J. Soc. Indust. Appl. Math.
Vol. 12, No. 4, December, 1964
Printed in U.S.A.

A SOLUTION OF THE SHANNON SWITCHING GAME*

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A winning play-by-play strategy is given for the graphical two-person "switching game" formulated by C. E. Shannon [7, 5, 4]. As this game models a breakdown-repair criterion of graphical connectivity, the solution also yields some novel results in graph theory. The arguments are in terms of a corresponding game played on matroids [15].

1. Description of the game. Two players, designated "cut" and "short", are presented with a finite unoriented-branch linear graph[1] in which two of the vertices are distinguished. The cut player, in his turn, deletes one of the branches of the graph, his object being to sever all paths between the two distinguished vertices. The short player alternates plays with the cut player. In his turn he makes an unplayed branch invulnerable to deletion, his object being to connect the two distinguished vertices by an invulnerable path. Both players have complete information. The game proceeds until one of the players reaches his goal.

In order that the game terminate, no "repairing" of deleted branches is permitted. Let n denote the number of branches. Since each branch is subject to only one play, the total number of plays cannot exceed n. Within these n plays either the distinguished vertices become separated or they become connected by an invulnerable path. Hence one of the players by reaching his goal will win.

In addition to the skill of the player, the outcome of the game is dependent upon the identity, cut or short, of the player who plays first. If a player can win playing second he can win, a fortiori, playing first. This can be done by making *any* initial play and then playing a strategy which would win if the first play had not been made. Whenever that strategy requires a play which has already been made, a substitute play can be made on any unplayed branch.

Thus a given graph with two distinguished vertices has exactly one of the following properties.

(i) The cut player plays first and the short player can win against all possible strategies of the cut player.

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* Received by the editors October 2, 1962, and in revised form November 7, 1963.

† Walter Reed Army Institute of Research, Washington, D. C. The work was sponsored in part by Case Institute of Technology, Cleveland, Ohio, and by the Mathematics Research Center, U. S. Army, Madison, Wisconsin, under Contract No. DA-11-022-ORD-2059.

[1] In [14] this is called an abstract graph. Branches are sometimes called edges, links or arcs; vertices are called nodes or terminals; and circuits are called cycles or loops.

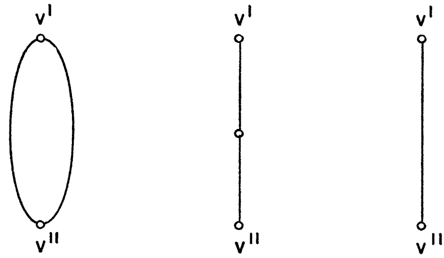
(ii) The short player plays first and the cut player can win against all possible strategies of the short player.

(iii) The player who plays first, regardless of identity, can win against all possible strategies of the other player.

Graphs having properties (i), (ii), or (iii) are said to yield *short*, *cut*, or *neutral games* respectively. Simple examples of such graphs, the distinguished vertices being denoted by v' and v'', are given in Fig. 1.

Another example of a neutral game is given by the graph of Fig. 2. The short player, playing first, can win by playing branch c and then countering any play by the cut player on a branch a_i or b_i by a play on the remaining branch of that letter. The cut player, playing first, can win by playing branch c and then countering any play by the short player on a branch of subscript 1 or 2 by a play on the remaining branch of that subscript.

Suppose a given graph with two distinguished vertices possesses a subgraph, other than the complete graph or a single branch, which is connected to the remaining graph at only two vertices. If the distinguished vertices are not internal vertices of the subgraph then the subgraph can be replaced in the given graph by any graph with the same property, cut, short, or neutral, without changing the property of the game on the original graph. If the game on the subgraph (alone) with respect to its connecting



Short game *Cut game* *Neutral game*

FIG. 1

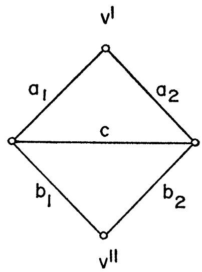


FIG. 2. *Neutral game*

vertices is neutral, it is most conveniently replaced by a single branch; if a cut game, by two disconnected vertices (an open circuit); and if a short game, by a single vertex (a short circuit). The validity of this reduction procedure can be verified from the description of the game. In general it can only be used to evaluate graphs, such as series-parallel, with little interconnectivity.

Any n-branch graph, with two distinguished vertices, can be classified as cut, short, or neutral by considering all of the $2 \cdot n!$ possible sequences of plays. This is both trivial and computationally unsatisfactory. As a principal result (see (29)) it is shown that a graph yields a short game if and only if there exist two disjoint trees which span the same subgraph and connect the distinguished vertices. This in turn yields criteria for neutral and cut games (see, for example, (30) and (31)). In general, the property of being cut, short, or neutral depends not only on the graph but also on the selected vertex pair. Nevertheless, for a given graph, the short player has a strategy (see (16)) which is independent of the choice of distinguished vertices and which will win with respect to any vertex pair for which the graph yields a short game. No such "global" strategy exists for the cut player.

2. Matroids. A generalization both of the circuit concept of graph theory and linear independence of matrix theory, called a matroid, was formulated by Whitney [15]. The connection between matroids and graphs has been further explored by Tutte [11, 12]. Matroids also form a natural setting for theorems relating to the Shannon game. While some knowledge of graph theory is assumed, the treatment of matroids will be essentially self-contained.

Each of the concepts rank, circuit, tree, and span can be used to define a matroid. The following definition, based on circuits, follows Robertson and Weston [10]. It will be shown to yield the ostensibly stronger postulates used by Whitney and Tutte. (The customary symbols $\{\ \}$, $\cup$, $\cap$, $-$, $\in$, and $\subset$ will be used to denote respectively: set formation, the set operations union, intersection, and difference, and the membership and inclusion relations. It is also assumed that the $\cup$ and $\cap$ operations are to be performed before the $-$ operation. Thus in the following definition, $C^{\#} \cup C^* - \{b\}$ denotes $(C^{\#} \cup C^*) - \{b\}$.)

A *finite matroid* M, $\mathfrak{M}$ is a collection $\mathfrak{M}$ of nonnull subsets C of a finite set M satisfying:

(1) *If* $C^{\#}, C^* \in \mathfrak{M}$, $C^{\#} \neq C^*$ *and* $b \in C^{\#} \cap C^*$ *then there exists* $C \in \mathfrak{M}$ such that $C \subset C^{\#} \cup C^* - \{b\}$.

(The nonminimal members of $\mathfrak{M}$ will later be deleted.)

As an example, let M be the set of rows of a matrix of real numbers. Consider a nonnull set of rows which are linearly dependent allowing only nonzero coefficients. Let $\mathfrak{M}$ be the collection of all such sets. Suppose $C^{\#}$ and C^* are two distinct members of $\mathfrak{M}$ each containing the row b. Each of the corresponding linear constraints can be solved for b and b can be eliminated between them. Collecting rows yields a linear dependence with nonzero coefficients. This latter constaint corresponds to a set C where $C \subset C^{\#} \cup C^* - \{b\}$ and C is a member of $\mathfrak{M}$.

In a finite matroid the members of M will be called *branches* and the members of $\mathfrak{M}$ will be called *circuits*, as the branches and circuits of any unoriented graph form a matroid. (This and the stronger property given by (39) can be shown by consideration of the group of 1-cycles (1-chains with zero boundary) of the graph. Chain groups are discussed in [11, part II].) For example, the six branches and seven circuits of the graph given by Fig. 3a correspond to the following matroid.

$$M = \{a_1, a_2, b_1, b_2, c, e\}$$

$$\mathfrak{M} = \{\{a_1, b_1, c\}, \{a_2, b_2, c\}, \{a_1, b_2, e\}, \{a_2, b_1, e\},$$
$$\{a_1, b_1, a_2, b_2\}, \{a_1, a_2, c, e\}, \{b_1, b_2, c, e\}\}.$$

Not every matroid, however, is the matroid of circuits of a graph, and two graphs may correspond to the same matroid. The vertices of a graph have no direct matroidal counterpart (see the comments following (17)).

The minimal members of $\mathfrak{M}$ themselves satisfy (1) and hence form a matroid. All of the matroidal concepts can be defined in terms of this latter matroid. Hence, following Whitney, it will also be assumed that *no member of $\mathfrak{M}$ properly contains any other member of $\mathfrak{M}$*. Likewise the graphical term *circuit* will be restricted to simple closed paths. With the assumption that all members of $\mathfrak{M}$ are minimal it is possible to prove (2), the remaining postulate of Whitney.

LEMMA. *Let M, $\mathfrak{M}$ be a finite matroid.*

(2) *If $C^{\#}, C^* \in \mathfrak{M}$, $a \in C^{\#} - C^*$ and $b \in C^{\#} \cap C^*$ then there exists $C \in \mathfrak{M}$ such that $a \in C \subset C^{\#} \cup C^* - \{b\}$.*

Proof. Assume that there exist a, b, $C^{\#}$, C^* for which (2) fails. For this a choose b, $C^{\#}$, C^* so that $C^{\#} \cup C^*$ is minimal. (A member of a given family of sets is minimal (or maximal) in that family if it does not contain (or is not contained in) any other member of that family. Since $\mathfrak{M}$ is finite, only finite families are being considered and hence a minimal (or maximal) member always exists.)
By (1) there exists $C^{\#\#} \subset C^{\#} \cup C^* - \{b\}$. Since $C^{\#\#} \not\subset C^{\#}$ there exists

$c \in (C^* - C^\#) \cap C^{\#\#}$. Again by (1) there exists $C^{**} \subset C^{\#\#} \cup C^* - \{c\}$. Since $C^{**} \not\subset C^*$ there exists $d \in (C^\# - C^*) \cap C^{**}$. If $a \in C^{**}$ then $a \in C^{\#\#}$ and, contrary to assumption, (2) holds for $a, b, C^\#, C^*$. Hence $a \in C^\# - C^{**}$, $c \in C^\# \cup C^* - C^\# \cup C^{**}$ and $C^\# \cup C^{**}$ is properly contained in $C^\# \cup C^*$. By the minimality assumption there exists C such that $a \in C \subset C^\# \cup C^{**} - \{d\}$. Since $a \in C \subset C^\# \cup C^*$ it follows that $b \in C$. Thus (2) fails for a, b, C, C^*; $d \in C^\# \cup C^* - C \cup C^*$ and $C \cup C^*$ is properly contained in $C^\# \cup C^*$. This contradicts the minimality of $C^\# \cup C^*$.

Clearly the property given by (2) implies that given by (1).

For fixed M, the matroid M, $\mathfrak{M}$ will be denoted simply by $\mathfrak{M}$. All matroids and graphs are assumed to be finite. Branches will be denoted by a, b, c, d, e, circuits by C's and arbitrary subsets of M by A's and B's. O will denote the null set. All of the matroidal results hold, a fortiori, for graphs.

$\mathcal{S}(A)$, the *span* of A, is the set of all members of M which are members of A or for which there exists $C \in \mathfrak{M}$ such that $a \in C \subset A \cup \{a\}$. Thus $\mathcal{S}(A)$ contains, in addition to A, any single branch which when adjoined to A, completes a circuit. Clearly,

(3) $$A \subset \mathcal{S}(A),$$

(4) $a \in \mathcal{S}(A) - A$ *if and only if there exists* C *such that* $C - A = \{a\}$,

and

(5) *if* $A \subset B$ *then* $\mathcal{S}(A) \subset \mathcal{S}(B)$.

Furthermore,[2]

(6) $$\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A).$$

Proof. Suppose $a \in \mathcal{S}(\mathcal{S}(A)) - \mathcal{S}(A)$. By (4) there exists $C^\#$ such that $C^\# - \mathcal{S}(A) = \{a\}$. Since M is finite $C^\#$ can be chosen so that $C^\# - A$ is minimal. Since $a \notin \mathcal{S}(A)$ there exists $b \in (C^\# - A) - \{a\}$. But $b \in \mathcal{S}(A) - A$ and hence there exists C^* such that $C^* - A = \{b\}$. Thus by (2) there exists C such that $a \in C \subset C^\# \cup C^* - \{b\}$. Consequently, $\{a\} \subset C - \mathcal{S}(A)$

[2] Define A to be *closed* if $\mathcal{S}(A) = A$, that is, if whenever A contains all but one of the branches of a circuit it contains the remaining branch. Then the set M is closed and the intersection of closed sets is closed. The null set is closed if there are no single branch circuits. Define A to be *open* if $M - A$ is closed. It follows that A is open if and only if it does not intersect any circuit in a single branch. The minimal nonnull open sets form the matroid dual to $\mathfrak{M}$ (see (17)). Thus the interior of A is the union of all circuits of the dual matroid which are contained in A. For infinite (M) matroids it might be desirable either to require an infinite form of circuit transitivity (2) or redefine $\mathcal{S}(A)$ to be the smallest closed set containing A and thus preserve property (6). In this connection see the postulates given in [6] and [9].

$\subset C^{\#} \cup C^* - \mathcal{S}(A) \subset \{a\}$ and $C - A \subset (C^{\#} \cup C^* - A) - \{b\} = (C^{\#} - A) - \{b\}$, which contradicts the minimality of $C^{\#} - A$. Hence, using (3), $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$.

It follows from (5) and (6) that

(7) *if* $A \subset \mathcal{S}(B)$ *then* $\mathcal{S}(A) \subset \mathcal{S}(B)$.

The spans $\mathcal{S}(A)$ of all subsets A of M form a lattice which characterizes the matroid [1]. It can be verified that $a \in \mathcal{S}(A) - \mathcal{S}(A - \{b\})$ implies that $b \in \mathcal{S}((A - \{b\}) \cup \{a\})$. This property together with (3), (5) and (6) can be shown to characterize the span operator. If $\mathfrak{M}$ is the matroid of circuits of a graph then $\mathcal{S}(A)$ is the set of all branches of the graph which are short-circuited as a result of short-circuiting the branches in A.

