MAT 303

MTH 303 ELEMENTARY DIFFERENTIAL EQUATIONS II (3 UNI

- SERIES SOLUTION OF SECOND-ORDER LINEAR EQUATIONS
- BESSEL, LEGENDRE, AND HYPERGEOMETRIC EQUATIONS AND FUNCTIONS
- GAMMA AND BETA FUNCTIONS
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- FOURIER, FOURIER-BESSEL, AND FOURIER-LEGENDRE SERIES
- FOURIER TRANSFORMATION
- SOLUTIONS OF LAPLACE'S, WAVE, AND HEAT EQUATIONS BY FOURIER METHOD

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Power Series

efin...on: An infinite series of the form

$$\sum_{n}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \quad (1)$$

...alled a power series in x, where $c_0, c_1, \ldots$ are constants. A series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots \quad (2)$$

a power series in $(x - x_0)$. By setting $z = (x - x_0)$, equation (2) becomes a power series in z. Hence, it suffices to tudy power series of the form (1).

f x is fixed in series (1), then it becomes a series of constants. If this series converges (i.e. equals a finite real number), hen the power series is said to converge at x. If it does not converge, it is said to diverge at x. If a power series onverges at every point of an interval I, it is said to converge over I. If the series of absolute values converges over , the power series is said to converge absolutely over I.

heorem: Given the power series $\sum_{n=0}^{\infty} c_n x^n$, then there exists a non-negative real number R such that

1. $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely if $|x| < R$;
2. $\sum_{n=0}^{\infty} c_n x^n$ diverges if $|x| > R$;
3. $\sum_{n=0}^{\infty} c_n x^n$ converges for all x, if $R = \infty$;
4. $\sum_{n=0}^{\infty} c_n x^n$ converges nowhere except at $x = 0$, if $R = 0$.

R is called the radius of convergence of the power series and is usually obtained from the limit:

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| \text{ or } R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|c_n|}}, \quad (3)$$

f this limit exists.

he above theorem does not give the convergence or divergence of the power series when $|x| = R$.

or a given power series, we must test for convergence or divergence at the endpoints $x = -R, R$. The interval over hich a power series converges is called its interval of convergence.

ample

nd the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

lution: Here $c_n = \frac{1}{n}$ and so $c_{n+1} = \frac{1}{n+1}$. We have

$$= \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1,$$

ich implies that the series converges absolutely over the interval $|x| < 1$ or $-1 < x < 1$.

en $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by alternating-series test.

en $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent harmonic series.

s, the interval of convergence is $-1 \le x < 1$.

cises

the radius of convergence of the series

ich implies

$' = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$

Employing these in the given differential equation yields

$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$

Replacing n with $n+2$ in the first term, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$

Extracting the value of $n = 0$ and summing from $n = 1$ yields

$(2!\,c_2 + c_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)c_{n+2} + (1-2n)c_n]\,x^n = 0.$

This implies (1)

$2!\,c_2 + c_0 = 0.$ (2)

$(n+2)(n+1)c_{n+2} + (1-2n)c_n = 0, \quad n = 1, 2, \ldots.$

Equation (2) is called a recurrence relation or a recursion formula. From equation (1), we get $c_2 = -\frac{1}{2}c_0$.

From equation (2), we see that (3)

$\frac{2(4)-1}{(4+2)(5)}$ $\frac{5}{6}$

$c_{n+2} = \frac{2n-1}{(n+2)(n+1)}c_n, \quad n = 1, 2, \ldots.$

From equation (3), we have

$c_3 = \frac{1}{3.2}c_1 = \frac{1}{3!}c_1; \quad c_4 = \frac{3}{4.3}c_2 = -\frac{3}{4.3.2}c_0 = -\frac{3}{4!}c_0; \quad c_5 = \frac{5}{5.4}c_3 = \frac{5}{5.4.3.2}c_1 = \frac{5}{5!}c_1;$

$c_6 = \frac{7}{6.5}c_4 = -\frac{21}{6.5.4.3.2}c_0 = -\frac{21}{6!}c_0; \quad c_7 = \frac{9}{7.6}c_5 = \frac{45}{7.6.5.4.3.2}c_1 = \frac{45}{7!}c_1; \ldots$

The general solution of the given equation is

$y = (c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \cdots) + (c_1x + c_3x^3 + c_5x^5 + c_7x^7 + \cdots)$, or

$y = c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 - \cdots\right) + c_1\left(x + \frac{1}{3!}x^3 + \frac{5}{5!}x^5 + \frac{45}{7!}x^7 + \cdots\right),$

where c_0 and c_1 are arbitrary constants.

Example 2

Find the general solution of the differential equation $(1-x^2)y'' - 6xy' - 4y = 0$.

