1. Hot to get from Marco’s Sraffa price system to a Leontief’s price model?

Marco’s original Sraffa-like price system (combining the depreciation and mark-up terms into one term for ease of exposition) is:

$$p_{nx\,1} = \hat{p}_{nxn}A_{nxn}(1+\mu_{nx\,1}) + \hat{w}_{nxn}l_{nx\,1} \qquad (1)$$

The Leontief price system is derived from the total output identity in monetary terms:

$$\hat{X}_{nxn}p_{nx\,1} = Z_{nxn}p_{nx\,1} + va_{nx\,1}$$

Dividing everything by total output ($X_{nx\,1}$) we have:

$$p_{nx\,1} = A_{nxn}p_{nx\,1} + v_{nx\,1}$$

And manipulating the equation we get the canonical expression for the Leontief price system:

$$p_{nx\,1} = (I_{nxn} - A_{nxn})^{-1}v_{nx\,1} \qquad (2)$$

Notation:

$^\wedge\, superscript\ denote\ diagonalized\ vector$

$p_{nx\,1} = sectoral\ price\ vector$

$I_{nxn} = Indentity\ matrix$

$A_{nxn} = tecnical\ coefficient\ matrix\ \ (= \hat{X}^{-1}{}_{nxn}Z_{nxn})$

$Z_{nxn} =\ sectoral\ intermediate\ inputs\ demand\ vector$

$\hat{X}^{-1}{}_{nxn} = diagonalized\ total\ output\ per\ sector$

$i_{nx\,1} = vector\ filled\ with\ 1's$

$\mu_{nx\,1} = gross\ sectoral\ profit\ rate\ (mark - up + depreciation)$[1]

$w_{nx\,1} = sectoral\ average\ wage\ rate\ vector$

l_{nxn}
$= diagonal\ matrix\ with\ labour\ coeficient$[2] $per\ sector\ vector\ (\hat{X}^{-1}{}_{nxn}\, l_{nx\,1})$

$v_{nx\,1} = value\ added\ (va)\ coefficient\ vector\ (= \hat{X}^{-1}{}_{nxn} va_{nx\,1})$

Now, how to get from one to another?

Manipulating (1) (and omitting dimensions subscripts) we get:

$$p = pA + pA\,\mu + wl \qquad (3)$$

Re-arranging terms in (3) we have:

$$p - pA = pA\mu + wl \qquad (4)$$

$$p(I - A) = pA\,\mu + wl \qquad (5)$$

$$p = (I - A)^{-1}(pA\,\mu + wl) \qquad (6)$$

So we have that:

$$v = (pA\,\mu + wl) \qquad (7)$$

Now recall that v_{1xn} is nothing but the vector sectoral valued added divided by the sectoral output, and that value added per sector is the sum of total wages and total profits in each sector j:

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[1] Combining depreciation and the mark-up from our code into one for simplification.

[2] The labour coefficient is the inverse of the labour productivity.

$$v_j = \frac{VA_j}{X_j} = \frac{\mu_j K_j + w_j L_j}{X_j}$$

Total wages is expressed as average wages (w_j) multiplied by total employment *(L_j)* and total profits as the multiplication of gross profit rate (μ_j) multiplied by the tock of capital *(K_j)*. So v can be further simplified into:

$$v_j = \frac{\mu_j K_j}{X_j} + \frac{w_j L_j}{X_j} = \mu_j \kappa_j + w_j l_j$$

Where κ_j is equal to the capital–output ratio and l_j is the labour coefficient (the inverse of labour productivity). As such the equivalence between the two systems (re–introducing dimension subscripts) is given by:

$$v = \mu\kappa + wl = pA\mu + wl$$

Which implies that there is an equivalency between $\mu\kappa = pA\mu$. As such, the total capital in each sector is equal to the total costs of intermediate inputs demanded in each sector:

$$\kappa = pA \rightarrow \frac{K}{X} = \frac{pZ}{X}$$

$$\boxed{K = pZ}$$

which is makes sense if we are considering a model only with circulating capital only.

However, in our model we have fixed capital. With fixed capital the more accurate way to represent the price system would be to have a matrix for capital-output (B) relationships akin to the role played by the A matrix in representing the ratio of intermediate inputs to total output in each sector. **Nevertheless, this is something for another time, as it would require some more fundamental changes into our model (we don't have this B matrix and it is not trivial to estimate one), which we don't have time to do now.**

2. **Introducing indirect taxes and tariffs:**

a) **Indirect (carbon) taxes on production:**

My proposal is that we re-write the price system in the following format, which avoids the need for the iteration. And, in principle, with appropriately calibrated values should yield all prices equal to 1 (without having to add another explicit normalization factor):

$$p_{nx\,1} = (I_{nxn} - A_{nxn})^{-1}(\hat{\mu}_{nxn}\kappa_{nx\,1} + \hat{w}_{nxn}l_{nx\,1})$$

$$p_{nx\,1} = (I_{nxn} - A_{nxn})^{-1}v_{nx\,1}$$

Let's define indirect (carbon) taxes on production on each sector as the carbon price charged on each good $(\hat{\theta}_{nxn})$, which can be set to zero for some goods and areas, multiplied by the emission intensities of each sector vector $(\beta^{e}_{nx\,1})$:

$$t_{nx\,1} = \hat{\theta}_{nxn}\beta^{e}_{nx\,1}$$

The price vector can be amended in the following format:

$$p_{nx\,1} = (I_{nxn} - A_{nxn})^{-1}\hat{v}_{nxn}(1 + t_{nx\,1})$$

Question: How should be the t vector populated if we want to add carbon tax only to some domestically produced goods of some sectors?