The following six lemmas concern formal properties of the span operator. They are proved preliminary to the principal theorem.

LEMMA. *Assume that* a, A, B *are such that* $a \in A - B$ *and* $\mathcal{S}(B) \subset \mathcal{S}(A)$.

(8) *Either* $\mathcal{S}(B) \subset \mathcal{S}(A - \{a\})$ *or there exists* $b \in B - A$ *such that* $\mathcal{S}(B) \subset \mathcal{S}(A \cup \{b\} - \{a\})$.

Proof. If $B - A \subset \mathcal{S}(A - \{a\})$ then $B \subset \mathcal{S}(A - \{a\})$ and thus by (7), $\mathcal{S}(B) \subset \mathcal{S}(A - \{a\})$. Otherwise there exists $b \in (B - A) - \mathcal{S}(A - \{a\})$. But $b \in \mathcal{S}(A)$. Hence there exists C such that $C - A = \{b\}$. Since $b \notin \mathcal{S}(A - \{a\})$ it follows that $a \in C$ and hence $a \in \mathcal{S}(A \cup \{b\} - \{a\})$. Thus $A \subset \mathcal{S}(A \cup \{b\} - \{a\})$ so that $\mathcal{S}(B) \subset \mathcal{S}(A) \subset \mathcal{S}(A \cup \{b\} - \{a\})$.

LEMMA. *Assume that* a, e, A, B *are such that* $a \in A - B$, $e \notin \mathcal{S}(A \cap B)$, *and* $e \in \mathcal{S}(A) = \mathcal{S}(B)$. *Then*

(9) *there exists* $b \in B - \mathcal{S}(A \cap B)$ *such that* $\mathcal{S}(A \cup \{b\} - \{a\}) = \mathcal{S}(B)$.

Proof. Since $e \in \mathcal{S}(B) - \mathcal{S}(A \cap B)$, $B - \mathcal{S}(A \cap B) \neq O$. If $\mathcal{S}(A - \{a\}) = \mathcal{S}(B)$ then $\mathcal{S}(A \cup \{b\} - \{a\}) = \mathcal{S}(B)$ for all $b \in B - \mathcal{S}(A \cap B)$. Otherwise $\mathcal{S}(A - \{a\}) \neq \mathcal{S}(B)$ and by (8) there exists $b \in B - A$ such that $\mathcal{S}(A \cup \{b\} - \{a\}) = \mathcal{S}(B)$. Furthermore, $b \in B - \mathcal{S}(A - \{a\}) \subset B - \mathcal{S}(A \cap B)$.

In the winning strategy for the short player, given in the proof of (14), the short player will play branch b of (9) in response to a play by the cut player on branch a.

LEMMA. *Assume that* A, B, A^*, B^* *are such that* $\mathcal{S}(A) = \mathcal{S}(B)$ *and* $\mathcal{S}(A^*) = \mathcal{S}(B^*)$. *Then*

(10) *there exist* $A^{\#}, B^{\#}$ *such that*

$$A^{\#} \subset A \cup A^*, \qquad B^{\#} \subset B \cup B^*,$$
$$A^{\#} \cap B^{\#} \subset (A \cap B) \cup (A^* \cap B^*), \quad \textit{and}$$
$$\mathcal{S}(A^{\#}) = \mathcal{S}(B^{\#}) = \mathcal{S}(A \cup A^*).$$

Proof. Let $A^{\#} = A \cup (A^* - B)$ and $B^{\#} = B \cup (B^* - A)$. Then $\mathcal{S}(A^{\#})$ contains $\mathcal{S}(A)$, $\mathcal{S}(B)$, $\mathcal{S}(A^*)$ and $\mathcal{S}(B^*)$; and $\mathcal{S}(B^{\#})$ contains $\mathcal{S}(B)$, $\mathcal{S}(A)$, $\mathcal{S}(B^*)$ and $\mathcal{S}(A^*)$.

The construction given by (10) is used in the proof of (16) to show the existence of a "global" strategy for the short player.

LEMMA. *Assume that a, e, A, B are such that $a \in A \cap B$, $e \notin A \cup B$ and $e \in \mathcal{S}(A) = \mathcal{S}(B)$. Then*

(11) *there exist $A^{\#}$, $B^{\#}$ such that*

$$A^{\#} \cup B^{\#} \subset A \cup B, \quad A^{\#} \cap B^{\#} \subset A \cap B - \{a\}, \quad \textit{and}$$
$$\mathcal{S}(A^{\#} \cup \{e\}) = \mathcal{S}(B^{\#}).$$

Proof. If $\{e\}$ is a circuit let $A^{\#}$ and $B^{\#}$ be null. Otherwise, for each $b \in (B - A) \cup \{e\}$, $b \in \mathcal{S}(A) - A$ and hence there exists C_b such that $C_b - A = \{b\}$. Similarly, for each $b \in (A - B) \cup \{e\}$, $b \in \mathcal{S}(B) - B$ and hence there exists C_b^* such that $C_b^* - B = \{b\}$.

Define the sequence $B_0, A_1, B_1, A_2, B_2, \cdots, A_i, B_i, \cdots$ by the following:

$$B_0 = O, \quad A_1 = C_e - \{e\},$$
$$A_{i+1} = A_i \cup ((B_i - \mathcal{S}(A_i)) \cap A) \cup (\bigcup_{b \in (B_i - \mathcal{S}(A_i)) - A} (C_b - \{b\})),$$
$$B_{i+1} = B_i \cup ((A_{i+1} - \mathcal{S}(B_i)) \cap B) \cap (\bigcup_{b \in (A_{i+1} - \mathcal{S}(B_i)) - B} (C_b^* - \{b\})).$$

Since M is finite and $A_i \subset A_{i+1}$, $B_i \subset B_{i+1}$, $A_\infty = \lim_{i \to \infty} A_i$ and $B_\infty = \lim_{i \to \infty} B_i$ exist and $e \in \mathcal{S}(A_\infty) = \mathcal{S}(B_\infty)$. Clearly $A_\infty \subset A$ and $B_\infty \subset B$.

If $a \notin A_\infty \cap B_\infty$, set $A^{\#} = A_\infty$ and $B^{\#} = B_\infty$.

If $a \in A_\infty \cap B_\infty$ the integer $|a|$ is defined by

$$|a| = \begin{cases} 1 & \text{if} \quad a \in A_1, \\ 2k & \text{if} \quad a \in B_k - A_k, \\ 2k+1 & \text{if} \quad a \in A_{k+1} - B_k. \end{cases}$$

If $|a| = 1$ set $A^{\#} = A - \{a\}$ and $B^{\#} = B$.

If $|a| = 2k > 1$ then there exists $b \in (A_k - \mathcal{S}(B_{k-1})) - B$ such that $a \in C_b^* - \{b\}$. Hence $e \in \mathcal{S}(A) = \mathcal{S}(B \cup \{b\} - \{a\})$. Thus b, A, $B \cup \{b\} - \{a\}$ satisfy the original assumptions given for a, A, B, while $|b|$, using the same C's (except that C_a^* is the previous C_b^*), is $2k - 1$.

If $|a| = 2k + 1 > 1$ then there exists $b \in (B_k - \mathcal{S}(A_k)) - A$ such that $a \in C_b - \{b\}$. Hence $e \in \mathcal{S}(A \cup \{b\} - \{a\}) = \mathcal{S}(B)$. Thus b, $A \cup \{b\} - \{a\}$, B satisfy the original assumptions given for a, A, B, while $|b|$, using the same C's (except that C_a is the previous C_b), is $2k$. Alternating applications of the previous two $|a|$ reduction procedures ($|a| = 2k$ and $|a| = 2k + 1$) yields an a, A, B such that $|a| = 1$. The conclusion follows.

The reshuffling of sets A and B as given by (11) allows, in the proof of (14), the substitution of the branch e for any deleted branch a.

LEMMA. *Assume that* a, e, A, B *are such that* $a \in A - B$, $e \notin A \cup B$, *and* $e \in \mathcal{S}(A) = \mathcal{S}(B)$. *Then*

(12) *there exist* $A^{\#}$, $B^{\#}$ *such that*

$$A^{\#} \cup B^{\#} \subset A \cup B - \{a\}, \quad A^{\#} \cap B^{\#} \subset A \cap B, \quad \text{and} \quad \mathcal{S}(A^{\#} \cup \{e\}) = \mathcal{S}(B^{\#}).$$

Proof. If $\mathcal{S}(A - \{a\}) = \mathcal{S}(B)$ set $A^{\#} = A - \{a\}$ and $B^{\#} = B$. Otherwise, by (8), there exists $b \in B - A$ such that $\mathcal{S}(A \cup \{b\} - \{a\}) = \mathcal{S}(B)$. Application of (11) to b, e, $A \cup \{b\} - \{a\}$, B yields the desired result.

The result (12) is used in the proof of (13).

LEMMA. *Assume for a fixed branch* e, *that* $M - \{e\} \neq O$ *and that for each* $a \in M - \{e\}$ *there exist* A_a, B_a *such that* a, $e \notin A_a \cup B_a$ *and* $\mathcal{S}(A_a \cup \{e\}) = \mathcal{S}(B_a)$. *Then*

(13) *there exist* A, B *such that* $e \notin A \cup B$,

$$A \cap B \subset \bigcup_{a \in M - \{e\}} (A_a \cap B_a), \text{ and } e \in \mathcal{S}(A) = \mathcal{S}(B).$$

Proof. If for some $a \in M - \{e\}$, $e \in \mathcal{S}(A_a)$ holds, let $A = A_a$ and $B = B_a$. Otherwise let a be any (fixed) member of $M - \{e\}$. Since $e \in \mathcal{S}(B_a) - B_a$ there exists C such that $C - B_a = \{e\}$. It follows that $\mathcal{S}(A_a \cup \{e\}) - \mathcal{S}(A_a) \supset (C - \{e\}) - \mathcal{S}(A_a) \neq O$. Let b be any member of $(C - \{e\}) - \mathcal{S}(A_a)$. There exist A_b, B_b such that b, $e \notin A_b \cup B_b$ and $\mathcal{S}(A_b \cup \{e\}) = \mathcal{S}(B_b)$. Thus the set $\mathcal{S}(\{e\} \cup A_b \cup ((B_a - \{b\}) - B_b))$ contains $\mathcal{S}(\{e\} \cup A_b)$, $\mathcal{S}(B_b)$, $\mathcal{S}(B_a - \{b\})$, $\mathcal{S}(B_a \cup \{e\} - \{b\})$, $\mathcal{S}(C)$, $\{b\}$, $\mathcal{S}(B_a)$, and $\mathcal{S}(A_a \cup \{e\})$. Similarly $\mathcal{S}(B_b \cup (A_a - A_b))$ contains $\mathcal{S}(B_b)$, $\mathcal{S}(A_b \cup \{e\})$, $\mathcal{S}(A_a)$, $\mathcal{S}(A_a \cup \{e\})$, and $\mathcal{S}(B_a)$. Thus $b \in \mathcal{S}(\{e\} \cup A_b \cup ((B_a - \{b\}) - B_b)) = \mathcal{S}(B_b \cup (A_a - A_b))$ and $b \notin \{e\} \cup A_b \cup ((B_a - \{b\}) - B_b)$, $b \notin B_b \cup (A_a - A_b)$.

By (12), with a, e replaced by e, b, there exist $A^{\#}$, $B^{\#}$ such that $A^{\#} \cup B^{\#} \subset A_a \cup B_a \cup A_b \cup B_b - \{b\}$, and $\mathcal{S}(A^{\#} \cup \{b\}) = \mathcal{S}(B^{\#})$. Let $A = A^{\#} \cup (B_a - B^{\#})$ and $B = B^{\#} \cup (A_a - A^{\#})$. Then $\mathcal{S}(A)$ contains $\mathcal{S}(A^{\#})$, $\{b\}$, $\mathcal{S}(A^{\#} \cup \{b\})$, $\mathcal{S}(B^{\#})$, $\mathcal{S}(B_a)$ and $\mathcal{S}(A_a \cup \{e\})$. Likewise $\mathcal{S}(B)$ contains $\mathcal{S}(B^{\#})$, $\mathcal{S}(A^{\#} \cup \{b\})$, $\mathcal{S}(A_a)$ and $\mathcal{S}(A_a \cup \{b\})$. Since $b \in \mathcal{S}(A_a \cup \{e\}) - \mathcal{S}(A_a)$, $\mathcal{S}(A_a \cup \{b\}) = \mathcal{S}(A_a \cup \{e\})$ and hence $\mathcal{S}(B)$ contains $\mathcal{S}(B_a)$. Thus $e \notin A \cup B$, $A \cap B \subset (A_a \cap B_a) \cup (A_b \cap B_b)$ and $e \in \mathcal{S}(A) = \mathcal{S}(B)$.

The result (13) is used to complete the induction in the second part of the proof of (14).

Both the matroid of circuits and the dual of a graph are primarily branch concepts. In order to carry the concept of distinguished vertices over to matroids it suffices to use a distinguished branch. This is a branch which, in the graph, connects the distinguished vertices. If the graph does not contain such a branch then it must be added. The additional branch is neither subject to play nor does it contribute to the connectivity in the Shannon game. Thus, in the corresponding circuit matroid, the distinguished branch is playable if and only if it corresponds to a (playable) branch of the original graph. If a graph, including the distinguished branch, is planar, the vertices of the distinguished branch are the distinguished vertices for the game on the dual graph (discussed in the next section). The basic result (see (21)) on duals applies only when all branches other than the distinguished branch are subject to play.

