Detail of Remark 4

I. Mean Square Convergence Analysis of OC-CKLMS

Here, we present the mean square convergence condition and steady-state mean square error (SS-MSE) analysis of the proposed OC-gCKLMS using the energy conservation relation [1], [2].

A. Mean Square Convergence Condition

By the weight update rule, the OC-gCKLMS does not always update the weight vectors, which complicates the analysis. Therefore, similar to [3]–[5], we first incorporate the censoring ratio P_{ce} into the weight update rule (Eq. (6) given in the original paper) as $\underline{\mathbf{w}}(i) = \underline{\mathbf{w}}(i-1) + \mu(1-P_{ce})\underline{\mathbf{\Phi}}^*\big(\mathbf{x}(i)\big)\,\underline{\mathbf{e}}(i)$. Then, the weight error vector $\tilde{\underline{\mathbf{w}}}(i) = \underline{\mathbf{w}}^o - \underline{\mathbf{w}}(i)$ can be obtained as

$$\tilde{\underline{\mathbf{w}}}(i) = \tilde{\underline{\mathbf{w}}}(i-1) - \mu(1-P_{ce})\underline{\mathbf{\Phi}}^*\left(\mathbf{x}(i)\right)\underline{\mathbf{e}}(i) \tag{1}$$

where $\underline{\mathbf{w}}^o$ is the optimal weight vector. Next, we can define the energy conservation relation [1], [2] by taking the squared norm of both sides of (1) as

$$\begin{aligned}\|\tilde{\underline{\mathbf{w}}}(i)\|^2 = \|\tilde{\underline{\mathbf{w}}}(i-1)\|^2 - 2\mu(1-P_{ce})\Re\left\{\underline{\mathbf{e}}_a^H(i)\underline{\mathbf{e}}(i)\right\}\\ + \mu^2(1-P_{ce})^2\left\|\underline{\mathbf{\Phi}}^*(\mathbf{x}(i))\underline{\mathbf{e}}(i)\right\|^2\end{aligned} \tag{2}$$

where a priori error $\underline{\mathbf{e}}_a(i) = \underline{\mathbf{\Phi}}^T(\mathbf{x}(i))\tilde{\underline{\mathbf{w}}}(i-1)$. Taking the expectation of $\|\tilde{\underline{\mathbf{w}}}(i)\|^2$ yields

$$\begin{aligned}E\left\{\|\tilde{\underline{\mathbf{w}}}(i)\|^2\right\} = E\left\{\|\tilde{\underline{\mathbf{w}}}(i-1)\|^2\right\}\\ -2\mu(1-P_{ce})E\left\{\Re\left\{\underline{\mathbf{e}}_a^H(i)\underline{\mathbf{e}}(i)\right\}\right\}\\ +\mu^2(1-P_{ce})^2E\left\{\left\|\underline{\mathbf{\Phi}}^*(\mathbf{x}(i))\underline{\mathbf{e}}(i)\right\|^2\right\}\end{aligned} \tag{3}$$

where the condition $E\left\{\|\tilde{\underline{\mathbf{w}}}(i)\|^2\right\} < E\left\{\|\tilde{\underline{\mathbf{w}}}(i-1)\|^2\right\}$ must be satisfied for convergence. To achieve this, it is required that the inequality $-2\mu(1-P_{ce})E\left\{\Re\left\{\underline{\mathbf{e}}^H(i)\underline{\mathbf{e}}_a(i)\right\}\right\} + \mu^2(1-P_{ce})^2E\left\{\|\underline{\mathbf{\Phi}}^*(\mathbf{x}(i))\underline{\mathbf{e}}(i)\|^2\right\} < 0$ holds, which leads to the bounds on the step size μ to be $0 < \mu < \frac{2E\left\{\Re\left\{\underline{\mathbf{e}}_a^H(i)\underline{\mathbf{e}}(i)\right\}\right\}}{(1-P_{ce})E\left\{\|\underline{\mathbf{\Phi}}^*(\mathbf{x}(i))\underline{\mathbf{e}}(i)\|^2\right\}}$.

B. Steady-State Mean Square Error Analysis

The SS-MSE is a key measure in the adaptive filtering concept, and to obtain it, we first define the following well-known assumptions [1], [6] as follows. *Assumption 1:* The measurement noise $\underline{\mathbf{n}}(i)$ is circular complex-valued, zero-mean, and independent and identically distributed (i.i.d.). It is also independent of the a priori error $\underline{\mathbf{e}}_a(i)$ and input signal $\mathbf{x}(i)$. *Assumption 2:* The error can be decomposed as $\underline{\mathbf{e}}(i) = \underline{\mathbf{e}}_a(i) + \underline{\mathbf{n}}(i)$, which implies $E\big\{\|\underline{\mathbf{e}}(i)\|^2\big\} = E\big\{\|\underline{\mathbf{e}}_a(i)\|^2\big\} + \sigma_n^2$, where σ_n^2 is the variance of the measurement noise $\underline{\mathbf{n}}(i)$. *Assumption 3:* At steady-state, $\|\underline{\mathbf{\Phi}}(\mathbf{x}(i))\|^2$ is independent of $\underline{\mathbf{e}}(i)$, which is commonly referred to as the separation principle.

