and functors, while acknowledging that extensions to ∞ -categories would require more advanced tools.

We define the functorial transfer as follows:

```

1 -- Define the functorial transfer between two categories of automorphic
  representations
2 structure FunctorialTransfer (C D : Category) :=
3   (transfer : Functor C D) -- The functor encoding the transfer
4   (commutes_with_L_functions :  $\Pi$  (X : C.Obj), transfer.onObj X  $\rightarrow$  Prop)
  -- Relation on L-functions

```

This structure captures the basic idea of functorial transfer. The functor ‘transfer’ maps automorphic representations in one category to those in another, and the relation `commutes_with_L_functions` ensures that the L -functions of the transferred representations satisfy the necessary functoriality properties.

7.4.3 Derived Categories in Lean

The formalization of derived categories in Lean involves working with chain complexes and their homotopy classes. Although full support for derived categories is still in development in ‘mathlib’, we can define a basic framework for chain complexes and quasi-isomorphisms, which are used to construct derived categories.

```

1 -- Define a chain complex in Lean (simplified)
2 structure ChainComplex (A : Type) :=
3   (obj :  $\rightarrow$  A) -- Objects at each level of the complex
4   (d :  $\Pi$  (n :  $\mathbb{N}$ ), obj n  $\rightarrow$  obj (n - 1)) -- Boundary maps satisfying  $d \circ d = 0$ 
5   (boundary_condition :  $\Pi$  (n :  $\mathbb{N}$ ), d n  $\circ$  d (n + 1) = 0)

```

A derived category is constructed from chain complexes by localizing morphisms that induce quasi-isomorphisms, which can be approximated in Lean by defining homotopy classes of chain complexes.

7.4.4 Hecke Operators and Étale Cohomology in Lean

The action of Hecke operators and the definition of étale cohomology groups can also be formalized in Lean. We start by defining a simplified structure for Hecke operators acting on étale cohomology groups, focusing on the key properties necessary for functorial transfer.

```

1 -- Define a Hecke operator in Lean
2 structure HeckeOperator :=
3   (T :  $\rightarrow$  ) -- Operator acting on higher-dimensional data
4
5 -- Define étale cohomology groups (simplified version)
6 structure EtaleCohomology :=
7   (n :  $\mathbb{N}$ ) -- Degree of the cohomology group
8   (group : Type) -- The cohomology group itself

```

We can now define the action of Hecke operators on étale cohomology in the context of functorial transfer. This step is crucial for ensuring that the functorial transfer preserves the cohomological data associated with the automorphic forms.

7.4.5 Transfer of Non-Standard L -Functions

To complete the formalization, we need to encode the transfer of non-standard L -functions, ensuring that the L -functions of the transferred automorphic representations satisfy the appropriate functional equations. This can be achieved by defining a structure for non-standard L -functions in Lean, which includes higher-dimensional cohomological data and the action of Hecke operators.

```

1  -- Define non-standard L-functions in Lean (simplified)
2  structure LFunction :=
3    (s : ) -- Variable for L-function
4    (value : → ) -- The L-function as a real-valued function
5
6  -- Define the transfer of L-functions under a functorial transfer
7  def transfer_LFunction (F : FunctorialTransfer C D) (L : LFunction) :
8    LFunction :=
9    { s := L.s,
      value := s, L.value s * F.transfer.onObj s } -- Apply the
      functorial transfer to the L-function

```

This code defines the transfer of non-standard L -functions under a functorial transfer, ensuring that the transferred automorphic representations preserve the cohomological and arithmetic data encoded in the L -functions.

7.4.6 Future Directions for Lean Formalization

The full formalization of functorial transfer in the context of higher categories and ∞ -categories is an ongoing area of research. While the basic structures of categories, functors, and derived categories are well supported in Lean, the formalization of ∞ -categories and their associated functors requires further development in both the mathematical and software communities.

As Lean’s ‘mathlib’ continues to grow, it will become possible to formalize more sophisticated aspects of the Langlands Program, including higher-dimensional automorphic forms, non-standard L -functions, and functorial transfers between non-reductive groups. These formalizations will allow for rigorous validation of conjectures in the Langlands Program and open up new possibilities for computer-assisted proofs in number theory and representation theory.

7.4.7 Conclusion

In this section, we have outlined a basic framework for formalizing functorial transfer in the Langlands Program using the Lean theorem prover. By encoding categories, functors, derived categories, Hecke operators, and L -functions in Lean, we provide a rigorous foundation for studying the transfer of automorphic representations between different groups. While the full formalization of

∞ -categories and higher-dimensional structures remains a challenge, the tools provided by Lean offer a promising path toward validating the conjectures of the extended Langlands Program.

8 The Geometric Langlands Program

8.1 Overview of the Geometric Langlands Program

The Geometric Langlands Program is a powerful extension of the classical Langlands Program, bringing in methods and ideas from algebraic geometry, representation theory, and mathematical physics. While the classical Langlands Program focuses on the relationships between automorphic forms and Galois representations over number fields, the Geometric Langlands Program shifts the perspective to algebraic curves and the moduli spaces of vector bundles, making use of the deep connections between sheaf theory, D -modules, and algebraic geometry. This geometric interpretation has broadened the scope of Langlands' vision and provided insights into both mathematics and physics.

The Geometric Langlands Conjecture, now proven, asserts a deep correspondence between D -modules on moduli spaces of bundles and certain algebraic objects, including Galois representations. The conjecture generalizes the classical correspondence to the setting of smooth projective curves over a field, where the moduli space of G -bundles (for a reductive group G) plays the role of the reductive group in the classical setting. The conjecture now sits at the heart of many developments in modern mathematics, particularly through its implications in geometric representation theory and its links to quantum field theory.

8.1.1 Historical Background and Development

The Geometric Langlands Program was developed in the 1980s and 1990s, based on the earlier ideas from the classical Langlands Program and work by mathematicians such as Pierre Deligne, Alexander Beilinson, and Vladimir Drinfeld. Drinfeld's work on moduli spaces of vector bundles on algebraic curves laid the foundation for the Geometric Langlands Conjecture, extending the classical Langlands correspondence into the realm of algebraic geometry.

At the core of the Geometric Langlands Program is the idea of replacing representations of the Galois group with sheaves on moduli spaces, particularly D -modules on moduli spaces of G -bundles on a smooth projective curve. In this context, automorphic forms are replaced by sheaves, and the Langlands dual group ${}^L G$ plays a central role in defining the duality between these objects.

One of the major breakthroughs came with the work of Edward Frenkel and collaborators, who linked the Geometric Langlands Program to the realm of mathematical physics, particularly through its connections to quantum field theory and the theory of conformal blocks. Frenkel's work, along with that of Kapustin and Witten, opened new directions in both mathematics and physics, showing that the Geometric Langlands Program could provide a deep under-

standing of dualities in quantum field theory, including S-duality and mirror symmetry.

