COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **35** (2) (2015) 153–180
DOI: 10.1007/s00493-014-2953-2

MATROID INTERSECTION, BASE PACKING AND BASE COVERING FOR INFINITE MATROIDS

NATHAN BOWLER, JOHANNES CARMESIN

Received March 5, 2012
Online First September 4, 2014

As part of the recent developments in infinite matroid theory, there have been a number of conjectures about how standard theorems of finite matroid theory might extend to the infinite setting. These include base packing, base covering, and matroid intersection and union. We show that several of these conjectures are equivalent, so that each gives a perspective on the same central problem of infinite matroid theory. For finite matroids, these equivalences give new and simpler proofs for the finite theorems corresponding to these conjectures.

This new point of view also allows us to extend, and simplify the proofs of some cases where these conjectures were known to be true.

1. Introduction

The well-known finite matroid intersection theorem of Edmonds states that for any two finite matroids M and N the size of a biggest common independent set is equal to the minimum of the rank sum $r_M(E_M)+r_N(E_N)$, where the minimum is taken over all partitions $E=E_M\dot\cup E_N$. The same statement for infinite matroids is true, but for a silly reason [10], which suggests that more care is needed in extending this statement to the infinite case.

Nash-Williams [3] proposed the following for finitary matroids.

Conjecture 1.1 (The Matroid Intersection Conjecture). *Any two matroids M and N on a common ground set E have a common independent set I admitting a partition $I=J_M\cup J_N$ such that $\mathrm{Cl}_M(J_M)\cup \mathrm{Cl}_N(J_N)=E$.*

Mathematics Subject Classification (2000): 05C63, 05B35, 05B40

For finite matroids this is easily seen to be equivalent to the intersection theorem, which is why we refer to Conjecture 1.1 as the Matroid Intersection Conjecture. If for a pair of matroids M and N on a common ground set there are sets I, J_M and J_N as in Conjecture 1.1, we say that M and N have the *Intersection property*, and that I, J_M and J_N *witness* this.

In [5], it was shown that this conjecture implies the celebrated Aharoni-Berger-Theorem [1], also known as the Erdős-Menger-Conjecture. Call a matroid *finitary* if all its circuits are finite and *co-finitary* if its dual is finitary. The conjecture is true in the cases where M is finitary and N is co-finitary [5].[1] Aharoni and Ziv [3] proved the conjecture for one matroid finitary and the other a countable direct sum of finite rank matroids.

In this paper we will demonstrate that the Matroid Intersection Conjecture is a natural formulation by showing that it is equivalent to several other new conjectures in unexpectedly different parts of infinite matroid theory.

Suppose we have a family of matroids $(M_k \mid k \in K)$ on the same ground set E. A *packing* for this family consists of a spanning set S_k for each M_k such that the S_k are all disjoint. Note that not all families of matroids have a packing. More precisely, the well-known finite base packing theorem states that if E is finite, then the family has a packing if and only if for every subset $Y \subseteq E$ the following holds.

$$\sum_{k\in K} r_{M_k.Y}(Y) \leq |Y|$$

The Aharoni-Thomassen graphs [2,11] show that this theorem does not extend verbatim to finitary matroids. However, the base packing theorem extends to finite families of co-finitary matroids [4]. This implies the topological tree packing theorems of Diestel and Tutte. Independently from our main result, we close the gap in between by showing that the base packing theorem extends to arbitrary families of co-finitary matroids (for example, topological cycle matroids).

Similar to packings are coverings: a *covering* for the family $(M_k \mid k \in K)$ consists of an independent set I_k for each M_k such that the I_k cover E. And analogously to the base packing theorem, there is a base covering theorem characterising the finite families of finite matroids admitting a covering.

We are now in a position to state our main conjecture, which we will show is equivalent to the intersection conjecture. Roughly, the finite base packing theorem says that a family has a packing if it is very dense. Similarly, the finite base covering theorem says roughly that a family has a covering if it is very sparse. Although not every family of matroids has a packing and not

[1] In fact in [5] the conjecture was proved for a slightly larger class.

every family has a covering, we could ask: is it always possible to divide the ground set into a "dense" part, which has a packing, and a "sparse" part, which has a covering?

Definition 1.2. *We say that a family of matroids $(M_k \mid k \in K)$ on a common ground set E, has the* Packing/Covering *property if E admits a partition $E = P \dot\cup C$ such that $(M_k \restriction_P \mid k \in K)$ has a packing and $(M_k.C \mid k \in K)$ has a covering.*

Conjecture 1.3. *Any family of matroids on a common ground set has the Packing/Covering property.*

Here $M_k \restriction_P$ is the restriction of M_k to P and $M_k.C$ is the contraction of M_k onto C. Note that if $(M_k \restriction_P \mid k \in K)$ has a packing, then $(M_k.P \mid k \in K)$ has a packing, so we get a stronger statement by taking the restriction here. Similarly, we get a stronger statement by contracting to get the family which should have a covering than we would get by restricting.

For finite matroids, we show that this new conjecture is true and implies the base packing and base covering theorems. So the finite version of Conjecture 1.3 unifies the base packing and the base covering theorem into one theorem.

For infinite matroids, we show that Conjecture 1.3 and the intersection conjecture are equivalent, and that both are equivalent to Conjecture 1.3 for pairs of matroids. In fact, for pairs of matroids, we show that (M, N) has the Packing/Covering property if and only if M and N^* have the Intersection property. As the Packing/Covering property is preserved under duality for pairs of matroids, this shows the less obvious fact that the Intersection property is also preserved under duality:

Corollary 1.4. *If M and N are matroids on the same ground set then M and N have the intersection property if and only if M^* and N^* do.*

Conjecture 1.3 also suggests a base packing conjecture and a base covering conjecture, which we show are equivalent to the intersection conjecture but not to the above mentioned rank formula formulation of base packing for infinite matroids.

The various results about when intersection is true, transfer via these equivalences to give results showing that these new conjectures also hold in the corresponding special cases. For example, while the rank-formulation of the covering theorem is not true for all families of co-finitary matroids, the new covering conjecture is true in that case. This yields a base covering theorem for the algebraic cycle matroid of any locally finite graph and the topological cycle matroid of any graph. Similarly, we immediately obtain in this way that the new packing and covering conjectures are true for finite families of finitary matroids. Thus we get packing and covering theorems for the finite cycle matroid of any graph.

For finite matroids, the proofs of the equivalences of these conjectures simplify the proofs of the corresponding finite theorems.

We show that Conjecture 1.3 might be seen as the infinite analogue of the rank formula of the matroid union theorem. It should be noted that there are two matroids whose union is not a matroid [4], so there is no infinite analogue of the finite matroid union theorem as a whole.

This new point of view also allows us to give a simplified account of the special cases of the intersection conjecture and even to extend the results a little bit. Our result includes the following:

Theorem 1.5. *Any family of matroids $(M_k \mid k \in K)$ on the same ground set E for which there are only countably many sets appearing as circuits of matroids in the family has the Packing/Covering property.*

This paper is organised as follows: In Section 2, we recall some basic matroid theory and introduce a key idea, that of exchange chains. After this, in Section 3, we restate our main conjecture and look at its relation to the infinite matroid intersection conjecture. In Section 4, we prove a special case of our main conjecture. In the next two sections, we consider base coverings and base packings of infinite matroids. In the final section, Section 7, we give an overview over the various equivalences we have proved.

2. Preliminaries

2.1. Basic matroid theory

Throughout, notation and terminology for graphs are that of [11], for matroids that of [13,8], and for topology that of [6]. M always denotes a matroid and $E(M)$, $\mathcal{I}(\mathcal{M})$, $\mathcal{B}(\mathcal{M})$, $\mathcal{C}(\mathcal{M})$ and $\mathcal{S}(M)$ denote its ground set and its sets of independent sets, bases, circuits and spanning sets, respectively.

Recall that the set $\mathcal{I}(\mathcal{M})$ is required to satisfy the following *independence axioms* [8]:

- (I1) $\emptyset \in \mathcal{I}(\mathcal{M})$.
- (I2) $\mathcal{I}(\mathcal{M})$ is closed under taking subsets.
- (I3) Whenever $I, I' \in \mathcal{I}(\mathcal{M})$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}(\mathcal{M})$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}(\mathcal{M})$, the set $\{I' \in \mathcal{I}(\mathcal{M}) \mid \mathcal{I} \subseteq \mathcal{I}' \subseteq \mathcal{X}\}$ has a maximal element.

The axiom (IM) for the dual M^* of M is equivalent to the following:

(IM*) Whenever $Y \subseteq S \subseteq E$ and $S \in \mathcal{S}(M)$, the set $\{S' \in \mathcal{S}(M) \mid Y \subseteq S' \subseteq S\}$ has a minimal element.