Solution: Since $x = 0$ is an ordinary point of the equation, we assume a series solution in the form

$y = \sum_{n=0}^{\infty} c_n x^n,$

which implies

$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$

Employing these in the given differential equation yields

$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 6nc_n x^n - \sum_{n=0}^{\infty} 4c_n x^n = 0.$

Replacing n with $n+2$ in the first term, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 6nc_n x^n - \sum_{n=0}^{\infty} 4c_n x^n = 0.$

Extracting the values of $n = 0$ and $n = 1$ and summing from $n = 2$ yields

$(2c_2 - 4c_0) + (6c_3 - 10c_1)x + \sum_{n=2}^{\infty}\{(n+2)(n+1)c_{n+2} - [n(n-1) + 6n + 4]c_n\}\,x^n = 0.$

This implies

$2c_2 - 4c_0 = 0. \quad \Rightarrow c_2 = 2c_0.$

$6c_3 - 10c_1 = 0. \quad \Rightarrow c_3 = \frac{5}{3}c_1.$

The recursion formula is

$(n+2)(n+1)c_{n+2} - [n(n-1) + 6n + 4]c_n = 0, \quad n = 2, 3, \ldots,$ or

$(n+2)(n+1)c_{n+2} - (n+1)(n+4)c_n = 0, \quad n = 2, 3, \ldots,$ or

$= \frac{n+4}{n+2} c_n, \ n = 2, 3, \ldots$

this recurrence relation, we have

$= \frac{6}{4} c_2 = 3c_0; \ c_5 = \frac{7}{5} c_3 = \frac{7}{3} c_1; \quad c_6 = \frac{8}{6} c_4 = 4c_0; \ c_7 = \frac{9}{7} c_5 = \frac{9}{3} c_1; \ldots$

general solution of the given equation is

$= (c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \cdots) + (c_1x + c_3x^3 + c_5x^5 + c_7x^7 + \cdots)$, or

$= c_0(1 + 2x^2 + 3x^4 + 4x^6 + \cdots) + \frac{1}{3} c_1(3x + 5x^3 + 7x^5 + 9x^7 + \cdots)$, or

$= c_0 \sum_{k=0}^{\infty} (k+1)x^{2k} + \frac{1}{3} c_1 \sum_{k=0}^{\infty} (2k+3)x^{2k+1}$,

where c_0 and c_1 are arbitrary constants.

Exercises

Find the power series solution of the following differential equations:

1. $y'' + xy' + y = 0$.
2. $(1 + x)y'' - y = 0$.
3. $(x^2 - 1)y'' + 2xy' - 2y = 0$.

Regular and Irregular Singular Points of a Second-order Linear Homogeneous Differential Equation

Definition: A singular point $x = x_0$ of the equation

$y'' + P(x)y' + Q(x)y = 0$

is said to be a regular singular point if both terms $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0. Otherwise, $x = x_0$ is an irregular singular point.

Note: If the factor $x - x_0$ appears at most to the 1st power in the denominator of $P(x)$ and at most to the 2nd power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.

Examples

Determine the singular points of each differential equation and classify each singular point as regular or irregular.

1. $x^3y'' + 4x^2y' + 3y = 0$.
2. $(x^2 - 9)^2 y'' + (x + 3)y' + 2y = 0$.
3. $(x^3 + 4x)y'' - 2xy' + 6y = 0$.

Solutions:

1. $x = 0$ is a singular point of the equation $x^3y'' + 4x^2y' + 3y = 0$.

   Here $P(x) = \frac{4x^2}{x^3} = \frac{4}{x}$ and $Q(x) = \frac{3}{x^3}$.

   Since x in the denominator of $Q(x)$ is to power 3, it follows that $x = 0$ is an irregular singular point.

2. $x = \pm 3$ are singular points of the equation $(x^2 - 9)^2 y'' + (x + 3)y' + 2y = 0$.

   Here $P(x) = \frac{x+3}{(x^2-9)^2} = \frac{1}{(x+3)(x-3)^2}$ and $Q(x) = \frac{2}{(x^2-9)^2} = \frac{2}{(x+3)^2(x-3)^2}$.

   Since $x + 3$ appears to power 1 in the denominator of $P(x)$ and to power 2 in the denominator of $Q(x)$, it follows that $x = -3$ is a regular singular point.

Since $x-3$ appears to power 2 in the denominator of $P(x)$, it follows that $x=3$ is an irregular singular point.

3. $x=0, x=\pm 2i$ are singular points of the equation $(x^3+4x)y''-2xy'+6y=0$.
Here $P(x)=-\frac{2x}{x^3+4x}=-\frac{2}{x^2+4}=-\frac{2}{(x+2i)(x-2i)}$ and $Q(x)=\frac{6}{x^3+4x}=\frac{6}{x(x^2+4)}=\frac{6}{x(x+2i)(x-2i)}$.
Since x is to power 0 and 1, respectively, in the denominators of $P(x)$ and $Q(x)$, it follows that $x=0$ is a regular singular point. Since $x+2i$ and $x-2i$ appear to power 1 each in the denominators of $P(x)$ and $Q(x)$, we conclude that $x=-2i$ and $x=2i$ are regular singular points.

