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Directed Switching Games on Graphs and Matroids

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We introduce directed versions of the Shannon Switching Game. In the Directed Shannon Switching Game two players, BLACK and WHITE, alternatively direct edges of a graph with two distinguished vertices x_0, x_1. WHITE wins if he connects x_0 to x_1 by a directed path. In the One-Way Game, WHITE is allowed to use edges directed by BLACK. Several related other games are also considered. All these games are particular cases of games played on oriented matroids. The main results of the paper yield complete classifications of directed switching games on graphic and cographic oriented matroids. © 1986 Academic Press, Inc.

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1. INTRODUCTION

The *Shannon Switching Game* in its original form is a two-player game with complete information played on a graph with two distinguished ver-

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tices. Two players, SHORT and CUT, play alternatively. A move of SHORT consists of making an unplayed edge invulnerable to deletion, his objective being to connect the two distinguished vertices by an invulnerable path. A move of CUT consists of deleting an unplayed edge, his objective being to prevent SHORT succeeding. The game proceeds until one of the players reaches his goal [12].

For a given graph and given distinguished vertices, the outcome of the game depends upon the identity SHORT or CUT of the first player. A classical argument (see below Proposition 2.1) shows that if a player can win playing second, then he can also win playing first. Hence exactly one of the following three cases occurs: SHORT playing second has a winning strategy, or CUT playing second has a winning strategy, or the first player, SHORT or CUT, has a winning strategy.

A complete solution of the Shannon Switching Game has been given by Lehman [12]: each type of game is characterized and winning strategies are described. Fundamental theorem: the Shannon Switching Game is winning for SHORT playing second if and only if the graph contains two edge-disjoint trees on a same subset of vertices containing the two distinguished vertices. This solution is given in [12] for a more general game played on matroids. The reader will find a survey of Lehman theory in Section 6.

Our purpose in the present paper is to introduce and solve several variants of the Shannon Switching Game. The general definitions are given in Section 8. We describe briefly in the Introduction the graphic case of the two main new games: the Directed Shannon Switching Game and the One-Way Game. In these two games the basic difference with the Shannon Switching Game is that players direct edges.

The complete rules of the Directed Shannon Switching Game are as follows. The game is a two-player game with complete information. The board is an undirected graph G with two distinguished vertices x_0 and x_1. Two players, BLACK and WHITE, play alternately.[1] A move of WHITE consists of directing an unplayed edge of G. A move of BLACK consists of marking an unplayed edge of G. The objective of WHITE is to form a directed path joining x_0 to x_1. The objective of BLACK is to prevent WHITE from meeting his objective. The game proceeds until one of the players reaches his goal.

The rules of the One-Way Game are those of the Directed Shannon Switching Game with one difference: BLACK also directs edges.

We point out that in the One-Way Game a winning configuration for WHITE, i.e., a directed path joining x_0 to x_1, may contain both black and

[1] We name the players WHITE (= SHORT) and BLACK (= CUT) as the meanings of the words "short" and "cut" are irrelevant in several games we consider. Also it is more convenient to consider that BLACK marks edges instead of deleting them.

white edges (here black means played by BLACK). In the Directed Shannon Switching Game a directed path joining x_0 to x_1 is necessarily constituted only of white edges (WHITE being the only player to direct edges).

Complete solutions of these games are given in Sections 9 and 12. Our results may be summarized by saying that the classifications of these directed games are identical to that of the Shannon Switching Game. More precisely the Directed Shannon Switching Game (resp. the One-Way Game) on a graph G with two distinguished vertices x_0 and x_1 is winning for WHITE playing second resp. for BLACK playing second, for the first player if and only if the Shannon Switching Game on G with respect to x_0 and x_1 has the same property.

The strategies used to derive the classifications in the directed case are different, and more elaborate, than Lehman classical strategy for the undirected case.

The Directed Shannon Switching Game and the One-Way Game have natural generalizations in the context of oriented matroids (see Sect. 8).

The main results presented in this paper have been announced in [7].

2. Notations and Terminology

All graphs and matroids considered in this paper are finite. By a graph we mean an undirected graph with possibly loops and multiple edges. We denote by $V(G)$ and $E(G)$, respectively, the vertex-set and the edge-set of a graph G. For $A \subseteq E(G)$ we denote by $V(A)$ the set of vertices of the edges in A.

Our matroid terminology is standard [13]. We denote by $E(M)$ the set of elements of a matroid M, and by M^* its dual (or orthogonal) matroid. For $A \subseteq E(M)$ we denote by $r_M(A)$ the rank of A in M. The matroids obtained from M by deleting (resp. contracting) A are denoted by $M \backslash A$ (resp. M/A). As usual we set $M(A) = M \backslash (E \backslash A)$ and we abbreviate $M \backslash \{e\}$ and $M/\{e\}$ by $M \backslash e$ and M/e for $e \in E(M)$.

We denote by $\mathbb{C}(G)$ (resp. $\mathbb{B}(G)$ the cycle matroid (resp. the cocycle matroid) of a graph G (cf. [13, 1.10, 2.4]). A matroid M is *graphic* (resp. *cographic*) if there is a graph G such that $M \approx \mathbb{C}(G)$ (resp. $M \approx \mathbb{B}(G)$). For $A \subseteq E(G)$ we denote by $G \backslash A$ and G/A, respectively, the graph obtained from G by deleting resp. contracting the edges in A.

Basic definitions concerning oriented matroids may be found in [2]. We recall that a *signed set* is a set X together with a partition into two distinguished subsets X^+, the positive part of X, and X^-, its negative part. Let M be an oriented matroid. For $A \subseteq E(M)$ we denote by ${}_{\bar{A}}M$ the oriented matroid obtained from M by *sign reversal* on A: the signed circuits of ${}_{\bar{A}}M$ are all signed sets ${}_{\bar{A}}X$ for X a signed circuit of M, where ${}_{\bar{A}}X$ is defined by $({}_{\bar{A}}X)^+ = (X^+\setminus A) \cup (X^- \cap A)$ and $({}_{\bar{A}}X)^- = (X^-\setminus A) \cup (X^+ \cap A)$ [2]. *Signing* an element $e \in E(M)$ means reversing sign or not on $\{e\}$. Clearly if M is the oriented cycle or cocycle matroid of a directed graph G, signing an element $e \in E(M)$, i.e., an edge of G, means reversing or not its direction.

Let M be a matroid and $e \in E(M)$. A subset $A \subseteq E(M)\setminus\{e\}$ is a *circuit of M broken at e* if $A \cup \{e\}$ is a circuit of M. If M is an oriented matroid, a signed set $A \subseteq E(M)\setminus\{e\}$ is a *positive circuit broken at e* if there is a positive circuit C of M such that $e \in C$ and $A = C\setminus\{e\}$.

3. Undirected and Directed Switching Games as Positional Games

Let E be a set and $\mathscr{C} \subseteq 2^E$ be a nonempty collection of subsets of E. Consider the following two-player game with complete information: Two players BLACK and WHITE play alternatively by marking unplayed elements of E. The objective of WHITE is to mark a subset of E belonging to $\mathscr{C}$, the objective of BLACK is to prevent WHITE from succeeding. We call this game a *positional game on E* with collection of (WHITE) *winning configurations $\mathscr{C}$*.

Actually the usual definition of positional games is more general. The above games are called positional games of type 1 in [1]. We use a simplified terminology for brevity.

The Shannon Switching Game on a graph G with respect to two vertices x_0, x_1 is clearly a positional game on $E(G)$, WHITE winning configurations are all edge-sets of paths of G joining x_0 to x_1.

The Directed Shannon Switching Game defined in the Introduction is not a positional game. However, the following model of this game is "almost" a positional game: Let $\vec{G}$ be any orientation of G. Denote by $\vec{\vec{G}}$ the graph obtained from $\vec{G}$ by inroducing a new edge e^* for every edge $e \in E(\vec{G})$, with e^* parallel to e and directed in the opposite direction. The Directed Shannon Switching Game is almost a positional game on $E \cup E^*$, $E = E(\vec{G})$, WHITE winning configurations being all edge-sets of directed paths of G joining x_0 to x_1. Only here moves are restricted: when an edge in a pair $\{e, e^*\}$ has been played, by either player, then the other edge is no longer subject to play. We call such a game a positional game with *move restrictions* or *restricted positional game.*

It suffices here to define move restrictions by a partition π of E. In a restricted positional game $(E, \mathscr{C}, \pi)$ the additional rule is that at most one element may be played in each class of π. The other rules of positional games are unchanged.

The One-Way Game is not a restricted positional game in the above sense. In the model played on $\vec{G}$ the objective of WHITE is to obtain a configuration in $\mathscr{C}$ that is completely played, both colors being allowed. We call an *uncolored restricted positional game* a game such as the One-Way Game, with the rules of a restricted positional game, except that the objective of WHITE is to obtain a winning configuration that is completely played. Of course this rule only makes sense when moves are actually restricted. By opposition, the first type of restricted positional games will occasionally be called *colored.*

We will show in Section 13 that if the One-Way Game on a given graph is winning for WHITE, then there is a strategy ensuring that WHITE can mark a white winning configuration against all strategies of BLACK. An uncolored restricted positional game with this property behaves like a colored game. We say that such a game is *reducible* to a (colored) restricted positional game.

Our purpose in this section is to establish two general properties of restricted positional games, which will be of repeated use in the sequel.

The first proposition is well-known for positional games.

PROPOSITION 3.1. *If a colored or reducible uncolored restricted positional game is winning for* WHITE *playing second then it is also winning for* WHITE *playing first.*

Proof. Let Σ_2 be a winning strategy for WHITE playing second producing a white winning configuration against all strategies of BLACK. A winning strategy Σ_1 for WHITE playing first is derived from Σ_2 played by WHITE in a fictitious game defined as follows.

In the fictitious game BLACK plays first. His first move is an arbitrary element b'_1 of E. The first move w_1 of WHITE is a response prescribed by Σ_2 to b'_1. Suppose inductively that the moves have been $w_1, w_2, \dots, w_i$ for WHITE, responss prescribed by Σ_2 to moves $b'_1, b'_2, \dots, b'_i$ of BLACK in the fictitious game, $i \geqslant 1$, and that BLACK has played $b_1, b_2, \dots, b_{i-1}$ in the real game. If there is no playable element in the fictitious game, then by the hypothesis on Σ_2 WHITE has won, hence $\{w_1, w_2, \dots, w_i\}$ contains a white winning configuration. It follows that WHITE has also won in the real game.

