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A note on biased and non-biased games[☆]

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Abstract

In this paper, we consider two spanning tree games played on the complete graph of order n. We also consider the connection between biased games and non-biased games.

1. Introduction

Let X be a finite set and $\mathscr{H} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of X. We say that the pair $(\mathscr{H}, X)$ is a *hypergraph*. Each $A \in \mathscr{H}$ is called an *edge* of $\mathscr{H}$, and X is the *vertex set* of $\mathscr{H}$. We assume that $X = \bigcup_{i=1}^{m} A_i$.

Given such X and $\mathscr{H}$ as above, we can define several games on X as follows:

(1) $[r, s; t]$-*game*: Two players, maker and breaker, alternately take previously untaken vertices of X, with the breaker going first, such that the breaker takes r vertices per move and the maker takes s vertices per move. The game continues until all the vertices of X have been taken. If the breaker (or maker) is the last player and the remaining vertices in X are fewer than r (or s), then he takes all of them. The maker's goal is to take a subset of X containing at least t pairwise disjoint edges of $\mathscr{H}$. The breaker's goal, on the other hand, is to prevent the maker from achieving his goal. The player who achieves his goal is the winner. If the maker has a winning strategy for the $[r, s; t]$-game, then we say that $\mathscr{H}$ is $[r, s; t]$-*achievable*, or $\mathscr{H}$ is $[r, s; t]$, for short.

(2) $[r, s; t]$-*avoidance game*: This is a counterpart of the $[r, s; t]$-game. Two players, antimaker and antibreaker, alternately take previously untaken vertices of X, with the antibreaker going first, such that the antibreaker takes r vertices per move and the antimaker takes s vertices per move. The game continues until all the vertices of X have been taken. If the antibreaker (or antimaker) is the last player and the

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remaining vertices in X are fewer than r (or s), then he takes all of them. The antimaker's goal here is to avoid achieving a subset of X containing t pairwise disjoint edges of $\mathscr{H}$ and the antibreaker's goal is to force the antimaker to achieve such a subset. If the antibreaker has a winning strategy for the $[r, s; t]$-avoidance game, then we say that $\mathscr{H}$ is $[r, s; t]$-*unavoidable.*

Remark 1. Of course, the $[r, s; t]$ and $[r, s; t]$-avoidance games on $\mathscr{H}$ are equivalent to the corresponding $[r, s; 1]$-games on the hypergraph $(\mathscr{H}', X)$, whose edges are all unions of t pairwise disjoint edges of $\mathscr{H}$. Nonetheless, we consider the present usage is more natural.

Remark 2. It is easy to see (as in the "strategy-stealing" argument) that if $\mathscr{H}$ is $[r, s; t]$-achievable, then the maker also wins if we modify the rules to allow the maker to take at most r but not 0 vertices per move, and the breaker at most s but not 0 vertices per move.

(3) $[r, s; \varepsilon, \mathscr{H}]$-*game*: Given $\frac{1}{2} > \varepsilon > 0$, two players, **I** and **II**, alternately take previously untaken vertices of X, with **I** playing first such that he takes r vertices per move and **II** takes s vertices per move. The game continues until all the vertices of X have been taken. If **I** (or **II**) is the last player and the remaining vertices in X are fewer than r (or s), then he takes all of them. We say **I** wins the game if there exists an $A \in \mathscr{H}$ such that **I** has taken at least $(1 - \varepsilon)|A|$ vertices of A. Otherwise, he is the loser. (Note that here the "maker" is player **I**.) If $r = s = 1$, then we just write $[\varepsilon, \mathscr{H}]$, for short.

(4) $[r, s; \varepsilon, \mathscr{H}]$-*avoidance game*: This is the counterpart of (3). In this game, **I** again plays first and his goal is to take fewer than $(1 - \varepsilon)|A|$ vertices A for each $A \in \mathscr{H}$. Once again, we call this $[\varepsilon, \mathscr{H}]$-avoidance game if $r = s = 1$.

In the $[r, s; t]$-game, if $r = s$, the game is called an *unbiased* game, otherwise, it is called *biased*. The concept of biased games was introduced by Chvátal and Erdős [6].