As the dual of any matroid is also a matroid, every matroid satisfies this. We need the following facts about circuits, the first of which is commonly referred to as the infinite circuit elimination axiom [8]:

(C3) Whenever $X \subseteq C \in \mathcal{C}(\mathcal{M})$ and $\{C_x \mid x \in X\} \subseteq \mathcal{C}(\mathcal{M})$ satisfies $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists a $C' \in \mathcal{C}(\mathcal{M})$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.
(C4) Every dependent set contains a circuit.

A matroid is called *finitary* if every circuit is finite.

Lemma 2.1. *A set S is M-spanning iff it meets every M-cocircuit.*

Proof. We prove the dual version where $I := E(M) \setminus S$.

(1) A set I is M^*-independent iff it does not contain an M^*-circuit.

Clearly, if I contains a circuit, then it is not independent. Conversely, if I is not independent, then by (C4) it also contains a circuit. ∎

Let 2^X denote the power set of X. If $M = (E, \mathcal{I})$ is a matroid, then for every $X \subseteq E$ there are matroids $M \restriction_X := (X, \mathcal{I} \cap \in^{\mathcal{X}})$ (called the *restriction* of M to X), $M \setminus X := M \restriction_{E \setminus X}$ (which we say is obtained from M by *deleting* X)[2], $M.X := (M^* \restriction_X)^*$ (which we say is obtained by *contracting onto* X) and $M/X := M.(E \setminus X)$ (which we say is obtained by *contracting* X). For $e \in E$, we will also denote $M/\{e\}$ by M/e and $M \setminus \{e\}$ by $M \setminus e$.

Given a base B of X (that is, a maximal independent subset of X), the independent sets of M/X can be characterised as those subsets I of $E \setminus X$ for which $B \cup I$ is independent in M.

Lemma 2.2. *Let M be a matroid with ground set $E = C \dot\cup X \dot\cup D$ and let o' be a circuit of $M' = M/C \backslash D$. Then there is an M-circuit o with $o' \subseteq o \subseteq o' \cup C$.*

[2] We use the notation $M \restriction_X$ rather than the conventional notation $M \mid X$ to avoid confusion with our notation $(M_k \mid k \in K)$ for families of matroids.

Proof. Let s be any M-base of C. Then $s \cup o'$ is M-dependent since o' is M'-dependent. On the other hand, $s \cup o' - e$ is M-independent whenever $e \in o'$ since $o' - e$ is M'-independent. Putting this together yields that $s \cup o'$ contains an M-circuit o, and this circuit must not avoid any $e \in o'$, as desired. ∎

For a family $(M_k \mid k \in K)$ of matroids, where M_k has ground set E_k, the *direct sum* $\bigoplus_{k\in K} M_k$ is the matroid with ground set $\bigcup_{k\in K} E_k \times \{k\}$, with independent sets the sets of the form $\bigcup_{k\in K} I_k \times \{k\}$ where for each k the set I_k is independent in M_k. Contraction and deletion commute with direct sums, in the sense that for a family $(X_k \subseteq E_k \mid k \in K)$ we have $\bigoplus_{k\in K}(M_k/X_k) = (\bigoplus_{k\in K} M_k)/(\bigcup_{k\in K} X_k \times \{k\})$ and $\bigoplus_{k\in K}(M_k \backslash X_k) = (\bigoplus_{k\in K} M_k) \backslash (\bigcup_{k\in K} X_k \times \{k\})$

Lemma 2.3. *Let M be a matroid and $X \subseteq E(M)$. If $S_1 \subseteq X$ spans $M \restriction_X$ and $S_2 \subseteq E \setminus X$ spans M/X, then $S_1 \cup S_2$ spans M.*

Proof. Let B be a maximal independent subset of S_1. Then B spans S_1 and S_1 spans X, so B spans X. Thus B is a base of X. Now let $e \in M \setminus X \setminus S_2$. Since $e \in \mathrm{Cl}_{M/X}(S_2)$, there is a set $I \subseteq E \backslash X$ such that I is M/X-independent but $I + e$ is not. Then $B \cup I$ is M-independent but $B \cup I + e$ is not, so that $e \in \mathrm{Cl}_M(S_1 + S_2)$, as witnessed by the set $B + I$. Any other element of E is either in S_2 or is in $X \subseteq \mathrm{Cl}_M(S_1)$, and so is in the span of $S_1 \cup S_2$. ∎

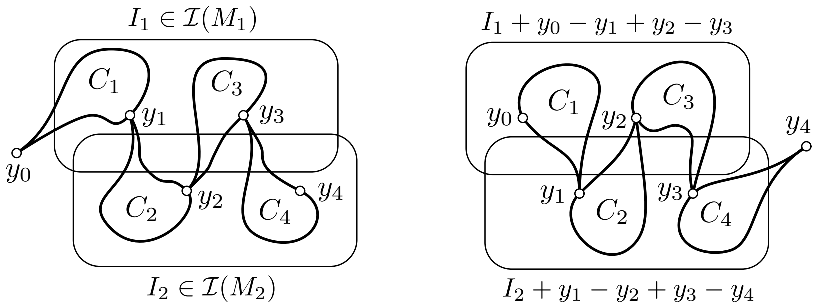
Lemma 2.4 ([9], Lemma 5). *Let M be a matroid with a circuit C and a co-circuit D, then $|C \cap D| \neq 1$.*

A particular class of matroids we shall employ is the *uniform* matroids $U_{n,E}$ on a ground set E, in which the bases are the subsets of E of size n. In fact, the matroids we will use are those of the form $U^*_{1,E}$, in which the bases are all those sets obtained by removing a single element from E. Such a matroid is said to consist of a single circuit, because $\mathcal{C}(\mathcal{U}^*_{\infty,\mathcal{E}}) = \{\mathcal{E}\}$. A subset is independent iff it isn't the whole of E. Note that for a subset X of E, $U^*_{1,E} \restriction_X$ is free (every subset is independent) unless X is the whole of E, and $U^*_{1,E}.X = U^*_{1,X}$ unless X is empty.

2.2. Exchange chains

Below, we will need a modification of the concept of exchange chains introduced in [4]. The only modification is that we need not only exchange chains for families with two members but more generally exchange chains for arbitrary families, which we define as follows: Let $(M_k \mid k \in K)$ be a family of matroids and let $B_k \in \mathcal{I}(\mathcal{M}_{||})$. A $(B_k \mid k \in K)$*-exchange chain (from* y_0

to y_n) is a tuple $(y_0,k_0;y_1,k_1;\ldots;y_n)$ where $B_{k_l}+y_l$ includes an M_{k_l}-circuit containing y_l and y_{l+1}. A $(B_k\,|\,k\in K)$-exchange chain from y_0 to y_n is called *shortest* if there is no $(B_k\,|\,k\in K)$-exchange chain $(y'_0,k'_0;y'_1,k'_1;\ldots;y'_m)$ with $y'_0=y_0$, $y'_m=y_n$ and $m<n$. A typical exchange chain is shown in Figure 1.



(a) Before the exchange (b) After the exchange

Figure 1. An (I_1,I_2)-exchange chain of length 4

Lemma 2.5. *Let $(M_k\,|\,k\in K)$ be a family of matroids and let $B_k\in\mathcal{I}(\mathcal{M}_{||})$. If $(y_0,k_0;y_1,k_1;\ldots;y_n)$ is a shortest $(B_k\mid k\in K)$-exchange chain from y_0 to y_n, then $B'_k\in\mathcal{I}(\mathcal{M}_{||})$ for every k, where*

$$B'_k := B_k \cup \{y_l \mid k_l = k\} \setminus \{y_{l+1} \mid k_l = k\}.$$

Moreover, $\mathrm{Cl}_{M_k} B_k=\mathrm{Cl}_{M_k} B'_k$.

Proof (Sketch). The proof that the B'_k are independent is done by induction on n and is that of Lemma 4.2 in [4]. To see the second assertion, first note that $\{y_l \mid k_l = k\} \subseteq \mathrm{Cl}_{M_k} B_k$ and thus $B'_k \subseteq \mathrm{Cl}_{M_k} B_k$. Thus it suffices to show that $B_k \subseteq \mathrm{Cl}_{M_k} B'_k$. For this, note that the reverse tuple $(y_n,k_{n-1};y_{n-1},k_{n-2};\ldots;y_0)$ is a B'_k-exchange chain giving back the original B_k, so we can apply the preceding argument again. ∎

Lemma 2.6. *Let M be a matroid and $I,B\in\mathcal{I}(\mathcal{M})$ with B maximal and $B\setminus I$ finite. Then $|I\setminus B|\le|B\setminus I|$.*

Lemma 2.7. *Let $(M_k \mid k\in K)$ be a family of matroids, let $B_k\in\mathcal{I}(\mathcal{M}_{||})$ and let C be a circuit for some M_{k_0} such that $C\setminus B_{k_0}$ only contains one element, e. If there is a $(B_k\mid k\in K)$-exchange chain from x_0 to e, then for every $c\in C$, there is a $(B_k\,|\,k\in K)$-exchange chain from x_0 to c.*

Proof. Let $(y_0=x_0, k_0; y_1, k_1; \ldots; y_n=e)$ be an exchange chain from x_0 to e. Then $(y_0=x_0, k_0; y_1, k_1; \ldots; y_n=e, k_0; c)$ is the desired exchange chain. ∎

3. The Packing/Covering conjecture

The matroid union theorem is a basic result in the theory of finite matroids. It gives a way to produce a new matroid $M=\bigvee_{k\in K} M_k$ from a finite family $(M_k \,|\, k\in K)$ of finite matroids on the same ground set E. We take a subset I of E to be M-independent iff it is a union $\bigcup_{k\in K} I_k$ with each I_k independent in the corresponding matroid M_k. The fact that this gives a matroid is interesting, but a great deal of the power of the theorem comes from the fact that it gives an explicit formula for the ranks of sets in this matroid:

$$r_M(X) = \min_{X=P\dot\cup C} \sum_{k\in K} r_{M_k}(P) + |C|. \tag{2}$$

Here the minimisation is over those pairs (P,C) of subsets of X which partition X.