A version of the Shannon game can be played on any finite matroid: Assume a given matroid $\mathfrak{M}$ with a distinguished branch e. This branch, which need not be subject to play, takes the place of the two distinguished vertices. As in the graphical game, the cut and short players alternate plays, playing, one at a time, only previously unplayed branches. *The short player's goal is to play a set of branches which together span e.* Thus if e is a playable branch which has not yet been played by the cut player, then the short player, in his turn, can win by playing e. Otherwise the short player can win only by playing the branches $C - \{e\}$ of some circuit C containing e. The cut player's goal is to prevent the short player from winning. This goal will be made explicit after the introduction of dual matroids (see (22)).

As in the case of graphs, one of the players must win and there is no advantage in playing second. Thus a matroid with a given distinguished branch has exactly one of the properties (i), (ii), or (iii) and the corresponding game is again called short, cut, or neutral. If $\mathfrak{M}$ is the matroid of circuits of a graph, then a set of branches spans e if and only if, in the graph, it connects the vertices of e. Thus the game on $\mathfrak{M}$ with respect to the branch e is in agreement with that played on the graph with respect to the vertices of e.

The matroidal game is described in terms of those (and only those) circuits C which contain the branch e. In (46) it is shown that these circuits uniquely determine all of the circuits with which they intersect. This collection of circuits forms the smallest matroid containing the circuits C (which contain e) and this matroid can be used for the determination of the class; cut, short or neutral, of the Shannon game. Those matroids, with a distinguished branch, which yield short games are characterized by the following theorem.

THEOREM. *Consider the Shannon game played on a matroid* $\mathfrak{M}$ *with respect to a branch* e, e *not being subject to play.*

(14) *The short player, playing second, can win against all possible strategies of the cut player (i.e., a short game) if and only if there exist* A *and* B *such that* $e \notin A \cup B$, $A \cap B = O$, *and* $e \in \mathcal{S}(A) = \mathcal{S}(B)$.

Proof. Sufficiency. Let A, B satisfy $e \notin A \cup B$, $A \cap B = O$ and $e \in \mathcal{S}(A) = \mathcal{S}(B)$. At any turn of the cut player let $A^{\#}$ consist of the unplayed branches of A plus those branches of $A \cup B$ already played by the short player. Let $B^{\#}$ consist of the unplayed branches of B plus those branches of $A \cup B$ already played by the short player. Thus at the beginning of the game $A^{\#} = A$ and $B^{\#} = B$.

Assume that $e \in \mathcal{S}(A^{\#}) = \mathcal{S}(B^{\#}) = \mathcal{S}(A)$. If $e \in \mathcal{S}(A^{\#} \cap B^{\#})$ then e has been spanned by an "invulnerable path" and the short player has won. Otherwise the cut player plays some (unplayed) branch $a \in M - A^{\#} \cap B^{\#}$.

If $a \in M - B^{\#}$ then, by (9) if $a \in A^{\#} - B^{\#}$ and trivially if $a \in M - A^{\#} \cup B^{\#}$, there exists $b \in B^{\#} - \mathcal{S}(A^{\#} \cap B^{\#})$ such that $e \in \mathcal{S}(A^{\#} \cup \{b\} - \{a\}) = \mathcal{S}(B^{\#})$. The short player now plays b. If $e \in \mathcal{S}((A^{\#} \cup \{b\} - \{a\}) \cap B^{\#})$ the short player has won. Otherwise the procedure is repeated using $A^{\#} \cup \{b\} - \{a\}$, $B^{\#}$ instead of the previous $A^{\#}$, $B^{\#}$.

If $a \in B^{\#} - A^{\#}$ then, by (9), there exists $b \in A^{\#} - \mathcal{S}(A^{\#} \cap B^{\#})$ such that $e \in \mathcal{S}(A^{\#}) = \mathcal{S}(B^{\#} \cup \{b\} - \{a\})$. The short player now plays b. If $e \in \mathcal{S}(A^{\#} \cap (B^{\#} \cup \{b\} - \{a\}))$ the short player has won. Otherwise the procedure is repeated using $A^{\#}$, $B^{\#} \cup \{b\} - \{a\}$ instead of the previous $A^{\#}$, $B^{\#}$. Thus each play by the short player increases the span of $A^{\#} \cap B^{\#}$. Since $A^{\#}$ and $B^{\#}$ are finite, after some finite number of plays $A^{\#} \cap B^{\#}$ will span e and the short player will win.

The above strategy depends only on A and B. e is not used to determine strategy but only to decide whether the short player has won. Furthermore the same strategy will eventually win the game with respect to any $a \in \mathcal{S}(A)$. This observation will be used in the proof of (16).

Necessity. Assume that the short player can win against any strategy of the cut player (i.e., a short game).

If card $M \leqq 3$ (card M being the number of elements (branches) in M), it can be shown by exhaustion that either $\{e\}$ is a circuit or M contains three branches a, b, e such that $\{a, b\}$, $\{a, e\}$, and $\{b, e\}$ are circuits. In the former case $A = B = O$ and in the latter case, $A = \{a\}$ and $B = \{b\}$ satisfy the conclusion of the theorem.

Assume that the theorem holds for card $M \leqq n$ when $n \geqq 3$ and consider a matroid $\mathfrak{M}$ such that card $M = n + 2$. For an initial cut play on some branch a, there exists a short play on some branch b ($\{b\}$ not a circuit) such that the resulting game on the remaining branches $M - \{a, b\}$ is still a short game. The matroid corresponding to this reduced n-branch game has circuits $C - \{b\}$ where $b \in C \subset M - \{a\}$ and circuits C where $C \subset M - \{a, b\}$ and C does not contain any of the previous $(C - \{b\})$'s.

By the inductive assumption there exist A_a, B_a such that $a, b, e \notin A_a \cup B_a$, $A_a \cap B_a = O$, and, in the reduced matroid, $e \in \mathcal{S}(A_a) = \mathcal{S}(B_a)$. Hence in the original matroid, $A_a^* = A_a \cup \{b\}$ and $B_a^* = B_a \cup \{b\}$ satisfy $a, e \notin A_a^* \cup B_a^*$, $A_a^* \cap B_a^* = \{b\}$, and $e \in \mathcal{S}(A_a^*) = \mathcal{S}(B_a^*)$. By (11), for each $a \in M - \{e\}$ there exist $A_a^\#, B_a^\#$ such that $A_a^\# \cup B_a^\# \subset A_a^* \cup B_a^*$ (and hence $a, e \notin A_a^\# \cup B_a^\#$), $A_a^\# \cap B_a^\# = O$, and $\mathcal{S}(A_a^\# \cup \{e\}) = \mathcal{S}(B_a^\#)$. Then by (13) there exist A, B such that $e \notin A \cup B$, $A \cap B = O$, and $e \in \mathcal{S}(A) = \mathcal{S}(B)$. This completes the induction.

In general neither A and B nor the resulting strategy is unique.

COROLLARY. *Consider the Shannon game played on a matroid $\mathfrak{M}$ with respect to a branch e, e being subject to play.*

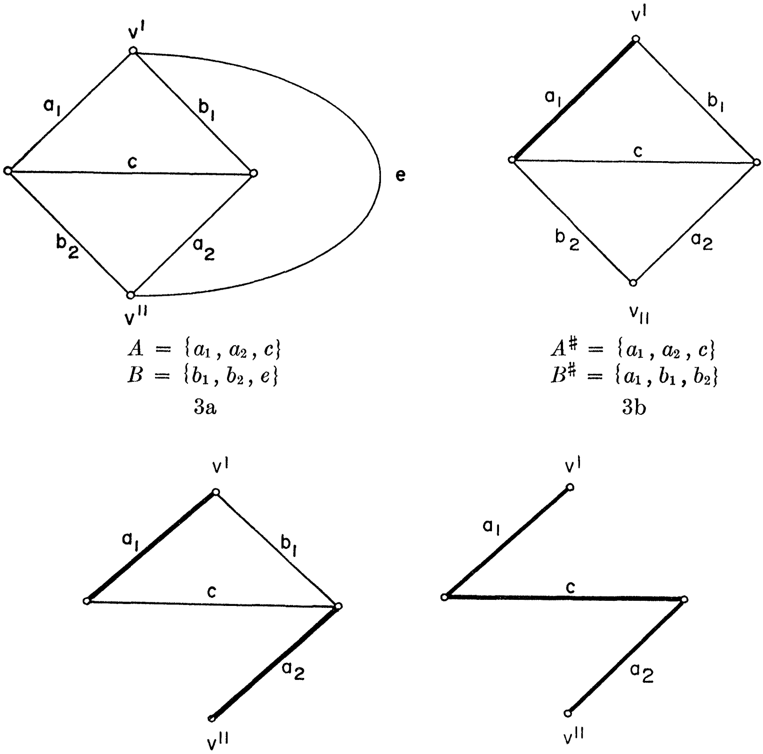
(15) *The short player can win against all possible strategies of the cut player if and only if there exist A, B such that $A \cap B = O$ and $e \in \mathcal{S}(A) = \mathcal{S}(B)$.*

Proof. Consider the game on the augmented matroid having branches $M \cup \{e'\}$, circuits C for all $C \in \mathfrak{M}$ and circuits $C \cup \{e'\} - \{e\}$ for all C such that $e \in C \in \mathfrak{M}$. Application of (14) with respect to the unplayable branch e' yields the desired result.

As an example of (15) let $\mathfrak{M}$ be the matroid of circuits of the graph given in Fig. 3a, where e is the branch connecting the distinguished vertices v' and v''. Since $A = \{a_1, a_2, e\}$ and $B = \{b_1, b_2, c\}$ are disjoint and have the same span, the graph yields a short game. (In this example A and B span M and hence the graph yields a short game with respect to every pair of vertices.) In the following sequence of plays the short player uses the strategy given in the first part of the proof of (14): The cut player plays first, deleting branch e of the graph of Fig. 3a to obtain that of Fig. 3b. The short player, by playing branch a_1, restores the span of the set A. The cut player then plays branch b_2 resulting in the graph of Fig. 3c and the short player is forced to play the branch a_2. The cut player plays branch b_1, resulting in the graph of Fig. 3d, and the short player plays branch c and wins.

The strategy given in the first part of the proof of (14) need not win in a minimum number of plays. For example, in the short game played on the graph of Fig. 4 (distinguished vertices v' and v'') the strategy of (14) based on $A = \{a_2, a_3, a_4, b_1, b_5\}$, $B = \{a_1, a_5, b_2, b_3, b_4\}$ might proceed as follows.

The cut player deletes a_1 forcing the short player to play either a_3 or a_4.



$A = \{a_1, a_2, c\}$
$B = \{b_1, b_2, e\}$
3a

$A^\# = \{a_1, a_2, c\}$
$B^\# = \{a_1, b_1, b_2\}$
3b

$A^\# = \{a_1, a_2, c\}$
$B^\# = \{a_1, a_2, b_1\}$
3c

$A^\# = \{a_1, a_2, c\}$
$B^\# = \{a_1, a_2, c\}$
3d

FIG. 3

The cut player then deletes a_5 forcing the short player to play either a_2 or the previously unplayed a_4 or a_3. The cut player can easily continue the game so that the short player wins only after making a total of five plays. The short player however can always win within three plays by a strategy of playing only b_i's (or a_i's) after an initial cut play on an a_j (or b_j).

The inefficiency of the strategy of (14) is partially offset by a property given in (16). No general strategy is known (other than that obtained by exhaustion) which will always win in a minimum number of plays. Further discussion of the strategy given in the proof of (14) is contained in the section on examples.

In the following corollary, the matroid $\mathfrak{M}$ may contain several unplayable branches. These branches are used solely as alternate distinguished branches with respect to which the game can be played.

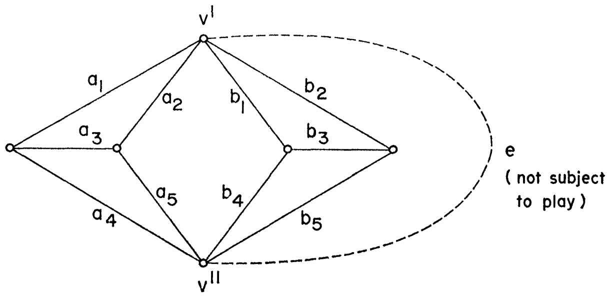


FIG. 4

COROLLARY. *Consider the Shannon game played on a matroid $\mathfrak{M}$ in which the branches belonging to $M^\# \subset M$ are subject to play.*

(16) *A strategy exists for the short player, playing second, which will win with respect to any branch of M for which the matroid yields a short game. This strategy is independent of the chosen distinguished branch and depends only on the matroid of circuits of playable branches (i.e., the matroid of circuits C, where $C \subset M^\#$).*

Proof. Construct two sets A, B such that $A \cup B \subset M^\#$, $A \cap B = 0$ and $\mathcal{S}(A) \cap M^\# = \mathcal{S}(B) \cap M^\#$, and such that $\mathcal{S}(A) \cap M^\#$ is maximal over all such sets A, B. Clearly A and B depend only on those circuits contained in $M^\#$. Furthermore, $\mathcal{S}(A) = \mathcal{S}(\mathcal{S}(A) \cap M^\#) = \mathcal{S}(\mathcal{S}(B) \cap M^\#) = \mathcal{S}(B)$. If for any $a \in M$ the game with respect to a is a short game, then by (14) and (15) there exist A^*, B^* such that $A^* \cup B^* \subset M^\#$, $A^* \cap B^* = 0$, and $a \in \mathcal{S}(A^*) \cap (M^\# \cup \{a\}) = \mathcal{S}(B^*) \cap (M^\# \cup \{a\})$ (or equivalently $a \in \mathcal{S}(A^*) = \mathcal{S}(B^*)$). The construction of (10) applied to A, B, A^*, B^* yields $A^\#$, $B^\#$ such that $A^\# \cup B^\# \subset M^\#$, $A^\# \cap B^\# = 0$ and $\mathcal{S}(A^\#) = \mathcal{S}(B^\#) = \mathcal{S}(A \cup A^*)$. Since $\mathcal{S}(A) \cap M^\#$ is maximal, $a \in \mathcal{S}(A^*) \subset \mathcal{S}(A^\#) = \mathcal{S}(\mathcal{S}(A^\#) \cap M^\#) \subset \mathcal{S}(\mathcal{S}(A) \cap M^\#) = \mathcal{S}(A)$. Using the strategy given for A and B in the first part of the proof of (14), the short player can win with respect to any $a \in \mathcal{S}(A)$.