Two general problems of interest are:

(1) For a given $\mathscr{H}$, find (estimate) the maximum t for which $\mathscr{H}$ is $[1, 1; t]$.

(2) For a given $\mathscr{H}$, find (estimate) the maximum r for which $\mathscr{H}$ is $[r, 1; 1]$.

One can also raise similar problems for the avoidance games. We shall consider a special case of problem (1) in Section 2, and here we give an answer to the following problem.

(3) Find the maximum value $f(r)$ such that if $\mathscr{H}$ is $[r, 1; 1]$, then $\mathscr{H}$ is $[1, 1; f(r)]$.

Theorem 1. $f(r) = \lceil r/2 \rceil$, *where* $\lceil x \rceil$ *denotes the least integer not less than* x.

Proof. We prove $f(r) \geqslant \lceil r/2 \rceil$ first. In other words, we shall prove the following statement: if $\mathscr{H}$ is $[r, 1; 1]$, then it is $[1, 1; \lceil r/2 \rceil]$.

By Remark 2 above, we only need to consider the case $r = 2k - 1$ for some positive integer k; that is, we must show that if $\mathscr{H}$ is $[2k - 1, 1; 1]$, then it is also $[1, 1; k]$.

We now consider the $[1,1;k]$-game, and denote the maker's $[k(i-1)+1]$th through (ki)th moves by $y_i^{(1)},\dots,y_i^{(k)}$, the breaker's moves are denoted simply by $x_1,x_2,\dots$.

The maker's strategy is just this: for the move $y_i^{(j)}$ he pretends (again using Remark 2) that he is playing the "modified" $[2k-1,1;1]$-game G_j in which his opponent may take $2k-1$ or fewer but not 0 vertices per move.

He regards his opponent's previous moves in this imagined game as being $x_1,\dots,x_{(i-1)k+j},y_1^{(1)},\dots,y_1^{(j-1)},y_1^{(j+1)},\dots,y_1^{(k)},y_2^{(1)},\dots,y_2^{(j-1)},y_2^{(j+1)},\dots,y_2^{(k)},\dots,y_i^{(1)},\dots,y_i^{(j-1)}$, while his own are $y_1^{(j)},y_2^{(j)},\dots,y_{i-1}^{(j)}$, and he chooses his ith move in g_j, which is $y_i^{(j)}$ in the real $[1,1;k]$-game, accordingly. Thus $y_1^{(j)},y_2^{(j)},\dots,$ is a sequence of moves chosen according to a winning strategy for the maker in some $[2k-1,1;1]$-game G_j on $\mathscr{H}$, and so $\{y_1^{(j)},y_2^{(j)},\dots\}$ contains an edge of $\mathscr{H}$.

On the other hand, we give an example to show that $f(r)=\lceil r/2\rceil$. Let $\mathscr{H}$ be the family of $r+1$ singletons. Then $\mathscr{H}$ is $[r,1;1]$. For this $\mathscr{H}$, it is easy to see that $f(r)=\lceil r/2\rceil$. $\square$

Remark 3. If we instead defined $[r,s;t]$-games to have the maker going first, then the quantity corresponding to $f(r)$ would be 1 for all r, as is easily seen by taking $\mathscr{H}$ to consist of all sets containing some fixed element of X.

The following theorem is very useful, although we do not need it in this paper.

Theorem 2 [4]. *If* $\frac{1}{2}>\varepsilon>0$ *and*

$$\sum_{A\in\mathscr{H}}(2(1-\varepsilon)^{1-\varepsilon}\varepsilon^{\varepsilon})^{-|A|}<\frac{1}{2(1-\varepsilon)},$$

then **II** *has a winning strategy for the* $[\varepsilon,\mathscr{H}]$*-game.*

For the $[\varepsilon,\mathscr{H}]$-avoidance game, we have a similar result.

Theorem 3 (Lu [9,10]). *If* $\frac{1}{2}>\varepsilon>0$, $|X|$ *is even, and*

$$\sum_{A\in\mathscr{H}}(2(1-\varepsilon)^{1-\varepsilon}\varepsilon^{\varepsilon})^{-|A|}<1,$$

then **I** *has a winning strategy for the* $[\varepsilon,\mathscr{H}]$*-avoidance game.*

2. Spanning tree games

We discuss two spanning tree games played on K_n, the complete graph of order n. For undefined graph terminology, see, for example, [5]. For more games of this type, see [1–4,6,9–11].