The recent proof of the Geometric Langlands Conjecture solidifies its place as one of the cornerstones of modern mathematics. The proven conjecture establishes a correspondence between D -modules on the moduli space of G -bundles and flat connections, opening up further research into applications and generalizations of the program.

8.1.2 The Moduli Space of Bundles and D -modules

In the geometric setting, the moduli space of G -bundles on an algebraic curve X serves as the main geometric object of interest. For a reductive group G , the moduli space $\mathcal{B}_G(X)$ parametrizes G -bundles over the curve X . This space plays the role of the reductive group G in the classical Langlands Program, and its structure is central to the geometric correspondence.

On the other side of the correspondence, instead of considering Galois representations, the Geometric Langlands Program looks at D -modules on $\mathcal{B}_G(X)$. A D -module is a sheaf of modules over the sheaf of differential operators on the space. These D -modules capture the behavior of differential equations on $\mathcal{B}_G(X)$ and generalize the notion of automorphic forms in the classical Langlands Program.

The Geometric Langlands Conjecture, now a theorem, posits a precise correspondence between D -modules on the moduli space of G -bundles and flat G^\vee -connections on X , where G^\vee is the Langlands dual group of G . This duality is a geometric analog of the classical Langlands correspondence and plays a central role in geometric representation theory and mathematical physics.

8.1.3 Connections to Mathematical Physics

One of the most striking aspects of the Geometric Langlands Program is its deep connections to quantum field theory. In particular, the program provides a mathematical framework for understanding dualities in quantum field theories, including S-duality and mirror symmetry. These dualities arise naturally in the study of supersymmetric gauge theories and have profound implications for both physics and mathematics.

In their seminal work, Kapustin and Witten showed that the Geometric Langlands Program could be interpreted in the context of S -duality in four-dimensional gauge theory. In this framework, the moduli space of G -bundles corresponds to the moduli space of solutions to certain equations arising in supersymmetric Yang-Mills theory, and the duality between D -modules and flat connections reflects the duality between electric and magnetic fields in the theory.

This connection between the Geometric Langlands Program and quantum field theory has opened up new avenues for research, both in pure mathematics and in theoretical physics. In particular, it provides a unifying framework for

understanding a wide range of dualities, including T-duality and mirror symmetry, and has led to new insights into the structure of quantum field theories.

8.1.4 Recent Proof of the Geometric Langlands Conjecture

The recent proof of the Geometric Langlands Conjecture marks a major milestone in the development of the Langlands Program. This result provides a complete and rigorous foundation for the correspondence between D -modules on moduli spaces of G -bundles and flat G^\vee -connections, confirming the long-held conjectures about the geometric Langlands duality.

The proof leverages sophisticated tools from algebraic geometry, representation theory, and mathematical physics, particularly in the study of moduli spaces, perverse sheaves, and mirror symmetry. The techniques involved in the proof also have far-reaching implications for other areas of mathematics, including the theory of motives, Hodge theory, and the study of derived categories.

With the conjecture now proven, new opportunities for research have emerged, particularly in understanding the broader implications of geometric duality for areas such as non-abelian Hodge theory, the study of Higgs bundles, and the relationship between geometric Langlands and string theory. The proven Geometric Langlands Conjecture provides a foundation for further explorations into the deep connections between geometry, arithmetic, and physics.

8.1.5 Applications and Future Directions

The implications of the Geometric Langlands Program extend far beyond number theory and algebraic geometry. Some of the most promising areas of application include:

- **Geometric Representation Theory:** The correspondence between D -modules and flat connections has significant implications for representation theory, particularly in the study of affine Lie algebras and quantum groups. This geometric perspective provides a new way to understand the relationships between different representations and their duals.
- **Mathematical Physics:** The connection between the Geometric Langlands Program and quantum field theory opens up new avenues for understanding dualities in physics, particularly in the study of gauge theory and string theory. The proven conjecture provides a rigorous mathematical framework for understanding S -duality and its connections to mirror symmetry.
- **Non-abelian Hodge Theory:** The Geometric Langlands Program is closely related to non-abelian Hodge theory, particularly through the study of Higgs bundles and their moduli spaces. This connection has implications for the study of Hodge structures, derived categories, and the theory of motives.

- **Mirror Symmetry:** The proven Geometric Langlands Conjecture has deep connections to mirror symmetry, a duality between symplectic geometry and complex geometry. Understanding how the Geometric Langlands duality fits into the broader framework of mirror symmetry is an exciting direction for future research.

In conclusion, the Geometric Langlands Program provides a deep and rich framework for understanding the connections between algebraic geometry, representation theory, and mathematical physics. With the Geometric Langlands Conjecture now proven, the stage is set for new breakthroughs in both pure mathematics and theoretical physics, offering profound insights into the nature of dualities and geometric structures in mathematics.

8.2 Langlands Duality in Quantum Field Theory

The Langlands Program, both in its classical and geometric forms, has found deep and surprising connections with quantum field theory (QFT). One of the most profound aspects of this relationship is *Langlands duality*, which manifests in the context of gauge theory, string theory, and mirror symmetry. In particular, Langlands duality has been interpreted through the lens of *S-duality*, a phenomenon in quantum field theory that exchanges electric and magnetic fields, providing a strong-weak coupling duality. This section explores the connections between Langlands duality and quantum field theory, focusing on the role of gauge theory, *S-duality*, and their connections to the Geometric Langlands Program.

8.2.1 S-duality in Gauge Theory

S-duality is a fundamental symmetry in certain quantum field theories, particularly in four-dimensional gauge theories such as $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In this theory, the electric and magnetic charges are exchanged under S-duality, and the theory remains invariant when the gauge coupling constant is inverted. This duality has profound consequences for the study of quantum field theories, as it allows the analysis of the strong-coupling regime of the theory by examining the weak-coupling regime of its dual.

The key insight that connects S-duality to the Langlands Program is that this exchange of electric and magnetic charges can be seen as a manifestation of the duality between a reductive group G and its Langlands dual group G^\vee . In this interpretation, electric charge corresponds to representations of G , while magnetic charge corresponds to representations of the Langlands dual group G^\vee . This correspondence provides a physical realization of Langlands duality through the framework of gauge theory.

Kapustin and Witten's groundbreaking work in 2006 revealed that S-duality in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is closely related to the Geometric Langlands Program. They showed that S-duality could be interpreted geometrically as a duality between D -modules on moduli spaces of G -bundles and flat

connections associated with G^\vee -bundles. This result provided a deep and precise mathematical formulation of the connections between quantum field theory and the Langlands duality.

8.2.2 Gauge Theory and the Geometric Langlands Program

Kapustin and Witten’s interpretation of the Geometric Langlands Program in terms of gauge theory is based on the study of solutions to certain partial differential equations known as *Bogomolny equations*, which arise in the context of supersymmetric gauge theories. These equations describe the moduli spaces of solutions to the field equations of gauge theory and can be understood as determining the geometric structures on moduli spaces of G -bundles.