Exercises

Locate the regular and/or irregular singular points of the differential equations

1. $(1-x)y''-y'+xy=0$.
2. $x^3(1-x^2)y''+(2x-3)y'+xy=0$.
3. $(x^2+x-6)y''+(x+3)y'+(x-2)y=0$.
4. $x^3(x^2-25)(x-2)^2y''+3x(x-2)y'+7(x+5)y=0$.

Series Solution of a Second-order Linear Homogeneous Differential Equation around a Regular Singular Point

To obtain the series solution of the differential equation

$a_2(x)y''+a_1(x)y'+a_0(x)y=0$ (1)

about a regular singular point, we employ Frobenius method.

Theorem: (Frobenius Method)

Let $x=x_0$ be a regular singular point of equation (1). Then there exists at least one series solution of the form

$y=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$, (2)

where $c_0\neq 0$ and the number r is a constant to be determined. The series converges at least on some interval $0<x-x_0<R$.

Indicial Equation: If $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ have power series expansions

$(x-x_0)P(x)=p_0+p_1(x-x_0)+p_2(x-x_0)^2+\cdots$,

$(x-x_0)^2Q(x)=q_0+q_1(x-x_0)+q_2(x-x_0)^2+\cdots$,

where $P(x)=\frac{a_1(x)}{a_2(x)}$, $Q(x)=\frac{a_0(x)}{a_2(x)}$, $a_2(x)$, then the quadratic equation

$r^2+(p_0-1)r+q_0=0$ (3)

is called the indicial equation of (1).

We shall consider 3 cases of the roots of the indicial equation (3):

Case I: r_1-r_2 is not an integer;

Case II: r_1-r_2 is a positive integer;

Case III: $r_1=r_2$.

Case I: Roots of the Indicial Equation do not differ by an Integer

Examples

Use the method of Frobenius to obtain two linearly independent series solutions around the regular singular point

, and form the general solution.

$2x^2y'' + (x - x^2)y' - y = 0.$

$2xy'' - (3 + 2x)y' + y = 0.$

tions:

1. By the method of Frobenius, we assume a series solution, around the regular singular point $x = 0$, of the equation

$2x^2y'' + (x - x^2)y' - y = 0$

to be

$y = \sum_{n=0}^{\infty} c_n x^{n+r},\ c_0 \neq 0.$

This is differentiated twice with respect to x to obtain

$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1},\quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$

Employing these in the given equation yields

$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r+1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$

$\Rightarrow \sum_{n=0}^{\infty}[(n+r)(2n+2r-1)-1]c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r+1} = 0.$

Extracting x^r and replacing n with $n-1$ in the last summation, we get

$x^r\{\sum_{n=0}^{\infty}[(n+r)(2n+2r-1)-1]c_n - \sum_{n=1}^{\infty}(n+r-1)c_{n-1}\}x^n = 0.$

Since $x^r \neq 0$, extracting the value of $n = 0$ from the first summation, and summing from $n = 1$, we obtain

$[r(2r-1)-1]c_0 + \sum_{n=1}^{\infty}\{[(n+r)(2n+2r-1)-1]c_n - (n+r-1)c_{n-1}\}x^n = 0.$ (1)

The indicial equation is $r(2r-1)-1 = 0$, or $2r^2 - r - 1 = 0$, since $c_0 \neq 0$, with roots $r_1 = 1,\ r_2 = -\frac{1}{2}$.

It should be noted that $r_1 - r_2 = \frac{3}{2} \neq$ integer.

We now obtain two linearly independent solutions corresponding to the two indicial roots.

$\underline{r = 1}$: From equation (1), the recurrence relation is

$[(n+1)(2n+1)-1]c_n - nc_{n-1} = 0,$ or

$c_n = \frac{n}{2n(n+1)+(n+1)-1}c_{n-1},\ n = 1, 2, \ldots,$ or

$c_n = \frac{1}{2n+3}c_{n-1},\ n = 1, 2, \ldots,$

which yields

$c_1 = \frac{1}{5}c_0,\quad c_2 = \frac{1}{7}c_1 = \frac{1}{5.7}c_0,\quad c_3 = \frac{1}{9}c_2 = \frac{1}{5.7.9}c_0,\ \ldots$

Generally,

$c_n = \frac{1}{5.7.9.....(2n+3)}c_0,\ n = 1, 2,\ \ldots$

Thus, one series solution of the given differential equation is

$y_1 = c_0\left[x + \sum_{n=1}^{\infty}\frac{x^{n+1}}{5.7.9.....(2n+3)}\right] = c_0 x\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{5.7.9.....(2n+3)}\right].$ (2)

$\underline{r = -\frac{1}{2}}$: The recursion formula, from equation (1), is

$\left[\left(n-\frac{1}{2}\right)(2n-2)-1\right]c_n - \left(n-\frac{3}{2}\right)c_{n-1} = 0,$ or

