Math Homework #5

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Questions: 7.1, 7.2, 7.4, 7.7, 7.13, 7.20, 7.21

Problem #7.1

Claim: if S is a nonempty subset of V, then $conv(S)$ is convex.
Proof: Given an arbitrary set of points in $\{x_i\}_1^n \in S$ we have $a_1 = \sum_{i=2}^n \lambda_i$ and $a_0 = \lambda_1$. Hence, we see that $x_1, \sum_{i=2}^n x_i \in conv(S)$ fits the definition of a convex component. As this was an arbitrary set, we have that $conv(S)$ is convex.

Question #7.2

Claim: A hyperplane is convex.
Proof: Assume that a hyperplane H is not convex. Then we have for some set $\{x_i\}_{i=1}^n \in H$ where $\sum_{i=1}^n \lambda_i x_i \notin H$. This implies for some $\{a_i\}_{i=1}^n \in V$ we have $\sum_{i=1}^n \lambda_i \langle a_i, x_i \rangle \neq b$. However, this directly contradicts that $\langle a_i, x_i \rangle = b$ as the definition of a hyperplane. Hence, a hyperplane is convex.

Claim: A half-space is convex.
Proof: Assume that a half-space S is not convex. Then we have for some set $\{x_i\}_{i=1}^n \in S$ where $\sum_{i=1}^n \lambda_i x_i \notin S$. This implies for some $\{a_i\}_{i=1}^n \in V$ we have $\sum_{i=1}^n \lambda_i \langle a_i, x_i \rangle \geq b$. However, this directly contradicts that $\langle a_i, x_i \rangle \leq b$ as the definition of a half-space. Hence, a half-space is convex.

Question #7.4

Claim: $\|x-y\|^2 = \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y \rangle$.
Proof:

$$\begin{aligned} \|x-y\|^2 = \|x-p+p-y\|^2 &= \langle x-p+p-y, x-p+p-y \rangle \\ &= \langle x-p, x-p \rangle + \langle p-y, p-y \rangle + 2\langle x-p, p-y \rangle \\ &= \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y \rangle \end{aligned} \tag{1}$$

Claim: $\|x-y\| > \|x-p\|$
Proof:

$$\begin{aligned} \|x-y\|^2 &= \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y \rangle. \\ &< \|x-p\|^2, \quad \text{as } \|p-y\|^2 + 2\langle x-p, p-y \rangle > 0 \end{aligned} \tag{2}$$

As squared functions are continuous and convex, we have that this statement holds.

Claim: This number space is convex.

Proof: For $z = \lambda y + (1-\lambda)p$ we have:

$$\begin{aligned} \|x-z\|^2 =& \|x - \lambda y - (1-\lambda)p\|^2 \\ &= \langle x - \lambda y - (1-\lambda)p, x - \lambda y - (1-\lambda)p\rangle \\ &= \langle x - \lambda y - p + \lambda p, x - \lambda y - p + \lambda p\rangle \\ &= \langle x-p, x-p\rangle + \langle \lambda p - \lambda y, \lambda p - \lambda y\rangle + 2\langle x-p, \lambda p - \lambda y\rangle \\ &= \langle x-p, x-p\rangle + \langle -\lambda(y-p), -\lambda(y-p)\rangle + 2\langle x-p, \lambda(p-y)\rangle \\ &= \|x-p\|^2 + \lambda^2\|y-p\|^2 + 2\lambda\langle x-p, p-y\rangle \end{aligned} \tag{3}$$

Claim: If p is a projection of x onto the convex set C, then $\langle x-p, p-y\rangle > 0 \ \forall y \in C$.
Proof: Assume $\exists y^* \in C$ we have $\langle x-p, p-y^*\rangle = 0$. However, by definition of the inner product $\langle a, b\rangle = 0 \iff a$ and b are orthogonal. However, this is not possible, as x and y are not both in C.

Question #7.7

Claim: For any convex set C, for any convex functions $f_1, ..., f_k$ taking C to $\mathbb{R}$, and for any $\lambda_1, ..., \lambda_k \in \mathbb{R}_+$, the function $f(x) = \sum_{i=1}^k \lambda_i f_i(x)$ is convex.
Proof: We use the definition that $f(\lambda z_1 + (1\ \lambda)z_2) \leq \lambda f(z_1) + (1-\lambda)f(z_2)$ for any convex function. Let $\phi \in [0,1]. We have:$

$$\begin{aligned} f(\phi x_1 + (1-\phi)x_2) =& \sum_{i=1}^k \lambda_i f_i(\phi x_1 + (1-\phi)x_2) \\ &\leq \sum_{i=1}^k \lambda_i \Big[\phi f_i(x_1) + (1-\phi)f_i(x_2)\Big] \\ &= \phi \sum_{i=1}^n \Big[\lambda_i f(x_1)\Big] + (1-\phi)\sum_{i=1}^n \Big[\lambda_i f(x_2)\Big] \\ &= \phi f(x_1) + (1-\phi)f(x_2) \end{aligned} \tag{4}$$

Question #7.13

Claim: if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above, then f is constant.
Proof: Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and bounded above. Further, let $x^*, x_0 \in \mathbb{R}^n$ where x^* is the supremum of f (and exists and is finite as f is bounded above). We have that $f(x_0) = f(x^*) + Df(x^*)(x_0 - x^*) = f(x^*) \geq f(x_0)$. This equality only holds where $\forall x \in \mathbb{R}, f(x) = f(x^*)$. Hence, f must be constant.

Question #7.20

Claim: if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $-f$ is also convex, then f is affine.
Proof: We have for some $x_1, x_2 \in \mathbb{R}^n$:

$$\begin{gathered} f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \\ \text{and} \\ -f(\lambda x_1 + (1-\lambda)x_2) \leq -\lambda f(x_1) - (1-\lambda)f(x_2) \\ f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2) \end{gathered} \tag{5}$$

thus, $f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2)$, which is true in general if and only if f is a linear transformation. Hence, f is affine.

Question #7.21

Claim: If $D \subset \mathbb{R}$ with $f : \mathbb{R}^n \to D$, and if $\phi : D \to \mathbb{R}$ is a strictly increasing function, then x^* is a local minimizer for the ϕ problem $\iff$ x^* is a local minimizer for the original problem.
Proof: ($\Rightarrow$) Assume x^* is a local minimizer for the ϕ problem. Then we have that as ϕ monotonically increasing, that the lowest value of $\phi f(x)$ is also the lowest value of $f(x)$ (To see this, assume the main problem has a different minimum $y \in \mathbb{R}^n$. This would imply $f(y) < f(x^*)$, as ϕ is strictly increasing, which forms a contradiction). However, this is exactly the claim the problem makes.

($\Leftarrow$) Let x^* be the minimum for the main problem, and assume $y \in \mathbb{R}^n$ is the minimum for the ϕ problem. From this we see that if y is different than x^*, then $f(y) < f(x^*)$ by definition of a strictly-increasing operator. However, this forms a contradiction.