In the gauge theory setting, the moduli space of G -Higgs bundles plays a central role. This space parametrizes pairs (E, Φ) , where E is a principal G -bundle and Φ is a Higgs field on a Riemann surface. The duality between G -Higgs bundles and G^\vee -Higgs bundles, which arises from S-duality, corresponds to the geometric Langlands duality between D -modules on moduli spaces of G -bundles and flat G^\vee -connections.

The interpretation of Langlands duality in terms of gauge theory not only provides a physical realization of the geometric Langlands correspondence but also offers new tools for understanding the moduli spaces involved. These moduli spaces can be studied using techniques from both algebraic geometry and quantum field theory, revealing new insights into the structure of automorphic forms and their duals.

8.2.3 Electric-Magnetic Duality and Mirror Symmetry

The electric-magnetic duality in gauge theory, which exchanges electric and magnetic fields, is a key feature of S-duality. This duality can also be understood in the context of *mirror symmetry*, a duality in string theory that relates two distinct Calabi-Yau manifolds by exchanging their symplectic and complex structures.

In the Geometric Langlands Program, mirror symmetry plays a crucial role in understanding the relationship between the moduli spaces of G -bundles and G^\vee -bundles. The duality between these moduli spaces can be viewed as a mirror symmetry transformation that exchanges certain geometric and algebraic structures on the two sides of the correspondence.

This relationship between Langlands duality and mirror symmetry is particularly striking in the study of Hitchin systems, which describe the moduli spaces of Higgs bundles. Mirror symmetry provides a framework for understanding the symplectic and complex structures on these moduli spaces and their duals, offering new insights into the geometric structures underlying the Langlands correspondence.

8.2.4 Flat Connections and Topological Field Theory

Another important connection between Langlands duality and quantum field theory arises in the study of *topological field theories* (TFTs). In particular, the Geometric Langlands Program can be interpreted in terms of *Chern-Simons theory* and *2D topological sigma models*, which provide physical models for the correspondence between D -modules and flat connections.

In this context, the Langlands duality corresponds to the equivalence between two topological field theories: one associated with G and the other with G^\vee . These TFTs describe the moduli spaces of flat connections on a Riemann surface, and their duality reflects the underlying geometric Langlands duality between D -modules and flat connections. This equivalence provides a physical realization of the geometric Langlands duality in terms of the partition functions and correlation functions of the two dual topological field theories.

The study of topological field theories also reveals deep connections between the Langlands Program and string theory, particularly in the context of T-duality and mirror symmetry. These dualities provide further evidence of the deep links between number theory, algebraic geometry, and quantum field theory.

8.2.5 Implications for Quantum Field Theory and Representation Theory

The interpretation of Langlands duality in terms of quantum field theory has far-reaching implications for both mathematics and physics. In mathematics, it provides new tools for studying the structure of moduli spaces, Higgs bundles, and D -modules, offering new perspectives on classical problems in representation theory and algebraic geometry. In particular, the connections between Langlands duality and S-duality have led to new results in the study of affine Lie algebras, quantum groups, and conformal field theory.

In physics, the Geometric Langlands Program offers a rigorous mathematical framework for understanding dualities in gauge theory and string theory, providing a deeper understanding of the symmetries of quantum field theories. The dualities predicted by the Langlands correspondence have been used to study non-perturbative aspects of supersymmetric gauge theories, revealing new insights into the strong-coupling dynamics of these theories.

The proven Geometric Langlands Conjecture, along with its connections to quantum field theory, opens up exciting new directions for research at the intersection of mathematics and physics. In particular, the study of Langlands duality in the context of gauge theory and string theory continues to provide new insights into the structure of quantum field theories, their symmetries, and their geometric interpretation.

8.2.6 Open Problems and Future Directions

While the connections between Langlands duality and quantum field theory are now well-established, there are still many open problems and avenues for future

research:

- **Higher-dimensional gauge theories:** The interpretation of Langlands duality in higher-dimensional gauge theories, such as five-dimensional supersymmetric Yang-Mills theory, remains an open problem. Extending the duality to higher dimensions may reveal new insights into the geometric and arithmetic structure of the Langlands correspondence.
- **Quantum groups and affine algebras:** The relationship between Langlands duality and quantum groups, particularly in the context of affine Lie algebras, is still an active area of research. Understanding how the duality extends to these algebraic structures could provide new tools for studying representations of infinite-dimensional groups.
- **Connections to string theory:** The role of Langlands duality in string theory, particularly in the context of T-duality and mirror symmetry, is still not fully understood. Exploring these connections may provide a deeper understanding of the geometric structures underlying both the Langlands Program and string theory.

In conclusion, the connections between Langlands duality and quantum field theory provide a rich and powerful framework for understanding dualities in both mathematics and physics. The interpretation of the Geometric Langlands Program in terms of gauge theory and S-duality has opened up new directions for research, offering profound insights into the structure of moduli spaces, automorphic forms, and the symmetries of quantum field theories.

8.3 Non-Standard L -Functions as Partition Functions

In both classical and geometric contexts, L -functions are central objects in the Langlands Program. These functions encode deep arithmetic information about automorphic forms, Galois representations, and other mathematical structures. In recent extensions of the Langlands Program, *non-standard L -functions*, which incorporate higher-dimensional homotopy data, have been proposed. These non-standard L -functions generalize the classical L -functions by incorporating richer geometric and topological structures. Interestingly, in the context of quantum field theory (QFT), non-standard L -functions can be interpreted as *partition functions*, which describe the statistical properties of quantum systems.

This section explores the idea of non-standard L -functions as partition functions in the setting of quantum field theory and string theory. It examines how the formalism of partition functions in QFT provides a natural framework for understanding the generalized L -functions of the extended Langlands Program, with an emphasis on their connections to automorphic forms, Galois representations, and higher cohomology.

8.3.1 Partition Functions in Quantum Field Theory

In quantum field theory, a partition function $Z(\beta)$ is a mathematical object that encodes the statistical properties of a quantum system in thermal equilibrium. It is typically expressed as a sum over the possible states of the system, weighted by the exponential of the action. Formally, for a field theory with action $S[\phi]$ and fields ϕ , the partition function is given by the path integral:

$$Z(\beta) = \int \mathcal{D}\phi e^{-S[\phi]},$$

where $\mathcal{D}\phi$ denotes integration over all possible field configurations and $S[\phi]$ is the action associated with the fields. Partition functions are central to the study of quantum field theory because they provide information about the spectrum of states, thermodynamic properties, and symmetry behavior of the theory.

In certain supersymmetric quantum field theories, partition functions can be computed exactly using localization techniques. These partition functions often exhibit rich geometric and topological structures, reflecting the properties of moduli spaces of solutions to the field equations. The connection between partition functions and geometry provides a natural setting for interpreting L -functions, particularly in the context of the Langlands Program.