For infinite matroids, or infinite families of matroids, this theorem is no longer true [4], in that M is no longer a matroid. However, it turns out, as we shall now show, that we may conjecture a natural extension of the rank formula to infinite families of infinite matroids.

First, we state the formula in a way which does not rely on the assumption that M is a matroid:

$$\max_{I_k\in\mathcal{I}(\mathcal{M}_{\|})} \left|\bigcup_{k\in K} I_k\right| = \min_{E=P\dot\cup C} \sum_{k\in K} r_{M_k}(P) + |C|. \tag{3}$$

Note that this is really only the special case of (2) with $X=E$. However, it is easy to deduce the more general version by applying (3) to the family $(M_k\restriction_X \,|\, k\in K)$.

Note also that no value $|\bigcup_{k\in K} I_k|$ appearing on the left is bigger than any value $\sum_{k\in K} r_{M_k}(P)+|C|$ appearing on the right. To see this, note that $|\bigcup_{k\in K}(I_k\cap P)| \le \sum_{k\in K} r_{M_k}(P)$ and $\bigcup_{k\in K}(I_k\cap C)\subseteq C$. So the formula is equivalent to the statement that we can find $(I_k \,|\, k\in K)$ and P and C with $P\dot\cup C=E$ so that

$$\left|\bigcup_{k\in K} I_k\right| = \sum_{k\in K} r_{M_k}(P) + |C|. \tag{4}$$

For this, what we need is to have equality in the two inequalities above, so we get

$$\left|\bigcup_{k\in K}(I_k \cap P)\right| = \sum_{k\in K} r_{M_k}(P) \text{ and } \bigcup_{k\in K}(I_k \cap C) = C. \tag{5}$$

The equation on the left can be broken down a bit further: it states that each $I_k \cap P$ is spanning (and so a base) in the appropriate matroid $M_k \restriction_P$, and that all these sets are disjoint. This is the familiar notion of a packing:

Definition 3.1. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E. A* packing *for this family consists of a spanning set S_k for each M_k such that the S_k are all disjoint.*

So the $I_k \cap P$ form a packing for the family $(M_k \restriction_P | k \in K)$. In fact, in this case, each $I_k \cap P$ is a base in the corresponding matroid. In Definition 3.1, we do not require the S_k to be bases, but of course if we have a packing we can take a base for each S_k and so obtain a packing employing only bases.

Dually, the right hand equation in (5) corresponds to the presence of a covering of C:

Definition 3.2. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E. A* covering *for this family consists of an independent set I_k for each M_k such that the I_k cover E.*

It is immediate that the sets $I_k \cap C$ form a covering for the family $(M_k \restriction_C | k \in K)$. In fact we get the stronger statement that they form a covering for the family $(M_k.C \,|\, k \in K)$ where we contract instead of restricting, since for each k we have that $I_k \cap P$ is an M_k-base for P, and we also have that I_k, which is the union of $I_k \cap C$ with $I_k \cap P$, is M_k-independent.

Putting all of this together, we get the following self-dual notion:

Definition 3.3. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E. We say this family has the* Packing/Covering property *iff there is a partition of E into two parts P (called the* packing side*) and C (called the* covering side*) such that $(M_k \restriction_P | \, k \in K)$ has a packing, and $(M_k.C \,|\, k \in K)$ has a covering.*

We have established above that this property follows from the rank formula for union, but the argument can easily be reversed to show that in fact Packing/Covering is equivalent to the rank formula, where that formula makes sense. However, Packing/Covering also makes sense for infinite matroids, where the rank formula is no longer useful. We are therefore led to the following conjecture:

Conjecture 1.3. *Every family of matroids on the same ground set has the Packing/Covering property.*

Because of this link to the rank formula, we immediately get a special case of this conjecture:

Theorem 3.4. *Every finite family of finite matroids on the same ground set has the Packing/Covering property.*

Packing/Covering for pairs of matroids is closely related to another property, which is conjectured to hold for all pairs of matroids.

Definition 3.5. *A pair (M, N) of matroids on the same ground set E has the* Intersection property *iff there is a subset J of E, independent in both matroids, and a partition of J into two parts J^M and J^N such that*

$$\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = E.$$

Conjecture 1.1. *Every pair of matroids on the same ground set has the Intersection property.*

We begin by demonstrating a link between Packing/Covering for pairs of matroids and Intersection.

Proposition 3.6. *Let M and N be matroids on the same ground set E. Then M and N have the Intersection property iff (M, N^*) has the Packing/Covering property.*

Proof. Suppose first of all that (M, N^*) has the Packing/Covering property, with packing side P decomposed as $S^M \dot{\cup} S^{N^*}$ and covering side C decomposed as $I^M \dot{\cup} I^{N^*}$. Let J^M be an M-base of S^M, and J^N an N-base of $C \setminus I^{N^*}$. $J = J^M \cup J^N$ is independent in M since $J^N \subseteq I^M$ is independent in $M.C$ and J^M is independent in $M\restriction_P$. Similarly J is independent in N since $J^M \subseteq P \setminus S^{N^*}$ is independent in $N.P$ and J^N is independent in $N\restriction_C$. But also

$$\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = \mathrm{Cl}_M(S^M) \cup \mathrm{Cl}_N(C \setminus I^{N^*}) \supseteq P \cup C = E.$$

Now suppose instead that M and N have the Intersection property, as witnessed by $J = J^M \dot{\cup} J^N$. Let $J^M \subseteq P \subseteq \mathrm{Cl}_M(J^M)$ and $J^N \subseteq C \subseteq \mathrm{Cl}_N(J^N)$ be a partition of E (this is possible since $\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = E$). We shall show first of all that $M\restriction_P$ and $N^*\restriction_P$ have a packing, with the spanning sets given by $S^M = J^M$ and $S^{N^*} = P \setminus J^M$. J^M is spanning in $M\restriction_P$ since $P \subseteq \mathrm{Cl}_M(J^M)$, so it is enough to check that $P \setminus J^M$ is spanning in $N^*\restriction_P$, or equivalently that J^M is independent in $N.P$. But this is true since J^N is an N-base of C and $J^M \cup J^N$ is N-independent.

Similarly, J^N is independent in $M.C$, and since $C \subseteq \mathrm{Cl}_N(J^N)$ J^N is spanning in $N \restriction_C$ and so $C \setminus J^N$ is independent in $N^*.C$. Thus the sets $I^M = J^N$ and $I^{N^*} = C \setminus J^N$ form a covering for $(M.C, N^*.C)$. ∎

Corollary 3.7. *If M and N are matroids on the same ground set, then (M, N) has the Packing/Covering property iff (M^*, N^*) does.* □

This corollary is not too hard to see directly. However, the following similar corollary is less trivial.

Corollary 1.4. *If M and N are matroids on the same ground set then M and N have the Intersection property iff M^* and N^* do.* □

Proposition 3.6 shows that Conjecture 1.1 follows from Conjecture 1.3, but so far we would only be able to use it to deduce that any pair of matroids has the Packing/Covering property from Conjecture 1.1. However, this turns out to be enough to give the whole of Conjecture 1.3.

Proposition 3.8. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E, and let $M = \bigoplus_{k \in K} M_k$, on the ground set $E \times K$. Let N be the matroid on the same ground set given by $\bigoplus_{e \in E} U^*_{1,K}$. Then the M_k have the Packing/Covering property iff M and N do.*

Proof. First of all, suppose that the M_k have the Packing/Covering property and let P, C, S_k and I_k be as in Definition 3.3. We can partition $E \times K$ into $P' = P \times K$ and $C' = C \times K$. Let $S^M = \bigcup_{k \in K} S_k \times \{k\}$, and let $S^N = P' \setminus S^M$. S^M is spanning in $M \restriction_{P'}$ by definition, and since the sets S_k are disjoint, there is for each $e \in P$ at most one $k \in K$ with $(e,k) \notin S^N$. Thus S^N is spanning in $N \restriction_{P'}$. Similarly, let $I^M = \bigcup_{k \in K} I_k \times \{k\}$ and let $I^N = C' \setminus I^M$. I^M is independent in $M.C'$ by definition, and since the sets I_k cover C there is for each $e \in E$ at least one $k \in K$ with $(e,k) \notin I^N$. Thus I^N is independent in $N.C'$.