Suppose there is at least one playable element in the fictitious game. Let b_i be the ith move of BLACK in the real game. If b_i is playable in the fictitious game we set $b'_{i+1} = b_i$. Otherwise we take for b'_{i+1} any playable element of the fictitious game. The $(i+1)$th move w_{i+1} of WHITE in the real game prescribed by Σ_1 is a response prescribed by Σ_2 to the move b'_{i+1} of BLACK in the fictitious game. ∎

It is intuitively clear, and this follows from Lehman theory [12], that if the Shannon Switching Game is winning for WHITE on a subgraph of G

then it is also winning on G. This property generalizes to restricted positional games.

PROPOSITION 3.2. *Let $(E, \mathscr{C}, \pi)$ be a restricted positional game (colored or uncolored). Let E' be a subset of E which is a union of classes of π. Set $\mathscr{C}' = \mathscr{C} \cap 2^{E'}$ and $\pi' = \pi \cap 2^{E'}$. Suppose in the uncolored case that the subgame $(E', \mathscr{C}', \pi')$ is reducible.*

If the subgame $(E', \mathscr{C}', \pi')$ is winning for WHITE *playing first resp. second then the game $(E, \mathscr{C}, \pi)$ is winning for* WHITE *playing first (resp. second).*

Proof. The proof is similar to that of Proposition 3.1. We consider the case WHITE playing second. The other case can be proved in the same way. Let Σ' be a winning strategy for WHITE in the game on E' producing a white winning configuration against all strategies of BLACK. A winning strategy Σ for WHITE in the game on E is derived from Σ' played by WHITE in a fictitious game on E' defined as follows.

Suppose inductively that the first i moves, $i \geqslant 0$, have been $b_1, b_2, ..., b_i$ played by BLACK and $w_1, w_2, ..., w_i \in E'$ played by WHITE, responses prescribed by Σ' to moves $b'_1, b'_2, ..., b'_i$ of BLACK in a fictitious game played on E'. If there is no playable element left in the fictitious game on E', then WHITE has won in this game, hence also in the real game on E.

Suppose there is at least one playable element in the fictitious game on E'. Let b_{i+1} be the $(i+1)$th move of BLACK. If b_{i+1} is a playable element in the fictitious game on E' we set $b'_{i+1} = b_{i+1}$. Otherwise we take for b'_{i+1} any playable element in the fictitious game on E'. The $(i+1)$th move w_{i+1}th move w_{i+1} of WHITE in the real game on E prescribed by Σ is a response prescribed by Σ' to the move b'_{i+1} of BLACK in the fictitious game on E'. ∎

Propositions 3.1 and 3.2 hold for a larger class of games (containing in particular all positional games in the usual sense). The generality considered here is sufficient for our applications.

The Graphic Unrooted Signing Game discussed in Section 14 is an example of an uncolored restricted positional game which is not reducible in general. Played on the graph constituted by two parallel edges, this game is obviously winning for WHITE playing second and losing for WHITE playing first.

For later use we recall that the *dual* game of a two-player game, played by BLACK and WHITE, is the game with the same rules except that the names BLACK and WHITE are interchanged.

4. The Matroid Union Theorem

Our main tool is the following particular case of the

Matroid Union Theorem. *Let M be a matroid on a set E. Then $\{I_1 \cup I_2 : I_1, I_2 \subseteq E$ independent in $M\}$ is the collection of independent sets of a matroid on E, denoted by $M \vee M$. The rank function of $M \vee M$ is given by $r_{M \vee M}(X) = \mathrm{Min}_{Y \subseteq X}\,(2r_M(Y) + |X \setminus Y|)$ for any $X \subseteq E$.*

In its full generality the Matroid Union Theorem, or more simply, the Union Theorem, deals with the union $M_1 \vee M_2 \vee \cdots \vee M_k$ of $k \geqslant 2$ not necessary equal matroids on E (see [13, Theorem 8.3.1]). For the history of this theorem we quote from [13, p. 134]: The idea [of union of matroids] originates in a paper of Nash-Williams [66]... The rank formula for the union of matroids is implicit in the paper of Edmonds and Fulkerson [65], though as pointed out in Welsh [70] it can easily be deduced from the theorem of Rado [42].

We complete the Union Theorem by several lemmas. Lemmas 4.1, 4.2, 4.3 are implicit in [3]. Lemmas 4.1, 4.2, 4.4 hold for unions of any $k \geqslant 2$ matroids. In all four lemmas M is a matroid on a set E.

Lemma 4.1. [3]. *For any two bases B_1, B_2 of M and any $X \subseteq E$ we have $|B_1 \cup B_2| \leqslant 2r_M(X) + |E \setminus X|$. The equality holds if and only if $E \setminus X \subseteq B_1 \cup B_2$ and $B_1 \cap X$, $B_2 \cap X$ are two disjoint bases of X.*

Proof. The inequality is the easy part of the Union Theorem. We have $|B_1 \cup B_2| = |(B_1 \cup B_2) \cap X| + |(B_1 \cup B_2)| \setminus X| \leqslant |B_1 \cap X| + |B_2 \cap X| + |E \setminus X| \leqslant 2r_M(X) + |E \setminus X|$.

The inequality $|B_1 \cup B_2| \leqslant 2r_M(X) + |E \setminus X|$ is a consequence of the four inequalities $|(B_1 \cup B_2) \cap X| \leqslant |B_1 \cap X| + |B_2 \cap X|$, $|B_1 \cap X| \leqslant r_M(X)$, $|B_2 \cap X| \leqslant r_M(X)$, $|(B_1 \cup B_2) \setminus X| \leqslant |E \setminus X|$. The equality $|B_1 \cup B_2| = 2r_M(X) + |E \setminus X|$ forces the equality in these four inequalities, clearly implying the "only if" part of Lemma 4.1. The converse is immediate. ∎

It follows from the Union Theorem that equality holds in $|B_1 \cup B_2| \leqslant 2r_M(X) + |E \setminus X|$ if and only if $|B_1 \cup B_2|$ is maximum and $2r_M(X) + |E \setminus X|$ is minimum, both being then equal to $r(M \vee M)$ and $B_1 \cup B_2$ being a base of $M \vee M$. Two bases B_1, B_2 of M such that $|B_1 \cup B_2|$ is maximal are said *maximally distant.*

Let $\mathscr{P}(M)$ denote the collection of subsets X of E achieving the minimum $r(M \vee M)$ of $2r_M(X) + |E \setminus X|$.

Lemma 4.2 [3]. *$\mathscr{P}(M)$ is closed under union and intersection.*

Proof. Let X, $Y \in \mathscr{P}(M)$. We have $r(M \vee M) \leqslant 2r(X \cap Y) + |E \backslash (X \cap Y)|$ and $r(M \vee M) \leqslant 2r(X \cup Y) + |E \backslash (X \cup Y)|$. Hence by the submodularity of the rank function $2r(M \vee M) \leqslant 2r(X \cap Y) + |E \backslash (X \cap Y)| + 2r(X \cup Y) + |E \backslash (X \cup Y)| \leqslant 2r(X) + |E \backslash X| + 2r(Y) + |E \backslash Y| = 2r(M \vee M)$. It follows that equality holds, implying $X \cap Y$ and $X \cup Y \in \mathscr{P}(M)$. ∎

LEMMA 4.3 [3]. $\mathscr{P}(M^*) = \{E \backslash X : X \in \mathscr{P}(M)\}$.

Proof. From the formula $r_{M^*}(X) = |X| + r_M(E \backslash X) - r(M)$ follows $2r_{M^*}(E \backslash X) + |X| = 2r_M(X) + |E \backslash X| + |E| - r(M)$. ∎

A set $X \subseteq E$ is called *cyclic* in M if X is a union of circuits of M.

LEMMA 4.4. *A set $X \in \mathscr{P}(M)$ is a cyclic flat of M and satisfies $r_{M \vee M}(X) = 2r_M(X)$.*

Proof. For any $e \in E \backslash X$ we have $2r(X \cup e) + |E \backslash (X \cup e)| = 2r(X \cup e) + |E \backslash X| - 1 \geqslant 2r(X) + |E \backslash X|$, hence $2r(X \cup e) \geqslant 2r(X) + 1$. It follows that $r(X \cup e) = r(X) + 1$. Hence X is a flat of M.

For any $e \in X$ we have $2r(X \backslash \{e\}) + |E \backslash \{e\}) = 2r(X \backslash \{e\}) + |E \backslash X| + 1 \geqslant 2r(X) + |E \backslash X|$. hence $2r(X \backslash \{e\}) \geqslant 2r(X) - 1$. It follows that $r(X \backslash \{e\}) = r(X)$. Hence X is cyclic.

The property $r_{M \vee M}(X) = 2r_M(X)$ follows from Lemma 4.1. ∎

5. BLOCKS IN MATROIDS AND GRAPHS

We say that a matroid M is a *block-matroid,* or simply a *block* if $E(M)$ is a union of two disjoint bases of M. By definition of a union of matroids M is a block if and only if $r(M \vee M) = 2r(M) = |E(M)|$. Hence by the Union Theorem we have

PROPOSITION 5.1. *A matroid M on a set E is a block if and only if*

(1) $|X| \leqslant 2r_M(X)$ *for all* $X \subseteq E$ *and*

(2) $|E| = 2r_M(E)$ $(= 2r(M))$.

In all statements of this section M is a matroid on a set E. We say that a subset $X \subseteq E$ is a *block of M* if the submatroid $M(X)$ is a block. We observe that if $r(M \vee M) = |E|$, hence in particular if M is a block, a subset X of E is a block of M if and only if $X \in \mathscr{P}(M)$ (with notations of Section 4). Specializing lemmas of Section 4 we get

PROPOSITION 5.2. *Suppose M is a block*

(i) *a block of M is a cyclic flat of M,*

(ii) *a subset* $X \subseteq E$ *is a block of* M *if and only if* $|X| = 2r_M(X)$

(iii) *the collection of blocks of* M *is closed under union and intersection. Hence for any* $A \subseteq E$ *there is a unique inclusion-maximal block contained in* A *(resp. a unique inclusion-minimal block containing* A*).*

PROPOSITION 5.3. (i) *M is a block if and only if M^* is a block.*

(ii) *Suppose M is a block. Then $X \subseteq E$ is a block of M if and only if $E \backslash X$ is a block of M^*.*

Proof. Property (i) is immediate since the complement of a base of M is a base of M^* and conversely. Property (ii) is a specialization of Lemma 4.3. ∎

PROPOSITION 5.4. *Suppose M is a block*

(i) *Let B_1, B_2 be two disjoint bases of M partitioning E. Then $X \subseteq E$ is a block of M if and only if $B_1 \cap X$ and $B_2 \cap X$ are two bases of X*

(ii) *For any block X of M, M/X is a block.*

Proof. Property (i) is a specialization of Lemma 4.1.

