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ON CRITICALLY h-CONNECTED SIMPLE GRAPHS

Yaha Ould HAMIDOUNE

Université Pierre et Marie Curie, U.E.R. 48, E.R. Combinatoire, 4 Place Jussieu 75230 Paris, France

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Chartrand, Kaugars and Lick proved that every critically h-connected graph contains a vertex of degree not exceeding $\frac{3}{2}h - 1$. We prove here that there are two such vertices. Moreover for every h ($h \geq 3$) there is a critically h-connected graph having exactly two vertices of degree not exceeding $\frac{3}{2}h-1$.

1. Introduction

We use terminology and notations of Berge [1]. A graph $G=(V, E)$ is *h-connected* if for every $A \subset V$ with $|A|<h$, G_{V-A} is connected (where G_{V-A} is the subgraph of G induced by $V-A$). The *connectivity* of G is

$$\kappa(G) = \mathrm{Max}(h\colon G \text{ is } h\text{-connected}).$$

The graph G is *critically* h-connected if $\kappa(G)=h$ and for every $x \in V$, $\kappa(G_{V-x}) = h-1$.

Let $G=(V, E)$ be a graph and $A \subset V$. We use the following notations.

$$N(A) = \Gamma(A) - A; \qquad \tilde{A} = V-(A \cup N(A)).$$

Suppose G is connected and noncomplete. The set A is a *cut-set* of G if G_{V-A} is not connected. A cut-set of G with cardinality $\kappa(G)$ is called a *minimum cut-set* of G. The set A is said to be a fragment of G if $\tilde{A} \neq \emptyset$ and $|N(A)| = \kappa(G)$ (i.e. if $N(A)$ is a minimum cut-set of G). Observe that A is a fragment of G if and only if $\tilde{A}$ is so. Let A be a fragment and let $N(A)=T$. We will say that A is a fragment with respect to T. A fragment containing no other fragment as a proper subset is called an *end*. A fragment with minimal cardinality is called an *atom*. A fundamental property of atoms is the following.

Theorem A (Mader [6]). *Let G be a connected graph, A be an atom of G and T be a minimum cut-set of G. If $A \cap T \neq \emptyset$, then $A \subset T$ and $|A| \leq \frac{1}{2}\kappa(G)$.*

A generalization of this theorem to the case of digraphs is proved in [4].

Chartrand, Kaugars and Lick proved in [2] that a critically h-connected graph contains at least a vertex of degree not exceeding $\frac{3}{2}h-1$. A short proof for this result is obtained by Mader in [5], using Theorem A. Nebesky proved in [7] that a critically 2-connected graph of order $\geqslant 6$ contains at least four vertices of degree 2. Entringer and Slater proved in [3] that every critically 3-connected graph contains at least two vertices of degree 3. The above two results generalize the result of Chartrand, Kaugars and Lick for $h=2$ and $h=3$ respectively. We prove in this paper that a critically h-connected graph contains at least two vertices of degree not exceeding $\frac{3}{2}h-1$. We construct a critically h-connected graph with exactly two vertices of degree not exceeding $\frac{3}{2}h-1$ for $h\geqslant 3$.

Our method uses, among many properties of fragments, Mader Theorem A and the following lemma.

Lemma B (Mader [5]). *Let A and B be two fragments of a connected graph G such that $A\cap B\neq\emptyset$. Then $|\bar{B}\cap N(A)|\leqslant|A\cap N(B)|$. Moreover if A is an end not contained in B, then the above inequality is strict.*

We note that Lemma B is only implicit in [5]. A generalization of this lemma to the directed case is proved in [4].

2. Some properties of fragments

We prove in this section some properties of fragments of graphs. We use some methods related to those of our paper [4] and those used by Mader [5, 6].