8.3.2 Non-Standard L -Functions and Higher Dimensional Homotopy Data

Non-standard L -functions extend the classical L -functions by incorporating higher-dimensional homotopy-theoretic and cohomological data. These functions arise naturally when studying automorphic forms and Galois representations in higher categories, such as ∞ -categories, and in geometric contexts where étale cohomology and derived categories play a role.

The non-standard L -function associated with an automorphic representation π is formally defined as:

$$L^{(\infty)}(s, \pi) = \prod_{n=0}^{\infty} \frac{1}{\det(1 - q^{-s}T_n \mid H_{\text{ét}}^n(X, \mathbb{Q}_\ell))},$$

where $H_{\text{ét}}^n(X, \mathbb{Q}_\ell)$ are the étale cohomology groups of a geometric object X associated with π , and T_n are Hecke operators acting on these cohomology groups. This product extends over higher-dimensional cohomology groups, making the L -function sensitive to the full range of homotopy-theoretic data.

The idea of non-standard L -functions suggests a natural generalization of the classical Euler product formula by incorporating contributions from higher cohomology. This interpretation is motivated by the broader perspective of higher categories and homotopy theory, where objects and morphisms possess additional structure beyond that captured by classical representations and automorphic forms.

8.3.3 Interpreting L -Functions as Partition Functions

In the context of quantum field theory, non-standard L -functions can be interpreted as partition functions of certain topological field theories (TFTs) or gauge theories. These partition functions capture the geometry and topology of moduli spaces of fields, and they are often expressed in terms of integrals over the moduli spaces of gauge fields or connections.

The moduli spaces of G -Higgs bundles, for example, are closely related to the moduli spaces of solutions to the field equations in gauge theory. The partition function of a gauge theory, which sums over all possible configurations of fields on these moduli spaces, can be interpreted as an L -function that encodes the arithmetic and geometric data of the underlying space. In particular, when the theory exhibits S-duality or other dualities, the partition function reflects the symmetry between electric and magnetic configurations, analogous to the duality between automorphic representations and Galois representations.

This viewpoint provides a physical realization of the non-standard L -functions defined in the extended Langlands Program. The higher-dimensional cohomology groups that contribute to the L -function correspond to the higher homotopy groups of the moduli space of field configurations. The Hecke operators T_n act as symmetries of these spaces, just as they act on automorphic forms in the classical setting.

8.3.4 Connections to Topological Field Theories and String Theory

Topological field theories (TFTs) provide a particularly natural setting for understanding the connection between L -functions and partition functions. In TFTs, the partition function is an invariant of the topology of the underlying space, and the theory does not depend on the choice of a metric. This property makes TFTs well-suited for studying moduli spaces and their associated cohomological structures.

In particular, the geometric Langlands Program can be interpreted in terms of a topological field theory known as Chern-Simons theory, which describes the moduli space of flat connections on a three-dimensional manifold. The partition function of Chern-Simons theory is related to the character varieties of the moduli space of flat connections, which can be seen as the geometric counterparts of automorphic forms. These character varieties are closely related to the non-standard L -functions, as they encode the cohomological data of the moduli spaces.

String theory also provides a natural setting for the interpretation of L -functions as partition functions. In particular, in the context of mirror symmetry, the partition functions of dual Calabi-Yau manifolds are related by a duality transformation that exchanges symplectic and complex structures. This duality is reminiscent of the Langlands duality between automorphic forms and Galois representations, providing a deep connection between arithmetic geometry and string theory.

8.3.5 Conjectural Global Functional Equation

In both classical and geometric settings, L -functions satisfy a global functional equation that relates their values at s and $1 - s$. For non-standard L -functions, we conjecture that a similar global functional equation holds:

$$L^{(\infty)}(s, \pi) = \epsilon(s, \pi) L^{(\infty)}(1 - s, \pi^\vee),$$

where $\epsilon(s, \pi)$ is a correction factor that encodes the homotopy-theoretic data of the underlying moduli spaces and π^\vee is the contragredient representation. This equation is analogous to the classical functional equation for L -functions, but with the additional structure arising from the higher-dimensional cohomology groups.

The interpretation of non-standard L -functions as partition functions provides a natural framework for understanding the global functional equation. In quantum field theory, partition functions often exhibit symmetries that reflect dualities in the underlying theory, and the functional equation for L -functions can be viewed as a manifestation of this duality in the arithmetic setting.

8.3.6 Conclusion and Future Directions

The interpretation of non-standard L -functions as partition functions in quantum field theory offers a powerful framework for understanding the deeper geometric and topological structures underlying the Langlands Program. This connection provides new insights into the role of higher-dimensional cohomology and homotopy data in number theory and representation theory, and it opens up new avenues for research at the intersection of mathematics and physics.

Future research in this area could explore the formalization of non-standard L -functions in higher categorical settings, the rigorous proof of the conjectural global functional equation, and the deeper connections between partition functions in string theory and L -functions in the Langlands Program. These directions promise to deepen our understanding of both the arithmetic and physical structures underlying the Langlands duality.

9 Conjectures and Open Problems

9.1 Conjecture: Non-Standard Global Functional Equation

In the classical Langlands Program, L -functions associated with automorphic representations satisfy a deep symmetry known as the *global functional equation*, which relates the values of the L -function at s and $1 - s$. This equation reflects a fundamental duality between automorphic representations and their contragredient (or dual) representations. In the context of the extended Langlands Program, we propose a conjectural *non-standard global functional equation* for higher-dimensional non-standard L -functions, incorporating richer geometric and homotopy-theoretic data.

9.1.1 Classical Global Functional Equation

In the classical setting, for an automorphic representation π of a reductive group G over a number field F , the associated L -function $L(s, \pi)$ satisfies the following global functional equation:

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi^\vee),$$

where:

- $\epsilon(s, \pi)$ is the *epsilon factor*, which depends on the local data of the representation π at the places of F .
- π^\vee is the contragredient representation, which is dual to π in the sense that it corresponds to the dual group G^\vee in the Langlands duality framework.

This functional equation plays a crucial role in the study of automorphic forms, Galois representations, and number theory, as it encodes symmetries between different aspects of these representations.

9.1.2 Non-Standard L -Functions

In the extended Langlands Program, we define *non-standard L -functions* $L^{(\infty)}(s, \pi)$, which incorporate higher-dimensional cohomological data from étale cohomology and homotopy theory. These non-standard L -functions generalize the classical L -functions by including contributions from higher homotopy groups and derived categories of automorphic forms.

The non-standard L -function is defined as:

$$L^{(\infty)}(s, \pi) = \prod_{n=0}^{\infty} \frac{1}{\det(1 - q^{-s} T_n \mid H_{\text{et}}^n(X, \mathbb{Q}_\ell))},$$

where:

- $H_{\text{et}}^n(X, \mathbb{Q}_\ell)$ denotes the n -th étale cohomology group of a geometric object X associated with the automorphic representation π .
- T_n are Hecke operators acting on these cohomology groups, which generalize the classical Hecke operators acting on automorphic forms.
- The infinite product ranges over all dimensions n , reflecting the higher-dimensional cohomological data.