(ii) Let B_1, B_2 be two bases of M partitioning E. If X is a block, by (i) $B_1 \cap X$ and $B_2 \cap X$ are two bases of X. Hence $B_1 \backslash X$ and $B_2 \backslash X$ are two bases of M/X. Since $B_1 \backslash X$ and $B_2 \backslash X$ partition $E \backslash X$, the matroid M/X is a block. ∎

PROPOSITION 5.5. *Suppose M is a block. Let $b, w \in E$. Then $M \backslash b/w$ is a block if and only if every block of M containing w contains also b.*

Proof. Suppose $M' = M \backslash b/w$ is not a block. We have $|E(M')| = |E(M)| - 2 = 2(r(M) - 1) = 2r(M')$. Hence by Proposition 5.1 there is $X \subset E \backslash \{b, w\}$ such that $|X| > 2r_{M'}(X)$. We have $r_{M'}(X) = r_M(X \cup \{w\}) - 1$, hence $|X \cup \{w\}| \geqslant 2r_M(X \cup \{w\})$. Since M is a block, it follows that $X \cup \{w\}$ is a block of M, by Proposition 5.1. The block $X \cup \{w\}$ contains w but does not contain b.

Conversely suppose $M' = M \backslash b/w$ is a block, and there is a block X of M with $w \in X \subseteq E \backslash \{b\}$. Since M' is a block and $X \backslash \{w\} \subseteq E(M')$ we have by Proposition 5.1, $|X \backslash \{w\}| \leqslant 2r_{M'}(X \backslash \{w\})$. Hence $|X| - 1 \leqslant 2(r_M(X) - 1)$, contradicting $|X| = 2r_M(X)$. ∎

By Proposition 5.2(iii), the condition in Proposition 5.5 can also be stated: $M \backslash b/w$ is a block if and only if w is not contained in the maximal block of $M \backslash b$, or, equivalently, if and only if b belongs to the minimal block of M containing w.

PROPOSITION 5.6. *Suppose M and $M\backslash b/w$ are blocks, $b, w \in E$. Then for any block X of M, $X\backslash\{b, w\}$ is a block of $M\backslash b/w$ if and only if $b, w \in X$ or $b, w \notin X$.*

Proof. The "only if" part is clear since a block has even cardinality. We prove the "if" part.

Suppose $b, w \in X$. We have

$$2r_{M\backslash b/w}(X\backslash\{b, w\}) = 2r_M(X\backslash\{b\}) - 2.$$

Since X is a block we have $r_M(X\backslash\{b\}) = r_M(X)$ (a block has no isthmuses). Hence $2r_{M\backslash b/w}(X\backslash\{b, w\}) = 2r_M(X) - 2 = |X\backslash\{b, w\}|$.

Suppose $b, w \notin X$. We have

$$2r_{M\backslash b/w}(X\backslash\{b, w\}) = 2r_M(X \cup \{w\}) - 2.$$

Since M and $M(X)$ are blocks we have $r_M(X \cup \{w\}) = r_M(X) + 1$ (a sub-block of a block M is closed in M). Hence $2r_{M\backslash b/w}(X\backslash\{b, w\}) = 2r_M(X) = |X| = |X\backslash\{b, w\}|$.

In both cases $X\backslash\{b, w\}$ is a block of $M\backslash b/w$ by Proposition 5.2(ii). ∎

We say that a graph G is a *block-graph* if its cycle matroid $\mathbb{C}(G)$ is a block-matroid: a connected graph G is a block if and only if its edge-set is a union of two edge-disjoint spanning trees of G. We define a *block* of a graph G as a block of $\mathbb{C}(G)$.

6. SWITCHING GAMES ON MATROIDS-LEHMAN SOLUTION OF THE SHANNON SWITCHING GAME

This section is a survey of the classical theory, mainly due to Lehman [12] (see also [6; 13, 19.4]).

The Shannon Switching Game on graphs has been generalized by Lehman to a game on matroids: The board is a matroid M with a distinguished element e not subject to play. Moves of both BLACK and WHITE consist of marking an unplayed element. The objective of WHITE is to form a white circuit of M broken at e. We call this game the *Lehman Switching Game* on M with respect to e.

Clearly the Shannon Switching Game on a graph G with respect to two vertices x_0, x_1 is equivalent to the Lehman Switching Game on the cycle matroid of the graph $G + e$ with respect to e, where e is an edge not in G joining x_0 and x_1.

THEOREM A (Lehman [12, Theorem 14]). *The Lehman Switching Game on a matroid M with respect to an element $e \in E(M)$ is winning for* WHITE *playing second if and only if there is a block of M spanning but not containing e.*

The "if" part of Theorem A is proved by describing explicitly a winning strategy for WHITE. The key observation is

LEMMA A.1. *Let b (resp. w) be a black (resp. white) element of $E(M)\backslash\{e\}$. Then M contains a white circuit broken at e if and only if $M\backslash b/w$ contains a white circuit broken at e.*

Hence, by induction, to obtain a winning strategy for WHITE it suffices to determine a response w of WHITE to a first move b of BLACK such that $M\backslash b/w$ satisfies again the hypothesis of Theorem A. By Proposition 3.2 we may suppose that $E(M)\backslash\{e\}$ is a block of M spanning e.

Lehman Strategy

(WHITE strategy for the Lehman Switching Game, BLACK playing first).

Let M be a matroid with a distinguished element e, such that $E(M)\backslash\{e\}$ is a block of M spanning e.

Let B_1, B_2 be two disjoint bases of $M\backslash e$. By hypothesis $B_1 \cup B_2 = E(M)\backslash\{e\}$.

(1) Let $b \in E(M)\backslash\{e\}$ be the first move of BLACK. Suppose $b \in B_1$ (say). Then WHITE marks any $w \in B_2$ such that b is contained in the unique circuit of M contained in $B_1 \cup \{w\}$.

(2) Suppose inductively that BLACK and WHITE have already played the elements $b_1, b_2, \ldots, b_i$ and $w_1, w_2, \ldots, w_i$, $i \geqslant 0$, respectively. Let $b_{i+1} \in E(M)\ \{e, b_1, b_2, \ldots, b_i, w_1, w_2, \ldots, w_i\}$ be the $(i+1)$th move of BLACK.

Then WHITE considers b_{i+1} as a first move of BLACK in the matroid $M' = M\backslash\{b_1, b_2, \ldots, b_i\}/\{w_1, w_2, \ldots, w_i\}$ and marks an element w_{i+1} given by (1) applied to M' and b_{i+1}.

In the situation of (1) as easily seen, $B_1' = B_1\backslash\{b\}$ and $B_2' = B_2\backslash\{w\}$ are two disjoint bases of $M' = M\backslash b/w$, and $E(M')\backslash\{e\}$ is a block of M' spanning e. The "if" part of Theorem A follows. The "only if" part of Theorem A is proved directly in [12, Theorem 14]. It is also a consequence of the following "Classification Theorem":

THEOREM B (Lehman [12, Theorem 32). *Let M be a matroid and $e \in E(M)$. Then exactly one of the following three cases occurs:*

(i) *there is a block of M spanning but not containing e,*

(ii) *there is a block of M^* spanning but not containing e,*

(iii) *in both* M *and* M^* *there are blocks containing* e.

The Lehman Switching Game on M *with respect to* e *is winning for* WHITE *playing second in case* (i) *resp. winning for* BLACK *playing second in case* (ii), *winning for the first player in case* (iii).

The first part of Theorem B implies its second part by combining with the "if" part of Theorem A and

LEMMA B.1 [12, Lemma 21]. *The Lehman Switching Game on a matroid* M *with respect to an element* $e \in E(M)$ *is winning for* BLACK *playing second if and only if the Lehman Switching Game on* M^* *with respect to* e *is winning for* WHITE *playing second.*

LEMMA B.2 [12, Lemma 25]. *The Lehman Switching Game on a matroid* M *with respect to an element* $e \in E(M)$ *is winning for the first player if and only if both Lehman Switching Games with respect to e in the matroids obtained from* M *and* M^* *by doubling e in parallel are winning for* WHITE *playing second.*

Lemma B.1 results from the fact that for any black/white partition of $E(M)\setminus\{e\}$ there is either a white circuit broken at e or a black cocircuit broken at e, but not both. Note that Lemma B.1 expresses that the Lehman Switching Game is self-dual up to the replacement of M by M^*.

Lemma B.2 follows from Lemma B.1 and the observation that the Lehman Switching Game on M with respect to $e \in E(M)$ is winning for WHITE playing first if and only if the Lehman Switching Game with respect to e on the matroid obtained from M by doubling e in parallel is winning for WHITE playing second (since in the new matroid BLACK loses immediately if his first move is not the new element).

Conversely the "only if" part of Theorem A and Lemmas B.1, B.2 imply the first part of Theorem B.

A stronger form of Theorem B has been given by Bruno–Weinberg in [3] by relating its three cases to the principal partition of M. The notion of principal partition, introduced by Kishi and Kajitani for graphs [10] and generalized to matroids by Bruno and Weinberg [3], is relevant in several applications of matroid theory. We refer the reader to [8] for a recent survey on the subject.

Let M be a matroid on a set E. With notations of Section 4 let $\mathscr{P}(M)$ be the set of subsets X of E achieving the minimum of $2r_M(X)+|E\setminus X|$. We set $P(M)=\bigcap_{X\in\mathscr{P}(M)} X$, $Q(M)=\bigcup_{X\in\mathscr{P}(M)} X$.

By Lemma 4.2 $P(M)$ and $Q(M)\in\mathscr{P}(M)$. By Lemma 4.3 we have $P(M^*)=E\setminus Q(M)$. It follows that $P=P(M)$ and $P^*=P(M^*)$ are two disjoint subsets of E. The *principal partition* of E in the sense of [3] is the partition $E=P+P^*+R$ where $R=E\setminus(P\cup P^*)$.

