A brief introduction to Agda

or, how I learned to stop worrying and love the typechecker

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Agenda

- **Introduction**
  About Agda
  Syntax
  Interactive development

- **Induction**
  Basic definitions
  Proving theorems

- **Co-induction**
  Basic definitions

About Agda

A bit of history..

- V1.0 (approx. 1999) by C. Coquand
- V2.0 (approx. 2007) is a complete rewrite by U. Norell
- draws inspiration from previous works such as Alf, Alfa, Automath, Cayenne, Coq, ...

..and about Agda itself.

- a **total** dependently typed programming language
- an extension of intuitionistic Martin-Löf type-theory
- a proof assistant (due to its dependent typing and Curry-Howard's correspondence)

Syntax

- note that almost any character (including unicode codepoints and “,”) except “(” and “)” is valid in identifiers! For example, `3::2::1::[]` is lexed as an identifier, and we must use spaces to make Agda parse it successfully. These are all valid identifiers:
  - `this+is*a-valid[identifier] : ℕ → ℕ`
  - `this,as→well : ℕ → ℕ`

- the character “_” has a special meaning in definitions, as it allows the definition of mixfix operators. For example: `if_then_else_` defines a function which can be used in the following ways:
  - `(if_then_else_) x y z`
  - `if x then y else z`
  - `(if x then_else_) y z`
  - `(if_then y else z) x`
  - `(if x then _ else z) y`
  - ...

Syntax

Let's see a simple definition of a function:

```
not : Bool → Bool
not false = true
not true = false
```

We can see that pattern matching, in Agda, is similar to that of Haskell. We also can use implicit arguments (notice the dependent typing):

```
id : {A : Set} → A → A
id a = a
```

We can then use `id` both as `(id {Bool} true)` and `(id true)`.

Interactive development

this would be better with live coding

Being a proof assistant, interaction between the developer and Agda itself is really important. The main points of interaction are, in a way similar to Coq, *goals* and *holes*. A hole is syntactically represented by a `?` in the source, and it represents an unknown value:

```
f : ℕ → ℕ
f = ?
```

Agda will track the `?` as representing an unknown value which must have type `ℕ → ℕ`, and graphically show it in the editor as an unsolved goal. We can then ask Agda to automatically solve the goal, refine it or split the definition of the function to match on the possible values of an argument.

```
f : ℕ → ℕ
f = λ z → z
```

For example, when asked to solve the previous goal automatically, Agda may choose to fill it with the identity function `f = λ z → z` .

Basic definitions

Let's start from the basic data types: red-black trees. Ok, no, natural numbers:

```
data Nat : Set where
 zero : Nat
 suc : Nat → Nat
```

In Agda, data structures need not be tagged with "inductive" or "coinductive" (albeit records do). Data definitions can be indexed...

```
data NatVec : ℕ → Set where
 ε    : Vec zero
 _::_ : ∀ {n : ℕ} → Nat → Vec n → Vec (suc n)
```

...and parametrized

```
data Vec (A : Set) : ℕ → Set where
 ε    : Vec A zero
 _::_ : ∀ {n : ℕ} → A → Vec A n → Vec A (suc n)
```

Basic definitions

We can also model relations as datatypes. The *mother* of all relations is **propositional equality**; what follows is the internal definition of (propositional, intensional, “definitional”) equality.

```
-- lib/Relation/Binary/PropositionalEquality/Core.agda
data _≡_ {a} {A : Set a} (x : A) : A → Set a where
  refl : x ≡ x

cong : ∀ {a b} {A : Set a}  {B : Set b} (f : A → B) {m  n}  → m ≡ n → f m ≡ f n
cong f  refl  = refl
```

The use of relations and the power of Agda’s dependent type system allows us to write proofs just like function definitions:

```
+-id : ∀ (x : Nat) → zero + x ≡ x
+-id x = ?
```

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```
+-id : ∀ (x : Nat) → zero + x ≡ x
+-id zero = refl
+-id (suc x) = ? -- must be a term of type  '(zero + suc x) ≡ suc x'
```

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The use of relations and the power of Agda’s dependent type system allows us to write proofs just like function definitions:

```
+-id : ∀ (x : Nat) → zero + x ≡ x
+-id zero = refl
+-id (suc x) = cong suc (+-id x)
```

of course, Agda is powerful enough to let this proof be more concise:

```
+-id : ∀ (x : Nat) → zero + x ≡ x
+-id x = refl
```

Equational reasoning

Of course, proofs can be much more complicated than that for `+-id` and might need *chains of equations* to express the various steps of a proof of a theorem. Ideally, every relation should have means to allow reasoning about it. Regarding the equality relation `≡`, the standard library gives us the following:

```
-- lib/Relation/Binary/PropositionalEquality/Core.agda
begin_ : ∀{x y : A} → x ≡ y → x ≡ y
begin_ x≡y = x≡y

_≡⟨⟩_ : ∀ (x {y} : A) → x ≡ y → x ≡ y
_ ≡⟨⟩ x≡y = x≡y

step-≡ : ∀ (x {y z} : A) → y ≡ z → x ≡ y → x ≡ z
step-≡ _ y≡z x≡y = trans x≡y y≡z

syntax step-≡  x y≡z x≡y = x ≡⟨ x≡y ⟩ y≡z

_∎ : ∀ (x : A) → x ≡ x
_∎ _ = refl
```

(this is generalized with `Setoid` and `PartialSetoid`)

Equational reasoning

Chains of equations can often get difficult to read and follow, so let's take it slow and prove the associativity of _+_:

```
+-assoc : ∀ (m n p : ℕ ) → (m + n) + p ≡ m + (n + p)
+-assoc m n p = ?
```

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```
+-assoc : ∀ (m n p : ℕ ) → (m + n) + p ≡ m + (n + p)
+-assoc zero n p = ? -- split on m
+-assoc (suc m) n p = ?
```

Equational reasoning

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```
+-assoc : ∀ (m n p : ℕ ) → (m + n) + p ≡ m + (n + p)
+-assoc zero n p = refl
+-assoc (suc m) n p = ? -- we must show ((suc m) + n) + p ≡ (suc m) + (n + p)
```

On paper, we would prove this in a way similar to this: suppose

$$(m + n) + p \equiv m + (n + p) \tag{1}$$

then

$$\begin{aligned}
& & ((\mathrm{suc}\ m) + n) + p &\equiv (\mathrm{suc}\ m) + (n + p) \\
&\to & (\mathrm{suc}\ (m + n)) + p &\equiv (\mathrm{suc}\ m) + (n + p) \\
&\to & \mathrm{suc}\ ((m + n) + p) &\equiv (\mathrm{suc}\ m) + (n + p) \\
&\to (\mathit{by}\ (1)) & \mathrm{suc}\ (m + (n + p)) &\equiv (\mathrm{suc}\ m) + (n + p) \\
&\to & (\mathrm{suc}\ m) + (n + p) &\equiv (\mathrm{suc}\ m) + (n + p)
\end{aligned}$$

We can directly map this chain of reasoning in Agda!

Equational reasoning

Chains of equations can often get difficult to read and follow, so let's take it slow and prove the associativity of _+_:

```
+-assoc : ∀ (m n p : ℕ ) → (m + n) + p ≡ m + (n + p)
+-assoc zero n p = refl
+-assoc (suc m) n p =
 begin
   suc m + n + p
  ≡⟨⟩
   suc (m + n) + p
  ≡⟨⟩
   ? -- goal has type suc (m + n) + p ≡ suc m + (n + p)
```

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+-assoc (suc m) n p =
 begin
   suc m + n + p
  ≡⟨⟩
   suc (m + n) + p
  ≡⟨⟩
   suc (m + n + p)
  ≡⟨⟩
   ? -- we now want to use the inductive hypothesis!
```

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+-assoc zero n p = refl
+-assoc (suc m) n p =
 begin
   suc m + n + p
  ≡⟨⟩
   suc (m + n) + p
  ≡⟨⟩
   suc (m + n + p)
  ≡⟨ cong suc (+-assoc m n p) ⟩
   suc (m + (n + p))
 ∎
```

Equational reasoning

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+-assoc : ∀ (m n p : ℕ ) → (m + n) + p ≡ m + (n + p)
+-assoc zero n p = refl
+-assoc (suc m) n p =
 begin
   suc m + n + p
  ≡⟨⟩
   suc (m + n) + p
  ≡⟨⟩
   suc (m + n + p)
  ≡⟨ cong suc (+-assoc m n p) ⟩
   suc (m + (n + p))
 ∎
```

Of course, this kind of proof technique can end up in something very difficult to follow...

```
-- lib/Relation/Binary/HeterogeneousEquality/Quotients/Examples.agda
+²-cong {a₁ , b₁} {c₁ , d₁} {a₂ , b₂} {c₂ , d₂} ab~cd₁ ab~cd₂ = begin
    (a₁ + c₁) + (b₂ + d₂) ≡⟨ ≡.cong (_+ (b₂ + d₂)) (+-comm a₁ c₁) ⟩
    (c₁ + a₁) + (b₂ + d₂) ≡⟨ +-assoc c₁ a₁ (b₂ + d₂) ⟩
    c₁ + (a₁ + (b₂ + d₂)) ≡⟨ ≡.cong (c₁ +_) (≡.sym (+-assoc a₁ b₂ d₂)) ⟩
  ...
```

Basics of (sized) co-induction

Most of the infrastructure for sized co-induction in Agda is based on the following record:

```
-- lib/Size.agda
SizedSet : (ℓ : Level) → Set (suc ℓ)
SizedSet ℓ = Size → Set ℓ

-- lib/Codata/Sized/Thunk.agda
record Thunk {ℓ} (F : SizedSet ℓ) (i : Size) : Set ℓ where
  coinductive
  field force : {j : Size< i} → F j
```

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record Thunk {ℓ} (F : SizedSet ℓ) (i : Size) : Set ℓ where
  coinductive
  field force : {j : Size< i} → F j
```

Other co-inductive (or inductive-co-inductive) data types are defined in terms of `Thunk`:

```
data Stream (A : Set a) (i : Size) : Set a where
  _∷_ : A → Thunk (Stream A) i → Stream A i

head : Stream A i → A
head (x ∷ xs) = x

tail : {j : Size< i} → Stream A i → Stream A j
tail (x ∷ xs) = xs .force

map : (A → B) → Stream A i → Stream B i
map f (x ∷ xs) = f x ∷ λ where .force → map f (xs .force)
```