Asymptotic regime for improperness tests of complex random vectors

ANR Ricochet, Grenoble
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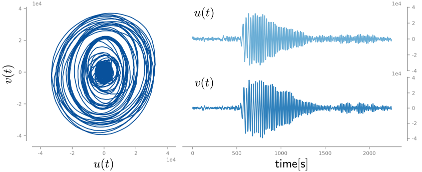
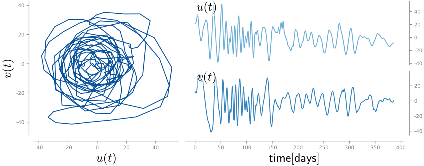
Complex valued signals

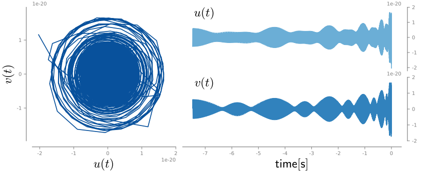
Seismic waves



Ocean currents



Gravitational waves



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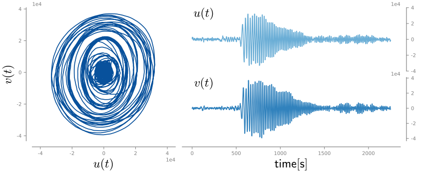

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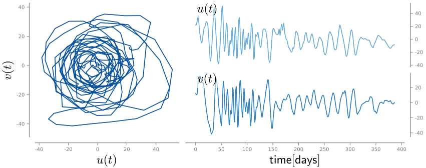

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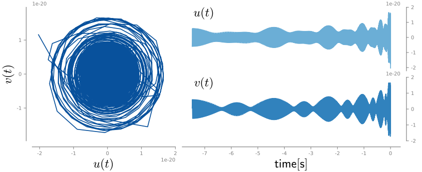

Properness/Circularity

- *Comon [1994]*,
- *Picinbono [1994]*.

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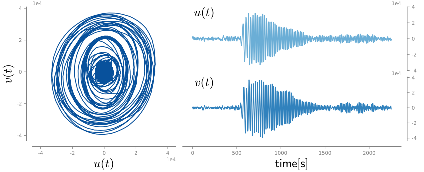

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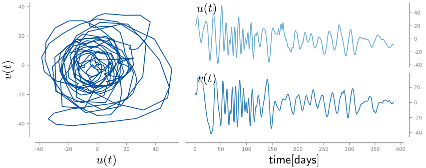

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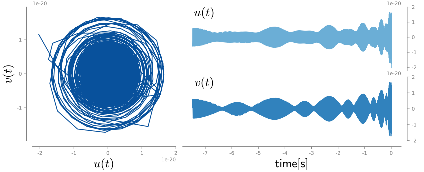

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Improperness testing

- GRLT:
  Olila et al. [2004],
  Schreier et al. [2006],
- Asymptotics:
  Delmas et al. [2011],
- Frequency domain:
  Walden et al. [2017].

Context & Applications

Assumptions

- Complex vector $\mathbf{z} \in \mathbb{C}^N \leftrightarrow N$-samples of $\mathbb{C}$-valued signal,
- Vectors/Signals are centered, *i.e.* $\mathbb{E}[\mathbf{z}] = 0$,
- Gaussian case,
- Sample size: M (number of independent observed signals),
- $M \geq 2N$ (sample covariance is not rank deficient).

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Applications†

- Symmetry deciphering (noise *vs.* signal),
- Low-rank improper signal in proper noise,
- Fluorescence imaging in complex media (Phase retrieval).

†: see "Statistical Signal Processing of Complex-Valued data. The theory of improper and noncircular signals" by *Schreier P. and Scharff, L.L., Cambridge Univ. Press, 2010.*

Improperness testing

Canonical Correlation distribution

Generalized Likelihood Ratio Test

Roy's test

Spiked model/Phase transition

Simulations

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Simulations

Representation of $\mathbf{z} \in \mathbb{C}^N$

Real representation

N-dimensional complex-valued Gaussian vector $\mathbf{z} = \mathbf{u} + \boldsymbol{i}\mathbf{v} \in \mathbb{C}^N$ with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ and $\boldsymbol{i}^2 = -1$ can be represented as:

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Statistics

Second-order statistics of $\mathbf{z}$ consists of the *real covariance matrix* $\mathbf{C} \in \mathbb{R}^{2N \times 2N}$ of $\mathbf{x}$ given by:

$$\mathbf{C} = \mathbb{E}[\mathbf{x}\mathbf{x}^T] = \begin{pmatrix} \mathbf{C}_{\mathbf{uu}}, & \mathbf{C}_{\mathbf{uv}} \\ \mathbf{C}_{\mathbf{vu}}, & \mathbf{C}_{\mathbf{vv}} \end{pmatrix}$$

with $\mathbf{C}_{\mathbf{ab}} = \mathbb{E}[\mathbf{a}\mathbf{b}^T] \in \mathbb{R}^{N \times N}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$.