THEOREM C (Bruno–Weinberg [3, Theorem 7.3]). *Let M be a matroid and e be an element of M. Let P, P^*, R be the principal partition of M (with the above notations).*

(i) *$e \in P$ if and only if there is a block of M spanning but not containing e.*

(ii) *$e \in P^*$ if and only if there is a block of M^* spanning but not containing e.*

(iii) *$e \in R$ if and only if there are blocks containing e in both M and M^*.*

For completeness we give a short proof of Theorem C (based on the proof in [3] simplified here by the use of the Union Theorem).

LEMMA C.1 [3, Theorem 5.5]. *$P(M)$ is the set of non-isthmuses of $M \vee M$.*

Proof. We have to show that $P(M)$ is the union of all sets $E\backslash B$ for B a basis of $M \vee M$. By Lemma 4.1 we have $X \supseteq E\backslash B$ for all $X \in \mathscr{P}(M)$ and B basis of $M \vee M$. Hence $P(M) \supseteq \bigcup_{B \text{ base of } M \vee M}(E\backslash B)$.

Suppose the equality does not hold. There is $e \in P(M)$ such that $e \in B$ for all bases B of $M \vee M$. In other words e is an isthmus of $M \vee M$: the rank of $(M \vee M)\backslash e = (M\backslash e) \vee (M\backslash e)$ is strictly less than the rank of $M \vee M$. Hence by the Union Theorem there is $X \subseteq E\backslash\{e\}$ such that

$$2r_{M\backslash e}(X) + |E\backslash\{e\}\backslash X| < r(M \vee M).$$

Equivalently we have $2r_M(X) + |E\backslash X| \leqslant r(M \vee M)$. Since on the other hand $2r_M(X) + |E\backslash X| \geqslant r(M \vee M)$, we have $X \in \mathscr{P}(M)$. Hence $P(M) \subseteq X$, contradicting $e \notin X$. ∎

Proof of Theorem C. "Only if" part of (i). Suppose $e \in P(M)$. By Lemma C.1 there is a basis B of $M \vee M$ such that $e \notin B$. Let B_1, B_2 be two bases of M such that $B = B_1 \cup B_2$. Since $P(M) \in \mathscr{P}(M)$, by Lemma 4.1 $B_1 \cap P(M)$ and $B_2 \cap P(M)$ are two disjoint bases of $P(M)$ partitioning $B \cap P(M)$. Hence $B \cap P(M)$ is a block of M spanning but not containing e.

First half of the "only if" part of (iii). Suppose $e \in Q(M)\backslash P(M) = R$. Let B be any basis of $M \vee M$. Let B_1, B_2 be two bases of M such that $B = B_1 \cup B_2$. Since $Q(M) \in \mathscr{P}(M)$ by Lemma 4.1, $B_1 \cap Q(M)$ and $B_2 \cap Q(M)$ are two disjoint bases of $Q(M)$ partitioning $B \cap Q(M)$. Hence $B \cap Q(M)$ is a block of M containing e.

We obtain the "only if" part of (ii) and the second half of the "only if" part of (iii) by considering M^* instead of M, since $Q(M)\backslash P(M) = Q(M^*)\backslash P(M^*)$.

The "if" parts follow from the classification of the Lehman Switching

Game. Direct proofs that the three conditions on blocks are mutually exclusive can also easily be given. ∎

From Theorem C and its proof we get the following characterization of $Q(M)$.

PROPOSITION 6.1. *$Q(M)$ is the union of all blocks of M plus all loops of M.* ∎

7. SWITCHING GAMES ON MATROIDS–EXTENDED AND UNROOTED GAMES

As noted by Lehman [12, Theorem 16] "[using this strategy] the short player playing second will win with respect to any branch [edge] of M for which the matroid yields a short game." This statement amounts to the "if" part of the following strengthening of Theorem A:

THEOREM A′ (implicit in [12]). *In the Switching Game on a matroid M,* WHITE *playing second can mark a base of M against all strategies of* BLACK *if and only if M contains a spanning block.*

The proof of the "if" part of Lehman Theorem (Sect. 6 Theorem A) yields a proof of the "if" part of Theorem A′ by means of the following variant of Lemma A.1: *Let b (resp. w) be a black (resp. white) element of a matroid M. Then M contains a white base if and only if $M\backslash b/w$ contains a white base.* As observed by Chvátal–Erdös in [5, Introduction] the necessity of the condition in Theorem A′ follows easily from the Union Theorem: If M contains no spanning block, i.e., $r(M \vee M) < 2r(M)$ then there is $A \subseteq E = E(M)$ such that $2r_M(A) + |E \backslash A| < 2r(M)$. Suppose BLACK plays by marking elements of $E\backslash A$ as long as there remains any. Since BLACK plays first, at most $[|E\backslash A|/2]$ elements of $E\backslash A$ can be marked by WHITE. Hence WHITE cannot mark a base of M since $r_M(A) + [|E\backslash A|/2] < r(M)$.

We call an *Extended Lehman Switching Game* a switching game on a matroid M where the objective of WHITE is to mark a base of M. Theorem A' can be rephrased as: The *Extended Lehman Switching Game on a matroid M is winning for* WHITE *playing second if and only if M contains a spanning block.*

The Extended Lehman Switching Game on a matroid is a positional game by definition. Clearly a matroid whose elements are colored black or white contains either a white base or a black cocircuit but not both. Hence the Dual Extended Lehman Swtiching Game on a matroid M is also a positional game, with winning configurations constituted by circuits of M^*.

We define the *Unrooted Lehman Switching* Game on a matroid M as a positional game on $E(M)$ with WHITE winning configurations constituted by circuits of M.

It follows from Proposition 3.1 that if the Extended Lehman Switching Game is winning for WHITE (resp. BLACK) playing second, then the game is also winning for WHITE (resp. BLACK) playing first.

The proof of Theorem A′ can easily be adapted to prove: *The Extended Lehman Switching Game on a matroid M is winning for* WHITE *playing first if and only if M contains a spanning diminished block*, where a *diminished block* is a submatroid $M(X)$ with X the union of two bases intersecting in just one element.

We define an *augmented block* (resp. a *k-augmented block*), $k \geqslant 1$, of a matroid M as a block X of M plus one element (resp. k) elements not contained in X and spanned by X. The complete classification of the Extended Lehman Switching Game is given by

THEOREM 7.1 (partially implicit in [12]). *Let M be a matroid. Exactly one of the following three cases hold*:

(i) *M contains a spanning block.*

(ii) *M^* contains a 2-augmented block.*

(iii) *M contains a spanning diminished block and M^* contains an augmented block.*

The Extended Lehman Switching Game is winning for WHITE *playing second in case* (i) *resp. winning for* BLACK *playing second in case* (ii), *winning for the first player in case* (iii).

A direct proof of the first part of Theorem 7.1 can easily be obtained by using the Union Theorem. Actually the three cases of Theorem 7.1 have simple characterizations in terms of $r(M \vee M)$:

PROPOSITION 7.2. *Let M be a matroid. Then*

(i) *M contains a spanning block if and only if $r(M \vee M) = 2r(M)$.*

(ii) *M contains a spanning diminished block and M^* contains an augmented block if and only if $r(M \vee M) = 2r(M) - 1$.*

(iii) *M^* contains a 2-augmented block if and only if $r(M \vee M) \leqslant 2r(M) - 2$.*

In Lemmas 7.3 and 7.4 M is a matroid on E and k is a nonnegative integer.

LEMMA 7.3. $r(M^* \vee M^*) = r(M \vee M) - 2r(M) + |E|$.

Proof. For any subsets A, $B \subseteq E$ we have $|(E\backslash A) \cup (E\backslash B)| = |A \cup B| - |A| - |B| + |E|$. Hence A, B are two maximally distant bases of M if and only if $E\backslash A$, $E\backslash B$ are two maximally distant bases of M^*. Lemma 7.3 follows.

LEMMA 7.4. *M contains a k-augmented block if and only if $r(M \vee M) \leqslant |E| - k$.*

Proof. Let X be a k-augmented block of M. Let A be a block of M contained in X such that $|X\backslash A| = k$ and A spans X in M. Clearly A is a basis of X in $M \vee M$. Let B be a basis of $M \vee M$ containing A. We have $B \cap (X\backslash A) = \varnothing$. Hence $|E| \geqslant |B| + |X\backslash A| = r(M \vee M) + k$.

Conversely suppose $r(M \vee M) \leqslant |E| - k$. We may suppose $k \geqslant 1$, the case $k = 0$ being trivial. Let $X \subseteq E$ be inclusion-minimal with the property $r_{M \vee M}(X) \leqslant |X| - k$. By minimality $r_{M \vee M}(X\backslash e) > |X\backslash e| - k$ for all $e \in X$. Since $X \neq \varnothing$, it follows that $r_{M \vee M}(X) = |X| - k$ and $r_{M \vee M}(X\backslash e) = r_{M \vee M}(X)$ for all $e \in X$, i.e. X has no isthmuses in $M \vee M$. Hence by Lemma C.1 Section 6 $X = P(M(X)) \in \mathscr{P}(M(X))$. By Lemma 4.1 any basis A of X in $M \vee M$ is a spanning block of $M(X)$. Since $r_{M \vee M}(X) = |X| - k$ we have $|X\backslash A| = k$. Hence X is a k-augmented block of M. ∎

Proof of Proposition 7.2. (i) Clearly M contains a spanning block if and only if $r(M \vee M) = 2r(M)$.

(ii) As easily seen, M contains a spanning diminished block if and only if $r(M \vee M) \geqslant 2r(M) - 1$. On the other hand by the case $k = 1$ of Lemma 7.4 and Lemma 7.3, M^* contains an augmented block if and only if $r(M \vee M) \leqslant 2r(M) - 1$.

(iii) In view of Lemma 7.3, this is the case $k = 2$ of Lemma 7.4. ∎

Using Proposition 7.2 we get the following alternate classification of the Extended Lehman Switching Game:

THEOREM 7.5. *Let M be a matroid. The Extended Lehman Switching on M is*

(i) *winning for* WHITE *playing second if and only if $r(M \vee M) = 2r(M)$*

(*resp.* (ii) *winning for the first player if and only if $r(M \vee M) = 2r(M) - 1$*

(*resp.* (iii) *winning for* BLACK *playing second if and only if $r(M \vee M) \leqslant 2r(M) - 2$.*

By definition the Unrooted Lehman Switching Game on a matroid M is equivalent to the dual Extended Lehman Switching Game on M^*. Its classification is obtained by dualizing Theorems 7.1, 7.5.