$2[(2n-1)(n-1)-1]c_n - (2n-3)c_{n-1} = 0,$ or

$2n(2n-3)c_n - (2n-3)c_{n-1} = 0,$ or

$c_n = \frac{1}{2n}c_{n-1},\ n = 1, 2, \ldots,$

$$A_1 x\left[1+\sum_{n=1}^{\infty}\frac{x^n}{5\cdot 7\cdot 9\cdots(2n+3)}\right]+A_2 x^{-\frac{1}{2}}\left[1+\sum_{n=1}^{\infty}\frac{x^n}{2\cdot 4\cdot 6\cdots 2n}\right],$$

re the second solution may be written as

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n n!}=x^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^n}{n!}=x^{-\frac{1}{2}}e^{\frac{x}{2}}$$

which gives

$c_1 = \frac{1}{2}c_0,\ \ c_2 = \frac{1}{4}c_1 = \frac{1}{2.4}c_0,\ \ c_3 = \frac{1}{6}c_2 = \frac{1}{2.4.6}c_0, \ldots.$

Generally,

$c_n = \frac{1}{2.4.6.\ldots.2n}c_0,\ \ n = 1, 2, \ldots.$

Hence, the second series solution of the differential equation is

$y_2 = c_0 x^{-\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{2.4.6.\ldots.2n}\right],$ (3)

and is linearly independent with solution (2).

The general solution of the given problem is

$y = A_1 x\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{5.7.9.\ldots.(2n+3)}\right] + A_2 x^{-\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{2.4.6.\ldots.2n}\right],$

where A_1 and A_2 are arbitrary constants, with $c_0 \equiv 1$.

2. Since $x = 0$ is a regular singular point of the equation

$2xy'' - (3 + 2x)y' + y = 0,$

we assume a series solution in the form

$y = \sum_{n=0}^{\infty} c_n x^{n+r},\ c_0 \neq 0.$

This is differentiated twice with respect to x to obtain

$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1},\ \ y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$

Employing these in the given equation yields

$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$

$\Rightarrow \sum_{n=0}^{\infty}[2(n+r)(n+r-1) - 3(n+r)]c_n x^{n+r-1} - \sum_{n=0}^{\infty}[2(n+r) - 1]c_n x^{n+r} = 0.$

Extracting x^r and replacing n with $n+1$ in the first summation, we get

$x^r\{\sum_{n=-1}^{\infty}(n+r+1)(2n+2r-3)c_{n+1} - \sum_{n=0}^{\infty}(2n+2r-1)c_n\}x^n = 0.$

Since $x^r \neq 0$, extracting the value of $n = -1$ from the first summation, and summing from $n = 0$, we obtain

$r(2r-5)c_0 x^{-1} + \sum_{n=0}^{\infty}\{(n+r+1)(2n+2r-3)c_{n+1} - (2n+2r-1)c_n\}x^n = 0.$ (1)

The indicial equation is $r(2r-5) = 0$, since $c_0 \neq 0$, with roots $r_1 = \frac{5}{2}$, $r_2 = 0$.

It should be noted that $r_1 - r_2 = \frac{5}{2} \neq$ integer.

We now obtain two linearly independent solutions corresponding to the two indicial roots.

$\underline{r = 0}$: From equation (1), the recurrence relation is

$(n+1)(2n-3)c_{n+1} - (2n-1)c_n = 0,$ or

$c_{n+1} = \frac{2n-1}{(n+1)(2n-3)}c_n,\ \ n = 0, 1, 2, \ldots,$

which yields

$c_1 = \frac{1}{3}c_0,\quad c_2 = -\frac{1}{2}c_1 = -\frac{1}{6}c_0,\quad c_3 = c_2 = -\frac{1}{6}c_0,\ \ldots.$

Thus, one series solution of the given differential equation is

$y_1 = c_0\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \cdots\right).$ (2)

$\underline{r = \frac{5}{2}}$: The recursion formula, from equation (1), is

$(n + \frac{7}{2})(2n+2)c_{n+1} - (2n+4)c_n = 0,$ or

$(2n+7)(n+1)c_{n+1} - 2(n+2)c_n = 0,$ or

$$\left(1+\frac{2\cdot 2}{7}x+\frac{2^2\cdot 3}{7\cdot 9}x^2+\frac{2^3\cdot 4}{7\cdot 9\cdot 11}x^3+\cdots\right)$$

$$x^{\frac{5}{2}}\left[1+\sum_{n=1}^{\infty}\frac{2^n(n+1)}{7\cdot 9\cdots\cdots(2n+5)}x^n\right].$$

$$y=A_1 J_{1/3}(x)+A_2 Y_{1/3}(x).$$

$$y=A_1 x^{\frac{7}{8}}\left(1-\frac{2}{15}x+\frac{2^2}{23\cdot 15\cdot 2}x^2-\frac{2^3}{31\cdot 23\cdot 15\cdot 3!}x^3+\cdots\right)$$

$$+A_2\left(1-2x+\frac{2^2}{9\cdot 2}x^2-\frac{2^3}{17\cdot 9\cdot 3!}x^3+\cdots\right).$$

$$y=A_1 x^{\frac{2}{3}}\left(1-\frac{1}{2}x+\frac{5}{28}x^2-\frac{1}{21}x^3+\cdots\right)$$

$$+A_2 x^{\frac{1}{3}}\left(1-\frac{1}{2}x+\frac{1}{5}x^2-\frac{7}{120}x^3+\cdots\right).$$

$$y=A_1 x\left[1+\frac{1}{15}\sum_{n=1}^{\infty}(2n+3)(2n+5)x^n\right]$$

$$+A_2 x^{\frac{1}{2}}\left[1+\frac{1}{2}\sum_{n=1}^{\infty}(n+1)(n+2)x^n\right].$$