In graphical terms (16) asserts the existence of a strategy for the short player, playing second, which will win where possible independent of the choice of distinguished vertices. Such a strategy does not exist for the cut player. Consider, for example, the game played on the graph of Fig. 5. Given either u', u'' or v', v'' to be the distinguished vertices, the cut player, playing second, can win against any strategy of the short player. No strategy for the cut player, even playing first, can force a win with respect to both vertex pairs in the same game.

In addition, the graph of Fig. 5 yields a neutral game with respect to

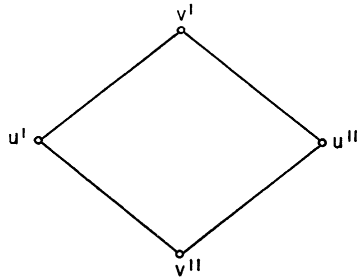


FIG. 5

both the vertex pairs u', v' and u'', v''. But no strategy for either player, playing first, can force a win with respect to both vertex pairs in the same game. Hence there is no general strategy (like that of (16)) independent of distinguished branches which will apply to neutral games.

A *pairing strategy* is a strategy, used by one player, in which the branches are paired so that a play on one branch of a pair by the opposing player is countered by a play on the remaining branch; the pairing being fixed throughout the game. A play on an unpaired branch, or where the remaining branch has already been played, is countered by an arbitrary play. The pairing type of strategy, while convenient in play, is fundamentally different from the strategy given in the first part of the proof of (14). Consider the complete graph consisting of four vertices and six branches. If two of the vertices are distinguished it is easy to find a pairing strategy which will win for the short player playing second. However no such strategy can also insure a win in the short game played with respect to the remaining pair of vertices. Thus for this example there exists no pairing strategy having the property given by (16).

Consider the Shannon game played with respect to the vertices v' and v'' of the graph of Fig. 6. It is a short game as the sets $A = \{a_1, \cdots, a_5\}$ and $B = \{b_1, \cdots, b_5\}$ satisfy the hypotheses of (14). Any winning pairing strategy for the short player must pair a_1 with b_1, and a_5 with b_5. Consideration of an initial cut play on a_2 requires that b_2 be paired with b_3 or that a_3 be paired with b_4. Similarly consideration of a cut play on a_4 requires that b_2 be paired with a_3 or that b_3 be paired with b_4. Since each of the four possible sets of pairings is inconsistent, no winning pairing strategy exists. Thus only a restricted class of short games can be won by pairing strategies. It is not known whether this class includes all games on *planar* graphs.

3. Duals. A graph has one or more duals if and only if it is planar [14]. Unlike graphs, each matroid has a unique matroid dual. By dualizing (14),

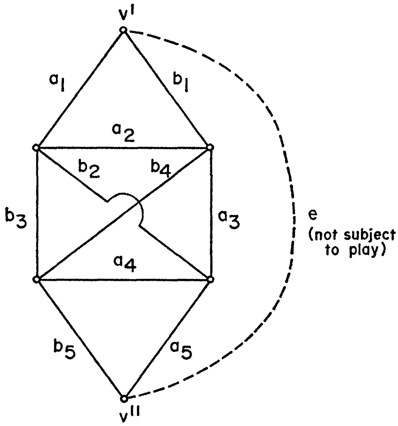


FIG. 6

the result for the short game, it is possible to characterize those matroids, with a distinguished branch (not subject to play), which yield a cut game.

Given $\mathfrak{M}$, $\mathfrak{M}'$ is defined to be the collection of minimal nonnull subsets A of M such that

(17) card $A \cap C \neq 1$ *for all* $C \in \mathfrak{M}$.

Anticipating the results that $\mathfrak{M}'$ is a matroid, (19), and that $\mathfrak{M}''$ is $\mathfrak{M}$, (20), we will call $\mathfrak{M}'$ the *dual* of $\mathfrak{M}$. (Regarding the span as a closure operator, it follows that A is open if and only if A satisfies card $A \cap C \neq 1$ for all $C \in \mathfrak{M}$. $\mathfrak{M}'$ consists of the minimal nonnull open subsets of M.)

If $\mathfrak{M}$ is the matroid of circuits of a graph then the collection of branches incident on a single vertex is a member of $\mathfrak{M}'$. In general, not all members of $\mathfrak{M}'$ correspond to vertices. $\mathfrak{M}'$, however, is generated by the vertex collections under the symmetric difference ($\oplus$) operation.

The following lemma yields a more tractable equivalent of (17).

LEMMA. *Assume that a is a member of A. Then*

(18) $A \in \mathfrak{M}'$ *if and only if A is a minimal set such that* $A \cap C - \{a\} \neq O$ *for all C such that* $a \in C \in \mathfrak{M}$.

Proof. Sufficiency. Assume there exists C such that $a \notin C$ and $A \cap C = \{b\}$. Since A is minimal there exists C^* such that $a \in C^*$, $A \cap C^* \neq \{a\}$, and $(A - \{b\}) \cap C^* = \{a\}$. Thus $A \cap C^* = \{a\} \cup \{b\}$. But by (2) there exists $C^\#$ such that $a \in C^\# \subset C \cup C^* - \{b\}$ and hence the contradiction $A \cap C^\# = \{a\}$. It follows that card $A \cap C \neq 1$ holds for all $C \in \mathfrak{M}$.

Assume that $b \in B \subset A - \{a\}$. Then as above there exists C^* such that

$A \cap C^* = \{a\} \cup \{b\}$ and hence card $B \cap C^* = 1$. Thus A is minimal in the property (17) and consequently $A \in \mathfrak{M}'$.

Necessity. If $a \in A \in \mathfrak{M}'$ then A has the property $A \cap C - \{a\} \neq O$ for all C containing a. The minimality of A with respect to this property follows from its minimality in the family $\mathfrak{M}'$.

(The assumption in (18), that a is a member of A, can be weakened as in the following result. For fixed $a, A \in \mathfrak{M}'$ and $A \cap (\bigcup_{a \in C \in \mathfrak{M}} C) \neq O$ hold if and only if A is a minimal nonnull set such that card $A \cap C \neq 1$ holds for all C such that $a \in C \in \mathfrak{M}$. The proof is similar to that of (18).)

The definition of $\mathfrak{M}'$ as expressed by (18) is used in (19) to show that $\mathfrak{M}'$ is a matroid.

LEMMA.

(19) *$\mathfrak{M}'$ is the collection of circuits of a matroid.*

Proof. Since the members of $\mathfrak{M}'$ are minimal (and nonnull), no member is properly contained in any other member. Assume $A, B \in \mathfrak{M}$ such that $a \in A - B$ and $b \in A \cap B$. For any $C \in \mathfrak{M}$ such that $a \in C$, $b \notin C$ implies $(A \cup B - \{b\}) \cap C - \{a\} \supset A \cap C - \{a\} \neq O$ and $b \in C$ implies $(A \cup B - \{b\}) \cap C - \{a\} \supset B \cap C - \{b\} \neq O$. Hence there exists $A^* \in \mathfrak{M}'$ such that $a \in A^* \subset A \cup B - \{b\}$. Thus $\mathfrak{M}'$ satisfies property (2).

Thus for any matroid M, $\mathfrak{M}$ there is a unique matroid M, $\mathfrak{M}'$ having branches M and circuits $\mathfrak{M}'$. This matroid will be denoted simply by $\mathfrak{M}'$. If $\mathfrak{M}$ is the circuit matroid of a planar graph, then $\mathfrak{M}'$ is the circuit matroid of any of its graphical duals. This is a consequence of the similarity of definitions of the graphical dual as given in [14] and the matroid dual as given in [15] (which is equivalent, see [12], to the definition used here). Hence duality in graphs agrees, where defined, with duality in matroids.

LEMMA.

(20) *$\mathfrak{M}''$, the dual of $\mathfrak{M}'$, is $\mathfrak{M}$.*

Proof. Assume that C is such that $a \in C \in \mathfrak{M}$. Then $C \cap C' - \{a\} \neq O$ for all $a \in C \in \mathfrak{M}'$. For any $b \in C - \{a\}$ and any $C^* \in \mathfrak{M}$, $C^* \not\subset C - \{a\}$ and hence $((M - C) \cup \{b\}) \cap C^* \neq O$. Consequently, by (19), $(M - C) \cup \{a\} \cup \{b\}$ contains some C_b' where $a \in C_b' \in \mathfrak{M}'$. Since $(C - \{b\}) \cap C_b' - \{a\} = O$ for all $b \in C - \{a\}$, C is minimal and thus $C \in \mathfrak{M}''$. Assume C'' such that $a \in C'' \in \mathfrak{M}''$. Since $C'' \cap C' - \{a\} \neq O$ for all C' where $a \in C' \in \mathfrak{M}'$, C'' contains some $C^{\#}$ where $a \in C^{\#} \in \mathfrak{M}$. However $C^{\#} \in \mathfrak{M}''$ and C'' is minimal so that $C^{\#} = C''$. Consequently $\mathfrak{M}'' = \mathfrak{M}$.

Consider the Shannon game played on a matroid $\mathfrak{M}$ with respect to a branch e, e not being subject to play. For each play by the cut or short player on a branch a of $\mathfrak{M}$ let there be a play by the short or cut player, respectively, on the same branch a of $\mathfrak{M}'$. The game played on $\mathfrak{M}'$ with respect to e is called the *dual game*. The outcomes of the primal and dual games are related in (21).

LEMMA. *Assume that all branches of $\mathfrak{M}$ except e are subject to play. Then*

(21) *the short player wins the game on $\mathfrak{M}$ if and only if the cut player wins the (dual) game on $\mathfrak{M}'$.*

Proof. The short player can win the game on $\mathfrak{M}$ only by playing a set of branches containing $C - \{e\}$ where C is a circuit of $\mathfrak{M}$ containing e. In the game on $\mathfrak{M}'$ these branches are played by the cut player. For any circuit C' of $\mathfrak{M}'$ which contains e, (18) yields $C \cap C' - \{e\} \neq O$. Hence in the dual game the short player cannot play every branch of $C' - \{e\}$ and consequently cannot win.

The cut player wins the game on $\mathfrak{M}'$ only if for every circuit C' of $\mathfrak{M}'$ containing e, $C' - \{e\}$ contains either a branch played by the cut player or a branch not subject to play. Since all branches except e *are* subject to play, the cut player wins the game on $\mathfrak{M}'$ only if the short player plays all of the branches $C - \{e\}$ of some circuit C of $\mathfrak{M}$ and hence wins the game on $\mathfrak{M}$.

If branches other than e are not subject to play it may be possible for the cut player to win both the primal and dual games. As a trivial example consider the matroid consisting of two branches a and e and a single circuit $\{a, e\}$, neither branch being subject to play. This matroid is its own dual and since no plays are allowed the cut player has won both games. In all subsequent results on dual games it will be assumed that all branches, except possibly the distinguished branch, are subject to play. In applying these results to graphs it should be noted that the graphical dual exists only when the graph, including the distinguished branch, is planar.

Assume that a game is played on a matroid $\mathfrak{M}$ with respect to a branch e, e not being subject to play. The condition that the game end in favor of the cut player is that

(22) *the span, in the dual matroid, of the branches played by the cut player, contains e.*

(This condition is an immediate consequence of (21).) Thus when e is in the span in $\mathfrak{M}$ of the branches played by the short player or is in the span in $\mathfrak{M}'$ of the branches played by the cut player, the game ends. Otherwise it is still possible, dropping the assumption of alternating plays, for either player to attain his goal.

By (21) a winning strategy for the short player on $\mathfrak{M}$ (or $\mathfrak{M}'$) yields a winning strategy for the cut player on $\mathfrak{M}'$ (or $\mathfrak{M}$). Hence

(23) *the Shannon game, played with respect to a branch not subject to play, is a short game if and only if the dual game is a cut game.* (The game played with respect to a playable branch can never be a cut game.)

The dual of a neutral game, being neither a cut game nor a short game, is neutral. Hence

(24) *the Shannon game, played with respect to a branch not subject to play, is neutral if and only if the game is in the same class—cut, short, or neutral—as its dual.*

LEMMA. *Assume that the Shannon game is played on a matroid $\mathfrak{M}$ with respect to a branch e, e not being subject to play. Then*

(25) *the game is a neutral game if and only if the corresponding games on both $\mathfrak{M}$ and $\mathfrak{M}'$, where e* is *subject to play, are short games.*

Proof. Assume that the game is neutral. In the game where e is subject to play, the short player, playing second can win whether or not the cut player, as a first play, plays e. By (24) the game on $\mathfrak{M}'$ where e is subject to play, is also a short game.

Assume that the games on $\mathfrak{M}$ and $\mathfrak{M}'$ where e is subject to play are short games. Then the short player playing first can win the corresponding games where e is not subject to play. By (21), the cut player playing first can win the game on $\mathfrak{M}$ where e is not subject to play. Hence this game is neutral.

Thus (25) yields a criterion for neutral games in terms of the criterion (14) for short games. A criterion for cut games can be found by combining (14) and (23). In the following theorem this criterion is formulated in terms of the given (primal) matroid. A corresponding strategy can be obtained by dualizing the strategy given in the first part of the proof of (14).

