PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

CRITICALLY n-CONNECTED GRAPHS

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ABSTRACT. The following result is proved. Every n-connected graph contains either a vertex whose removal results in a graph which is also n-connected or a vertex of degree less than $(3n-1)/2$.

Introduction. A graph G is said to be *n-connected* if the removal of fewer than n vertices from G neither disconnects it nor reduces it to the trivial graph consisting of a single vertex. The maximum value of n for which a graph G is n-connected is called its *connectivity* and is denoted by $\kappa(G)$. The minimum degree of G is designated by $\delta(G)$; the inequality $\kappa(G) \leqq \delta(G)$ is well known.

A graph G is said to be *critically n-connected* if $\kappa(G)=n$ and $\kappa(G-v)=n-1$ for each vertex v of G. Analogously, a graph G is *minimally n-connected* if $\kappa(G)=n$ and for each edge e of G, $\kappa(G-e)=n-1$. The object of this article is to present a necessary condition for a graph to be critically n-connected and to discuss related topics.

Since 1-connected graphs are the nontrivial connected graphs and since every nontrivial connected graph G has at least two vertices u and v such

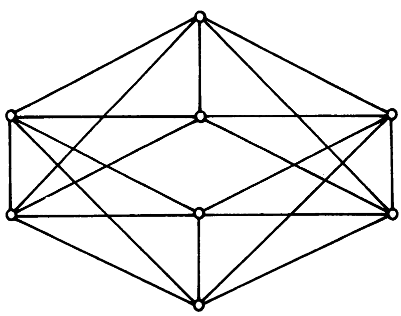

FIGURE 1

Received by the editors August 8, 1968 and, in revised form, April 17, 1969.
AMS 1969 *subject classifications*. Primary 0540.
Key words and phrases. Graph, connectivity, critically n-connected.
[1] Research supported in part by grant GP-9435 from the National Science Foundation.
[2] Research supported in part by a Faculty Research Fellowship from Western Michigan University.

that each of $G-u$ and $G-v$ is connected, it follows that the only critically 1-connected graph is the complete graph of order two. It is also easily observed that a graph is minimally 1-connected if and only if it is a nontrivial tree; thus if G is a graph which is either critically 1-connected or minimally 1-connected, then $\delta(G)=1$. Dirac **[2]** and Plummer **[7]** have shown that if G is minimally 2-connected then $\delta(G)=2$. Recently, Halin **[4]** extended this result so that if G is a minimally n-connected graph, $n\geqq 1$, then $\delta(G)=n$. It was shown in **[5]** that every critically 2-connected graph has minimum degree 2. The graph in Fig. 1 shows that no theorem on critically n-connected graphs analogous to Halin's theorem on minimally n-connected graphs is possible. The graph G of Fig. 1 is critically 4-connected but $\delta(G)=5$.

We shall prove that every critically n-connected graph, $n\geqq 2$, has a vertex of degree less than $(3n-1)/2$ and that the number $(3n-1)/2$ cannot be improved.

Preliminaries. Before proceeding further, it is convenient to give a few definitions and establish some notation. All terms not defined here may be found in Harary **[3]**.

If U is a nonempty subset of the vertex set $V(G)$ of G, then the *subgraph* H *induced by* U, written $H=\langle U\rangle$, is the subgraph whose vertex set is U and where two vertices are adjacent if and only if these vertices are adjacent in G. A set S of vertices of G is called a *cut set* of G if the (induced) subgraph $G-S=\langle V(G)-S\rangle$ is disconnected; S is an *n-cut set* if $|S|=n$. Two paths of G are said to be *disjoint* if they have no vertices in common except possibly end vertices.

Two special classes of graphs which we shall encounter are the complete graphs and the complete bipartite graphs. The *complete graph* K_p has p vertices every two of which are adjacent. The *complete bipartite* graph $K(m, n)$ has its vertex set V partitioned into two subsets V_1 and V_2, where $|V_1|=m$ and $|V_2|=n$, such that two vertices u and v are adjacent if and only if $u\in V_i$ and $v\in V_j$, $i\neq j$.

The concepts of "critically n-connected" and "minimally n-connected" are independent in the sense that neither property implies the other. For

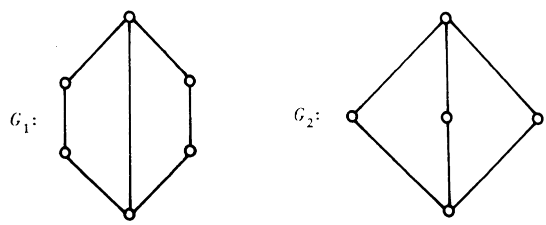

FIGURE 2

example, the graph G_1 of Fig. 2 is critically 2-connected but not minimally 2-connected while $G_2=K(2, 3)$ is minimally 2-connected and not critically 2-connected. In general, the graph $K(n, n+1)$ is minimally n-connected but not critically n-connected. For $n\geqq 3$, the graph obtained by adding an extra edge to $K(n, n)$ is critically n-connected but not minimally n-connected.