Let $\mathscr{F} = \{F_1, F_2, \dots\}$ be a family of graphs, and G be a graph of order n. Let $\mathscr{F}_G$ be the family of all those subgraphs G^* of G such that G^* is isomorphic to some $F \in \mathscr{F}$. Let $X = E(G)$ be our game board. A winning set is an edge set $A \subseteq E$ such that $G(A) \in \mathscr{F}_G$, where $G(A)$ is the subgraph of G induced by A. If we let $\mathscr{H}(G, \mathscr{F})$ be the family of all such winning sets A, then we have the corresponding $[r, s; t]$-achievement game and $[r, s; t]$-avoidance game. We say $\mathscr{F}$ is $[r, s; t]$ on G if $\mathscr{H}(G, \mathscr{F})$ is $[r, s; t]$. Similarly, we say that $\mathscr{F}$ is $[r, s; t]$-*unavoidable* on G if $\mathscr{H}(G, \mathscr{F})$ is.

Let $V = V(K_n)$ be the vertex set of K_n, and $\mathscr{T}_n$ be the set of all spanning trees of K_n. For $A \subseteq V$, let $\bar{A} = V - A$, and $[A, \bar{A}] = \{xy \mid x \in A, y \in \bar{A}\}$.

2.1. The achievement game

Theorem 4. *$\mathscr{T}_n$ is $[1, 1; \lceil n/4 \rceil]$ on K_n.*

To prove Theorem 4, we need the following two well-known results. The first one is due to Lovász [8].

Theorem 5. *A graph of order n can be covered by $\lceil n/2 \rceil$ edge-disjoint paths and cycles.*

Corollary 1. *K_{2k} can be covered by k edge-disjoint Hamiltonian paths and K_{2k+1} by k Hamilton cycles.*

Corollary 2. *K_n has $\lceil n/2 \rceil$ edge-disjoint Hamiltonian paths.*

The next result is due to Lehman [7]. Let $G = (V, E)$ be a multigraph, $\mathscr{T}_G$ be the set of all spanning trees of G. Lehman proved the following theorem.

Theorem 6. *$\mathscr{T}_G$ is $[1, 1; 1]$ if and only if G has two edge-disjoint spanning trees.*

Proof of Theorem 4. By Corollary 2, K_n has $\lceil n/2 \rceil$ edge-disjoint Hamiltonian paths, which are, in particular, spanning trees of K_n. Let $P_1, P_2, \dots, P_{[n/2]}$ be such a family of $[n/2]$ paths. Let $G_i = P_{2i-1} \cup P_{2i}$ for $i = 1, 2, \dots, [n/4]$. Then by Theorem 6, each $\mathscr{T}_{G_i}$ is $[1, 1; 1]$ since G_i has two edge-disjoint spanning trees. In other words, the maker can obtain a spanning tree on each G_i, thus $[n/4]$ edge-disjoint spanning trees in total. Therefore, the maker can use the following explicit winning strategy. Before the game, he finds $[n/4]$ edge-disjoint subgraphs $G_1, \dots, G_{[n/4]}$. Now, the original game is divided into $[n/4]$ sub-games. In each sub-game, he has a winning strategy as in [7], and he may continue these as above to win the full game. Thus $\mathscr{T}_n$ is $[1, 1; [n/4]]$. □

This result is the best possible since it is impossible to obtain $[n/4] + 1$ edge-disjoint spanning trees, and it is the first nontrivial hypergraph for which we know the exact maximum t for which the hypergraph is $[1, 1; t]$.

2.2. The avoidance game

In this subsection, we consider the avoidance game for the spanning tree problem. We are going to show the following result.

Theorem 7. *Given any $1 > \eta > 0$, there exists an $N = N(\eta)$, such that if $n \geqslant N$, then $\mathscr{T}_n$ is $[1, 1; [\frac{1}{4}(1 - \eta)n]]$-unavoidable on K_n.*

To prove this theorem, we need a result from graph theory.