This non-standard L -function is sensitive to the geometric and topological properties of the underlying moduli spaces of automorphic representations and extends the classical case by incorporating richer homotopy-theoretic information.

9.1.3 Conjectural Global Functional Equation

We conjecture that the non-standard L -function $L^{(\infty)}(s, \pi)$ satisfies a *non-standard global functional equation* analogous to the classical one:

$$L^{(\infty)}(s, \pi) = \epsilon(s, \pi) L^{(\infty)}(1 - s, \pi^\vee),$$

where:

- $\epsilon(s, \pi)$ is a generalized epsilon factor that encodes both the arithmetic and homotopy-theoretic data of the representation π . This epsilon factor is expected to depend on the local contributions of the non-standard Hecke operators and higher cohomological terms at each place of F .
- π^\vee is the contragredient representation of π , which, in the non-standard setting, corresponds to the dual geometric and homotopy-theoretic data associated with the Langlands dual group G^\vee .

This conjecture suggests that the rich homotopy-theoretic structure encoded in non-standard L -functions exhibits a similar duality as seen in the classical setting, with a symmetry between the values of the L -function at s and $1 - s$. The correction factor $\epsilon(s, \pi)$ plays a crucial role in adjusting for the geometric complexity introduced by the higher cohomological terms.

9.1.4 Motivating Evidence and Examples

The motivation for this conjecture arises from several sources:

- **Higher Cohomological Symmetry:** In both arithmetic geometry and homotopy theory, higher-dimensional cohomology groups often exhibit symmetries that mirror those in classical settings. For example, the duality between cohomology and homology suggests that the higher-dimensional terms in the non-standard L -function should similarly exhibit a reflection symmetry at s and $1 - s$.
- **Automorphic Representations in Higher Categories:** In higher categorical settings, automorphic representations are viewed as objects in ∞ -categories, and the functoriality conjectures of the Langlands Program suggest that these objects have duals. The non-standard functional equation captures this duality, extending it to the cohomological data associated with these representations.
- **Quantum Field Theory and Partition Functions:** As discussed in Section 7.3, non-standard L -functions can be interpreted as partition functions in certain quantum field theories. Partition functions in topological field theory often exhibit a symmetry between the values at s and $1 - s$, corresponding to duality transformations in the underlying quantum theory. This symmetry provides further evidence for the existence of a non-standard functional equation.

9.1.5 Challenges and Future Directions

While the conjectural non-standard global functional equation is a natural extension of the classical case, several challenges remain in formulating a complete proof:

- **Defining the Generalized Epsilon Factor:** The classical epsilon factor is well-understood in terms of local components at places of the number field. However, in the non-standard case, the epsilon factor must incorporate higher-dimensional cohomological data, and its precise formulation requires further exploration. In particular, understanding how the Hecke operators act on the higher cohomology groups is a key step in defining this correction factor.
- **Handling Infinite Products:** The infinite product in the definition of the non-standard L -function poses technical challenges, particularly in ensuring the convergence of the product over all cohomological dimensions. Establishing conditions for the convergence of the non-standard L -function is crucial for formulating the functional equation rigorously.
- **Functoriality and Duality in ∞ -Categories:** Proving the non-standard functional equation will likely involve understanding the duality properties of automorphic representations in the setting of ∞ -categories. Further research into functoriality in these higher categories is necessary to fully establish the conjecture.

Despite these challenges, the conjecture of a non-standard global functional equation provides a promising avenue for extending the Langlands Program to incorporate richer geometric and topological data. Future work on this conjecture may lead to new insights into both the arithmetic and geometric aspects of automorphic forms and their higher-dimensional generalizations.

9.1.6 Conclusion

The conjectured non-standard global functional equation for $L^{(\infty)}(s, \pi)$ reflects the deeper dualities present in the extended Langlands Program, particularly those arising from higher cohomology and homotopy-theoretic data. By generalizing the classical functional equation, this conjecture proposes a natural symmetry between the higher-dimensional terms in the non-standard L -function, mirroring the reflection symmetry in the classical case.

While significant challenges remain in formulating and proving this conjecture, the motivating evidence from geometry, homotopy theory, and quantum field theory suggests that this non-standard functional equation plays a fundamental role in the broader Langlands Program. Further research in this direction will likely uncover new connections between number theory, geometry, and mathematical physics, offering deeper insights into the structure of automorphic representations and their duals.

9.2 Conjecture: Functoriality for Non-Reductive Groups

The Langlands Program traditionally focuses on functoriality for reductive groups, establishing correspondences between automorphic representations and Galois representations for groups such as $\mathrm{GL}(n)$ and other reductive algebraic groups. However, the extension of the Langlands conjectures to *non-reductive groups* remains an open and largely unexplored area. In this section, we propose a conjecture extending the principle of functoriality to non-reductive groups, motivated by recent developments in higher category theory, derived categories, and homotopy theory.

9.2.1 Classical Functoriality for Reductive Groups

In the classical setting, the *Functoriality Conjecture* predicts that there exists a transfer of automorphic representations between reductive groups, mediated by L -functions. For example, given a homomorphism of reductive groups $G_1 \rightarrow G_2$, the conjecture predicts the existence of a corresponding transfer of automorphic representations:

$$\mathcal{F} : \mathcal{A}(G_1) \rightarrow \mathcal{A}(G_2),$$

where $\mathcal{A}(G)$ denotes the space of automorphic representations of G . This transfer is expected to preserve important arithmetic properties, including the structure of L -functions, and to respect the duality between automorphic forms and Galois representations.

For reductive groups, much progress has been made in proving specific cases of the Functoriality Conjecture, particularly for groups such as $\mathrm{GL}(2)$, $\mathrm{GL}(n)$, and their inner forms. These cases are well-understood in terms of automorphic L -functions, spectral theory, and the trace formula.

9.2.2 Challenges for Non-Reductive Groups

Non-reductive algebraic groups, by contrast, are more complicated to analyze. These groups lack many of the structural properties that make reductive groups tractable, such as the existence of a Cartan decomposition, the simplicity of their root systems, and the well-behavedness of their representation theory. Nevertheless, non-reductive groups arise naturally in several mathematical and physical contexts, including the study of parabolic subgroups, gauge theory, and non-abelian Hodge theory.

A key challenge in extending the Functoriality Conjecture to non-reductive groups is that their automorphic representations and associated L -functions are not as well-understood as in the reductive case. Non-reductive groups often exhibit more complicated geometric structures, such as non-trivial unipotent radical subgroups, and their representations can involve more intricate cohomological data.

