$$P(T_{G,w} \mid Y, \Theta) \propto \prod_{i=1}^{N} N(\theta_i, \beta T_{G_i w_i}, \Sigma)\, N(T_{G,w}, \nu_G, \phi_G^2)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}(\theta_i - \beta T_{G_i w_i})^T \Sigma^{-1}(\theta_i - \beta T_{G_i w_i}) + (T_{G,w}-\nu_G)^2 \frac{1}{\phi^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N} T_{G_i w_i}^T \beta^T \Sigma^{-1}\beta T_{G_i w_i} - 2\beta T_{G_i w_i}\theta_i^T + (T_{G,w}^2 - 2\nu_G T_{Gw})\frac{1}{\phi^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[T_{Gw}^2\left(\frac{1}{\phi^2} + N_{Gw}\beta^T\Sigma^{-1}\beta\right) - 2T_{Gw}\left(\sum_{i:G_i=G \wedge w_i=w} \beta\Sigma^{-1}\theta_i^T + \frac{\nu_G}{\phi^2}\right)\right]\right\}$$

$\propto N(\mu_{Gw}, V_{Gw}^{-1})$ where $V_{Gw} = \left(\frac{1}{\phi^2} + N_{Gw}\beta^T\Sigma^{-1}\beta\right)$

$\mu_{Gw} = V_{Gw}^{-1}\left(\sum_{i:G_i=G, w_i=w}\beta\Sigma^{-1}\theta_i + \frac{\nu_G}{\phi^2}\right)$ and N_{Gw} = # of INDIVIDUALS IN GROUP G AT WAVE w.

$$P(\sigma^2 \mid Y,\Theta) \propto \prod_{i=1}^{N}\prod_{k=1}^{K} N(\mu_{ik}, b_k(\lambda\theta_i - a_k), \sigma^2)\, IGa(\sigma^2, a_\sigma, b_\sigma)$$

$$\propto (\sigma^2)^{-\frac{NK}{2}} \exp\left\{\sigma^{2^{-1}} \cdot \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{K}(\mu_{ik} - b_k(\lambda\theta_i - a_k))^2\right\} (\sigma^2)^{-a-1} \exp\left\{-b\,\sigma^{2^{-1}}\right\}$$

$$\propto (\sigma^2)^{-\frac{NK}{2}-a-1} \exp\left\{\sigma^{2^{-1}}\left(b + \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{K}(\mu_{ik} - b_k(\lambda\theta_i - a_k))^2\right)\right.$$

$\propto IGa(\alpha_\sigma, \beta_\alpha)$ where $\alpha_\sigma = a_\sigma + \frac{NK}{2}$; $\beta_\sigma = b_\sigma + \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{K}(\mu_{ik} - b_k(\lambda\theta_i - a_k))^2$

$$P(\tau^2 \mid Y,\Theta) \propto \prod_{k=1}^{K} N(b_k, 0, \tau^2)\, IGa(\tau^2, a_\tau, b_\tau)$$

$$\propto (\tau^2)^{-\frac{K}{2}} \exp\left\{\tau^{2^{-1}} \frac{1}{2}\sum_{k=1}^{K}(b_k)^2\right\} (\tau^2)^{-a-1} \exp\left\{-b_\tau \tau^{2^{-1}}\right)$$

$$\propto IGa\left(a_\tau + \frac{K}{2}, b_\tau + \frac{1}{2}\sum_{k=1}^{K} b_k^2\right)$$

$$P(\omega^2 \mid Y,\Theta) \propto IGa\left(a_a + \frac{K}{2}; b_a + \frac{1}{2}\sum_{i=1} a_k^2\right)$$

$$P(\kappa^2 \mid Y,\Theta) \propto IGa\left(a_\kappa + \frac{P}{2}; b_\kappa + \frac{1}{2}\sum_{i=1}^{P} \lambda_i^2\right)$$

NOTATION:

N = # OF INDIVIDALS

W = # OF WAVES (TIMES)

