Math Homework #4

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Questions: 6.1, 6.5, 6.6, 6.11, 6.14

Question #6.1

Question: Write an optimization problem in standard form that is equivalent to given $x, y \in \mathbb{R}^n,\ a, b \in \mathbb{R}$, and $A \in M_n(\mathbb{R})$, choose $w \in \mathbb{R}^n$ such that it maximizes and constrains.

$$\begin{aligned} &\textit{minimize} && -e^{-w^T x} \\ &\textit{subject to} && w^T A w - w^T A y - w^T x \leq -a \\ &&& y^T w - w^T x = b \end{aligned} \tag{1}$$

Question #6.5

A plastics company makes two products: knobs for electronic products and milk cartons. The primary poduction expenses for each are labor and the raw plastic.

Each milk bottle requires 4 grams of plastic and 2 minutes of labor. Each knob takes 3 grams of plastic and 1 minute of labor. During the current production period, the copmay has 240 kilograms of plastic and 100 hours of labor. Each milk bottle yields a profit of \$0.07 and each knob \$0.05.

Question: Write an optimization problem in standard form that is equivalent to finding the amount the company should product of each product in order to maximize its profits.

Let $p \in [0, \infty] \subset \mathbb{R}$ represent profit, $L \in [0, \infty] \subset \mathbb{R}$

$$\begin{aligned} &\textit{minimize} && -\Big[0.07(4P_m + 2L_m) + 0.05(3P_k + L_k)\Big] \\ &\textit{subject to} && \\ &&& P_m + P_k = 240 \\ &&& L_m + L_k = 100 \end{aligned} \tag{2}$$

Question #6.6

Problem: Find and identify all critical points of the function $F(x, y)$. Determine whether each is a min/max, local/global, or saddlepoint.

$$f(x, y) = 3x^2 y + 4xy^2 + xy \tag{3}$$

Solution:

$$Df(x, y) = [6xy + 4y^2 + y, 3x^2 + 8xy + x] \tag{4}$$

From this, we have one real root: $(0,0)$. The nature of these critical points can be found using the Hessian matrix:

$$H = \begin{bmatrix} 6y & 6x+8y+1 \\ 6x+8y+1 & 8x \end{bmatrix}$$

We then have the following:

$$det(H(0,0)) = 6(0)\cdot 8(0) - (6(0)+8(0)+1)^2 = -1 < 0 \tag{5}$$

We see then that $f(0,0)$ is a not a local max or min. Upon inspection of the graph, it appears to be a saddle point.

Question #6.11

Consider a quadratic function $f(x) = ax^2 + bx + c$, where $a > 0$ and $b, c \in \mathbb{R}$.

Claim: For any initial guess $x_0 \in \mathbb{R}$, one iteration of Newton's method lands at the unique minimizer of f.
Proof: For any arbitrary x_0:

$$x_0 - \frac{2ax_0 + b}{2a} = x_0 - \left(x_0 + \frac{b}{2a}\right) = \frac{-b}{2a} \tag{6}$$

which is precisely the minimizer of $f(x)$.

Question #6.14

```
import sympy as sy
from numba import jit
import time

def testf(x):
    return np.random.random()*x**2 + np.random.random()*x + np.random.random()

def fprime(f):
    x = sy.symbols('x')
    y = f(x)
    yprime = y.diff(x)
    return yprime

def fdbprime(f):
    x = sy.symbols('x')
    y = f(x)
    yprime = y.diff(x)
    ydbprime = yprime.diff(x)
    return ydbprime

def Newton(f, tol=0.001):
    x = sy.symbols('x')
    t = 0
    e = 1
    g = 0
    start = time.time()
    while e > tol and t < 100000:
```

```
        f1 = fprime(f)
        f2 = fdbprime(f)
        xt = g - (f1.subs(x,g)/(f2.subs(x,g)))
        e = abs(g - xt)
        t = t + 1
        g = xt
    elapse = time.time() - start
    return xt,t,elapse

minimum, iterations, elapsed = Newton(testf)
print("Local Minimum              :",minimum)
print("Total No. of Iterations    :",iterations)
print("Time Elapsed               :",elapsed," seconds")
```