Lemma 2.1. *Let $G=(V,E)$ be a graph, A and B be two subsets of V such that $N(A)=N(\bar{A})$ and $N(B)=N(\bar{B})$. Then*

$$|N(A\cap B)|+|N(\bar{A}\cap\bar{B})|\leqslant|N(A)|+|N(B)|.$$

Proof. We verify easily that

$$\begin{aligned} N(A\cap B)&\subset((A\cup N(A))\cap(B\cup N(B))-(A\cap B)\\ &=(N(A)-\bar{B})\cup(A\cap N(B)). \end{aligned}$$

Similarly,

$$N(\bar{A}\cap\bar{B})\subset(N(\bar{A})\cap\bar{B})\cup(N(\bar{B})-A).$$

The lemma follows from the above two relations easily.

Lemma 2.2. *Let A be an end of a noncomplete graph $G=(V,E)$ and F be a fragment of G not containing A. Then $A\cap F=\emptyset$ or $\bar{A}\cap\bar{F}=\emptyset$.*

Proof. Suppose $A\cap F\neq\emptyset$. Then $|N(A\cap F)|>\kappa(G)$, otherwise $A\cap F$ would be a fragment of G (observe that $\overline{A\cap F}\supset\bar{A}\neq\emptyset$), which is a contradiction since A is an end and $A \cap F$ is a proper subset of A. By Lemma 2.1, we have

$$|N(A \cap F)| + |N(\bar{A} \cap \bar{F})| \leq |N(A)| + |N(F)| = 2\kappa(G).$$

Using the above two relations, we have $|N(\bar{A} \cap \bar{F})| < \kappa(G)$. But $\overline{\bar{A} \cap \bar{F}} \supset \bar{A} \neq \emptyset$. Thus $\bar{A} \cap \bar{F} = \emptyset$ (observe that $\kappa(G) = \mathrm{Min}\,(|N(A)| : A \neq \emptyset$ and $\bar{A} \neq \emptyset))$.

We say that a fragment A is a *proper* fragment if $|A| \leq |\bar{A}|$. The proper fragments have the following property.

Proposition 2.3. *Let $G = (V, E)$ be a connected noncomplete graph, A be a proper end of G and F be a proper fragment of G not containing A. Then $A \cap F = \emptyset$.*

Proof. Suppose $A \cap F \neq \emptyset$. By Lemma B, we have $|\bar{F} \cap N(A)| \leq |A \cap N(F)|$ and $|\bar{A} \cap N(F)| \leq |F \cap N(A)|$. Using Lemma 2.2, we have $\bar{A} \cap \bar{F} = \emptyset$. It follows that

$$|\bar{F}| = |\bar{F} \cap A| + |\bar{F} \cap N(A)| \leq |\bar{F} \cap A| + |A \cap N(F)| = |A - F| < |A|$$

(observe that $A \cup N(A) \cup \bar{A}$ is a partition of V).

Similarly $|\bar{A}| < |F|$. By addition we have $|\bar{A}| + |\bar{F}| < |A| + |F|$, which is a contradiction since $|A| \leq |\bar{A}|$ and $|F| \leq |\bar{F}|$.

Lemma 2.4. *Let A and F be two fragments of a noncomplete graph $G = (V, E)$ such that $A \subset N(F)$. If $\bar{A} \cap F \neq \emptyset$ and $\bar{A} \cap \bar{F} \neq \emptyset$, then $|A| \leq \frac{1}{2}\kappa(G)$.*

Proof. By Lemma B, applied to $\bar{A}$ and $\bar{F}$, we have $|A| = |A \cap N(F)| \leq |\bar{F} \cap N(A)|$. Similarly $|A| = |A \cap N(F)| \leq |F \cap N(A)|$. By addition we have

$$2|A| \leq |\bar{F} \cap N(A)| + |F \cap N(A)| \leq |N(A)| = \kappa(G).$$

Lemma 2.5. *Let $G = (V, E)$ be a connected noncomplete graph, A be a fragment of G and T be a minimum cut-set of G containing A. If $|A| > \frac{1}{2}\kappa(G)$, then there is a fragment B with respect to T such that $|B| \leq \frac{1}{2}\kappa(G)$.*