Now suppose, instead, that M and N have the Packing/Covering property, with packing side P decomposed as $S^M \dot\cup S^N$ and covering side C decomposed as $I^M \dot\cup I^N$. First we modify these sets a little so that the packing and covering sides are given by $\overline{P} \times K$ and $\overline{C} \times K$ for some sets $\overline{P}$ and $\overline{C}$. To this end, we let $\overline{P} = \{e \in E \mid (\forall k \in K)(e,k) \in P\}$, and $\overline{C} = \{e \in E \mid (\exists k \in K)(e,k) \in C\}$, so that $\overline{P}$ and $\overline{C}$ form a partition of E. Let $\overline{S}^N = S^N \cap (\overline{P} \times K)$ and $\overline{I}^N = I^N \cup ((\overline{C} \times K) \setminus C)$. We shall show that

$(S^M,\overline{S}^N)$ is a packing for $(M\restriction_{\overline{P}\times K},N\restriction_{\overline{P}\times K})$ and $(I^M,\overline{I}^N)$ is a covering for $(M.(\overline{C}\times K),N.(\overline{C}\times K))$.

For any $e\in\overline{C}$, the restriction of the corresponding copy of $U^*_{1,K}$ to $P\cap(\{e\}\times K)$ is free, and so since the intersection of S^N with this set is spanning there, it must contain the whole of $P\cap(\{e\}\times K)$. So since $S^M\subseteq P$ is disjoint from S^N, it can't contain any (e,k) with $e\in\overline{C}$. That is, $S^M\subseteq\overline{P}\times K$. It also spans $\overline{P}\times K$ in M, since it spans the larger set P. For each $e\in\overline{P}$, $\overline{S}^N\cap(\{e\}\times K)=S^N\cap(\{e\}\times K)$ N-spans $\{e\}\times K$. Thus $\overline{S}^N$ N-spans $\overline{P}\times K$, so $(S^M,\overline{S}^N)$ is a packing for $(M\restriction_{\overline{P}\times K},N\restriction_{\overline{P}\times K})$.

To show that $(I^M,\overline{I}^N)$ is a covering for $(M.(\overline{C}\times K),N.(\overline{C}\times K))$, it suffices to show that $\overline{I}^N$ is $N.(\overline{C}\times K)$-independent. For each $e\in\overline{C}$, the set $C\cap(\{e\}\times K)$ is nonempty, so the contraction of the corresponding copy of $U^*_{1,K}$ to this set consists of a single circuit, so there is some point in this set but not in I^N. Then that same point is also not in $\overline{I}^N$, and so $\overline{I}^N\cap(\{e\}\times K)$ is independent in the corresponding copy of $U^*_{1,K}$, so $\overline{I}^N$ is indeed $N.(\overline{C}\times P)$-independent.

Now that we have shown that $\overline{P}\times K$, $\overline{C}\times K$, $(S^M,\overline{S}^N)$ and $(I^M,\overline{I}^N)$ also witness that M and N have the Packing/Covering property, we show how we can construct a packing and a covering for $(M_k\restriction_{\overline{P}}|\,k\in K)$ and $(M_k.\overline{C}\,|\,k\in K)$ respectively.

For each $k\in K$ let $I_k=\{e\in E\,|\,(e,k)\in I^M\}$. Since, as we saw above, I^M meets each of the sets $\{e\}\times K$ with $e\in\overline{C}$, the union of the I_k is $\overline{C}$. Since also each I_k is independent in $M_k.\overline{C}$, they form a covering for $(M_k.\overline{C}\mid k\in K)$. Similarly, let $S_k=\{e\in E\mid(e,k)\in S^M\}$. Since the intersection of $\overline{S}^N$ with $\{e\}\times K$ is spanning in the corresponding copy of $U^*_{1,k}$ for any $e\in\overline{P}$, it follows that for such e it misses at most one point of this set, so that there can be at most one point in $S^M\cap(\{e\}\times K)$, so the S_k are disjoint. Thus they form a packing of $(M_k\restriction_{\overline{P}}|\,k\in K)$. ∎

Corollary 3.9. *The following are equivalent:*

(a) Any two matroids have the Intersection property (Conjecture 1.1).
(b) Any two matroids in which the second is a direct sum of copies of $U_{1,2}$ have the Intersection property.
(c) Any pair of matroids has the Packing/Covering property.
(d) Any pair of matroids in which the second is a direct sum of copies of $U_{1,2}$ has the Packing/Covering property.
(e) Any family of matroids has the Packing/Covering property (Conjecture 1.3).

Proof. We shall prove the following equivalences.

$$\begin{array}{ccccc} (b) & \longleftrightarrow & (d) & & \\ & & \updownarrow & & \\ (a) & \longleftrightarrow & (c) & \longleftrightarrow & (e) \end{array}$$

The equivalences of (a) with (c) and (b) with (d) both follow from Proposition 3.6. (c) evidently implies (d), but we can also get (c) from (d) by applying Proposition 3.8. Similarly, (e) evidently implies (c) and we can get (e) from (c) by applying Proposition 3.8. ∎

4. A special case of the Packing/Covering conjecture

In [3], Aharoni and Ziv prove a special case of the intersection conjecture. Here we employ a simplified form of their argument to prove a special case of the Packing/Covering conjecture. Our simplification also yields a slight strengthening of their theorem.

Key to the argument is the notion of a wave.

Definition 4.1. *Let $(M_k\,|\,k\in K)$ be a family of matroids all on the ground set E. A* wave *for this family is a subset P of E together with a packing $(S_k \mid k \in K)$ of $(M_k \restriction_P |\; k \in K)$. In a slight abuse of notation, we shall sometimes refer to the wave just as P or say that elements of P are in the wave. A wave is a* hindrance *if the S_k don't completely cover P. The family is* unhindered *if there is no hindrance, and* loose *if the only wave is the empty wave.*

Remark 4.2. Those familiar with Aharoni and Ziv's notion of wave should observe that if $(P,(S_1,S_2))$ is a wave as above and we let F be an M_2-base of S_2, then F is not only M_2-independent, but also $M_1^*.P$-independent, since $S_1\subseteq P\setminus F$ is $M_1\restriction_P$-spanning. Now since $P\subseteq \mathrm{Cl}_{M_2}(F)$, we get that F is also $M_1^*.\mathrm{Cl}_{M_2}(F)$-independent. Thus F is a wave in the sense of Aharoni and Ziv for the matroids M_1^* and M_2. There is a similar correspondence of the other notions defined above.

Similarly, they say that the pair (M_1,M_2) is *matchable* iff there is a set which is M_1-spanning and M_2-independent. Those interested in translating between the two contexts should note that there is a covering for (M_1,M_2) iff (M_1^*,M_2) is matchable.

We define a partial order on waves by $(P,(S_k\,|\,k\in K))\le(P',(S'_k\,|\,k\in K))$ iff $P\subseteq P'$ and for each $k\in K$ we have $S_k\subseteq S'_k$. We say a wave is *maximal* iff it is maximal with respect to this partial order.

Lemma 4.3. *For any wave P there is a maximal wave $P_{\max} \geq P$.*

Proof. This follows from Zorn's Lemma since for any chain $((P_i, (S^i_k \mid k \in K)) \mid i \in I)$ the union $(\bigcup_{i \in I} P_i, (\bigcup_{i \in I} S^i_k \mid k \in K))$ is a wave. ∎

Lemma 4.4. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E, and let $(P, (S_k \mid k \in K))$ and $(P', (S'_k \mid k \in K))$ be two waves. Then $(P \cup P', (S_k \cup (S'_k \setminus P) \mid k \in K))$ is a wave.*

Proof. Clearly, the $S_k \cup (S'_k \setminus P)$ are disjoint and $cl_{M_k} S_k$ includes $S'_k \cap P$ and hence $cl_{M_k}(S_k \cup (S'_k \setminus P))$ includes $P \cup P'$, as desired. ∎

Corollary 4.5. *If $P_{\max}$ is a maximal wave then anything in any wave P is in $P_{\max}$.*

Proof. We apply Lemma 4.4 to the pair $(P_{\max}, P)$. ∎

Lemma 4.6. *For any $e \in E$ and $k \in K$, any maximal wave P satisfies $e \in \mathrm{Cl}_{M_k} P$ whenever there is any wave P' with $e \in \mathrm{Cl}_{M_k} P'$.*

In particular, if e is not contained in any wave, there are at least two k such that, for every wave P', $e \notin \mathrm{Cl}_{M_k} P'$.