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$c_{n+1} = \frac{2(n+2)}{(n+1)(2n+7)} c_n, \ \ n = 0, 1, 2, \ldots,$

which gives

$c_1 = \frac{2.2}{7} c_0, \ \ c_2 = \frac{2.3}{2.9} c_1 = \frac{2^2.3}{7.9} c_0, \ \ c_3 = \frac{2.4}{3.11} c_2 = \frac{2^3.4}{7.9.11} c_0, \ldots.$

Hence, the second series solution of the differential equation is

$$y_2 = c_0 x^{\frac{5}{2}} \left(1 + \frac{2.2}{7} x + \frac{2^2.3}{7.9} x^2 + \frac{2^3.4}{7.9.11} x^3 + \cdots \right), \qquad (3)$$

and is linearly independent with solution (2).

From equations (2) and (3), the general solution of the given problem is

$$y = A_1 \left(1 + \frac{1}{3} x - \frac{1}{6} x^2 - \frac{1}{6} x^3 - \cdots \right) + A_2 x^{\frac{5}{2}} \left(1 + \frac{2.2}{7} x + \frac{2^2.3}{7.9} x^2 + \frac{2^3.4}{7.9.11} x^3 + \cdots \right),$$

where A_1 and A_2 are arbitrary constants, with $c_0 \equiv 1$.

xercises

ind the two linearly independent series solutions about the point $x_0 = 0$, and form the general solution.

1. $x^2 y'' + x y' + \left(x^2 - \frac{1}{9}\right) y = 0.$
2. $4xy'' + \frac{1}{2} y' + y = 0.$
3. $9x^2 y'' + 9x^2 y' + 2y = 0.$
4. $2x^2(1 - x)y'' - x(1 + 7x)y' + y = 0.$
5. Show that the differential equation
$$(1 - x^2)y'' - 3xy' + (k^2 - 1)y = 0,$$
where k is a constant, has solutions of the form $\sum_{n=0}^{\infty} c_n x^{n+r}$, provided $r = 0$ or 1. Hence, obtain two linearly independent solutions, one of which is of the form
$y = 1 + \sum_{n=1}^{\infty} \frac{(1-k^2)(3^2-k^2)\ldots[(2n-1)^2-k^2]}{(2n)!} x^{2n},$
and write down the general solution.

ase II: Roots of the Indicial Equation differ by a Positive Integer

$r_1 - r_2 = N$, where N is a positive integer, then there exist two linearly independent solutions of the differential quation

$_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

the form

$_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \ \ c_0 \neq 0,$

$_2 = A y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \ \ b_0 \neq 0,$

here A is a constant that could be zero.

ote: One method of finding y_2 after obtaining y_1 is the reduction of order method. By this method, if the equation $'' + P(x)y' + Q(x)y = 0$ is to be solved and y_1 is known, we compute y_2 from the formula

$_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx.$

xample

ompute two linearly independent solutions of the differential equation

$^2 y'' - (x + 2)y = 0$

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$$y = A_1 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + A_2 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

$$= \frac{1}{x}\left(A_1 \cosh x + A_2 \sinh x\right).$$

$$y = A_1\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) + A_2 \sum_{n=0}^{\infty} (n+1) x^{n+4}$$

$$y = A_1 y_1 + A_2\left[y_1 \ln x + y_1\left(2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots\right)\right],$$

where $y_1 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

$$\therefore y = A_1 x e^{-x} + A_2 x e^{-x}\left(\ln x + x + \frac{1}{4}x^2 + \frac{1}{3\cdot 3!}x^3 + \cdots\right).$$

nd the point $x_0 = 0$.

tion: Since $x_0 = 0$ is a regular singular point, we assume a solution in the form

$= \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$,

ich is differentiated twice with respect to x to get

$= \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$

nploying these in the given equation yields

$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} - \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0.$

$\Rightarrow \sum_{n=0}^{\infty}[(n+r)(n+r-1)-2]c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0.$

xtracting x^r and replacing n with $n-1$ in the last summation, we get

$^r\{\sum_{n=0}^{\infty}[(n+r)(n+r-1)-2]c_n - \sum_{n=1}^{\infty} c_{n-1}\}x^n = 0.$

ince $x^r \neq 0$, extracting the value of $n = 0$ from the first summation, and summing from $n = 1$, we obtain

$r^2 - r - 2)c_0 + \sum_{n=1}^{\infty}\{[(n+r)(n+r-1)-2]c_n - c_{n-1}\}x^n = 0.$ (1)

he indicial equation is $r^2 - r - 2 = 0$, since $c_0 \neq 0$, with roots $r_1 = 2,\ r_2 = -1$.

t should be noted that $r_1 - r_2 = 3$ which is a positive integer.