Properness/Improperness

Definition

The complex vector $\mathbf{z} \in \mathbb{C}^N$ is called *proper*† *iff*:

$$\mathbf{C_{uu}} = \mathbf{C_{vv}} \qquad \text{and} \qquad \mathbf{C}_{\mathbf{uv}}^T = -\mathbf{C_{uv}} \tag{1}$$

or *improper* otherwise.

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$\dagger$: Properness $\leftrightarrow$ invariance by rotation $\leftrightarrow$ decorrelation of $\mathbf{z}$ and $\mathbf{z}^H$

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Testing problem

$$\begin{cases} H_0: & \mathbf{z} \text{ is } \text{proper} \ \text{ if condition (1) holds} \\ H_1: & \mathbf{z} \text{ is } \text{improper} \text{ otherwise} \end{cases}$$

Invariant Parameters I

Definition

Let $\mathcal{G}$ be the set of non-singular matrices $\mathbf{G} \in \mathbb{R}^{2N \times 2N}$ s.t.

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Properties

- Null hypothesis H_0 is equivalent to $\mathbf{C} \in \mathcal{T} = \mathcal{S} \cap \mathcal{G}$.
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- $\mathcal{G}$ acts transitively on $\mathcal{T}$ under the action $(\mathbf{G}, \mathbf{T}) \in \mathcal{G} \times \mathcal{T} \mapsto \mathbf{G}\mathbf{T}\mathbf{G}^T \in \mathcal{T}$

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$\Rightarrow$ Parameters to be tested should be the same for $\mathbf{C}$ and $\mathbf{G}\mathbf{C}\mathbf{G}^T$ for any $\mathbf{G} \in \mathcal{G}$

Invariant Parameters II

Definition

Consider the real symmetric $2N \times 2N$ matrix:

$$\mathbf{\Gamma}(\mathbf{C}) = \dot{\mathbf{C}}^{-\frac{1}{2}} \ddot{\mathbf{C}} \dot{\mathbf{C}}^{-\frac{1}{2}}.$$

with $\mathbf{C} = \dot{\mathbf{C}} + \ddot{\mathbf{C}}$ and

$$\dot{\mathbf{C}} = \frac{1}{2} \begin{pmatrix} \mathbf{C_{uu}} + \mathbf{C_{vv}} & \mathbf{C_{uv}} - \mathbf{C_{vu}} \\ \mathbf{C_{vu}} - \mathbf{C_{uv}} & \mathbf{C_{uu}} + \mathbf{C_{vv}} \end{pmatrix} \in \mathcal{G},$$
$$\ddot{\mathbf{C}} = \frac{1}{2} \begin{pmatrix} \mathbf{C_{uu}} - \mathbf{C_{vv}} & \mathbf{C_{uv}} + \mathbf{C_{vu}} \\ \mathbf{C_{uv}} + \mathbf{C_{vu}} & \mathbf{C_{vv}} - \mathbf{C_{uu}} \end{pmatrix}.$$

Invariant Parameters III

Lemma (Invariant parameters (Andersson 1975))

Any matrix $\mathbf{C} \in \mathcal{S}$ *can be written as:*

$$\mathbf{C} = \mathbf{G} \begin{pmatrix} \mathbf{I}_N + \mathbf{D}_\lambda & 0 \\ 0 & \mathbf{I}_N - \mathbf{D}_\lambda \end{pmatrix} \mathbf{G}^T,$$

where

- $\mathbf{G} \in \mathcal{G}$,
- $\mathbf{D}_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$,
- λ_n*: non-negative eigenvalues of* $\mathbf{\Gamma}(\mathbf{C})$.
- $\lambda_n \in [0, 1]$ *and ordering:* $1 \geq \lambda_1 \geq \ldots \geq \lambda_N \geq 0$.