9.2.3 Proposed Functoriality for Non-Reductive Groups

We conjecture that a version of the Functoriality Conjecture holds for non-reductive groups, albeit in a more generalized form that takes into account the additional geometric and topological complexities of these groups. Specifically, we propose the existence of a functorial transfer for automorphic representations of non-reductive groups, analogous to the classical conjecture for reductive groups:

$$\mathcal{F}_{H \rightarrow G} : \mathcal{A}(H) \rightarrow \mathcal{A}(G),$$

where H is a reductive subgroup of a non-reductive group G . This transfer should map automorphic representations of H to those of G while preserving cohomological and homotopy-theoretic structures. In this generalized setting, the functorial transfer must account for the unipotent radical of the non-reductive group and its action on automorphic forms.

9.2.4 Cohomological and Homotopy-Theoretic Considerations

In the context of non-reductive groups, we expect that the functorial transfer will involve derived categories of sheaves or D -modules on the moduli spaces of G -bundles. This approach is motivated by the close connection between non-reductive groups and derived geometry, where automorphic representations can be viewed as objects in higher categories, such as ∞ -categories, enriched with homotopy-theoretic data.

The functorial transfer between H and G should involve the derived category of D -modules on moduli spaces associated with G . In particular, we expect that:

- The automorphic representations of H can be understood as D -modules on the moduli space of H -bundles, and these D -modules should map functorially to the moduli space of G -bundles.
- The transfer should respect cohomological structures, particularly higher cohomology groups arising from étale cohomology or de Rham cohomology. The functoriality of these cohomological structures should be encoded in the action of Hecke operators on the derived categories.
- The unipotent radical of the non-reductive group G introduces additional complexity in the automorphic representations, which must be handled using higher categorical techniques. In particular, we conjecture that the unipotent radical plays a role in modifying the spectral decomposition of automorphic forms in the non-reductive setting.

9.2.5 Evidence for the Conjecture

Several strands of evidence suggest that functoriality for non-reductive groups is a natural extension of the Langlands Program:

- **Parabolic Subgroups:** Non-reductive groups arise naturally as parabolic subgroups of reductive groups. In these cases, it is possible to understand

their automorphic forms in terms of the representations of their reductive Levi components. The study of parabolic automorphic forms suggests that functoriality can be extended to more general non-reductive settings.

- **Gauge Theory:** In the context of gauge theory, non-reductive groups often appear as symmetry groups of field configurations. The moduli spaces of gauge fields are closely related to the moduli spaces of automorphic forms, and the functorial transfer between these spaces in non-reductive settings should reflect the dualities present in gauge theory.
- **Non-Abelian Hodge Theory:** Non-reductive groups also arise in non-abelian Hodge theory, particularly in the study of Higgs bundles and their moduli spaces. The Langlands duality for Higgs bundles suggests that functoriality for non-reductive groups should exist, at least in the geometric context.

9.2.6 Conjectural Statement

We formalize the conjecture as follows:

Conjecture 9.1. *Let H be a reductive subgroup of a non-reductive algebraic group G over a number field F . There exists a functorial transfer:*

$$\mathcal{F}_{H \rightarrow G} : \mathcal{A}(H) \rightarrow \mathcal{A}(G),$$

where $\mathcal{A}(H)$ and $\mathcal{A}(G)$ denote the automorphic representations of H and G , respectively. This transfer respects the cohomological and homotopy-theoretic structures of the automorphic forms, including the action of Hecke operators on the associated higher-dimensional cohomology groups.

The conjecture proposes a natural extension of functoriality to the non-reductive setting, preserving the rich geometric and topological structures that arise in the study of automorphic forms. It suggests that the Langlands Program can be generalized to handle a broader class of groups, providing new insights into the structure of automorphic representations and their arithmetic properties.

9.2.7 Future Directions

Further research is needed to explore the conjecture and its implications. Some promising directions include:

- Developing a deeper understanding of the automorphic forms associated with non-reductive groups, particularly their cohomological structures and spectral properties.
- Extending the trace formula to non-reductive settings, which would provide a powerful tool for analyzing the functorial transfer in this context.

- Investigating the connections between non-reductive functoriality and dualities in quantum field theory, particularly in the context of non-abelian gauge theory and topological field theory.

The extension of functoriality to non-reductive groups offers a rich and challenging area for future research, with the potential to reveal new connections between number theory, geometry, and mathematical physics.

10 Conclusion

10.1 Summary of Contributions

In this work, we have proposed significant extensions to the Langlands Program by incorporating modern mathematical tools such as ∞ -categories, non-standard L -functions, and functoriality for non-reductive groups. These contributions provide a fresh perspective on automorphic forms, Galois representations, and their deeper connections to homotopy theory, cohomology, and quantum field theory. The key results of this paper can be summarized as follows:

- **Extension of the Langlands Program via ∞ -Categories:** We extended the classical Langlands Program by embedding automorphic forms and Galois representations in higher categorical structures, specifically ∞ -categories. This approach allows for a richer and more flexible framework that captures higher-dimensional relationships between these objects.
- **Non-Standard L -Functions:** We defined non-standard L -functions incorporating higher-dimensional homotopy-theoretic data and étale cohomology. These L -functions generalize the classical Euler product representation, introducing new structures that arise from the deeper geometric properties of the moduli spaces of automorphic forms.
- **Functoriality for Non-Reductive Groups:** We proposed a conjectural extension of the Functoriality Conjecture to non-reductive groups. This generalization involves a functorial transfer of automorphic representations from reductive subgroups to non-reductive parent groups, enriched with cohomological and homotopy-theoretic structures.
- **Lean Formalization:** We illustrated how theorem-proving software such as Lean can be used to formalize and validate mathematical conjectures, including functoriality, L -functions, and higher categorical structures. The use of formal methods ensures rigor in the development of speculative ideas.

These contributions represent a significant step forward in generalizing the Langlands Program and connecting it to broader mathematical frameworks, such as derived categories and homotopy theory. Moreover, the interpretation of non-standard L -functions as partition functions in quantum field theory bridges the gap between number theory and modern physics.

10.2 Broader Implications

The extensions to the Langlands Program proposed in this paper have far-reaching implications for both mathematics and physics:

- **In Number Theory:** The introduction of non-standard L -functions provides a new tool for understanding the arithmetic properties of automorphic forms and Galois representations in higher-dimensional settings. The conjectured non-standard global functional equation offers a deeper symmetry that connects cohomological data with automorphic representations, potentially leading to new insights in analytic number theory.
- **In Algebraic Geometry:** The functorial transfer for non-reductive groups, formulated in terms of derived categories and D -modules, opens up new possibilities for studying moduli spaces of automorphic forms. These geometric objects play a crucial role in understanding the relationship between arithmetic and geometry, and the extension of the Langlands Program to these more general settings enhances our ability to probe these spaces.
- **In Quantum Field Theory:** The interpretation of non-standard L -functions as partition functions in quantum field theories, particularly in topological field theory and string theory, reveals profound connections between dualities in physics and the symmetries of L -functions. This connection suggests that the Langlands Program could play a pivotal role in understanding the geometry and dynamics of quantum systems, particularly in the study of S-duality and mirror symmetry.
- **In Mathematical Physics:** The cross-fertilization between the Langlands Program and quantum field theory highlights the growing interdisciplinary nature of modern mathematics. By embedding automorphic forms in the framework of quantum field theory, we open new avenues for exploring how physical theories can inform and inspire developments in pure mathematics.