Proof. Let C be a fragment with respect to T. Using Lemma 2.4, we have $C \cap \bar{A} = \emptyset$ or $\bar{C} \cap \bar{A} = \emptyset$. We assume, without loss of generality, $C \cap \bar{A} = \emptyset$. Therefore $C \subset N(A)$. If $\bar{C} \cap \bar{A} = \emptyset$, then $\bar{C} \subset N(A)$. Hence $C \cup \bar{C} \subset N(A)$. It follows that $|C| \leq \frac{1}{2}\kappa(G)$ or $|\bar{C}| \leq \frac{1}{2}\kappa(G)$. Suppose $\bar{C} \cap \bar{A} \neq \emptyset$. By Lemma B, applied to $\bar{C}$ and $\bar{A}$, we have

$$\begin{aligned} |A| &= |A \cap T| = |A \cap N(C)| \leq |\bar{C} \cap N(A)| \\ &\leq |N(A) - C| = |N(A)| - |C| = \kappa(G) - |C|. \end{aligned}$$

It follows that $|C| < \frac{1}{2}\kappa(G)$.

3. Critically h-connected graphs

We verify easily that a graph G is critically h-connected if and only if $\kappa(G)=h$ and the family of minimum cut-sets covers the vertex set of G. A graph $G=(V,E)$ is said to be a *maximal* critically h-connected graph if G is critically h-connected and if the graph obtained by addition of any edge not belonging to E is not critically h-connected. The following lemma implies that every critically h-connected graph is a spanning subgraph of a maximal critically h-connected graph.

Lemma 3.1. *Let G be a critically h-connected graph and G' be the graph obtained by addition of one edge to G. Then $\kappa(G')=h$.*

Proof. We have clearly $\kappa(G')\geqslant\kappa(G)=h$. Let T be a minimum cut-set of G containing a vertex incident to the added edge. We see easily that T is a cut-set of G'. Therefore $\kappa(G')\leqslant|T|=h$.

Lemma 3.2. *Let $G=(V,E)$ be a maximal critically h-connected graph, x and y be two nonadjacent vertices of G. Then there is a fragment F of G such that $x\in F$ and $y\in\bar{F}$.*

Proof. By definition the graph $G'=(V,E\cup\{\{x,y\}\})$ is not critically h-connected. We see, using Lemma 3.1, that there is a vertex $z\in V$ which belongs to no minimum cut-set of G'. Let T be a minimum cut-set of G containing z. We have $T\cap\{x,y\}=\emptyset$, otherwise T would be a cut-set of G'. Let F be a fragment of G such that $x\in F$ and $N(F)=T$. Then $y\in\bar{F}$, otherwise T would a cut-set of G'.

Lemma 3.3. *Let $G=(V,E)$ be a maximal critically h-connected graph, and B be an end of G. Then G_B is a complete graph or $|\bar{B}|\leqslant\frac{1}{2}h$.*

Proof. Suppose G_B not complete. By Lemma 3.2, there is a fragment F such that $F\cap B\neq\emptyset$ and $\bar{F}\cap B\neq\emptyset$. By Lemma 2.2, we have $\bar{F}\cap\bar{B}=\emptyset$ and $F\cap\bar{B}=\emptyset$. Thus $\bar{B}\subset N(F)$. Using Lemma 2.4, we have $|\bar{B}|\leqslant\frac{1}{2}\kappa(G)=\frac{1}{2}h$.

Proposition 3.4. *Let $G=(V,E)$* be a maximal critically *h-connected graph and A be an atom of G. Then there is a fragment F of G such that $F\neq A$ and $|F|\leqslant\frac{1}{2}\kappa(G)$.*

Proof. Suppose the contrary. We prove the following.

(1) There is an end B such that $B\not\subset N(A)$, $B\neq A$ and $B\neq\bar{A}$.

