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A SOLUTION TO THE MISÈRE SHANNON SWITCHING GAME

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Let G be a graph and x_0, x_1 be two different vertices of G. Two players, Black and White, mark alternately non marked edges of G. White loses if and only if he marks all edges of a path connecting x_0 and x_1. This game is the misère version of the well-known Shannon Switching Game. We give its classification as a particular case of the classification of a more general game played on a matroid.

1. Introduction

Let M be a matroid and $e \in E(M)$. Two players, Black and White, mark alternately non marked elements of $E(M) - e$. In the ordinary Shannon Switching Game, White wins if he marks all elements of a circuit broken at e; otherwise Black wins. We recall that a *circuit broken at e* is a set of the form $C - e$, where C is a circuit of M containing e. We define the *Misère Switching Shannon Game on M with respect to e* as follows. The rules are the same as above except that White loses if he marks all elements of a circuit broken at e and wins otherwise.

The Shannon Switching Game was introduced by Shannon for graphs and generalized to matroids by Lehman in [4]. Lehman has given a complete classification of this game [4].

The Misère Shannon Switching Game is different from the misère game considered by Kano for graphs [5] and generalized to matroids in [3], where White loses if and only if he marks a circuit of M.

A matroid M is called a *block* if $E(M)$ is the union of two disjoint bases. A block of M is a subset $X \subseteq E$ such that the induced matroid $M(X)$ is a block.

Theorem 1A (Lehman [5]). *Let M be a block. Then White playing second can mark a base of M.*

Theorem 1B (Lehman [5]). *Let M be a matroid and $e \in E(M)$. The Shannon Switching Game with respect to e has the following classification.*

(i) *White wins playing second if and only if there is a block of M not containing e but spanning e in M.*

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(ii) *The first player wins if and only if there are blocks containing e in both M and M^*.*

(iii) *Black wins playing second if and only if there is a block of M^* not containing e but spanning e in M^*.*

A survey of Lehman theory and short proofs of Theorem 1A and 1B are contained in [2].

2. A solution to the Misere Shannon Switching Game

We use the following lemma.

Lemma 2.1. *Let M be a matroid and X be a block of M. Two players Black and White play alternatively by marking elements of M. Then White playing first resp. second can force Black to mark a basis of $M(X)$.*

Proof. Since $M(X)$ is a block, then so is $(M(X))^*$. By Theorem 1A there is a strategy $\mathcal{S}$ for White playing second (resp. first) for marking a base of $(M(X))^*$.

White playing second uses the following strategy:

(1) If Black marks an element of $E(M)-X$, then White marks any element of $E(M)-X$ if there remains any. Otherwise White considers a fictitious move on X played by Black and answers according to $\mathcal{S}$.

(2) Black marks an element of X which is not a fictitious move. Then White answers according to $\mathcal{S}$.

(3) Blacks plays a fictitious move (i.e. an element chosen by White as a fictitious move of Black in the previous course of the game). Then White choses any non marked element if there remains any as a new fictitious move of Black and answers according to $\mathcal{S}$. If there is no such element the game is over.

Observe that up to the order of moves, on X White answered to all moves of Black according to $\mathcal{S}$. Hence White has marked a base of $(M(X))^*$. It follows that Black has marked a base of $M(X)$.

The case White playing first reduces to the case playing second by considering the game on M plus a loop played as first move by Black. $\square$

The following definition is given in [1].

Let E be a set and $\mathscr{C} \in 2^E$. Two players, Black and White, mark alternately non marked elements of E. White wins if he marks all elements of a set in $\mathscr{C}$. Otherwise Black wins.

This game is a *positional game of type* 1 *with set of winning configurations* $\mathscr{C}$. We denote it by $(E, \mathscr{C})$.

Proposition 2.2. *Let $\mathscr{G}=(E, \mathscr{C})$ be a positional game of type* 1. *Suppose that White has a winning strategy playing second. Then White has a winning strategy playing first an arbitrary element $e \in E$.*

Proposition 2.2 is an easy refinement of the well-known fact that if White has a winning strategy playing second, then he has also a winning strategy playing first (see [1, 4]).

Theorem 2.3. *The Misère Shannon Switching Game on a matroid M with respect to an element $e \in E(M)$ has the following classification:*

(i) *If there is a block of M not containing e but spanning e in M then this game is winning for Black.*

(ii) *If there is a block of M and a block of M^* containing e, and $|E|$ is odd, then the first player wins.*

(iii) *If there is a block of M and a block of M^* containing e, and $|E|$ is even, then the second player wins.*

(iv) *If there is a block of M^* not containing e but spanning e in M^* then White wins.*

Proof. (i) Let X be a block of M spanning but not containing e. By Lemma 2.1, Black can force White to mark a base of $M(X)$. Then White loses.

(ii) Let X and X' be blocks of M and M^* containing e respectively. Note that $|E| - |X|$ and $|E| - |X'|$ are both odd.

(1) White playing first uses the following strategy: The first move is any element of $E \backslash X'$.

If Black marks an element of $E - X'$, then White marks any non marked element of $E - X'$ (always possible for parity reason).

If Black marks an element of X', then White plays according to the strategy given by Proposition 2.2 in the game on $(M^*(X'))^*$ with $\mathscr{C}$ the set of bases of $(M^*(X'))^*$ and e being his first move. Then when the game is over White has marked a base of $(M^*(X'))^*/e$, hence Black has marked a base of $M^*(X')$ not containing e but spanning e in M^*. It follows that there is no white circuit broken at e in M.

(2) Black playing first.

As in (1), Black forces White to mark a base of X. Then White marks a white circuit broken at e.

(iii) There is a block X of M and a block X' of M^* containing e and $|E|$ is even.

Note that $|E| - |X|$ and $|E| - |X'|$ are both even.

(1) White playing first.

Black uses the following strategy: If White marks an element of X, then Black plays according to the strategy given by Proposition 2.2 in the game on $(M(X))^*$ with $\mathscr{C}$ the set of bases of $(M(X))^*$ and e being his first move.

If White marks an element of $E - X$, then Black plays any element of $E - X$.

Using this strategy, Black forces White to mark a base of X. Hence White marks a circuit broken at e.

(2) Black playing first.

White uses the strategy in (1), where X is replaced by X' and $M(X)$ by $M^*(X')$.

(iv) Let X' be a block of M^* spanning but not containing e. by Lemma 2.1 White can force Black to mark a base of $M^*(X)$. It follows that there is no white circuit broken at e when the game is over. $\square$

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