Then, we define the SS-MSE expression as

$$\text{SS-MSE} = \lim_{i\to\infty} E\big\{\|\underline{\mathbf{e}}(i)\|^2\big\} = \lim_{i\to\infty} E\big\{\|\underline{\mathbf{e}}_a(i)\|^2\big\} + \sigma_n^2 \tag{4}$$

where the expression $\zeta = \lim_{i\to\infty} E\big\{\|\underline{\mathbf{e}}_a(i)\|^2\big\}$ is the steady-state excess MSE (SS-EMSE). At steady-state, it holds that $\lim_{i\to\infty} E\big\{\|\tilde{\underline{\mathbf{w}}}(i)\|^2\big\} = \lim_{i\to\infty} E\big\{\|\tilde{\underline{\mathbf{w}}}(i-1)\|^2\big\}$. Thus, we can simplify (3) as

$$\begin{aligned}\lim_{i\to\infty} E\left\{\Re\left\{\underline{\mathbf{e}}_a^H(i)\underline{\mathbf{e}}(i)\right\}\right\}\\ = \frac{\mu(1-P_{ce})}{2}\lim_{i\to\infty} E\left\{\|\underline{\mathbf{\Phi}}^*(\mathbf{x}(i))\underline{\mathbf{e}}(i)\|^2\right\}.\end{aligned} \tag{5}$$

Under *Assumptions 1*, *2*, and *3*, we first obtain (5) as $\zeta = \frac{\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}{2}\left(\zeta + \sigma_n^2\right)$, where $\mathrm{Tr}(\mathbf{R}) = E\left\{\|\underline{\mathbf{\Phi}}(\mathbf{x}(i))\|^2\right\}$ and $\mathbf{R}$ is the augmented covariance matrix. Then, we define the SS-EMSE expression as

$$\zeta = \frac{\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}{2-\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}\sigma_n^2. \tag{6}$$

Taking into account *Assumptions 2* and (6), we can also define the SS-MSE as

$$\text{SS-MSE} = \frac{2}{2-\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}\sigma_n^2. \tag{7}$$

To further measure the performance of the OC-gCKLMS, similarly to [1], [6], we can also express the misadjustment $\mathcal{M}$ as

$$\mathcal{M} \triangleq \frac{\zeta}{\sigma_n^2} = \frac{\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}{2-\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R})}. \tag{8}$$

From (8), it follows that, to guarantee convergence in the mean square sense, the misadjustment $\mathcal{M}$ must remain bounded and strictly positive [7], i.e., $2-\mu(1-P_{ce})\mathrm{Tr}(\mathbf{R}) > 0$. Consequently, the mean square error $E\big\{\|\underline{\mathbf{e}}(i)\|^2\big\}$ asymptotically approaches the measurement variance σ_n^2, provided that

$$0 < \mu < \frac{2}{(1-P_{ce})\mathrm{Tr}(\mathbf{R})}. \tag{9}$$

Remark: When $P_{ce} = 0$, the SS-EMSE, SS-MSE, misadjustment, and step bound of the OC-gCKLMS reduce to those of the standard gCKLMS [8], which is also similar to those of LMS [1], [6], KLMS [9], and OC-CLMS [5]. Furthermore, for $P_{ce} > 0$, since the OC strategy in the OC-gCKLMS scales μ with the parameter $(1-P_{ce})$, which actually corresponds to a smaller step size when compared to the gCKLMS. Therefore, the OC-gCKLMS produces lower SS-EMSE, SS-MSE, and misadjustment values, while yielding a slow convergence rate compared to the gCKLMS. On the other hand, as observed (9), the upper bound of μ increases with P_{ce}, allowing the use of a larger step size. This makes the OC-gCKLMS converge faster.

References

[1] A. H. Sayed, *Adaptive Filters*. Hoboken, NJ, USA: Wiley, 2011.

[2] N. Yousef and A. Sayed, "A unified approach to the steady-state and tracking analyses of adaptive filters," *IEEE Transactions on Signal Processing*, vol. 49, no. 2, pp. 314–324, 2001.

[3] P. S. Diniz and S. Werner, "Set-membership binormalized data-reusing LMS algorithms," *IEEE Transactions on Signal Processing*, vol. 51, no. 1, pp. 124–134, 2003.

[4] M. V. Lima and P. S. Diniz, "Steady-state mse performance of the set-membership affine projection algorithm," *Circuits, Systems, and Signal Processing*, vol. 32, pp. 1811–1837, 2013.

[5] E. C. Mengüç, M. Xiang, and D. P. Mandic, "Online censoring based complex-valued adaptive filters," *Signal Processing*, vol. 200, p. 108638, 2022.

[6] S. Haykin, *Adaptive Filter Theory*. Upper Saddle River, NJ: Prentice Hall, 2002.

[7] D. P. Mandic and V. S. L. Goh, *Complex Valued Nonlinear Adaptive Filters: Noncircularity, widely Linear and Neural Models*. Wiley Publishing, 2009.

[8] R. Boloix-Tortosa, J. J. Murillo-Fuentes, and S. A. Tsaftaris, "The generalized complex kernel least-mean-square algorithm," *IEEE Transactions on Signal Processing*, vol. 67, no. 20, pp. 5213–5222, 2019.

[9] W. Liu, P. P. Pokharel, and J. C. Principe, "The kernel least-mean-square algorithm," *IEEE Transactions on Signal Processing*, vol. 56, no. 2, pp. 543–554, 2008.