THEOREM. *Consider the Shannon game on a matroid $\mathfrak{M}$ played with respect to a branch e, e not being subject to play.*

(26) *The cut player, playing second, can win against all possible strategies of the short player (i.e., a cut game) if and only if there exist A, B such that*
$e \notin A \cup B, \quad A \cap B = O,$
$e \in C \in \mathfrak{M}$ *implies* $A \cap C \neq O$, *and*
$C \in \mathfrak{M}$ *implies* $A \cap C \neq O$ *if and only if* $B \cap C \neq O$.

Proof. The Shannon game is a cut game if and only if the dual game is a short game; that is, if and only if there exist A, B such that $A \cap B = O$, $e \notin A \cup B$, and $e \in \mathcal{S}'(A) = \mathcal{S}'(B)$ where $\mathcal{S}'$ denotes the span in the matroid $\mathfrak{M}'$.

Necessity. Assume a cut game. If $C \cap A \neq O$ or $e \in C$ thee there exists a such that $a \in C \cap (A \cup \{e\}) \subset \mathcal{S}'(A) = \mathcal{S}'(B)$. Since $a \in \mathcal{S}'(B) - B$, there exists $C' \in \mathfrak{M}'$ such that $C' - B = \{a\}$ and hence $C \cap B \supset C \cap C' - \{a\} \neq O$. By symmetry, $C \cap B \neq O$ implies $C \cap A \neq O$.

Sufficiency. Assume sets A, B satisfying the conditions of the theorem. If $a \in A \cup \{e\}$ then $a \in C \in \mathfrak{M}$ implies $(B \cup \{a\}) \cap C - \{a\} = B \cap C \neq O$ and hence there exists $C' \in \mathfrak{M}'$ such that $C' - B = \{a\}$. Consequently $A \cup \{e\} \subset \mathcal{S}'(B)$ and by symmetry $B \cup \{e\} \subset \mathcal{S}(A)$. It follows that $e \in \mathcal{S}'(A) = \mathcal{S}'(B)$ and thus $\mathfrak{M}$ yields a cut game.

Thus, by (26), a graph with two distinguished vertices yields a cut game if and only if there exist two disjoint sets of branches such that every path connecting the two distinguished vertices contains at least one branch of each set, and every circuit which contains a branch of one set contains a branch of the other.

4. Trees. In graph theory, a tree is a connected set of branches which does not contain a circuit [14]. Equivalently, a tree is a minimal set of branches which connects a given set of vertices. These vertices determine its span; in a drawn graph they are readily found by inspection. Several of the previous results will be restated in terms of trees.

Let $\mathfrak{M}$ be a given matroid. A *tree* T is a subset of M such that

(27) $A \subset T \subset \mathcal{S}(A)$ *implies* $A = T$.

Thus a tree is a minimal set having a given span. Alternatively, by (28), a tree is a set of branches containing no circuits. (In graph theory a tree is by definition connected; otherwise a set of branches containing no circuits is called a forest. As this connectivity distinction cannot be made in matroids, only the term tree will be used. Nevertheless, if $\mathfrak{M}$ is the matroid of circuits of a graph, the trees used in (29), for a given branch e, can be chosen so that they are connected in the graph.)

LEMMA. *Let T be a set of branches of a matroid $\mathfrak{M}$.*

(28) *T is a tree if and only if $C - T \neq O$ for all $C \in \mathfrak{M}$.*

Proof. The null set is a tree and contains no circuits. If T is not null and $C - T \neq O$ for all $C \in \mathfrak{M}$ then for any $a \in T$ it follows that $a \notin \mathcal{S}(T - \{a\})$ and hence, by (27), T is a tree.

If for some circuit C, $C \subset T$, then $\mathcal{S}(T) = \mathcal{S}(T - \{a\})$ for all $a \in C$ and hence T is not a tree.

Trees will be denoted by T's. Two sets A and B are said to be *cospanning* if they have the same span, that is, if $\mathcal{S}(A) = \mathcal{S}(B)$. Thus two trees of a graph are cospanning if and only if they connect the same set of vertices.

The sets A and B, occurring in (14), (15), and the proof of (16), are cospanning. If they are chosen to be minimal then they are trees. Thus (14), (15), and (16) are summarized in the following principal result.

THEOREM.

(29) *The Shannon game on a matroid* $\mathfrak{M}$ *with respect to a distinguished branch e is a short game if and only if there exist two disjoint cospanning trees which span e. Neither tree is to contain e unless e is a playable branch. Furthermore the trees, and hence the strategy of the short player, can be chosen without knowing which is the distinguished branch. (These trees will span a branch if and only if the game with respect to that branch is a short game.)*

Under the assumption that A and B are trees, the winning strategy of the short player, developed in the first part of the proof of (14), will be called the *two tree strategy.*

The sets A and B of (26) contain no circuits and hence are trees. They however cannot be cospanning as a matroid cannot yield both cut and short games with respect to the same fixed branch. In a graph neither set is necessarily connected (i.e., they are forests). Thus the minimal set pairs A, B, which in the dual matroid are cospanning trees, do not correspond to any obvious graphical entity.

The criterion for short games yields the following criteria for cut and neutral games.

LEMMA. *Consider the Shannon game played on a matroid* $\mathfrak{M}$ *with respect to a branch e, e not being subject to play*:

(30) *The game is a cut game if and only if the corresponding game where e is subject to play is* not *a short game.*

(31) *The game is a neutral game if and only if it is* not *a short game and the corresponding game where e is subject to play* is *a short game.*

Proof. If the game where e is not subject to play is a cut game then the cut player, playing first (on branch e), can win the game where e is subject to play. This proves the necessity of the criterion of (30) and the sufficiency of the criterion of (31). The necessity of (31) follows from (25) and the sufficiency of (30) follows from (31).

Together (29), (30), and (31) yield criteria for short, cut, and neutral games. Unfortunately, the resulting criteria for cut and neutral games are nonconstructive in that they involve the nonexistence of cospanning trees. For this reason the criterion for cut games resulting from (23) and (29) and that for neutral games resulting from (25) and (29) may prove more satisfactory.

The sequential classification of Shannon games given by (i), (ii), (iii) together with the criteria of (23), (25), and (29) yields a corresponding classification of matroids:

THEOREM. *Let $\mathfrak{M}$ be a matroid and e a fixed branch.*

(32) *Exactly one of the following statements holds:*

(i′) *$\mathfrak{M}$ contains two disjoint cospanning trees spanning but not containing e.*

(ii′) *$\mathfrak{M}'$ contains two disjoint cospanning trees spanning but not containing e.*

(iii′) *$\mathfrak{M}$ and $\mathfrak{M}'$ each contain two disjoint cospanning trees, in each case e being a member of one tree.*

If $\mathfrak{M}$ is assumed to be the matroid of circuits of a graph the trees of (32) can be chosen so that they are connected in the graph. If in addition the graph is assumed to be planar, then (32) constitutes a result in graph theory. Examples of graphs whose circuit matroids have properties (i′), (ii′), and (iii′) are given in Fig. 7. Thus the graph i′ of Fig. 7 contains

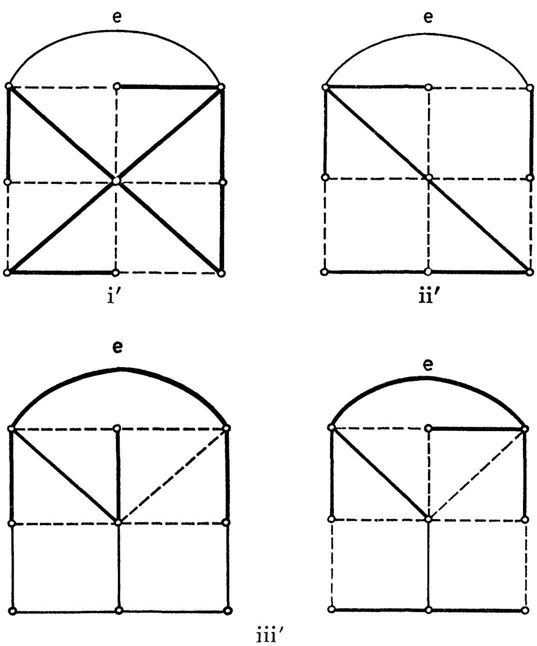


FIG. 7

two disjoint cospanning trees (indicated by solid and dashed lines) which span but do not contain e and hence the graph yields a short game. The graph ii$'$ contains two disjoint branch sets which in the dual graph are cospanning trees which span but do not contain e. Hence the graph yields a cut game. The graph iii$'$ has both properties except that the branch e is allowed to be a member of one of the disjoint sets. This graph yields a neutral game. (Each game is with respect to the vertices of e. e is not subject to play.) Any planar graph with a distinguished branch has one of these properties.

Among sets having a given span, a tree contains a minimum number of branches. This is a special case of (33).

LEMMA.

(33) *If* $\mathcal{S}(T) \subset \mathcal{S}(A)$ *then* card $T \leqq$ card A.

Proof. Choose B such that $\mathcal{S}(T) \subset \mathcal{S}(B)$, card $B =$ card A, and $T - B$ is minimal. If $T \subset B$ then the assertion holds. Otherwise, since T is a tree, there exists $a \in B - T$. Then by (8) there exists $b \in T - B$ such that $\mathcal{S}(T) \subset \mathcal{S}(B \cup \{b\} - \{a\})$ and hence $B \cup \{b\} - \{a\}$ contradicts the minimality assumption.

From (33) it follows that

(34) *If* $\mathcal{S}(T^{\#}) = \mathcal{S}(T^*)$ *then* card $T^{\#} =$ card T^*.

A tree is said to be *maximal* if it is not properly contained in any other tree. Consequently,

(35) *a tree is maximal if and only if it spans* M.

(Maximal trees are sometimes called spanning trees. In [15] a maximal tree is called a base.)

The *rank* of a matroid $\mathfrak{M}$ is defined to be card T where T is any maximal tree of the matroid. Since all maximal trees of a given matroid contain the same number of branches the rank is unique. Alternatively the rank is the minimum number of branches required for a set to span M.

A maximal tree of a connected graph is a tree which connects all of the vertices. This definition is in agreement with that of the maximal tree of a matroid. The rank of the circuit matroid of such a graph is the number of vertices minus one [14, 15]. For a graph which is not connected (the corresponding matroid is called *separable*) no maximal tree (i.e., a forest) is connected. The rank of the circuit matroid of such a graph is the number of vertices minus the number of maximal connected subgraphs.

A matroid (or graph) will be called *well-connected* if it contains two disjoint maximal trees (or forests). Thus a matroid is well-connected if and only if the short player, playing second, can win the Shannon game with respect to each branch, all branches being subject to play. (By (16) this can be done by a single strategy.) A well-connected graph is not necessarily connected. The graphs of Figs. 4, 6, 10, 12, and 15 are however both connected and well-connected.

The following lemma characterizes a class of minimally well-connected matroids in terms of neutral games.

LEMMA. *Let $\mathfrak{M}$ be a given matroid.* (*Except as noted, all branches of M will be subject to play.*)

(36) *M can be partitioned into two disjoint maximal trees if and only if the game with respect to each branch a of M, a not being subject to play, can always be won by the first player* (*i.e., a neutral game*).

Proof. Sufficiency. Assume that $\mathfrak{M}$ yields a neutral game with respect to each branch. By (25) the game with respect to each branch a, a subject to play, is a short game. Then by (29), $\mathfrak{M}$ contains two disjoint maximal trees $T^{\#}$, T^*. If $a \in M - T^{\#} \cup T^*$ then the game with respect to a, a not subject to play, is a short game. Thus by contradiction, $M = T^{\#} \cup T^*$.

Necessity. Let $T^{\#}$, T^* be maximal trees which partition M. Assume for some distinguished branch a, not subject to play, that $\mathfrak{M}$ yields a short game. Then there exist two disjoint cospanning sets A, B which span but do not contain a. Define $A^{\#} = A \cup (T^{\#} - B \cup \{a\})$ and $B^{\#} = B \cup (T^* - A \cup \{a\})$. Then $A^{\#}$ and $B^{\#}$ are disjoint and do not contain a. Furthermore $\mathcal{S}(A^{\#})$ contains $\mathcal{S}(A)$, $\mathcal{S}(B)$, $\{a\}$, $\mathcal{S}(T^{\#})$; and $\mathcal{S}(B^{\#})$ contains $\mathcal{S}(B)$, $\mathcal{S}(A)$, $\{a\}$, $\mathcal{S}(T^*)$. Hence $\mathcal{S}(A^{\#}) = \mathcal{S}(B^{\#}) = M$. But by (33), $\text{card } M = \text{card } T^{\#} + \text{card } T^* \leqq \text{card } A^{\#} + \text{card } B^{\#} < \text{card } M$, thus contradicting the short game assumption. Since the corresponding game where a *is* subject to play *is* a short game, the result follows by (31).

It is easily shown that the assumptions of (36) are self-dual, that is, that $T^{\#}$ and T^* are maximal trees of $\mathfrak{M}'$. Furthermore the game can be won by a slight variation of the two tree strategy in which the first play depends upon the distinguished branch.

In attempting to determine whether a given branch is spanned by two disjoint cospanning trees, it may happen (see (37)) that two such trees are found which do not span the given branch. However these trees, denoted by $T^{\#}$ and T^*, can be used to simplify the problem. The matroid can be replaced by a new matroid whose circuits are the minimal nonnull sets of the form $C - T^{\#} \cup T^*$ where C is any circuit of the given matroid. If in the new matroid the given branch is spanned by two disjoint cospanning trees, then by the technique of (10), these trees together with $T^{\#}$ and T^* can be used to construct the required trees of the original matroid. If no such trees exist in the new matroid then the given branch cannot be spanned by two disjoint cospanning trees of the original matroid.