Proof. Let $(P, (S_k \mid k \in K))$ be a maximal wave. By Corollary 4.5 for any wave $(P', (S'_k \mid k \in K))$ we have $S'_k \subseteq \mathrm{Cl}_{M_k} S_k$. Thus $e \in \mathrm{Cl}_{M_k} P' = \mathrm{Cl}_{M_k} S'_k$ implies $e \in \mathrm{Cl}_{M_k} P$, as desired.

For the second assertion, assume toward contradiction that there is at most one k_0 such that, for every wave P', $e \notin \mathrm{Cl}_{M_{k_0}} P'$. Then $e \in \mathrm{Cl}_{M_k} P$ for all $k \neq k_0$. But then the following is a wave and contains e: $X := (P + e, (\overline{S}_k \mid k \in K))$ where $\overline{S}_{k_0} = S_{k_0} + e$ and $\overline{S}_k = S_k$ for other values of k. This is a contradiction. ∎

Lemma 4.7. *Let $(P, (S_k \mid k \in K))$ be a wave for a family $(M_k \mid k \in K)$ of matroids. Let $(P', (S'_k \mid k \in K))$ be a wave for the family $(M_k/P \mid k \in K)$. Then $(P \cup P', (S_k \cup S'_k \mid k \in K))$ is a wave for the family $(M_k \mid k \in K)$. If either P or P' is a hindrance, then so is $P \cup P'$.*

Remark 4.8. In fact, though we will not need this, a similar statement can be shown for an ordinal indexed family of waves P^β, with P^β a wave for the family $(M_k / \bigcup_{\gamma < \beta} P^\gamma \mid k \in K)$.

Proof. For each k, the set S'_k is spanning in $M_k\restriction_{P\cup P'}/P$ and S_k is spanning in $M_k\restriction_{P\cup P'}\restriction P$, so by Lemma 2.3 each set $S_k\cup S'_k$ spans $P\cup P'$, and they are clearly disjoint. If the S_k don't cover some point of P then the $S_k\cup S'_k$ also don't cover that point, and the argument in the case where P' is a hindrance is similar. ∎

Corollary 4.9. *For any maximal wave $P_{\max}$, the family $(M_k/P_{\max}\,|\,k\in K)$ is loose.*

We are now in a position to present another Conjecture equivalent to the Packing/Covering Conjecture. It is for this new form that we shall present our partial proof.

Conjecture 4.10. *Any unhindered family of matroids has a covering.*

Proposition 4.11. *Conjecture 4.10 and Conjecture 1.3 are equivalent.*

Proof. First of all, suppose that Conjecture 1.3 holds, and that we have an unhindered family $(M_k \mid k\in K)$ of matroids. Using Conjecture 1.3, we get P, C, S_k and I_k as in Definition 3.3. Then $(P,(S_k\mid k\in K))$ is a wave, and since it can't be a hindrance the sets S_k cover P. They must also all be independent, since otherwise we could remove a point from one of them to obtain a hindrance. So the sets $S_k\cup I_k$ give a covering for $(M_k\,|\,k\in K)$.

Now suppose, instead, that Conjecture 4.10 holds, and let $(M_k\mid k\in K)$ be any family of matroids on the ground set E. Then let $(P,(S_k\mid k\in K))$ be a maximal wave. By Conjecture 4.9, $(M_k/P\,|\,k\in K)$ is loose, and so in particular this family is unhindered. So it has a covering $(I_k\,|\,k\in K)$. Taking covering side $C=E\setminus P$, this means that the M_k have the Packing/Covering property. ∎

Lemma 4.12. *Suppose that we have an unhindered family $(M_k\,|\,k\in K)$ of matroids on a ground set E. Let $e\in E$ and $k_0\in K$ such that for every wave P we have $e\notin \mathrm{Cl}_{M_{k_0}} P$. Then the family $(M'_k\,|\,k\in K)$ on the ground set $E-e$ is also unhindered, where $M'_{k_0}=M_{k_0}/e$ but $M'_k=M_k\backslash e$ for other values of k.*

Proof. Suppose not, for a contradiction, and let $(P,(S_k\mid k\in K))$ be a hindrance for $(M'_k\mid k\in K)$. Without loss of generality, we assume that the S_k are bases of P. Let $\overline{S}_k$ be given by $\overline{S}_{k_0}=S_{k_0}+e$ and $\overline{S}_k=S_k$ for other values of k. Note that $\overline{S}_{k_0}$ is independent because otherwise, by the M_{k_0}/e-independence of S_{k_0}, we must have $e\in\mathrm{Cl}_{M_{k_0}}(S_{k_0})$ (in fact, $\{e\}$ must be an M_{k_0}-circuit), so that $P\subseteq\mathrm{Cl}_{M_{k_0}}(S_{k_0})$, and thus $(P,(S_k\,|\,k\in K))$ is a wave for the M_k with $e \in \mathrm{Cl}_{M_{k_0}} P$. Let P' be the set of $x \in P$ such that there is no $(\overline{S}_k \mid k \in K)$-exchange chain from x to e.

Let $x_0 \in P \setminus \bigcup_{k \in K} S_k$. If $x_0 \in P'$, then we will show that $(P', (P' \cap \overline{S}_k \mid k \in K))$ is a wave containing x_0. This contradicts the assumption that $(M_k \mid k \in K)$ is unhindered. We must show for every k that every $x \in P' \setminus P' \cap \overline{S}_k$ is M_k-spanned by $P' \cap \overline{S}_k$. Since $e \notin P'$, we cannot have $x = e$. Let C be the unique circuit contained in $x + \overline{S}_k$. If $x \in P'$, then $C \subseteq P'$ by Lemma 2.7, so $x \in \mathrm{Cl}_{M_k}(P' \cap \overline{S}_k)$, as desired.

If $x_0 \notin P'$, there is a shortest $(\overline{S}_k \mid k \in K)$-exchange chain

$$(y_0 = x_0, k_0; y_1, k_1; \ldots; y_n = e)$$

from x_0 to e. Let $\overline{S}'_k := \overline{S}_k \cup \{y_l \mid k_l = k\} \setminus \{y_{l+1} \mid k_l = k\}$. By Lemma 2.5, $\overline{S}'_k$ is M_k-independent and $\mathrm{Cl}_{M_k} \overline{S_k} = \mathrm{Cl}_{M_k} \overline{S_k}'$ for all $k \in K$. Thus each $\overline{S}'_k$ M_k-spans P but avoids e, in other words: $(P, (\overline{S}'_k \mid k \in K))$ is an $(M_k \mid k \in K)$-wave. But also $e \in \mathrm{Cl}_{M_{k_0}} P$ since $e \in \overline{S}_{k_0}$, a contradiction. ∎

We will now discuss those partial versions of Conjecture 4.10 which we can prove. We would like to produce a covering of the ground set by independent sets - and that means that we don't want any of the sets in the covering to include any circuits for the corresponding matroid. First of all, we show that we can at least avoid *some* circuits. In fact, we'll prove a slightly stronger theorem here, showing that we can specify a countable family of sets, which are to be avoided whenever they are dependent. In all our applications, the dependent sets we care about will be circuits.

Theorem 4.13. *Let $(M_k \mid k \in K)$ be an unhindered family of matroids on the same ground set E. Suppose that we have a sequence of subsets o_n of E. Then there is a family $(I_k \mid k \in K)$ whose union is E and such that for no $k \in K$ and $n \in \mathbb{N}$ do we have both $o_n \subseteq I_k$ and o_n dependent in M_k.*

Proof. If some wave includes the whole ground set, then as the family is unhindered, this wave would yield the desired covering. Unfortunately, we may not assume this. Instead, we recursively build a family $(J_k \mid k \in K)$ of disjoint sets such that some wave $(P, (S_k \mid k \in K))$ for the $M_k / J_k \setminus \bigcup_{l \neq k} J_l$ includes enough of $E \setminus \bigcup_k J_k$ that any family $(I_k \mid k \in K)$ whose union is E and with $I_k \cap (P \cup \bigcup_{k \in K} J_k) = S_k \cup J_k$ will work.

We construct J_k as the nested union of some $(J_k^n \mid n \in \mathbb{N} \cup \{0\})$ with the following properties. Abbreviate $M_k^n := M_k / J_k^n \setminus \bigcup_{l \neq k} J_l^n$.

(a) J_k^n is independent in M_k.
(b) For different k, the sets J_k^n are disjoint.

(c) $(M_k^n \,|\, k \in K)$ is unhindered.
(d) Either the set $o_n \setminus \bigcup_{k\in K} J_k^n$ is included in some $(M_k^n \,|\, k \in K)$-wave or there are distinct l, l' such that there is some $e \in o_n \cap J_l^n$ and some $e' \in o_n \cap J_{l'}^n$.

Put $J_k^0 := \emptyset$ for all k. These satisfy (a)-(c), and (d) is vacuous since there is no term o_0 (we are following the convention that 0 is not a natural number). Assume that we have already constructed J_k^n satisfying (a)-(d).