Let $G = (V, E)$ be a multigraph and let P be a partition of V into p nonempty subsets $V_1, \ldots, V_p$. Let $e(G) = |E|$ for any graph G. Put $|P| = p$ and denote by $G \mid P$ the multigraph of order p obtained from G by contracting each V_i into a vertex v_i. (We delete all those edges with both ends in V_i for each i and keep all other edges. Thus $G \mid P$ is loopless.) We use t_i to denote $|V_i|$. If G contains k edge-disjoint spanning trees, then clearly

$$e(G \mid P) \geqslant k(p - 1) = k(|P| - 1).$$

Tutte and Nash-Williams proved the converse is also true.

Theorem 8 (Nash-Williams [12] and Tutte [13]). *Let G be a loopless multigraph such that $e(G \mid P) \geqslant k(|P| - 1)$ for every partition P of V. Then G contains k edge-disjoint spanning trees.*

Let $V = V(K_n)$ and

$$\mathscr{H} = \{[A, \bar{A}] \mid \emptyset \neq A \subset V\}.$$

Lemma 1. *Given $\frac{1}{2} > \varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that, if $n \geqslant N$, then I has a winning strategy for the $[\varepsilon, \mathscr{H}]$-avoidance game.*

Proof. By Theorem 3, we only need to show that

$$I_n = \sum_{k=1}^{n-1} \binom{n}{k} c^{-k(n-k)} \to 0,$$

where

$$c = 2(1 - \varepsilon)^{(1-\varepsilon)} \varepsilon^{\varepsilon} > 1,$$

and this can be verified quite easily. □

Remark 4. In the proof of Lemma 1, we have not mentioned the parity of $|X| = \binom{n}{2}$. It is easy to see that this does not invalidate the proof since $I_n \to 0$. For a detailed remark, see [11, Remark 9, p. 42].

Proof of Theorem 7. Let $\varepsilon = \frac{1}{2}(1-\eta)$. We assume $n \geqslant N$, where N is defined in Lemma 1. By Lemma 1, the antibreaker as the first player has a winning strategy for the $[\varepsilon, \mathscr{H}]$-avoidance game. We claim that this is also a winning strategy for the $[1,1;[(\varepsilon/2)n]]$-avoidance game for $\mathscr{T}_n$. Let $V_1 \cup \cdots \cup V_k = V$ be a partition P of $V = V(K_n)$. We want to show that (M is the antimaker's final graph, the subgraph of K_n induced by all edges taken by the antimaker)

$$e(M \mid P) \geqslant \left[\frac{\varepsilon}{2}(k-1)n\right].$$

We may assume that $k \geqslant 2$. In the graph $M \mid P$, we have $d(v_i) \geqslant \varepsilon t_i(n-t_i)$, by Lemma 1, where $d(v_i)$ denotes the degree of v_i in $M \mid P$. So

$$\begin{aligned} e(M \mid P) &= \frac{1}{2}\sum_{i=1}^{k} d(v_i) \\ &\geqslant \frac{\varepsilon}{2}\sum_{i=1}^{k} t_i(n-t_i) \\ &= \frac{\varepsilon}{2}\left(n^2 - \sum_{i=1}^{k} t_i^2\right). \end{aligned}$$

If k is fixed, let $f(t_1,\ldots,t_k) = n^2 - \sum_{i=1}^{k} t_i^2$. Subject to $\sum_{i=1}^{k} t_i = n$, then

$$\begin{aligned} f(t_1,\ldots,t_k) &\geqslant f(1,\ldots,1,n-k+1) \\ &= (k-1)(2n-k), \end{aligned}$$

and hence

$$\begin{aligned} e(M \mid P) &\geqslant \frac{\varepsilon}{2}(k-1)(2n-k) \\ &\geqslant \frac{\varepsilon}{2}(k-1)n \\ &= \frac{1}{4}(1-\eta)(k-1)n. \end{aligned}$$

By Theorem 8, M has at least $[(\varepsilon/2)n] = [\frac{1}{4}(1-\eta)n]$ edge-disjoint spanning trees. Thus we proved that $\mathscr{T}_n$ is $[1,1;[\frac{1}{4}(1-\eta)n]]$-unavoidable. □

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