We note that it is rarely easy to ascertain whether a given graph is critically n-connected for some n. Despite this fact, such graphs are quite numerous; indeed if G is n-connected and G' is a subgraph of G containing the minimum number of vertices such that G' is n-connected, then G is critically n-connected.

A necessary condition for critically n-connected graphs. We now present the main result of this article.

THEOREM. *If G is a critically n-connected graph, $n\geqq 2$, then $\delta(G)<(3n-1)/2$ and the number $(3n-1)/2$ cannot be improved.*

PROOF. Suppose the theorem to be false so that there exists a graph G of order p having $\kappa(G)=n$ and $\delta(G)\geqq(3n-1)/2$ such that for every $v\in V(G)$, $\kappa(G-v)=n-1$. We note that since $\delta(G)\geqq(3n-1)/2$, G is not complete. This implies that every vertex of G belongs to some n-cut set of G.

Among all n-cut sets S' of G, let S be one such that $G-S$ contains a component G_1 of smallest order; denote the order of G_1 by m. Furthermore, let $G_2=G-S-V(G_1)$.

Let $v\in V(G_1)$ and $u\in V(G_2)$. By a result of Whitney [**8**] there exist n disjoint $u-v$ paths in G; necessarily, each such path contains precisely one vertex of S. Hence there exist n disjoint paths joining u and S (and also v and S).

Let $w\in V(G_1)$, and let S^* be an n-cut set of G containing w. Define $G^*=G-S^*$ and, furthermore, let $V_1=V(G_1)\cap S^*$, $V_2=V(G_2)\cap S^*$, and $V_3=S\cap S^*$, where $|V_i|=n_i$, $i=1, 2, 3$. We note that $n_1+n_2+n_3=n$ and $n_1\geqq 1$.

We now show that $n_2\geqq n_1$. If $S^*\supseteq V(G_2)$, then this is obvious. Assume therefore that $V(G_2)-V_2\neq\varnothing$. We have already noted that for each $u\in V(G_2)$, there exists in G a set of n disjoint paths joining u and S. If $u\in V(G_2)-V_2$, then at least $n-n_2-n_3=n_1$ of these paths contain no vertices of $V_2\cup V_3$. In this case, denote the set of end vertices in S of these n_1 (or more) paths by $R(u)$. Thus for each $u\in V(G_2)-V_2$, there exists a set $R(u)\subset S-V_3$ such that there are disjoint paths containing no elements of $V_2\cup V_3$ which join u and $R(u)$ where $|R(u)|\geqq n_1$. If there exist vertices u_1, $u_2\in V(G_2)-V_2$ such that $R(u_1)\cap R(u_2)=\varnothing$, then $|S-V_3|\geqq 2n_1$ so that $n-n_3\geqq 2n_1$ and $n_2\geqq n_1$. Otherwise, let $R=\bigcup R(u)$, the union taken over all

$u \in V(G_2) - V_2$, and let $G' = \langle R \cup (V(G_2) - V_2) \rangle$. It is now easy to verify that every two vertices of G' are connected so that G' itself is connected. Hence G' is a subgraph of a component of G^*. Since the order of G' is at least $n_1 + (p-m-n) - n_2$, there must be a component of G^* of order at most $m + n_2 - n_1$. Therefore, $m \leqq m + n_2 - n_1$ so that $n_2 \geqq n_1$. Thus in any case, $n_2 \geqq n_1$.

The inequality $n_2 \geqq n_1$ implies that $n_1 \leqq n/2$. We next verify that $V(G_1) - V_1 \neq \varnothing$ or, equivalently, that $n_1 < m$. Assume that $n_1 = m$ so that $V(G_1) = V_1$. Hence for each $v \in V(G_1)$,

$$\deg v \leqq (n_1 - 1) + n \leqq (3n - 2)/2,$$

which contradicts the fact that $\delta(G) \geqq (3n-1)/2$. We conclude therefore that $n_1 < m$ and $V(G_1) - V_1 \neq \varnothing$.

Let $F = \langle (V(G_1) - V_1) \cup (S - V_3) \rangle$. We show that F is disconnected. Suppose, to the contrary, that F is a connected subgraph of G^*. Since G^* is not connected, $V(G_2) - V_2 \neq \varnothing$. Because each $u \in V(G_2) - V_2$ is joined to $S - V_3$ by at least n_1 paths in G^*, it follows that G^* is connected which is impossible. Thus F is disconnected.

Denote the components of F by F_t, $t = 1, 2, \cdots, k$, where $k \geqq 2$. Furthermore, for each $t = 1, 2, \cdots, k$, denote by W_t the set of vertices of F_t in S, where $|W_t| = s_t$. We note that each $W_t \neq \varnothing$; for otherwise there would exist a component of F of order less than m contained in $\langle V(G_1) - V_1 \rangle$ which would also be a component of G^*.