G = # OF GROUPS

K = # OF QUESTIONS

P = # OF FACTORS (3 IN OUR CASE)

$$P(\Sigma \mid Y, \Theta) \propto \prod_{i=1}^{N} N(\theta_i, \beta T_{G i \omega i}, \Sigma)\, IW(s_0, S_0)$$

$$\propto IW\left(s_0 + N; \left[S_0 + \sum_{i=1}^{N} (Y_i - \theta_i)(Y_i - \theta_i)^T\right]^{-1}\right)$$

$$P(\xi^2 \mid Y, \Theta) \propto IGa\left(a_\xi + \frac{P}{2}; b_\xi + \frac{1}{2}\sum_{i=1}^{P} \beta_i^2\right)$$

$$P(\nu_G \mid Y, \Theta) \propto \prod_{\omega=1}^{W} N(T_{G,\omega}, \nu_G, \phi^2)\, N(\nu_G, 0, h^2)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\phi^2} \sum_{\omega=1}^{W} (T_{G,\omega} - \nu_G)^2 + \nu_G^2 \frac{1}{h^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{-2}{\phi^2} \sum_{\omega=1}^{W} T_{G\omega} \nu_G + \frac{1}{\phi^2}\nu_G^2 + \frac{1}{h^2}\nu_G^2\right]\right\}$$

$$\propto N(M_{\nu_G}, V_{\nu_G}^{-1}) \text{ where } V_{\nu_G} = \left(\frac{1}{\phi^2} + \frac{1}{h^2}\right); M_{\nu_G} = V_{\nu_G}^{-1}\left(\frac{1}{\phi^2}\sum_{\omega=1}^{W} T_{G,\omega}\right)$$

$$P(\phi^2 \mid Y, \Theta) \propto IGa\left(a_\phi + \frac{GW}{2}; b_\phi + \frac{1}{2}\sum_{G=1}^{6}\sum_{\omega=1}^{W} (T_{G\omega} - \nu_G)^2\right)$$

GENERAL ALGORITHM:

1. FOR $P \in \Theta$:
   - INITIALIZE P BY SAMPLING FROM PRIOR $(P_0(P))$.
2. FOR IT=1; IT <= N_ ITERATIONS:
   - FOR $P \in \Theta$:
     - UPDATE P by SAMPLING FROM $P(P \mid Y, \Theta)$

$$Y_{ik} = \begin{cases} 1 & \text{if } \mu_{ik} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{ik} \overset{iid}{\sim} N(b_k(\lambda\theta_i - a_k), \sigma^2)$$

$$a_k \sim N(0,\omega^2),\ b_k \sim N(0,\tau^2)$$

$$\lambda_p \sim N(0,\kappa^2),\ \theta_i \sim N_3(\beta T_{G,w_i}, \Sigma)$$

$$\beta_p \sim N(0,\xi^2);\ T_{sw} \sim N(\nu_s, \phi^2)$$

$$\sigma^2 \sim IGa(a_\sigma, b_\sigma);\ \tau^2 \sim IGa(a_\tau, b_\tau)$$

$$\omega^2 \sim IGa(a_\omega, b_\omega);\ \kappa^2 \sim IGa(a_\kappa, b_\kappa)$$

$$\Sigma \sim IW(S_0, \nu_0);\ \xi^2 \sim IGa(a_\xi, b_\xi)$$

$$\nu_s \sim N(0, h^2);\ \phi^2 \sim IGa(a_\phi, b_\phi)$$

$$P(\mu_{ik}|Y, \Theta) \propto [Y_{ik} 1(\mu_{ik} \geq 0) + (1-Y_{ik}) 1(\mu_{ik} \leq 0)] N(b_k(\lambda\theta_i - a_k), \sigma^2)$$

$$\propto N(b_k(\lambda\theta_i - a_k), \sigma^2) Y_{ik} 1(\mu_{ik} \geq 0) + (1-Y_{ik}) N(b_k(\lambda\theta_i - a_k), \sigma^2) 1(\mu_{ik} < 0)$$

$$= \begin{cases} N_{[0,\infty]}(b_k(\lambda\theta_i - a_k), \sigma^2) & \text{if } Y_{ik} = 1 \\ N_{[-\infty,0]}(b_k(\lambda\theta_i - a_k), \sigma^2) & \text{if } Y_{ik} = 0 \end{cases}$$

$N[a,b]$ = TRUNCATED NORMAL DISTRIBUTION.

UPDATE FOR μ_{ik}:

IF $Y_{ik} == 1$: DRAW $\mu_{ik}^{(T+1)} \sim N_{[0,\infty]}(b_k(\lambda\theta_i - a_k), \sigma^2)$