The dual of Theorem 7.5 in the case when M is the cycle matroid of a graph is given by Kano in [9] Theorem 3.

8. Directed Switching Games and Signing Games on Oriented Matroids

Matroids provide a natural context for switching games as shown by Lehman theory. Analogously a natural context for directed switching games is provided by oriented matroids.

We introduce in this section several directed switching games on oriented matroids. The first two games generalize the Directed Shannon Switching Game and the One-Way Game defined in the introduction.

All directed switching games are two-player games with complete information. We define a game by describing its board, BLACK and WHITE moves and WHITE winning configurations. In each game the objective of WHITE is to form a winning configuration and the objective of BLACK is to prevent WHITE from succeeding. We recall that *marking* an element means assigning a color to it (black or white) and that *signing* an element means marking it and in addition reversing or not its sign.

I. *The Directed Switching Game*

The board is an oriented matroid M with a distinguished element e not subject to play. A move of BLACK consists of marking an unplayed element, a move of WHITE consists of signing an unplayed element. A WHITE winning configuration is a white positive circuit of M broken at e.

II. *The Signing Game*

The board is an oriented matroid M with a distinguished element e not subject to play. Moves of both BLACK and WHITE consist of signing an unplayed element. A WHITE winning configuration is a marked positive circuit of M broken at e.

III. *The Unrooted Directed Switching Game*

The board is an oriented matroid M. A move of BLACK consists of marking an unplayed element. A move of WHITE consists of signing an unplayed element. A WHITE winning configuration is a white positive circuit of M.

IV. *The Unrooted Signing Game*

The board is an oriented matroid M. Moves of both BLACK and WHITE consist of signing an unplayed element. A WHITE winning configuration is a marked positive circuit of M.

V–VIII.

For each game I–IV another game is obtained by not allowing the players to reverse element signs: moves of both BLACK and WHITE consist of marking elements.

In Sections 9–13 we solve the first three games in the cases of graphic and cographic oriented matroids. In graphic games a third WHITE winning configuration is considered (Sect. 13) namely a spanning arborescence rooted at a given vertex (the game is called *extended*). Unlike the undirected case there is no evident generalization to (oriented) matroids.

Our results may be summarized by saying that the classifications of the graphic and cographic Directed Switching Game, Signing Game and Unrooted Directed Switching Game are identical to the classification of the corresponding undirected switching game on the same board (see Theorems 9.3, 10.2, 12.1, 13.1, 13.2).

We conjecture that this property generalizes to any oriented matroid for Games I, II, III:

CONJECTURE 8.1. *The Directed Switching Game on an oriented matroid M with respect to a given element e is winning for* WHITE *playing second if (and only if) there is a block of M spanning but not containing e.*

The "only if" part of Conjecture 8.1 follows from Lehman's results. If the "if" part is true, the classification of the Directed Switching Game can easily be achieved (see Sects. 9, 10). The classifications of the Signing Game and of the Unrooted Directed Switching Game follow as corollaries (see Sects. 12, 13). Partial results, in particular in the case of regular matroids, will be the subject of another paper.

The classifications of games IV–VIII are different in general from the classifications of the corresponding undirected game on the same board (see Remark 9.5 and Sect. 14).

Finally we mention that the above games are closely related to convexity. In Games I, II we get clearly an equivalent version by reversing the sign of the distinguished element e. A WHITE winning configuration is then a signed circuit C with $C^- = \{e\}$ broken at e. By definition of convexity in oriented matroids, $C^- = \{e\}$ means that e is in the convex hull of $C \setminus \{e\}$ [11]. In other words the objective of WHITE in the (modified) Directed Switching Game resp. Signing Game is to capture e in the convex hull of white (resp. marked elements). Positive circuits considered in other games have also a natural interpretation in terms of convexity.

9. The Directed Switching Game (Graphic Case)

The Graphic Directed Switching Game is equivalent to the Directed Shannon Switching Game defined in the introduction (Section 1). The game is played on an undirected graph G with two distinguished vertices x_0 and x_1. A move of WHITE consists of directing an unplayed edge. A move of BLACK consists of marking an unplayed edge. The objective of WHITE is to form a white directed path joining x_0 to x_1, the objective of BLACK is to preven WHITE from meeting his objective. (In view of WHITEs objective we may equivalently consider that BLACK deletes edges as in the original Shannon game).

A first idea to solve this game would be to direct Lehman strategy. The graph in Fig. 1 with the indicated trees T_1 and T_2 shows that this idea does not work.

Suppose BLACK marks the edge b at his first move. Lehman strategy with respect to T_1 and T_2 prescribes one of the three edges marked w as a response for WHITE. The reader can easily convince himself that these three edges with any of the two possible directions are losing for WHITE, despite the fact that the game is winning for WHITE playing second (see below).

A difficulty arises from the fact that Lemma A.1 (see Sect. 6) which is fundamental for Lehman Strategy does not hold in the directed case in general. Given a black edge b and a white directed edge w, the existence of a white directed path joining x_0 and x_1 in $G\backslash b/w$ does not imply the existence of such a path in G.

This failure can be repaired in the graphic (and cographic) case on the basis of the observation that two adjacent directed edges behave nicely with respect to orientation. With some elaboration this idea leads to the desired strategy.

The basic situation to consider is that of a connected block-graph. We prove the directed counterpart (in the graphic case) of Lehman Theorem for the Extended Lehman Switching Game (Theorem A′, Sect. 7). Here the objective of WHITE is to direct a spanning arborescence of G rooted at a

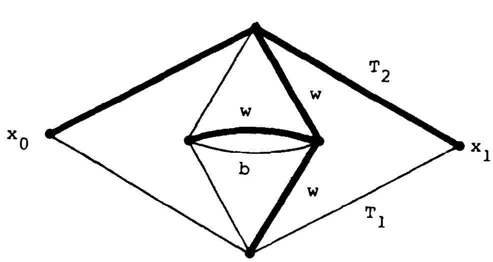

Figure 1

given vertex. We point out that, like Lehman Theorems A and A′, this theorem is actually equivalent to the (apparently weaker) theorem dealing with directed paths joining two distinguished vertices.

We recall from Section 5 that the collection of blocks of a block-graph is closed under union and intersection. Hence any subset of edges contains a maximal block (possibly empty) and is contained in a minimal block.

LEMMA 9.1. *Let G be a connected block-graph, $x_0 \in V(G)$ and $b \in E(G)$. There is a unique inclusion-minimal connected block of G incident to x_0 and containing b.*

Proof. Let T_1, T_2 be two edge-disjoint spanning trees of G partitioning $E(G)$. Suppose notations such that $b \in T_1$. Let A be the edge-set of the path of T_1 joining x_0 to a vertex of b and containing b.

Consider a connected block X of G incident to x_0 and containing b. It follows from Lemma 5.4, X being connected, that $T_1 \cap X$ and $T_2 \cap X$ are two spanning trees of X. Since X is incident to x_0 and contains b, we have $A \subseteq X$. Hence the unique inclusion-minimal block of G containing A (clearly connected, incident to x_0 and containing b) is also the unique inclusion-minimal connected block of G incident to x_0 and containing b. ∎

Strategy I

(WHITE strategy for the Extended Graphic Directed Switching Game and the Directed Shannon Switching Game, BLACK playing first.

Let G be a connected block-graph with a distinguished vertex x_0.

(1) Let $b \in E(G)$ be the first move of BLACK.

Let P be the minimal connected block of G incident to x_0 and containing b. Let Q be the maximal connected block of G incident to x_0 and not containing b (Q may be empty). WHITE directs an edge $w \in P \setminus (\{b\} \cup Q)$ incident to $\{x_0\} \cup V(Q)$ in the direction outgoing from $\{x_0\} \cup V(Q)$.

(2) Suppose BLACK and WHITE have already played the edges $b_1, b_2, \ldots, b_i$ and $w_1, w_2, \ldots, w_i$, $i \geqslant 0$, respectively. Let $b_{i+1} \in E(G) \setminus \{b_1, b_2, \ldots, b_i, w_1, w_2, \ldots, w_i\}$ be the $(i+1)$th move of BLACK.

Then WHITE considers b_{i+1} as a first move of BLACK in the graph $G' = G \setminus \{b_1, b_2, \ldots, b_i\} / \{w_1, w_2, \ldots, w_i\}$ and directs an edge w_{i+1} given by (1) applied to G' and b_{i+1}.

THEOREM 9.2. *Let G be a connected block-graph with a distinguished vertex x_0. In the directed Switching Game on G,* WHITE *playing second with Strategy* I *directs a spanning arborescence of G rooted at x_0 against all strategies of* BLACK.

Proof. (1) Strategy I is well defined (i.e., provides at least one response for WHITE to a move of BLACK for any game on G).

Let $b \in E(G)$ be the first move of BLACK. Since Q is a connected block of the block-graph G, every edge of G with both vertices in $V(Q)$ belongs to Q (see Sect. 5). Hence $V(P)\backslash(\{x_0\} \cup V(Q)) \neq \varnothing$, since this set contains at least one vertex of b. On the other hand $x_0 \in V(P)$. The block P is connected, hence 2-edge connected. It follows that at least two edges of $P\backslash Q$ are incident to $\{x_0\} \cup V(Q)$. At least one, w, is different from b.

Clearly w is not in the maximal block of G not containing b (since Q is a connected component of this block). Hence by Proposition 5.5 $G\backslash b/w$ is again a (connected) block. By induction on $|E(G)|$, Part (2) of the definition of Strategy I implies that this strategy is well defined.

(2) Let $b_1, b_2, \ldots, b_m$ and $w_1, w_2, \ldots, w_m$, $2m = |E(G)|$, be the lists of edges played by BLACK and WHITE, respectively, in a game, WHITE playing Strategy I. We prove by induction on $m \geqslant 1$ that $w_1, w_2, \ldots, w_m$ constitute the edges of a spanning arborescence of G rooted at x_0, with the additional property that the restriction of this arborescence to any connected block X of G incident to x_0 is a spanning arborescence of $G[X]$ rooted at x_0.