We claim that precisely one of the subgraphs F_t contains elements of $V(G_1) - V_1$. Assume this is not the case so that there are two subgraphs F_i and F_j, $i \neq j$, containing elements of $V(G_1) - V_1$. Let $W_i' = \bigcup W_t$, $t \neq i$, where $|W_i'| = s_i'$. Each of the sets $V_1 \cup V_3 \cup W_i$ and $V_1 \cup V_3 \cup W_i'$ is a cut set of G, for in each case the removal of the set from G produces a graph having a component contained in $\langle V(G_1) - V_1 \rangle$. This implies that $n_1 + n_3 + s_i \geqq n$ and $n_1 + n_3 + s_i' \geqq n$ so that $s_i \geqq n_2$ and $s_i' \geqq n_2$. However, the equality $n_1 + n_2 + n_3 = s_i + s_i' + n_3 = n$ together with the inequality $n_2 \geqq n_1$ yield $s_i = s_i' = n_1 = n_2$. Therefore, $V_1 \cup V_3 \cup W_i$ is an n-cut set of G, but the graph $G - (V_1 \cup V_3 \cup W_i)$ has a component of order less than m. This produces a contradiction; hence exactly one of the subgraphs F_t contains elements of $V(G_1) - V_1$. Let F_1 be the subgraph with this property.

Now $V_1 \cup V_3 \cup W_1$ is a cut set of G so that $n_1 + n_3 + s_1 \geqq n$ or $s_1 \geqq n_2$. Let G_1^* be a component of G^* which contains vertices of W_1'. If $V(G_1^*) \subseteq W_1'$, then $s_1' \geqq m$, but this implies that

$$n = s_1 + s_1' + n_3 \geqq n_2 + m + n_3 > n_2 + n_1 + n_3 = n,$$

which is impossible. Therefore, G_1^* contains vertices of $V(G_2) - V_2$, which incidentally shows that $V(G_2) - V_2 \neq \varnothing$.

We show next that $V_2 \cup V_3 \cup W_1'$ is a cut set of G. Suppose this is not so. Then $G' = G - (V_2 \cup V_3 \cup W_1')$ is connected. Since F_1 is connected, the graph $G'' = G' - V_1$ is also connected. However, $G^* = \langle V(G'') \cup W_1' \rangle$ is disconnected; therefore, G^* has a component which is a subgraph of $\langle W_1' \rangle$, but we have seen that every component of G^* which contains elements of W_1' also contains elements of $V(G_2) - V_2$. Hence $G - (V_2 \cup V_3 \cup W_1')$ is disconnected so that $V_2 \cup V_3 \cup W_1'$ is a cut set of G. This produces the inequality $n_2 + n_3 + s_1' \geqq n$ or $s_1' \geqq n_1$.

We now know that $s_1 + s_1' = n_1 + n_2$, $s_1 \geqq n_2$, and $s_1' \geqq n_1$. From this we conclude that $s_1 = n_2$ and $s_1' = n_1$. Returning to the cut set $V_1 \cup V_3 \cup W_1$, we note that this is an n-cut set. However, $G - (V_1 \cup V_3 \cup W_1)$ contains a component of order less than m. This produces a contradiction, and the desired result follows.

Using the construction in [6], we show that the number $(3n-1)/2$ cannot be improved, i.e., for each positive integer n and positive integer $m < (3n-1)/2$, there is a critically n-connected graph G with $\delta(G) = m$. Before giving the construction, we define the join of two graphs. The *join* of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the union $G_1 \cup G_2$ of G_1 and G_2 together with all edges of the type $v_1 v_2$ where v_i is a vertex of G_i, $i = 1, 2$.

For $n \geqq 2m+2$, define the collection $\{H_{n,m}\}$ of graphs as follows:

$$\begin{aligned} H_{n,m} &= 2K_{m+1}, && \text{for } n = 2m+2, \\ &= K_{n-2m-2} + 2K_{m+1}, && \text{for } n > 2m+2. \end{aligned}$$

It is easily seen that $H_{n,m}$ has order n and $\delta(H_{n,m}) = n-m-2$. Using a result in [1], the equality $\kappa(H_{n,m}) = n-2m-2$ follows.

For $n < m < (3n-1)/2$, define

$$G_{n,m} = H_{n,m-n} + 2K_{m-n+1}.$$

The graph G given in Fig. 1 is $G_{4,5}$. From the information obtained about $H_{n,m}$, it follows that $\delta(G_{n,m}) = m$, and with the aid of the above-mentioned result in [1], $\kappa(G_{n,m}) = n$. Let v be a vertex of $G_{n,m}$. If v belongs to $H_{n,m-n}$, then the removal of v and the remaining $n-1$ vertices of $H_{n,m-n}$ results in a disconnected graph; thus, $\kappa(G_{n,m} - v) = n-1$. If v belongs to $2K_{m-n+1}$, then the removal of the vertices of $2K_{m-n+1}$ together with a $(3n-2m-2)$-cut set of $H_{n,m-n}$ gives a disconnected graph. Hence, here too we have $\kappa(G_{n,m} - v) = n-1$. The graph $G_{n,m}$ is therefore critically n-connected.

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