If (d) with o_{n+1} in place of o_n is already satisfied by the $(J_k^n \,|\, k \in K)$, we can simply take $J_k^{n+1} := J_k^n$ for all k.

Otherwise, if we let P_{max} be a maximal wave, there is some $e \in o_{n+1} \setminus \bigcup_{k\in K} J_k^n$ not in P_{max} and so not in any $(M_k^n \,|\, k \in K)$-wave. By Lemma 4.6, there are at least two $k \in K$ such that $e \notin \mathrm{Cl}_{M_k^n} P'$ for every wave P'. In particular, e is not a loop ($\{e\}$ is independent) in M_k^n for those two k. Let l be one of these two values of k. Now let $\overline{J_l^{n+1}} := J_l^n + e$ and $\overline{J_k^{n+1}} := J_k^n$ for $k \neq l$. Then the $\overline{J_k^{n+1}}$ satisfy (a) and (b). By Lemma 4.12 and the choice of e, we also have (c).

If the $\overline{J_k^{n+1}}$ already satisfy (d), then we are done. Else, to obtain (d), repeat the induction step so far and find $e' \in o_{n+1} \setminus \bigcup_{k\in K} \overline{J_k^{n+1}}$ not in any $(\overline{M_k^n} \,|\, k \in K)$-wave. Here $\overline{M_k^n}$ is M_k^n/e if $k = l$ and $M_k^n \setminus e$ otherwise. Further we find, $l' \neq l$ such that $\{e'\}$ is independent in $\overline{M_{l'}^n}$ and $e' \notin \mathrm{Cl}_{M_l} P'$ for every wave P'. Now let $J_{l'}^{n+1} := \overline{J_{l'}^{n+1}} + e'$ and $J_k^{n+1} := \overline{J_k^{n+1}}$ for $k \neq l'$. Then the J_k^{n+1} satisfy (a) and (b) and now also (d). By Lemma 4.12 and the choice of e', we also have (c).

We now define a new family of matroids by $M'_k := M_k/J_k \setminus \bigcup_{l\neq k} J_l$, and we construct an $(M'_k \mid k \in K)$-wave $(P, (S_k \mid k \in K))$. We once more do this by taking the union of a recursively constructed nested family. Explicitly, we take $S_k = \bigcup_{n\in\mathbb{N}} S_k^n$ and $P = \bigcup_{n\in\mathbb{N}} P^n$, where for each n the wave $W^n = (P^n, (S_k^n \,|\, k \in K))$ is a maximal wave for $(M_k^n \,|\, k \in K)$ and the S_k^n are nested. We can find such waves using Lemma 4.3: for each n we have that W^n is also a wave for $(M_k^{n+1} \mid k \in K)$ since in our construction we never contract or delete anything which is in a wave.

Now let $(I_k \mid k \in K)$ be chosen so that $\bigcup I_k = E$ and for each $k_0 \in K$ we have $I_{k_0} \cap (P \cup \bigcup_{k\in K} J_k) = S_{k_0} \cup J_{k_0}$. Suppose for a contradiction that for some pair (k_0, n) we have $o_n \subseteq I_{k_0}$ and o_n is dependent in M_{k_0}. Then by (d), either the set $o_n \setminus \bigcup_{k\in K} J_k^n$ is included in some $(M_k^n \mid k \in K)$-wave or there are distinct l, l' such that there is some $e \in o_n \cap J_l^n$ and some $e' \in o_n \cap J_{l'}^n$. In the second case, clearly $o_n \not\subseteq I_{k_0}$.

In the first case, we will find a hindrance for $(M_k^n \mid k \in K)$, which contradicts (c). It suffices to show that $S_{k_0}^n$ is dependent in $M_{k_0}^n$, since then we can obtain a hindrance by removing a point from $S^n_{k_0}$ in W^n. Let $o = o_n \setminus \bigcup_{k\in K} J^n_k = o_n \setminus J^n_{k_0}$. Note that o is dependent in $M^n_{k_0}$, since o_n is dependent in $M^n_{k_0}$ but $J^n_{k_0}$ is not by (a). By assumption, $o \subseteq P^n$, and so since also $o \subseteq o_n \subseteq I_{k_0}$ we have $o \subseteq I_{k_0} \cap P^n = S^n_{k_0}$, so that $S^n_{k_0}$ is $M^n_{k_0}$-dependent as required. ∎

Note that, in particular, if we have a countable family of matroids each with only countably many circuits, then Theorem 4.13 applies in order to prove Conjecture 1.3 in that special case. Requiring only countably many circuits might seem quite restrictive, but there are many cases where it holds:

Proposition 4.14. *A matroid of any of the following types on a countable ground set has only countably many circuits:*

(a) A finitary matroid.
(b) A matroid whose dual has finite rank.
(c) A direct sum of matroids each with only countably many circuits.

Proof. (a) follows from the fact that the countable ground set has only countably many finite subsets. For (b), since every base B has finite complement, there are only countably many bases. As every circuit is a fundamental circuit for some base, there can only be countably many circuits, as desired. For (c), there can only be countably many nontrivial summands in the direct sum since the ground set is countable, and the result follows. ∎

In particular, Theorem 4.13 applies to any countable family of matroids each of which is a direct sum of matroids that are finitary or whose duals have finite rank. This includes the main result of Aharoni and Ziv in [3], if the ground set E is countable, by Proposition 3.6.

If we have a family of sets $(I_k \mid k \in K)$ which does not form a covering, because some elements aren't independent, how might we tweak it to make them more independent? Suppose that the reason why I_k is dependent is that it contains a circuit o of M_k, but that o also includes a cocircuit for another matroid $M_{k'}$ from our family. Then we could move some point from I_k into $I_{k'}$ to remove this dependence without making $I_{k'}$ any more dependent.[3] We are therefore not so worried about circuits including cocircuits in this way as we are about other sorts of circuits. Therefore we now consider cases where most circuits do include such cocircuits:

Definition 4.15. *Let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E. For each $k \in K$ we let W_k be the set of all M_k-circuits that*

[3] We may assume that the I_k are disjoint. Then any new circuits in $I_{k'}$ would have to meet the cocircuit in just one point, which is impossible by Lemma 2.4.

do not contain an $M_{k'}$-cocircuit with $k' \neq k$. Call the family $(M_k \mid k \in K)$ of matroids at most countably weird *if $\bigcup W_k$ is at most countable.*

Note that if E is countable then $(M_k \mid k \in K)$ is at most countably weird if and only if $\bigcup W_k^\infty$ is countable where W_k^∞ is the subset of W_k consisting only of the infinite circuits in W_k.

Theorem 4.16. *Any unhindered and at most countably weird family $(M_k \mid k \in K)$ of matroids has a covering.*

Proof. Apply Theorem 4.13 to $(M_k \mid k \in K)$ where the o_n enumerate $\bigcup W_k$ where the W_k are defined as in Definition 4.15.

So far $(I_k \mid k \in K)$ is not necessarily a covering since each I_k might still contain circuits. But by the choice of the family of circuits each circuit contained in I_k contains an $M_{k'}$-cocircuit with $k' \neq k$.

In the following, we tweak $(I_k \mid k \in K)$ to obtain a covering $(L_k \mid k \in K)$. First extend I_k into a minimal M_k-spanning set B_k by $(IM)^*$. We obtain L_k from B_k by removing all elements in $I_k \cap \bigcup_{l \neq k} B_l$. We can suppose without loss of generality $(I_k \mid k \in K)$ was a partition of E, and so the family $(L_k \mid k \in K)$ covers E. It remains to show that L_k is independent. For this, assume for a contradiction that L_k contains an M_k-circuit C. By the choice of B_k, the circuit C is contained in I_k. In particular, C contains an M_l-cocircuit X for some $l \neq k$. By construction B_l meets X and thus C. As $C \subseteq I_k$, the circuit C is not contained in L_k, a contradiction. So $(L_k \mid k \in K)$ is the desired covering. ∎

Theorem 4.17. *Any at most countably weird family $(M_k \mid k \in K)$ of matroids has the Packing/Covering property.*

Proof. For each $k \in K$, let W_k be the set of all M_k-circuits that do not contain an $M_{k'}$-cocircuit with $k' \neq k$. Let $(P, (S_k \mid k \in K))$ be a maximal wave. We may assume that each S_k is a base of P. It suffices to show that the family $(M_k/P \mid k \in K)$ has a covering.

By Theorem 4.16, it suffices to show that the family $(M_k/P \mid k \in K)$ is at most countably weird. Let $\overline{W}_k$ be the set of M_k/P-circuits that do not include some $M_{k'}/P$-cocircuit for some $k' \neq k$. By Lemma 2.2, for each $o \in \overline{W}_k$, there is an M_k-circuit $\hat{o}$ included in $o \cup S_k$ with $o \subseteq \hat{o}$.