As noted in (1) $G' = G\backslash b_1/w_1$ is a connected block-graph. Clearly $b_2, \ldots, b_m$ and $w_2, \ldots, w_m$ constitute the lists of edges played by BLACK and WHITE, respectively, in a game on G', WHITE playing Strategy I. Set $A' = \{w_2, \ldots, w_m\}$. By the induction hypothesis $A' \cap X'$ is a spanning arborescence of $G'[X']$ rooted at x_0 for every connected block X' of G' incident to x_0.

Set $A = \{w_1, w_2, \ldots, w_m\}$. Let X be a connected block of G incident to x_0. We show that $A \cap X$ is a spanning arborescence of $G[X]$ rooted at x_0. We distinguish two cases.

(2.1) If $b_1 \notin X$ then $X \subseteq Q$, hence $w_1 \notin X$. By Proposition 5.6 X is a block of G'. By the induction hypothesis $A \cap X = A' \cap X'$ is a spanning arborescence of $G[X] = G'[X']$ rooted at x_0.

(2.2) If $b_1 \in X$ then $P \subseteq X$, hence $w_1 \in X$. By Proposition 5.6 $X' = X\backslash\{b_1, w_1\}$ is a block of G'. By the induction hypothesis $A' \cap X'$ is a spanning arborescence of $G'[X']$ rooted at x_0. If $A \cap X$ is not a spanning arborescence of $G[X]$ rooted at x_0 then necessarily the terminal vertex $x \notin \{x_0\} \cup V(Q)$ of w_1 in G is also the terminal vertex of an edge w_i, $i \geqslant 2$, in G. The initial vertex of w_i in G is not in $\{x_0\} \cup V(Q)$ otherwise $Q \cup \{w_1, w_i\}$ would be a block of G contradicting the definition of Q. It follows that in G' the edge w_i enters $\{x_0\} \cup V(Q)$ at x. Hence $A' \cap Q$ is not a spanning arborescence of $G'[Q]$ rooted at x_0, a final contradiction since Q is a block of G' by Proposition 5.6. ∎

THEOREM 9.3. *The classification of the Directed Shannon Switching Game on a graph G with respect to two distinguished vertices* x_0, x_1 *is iden-*

tical to the classification of the (undirected) Shannon Switching Game on G with respect to x_0 and x_1.

In particular, the Directed Shannon Switching Game on G with respect to x_0 and x_1 is winning for WHITE *playing second if and only if there are two edge-disjoint trees of G on a same set of vertices containing x_0 and x_1.*

Proof. Let M be the cycle matroid of the graph $G+e$, where e is an edge not in G joining x_0 and x_1. By Lehman Theorem B there are three cases:

(1) M contains a block spanning but not containing e. Equivalently by Lehman Theorem B, the Shannon Switching Game is winning for WHITE playing second.

The Directed Shannon Switching Game is a restricted positional game (see Sect. 2). Hence by Theorem 9.2 and Proposition 3.2 applied to a block of M spanning but not containing e, WHITE playing second has a winning strategy.

(2) M^* contains a block spanning but not containing e. Equivalently by Lehman Theorem B, the Shannon Switching Game is winning for BLACK playing second.

BLACK playing second with Lehman strategy can separate x_0 and x_1 by a black cocycle. Hence WHITE cannot join x_0 to x_1 by a white directed path. The Directed Shannon Switching Game is winning for BLACK playing second.

(3) There is a block of M containing e, and a block of M^* containing e. Equivalently by Lehman Theorem B the Shannon Switching Game is winning for the first player.

It follows clearly from cases (1) and (2) that WHITE playing first can join x_0 to x_1 by a white directed path and that BLACK playing first can separate x_0 and x_1. The first player has a winning strategy in the Directed Shannon Switching Game. ∎

Theorem 9.2 means that WHITE can join x_0 to x_1 by a directed path (against all strategies of BLACK) if and only if he can join x_0 and x_1 by an undirected path. Equivalently BLACK can bar a white directed path joining x_0 to x_1 if and only if he can separate x_0 and x_1 (by a black cocycle).

Remark 9.4. The above solution of the Directed Shannon Switching Game is not complete in the following sense: In the undirected game one can easily decide, using Lehman theory, whether a partially played game i.e. a *position* is winning or losing for a given player. No information on the previously used strategies is needed. It suffices to use the following obvious extension of

LEMMA A.1. *For any subsets B of black edges and W of white edges, G contains a white path joining x_0 and x_1 if and only if $G\backslash B/W$ contains a white path joining x_0 and x_1.*

For instance, a position $b_1, b_2, ..., b_i, w_1, w_2, ..., w_i$ $i \geqslant 1$ is winning for WHITE playing second if and only if $G\backslash\{b_1, b_2, ..., b_i\}/\{w_1, w_2, ..., w_j$ contains a block incident to x_0 and x_1.

In the directed case Theorem 9.3 recognizes winning/losing *starting* positions (no edges already played). However, since the directed version of Lemma A.1 does not hold, no simple consequence results for general positions. We ask:

PROBLEM. *Find a theory to recognize winning/losing general positions in the directed Shannon Switching Game.*

Remark 9.5. The graphic case of Game V, defined in Section 8 for oriented matroids, is played on a directed graph $\vec{G}$ with two distinguished vertices x_0, x_1. Both BLACK and WHITE mark unplayed edges. The objective of WHITE is to form a directed path joining x_0 to x_1.

There are connected block-graphs G such that Game V is losing for WHITE playing second on $\vec{G}$ for any orientation $\vec{G}$ of G (examples with 2-cocycles are easily obtained; there are also 3-edge-connected ones).

In other words, in general, there is no winning strategy for the Directed Shannon Switching Game on a block-graph such that edge directions do not depend on the previous course of the game.

10. THE DIRECTED SWITCHING GAME (COGRAPHIC CASE)

The rules of the Cographic (Rooted) Directed Switching Game are those of the Graphic (Rooted) Directed Switching Game (equivalently the Directed Shannon Switching Game) except for the objective of WHITE. The board is a graph G with two distinguished vertices x_0 and x_1, $x_0 \neq x_1$. WHITE directs edges, BLACK marks edges. The objective of WHITE is to separate x_0 and x_1 by an elementary cocycle with all edges directed from the component of x_0 to the component of x_1.

The Cographic Directed Switching Game is not equivalent to the Dual Graphic Directed Switching Game. On the other hand Strategy I for the graphic game cannot be adapted to the cographic game. We construct a different strategy.

The basic situation to consider is that of a graph G with two distinguished vertices x_0 and x_1, $x_0 \neq x_1$, such that the graph obtained from G by identifying x_0 and x_1 is a block-graph. Let M be the cocycle matroid of the graph obtained from G by adding a new edge e joining x_0 and x_1:

$M = B(G+e)$. The above hypothesis on G says that $E(M)\backslash\{e\}$ is a block of M spanning e. This condition is expressed conveniently by considering the graph H obtained from G by adding two new edges e and e' joining x_0 and x_1: the hypothesis on G holds if and only if H is a block-graph.

It follows from Section 5 that if H is a block-graph, the collection of blocks of H is closed under union and intersection. Hence any subset of $E(H) = E(G) \cup \{e, e'\}$ contains a maximal (possibly empty) and is contained in a minimal block.

Strategy II

(WHITE strategy for the Cographic Directed Switching Game, BLACK playing first).

Let G be a graph with two distinguished vertices x_0 and x_1, $x_0 \neq x_1$, such that the graph obtained by identifying x_0 and x_1 is a block-graph. Let H be the graph obtained from G by adding two new edges joining x_0 and x_1.

(1) Let $b \in E(G)$ be the first move of BLACK. Let P be the minimal block of H containing b. We distinguish two cases:

(1.1) $P \subseteq E(G)$ (i.e., there is a block of G containing b).

Let Q be the maximal block of G not containing b. WHITE directs arbitrarily an edge $w \in P\backslash(\{b\} \cup Q)$.

(1.2) $P \nsubseteq E(G)$.

Let Q_0 and Q_1 be the maximal connected blocks of G incident to x_0 and x_1, respectively. WHITE directs an edge of G $w \in P\backslash(\{b\} \cup Q_0 \cup Q_1)$ incident to $\{x_0\} \cup V(Q_0)$ or to $\{x_1\} \cup V(Q_1)$, in the direction outgoing from $\{x_0\} \cup V(Q_0)$ in the first case respectively entering $\{x_1\} \cup V(Q_1)$ in the second case.

(2) Suppose BLACK and WHITE have already played the edges $b_1, b_2, ..., b_i$ and $w_1, w_2, ..., w_i$ $i \geqslant 0$, respectively. Let $b_{i+1} \in E(G)\backslash\{b_1, b_2, ..., b_i, w_1, w_2, ..., w_i\}$ be the $(i+1)$th move of BLACK.

Then WHITE considers b_{i+1} as a first move of BLACK in the graph $G' = G/\{b_1, b_2, ..., b_i\}\backslash\{w_1, w_2, ..., w_i\}$ and directs an edge w_{i+1} given by (1) applied to G' and b_{i+1}.

THEOREM 10.1. *Let G be a graph with two distinguished vertices x_0 and x_1, $x_0 \neq x_1$, such that the graph obtained from G by identifying x_0 and x_1 is a block-graph.*

Strategy II *is a winning strategy for* WHITE *playing second in the Cographic Directed Switching Game on G with respect to x_0 and x_1.*

Proof. (1) Strategy II is well defined for any game on G. Let $b \in E(G)$ be the first move of BLACK.

Case 1. $P \subseteq E(G)$. Since P and $P \cap Q$ are blocks $P \backslash G$ has even cardinality. Hence $b \in P \backslash Q$ implies that $P \backslash (\{b\} \cup Q)$ is not empty.

Case 2. $P \nsubseteq E(G)$. Since there is no block of G containing b, necessarily P contains e and e' hence P is incident to x_0 and x_1. We have $b \notin Q_0 \cup Q_1$ hence P is not contained in $Q_0 \cup Q_1 \cup \{e, e'\}$. The block P being minimal is connected, hence 2-edge-connected. It follows that at least two edges of $P \backslash (Q_0 \cup Q_1)$ are incident to $\{x_0, x_1\} \cup V(Q_0) \cup V(Q_1)$. At least one is different from b.

We have $V(Q_0) \cap V(Q_1) = \varnothing$ otherwise the definition of Q_0, Q_1 immediately implies $Q_0 = Q_1$ contradicting the hypothesis $H = G + \{e, e'\}$ is a block. By definition of a block, it follows that no edge of G is incident to both $\{x_0\} \cup V(Q_0)$ and $\{x_1\} \cup V(Q_1)$. In particular Strategy II unambigously defines a direction for any edge in $P \backslash (\{b\} \cup Q_0 \cup Q_1)$ incident to $\{x_0, x_1\} \cup V(Q_0) \cup V(Q_1)$.