Back to the improperness test

Invariant parametrization

- Invariant parameterization of $\mathbf{C}$ for the group action of $\mathcal{G}$ depends only on the N (non-negative) eigenvalues $1 \geq \lambda_1 \geq \ldots \geq \lambda_N \geq 0$ of $\mathbf{\Gamma}(\mathbf{C})$.
- Eigenvalues λ_n are termed *maximal invariant parameters* or population canonical correlation coefficients.
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Test reformulation

The invariant properness testing problem is thus:

$$\begin{cases} H_0 : & \textbf{z} \text{ is } \textit{proper} \text{ if } \lambda_1 = 0 \\ H_1 : & \textbf{z} \text{ is } \textit{improper} \text{ otherwise} \end{cases} \tag{2}$$

Invariant statistics

Dataset

Consider we are given a sample of size $M \geq 2N$, denoted $\{\mathbf{x}_m\}_{m=1}^{M}$ where $\mathbf{x}_m = [\mathbf{u}_m^T, \mathbf{v}_m^T]^T$ are $2N$-dimensional i.i.d. Gaussian real vectors with zero mean and covariance matrix $\mathbf{C}$.

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Sample canonical correlation coefficients

The sample covariance matrix $\mathbf{S} \in \mathbb{R}^{2N \times 2N}$ is:

$$\mathbf{S} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{x}_m \mathbf{x}_m^T = \begin{pmatrix} \mathbf{S}_{\mathbf{uu}}, & \mathbf{S}_{\mathbf{uv}} \\ \mathbf{S}_{\mathbf{vu}}, & \mathbf{S}_{\mathbf{vv}} \end{pmatrix} = \dot{\mathbf{S}} + \ddot{\mathbf{S}}$$

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Sample canonical correlation coefficients $r_n = l_n^2$

Canonical correlation distribution I

Limiting marginal empirical distribution under H_0

As M, $N \to \infty$ with the ratio $M/N \to \gamma \in [2, +\infty)$ being finite, the marginal empirical distribution of the (unsorted) r_n converges, under H_0, to the probability measure with density:

$$f(r) = \frac{1}{2\pi(1-r)} \sqrt{4(\gamma-1)\frac{1-r}{r} - (\gamma-2)^2}$$

on its support $r \in (0, c)$, with $c = \frac{4(\gamma-1)}{\gamma^2} \in (0, 1]$.

PS: Joint distribution of vector $(r_1, \ldots, r_N)$ known (eigenvalues of matrix-variate beta) but no closed-form.

Canonical correlation distribution II

Moments

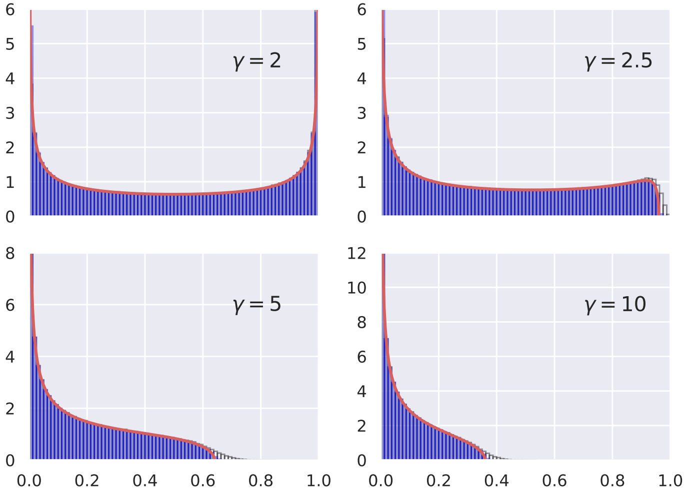
Under H_0, mean and variance of the limiting distribution of the unsorted r_n are:

- $E[r_n] = 1/\gamma$,
- $\mathrm{var}(r_n) = (\gamma - 1)/\gamma^3$.

Remarks

- When $\gamma \to +\infty$, $r_n \xrightarrow{a.s.} 0$ (usual behavior in small dimension, *i.e.* fixed N and $M \to +\infty$)
- When $\gamma = 2$, $r_n \xrightarrow{d} \mathcal{B}(\frac{1}{2}, \frac{1}{2})$ (arcsine law)

H_0: Limiting empirical distribution of r_n



Blue: $N = 100$, White: $N = 10$, Red: Limiting distribution $f(r)$.

Generalized Likelihood Test I

GLRT

Given the sample $\mathbf{X} = \{\mathbf{x}_m\}_{m=1}^{M}$, the GLRT statistic is defined as:

$$T \propto \frac{\sup_{\mathbf{C} \text{ s.t. } H_0} \quad p(\mathbf{X}\,;\,\mathbf{C})}{\sup_{\mathbf{C} \text{ s.t. } H_1} \quad p(\mathbf{X}\,;\,\mathbf{C})},$$

where $p(\mathbf{X}\,;\,\mathbf{C})$ is the multivariate normal pdf of the sample $\mathbf{X}$ composed of M i.i.d. $2N$-dimensional real Gaussian vectors with zero mean and covariance matrix $\mathbf{C}$.