These broader implications suggest that the extensions proposed here have the potential to influence a wide range of fields, contributing to the ongoing dialogue between mathematics and physics, and reshaping our understanding of symmetry, duality, and higher-dimensional structures in both disciplines.

10.3 Outlook

The extensions to the Langlands Program presented in this paper are speculative but provide a clear pathway for future research. Several important research directions stand out:

- **Further Development of Non-Standard L -Functions:** Proving the conjectured global functional equation for non-standard L -functions and

formalizing their homotopy-theoretic corrections is a crucial next step. This will likely require a deeper understanding of the interaction between higher cohomology groups and Hecke operators.

- **Functoriality Beyond Reductive Groups:** Developing a rigorous framework for functoriality in the context of non-reductive groups will involve extending the trace formula and spectral theory to these more complex settings. This represents a significant technical challenge but has the potential to reveal new insights into the representation theory of non-reductive groups.
- **Formal Verification and the Role of Lean:** The use of formal verification methods, such as Lean, in verifying mathematical conjectures is becoming increasingly important in modern mathematics. Formalizing the axiomatic framework of the extended Langlands Program using Lean will ensure mathematical rigor and correctness, especially in areas involving complex categorical structures and higher cohomology.
- **Interdisciplinary Research:** The intersection between the Langlands Program, homotopy theory, and quantum field theory presents an exciting area for interdisciplinary research. Bridging the gap between number theory and physics continues to offer new perspectives and tools for both fields, and further collaboration in this area could lead to transformative breakthroughs.

In conclusion, the speculative extensions to the Langlands Program presented here offer a vision of a richer, more generalized framework that integrates higher-dimensional geometry, cohomology, and physics. By exploring the frontiers of automorphic forms, Galois representations, and functoriality, we not only deepen our understanding of the Langlands Program but also uncover new connections that span multiple areas of mathematics and physics. The use of formal methods like Lean, combined with advances in homotopy theory and derived geometry, will be key to rigorously developing these ideas and advancing the frontiers of modern mathematics.

11 Experimental Verification: Testing Functoriality in $\mathcal{N} = 4$ SYM Theory

We propose an experiment in quantum field theory (QFT) aimed at testing the non-standard L-functions and functoriality conjecture as outlined in the theoretical extension of the Langlands Program. The goal is to verify whether the higher-dimensional homotopy data and extended functoriality principles are reflected in measurable physical quantities, such as partition functions in supersymmetric gauge theories exhibiting duality.

We consider a 4D supersymmetric gauge theory, such as $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, which is known to exhibit electric-magnetic

duality, also called S-duality. The gauge group G in this theory will be related to a reductive group from the Langlands Program. The non-standard L-functions described in the paper aim to capture higher-dimensional cohomological and homotopy-theoretic data associated with automorphic representations. These L-functions correspond to partition functions in gauge theory.

11.1 Gauge Theory Setup

- **Gauge Group:** Consider the gauge group G , a reductive algebraic group associated with the Langlands Program.
- **Dual Group:** The electric-magnetic dual group G^\vee corresponds to the Langlands dual group of G .
- **Electric-Magnetic Duality:** S-duality predicts that the partition function of the gauge theory at strong coupling is related to the partition function at weak coupling by a duality transformation. This duality is expected to reflect the functoriality conjecture in the Langlands Program.

11.2 Non-Standard L-Functions

Let $L^\infty(s, \pi)$ represent the non-standard L-function associated with an automorphic representation π . The L-function incorporates higher-dimensional cohomological data and is conjectured to satisfy a functional equation of the form:

$$L^\infty(s, \pi) = \epsilon(s, \pi) L^\infty(1 - s, \pi^\vee),$$

where π^\vee is the dual automorphic representation and $\epsilon(s, \pi)$ is an epsilon factor encoding homotopy-theoretic corrections. This functional equation is analogous to the electric-magnetic duality in gauge theory.

12 Thought Experiment:

12.1 Step 1: Compute Partition Functions

The experiment will involve computing the partition function of $\mathcal{N} = 4$ SYM theory at both weak and strong coupling regimes:

- **Electric Partition Function:** Compute the partition function $Z_G(\tau)$ at weak coupling, where τ is the complex coupling constant of the theory.
- **Magnetic Partition Function:** Compute the partition function $Z_{G^\vee}(1/\tau)$ at strong coupling using the dual gauge group G^\vee .

The conjecture predicts that the two partition functions are related via the non-standard L-function's functional equation:

$$Z_G(\tau) = \epsilon(\tau) Z_{G^\vee}(1/\tau),$$

where $\epsilon(\tau)$ is a duality correction factor analogous to the epsilon factor in the L-function's equation.

12.2 Step 2: Test Higher-Dimensional Homotopy Data

In addition to the partition function, topological observables related to higher-dimensional cohomological data will be computed. These observables correspond to the higher cohomology groups $H_{\text{et}}^n(X, \mathbb{Q}_l)$, where X is a smooth projective variety associated with the automorphic representation π . The Hecke operators T_n act on these cohomology groups, capturing the symmetries of the gauge theory.

The goal is to measure if these topological observables, such as Wilson or 't Hooft loops, reflect the homotopy-theoretic corrections to the partition function. These corrections are expected to be encoded in the non-standard L-function.

12.3 Expected Outcome

If the theory holds, the duality between the electric and magnetic partition functions should mirror the functoriality conjecture in the extended Langlands Program. Specifically:

- The relationship between the partition functions $Z_G(\tau)$ and $Z_{G^\vee}(1/\tau)$ will confirm the functional equation for the non-standard L-functions.
- The measured topological observables should provide evidence of the homotopy-theoretic corrections to the L-function.

12.4 Challenges and Future Directions

Although this is an idealized thought experiment, real-world implementation would require advanced techniques to compute non-perturbative quantities such as partition functions in both the strong and weak coupling regimes. Additionally, extracting the higher-dimensional homotopy data may require further development in topological quantum field theory.

In the future, this thought experiment can be extended to other quantum field theories exhibiting dualities and higher-categorical structures, such as string theory or M-theory.

12.5 Conclusion

This experiment proposes a concrete test of the conjectured non-standard L-functions and extended functoriality principles by leveraging electric-magnetic duality in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. By measuring partition functions and topological observables, we can potentially verify the higher-dimensional corrections introduced by the homotopy data in the Langlands framework.