ELSE $Y_{ik} == 0$: " $\mu_{ik}^{(T+1)} \sim N_{[-\infty,0]}(b_k(\lambda\theta_i - a_k), \sigma^2)$

$$P(a_k | Y, \Theta) \propto \prod_{i=1}^{N} N(\mu_{ik}, b_k(\lambda\theta_i - a_k), \sigma^2) N(a_k, 0, \omega^2)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N} (\mu_{ik} - b_k\lambda\theta_i + b_k a_k)^2 \frac{1}{\sigma^2} + \frac{1}{\omega^2} a_k^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{N}{\sigma^2} b_k^2 a_k^2 - \frac{2}{\sigma^2} a_k b_k \sum_{i=1}^{N} (b_k\lambda\theta_i - \mu_{ik}) + \frac{1}{\omega^2} a_k^2\right]\right\}$$

$$\propto N(M_{a_k}, V_{a_k}^{-1}) \text{ where: } V_{a_k} = \left(\frac{N b_k^2}{\sigma^2} + \frac{1}{\omega^2}\right);\ M_{a_k} = V_{a_k}^{-1}\left(\frac{b_k}{\sigma^2}\sum_{i=1}^{N}(b_k\lambda\theta_i - \mu_{ik})\right)$$

UPDATE FOR a_k:

DRAW: $a_k^{(T+1)} \sim N(M_{a_k}, V_{a_k}^{-1})$. $\forall k \in 1 \ldots K$

$$P(b_k | Y, \Theta) \propto \prod_{i=1}^{N} N(\mu_{ik}, b_k(\lambda\theta_i - a_k), \sigma^2) N(b_k, 0, \tau^2)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{N}(\mu_{ik} - b_k\lambda\theta_i + b_k a_k)^2 + \frac{1}{\tau^2} b_k^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2} b_k^2 \sum_{i=1}^{N}(\lambda\theta_i - a_k)^2 - \frac{2}{\sigma^2} b_k \sum_{i=1}^{N}\mu_{ik}(\lambda\theta_i - a_k) + \frac{1}{\tau^2} b_k^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[b_k^2\left(\frac{1}{\sigma^2}\sum_{i=1}^{N}(\lambda\theta_i - a_k)^2 + \frac{1}{\tau^2}\right) - 2b_k\left(\frac{1}{\sigma^2}\sum_{i=1}^{N}\mu_{ik}(\lambda\theta_i - a_k)\right)\right]\right\}$$

$$= N(M_{b_k}, V_{b_k}^{-1})$$

$$V_{b_k} = \left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\sum_{i=1}^{N}(\lambda\theta_i - a_k)^2\right)$$

$$M_{b_k} = V_{b_k}^{-1} \cdot \left(\frac{1}{\sigma^2}\sum_{i=1}^{N}\mu_{ik}(\lambda\theta_i - a_k)\right)$$

UPDATE SAME AS a_k.

$$P(\lambda \mid Y, \Theta) \propto \prod_{i=1}^{N} \prod_{k=1}^{K} N(\mu_{ik}; b_k(\lambda\theta_i - a_k), \sigma^2)\, N(\lambda, 0, \kappa^2 I)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{k=1}^{K}(\mu_{ik} - b_k\lambda\theta_i + b_k a_k)^2 + \frac{1}{\sigma^2}\lambda^T\lambda\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{b_k^2}{\sigma^2}\theta^T\lambda^T\lambda\theta - \frac{2}{\sigma^2}\lambda b_k \theta_i(\mu_{ik} - b_k a_k) + \frac{1}{\kappa^2}\lambda^T\lambda\right]\right\}$$

$$\propto N(M_\lambda, V_\lambda^{-1}) \text{ where } V_\lambda = \left(\sum_{i=1}^{N}\sum_{k=1}^{K}\frac{b_k^2}{\sigma^2}\theta_i^T\theta_i + \frac{1}{\kappa^2}\right);\ V_\lambda^{-1}\left(\sum_{i=1}^{N}\sum_{k=1}^{K}\frac{1}{\sigma^2}b_k\theta_i(\mu_{ik} - b_k a_k)\right)$$

$$P(\theta_i \mid Y, \Theta) \propto \prod_{k=1}^{K} N(\mu_{ik}, b_k(\lambda\theta_i - a_k), \sigma^2)\, N(T_{Gi}\omega_i\beta, \Sigma)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{N}(\mu_{ik} - b_k\lambda\theta_i + b_k a_k)^2 + (\theta_i - T_{Gi}\omega_i\beta)^T\Sigma^{-1}(\theta_i - T_{Gi}\omega_i\beta)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}b_k^2\theta_i^T\lambda^T\lambda\theta_i - \frac{2}{\sigma^2}b_k\lambda\theta_i(\mu_{ik} + b_k a_k) + \theta_i^T\Sigma^{-1}\theta_i - 2\theta_i^T\Sigma^{-1}T_{Gi}\omega_i\beta\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_i^T\left(\frac{b_k^2}{\sigma^2}\lambda^T\lambda + \Sigma^{-1}\right)\theta_i - 2\theta_i^T\left(\frac{b_k}{\sigma^2}\lambda^T(\mu_{ik} + b_k a_k) + \Sigma^{-1}T_{Gi}\omega_i\beta\right)\right]\right\}$$