$\hookrightarrow$ H_0 is rejected if $T > \eta_\alpha$ $\qquad$ η_α : H_0 law $+$ PFA

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GLRT Statistics

$$T = \prod_{n=1}^{N} (1 - r_n)$$

with r_n the N sample covariance correlation coefficients.

Generalized Likelihood Test II

Theorem

The GLRT statistics T is distributed under H_0 as the following Wilks lambda distribution:

$$T \sim \Lambda(N, M-N, N+1).$$

Moreover this statistics can be expressed under H_0 as:

$$T = \prod_{n=1}^{N} u_n,$$

where the u_n are independent beta-distributed random variables† *such that $u_n \sim \mathcal{B}\left(\frac{M-N-n+1}{2}, \frac{N+1}{2}\right)$, for $1 \leq n \leq N$.*

$\dagger$: Allows efficient sampling ($O(N)$) from the null hypotesis of T.

Generalized Likelihood Test III

Theorem (Central limit theorem in high dimension)

Let $T' = -\ln T$ *where* T *is the GLRT statistic. Assume that* $M, N \to \infty$ *so that the ratio* $M/N \to \gamma \in (2, +\infty)$*. Under* H_0*, the following asymptotic normal distribution is obtained for* T'*:*

$$\frac{1}{s}\left(T' - m\right) \xrightarrow{d} \mathcal{N}(0,1)$$

where:

- $m = M\left[\ln \frac{\gamma}{\gamma-1} + \frac{\gamma-2}{\gamma}\ln\frac{\gamma-2}{\gamma-1}\right] + \frac{1}{2}\ln\frac{\gamma}{\gamma-2}$,
- $s^2 = 2\left[\ln\frac{(\gamma-1)^2}{\gamma(\gamma-2)} + \frac{1}{M}\frac{1}{\gamma-2}\right]$.

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Remark

This CLT theorem provides a debiased version of the *low dimensional* Bartlett approximation (N fixed, $M \to \infty$).

Roy's test

Principle

This test relies on the statistics of the largest eigenvalue: here, the statistics of the largest squared canonical correlation $r_1 = l_1^2$.

$$\Rightarrow \text{reject } H_0 \text{ as soon as } r_1 > \eta_\alpha$$

Threshold η_α tuned according to the law of r_1 under H_0 and PFA α.

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Theorem (Limiting null distribution for Roy's test)

As $M, N \to \infty$ such that the ratio $M/N \to \gamma \in [2, +\infty)$ is finite, let $W = \log\left(r_1/(1-r_1)\right)$ be the logit *transform of r_1. Under H_0, the asymptotic law of W converges towards a first order Tracy-Widom law denoted as $\mathcal{TW}_1$:*

$$\frac{W - \mu}{\sigma} \to \mathcal{TW}_1,$$

with appropriate/tedious parameters $\mu(M, N)$ and $\sigma(M, N)$.

Spiked correlation model

Phase transition threshold

Recalling that $\gamma = M/N$, assume that $\lambda_1 \geq \cdots \geq \lambda_k > 0$, and $\lambda_{k+1} = \cdots = \lambda_N = 0$ (fixed k), then for $1 \leq n \leq k$:

$$\begin{aligned} &\text{if } \lambda_n^2 \leq \rho_c, \qquad r_n \xrightarrow{a.s.} c \\ &\text{if } \lambda_n^2 > \rho_c, \qquad r_n \xrightarrow{a.s.} \overline{\rho}_n \end{aligned}$$

- $\rho_c = \frac{1}{\gamma - 1}$ is the phase transition threshold,
- $\overline{\rho}_n = \lambda_n^2 \left(\frac{\gamma-1}{\gamma} + \frac{1}{\gamma \lambda_n^2} \right)^2$ is the limiting value,
- c is the right edge of the H_0 bulk.

Simulation Scenario I: Equi-correlated model

Equal canonical correlation coefficients

Real and imaginary part of z_m are:

$$\mathbf{u}_m = \mathbf{s}_m + \sqrt{\theta}\mathbf{q}_m,$$
$$\mathbf{v}_m = \mathbf{t}_m + \sqrt{\theta}\mathbf{q}_m,$$

where $\theta > 0$, $\mathbf{s}_m$, $\mathbf{t}_m$ and $\mathbf{q}_m$ are i.i.d. Gaussian vectors in $\mathbb{R}^N$, for $1 \leq m \leq M$.

$\Rightarrow$ Population canonical correlations $\lambda_1, \ldots, \lambda_N$, are all equal to $\lambda \equiv \frac{\theta}{1+\theta}$.