Next we show that if $\hat{o}$ includes some $M_{k'}$-cocircuit b, then $b \subseteq o$. In particular o includes some $M_{k'}/P$-cocircuit. Indeed, otherwise $b \cap P$ is nonempty and includes some $M_{k'} \restriction_P$-cocircuit. This cocircuit would be included in S_k, which is impossible since $S_{k'}$ spans P, and is disjoint from S_k. Thus if $\hat{o}$ is in W_k, then o is in $\overline{W}_k$.

For each $o \in \bigcup \overline{W}_k$, we pick some $k \in K$ such that $o \in \overline{W}_k$, and let $\iota(o) = \hat{o}$. Then $\iota : \bigcup \overline{W}_k \to \bigcup W_k$ is an injection since if $\iota(o) = \iota(q)$, then $o = \iota(o) \setminus P = \iota(q) \setminus q = q$. Thus $(M_k/P \mid k \in K)$ is at most countably weird and so $(M_k/P \mid k \in K)$ has a covering by Theorem 4.16, which completes the proof. ∎

However, there are still some important open questions here.

Definition 4.18 ([5]). *The* finitarisation of a matroid M *is the matroid* M^{fin} *whose circuits are precisely the finite circuits of* M.[4] *A matroid is called* nearly finitary *if every base misses at most finitely many elements of some base of the finitarisation.*

From Proposition 3.6 and the corresponding case of Matroid Intersection [5] we obtain the following:

Corollary 4.19. *Any pair of nearly finitary matroids has the Packing/Covering property.*

By Proposition 3.8 Corollary 4.19 implies that any finite family of nearly finitary matroids has the Packing/Covering property. Since every countable set has only countably many finite subsets, any family of finitary matroids supported on a countable ground set is at most countably weird, and thus has the Packing/Covering property by Theorem 4.17. On the other hand any family of two cofinitary matroids has the Packing/Covering by Corollary 4.19 since the pairwise Packing/Covering Property is self-dual. By Proposition 3.8, this implies that any family of cofinitary matroids has the Packing/Covering property. We sum up these results in the following table.

Type of family	cofinitary	finitary	nearly finitary
finite	✓	✓	✓
countable ground set	✓	✓	?
arbitrary	✓	?	?

In particular, we do not know the answer to the following open questions.

Open Question 4.20. *Must every family of nearly finitary matroids on a countable common ground set have the Packing/Covering property?*

Open Question 4.21. *Must every family of finitary matroids have the Packing/Covering property?*

[4] It is easy to check that M^{fin} is indeed a matroid [5].

5. Base covering

The well-known base covering theorem reads as follows.

Theorem 5.1. *Any family of finite matroids $(M_k \mid k \in K)$ on a finite common ground set E has a covering if and only if for every finite set $X \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k}(X) \geq |X|$$

Taking the family to contain only one matroid, consisting of one infinite circuit, we see that this theorem does not extend verbatim to infinite matroids. However, Theorem 5.1 extends verbatim to finite families of finitary matroids by compactness [4].[5] The requirement that the family is finite is necessary as $(U_k = U_{1,\mathbb{R}} \mid k \in \mathbb{N})$ satisfies the rank formula but does not have a covering.

In the following, we conjecture an extension of the finite base covering theorem to arbitrary infinite matroids. Our approach is to replace the rank formula by a condition that for finite sets X is implied by the rank formula but is still meaningful for infinite sets. A first attempt might be the following:

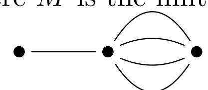
(6) Any packing for the family $(M_k \restriction_X \mid k \in K)$ is already a covering.

Indeed, for finite X, if $(M_k \restriction_X \mid k \in K)$ has a packing and there is an element of X not covered by the spanning sets of this packing, then this violates the rank formula. However, there are infinite matroids that violate (6) and still have a covering, see Figure 2.

We propose to use instead the following weakening of (6).

(7) If $(M_k \restriction_X \mid k \in K)$ has a packing, then it also has a covering.

To see that (7) does not imply the rank formula for some finite X, consider the family (M, M), where M is the finite cycle matroid of the graph



This graph has an edge not contained in any cycle (so that (M, M) does not have a packing) but enough parallel edges to make the rank formula false.

Using (7), we obtain the following:

Conjecture 5.2 (Covering Conjecture). *A family of matroids $(M_k \mid k \in K)$ on the same ground set E has a covering if and only if (7) is true for every $X \subseteq E$.*

[5] The argument in [4] is only made in the case where all M_k are the same but it easily extends to finite families of arbitrary finitary matroids.

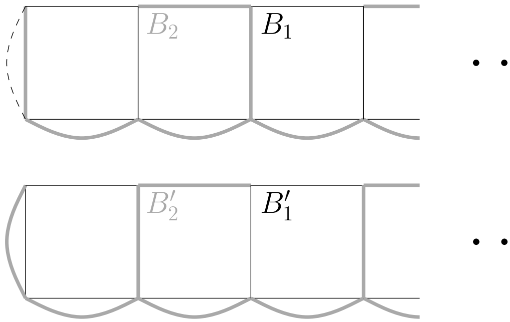


Figure 2. Above is a base packing which isn't a base covering. Below that is a base covering for the same matroids, namely the finite cycle matroid for the graph, taken twice

Proposition 5.3. *Conjecture 1.3 and Conjecture 5.2 are equivalent.*

Proof. For the "only if" direction, note that Conjecture 5.2 implies Conjecture 4.10, which by Proposition 4.11 implies Conjecture 1.3.

For the "if" direction, note that by assumption we have a partition $E = P \dot\cup C$ such that there exist disjoint $M_k \restriction_P$-spanning sets S_k and $M_k.C$-independent sets I_k whose union is C. By (7), $(M_k \restriction_P | k \in K)$ has a covering with sets B_k, where $B_k \in \mathcal{I}(\mathcal{M}_{\|} \restriction_{\mathcal{P}})$. As $I_k \cup B_k \in \mathcal{I}(\mathcal{M}_{\|})$, the sets $I_k \cup B_k$ form the desired covering. ∎

As Packing/Covering is true for finite matroids, Proposition 5.3 implies the non-trivial direction of Theorem 5.1. By Theorem 4.17 we obtain the following applications.

Corollary 5.4. *Any at most countably weird family of matroids $(M_k | k \in K)$ has a covering if and only if (7) is true for every $X \subseteq E$.*

Let us now specialise to graphs. A good introduction to the algebraic and the topological cycle matroids of infinite graphs is [7]. We rely on the fact that the algebraic cycle matroid of any locally finite graph and the topological cycle matroid of any graph are co-finitary.

Definition 5.5. *The bases of the topological cycle matroid are called* topological trees *and the bases of the algebraic cycle matroid are called* algebraic trees. *Using this we define* topological tree-packing, topological tree-covering, algebraic tree-packing, algebraic tree-covering.

Corollary 5.6 (Base covering for the topological cycle matroids). *A family of multigraphs $(G_k \mid k \in K)$ on a common ground set E has a topological tree-covering if and only if the following is true for every $X \subseteq E$.*

(8) *If $(G_k[X] \mid k \in K)$ has a topological tree-packing, then it also has a topological tree-covering.*

Corollary 5.7 (Base covering for the algebraic cycle matroids of locally finite graphs). *A family of locally finite multigraphs $(G_k \mid k \in K)$ on a common ground set E has an algebraic tree-covering if and only if the following is true for every $X \subseteq E$.*

(9) *If $(G_k[X] \mid k \in K)$ has an algebraic tree-packing, then it also has an algebraic tree-covering.*

6. Base packing

The well-known base packing theorem reads as follows.

Theorem 6.1. *Any family of finite matroids $(M_k \mid k \in K)$ on a finite common ground set E has a packing if and only if for every finite set $Y \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k.Y}(Y) \leq |Y|$$

Aigner-Horey, Carmesin and Fröhlich [4] extended this theorem to families consisting of finitely many copies of the same co-finitary matroid. We extend this to arbitrary co-finitary families.

Theorem 6.2. *Any family of co-finitary matroids $(M_k \mid k \in K)$ on a common ground set E has a packing if and only if for every finite set $Y \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k.Y}(Y) \leq |Y|$$

Proof by a compactness argument. We will think of partitions of the ground set E as functions from E to K - such a function f corresponds to a partition $(S_k^f \mid k \in K)$, given by $S_k^f = \{e \in E \mid f(e) = k\}$. Endow K with the co-finite topology where a set is closed iff it is finite or the whole of K. Then endow K^E with the product topology, which is compact since the topology on K is compact.

By Lemma 2.1 a set S is spanning for a matroid M iff it meets every cocircuit of that matroid. So we would like a function f contained in each of the sets $C_{k,B}=\{f \mid S_k^f \cap B \neq \emptyset\}$, where B is a cocircuit for the matroid M_k. We will prove the existence of such a function by a compactness argument: we need to show that each $C_{k,B}$ is closed in the topology given above and that any finite intersection of them is nonempty.