Example with 3 industries (Z1 produces good 1 and 2, and Z2 produces good 3):

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1-a_{33} \end{pmatrix}^{-1} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} \begin{pmatrix} 1+t_1 \\ 1+t_2 \\ 1+t_3 \end{pmatrix}$$

Substituting the terms inside the Leontief inverse matrix $(I_{nxn} - A_{nxn})^{-1}$ by b_{ij} to simplify notation and doing the first step of the multiplication:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1 & b_{12}v_2 & b_{13}v_3 \\ b_{21}v_1 & b_{22}v_2 & b_{23}v_3 \\ b_{31}v_1 & b_{32}v_2 & b_{33}v_3 \end{pmatrix} \begin{pmatrix} 1+t_1 \\ 1+t_2 \\ 1+t_3 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1(1+t_1) + b_{12}v_2(1+t_2) + b_{13}v_3(1+t_3) \\ b_{21}v_1(1+t_1) + b_{22}v_2(1+t_2) + b_{23}v_3(1+t_3) \\ b_{31}v_1(1+t_1) + b_{32}v_2(1+t_2) + b_{33}v_3(1+t_3) \end{pmatrix}$$

If we want to introduce the tax only in industry 1, we set $t_2 = t_3 = 0$ which simplify this equation to would simplify to:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1(1+t_1) + b_{12}v_2 + b_{13}v_3 \\ b_{21}v_1(1+t_1) + b_{22}v_2 + b_{23}v_3 \\ b_{31}v_1(1+t_1) + b_{32}v_2 + b_{33}v_3 \end{pmatrix}$$

b) <u>(Carbon) tariffs (carbon border adjustment mechanism):</u>

To do so we need to do following multiplication:

$$p_{nx1} = (I_{nxn} - A_{nxn})^{-1}\hat{v}_{nxn}(1 + cbam_{nx1}) - q_{nx1}$$

Where $cbam_{nx1}$ is vector filled with 1 for non-tariffed goods and $1 + tr_j$ for good j which is subjected to a tariff. Considering the multiregional setting, ***we need to enter an adjustment factor vector (q_{nx1}), to avoid that domestic prices in Z2 for the good subjected to import tariff's in Z1 are affected by the tariff.***

Now in the 2-region, 3-goods example, if we want to add a trade tariff imposed on imported inputs of Z2, that is in good 3. We want to arrive at the following expression:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1 + b_{12}v_2 + b_{13}v_3(1 + cbam_3) \\ b_{21}v_1 + b_{22}v_2 + b_{23}v_3(1 + cbam_3) \\ b_{31}v_1 + b_{32}v_2 + b_{33}v_3 \end{pmatrix}$$

where tar_3 is the trade tariff imposed in the imported good 3 used as an input in the production of goods 1 and 2, which are produced by Z1. To do so we would have to do the following multiplication:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{pmatrix} I - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & I - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & I - a_{33} \end{pmatrix}^{-1}}_{(I_{nxn}-A_{nxn})^{-1}} \underbrace{\begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}}_{\hat{v}_{nxn}} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 + tcbam_3 \end{pmatrix}}_{1+tar_{nx1}} - \underbrace{\begin{pmatrix} 0 \\ 0 \\ b_{33}v_3cbam_3 \end{pmatrix}}_{q_{nx1}}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1 & b_{12}v_2 & b_{13}v_3 \\ b_{21}v_1 & b_{22}v_2 & b_{23}v_3 \\ b_{31}v_1 & b_{32}v_2 & b_{33}v_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 + cbam_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ b_{33}v_3cbam_3 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1 + b_{12}v_2 + b_{13}v_3(1 + cbam_3) \\ b_{21}v_1 + b_{22}v_2 + b_{23}v_3(1 + cbam_3) \\ b_{31}v_1 + b_{32}v_2 + b_{33}v_3(1 + cbam_3) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ b_{33}v_3cbam_3 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_{11}v_1 + b_{12}v_2 + b_{13}v_3(1 + cbam_3) \\ b_{21}v_1 + b_{22}v_2 + b_{23}v_3(1 + cbam_3) \\ b_{31}v_1 + b_{32}v_2 + b_{33}v_3 \end{pmatrix}$$

The Leontief price system is derived from the total output identity in monetary terms:

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Dividing everything by total output X:

$$p_{nx\,1} = (I_{nxn} - A_{nxn})^{-1}v_{nx\,1} \qquad (2)$$