$$\propto N(M_{\theta_i}, V_{\theta_i}^{-1}) \text{ where: } V_{\theta_i} = \left(\sum_{k=1}^{K}\frac{b_k^2}{\sigma^2}\lambda^T\lambda + \Sigma^{-1}\right)$$

$$M_{\theta_i} = V_{\theta_i}^{-1}\left(\sum_{k=1}^{K}\frac{b_k^2}{\sigma^2}\lambda^T(\mu_{ik} + b_k a_k) + \Sigma^{-1}T_{Gi}\omega_i\beta\right).$$

$$P(\beta \mid Y, \Theta) \propto \prod_{i=1}^{N} N(\theta_i, \beta T_{Gi}\omega_i, \Sigma)\, N(\beta, 0, \xi^2 I)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}(\theta_i - \beta T_{Gi}\omega_i)^T\Sigma^{-1}(\theta_i - \beta T_{Gi}\omega_i) + \frac{1}{\xi^2}\beta^T\beta\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}(T_{Gi}\omega_i^T\beta^T\Sigma^{-1}\beta T_{Gi}\omega_i - 2\beta T_{Gi}\omega_i\theta_i^T) + \frac{1}{\xi^2}\beta^T\beta\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\beta^T\left(\sum_{i=1}^{N}T_{Gi}\omega_i^T\Sigma^{-1}T_{Gi}\omega_i + \frac{1}{\xi^2}\right)\beta - 2\beta\left(\sum_{i=1}^{N}T_{Gi}\omega_i\theta_i^T\right)\right]\right\}$$

$$\propto N(M_\beta, V_\beta^{-1}) \text{ where } V_\beta = \left(\sum_{i=1}^{N}T_{Gi}\omega_i^T\Sigma^{-1}T_{Gi}\omega_i + \frac{1}{\xi^2}\right)$$

$$M_\beta = V_\beta^{-1}\left(\sum_{i=1}^{N}T_{Gi}\omega_i\theta_i^T\right)$$

MODEL:

$P_0(\mu_{ik})$ = PRIOR.

$$Y_{ik} = \begin{cases} 1 & \text{if } \mu_{ik} \geq 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$\mu_{ik} \overset{iid}{\sim} N(b_k(\lambda \theta_i - a_k), \sigma^2)$ $\forall i \in 1 \dots N, k \in 1 \dots k$

$a_k \overset{iid}{\sim} N(0, \omega^2)$ $\forall k \in 1 \dots k$

$b_k \overset{iid}{\sim} N(0, \tau^2)$ $\forall k \in 1 \dots k$

$\lambda_p \overset{iid}{\sim} N(0, \kappa^2)$ $\forall p \in 1 \dots P$ (P=3 IN OUR CASE)

$\theta_i \overset{iid}{\sim} N_p(\beta T_{G_i w_i}, \Sigma)$ $\forall i \in 1 \dots N$

$\beta_p \overset{iid}{\sim} N(0, \xi^2)$ $\forall p \in 1 \dots P$

$T_{G,w} \overset{iid}{\sim} N(\nu_G, \phi^2)$ $\forall G \in 1 \dots G$; $w \in 1 \dots W$

$\sigma^2 \sim IGa(a_\sigma, b_\sigma)$

$\omega^2 \sim IGa(a_\omega, b_\omega)$

$\tau^2 \sim IGa(a_\tau, b_\tau)$

$\Sigma \sim IW(s_0, S_0)$

$\xi^2 \sim IGa(a_\xi, b_\xi)$

$\nu_G \sim N(0, h^2)$

$\phi^2 \sim IGa(a_\phi, b_\phi)$

HYPERPARAMETERS:

$a_\sigma, b_\sigma, a_\omega, b_\omega, a_\tau, b_\tau, s_0, S_0, a_\xi, b_\xi, h^2, a_\phi, b_\phi$

FIXED QUANTITIES WE CHOOSE PRIOR TO INFERENCE.
ALL ARE SCALARS > 0 EXCEPT FOR:
S_0 which IS A POSITIVE-DEFINITE MATRIX.
TRY A FEW DIFFERENT VALUES. YOU CAN START WITH
$a = 0.1$, $b = 0.1$ $\forall a, b$. AND $h^2 = 1$; $s_0 = P$; $S_0 = I_{P \times P}$.