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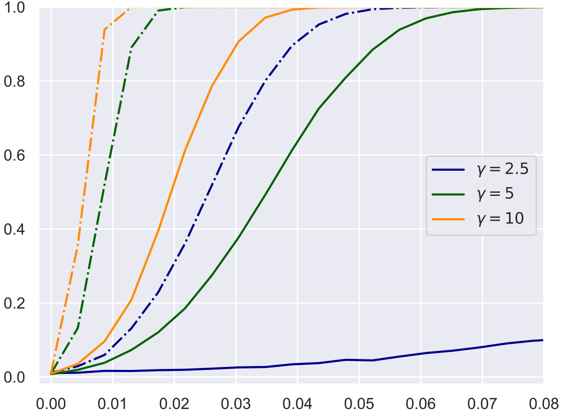


Power of Roy's test [solid line] and GLRT [dashdotted line] vs λ^2 under the equi-correlated model (N=100, PFA $\alpha = 0.01$).

Simulation Scenario II: Spiked model

Spike correlation model

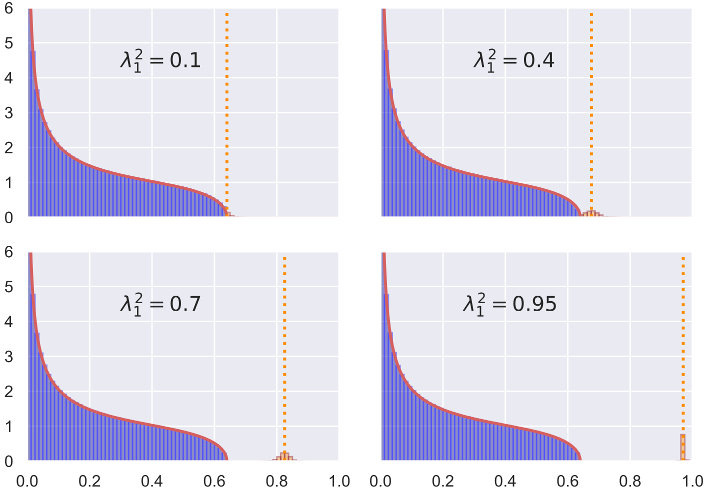
Real and imaginary parts of z_m have a common contribution of rank one: Low-rank improper signal corrupted by proper noise, for $1 \leq m \leq M$:

$$\mathbf{u}_m = \mathbf{s}_m + \sqrt{\theta} w_m \boldsymbol{\varphi}$$
$$\mathbf{v}_m = \mathbf{t}_m + \sqrt{\theta} w_m \boldsymbol{\varphi}$$

- $\theta > 0$, $\boldsymbol{\varphi} \in \mathbb{R}^N$ is a normed vector s.t. $||\varphi||_2 = 1$,
- w_m are i.i.d. Gaussian centered random variables with unit variance,
- $\mathbf{s}_m$, $\mathbf{t}_m$ are Gaussian i.i.d . vectors in $\mathbb{R}^N$,

$$\Longrightarrow \lambda_1 = \frac{\theta}{1+\theta} \text{ (\textit{spike})}, \ \lambda_2 = \cdots = \lambda_N = 0.$$

Simulation Scenario II: Spiked model



Histograms of r_n under the spiked Gaussian model for different spike magnitude λ_1. ($N = 100$, $\gamma = 5$ and $\rho_c = 0.25$.)

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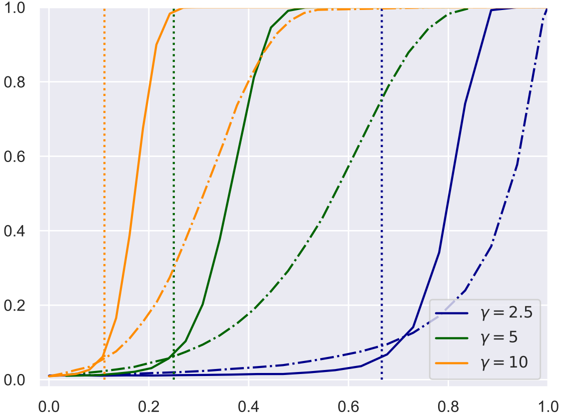

Power of Roy's test [solid line] and GLRT [dashdotted line] vs λ_1^2 under the spiked correlation model ((N=100, PFA $\alpha = 0.01$)).

Concluding remarks

Results for M, $N \to \infty$ and finite ratio $M/N \to \gamma \in [2, +\infty)$

- Statistics/CLT for the GLRT,
- Statistics of the Roy's test,
- Spiked model and phase transition.

Future work

- Quaternion-valued vectors/signals
- Random/Structured approximation