Thus Strategy II provides at least one response w for WHITE to the first move b of BLACK. Since in both cases $w \in P$ it follows from Proposition 5.5 that $H/b\backslash w$ is again a block. By induction on $|E(G)|$, Part (2) of the definition of Strategy II implies that this strategy is well defined.

(2) Strategy II is winning for WHITE.

It suffices to show that Strategy II is winning in a game where all edges have been played. Let $b_1, b_2, ..., b_m$ and $w_1, w_2, ..., w_n$, $2m = |E(G)|$, be the lists of edges played by BLACK and WHITE, respectively, in a game on G, WHITE playing Strategy II. We prove by induction on $m \geqslant 1$ that $\{w_1, w_2, ..., w_m\}$ contains an elementary cocycle of G with all edges directed from the component of x_0 to the component of x_1 and with the additional property that the intersection of this cocycle with every block of G is empty.

As noted in (1) $G' = G/b_1\backslash w_1$ satisfies the hypothesis of Theorem 10.1. Clearly $b_2, ..., b_m$ and $w_2, ..., w_m$ constitute the lists of edges played by BLACK and WHITE, respectively, in a game on G', WHITE playing Strategy II. Hence by the induction hypothesis $\{w_2, ..., w_m\}$ contains an elementary cocycle C' of G' with the desired properties. Let C be the unique elementary cocycle of G such that $C' = C \backslash \{w_1\}$.

Let X be a block of G. If $b_1 \in X$ we have $P \subseteq X$ hence $w_1 \in X$. If $b_1 \notin X$ we have $X \subseteq Q$ hence $w_1 \notin X$ (since Q_0 and Q_1 are connected components of Q). Since H and $H' = H/b_1\backslash w_1$ are blocks, it follows from Proposition 5.6 that $X' = X \backslash \{b_1, w_1\}$ is a block of H', hence also a block of G'. We have $X' \cap C' = \varnothing$ by the induction hypothesis. Hence $X \cap C \subseteq \{w_1\}$, implying $X \cap C = \varnothing$ since a cocycle always meet a block in 0 or $\geqslant 2$ elements.

It remains to verify that C has the right orientation. In the first case, since $P \subseteq E(G)$, we have $P \cap C = \varnothing$, hence $w_1 \notin C$. In the second case we have $Q_0 \cap C = \varnothing$ and $Q_1 \cap C = \varnothing$. Hence $\{x_0\} \cup V(Q_0)$ and $\{x_1\} \cup V(Q_1)$ inducing connected subgraphs of G are contained respectively in the components of x_0 and x_1 in $G \backslash C$. The desired property follows. ∎

THEOREM 10.2. *The classification of the Cographic Directed Switching Game on a graph G with respect to two distinguished vertices x_0, x_1 is identical to the classification of the (undirected) Dual Shannon Switching Game on G with respect to x_0 and x_1.*

The proof is left to the reader.

11. ALGORITHMS

Strategies I and II depend on constructions of blocks. We describe in this section algorithms for that purpose. The fundamental construction is that of two maximally distant bases of a matroid M, i.e., of a basis of $M \vee M$. This problem is equivalent to the Matroid Intersection Problem (see [13, 8.5]), hence Edmonds Algorithm can be used. We recall briefly a specific algorithm given in [3].

Let B_1, B_2 be two bases of a matroid M (two spanning trees in the case of a connected graph). Let $e \in E = E(M)$. If $e \in E \backslash B_1$ we denote by $s_1(e)$ the unique circuit of M contained in $B_1 \cup \{e\}$. If $e \in B_1$ we set $s_1(e) = \{e\}$. We define similarly $s_2(e)$ with respect to B_2 and we set $s(e) = s_1(e) \cup s_2(e)$. Given $A \subseteq E$ let $s(A) = \bigcup_{e \in A} s(e)$ and let $s^i(A)$ be defined inductively by $s^1(A) = s(A)$ and $s^{i+1}(A) = s(s^i(A))$. Since M is finite and $s^i(A) \subseteq s^{i+1}(A)$, for i large enough we have $s^i(A) = s^{i+1}(A) = \cdots$. We set $\hat{s}(A) = s^i(A) = s^{i+1}(A) = \cdots$.

If there is $e \in E \backslash (B_1 \cup B_2)$ such that $\hat{s}(e) \cap B_1 \cap B_2 \neq \varnothing$, then $B_1 \cup B_2$ can be *augmented*. Let i_1 be the smallest index i such that $s^i(e) \cap B_1 \cap B_2 \neq \varnothing$. Let $e_1 \in s^{i_1}(e) \cap B_1 \cap B_2$. Define inductively $i_1, i_2, \ldots,$ and $e_1, e_2, \ldots,$ by the properties i_{j+1} is the smallest index i such that $e_{j+1} \in s^i(e)$, $e_j \in s(e_{j+1})$. For j large enough the construction stops with $e_k = e$. Note that alternately $e_j \in s_1(e_{j-1})$ and $e_{j+1} \in s_2(e_j)$, depending on the parity of j and on the situation for e_1 and e_2. Suppose, for instance, $e_1 \in s_1(e_2)$ and k is even. We have $e_3, e_5, \ldots, e_{k-1} \in B_1 \backslash B_2$ and $e_2, e_4, \ldots, e_{k-2} \in B_2 \backslash B_1$. Then $B_1' = B_1 \backslash \{e_1, e_3, \ldots, e_{k-1}\} \cup \{e_2, e_4, \ldots, e_k\}$ and $B_2' = B_2 \backslash \{e_2, e_4, \ldots, e_{k-2}\} \cup \{e_3, e_5, \ldots, e_{k-1}\}$ are two bases of M, and $B_1' \cup B_2' = B_1 \cup B_2 \cup \{e\}$ [3].

Suppose $\hat{s}(e) \cap B_1 \cap B_2 = \varnothing$ for all $e \in E \backslash (B_1 \cup B_2)$, i.e., $\hat{s}(E \backslash (B_1 \cup B_2)) \cap B_1 \cap B_2 = \varnothing$. Set $X = \hat{s}(E \backslash (B_1 \cup B_2))$. Clearly $X \cap (B_1 \cup B_2)$ spans $X \backslash (B_1 \cup B_2) = E \backslash (B_1 \cup B_2)$, $X \cap B_1$ spans $X \cap B_2$ and $X \cap B_2$ spans $X \cap B_1$. In other words $X \cap B_1$ and $X \cap B_2$ are two distant bases of X. Hence by

Lemma 4.1 B_1, B_2 are two maximally distant bases of M and $2r_M(X)+|E\backslash X|=r(M \vee M)$ (i.e., $X\in\mathscr{P}(M)$ with notations of Sect. 4). Thus B_1, B_2 are two maximally distant bases of M if and only if $\hat{s}(E\backslash(B_1\cup B_2))\cap B_1\cap B_2=\varnothing$.

Let B_1, B_2 be two maximally distant bases of M. By Lemma 4.1 we have $\hat{s}(E\backslash(B_1\cup B_2))\subseteq P(M)$ (notations of Sect. 6). Hence $\hat{s}(E\backslash(B_1\cup B_2))=P(M)$ by minimality [3]. As is easily seen $Q(M)=P(M)\cup\bigcup_x \hat{s}(x)$, where the union is over all $x\in B_1\cup B_2$ such that $\hat{s}(x)\cap B_1\cap B_2=\varnothing$ (alternately, by duality $Q(M)=E\backslash\widehat{s^*}(B_1\cap B_2)$, where $\widehat{s^*}$ is defined in M^* with respect to $E\backslash B_1$ and $E\backslash B_2$) [3].

Suppose M is a block-matroid, i.e., $E(M)=B_1\cup B_2$, B_1, B_2 two disjoint bases of M. In this case by the above discussion, for any $A\subseteq E=E(M)$, $\hat{s}(A)$ is the inclusion-minimal block of M containing A. The inclusion-maximal block of M contained in A is $\bigcup_e \hat{s}(e)$, where the union is over all $e\in A$ such that $\hat{s}(e)\subseteq A$.

The Lehman Strategy for the (Extended) Switching Game on M is, given a BLACK move $b\in B_1$, that WHITE should respond any $w\in B_2$ such that $b\in s_1(w)$. Then $M\backslash b/w$ is again a block (see Sect. 6). By Proposition 5.5 $M\backslash b/w$ is a block if and only if $w\notin Q$ the maximal block of $M\backslash b$. Hence given B_1, B_2, the most general winning strategy for WHITE in the Lehman Switching Game on M is to play $w\in E\backslash\{b\}$ such that $b\in\hat{s}(w)$ (note that, by definition, for $w\in B_2$ we have $s_1(w)\subseteq\hat{s}(w)$).

Let G be a (connected) block-graph with a given vertex x_0. Let $b\in E(G)$ be the first move of BLACK. In Strategy I we have to construct (1) the (unique) connected block Q incident to x_0, not containing b and inclusion-maximal with these properties, and (2) the (unique) connected block P incident to x_0, containing b and inclusion-minimal with these properties.

The constructions are based on two given edge-disjoint spanning trees T_1, T_2 of G such that $T_1\cup T_2=E=E(G)$. Let $\hat{s}$ be defined as above with respect to T_1 and T_2.

(1) Let $X_1=\bigcup_e \hat{s}(e)$, the union being over all edges e of G incident to x_0 such that $b\notin\hat{s}(e)$. If $X_1=\varnothing$ then $Q=\varnothing$. If $X_1\neq\varnothing$, let $X_2=\bigcup_e \hat{s}(e)$, the union being over all edges e of G incident to $V(X_1)$ such that $b\notin\hat{s}(e)$. Going on inductively we define a sequence $X_1, X_2, \ldots$. Since $X_i\subseteq X_{i+1}$, for i large enough we have $X_i=X_{i+1}=\cdots$. Then $Q=X_i=X_{i+1}=\cdots$.

(2) Suppose $b\in T_1$. Let A be the edge-set of the (unique) path of T_1 containing b joining x_0 to a vertex of b. Then $P=\hat{s}(A)$.

The justifications of these constructions are left to the reader (for the second construction see the proof of Lemma 9.1).