Ve now obtain one solution corresponding to the larger indicial root $r = 2$.

$= 2$: From equation (1), the recurrence relation is

$(n+2)(n+1) - 2]c_n - c_{n-1} = 0,$ or

$_n = \frac{1}{n(n+3)} c_{n-1},\ n = 1, 2, \ldots,$

vhich yields

$_1 = \frac{1}{4} c_0, \quad c_2 = \frac{1}{2.5} c_1 = \frac{1}{40} c_0, \quad c_3 = \frac{1}{3.6} c_2 = \frac{1}{720} c_0, \quad \ldots.$

hus, one series solution of the given differential equation, with $c_0 = 1$, is

$_1 = x^2\left(1 + \frac{1}{4}x + \frac{1}{40}x^2 + \frac{1}{720}x^3 + \cdots\right).$ (2)

Ve employ the reduction of order method to find the second solution y_2 since y_1 is known, for the differential quation $y'' + P(x)y' + Q(x)y = 0$. By this method,

$_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx.$

n our problem, $P(x) = 0$. From equation (2), we obtain

$y_2 = y_1 \int \frac{dx}{x^4\left(1+\frac{1}{4}x+\frac{1}{40}x^2+\frac{1}{720}x^3+\cdots\right)^2} = y_1 \int \frac{dx}{x^4\left(1+\frac{1}{2}x+\frac{9}{80}x^2+\frac{11}{720}x^3+\cdots\right)} = y_1 \int \frac{1}{x^4}\left(1 - \frac{1}{2}x + \frac{11}{80}x^2 - \frac{1}{36}x^3 + \cdots\right) dx =$

$_1 \int \left(\frac{1}{x^4} - \frac{1}{2x^3} + \frac{11}{80x^2} - \frac{1}{36x} + \cdots\right) dx = y_1\left(-\frac{1}{3x^3} + \frac{1}{4x^2} - \frac{11}{80x} - \frac{1}{36}\ln x + \cdots\right),$

nd so

$_2 = -\frac{1}{36} y_1 \ln x + y_1\left(-\frac{1}{3x^3} + \frac{1}{4x^2} - \frac{11}{80x} + \cdots\right)$

$= -\frac{1}{36} y_1 \ln x + \left(x^2 + \frac{1}{4}x^3 + \frac{1}{40}x^4 + \frac{1}{720}x^5 + \cdots\right)\left(-\frac{1}{3x^3} + \frac{1}{4x^2} - \frac{11}{80x} + \cdots\right)$ or

$y_2 = -\frac{1}{3}\left(\frac{1}{12} y_1 \ln x + \frac{1}{x} - \frac{1}{2} + \frac{1}{4}x - \frac{23}{2880}x^2 + \cdots\right).$ (3)

Ve take the linear combination of solutions (2) and (3) to obtain the general solution of the given equation as

$= B_1 x^2\left(1 + \frac{1}{4}x + \frac{1}{40}x^2 + \frac{1}{720}x^3 + \cdots\right) + B_2\left(\frac{1}{12} y_1 \ln x + \frac{1}{x} - \frac{1}{2} + \frac{1}{4}x - \frac{23}{2880}x^2 + \cdots\right),$

vhere B_1 and B_2 are arbitrary constants.

lote I: The form of y_2 is as expected, since

$Ay_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n-1},\ b_0 \neq 0,$ with $n - r_2 = n - 1$, or

$Ay_1 \ln x + \frac{b_0}{x} + b_1 + b_2 x + b_3 x^2 + \cdots$

re $A = \frac{1}{12},\ b_0 = 1,\ b_1 = -\frac{1}{2},\ b_2 = \frac{1}{4},\ b_3 = -\frac{23}{2880}, \ldots.$

te II: Use was made of the binomial expansions below.

$\mp A)^{-1} = 1 \pm A + A^2 \pm A^3 + \cdots.$

$A + B + C)^2 = A^2 + B^2 + C^2 + 2(AB + AC + BC).$

$A + B + C + D)^2 = A^2 + B^2 + C^2 + D^2 + 2(AB + AC + AD + BC + BD + CD).$

$A + B + C)^3 = A^3 + B^3 + C^3 + 3(A^2B + A^2C + AB^2 + AC^2 + B^2C + BC^2) + 6ABC.$

ase III: Roots of the Indicial Equation are Equal

$r_1 = r_2$, then there exist two linearly independent solutions of the differential equation

$_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

the form

$_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1},\ c_0 \neq 0,$

$_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1},\ b_0 \neq 0.$

lote: The procedure for this case is the same as that for Case II.

xercises

ind two linearly independent series solutions of the following differential equations.