Consider a minimum cut-set T of G such that $A\cap T\neq\emptyset$. By Mader Theorem A, $A\subset T$. Let C be a fragment with respect to T, B_1 and B_2 be two ends contained in C and $\bar{C}$ respectively. We have $B_1\not\subset N(A)$ or $B_2\not\subset N(A)$, since

$|B_1|>\frac{1}{2}\kappa(G)$ and $|B_2|>\frac{1}{2}\kappa(G)$. We may assume without loss of generality $B=B_1$. From the above construction, we see that $B\neq A$ and $\bar{B}\neq A$.

(2) G_B is a complete graph.

By the last remark of (1) and Lemma 3.3, G_B is complete otherwise G would contain a fragment distinct from A with cardinality $\leq\frac{1}{2}\kappa(G)$.

Let $(T_i)_{1\leq i\leq k}$ be a minimum covering of B with minimum cut-sets.

(3) $k\geq 2$.

Suppose $B\subset T_1$. By Lemma 2.5 (observe that $|B|>\frac{1}{2}\kappa(G)$), there is a fragment A_1 with respect to T_1 such that $|A_1|\leq\frac{1}{2}\kappa(G)$. This relation contradicts our hypothesis since $A_1\neq A$ (observe that $B\subset N(A_1)$ and $B\not\subset N(A)$ by (1)).

(4) Let A_i be a fragment with respect to T_i such that $\bar{A}_i\cap B=\emptyset$: such a fragment exists since G_B is a complete graph, $i=1,\ldots,k$. Thus $B\subset(A_i\cup T_i)$, $1\leq i\leq k$. Hence $B\cap A_i\neq\emptyset$, otherwise $B\subset T_i$ contradicting the minimality of the covering. We have also $B\not\subset A_i$, otherwise $(T_j)_{j\neq i}$ would be a covering of B. It follows, using Lemma 2.2, that $\bar{B}\cap\bar{A}_i=\emptyset$, $i=1,\ldots,k$. Hence $\bar{A}_i\subset N(B)$. There is p, $1\leq p\leq k$, such that $T_p\neq N(A)$ (observe that $T_i\neq T_j$ for $i\neq j$, by the minimality of the covering). By our hypothesis $|\bar{A}_p|>\frac{1}{2}\kappa(G)$. By Lemma 2.2, there is a fragment with respect to $N(B)$ having a cardinality $\leq\frac{1}{2}\kappa(G)$. Therefore $|B|\leq\frac{1}{2}\kappa(G)$ or $|\bar{B}|\leq\frac{1}{2}\kappa(G)$ (since B is an end), a contradiction. This contradiction proves the proposition.

Theorem 3.5. *Let G be a critically h-connected graph. Then G contains two vertices of degree not exceeding $\frac{3}{2}h-1$.*

Proof. It is sufficient to prove the theorem in the case of a maximal critically h-connected graph. Let A be an atom of G. By Proposition 3, there is a fragment $F\neq A$ such that $|F|\leq\frac{1}{2}\kappa(G)$. Let $x\in A$ and $y\in F-A$. We have $d(y)\leq |F|-1+|N(F)|\leq\frac{3}{2}h-1$. Similarly we see that $d(x)\leq\frac{3}{2}h-1$.

Let X and Y be two disjoint sets such that $|X|=|Y|=h-1$ ($h\leq 3$), x and y be two distinct vertices such that $\{x,y\}\cap(X\cup Y)=\emptyset$. Consider a complete graph on $X\cup Y$. We construct a graph G by adding to the above complete graph the vertices x and y such that

$$N(x)=X\cup\{y\};\qquad N(y)=Y\cup\{x\}.$$

The graph G is critically h-connected, since its vertex-set can be covered by two minimum cut-sets ($X\cup\{y\}$ and $Y\cup\{x\}$). The only vertices of G with degree not exceeding $\frac{3}{2}h-1$ are x and y.

Corollary 3.6 (Entringer and Slater [3]). *In any critically 3-connected graph there are at least two vertices of degree 3.*

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