The previous technique is applied more readily to graphs. Assume that it is desired to find two disjoint cospanning trees which connect two given vertices of a graph. If two such trees are found which do not connect the vertices then the problem can be reduced to finding two such trees in the graph formed by condensing to a single vertex (i.e., short-circuiting) each set of vertices which are mutually connected by the trees.

If a matroid is well-connected and of rank r then it must contain at least $2r$ branches. These branches may be distributed in a variety of ways. It is a consequence of (37) that any matroid of rank r and at least $2r$ branches yields a short game with respect to some branch. Thus $2r$ branches of a matroid of rank r either form two disjoint maximal trees or they are "bunched" so that they contain two disjoint trees both spanning a submatroid. A related theorem on graphs is given in papers by Tutte [13] and Nash-Williams [8] (these references were supplied by the referee). Apparently little else is known concerning the existence and construction of pairs of disjoint cospanning trees.

THEOREM. *Consider a matroid of rank r and having n branches.*

(37) *If $n \geqq 2r > 0$ then there exist two disjoint cospanning trees spanning at least $n - 2r + 1$ branches.*

Proof. If $\mathfrak{M}$ is of rank 1, either there exist distinct branches a and b such that $\{a\}$ and $\{b\}$ are maximal trees or there exists only one branch which is not a circuit. In the first case $\{a\}$ and $\{b\}$ span all of the branches and in the second case the null set spans all but one of the branches. Thus the theorem holds for $r = 1$.

Assume that the theorem holds for $r \leqq r_0$. Let $\mathfrak{M}$ be a matroid of rank $r = r_0 + 1$ and having n branches where $n \geqq 2(r_0 + 1)$. If the game with respect to each branch (all branches being subject to play) is a short game, then by construction used in the proof of (16), there exist two disjoint maximal trees and the theorem holds. Otherwise, by (30), there exists a branch e such that the game played with respect to e, e not being subject to play, is a cut game. If $\mathfrak{M}$ has no circuits containing e then the matroid having branches $M - \{e\}$ and the circuits of $\mathfrak{M}$ is of rank r_0 and has $n - 1$ branches. By the inductive assumption there exist two disjoint nonnull cospanning trees which span at least $(n - 1) - 2r + 2$ branches in this matroid and hence span the same branches in M, $\mathfrak{M}$. If e is contained in some circuit of $\mathfrak{M}$ then by (26) there exist two disjoint nonnull sets A, B

(labeled so that card $A \geqq$ card B) such that $e \notin A \cup B$ and for each $C \in \mathfrak{M}$, $A \cap C \neq O$ if and only if $B \cap C \neq O$. Define the matroid $\mathfrak{M}^\#$ to have branches $M^\# = M - A \cup B$ and circuits C where $C \subset M^\#$ and $C \in \mathfrak{M}$. Let $T^\#$ be any maximal tree of $\mathfrak{M}^\#$. If $C \in \mathfrak{M}$ and $C \subset T^\# \cup A$ then $C \cap A \neq O$ and hence $C \cap B \neq O$. But B and C are disjoint. Thus $T^\# \cup A$ is a tree of $\mathfrak{M}$ and rank $\mathfrak{M} \geqq$ card $(T^\# \cup A) =$ rank $\mathfrak{M}^\# +$ card A. Since $0 <$ rank $\mathfrak{M}^\# <$ rank $\mathfrak{M}$ and card $M^\# \geqq$ card $M - 2$ card A $\geqq$ card $M + 2$ (rank $\mathfrak{M}^\# -$ rank $\mathfrak{M}$) $= n - 2r + 2$ rank $\mathfrak{M}^\# \geqq 2$ rank $\mathfrak{M}^\#$, the inductive assumption yields two disjoint cospanning trees of $\mathfrak{M}^\#$ which span at least card $M^\# - 2$ rank $\mathfrak{M}^\# + 1 \geqq n - 2r + 1$ branches. Since these trees are also disjoint cospanning trees of $\mathfrak{M}$ spanning at least the same branches, the induction is complete.

For example, consider any graph of $2r > 0$ branches and $r + 1$ vertices such that each branch connects two distinct vertices. It is a consequence of (37) that at least two vertices are connected by a pair of disjoint cospanning trees.

Corollary.

(38) *If* $n \geqq 2r > 0$ *then there exist two disjoint cospanning* sets *containing* (*together*) *at least* $n - 2r + 1$ *branches.*

The graph given in Fig. 8 is of rank r ($r + 1$ vertices) and contains $2r - 1$ branches. If any pair of distinct vertices is connected by an additional branch then the branches of the resulting graph can be partitioned into two maximal trees. Thus by (36) the cut player, playing first in the game on the given graph, can win with respect to any distinct pair of vertices. It follows that the given graph contains no pair of disjoint nonnull cospanning trees. Consequently (37) cannot be extended to the case $n < 2r$.

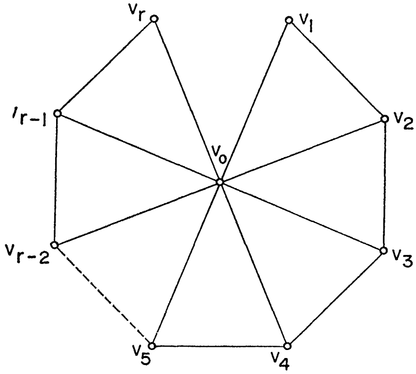


Fig. 8

5. Examples. The winning strategy for the short player developed for the game on matroids (see (14) and (16)) can be stated in terms of graphs and trees:

Suppose a given graph with two distinguished vertices yields a short game. Then by (29) there exist two disjoint cospanning trees which connect the two distinguished vertices. Branches of either tree when played by the cut player will be deleted from that tree. Branches of either tree when played by the short player will be added to the remaining tree. Each play of the short player is made so as to yield two cospanning trees whose span is that of the original trees and whose intersection (by definition) consists of those branches which are invulnerable to deletion.

Assume that the cut player plays first. If the deleted branch is a member of one of the trees then the short player should play a branch of the remaining tree such that its addition to the first (deleted) tree restores the span of that tree. Such a branch always exists but it is not in general unique. If the deleted branch is not a member of either tree then the short player should play a branch of one of the trees. Since this branch is considered to be added to the remaining tree some other circuit-forming branch of that tree is to be deleted (that is, deleted from the tree but not from the graph). This deletion is necessary as all trees with a given span contain the same number of branches.

If at any stage of the game the cut player deletes a branch which is a member of one tree but not a member of the other then the short player should play a branch of the remaining tree such that its addition to the first tree restores the span of that tree. At least one such branch of the remaining tree, not contained in the first tree, always exists. If the deleted branch is not a member of either tree then the short player should play some branch which is a member of one but not both of the trees. Again, this branch is considered as added to the remaining tree and some unplayed circuit-forming branch of that tree is to be deleted. The game is won by the short player whenever the set of branches common to both trees contains a path connecting the two distinguished vertices. Any branch of such a path is invulnerable to deletion. *This same two-tree strategy can be used by the short player to win the game with respect to any pair of vertices which are connected by the trees.*

If a graph is planar and yields a cut game, a winning strategy for the cut player can be obtained from that of the short player played on the dual graph. It is assumed that the distinguished vertices of the graph are connected by a branch e, the only branch not subject to play. By (21), the game on the dual graph, with respect to the vertices of e, is a short game and the winning strategy for the short player on this game yields a winning strategy for the cut player on the original graph. If the original graph is nonplanar a strategy can be obtained, though in a less convenient manner, from the matroid dual.

A neutral game can be converted to a short game by the addition of a playable branch connecting the two distinguished vertices. A winning strategy for the short player playing first in the neutral game can be then obtained from the strategy of the short player playing second in the short game. By duality (the graphical dual for planar graphs and the matroid dual for nonplanar graphs) a winning strategy can be obtained for the cut player playing first in a neutral game.

To avoid unnecessary complication the following examples were chosen so that the cospanning trees connect all vertices and together contain all branches.

The configuration of Fig. 9 is of rank 8 (9 vertices) and has 12 branches. Thus 4 additional branches are required for the existence of two disjoint maximal trees. One possible addition of branches and a resulting pair of trees is given in Fig. 10. (Another pair of trees is given in Fig. 7i′.) The sequence of plays of three actual games is given in Fig. 11. These games were played with respect to the vertices v' and v'' and in accordance with the two-tree strategy based on the trees of Fig. 10. The branch numbers denote the sequence in which the plays were made, odd numbers denoting plays by the cut player and even numbers plays by the short player. The short player playing second won the three games in 6, 7, and 8 plays re-

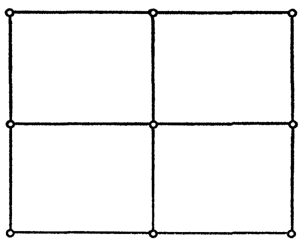


FIG. 9

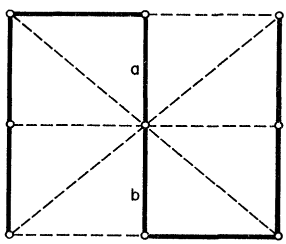


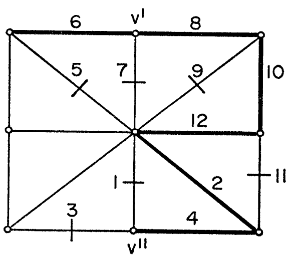
FIG. 10

spectively. This variation is due to the inefficiency of the two-tree strategy as in fact the short player can always win within 5 plays. This can be done by a strategy which pairs branches a and b of Fig. 10; that is, the short player plays a or b in response to a play by the cut player on b or a respectively. In addition the short player avoids play on that side of the a,b line first played by the cut player.

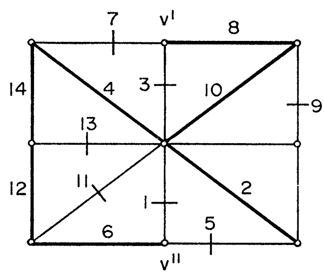
A different addition of branches to the configuration of Fig. 9 results in the configuration and tree pair of Fig. 12. The sequence of plays of an actual game based on the trees of Fig. 12 is given in Fig. 13.

The game of "Gale" or "Bridg-it" described in [3] is equivalent to the Shannon game played on the graph, with distinguished vertices v' and v'', given by Fig. 14. This graph contains 22 vertices (rank 21) and 41 branches. If an additional branch e, not subject to play, is used to connect the distinguished vertices, the resulting 42 branches can be partitioned, as shown in Fig. 15, into two maximal trees. Hence, by (36), the game of "Bridg-it" can be won by the first player. Since the dual of the graph, including the distinguished vertices, has the same form as the given graph (Fig. 15), this result also follows from (24).

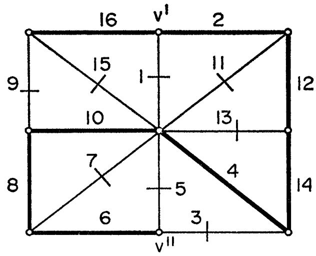
Assuming that the cut player has already deleted branch e, the two trees of Fig. 15 yield a winning strategy for the short player playing first on



Game 1a



Game 1b



Game 1c

FIG. 11

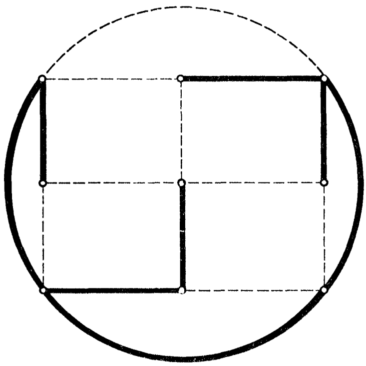


FIG. 12. *Trees for game* 2

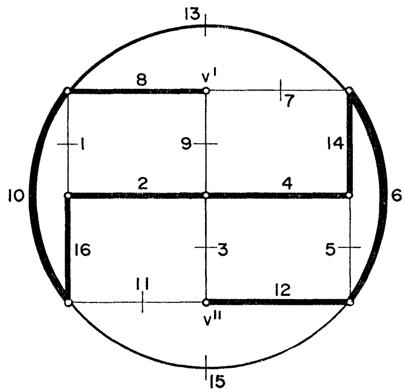


FIG. 13. *Game* 2

the Bridg-it graph of Fig. 14. The corresponding strategy for the cut player can be found from the graphical dual or from the diagonal symmetry of the original game.

The initial plays of the Bridg-it pairing strategy of O. Gross, given in [3] and indicated by arrows in Fig. 15, are in accordance with the strategy based on the trees of Fig. 15. However it is possible, using the pairing strategy of Gross, for the cut player to delete all of the branches incident upon the vertex u of Fig. 14 and of Fig. 15 and thus isolate that vertex. This is not possible by any two-tree strategy based on maximal trees.

APPENDIX

6. An extension of the Shannon game. The Shannon game admits a generalization from a game on matroids to one on Boolean functions. This generalization includes the unsolved game of "Hex" [2].