1. $xy'' + 2y' - xy = 0.$
2. $x(x-1)y'' + 3y' - 2y = 0.$
3. $xy'' + y' + y = 0.$
4. $x^2y'' + x(x-1)y' + y = 0.$

Gamma Function

he gamma function $\Gamma(\alpha)$ is defined by the integral

$$(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\, dt. \qquad (1)$$

onvergence of this integral requires that $\alpha - 1 > -1$ or $\alpha > 0$. (If α is complex, equation (1) is meaningful only for hose α whose real parts are positive.)

amma function is also called Euler's equation of the second kind. Integration of equation (1) by parts yields

$(\alpha) = \left[\frac{1}{\alpha} e^{-t} t^{\alpha}\right]_0^{\infty} + \frac{1}{\alpha}\int_0^{\infty} e^{-t} t^{\alpha-1}\, dt = \frac{1}{\alpha}\Gamma(\alpha+1),$ or

$$(\alpha+1) = \alpha\Gamma(\alpha). \qquad (2)$$

rom equation (1), $\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = -[e^{-t}]_0^{\infty} = 1.$

Jsing equation (2), we get

$(2) = 1\Gamma(1) = 1,\ \Gamma(3) = 2\Gamma(2) = 2.1 = 2!,\ \Gamma(4) = 3\Gamma(3) = 3.2! = 3!.$

lence, if α is a non-negative integer, say n, then

$$(n+1) = n!,\ n = 0, 1, \ldots. \qquad (3)$$

his shows that the gamma function may be regarded as a generalized factorial function.

peated application of equation (2), we get

$$) = \frac{\Gamma(\alpha+1)}{\alpha} = \frac{1}{\alpha}\cdot\frac{\Gamma(\alpha+2)}{\alpha+1} = \frac{1}{\alpha(\alpha+1)}\cdot\frac{\Gamma(\alpha+3)}{\alpha+2} = \cdots = \frac{\Gamma(\alpha+n+1)}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)},$$ where n is an integer.

s,

$$\iota) = \frac{\Gamma(\alpha+n+1)}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)},\ \alpha \neq 0,-1,-2,\dots \quad (4)$$

ay be used for defining the gamma function for non-positive α $(\neq 0,-1,-2,\dots)$, where n is chosen to be the nallest integer such that $\alpha+n+1>0$. Equations (1) and (4) then give a definition of $\Gamma(\alpha)$ for all $(\neq 0,-1,-2,\dots)$.

Ve frequently encounter $\Gamma(\frac{1}{2})$ in problems involving the gamma function. We, therefore, compute it. Using equation 1), we obtain

$$\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt.$$

et $t=u^2$. $\Rightarrow dt = 2udu$. $t=0 \Rightarrow u=0$, $t=\infty \Rightarrow u \Rightarrow \infty$.

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du.$$

ince $\int_0^\infty e^{-u^2}\,du = \int_0^\infty e^{-v^2}\,dv$, we get

$$\Gamma\left(\frac{1}{2}\right)\Big]^2 = \left(2\int_0^\infty e^{-u^2}\,du\right)\left(2\int_0^\infty e^{-v^2}\,dv\right) = 4\int_0^\infty e^{-u^2}\int_0^\infty e^{-v^2}\,dudv.$$

Ve introduce polar coordinates (r,θ) so that $u=r\cos\theta$, $v=r\sin\theta$. To transform $dudv$ to $drd\theta$, we have that

vhere $J = \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

s the Jacobian of the transformation. Since $u^2+v^2=r^2$, we get

$$\Gamma\left(\frac{1}{2}\right)\Big]^2 = 4\int_0^\infty re^{-r^2}\int_0^{\frac{\pi}{2}} d\theta dr = 2\int_0^\infty e^{-z}\int_0^{\frac{\pi}{2}} d\theta dz = 2\int_0^{\frac{\pi}{2}} d\theta = 2\theta\Big|_0^{\frac{\pi}{2}} = \pi,$$

vhere $z=r^2$. Therefore,

$$\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (5)$$

his is obtained considering that $0<u<\infty, 0<v<\infty \Rightarrow 0<r<\infty, 0<\theta<\frac{\pi}{2}$.

lote: We state, without proof, that $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$.

xample

valuate the following.

1. $\Gamma(7)$.
2. $\Gamma\left(-\frac{5}{2}\right)$.

olution:

1. $\Gamma(\alpha) = 6! = 720$, using equation (3).
2. Since $\alpha = -\frac{5}{2}$, then $n=2$ is the smallest integer such that $\alpha+n+1\left(=\frac{1}{2}\right)>0$. Using equation (4), we get

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{-\frac{15}{8}} = -\frac{8}{15}\sqrt{\pi}.$$

nples

uate the following using the definition of the gamma function

3. $\int_0^\infty e^{-h^2x^2}\,dx$.
4. $\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx$.

olution:

3. Let $t = h^2x^2$. $\Rightarrow dt = 2h^2x\,dx$. $\Rightarrow x = \frac{t^{\frac{1}{2}}}{h}$. $dx = \frac{1}{2h}t^{-\frac{1}{2}}dt$.

   $x = 0 \Rightarrow t = 0$, $x = \infty \Rightarrow t = \infty$.