To show that $C_{k,B}$ is closed, we rewrite it as $\bigcup_{e\in B}\{f \mid f(e)=k\}$. Each of the sets $\{f \mid f(e)=k\}$ is closed since their complements are basic open sets, and the union is finite since M_k is co-finitary.

Now let $(k_i \mid 1 \leq i \leq n)$ and $(B_i \mid 1 \leq i \leq n)$ be finite families with each B_i a cocircuit in M_{k_i}. We need to show that $\bigcap_{1\leq i\leq n} C_{k_i,B_i}$ is nonempty. Let $X=\bigcup_{1\leq i\leq n} B_i$. Since the rank formula holds for each subset of X, we have by the finite version of the base packing Theorem a packing $(S_k \mid k \in K)$ of $(M_k.X \mid k \in K)$. Now any f such that $f(e) = k$ for $e \in S_k$ will be in $\bigcap_{1\leq i\leq n} C_{k_i,B_i}$ by Lemma 2.1, since each B_i is an $M_{k_i}.X$-cocircuit. This completes the proof. ∎

Theorem 6.1 does not extend verbatim to arbitrary infinite matroids. Indeed, for every integer k there exists a finitary matroid M on a ground set E with no three disjoint bases yet satisfying $|Y| \geq k r_{M.Y}(Y)$ for every finite $Y \subseteq E$ [2,11].

In the following we conjecture an extension of the finite base packing theorem to arbitrary infinite matroids. This extension uses the following condition, which for finite sets Y is implied by the rank formula of the base packing theorem but is still meaningful for infinite sets:

(10) If $(M_k.Y \mid k \in K)$ has a covering, then it also has a packing.

Indeed, if $(M_k.Y \mid k \in K)$ has a covering and there is an element of Y contained in several of the corresponding independent sets, then this violates the rank formula.

Using our new condition, we obtain the following:

Conjecture 6.3 (Packing Conjecture). *A family of matroids $(M_k \mid k \in K)$ on the same ground set E has a packing if and only if (10) is true for every $Y \subseteq E$.*

Proposition 6.4. *Conjecture 1.3 and Conjecture 6.3 are equivalent.*

Proof. Since by Lemma 2.1 condition (10) for a pair of matroids is equivalent to (7) for the duals of those matroids and a pair of matroids have a packing if and only if their duals have a covering, Conjecture 6.3 implies that any pair of matroids satisfying (7) has a covering, and in particular that any unhindered pair of matroids has a covering. As in the proof of (4.11), this implies that any pair of matroids has the Packing/Covering property, which implies Conjecture 1.3 by Conjecture 3.9.

The converse is proved as in the proof of Proposition 5.3. ∎

As Packing/Covering is true for finite matroids, Proposition 6.4 implies the non-trivial direction of Theorem 6.1. By Theorem 4.17 we obtain the following applications.

Corollary 6.5. *Any at most countably weird family of matroids on ground set E has a packing if and only if (10) is true for every $Y \subseteq E$.*

Now let us specialise to graphs. The question if there is a packing theorem for the finite cycle matroid of an infinite graph was raised by Nash-Williams in 1967 [12] , who suggested that a countable graph G has k edge-disjoint spanning trees if and if $k{\cdot}r_{M.Y}(Y) \leq |Y|$ for every finite edge set Y. Here M is the finite cycle matroid of G. However, Aharoni and Thomassen constructed a counterexample in 1989 [2]. Our approach gives the following two packing theorems for finite cycle matroids of infinite graphs. We rely on the fact that the finite cycle matroid of any graph is finitary.

Corollary 6.6 (Base packing theorem for the finite cycle matroid). *Any family of countable multigraphs $(G_k \mid k \in K)$ with a common edge set E has a tree-packing if and only if (11) is true for every $Y \subseteq E$.*

(11) *If $(M_k.Y \mid k \in K)$ has a tree-covering, then it also has a tree-packing.*

Corollary 6.7 (Base packing theorem for the finite cycle matroid). *Any finite family of multigraphs $(G_k \mid k \in K)$ with common edge set E has a tree-packing if and only if (11) is true for every $Y \subseteq E$.*

A similar result was obtained by Aharoni and Ziv [3]. However, their argument is different and they have the additional assumption that the ground set is countable.

Note that the covering conjecture for arbitrary finitary families is still open and equivalent to Open Question 4.21.

7. Overview

We have shown that a great many natural conjectures are equivalent, which we will review in this section. We are indebted to a reviewer for pointing out the importance of the fact that many of the equivalences we have proved specialise to smaller classes than the class of all matroids. We therefore consider the following conjectures, each of which could be made relative to a class $\mathcal{M}$ of matroids.

The Intersection conjecture: Any two matroids in $\mathcal{M}$ on the same ground set have the Intersection property

The pairwise Packing/Covering conjecture: Any pair of matroids from $\mathcal{M}$ on the same ground set has the Packing/Covering property

The Packing/Covering conjecture: Any family of matroids from $\mathcal{M}$ on the same ground set has the Packing/Covering property

The Packing conjecture: A family of matroids $(M_k \in \mathcal{M} \mid \| \in \mathcal{K})$ on the same ground set E has a packing if and only if the following condition is true for every $Y \subseteq E$:

$$\text{If } (M_k.Y \mid k \in K) \text{ has a covering, then it also has a packing.}$$

The Covering conjecture: A family of matroids $(M_k \in \mathcal{M} \mid \| \in \mathcal{K})$ on the same ground set E has a covering if and only if the following condition is true for every $Y \subseteq E$:

$$\text{If } (M_k \restriction_Y \mid k \in K) \text{ has a packing, then it also has a covering.}$$

Most crudely, if $\mathcal{M}$ is a class of matroids containing all matroids $U^*_{1,K}$ and closed under duality, minors and direct sums then all of the above conjectures are equivalent to each other, with proofs exactly as in this paper. However, particular equivalences only depend on weaker conditions on the class $\mathcal{M}$. For the equivalence of the Intersection conjecture with the pairwise Packing/Covering conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under duality. For the equivalence of the pairwise Packing/Covering conjecture with the Packing/Covering conjecture, we just need that $\mathcal{M}$ contains all the matroids $U^*_{1,K}$ and is closed under direct sums. This equivalence also holds for classes of matroids of bounded size:

Lemma 7.1. *Let $\mathcal{M}_{<\kappa}$ be the class of all matroids on ground sets of cardinality less than κ for some regular*[6] *cardinal κ. Then the pairwise Packing/Covering conjecture for $\mathcal{M}_\kappa$ is equivalent to the Packing/Covering conjecture for $\mathcal{M}_\kappa$.*

Proof (assuming the axiom of choice). It is clear that the pairwise Packing/Covering conjecture follows from the Packing/Covering conjecture. For the converse, suppose the pairwise Packing/Covering conjecture holds, and let $(M_k \mid k \in K)$ be a family of matroids on the same ground set E of cardinality less than κ. For each $e \in E$, let K_e be the set of $k \in K$ for which $\{e\}$ is independent in M_k. Let $E' = \{e \in E \mid \#(K_e) < \kappa\}$, and let

[6] Recall that an infinite cardinal κ is *regular* if and only if no set of cardinality κ can be expressed as a union of fewer than κ sets, all of cardinality less than κ.

$K' = \bigcup_{e \in E'} K_e$. Then K' has cardinality less than κ, so by Proposition 3.8 the family $(M_k \restriction_{E'} | \, k \in K')$ has the Packing/Covering property: call the packing side P and the covering side C, and let the packing and the covering be $(I_k \,|\, k \in K')$ and $(S_k \,|\, k \in K')$.

Let $C' = E \setminus P$, and for any $k \in K \setminus K'$ let $S_k = \emptyset$, which is spanning in $M_k \restriction E'$ by the definition of K'. Using some well-ordering of $E \setminus E'$, we can choose recursively for each $e \in E \setminus E'$ an element $k(e)$ of K_e such that all of the $k(e)$ are distinct. For each $k \in K \setminus K'$, we now set $I_k = \{e \in E \setminus E' \,|\, k(e) = k\}$, which is either empty or has size 1 and is independent in M_k. Then the S_k form a packing of P and the I_k form a covering of C', so $(M_k \,|\, k \in K)$ has the Packing/Covering property. ∎

For the equivalence of the Packing/Covering conjecture with the Covering conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under contraction. For the equivalence of the Packing/Covering conjecture with the Packing conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under deletion. To see this, it is not enough to use the argument in the proof of Proposition 6.4, for that argument goes via the pairwise Packing/Covering conjecture. Instead, an argument dual to that for the Covering conjecture must be used, relying on the existence of maximal cowaves, where a cowave is a pair $(C, (I_k \,|\, k \in K))$ with the I_k forming a covering of $(M_k.C \,|\, k \in K)$. The existence of maximal cowaves can be demonstrated by an argument dual to that for Lemma 4.3.

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Nathan Bowler, Johannes Carmesin

Universität Hamburg

Department of Mathematics

Bundesstr. 55, 20146 Hamburg Germany

{n.bowler1729,johannes.carmesin}@gmail.com