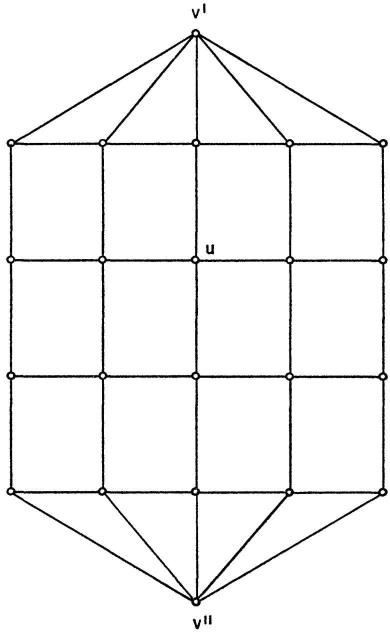


FIG. 14. *Bridg-it*

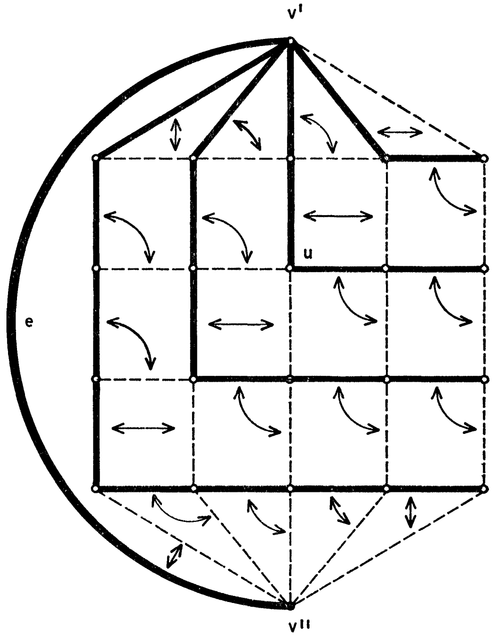


FIG. 15. *Trees for Bridg-it*

6.1. The game. Let $\mathfrak{f}$ be a function of n variables; the variables, denoted by $x_1, \cdots, x_n$, and $\mathfrak{f}$ taking values in the two element set $\{\theta, \phi\}$. $\mathfrak{f}$ is said to be *isotone* if a change in the value of any x_i from θ to ϕ cannot change the value of $\mathfrak{f}$ from ϕ to θ. Such a function can be expressed as a polynomial in the Boolean operations $\cup$ and $\cap$ $\quad(\theta \cup \phi = \phi$ and $\theta \cap \phi = \theta)$ and variables $x_1, \cdots, x_n$.

Two players, denoted by "θ" and "ϕ", are presented with an isotone function $\mathfrak{f}$. They alternate in assigning values, one per play, to the remaining unassigned variables. After a total of n such plays all of the variables have assigned values and the winner of the game is given by the value of $\mathfrak{f}$.

If a player can win playing second, he can win playing first. The argument is the same as that for the Shannon game. Hence $\mathfrak{f}$ has exactly one of the following properties.

(i) The ϕ player plays first and the θ player can win against all possible strategies of the ϕ player.

(ii) The θ player plays first and the ϕ player can win against all possible strategies of the θ player.

(iii) The player who plays first, regardless of identity, can win against all possible strategies of the other player.

Since $\mathfrak{f}$ is isotone, the θ and ϕ players can be assumed to play only their respective values without changing the property, (i), (ii), or (iii), of $\mathfrak{f}$.

Given a matroid $\mathfrak{M}$ with a distinguished branch e, the *associated isotone function* $\mathfrak{f}$ is defined as follows. Let $a, b, c, \cdots$ be the set of branches $M - \{e\}$ and let $x_a, x_b, x_c, \cdots$ be any set of values where each x is in the set $\{\theta, \phi\}$. For each $x_a, x_b, x_c, \cdots$, denote the set of subscripts (indices) of those x's having value θ by A_θ and define

$$\mathfrak{f}(x_a, x_b, x_c, \cdots) = \begin{cases} \theta \text{ if } e \in \mathcal{S}(A_\theta), \\ \phi \text{ if } e \notin \mathcal{S}(A_\theta). \end{cases}$$

Thus $\mathfrak{f}$ has the value θ if and only if there exists a circuit C such that $C - A_\theta = \{e\}$. If $\mathfrak{M}$ is the circuit matroid of a graph, the branches $C - \{e\}$ constitute a path connecting the vertices of e. Hence $\mathfrak{f}$ is the electrical "switching function" for the vertices of e.

Let $\mathfrak{f}$ be the isotone function associated with the matroid $\mathfrak{M}$ with distinguished branch e. The correspondence of the ϕ and θ players with the cut and short players, respectively, yields a correspondence (isomorphism) between the Shannon game on $\mathfrak{f}$ and that on $\mathfrak{M}$ with respect to e, e not being subject to play. Hence the Shannon game criteria and strategy (for fixed e), such as (14) and (26), will apply to the game on $\mathfrak{f}$.

Unfortunately, not all isotone functions are associated with matroids. The general problem of finding a simple (nonsequential) criterion for those isotone functions $\mathfrak{f}$ having property (i) is still unsolved. By interchanging the symbols θ and ϕ (duality) such a criterion yields a criterion for property

(ii) and by exhaustion, one for property (iii). It is to be expected that these criteria will also yield winning strategies for the appropriate player. In any case, knowing criteria for properties (i), (ii), and (iii), a player can make each of his plays so that the resulting function of one less variable is of the most favorable type. Clearly this primitive strategy will win for that player for whom a winning strategy exists.

6.2. Binary matroids. The *dual* of an isotone function $\mathfrak{f}$ is defined by dual $\mathfrak{f}(x_1, \cdots, x_n) = \mathfrak{f}'(x_1', \cdots, x_n')$ where $'$ denotes interchanging the values θ and ϕ. The dual of an isotone function is also isotone. If $\mathfrak{f}$ is expressed as a Boolean function in $\cup$ and $\cap$ then the dual of $\mathfrak{f}$ is the corresponding function in $\cap$ and $\cup$.

The definition of the dual of $\mathfrak{f}$ amounts to an interchange of the θ and ϕ values. Thus the dual of $\mathfrak{f}$ has property (i), (ii), or (iii) if and only if $\mathfrak{f}$ has property (ii), (i), or (iii), respectively. By the duality of Boolean functions and the duality of matroids as given by (18), a function $\mathfrak{f}$ is associated with the matroid $\mathfrak{M}$ if and only if the dual of $\mathfrak{f}$ is associated with $\mathfrak{M}'$, both matroids having the same distinguished branch.

A *path* of an isotone function $\mathfrak{f}$ of variables $x_1, \cdots, x_n$ is a minimal set of indices k such that if each x_k has the value θ then $\mathfrak{f}(x_1, \cdots, x_n) \equiv \theta$. A *cut* is a minimal set of indices k such that if each x_k has the value ϕ then $\mathfrak{f}(x_1, \cdots, x_n) \equiv \phi$. (These sets are often called minimal paths and minimal cuts.) It is easily seen that the paths and cuts of the dual of $\mathfrak{f}$ are respectively the cuts and paths of $\mathfrak{f}$.

In order to apply the matroidal results to the game on an isotone function it is necessary to verify that that function can be derived from a matroid. A principal result (see (47)) is that an isotone function $\mathfrak{f}$ of $x_1, \cdots, x_n$ is associated with a binary matroid (defined below) if and only if each path and cut intersect in an odd number of indices. No corresponding criterion is known for those functions associated with the class of *all* matroids.

A finite *binary matroid* M, $\mathfrak{M}$ is a collection $\mathfrak{M}$ of nonnull subsets C of a finite set M, such that no subset is properly contained in any other subset and

(39) *If* $C^\#, C^* \in \mathfrak{M}$ *and* $C^\# \neq C^*$, *then there exists* $C \in \mathfrak{M}$ *such that* $C \subset C^\# \cup C^* - C^\# \cap C^*$.

If M, $\mathfrak{M}$ is a binary matroid then it satisfies (1) and is thus a matroid. The circuits of any unoriented-branch linear graph form a binary matroid but not all matroids are binary. In (41) it will be shown that binary matroids satisfy a property ostensibly stronger than (39). It is this property which is used in the corresponding definition of Whitney [15].

The symmetric difference operation $\oplus$ is defined by $C^{\#} \oplus C^* = C^{\#} \cup C^* - C^{\#} \cap C^*$. Thus (39) states that the symmetric difference of any two distinct members of $\mathfrak{M}$ contains a member of $\mathfrak{M}$.

LEMMA. *Let* $\mathfrak{M}$ *be a binary matroid. Assume that* $C \in \mathfrak{M}$ *and* $C' \in \mathfrak{M}'$. *Then*

(40) card $(C \cap C')$ *is even.*

Proof. Assume that C, C' are such that card $(C \cap C')$ is the smallest integer for which (40) fails. Since card $(C \cap C') \neq 1$, there exist a, b, $c \in C \cap C'$ such that $a \neq b \neq c \neq a$. C' is a minimal set such that $C^{\#} \cap C' - \{a\} \neq O$ for all $C^{\#}$ such that $a \in C^{\#} \in \mathfrak{M}$. Hence there exists C^* such that $C^* \cap C' = \{a\} \cup \{c\}$. Since $a \in C \cap C^*$ and $b \in C - C^*$ there exists $C^{\#\#}$ such that $b \in C^{\#\#} \subset C \cup C^* - \{a\}$. In addition choose $C^{\#\#}$ so that $C \cup C^{\#\#}$ is minimal. Let C^{**} be any circuit satisfying $a \in C^{**} \subset C \cup C^{\#\#} - \{b\}$ and $C^{\#*}$ be any circuit satisfying $b \in C^{\#*} \subset C \cup C^{**} - \{a\}$. By the minimality of $C \cup C^{\#\#}$, $C \cup C^{\#\#} \subset C \cup C^{\#*} \subset C \cup C^{**} \subset C \cup C^{\#\#}$. Thus $C^{\#\#} - C = C^{**} - C$ and hence $C^{\#\#} \oplus C^{**} \subset C$. Since $C^{\#\#} \neq C^{**}$, it follows from (39) that $C^{\#\#} \oplus C^{**}$ contains a circuit. Consequently $C^{\#\#} \oplus C^{**} = C$. By construction, card $(C^{\#\#} \cap C')$ and card $(C^{**} \cap C')$ are positive and less than card $(C \cap C')$. From $C \oplus C^{\#\#} = C^{**}$ it follows that either card $(C^{\#\#} \cap C')$ or card $(C^{**} \cap C')$ is odd and contradicts the minimality of card $(C \cap C')$.

If $\mathfrak{M}$ is not binary, card $(C \cap C')$ can be odd. For example, let $M = \{a, b, c, d\}$ and $\mathfrak{M} = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Then $\mathfrak{M}' = \mathfrak{M}$ and letting $C = C' = \{a, b, c\}$ yields card $(C \cap C') = 3$.

LEMMA. *Let* $\mathfrak{M}$ *be a matroid and* $\mathfrak{M}'$ *be its dual. If* card $(C \cap C')$ *is even for all* $C \in \mathfrak{M}$ *and* $C' \in \mathfrak{M}'$ *then*

(41) *any symmetric difference of circuits is equal to a sum* (*union*) *of disjoint circuits.*

Proof. Assume that $C_1, \cdots, C_k$ are such that $a \in C_2 \oplus \cdots \oplus C_k$. (Otherwise $C_1 \oplus \cdots \oplus C_k$ is null and is the vacuous union of circuits.) For any C' such that $a \in C' \in \mathfrak{M}'$, card $(C_1 \cap C'), \cdots$, card $(C_k \cap C')$ are even. Thus card $((C_1 \oplus \cdots \oplus C_k) \cap C')$ is even and hence $(C_1 \oplus \cdots \oplus C_k) \cap C' - \{a\} \neq O$. Since $\mathfrak{M}'' = \mathfrak{M}$ it follows from (18) that $C_1 \oplus \cdots \oplus C_k$ contains a circuit C. If $C \neq C_1 \oplus \cdots \oplus C_k$ the argument can then be applied to $C_1 \oplus \cdots \oplus C_k \oplus C$. Eventually this construction terminates and yields a finite collection of disjoint circuits whose union is $C_1 \oplus \cdots \oplus C_k$.

Clearly the property expressed by (41) implies that expressed by (39).

Thus (40) and (41) yield the following characterization of binary matroids.

LEMMA. *Let* $\mathfrak{M}$ *be a matroid and* $\mathfrak{M}'$ *be its dual.*

(42) $\mathfrak{M}$ *is a binary matroid if and only if* card $(C \cap C')$ *is even for all* $C \in \mathfrak{M}$ *and* $C' \in \mathfrak{M}'$.

Since the property, card $(C \cap C')$ is even, is self-dual, (42) yields

(43) $\mathfrak{M}$ *is binary if and only if* $\mathfrak{M}'$ *is binary.*

THEOREM. *Let* $\mathfrak{A}$ *be a collection of nonnull subsets* A *of a finite set* M *such that no subset is properly contained in any other subset and each subset contains* e, *a fixed member of* M. *Define* $\mathfrak{A}'$ *to be the collection of minimal nonnull subsets* A', *each containing* e *and satisfying* $A \cap A' - \{e\} \neq O$ *for all* A *in* $\mathfrak{A}$. *Then*

(44) card $(A \cap A')$ *is even for all* $A \in \mathfrak{A}$ *and* $A' \in \mathfrak{A}'$ *if and only if* $\mathfrak{A}$ *is the collection of circuits containing* e *of some binary matroid* $\mathfrak{M}$.

Proof. Form the collection of sets generated by the members of $\mathfrak{A}$ under the symmetric difference ($\oplus$) operation. Let $\mathfrak{M}$ denote the set of all minimal nonnull members of this collection. If $B^{\#}$ and B^* are any two distinct members of $\mathfrak{M}$, then $B^{\#}$, B^* and hence $B^{\#} \oplus B^*$ can each be expressed as a symmetric difference of members of $\mathfrak{A}$. Consequently $B^{\#} \oplus B^*$ contains a member of $\mathfrak{M}$ and thus, by (39), $\mathfrak{M}$ is a binary matroid.