The WHITE response prescribed by Strategy I is an edge $w\in P\backslash(\{b\}\cup Q)$ incident to $\{x_0\}\cup V(Q)$. In practice if we are given T_1 and T_2 with $b\in T_1$ the construction of P is not necessary: At least one vertex of b is not in $\{x_0\} \cup V(Q)$. Then the path of T_2 joining x_0 to a vertex of b not in $\{x_0\} \cup V(Q)$ is contained in P. The first edge leaving $\{x_0\} \cup V(\Omega)$ on this path is a possible response w for WHITE.

Using this last construction we have $b \in T_1$ (say) and $w \in T_2$. Then $T_2' = T_2 \backslash \{w\}$ is a spanning tree of the graph $G' = G \backslash b/w$. However $T_1' = T_1 \backslash \{b\}$ is either a spanning tree of G' (if $b \in s_1(w)$) or consists of two components, a tree and a unicyclic graph, partitioning $V(G')$ (if $b \notin s_1(w)$). To apply the construction in G' we need two edge-disjoint spanning trees of G'. In the second case let e be an edge of the cycle of T_1' and f be an edge of T_2' joining the two components of T_1'. Then $T_1'' = (T_1' \backslash \{e\}) \cup \{f\}$ is a spanning tree of G'. Applying the algorithm of Bruno–Weinerg to T_1'', T_2', and e, we get two edge-disjoint spanning trees of G' in exactly one augmentation.

Similar constructions can be given for Strategy II, using two edge-disjoint spanning trees of $H = G \cup \{e, e'\}$.

All algorithms discussed in this section have a polynomial complexity.

12. The Signing Game (Graphic and Cographic Cases)— The One-Way Game

The Graphic Signing Game is equivalent to the One-Way Game defined in Section 1.

We recall that the One-Way Game is played on a graph G with two distinguished vertices x_0, x_1. A move (of both BLACK and WHITE consists of signing an unplayed edge. The objective of WHITE is to form a directed path joining x_0 to x_1 (which may contain edges played by BLACK). The objective of BLACK is to prevent WHITE from meeting his objective.

Let $\vec{G}$ be any orientation of G and let M be the oriented cycle matroid of the directed graph $\vec{G} + \vec{e}$, where $\vec{e}$ is an edge not in G joining x_0 to x_1. The Graphic Signing Game on M with respect to e is clearly equivalent to the One-Way Game on G with respect to x_0 and x_1.

The Cographic Signing Game is equivalent to the Dual Graphic Signing Game. More generally the Signing Game on an oriented matroid M with respect to an element e is equivalent to the Dual Signing Game on M^* with respect to e (proof: in an oriented matroid M a given element e is either contained in a positive circuit of M or in a positive cocircuit but not in both [2, Proposition 3.4].

The classification of the Graphic Signing Game (hence equivalently of the Cographic Signing Game) follows from the results of Sections 9 and 10.

THEOREM 12.1. *The classification of the One-Way game on a graph G with respect to two distinguished vertices x_0, x_1 is identical to the* *classification of the (undirected) Shannon Switching Game on G with respect to x_0 and x_1.*

In particular the One-Way Game on G with respect to x_0 and x_1 is winning for WHITE *playing second if and only if there are two edge-disjoint trees of G on a same set of vertices containing x_0 and x_1.*

Proof. Let M be the cycle matroid of the graph $G+e$, where e is an edge not in G joining x_0 and x_1. By Lehman Theorem B there are three cases:

(1) M contains a block spanning but not containing e.

By Theorem 9.2 WHITE playing second (with Strategy I possibly modified by "fictitious moves") can sign a (white) directed path joining x_0 to x_1 against all strategies of BLACK. The One-Way Game is winning for WHITE playing second.

(2) M^* contains a block spanning but not containing e.

By Theorem 10.2 BLACK playing second (with Strategy II possibly modified by "fictitious moves") can sign a (black) elementary cocycle with all edges directed from the component of x_1 to the component of x_0. Therefore WHITE cannot form a directed path joining x_0 to x_1, even by including edges played by BLACK. The One-Way Game is winning for BLACK playing second.

(3) There is a block of M containing e, and a block of M^* containing e.

It follows clearly from cases (1) and (2) that the One-Way Game is winning for the first player. ▮

The proof of Theorem 12.1 shows that both the One-Way Game and the Dual One-Way Game, which are uncolored restricted positional games in the terminology of Section 2, actually reduce to *colored* restricted positional games.

13. Extended and Unrooted Directed Switching Games

We derive in this section the classifications of the Extended Directed Shannon Switching Game and the graphic and cographic Unrooted Directed Switching Game as corollaries of the results of Sections 9 and 10. Unlike the undirected case the Extended Directed Shannon Switching Game and the cographic Unrooted Directed Switching Game are not dual games.

THEOREM 13.1. *The classification of the Extended Directed Shannon Switching Game on a graph G with respect to any vertex of G is identical to the classification of the Extended Lehman Switching Game on* $\mathbb{C}(G)$ (see Theorems 7.1, 7.5).

Proof. The proof of Theorem 13.1 from Theorem 9.2 is similar to the proof of Therem 7.1 from Theorem A' (Strategy I replacing Lehman Strategy). ∎

THEOREM 13.2. *The classification of the Unrooted Directed Switching Game in the graphic and cographic cases is identical to the classification of the undirected game.*

Proof. (1) Graphic case: Let G be a graph and $M = \mathbb{C}(G)$ be the cycle matroid of G. By Theorem 7.1 there are three mutually exclusive cases:

(1.1) M contains a 2-augmented block. We show that the game is winning for WHITE playing second.

By Proposition 3.2 we may suppose that M is an inclusion-minimal 2-augmented block matroid. Let $b_1 \in E = E(M)$ be the first move of BLACK. It is easily seen using the Union Theorem that $E\setminus\{b_1\}$ is an augmented block. Let any element $w_1 \in E\setminus\{b_1\}$ be the first move of WHITE. Then $E\setminus\{b_1, w_1\}$ is a block of $M\setminus b_1$ spanning w_1. In the Directed Shannon Switching Game on $G\setminus\{b_1, w_1\}$ WHITE playing second (with Strategy I) can mark a directed path joining the terminal vertex of w_1 to its initial vertex by Theorem 9.2. Thus WHITE playing second can sign a directed circuit of G against all strategies of BLACK.

(1.2) M^* contains a spanning block.

By Lehman Theorem A' BLACK playing second can mark a cobase of M against all strategies of WHITE. Hence WHITE cannot mark any circuit, and a fortiori cannot sign a directed circuit. The game is winning for BLACK playing second.

(1.3) M contains an augmented block and M^* contains a spanning diminished block.

The arguments used in (1.1) and (1.2) are easily adapted with WHITE (resp. BLACK) playing first. The game is winning for the first player.

(2) Cographic case.

The proof is analogous with Theorem 10.2 and Strategy II in place of Theorem 9.3 and Strategy I. ∎

We recall that the Unrooted Lehman Switching Game is the dual game of the Extended Lehman Switching Game up to matroid duality (see Sect. 7). The classification of the Unrooted Lehman Switching Game on a matroid M is given by the dual of Theorem 7.1.

14. Unrooted Signing Games

The graphic Unrooted Signing Game is played on a graph G, moves of both players consist of signing unplayed edges. The objective of WHITE is to form a marked directed cycle of G.

We do not have a solution of this game. If G is a block then WHITE playing second has a winning strategy: Let x_0 and x_1 be the terminal and initial vertices, respectively, of the first edge b_1 played by BLACK. Then WHITE playing Strategy I can form a white directed path joining x_0 to x_1 against all strategies of BLACK by results of Section 9. Hence WHITE wins the Unrooted Signing Game.

However WHITE playing first on a block may lose. A trivial example is provided by two parallel edges. More generally suppose G is a connected block with a 2-cocycle $\{e_1, e_2\}$. Then $G\backslash\{e_1, e_2\}$ has two components G_1, G_2, both blocks. Clearly WHITE has a winning strategy in the game on G if and only if WHITE has a winning strategy in at least one of G_1 or G_2.

We conjecture that the Graphic Unrooted Signing Game (and also the Cographic Unrooted Signing Game) on a 3-edge connected block graph is winning for WHITE playing first.

We point out that this conjecture if true does not cover all cases: the Unrooted Signing Game on $K_3 \times K_2$, which is a diminished block, is winning for WHITE playing first. Problem: what is the classification of Game IV (and also of Games V–VIII)?

APPENDIX: Signed Circuits Containing a Given Element in an Oriented Matroid

Clearly the Lehman Switching Game on a matroid M with respect to a given element e depends only on the collection of circuits of M containing e. Actually Lehman has shown [10, Theorem 46] that if M is connected, the circuits containing e completely detemine M. Similarly the Directed Switching Game and the Signing Game on an oriented matroid M depend only on the collection of signed circuits of M containing e.

Proposition 15.1. *A connected oriented matroid is uniquely determined by the collection of signed circuits containing a given element.*

Proof. In view of Lehman's theorem it is sufficient to show that the signatures of the circuits not containing a given element e in a connected oriented matroid M are uniquely determined by the signed circuits containing e. We prove this property by induction on the cardinality of the set of elements $E = E(M)$ of M.

Let C be a circuit of M such that $e \notin C$. We distinguish three cases.

(1) There is $p \in E \backslash (C \cup \{e\})$ such that $M \backslash p$ is connected.

By the induction hypothesis applied to $M \backslash p$, the signature of C is uniquely determined.

(2) There is $p \in E \backslash (C \cup \{e\})$ but $M \backslash p$ is not connected.

By the results of [4, Sect. 4], M/p is connected and C is a circuit of M/p. Since the signed circuits of M/p containing e are uniquely determined by the signed circuits of M containing e [2], it follows from the induction hypothesis applied to M/p that the signature of C is uniquely determined.

(3) $E = C \cup \{e\}$.

Since M is connected every pair of elements of M is contained in a circuit.

Consider $x, y \in C$, $x \neq y$. If x and y are in series in M, let X be a signed circuit of M containing e and x. Then X also contains y. By the signed elimination property we have $sg_C(x) \cdot sg_C(y) = sg_X(x) \cdot sg_X(y)$. If x and y are not in series, there is a signed circuit X of M containing e and x but not y, and a signed circuit Y of M containing e and y but not x. By the signed elimination property we have $sg_C(x) \cdot sg_C(y) = -sg_X(x) \cdot sg_Y(y) \cdot sg_X(e) \cdot sg_Y(e)$. ∎

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