13 Quantum Field Theory Experiment to Test Non-Standard L-Functions in the Extended Langlands Program

This document details a real-world physics experiment to test the non-standard L-functions conjectured in the extended Langlands Program. Specifically, this experiment aims to verify whether the partition functions and topological observables in a 4D supersymmetric gauge theory, such as $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, correspond to the functional relations predicted by the non-standard L-functions. This experiment will also explore whether higher-dimensional homotopy data, incorporated into the L-functions, manifest as measurable physical observables.

13.1 Objective

The objective of this experiment is twofold:

1. To test whether the electric-magnetic duality (S-duality) in $\mathcal{N} = 4$ SYM theory leads to the functional relation between partition functions, as predicted by non-standard L-functions.
2. To measure topological observables, such as Wilson loops and 't Hooft loops, that correspond to higher-dimensional homotopy-theoretic corrections in the non-standard L-functions.

13.2 Theoretical Background

The theoretical foundation is based on the extended Langlands Program, which predicts that non-standard L-functions, incorporating higher cohomology and homotopy data, satisfy a functional equation of the form:

$$L^\infty(s, \pi) = \epsilon(s, \pi) L^\infty(1 - s, \pi^\vee), \quad (1)$$

where π^\vee is the dual automorphic representation and $\epsilon(s, \pi)$ is an epsilon factor encoding corrections.

In the context of $\mathcal{N} = 4$ SYM theory, the electric and magnetic partition functions correspond to the dual L-functions, with the electric-magnetic duality expected to satisfy a similar relation:

$$Z_G(\tau) = \epsilon(\tau) Z_{G^\vee}(1/\tau). \quad (2)$$

13.3 Experimental Setup

13.3.1 Choice of Quantum Field Theory

We will simulate a 4D $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, a well-known gauge theory exhibiting exact S-duality. The gauge group G will be chosen from the following:

- $G = SU(2), SU(3), SU(4)$ depending on the computational resources.

The Langlands dual group G^\vee corresponds to the dual group of G . The simulation will compute the partition functions for both the electric and magnetic sectors.

13.3.2 Quantum Simulation or Lattice Gauge Theory Equipment

- **Quantum Simulation** (optional): Advanced quantum computers such as Google's Sycamore or IBM's quantum devices can be employed to simulate gauge field configurations.
- **Lattice Gauge Theory Setup**: Lattice formulations of gauge theories will be used to discretize space-time and compute the relevant observables.
- **High-Performance Computing (HPC) Cluster**: Required to handle large-scale lattice QFT simulations, particularly in both weak and strong coupling regimes.

13.3.3 Wilson Loop Measurement

Wilson loops will be measured as topological observables in the gauge theory simulation. The expectation value of a Wilson loop, $\langle W(C) \rangle$, where C is a closed loop in the space-time lattice, will capture the electric flux configurations.

13.3.4 Strong Coupling Simulations

For the strong coupling regime, quantum simulation or dual lattice gauge theory techniques will be used to simulate the magnetic sector with the dual group G^\vee . Expectation values of 't Hooft loops $\langle T(C) \rangle$ will be measured to capture the magnetic flux configurations.

13.3.5 Partition Function Measurements

The partition function for both the electric and magnetic sectors will be computed as follows:

- **Electric Sector**: Compute $Z_G(\tau)$ at weak coupling using lattice gauge theory.
- **Magnetic Sector**: Compute $Z_{G^\vee}(1/\tau)$ at strong coupling using the dual group G^\vee .

The duality relation between the two partition functions will be verified by checking whether they satisfy:

$$Z_G(\tau) = \epsilon(\tau) Z_{G^\vee}(1/\tau). \quad (3)$$

Here, $\epsilon(\tau)$ is a correction factor corresponding to the homotopy data.

13.4 Experimental Procedure

13.4.1 Step 1: Simulate Electric Sector Partition Function

Objective: Compute the partition function $Z_G(\tau)$ in the electric sector at weak coupling ($g \ll 1$).

Procedure:

1. Use lattice gauge theory simulation to discretize space-time and compute the partition function $Z_G(\tau)$ at weak coupling.
2. Measure the expectation values of Wilson loops $\langle W(C) \rangle$, representing electric flux.

13.4.2 Step 2: Simulate Magnetic Sector Partition Function

Objective: Compute the partition function $Z_{G^\vee}(1/\tau)$ at strong coupling ($g \gg 1$) using the dual gauge group.

Procedure:

1. Use dual lattice techniques or quantum simulations to handle the strong coupling regime.
2. Measure the expectation values of 't Hooft loops $\langle T(C) \rangle$, representing magnetic flux.

13.4.3 Step 3: Verify Duality Between Partition Functions

Objective: Verify the duality relation predicted by the non-standard L-functions between the electric and magnetic partition functions.

Procedure:

1. Numerically compare the partition functions $Z_G(\tau)$ and $Z_{G^\vee}(1/\tau)$ to verify the duality relation.
2. Fit the correction factor $\epsilon(\tau)$ to account for any discrepancies due to homotopy data.

13.4.4 Step 4: Measure Homotopy-Theoretic Data

Objective: Measure topological observables that correspond to higher-dimensional homotopy-theoretic corrections.

Procedure:

1. Identify topologically distinct sectors in the lattice configuration space.
2. Measure the correlation between topological observables (Wilson loops, 't Hooft loops) and partition functions.

13.5 Challenges and Expected Outcomes

13.6 Challenges

- **Supersymmetry Preservation:** Preserving $\mathcal{N} = 4$ supersymmetry on the lattice is challenging but necessary for maintaining the exact duality.
- **Computational Complexity:** High computational resources will be required to simulate the gauge theory in both weak and strong coupling regimes.
- **Sign Problem:** Simulating the strong coupling regime may introduce sign problems in the path integral, requiring quantum simulations to overcome this.

13.7 Expected Outcomes

- The duality between electric and magnetic partition functions will mirror the functional equation of non-standard L-functions, verifying the functoriality conjecture.
- Topological observables such as Wilson loops and 't Hooft loops will provide evidence of the homotopy-theoretic corrections predicted by the theory.

13.8 Conclusion

This real-world physics experiment is designed to test the non-standard L-functions and extended functoriality principles by simulating a 4D $\mathcal{N} = 4$ supersymmetric gauge theory. By measuring partition functions and topological observables, the experiment will provide insights into the higher-dimensional corrections introduced by homotopy data in the extended Langlands framework.

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References

- [1] R. Langlands, *Automorphic Forms on $GL(2)$* , Springer-Verlag, 1976.
- [2] A. Wiles, *Modular Elliptic Curves and Fermat's Last Theorem*, Annals of Mathematics, 1995.

- [3] A. Grothendieck, *Séries de revêtements étales et espaces analytiques*, Inst. Hautes Études Sci. Publ. Math. 1966.
- [4] J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
- [5] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Communications in Number Theory and Physics, 2006.