Assume that card $(A \cap A')$ is even for all $A \in \mathfrak{A}$ and $A' \in \mathfrak{A}'$. Let $A_1, \cdots, A_{2k+1}$ be members of $\mathfrak{A}$. Since card $(A_i \cap A' - \{e\})$ is odd for $i = 1, \cdots, 2k+1$, it follows that card $((A_1 \oplus \cdots \oplus A_{2k+1}) \cap A' - \{e\})$ is also odd. By the method used in the proof of (20), it can be shown that the collection of minimal subsets A'', each containing e and such that $A'' \cap A' - \{e\} \neq O$ for all $A' \in \mathfrak{A}'$, is precisely $\mathfrak{A}$. Thus $(A_1 \oplus \cdots \oplus A_{2k+1}) \cap A' - \{e\} \neq O$ for all $A' \in \mathfrak{A}$ implies that $A_1 \oplus \cdots \oplus A_{2k+1}$ contains a member of $\mathfrak{A}$. Consequently each member of $\mathfrak{A}$ is a member of $\mathfrak{M}$ and each member of $\mathfrak{M}$ which contains e is a member of $\mathfrak{A}$. (The same construction applied to $\mathfrak{A}'$ yields $\mathfrak{M}'$.)

Thus the condition, card $(A \cap A')$ is even, is sufficient for $\mathfrak{A}$ to yield a binary matroid. By (40) it is also necessary.

COROLLARY.

(45) *The property*
card $(A \cap A')$ *is even for all* $A \in \mathfrak{A}$ *and* $A' \in \mathfrak{A}'$
is equivalent to the property
for any $A_1, \cdots, A_{2k+1} \in \mathfrak{A}$ *there exists* $A \in \mathfrak{A}$ *such that* $A \subset A_1 \oplus \cdots \oplus A_{2k+1}$.

Proof. This result follows from (41) and the proof of (44).

If, in the statement of (44), M is defined to be the union of the members A of $\mathcal{A}$, then the matroid $\mathfrak{M}$ is unique. This is shown in (46) for general (not necessarily binary) matroids.

LEMMA. *Let $\mathfrak{M}$ be a matroid and e be a fixed member of M. If the union of all circuits containing e covers all of the circuits of $\mathfrak{M}$ (i.e., $\bigcup_{e \notin C \in \mathfrak{M}} C \subset \bigcup_{e \in C \in \mathfrak{M}} C$), then*

(46) *the collection of all circuits C which contain e uniquely determines $\mathfrak{M}$.*

Proof. Let $C^\#$ and C^* be any two distinct circuits containing e and let $C_1, \cdots, C_k$ be all of the distinct circuits C_i satisfying $e \in C_i \subset C^\# \cup C^*$. Choose $C \subset C^\# \cup C^* - \{e\}$. For any $a \in C \cap C^\#$ there exists i such that $C_i \subset C \cup C^\# - \{a\}$. Consequently $C \cap C_1 \cap \cdots \cap C_k = O$ and hence $C^\# \cup C^* - C_1 \cap \cdots \cap C_k$ contains the circuit C.

Now let C be any circuit not containing e. By hypothesis there exists $C^\#$ such that $e \in C^\#$ and $C^\# \cap C \neq O$. In addition choose $C^\#$ so that $C^\# \cup C$ is minimal. Then there exists C^* such that $e \in C^* \subset C^\# \cup C$ and $C^* \neq C^\#$. Define $C_1, \cdots, C_k$ as before. By the minimality of $C^\# \cup C$, $C^\# \cup C \subset C_i \cup C$ holds for all i. Thus $(C_1 \cap \cdots \cap C_k) \cup C \supset C^\# \cup C \supset C^\# \cup C^* \cup C$. It follows that C contains and hence equals $C^\# \cup C^* - C_1 \cap \cdots \cap C_k$. Thus the minimal sets of the form $C^\# \cup C^* - C_1 \cap \cdots \cap C_k$ are the circuits of $\mathfrak{M}$ which do not contain e.

If $\mathfrak{M}$ is a binary matroid then $C^\# \oplus C^*$ (where $C^\# \neq C^*$) will contain a circuit and hence, in the proof of (46), $C^\# \cup C^* - C_1 \cap \cdots \cap C_k$ can be replaced by $C^\# \oplus C^*$. The proof of (46) further shows that any circuit which intersects a circuit containing e is contained in the union of circuits containing e. This property is the basis for the decomposition of a matroid into independent components as described by Whitney in [15].

The results (44) and (45) yield (47) and (48), results which characterize those isotone functions $\mathfrak{f}$ associated with binary matroids.

THEOREM. *Let $\mathfrak{f}$ be an isotone function of $x_1, \cdots, x_n$.*

(47) *$\mathfrak{f}$ is associated with some binary matroid with a distinguished branch if and only if each path of $\mathfrak{f}$ and each cut of $\mathfrak{f}$ intersect in an odd number of indices.*

Proof. Define $M = \{1, \cdots, n, e\}$ and $\mathcal{A}$ to be the collection of all minimal nonnull subsets A of M each containing e and such that $x_k = \theta$ for all $k \in A - \{e\}$ implies $\mathfrak{f}(x_i, \cdots, x_n) \equiv \theta$. From $\mathcal{A}$ construct $\mathfrak{M}$ as in the proof of (44). Then $\mathfrak{f}$ is associated with the binary matroid with distinguished branch e if and only if card $(A \cap A')$ is even for all $A \in \mathcal{A}$ and

$A' \in \mathfrak{A}'$ ($\mathfrak{A}'$ being defined as in (44)) and this is the case if and only if the paths and cuts of $\mathfrak{f}$ intersect in an odd number of branches.

COROLLARY.

(48) *$\mathfrak{f}$ is associated with some binary matroid with a distinguished branch if and only if each symmetric difference of an odd number of paths of $\mathfrak{f}$ contains a path.*

Proof. (48) follows from (45) and (47).

The previously established criteria for cut, short, and neutral games can be formulated in terms of the paths and cuts of any isotone function associated with a binary matroid. The resulting criteria, given in (48), are analogous to those, given in (32), using the concept of cospanning trees.

THEOREM. *Let $K_1, \cdots, K_l$ and $P_1, \cdots, P_m$ be respectively the collection of cuts and paths of a finite isotone function $\mathfrak{f}$. Assume further that each cut and path intersects in an odd number of indices. Then:*

(49) *$\mathfrak{f}$ has property* (i) *if and only if there exist A, B (collections of indices) such that $A \cap B = O$ and for each i, j ($i, j = 1, \cdots, l$),*
$A \cap K_i \neq O$, $B \cap K_i \neq O$, and
$A \cap (K_i \oplus K_j) \neq O$ if and only if $B \cap (K_i \oplus K_j) \neq O$.

$\mathfrak{f}$ has property (ii) *if and only if there exist A, B such that $A \cap B = O$ and for each $i, j (i, j = 1, \cdots, m)$, $A \cap P_i \neq O$, $B \cap P_i \neq O$, and $A \cap (P_i \oplus P_j) \neq O$ if and only if $B \cap (P_i \oplus P_j) \neq O$.*

$\mathfrak{f}$ has property (iii) *if and only if there exist $A^{\#}$, $B^{\#}$, A^*, B^* such that $A^{\#} \cap B^{\#} = O$, $A^* \cap B^* = O$, and for each i, j,*
$A^{\#} \cap K_i \neq O$, $A^* \cap P_i \neq O$,
$A^{\#} \cap (K_i \oplus K_j) \neq O$ if and only if $B^{\#} \cap (K_i \oplus K_j) \neq O$, and
$A^* \cap (P_i \oplus P_j) \neq O$ if and only if $B^* \cap (P_i \oplus P_j) \neq O$.

Proof. Consider property (ii). By (47), $\mathfrak{f}$ is associated with some binary matroid $\mathfrak{M}$ with a distinguished branch. By (41), $P_i \oplus P_j$ is a union of circuits of $\mathfrak{M}$. Thus the condition given in (49) is a restatement of that given in (26).

Consider property (i). $\mathfrak{f}$ has property (i) if and only if the dual of $\mathfrak{f}$ has property (ii), and the cuts of $\mathfrak{f}$ are the paths of the dual of $\mathfrak{f}$. Hence the given conditions for property (i) are a consequence of the conditions for property (ii).

Consider property (iii). An additional fixed index added to each cut (or path) transforms a game with property (iii) into a game with property (i) (or (ii)), a play on the additional index being a forced first play by the ϕ (or θ) player. Hence the necessity of the given conditions is a consequence of the conditions for (i) and (ii). The conditions are also sufficient since they enable the first player to win.

The requirement in (49) that the paths and cuts intersect in an odd number of indices can be omitted if it is assumed that $\mathfrak{f}$ is associated with a matroid $\mathfrak{M}$ with a distinguished branch. Since the matroid need not be binary it is also necessary to replace each occurrence of "$K_i \oplus K_j$" by "$K_i \cup K_j - \bigcap_{K_k \subset K_i \cup K_j} K_k$" and each occurrence of "$P_i \oplus P_j$" by "$P_i \cup P_j - \bigcap_{P_k \subset P_i \cup P_j} P_k$". In the manner of the first part of the proof of (46), it is easily shown that the substituted expressions are unions of circuits of $\mathfrak{M}'$ and $\mathfrak{M}$ respectively.

6.3. Hex. Suppose that a finite isotone function $\mathfrak{f}$ is self-dual in the sense that the dual of $\mathfrak{f}$ can be obtained from $\mathfrak{f}$ by a permutation of indices. Then any winning strategy for the game on $\mathfrak{f}$, when permuted, yields a winning strategy for the game on the dual of $\mathfrak{f}$ and hence a winning strategy for the opposing player on the game on $\mathfrak{f}$. It follows that both $\mathfrak{f}$ and its dual must have property (iii), that is, the game can be won by the first player. If $\mathfrak{f}$ is self-dual without permutation, then the same winning strategy can be used by either player playing first. Such functions are very uncommon.

A class of self-dual functions can be derived from n by n hex (hexagon) configurations. In particular consider the function $\mathfrak{f}$ derived from the 3 by 3 hex configuration of Fig. 16. $\mathfrak{f}$ is defined by $\mathfrak{f}(x_1, \cdots, x_n) = \theta$ if

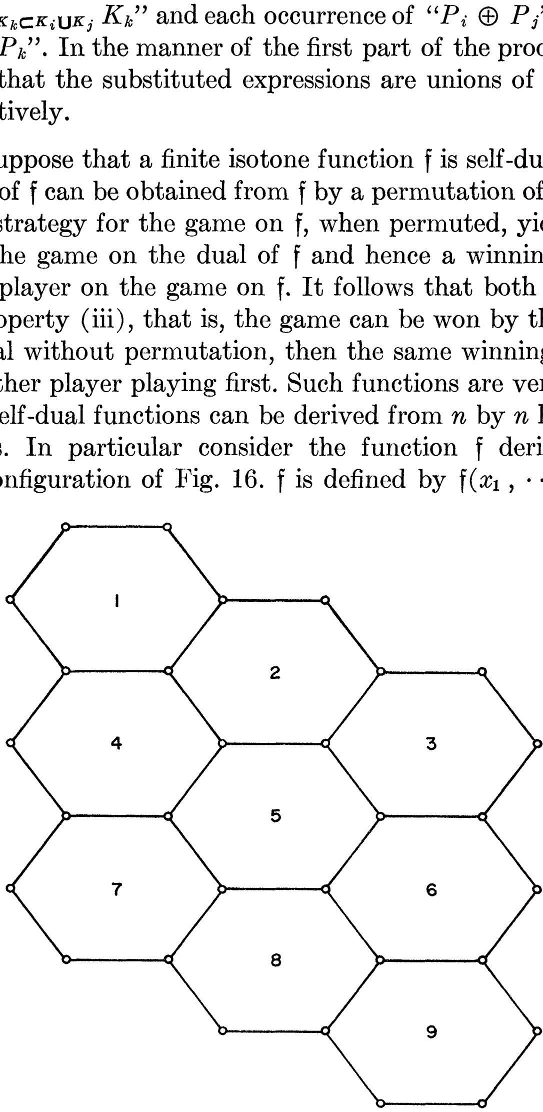


FIG. 16. 3 *by* 3 *Hex configuration*

and only if the indices k for which $x_k = \theta$ correspond to a set of hexagons which connect the right and left sides of the configuration. The dual of $\mathfrak{f}$ can be obtained from $\mathfrak{f}$ by interchanging the indices 2 and 4, 3 and 7, and 6 and 8. It can also be derived from the definition of $\mathfrak{f}$ by changing "right" and "left sides" to "top" and "bottom". Thus the indices 1, 2, 5, 6 constitute a path of $\mathfrak{f}$ while the indices 2, 5, 8 constitute a cut. Since the path and cut have an even number of indices in common, $\mathfrak{f}$ is not associated with a binary matroid. In fact it can easily be shown, by the circuit-generating technique used in the proof of (46), that $\mathfrak{f}$ is not associated with any matroid. These comments also apply to larger n by n hex configurations.

The game of "Nash" or "Hex", mentioned in [7] and described in [2], corresponds to the Shannon game on $\mathfrak{f}$ where $\mathfrak{f}$ is derived from an appropriate n by n hex configuration. Since $\mathfrak{f}$ is self-dual the game can be won by the first player. However except for small values of n no usable winning strategy is known.

It is natural to ask whether there exists a generalization of the concept of cospanning sets which would extend the short game criterion of (14) to any finite isotone function $\mathfrak{f}$. As an initial step, this criterion has been stated, in (49), in terms of cuts. If it were not for the fact that in Hex $\mathfrak{f}$ is not associated with a matroid, the set $A^{\#} = A^*$ of the odd numbered hexagons of Fig. 16 and the set $B^{\#} = B^*$ of the even numbered hexagons would satisfy the conditions of (49) for property (iii) and provide a strategy for Hex.

Thus far no generalization of (49) to an arbitrary isotone function $\mathfrak{f}$ has been found. Nor has any attempt to generalize (9) succeeded. If $\mathfrak{f}$ is not associated with a matroid then the generalized span must be with respect to the counterpart of a fixed distinguished branch and no counterpart of (16) is possible.

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