   Hence

   $$\int_0^\infty e^{-h^2x^2}\,dx = \frac{1}{2h}\int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt.$$

   Here $\alpha - 1 = -\frac{1}{2}$. $\Rightarrow \alpha = \frac{1}{2}$. Thus

   $$\int_0^\infty e^{-h^2x^2}\,dx = \frac{1}{2h}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2h}.$$

4. Let $y = \ln\frac{1}{x} = \ln x^{-1} = -\ln x$. $\Rightarrow \ln x = -y$ or $x = e^{-y}$. $\Rightarrow dx = -e^{-y}dy$.

   $x = 0 \Rightarrow y = \infty$; $x = 1 \Rightarrow y = 0$.

   Hence

   $$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx = \int_\infty^0 e^{-2y}y^3.-e^{-y}\,dy = \int_0^\infty e^{-2y}y^3\,dy.$$

   Let $t = 3y$. $\Rightarrow dt = 3dy$.

   $y = 0 \Rightarrow t = 0$, $y = \infty \Rightarrow t = \infty$.

   Thus

   $$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx = \int_0^\infty e^{-t}\left(\frac{t}{3}\right)^3\frac{dt}{3} = \frac{1}{3^4}\int_0^\infty e^{-t}t^3\,dt.$$

   Here $\alpha - 1 = 3$. $\Rightarrow \alpha = 4$. Therefore

   $$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx = \frac{1}{3^4}\Gamma(4) = \frac{3!}{3^4} = \frac{3.2}{3^4} = \frac{2}{3^3} = \frac{2}{27}.$$

Exercises

Evaluate

1. $\Gamma(5)$.
2. $\Gamma\left(-\frac{1}{2}\right)$.
3. $\Gamma\left(-\frac{3}{2}\right)$.
4. $\Gamma\left(\frac{13}{2}\right)$.

Evaluate each integral using the gamma function.

5. $\int_0^1 (\ln x)^{\frac{1}{3}}\,dx$.
6. $\int_0^\infty x^{\frac{3}{4}}e^{-x}\,dx$.
7. $\int_0^\infty \frac{x^a}{a^x}dx$, $a > 1$. [Hint: Recall that $a^x = e^{x\ln a}$.]

Beta Function

beta function is defined by the integral

$$p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \tag{1}$$

ıere $p > 0, q > 0$. Beta function is also called Euler's integral of the first kind. By the transformation $t = 1 - z$, it ın be shown that

$$(p, q) = \mathrm{B}(q, p), \tag{2}$$

vhich shows that the beta function is symmetric.

$\mathrm{B}(p, q)$ may be expressed by the gamma function as

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{3}$$

To prove this, we consider the product

$$\Gamma(p)\Gamma(q) = \left(\int_0^\infty e^{-u} u^{p-1}\,du\right)\left(\int_0^\infty e^{-v} v^{q-1}\,dv\right).$$

Putting $u = x^2, \;\; v = y^2$ leads to

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-x^2} x^{2p-1}\,dx \int_0^\infty e^{-y^2} y^{2q-1}\,dy = \iint_0^\infty e^{-(x^2+y^2)} x^{2p-1} y^{2q-1}\,dx\,dy,$$

since $du = 2x\,dx, \;\; dv = 2y\,dy$.

$u = v = 0 \;\; \Rightarrow x = y = 0, \;\; u = v = \infty \;\; \Rightarrow x = y = \infty.$

Employing the polar coordinates (r, θ), represented by $x = r\cos\theta\,, y = r\sin\theta$, we obtain

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-r^2} r^{2(p+q-\frac{1}{2})}\,dr \int_0^{\frac{\pi}{2}} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\,d\theta. \tag{4}$$

Let $r^2 = t. \Rightarrow 2r\,dr = dt. \;\; \Rightarrow dr = \frac{1}{2} t^{-\frac{1}{2}} dt.$

$r = 0 \;\Rightarrow t = 0, r = \infty \;\Rightarrow t = \infty.$

These are substituted into the first integral in equation (4) to get

$$\int_0^\infty e^{-r^2} r^{2p+2q-1}\,dr = \frac{1}{2}\int_0^\infty e^{-t} t^{p+q-1}\,dt = \frac{1}{2}\Gamma(p+q). \tag{5}$$

n the second integral in equation (4), we let $\cos\theta = z. \;\Rightarrow\; -\sin\theta\,d\theta = dz. \;\; \Rightarrow d\theta = -\left(1 - z^2\right)^{-\frac{1}{2}} dz.$

$\theta = 0 \;\; \Rightarrow z = 1, \;\; \theta = \frac{\pi}{2} \;\; \Rightarrow z = 0.$

Also $\sin\theta = \left(1 - z^2\right)^{\frac{1}{2}}$. Hence

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\,d\theta = -\int_1^0 z^{2p-1}\left(1 - z^2\right)^{q-\frac{1}{2}}\left(1 - z^2\right)^{-\frac{1}{2}} dz = \int_0^1 z^{2p-1}\left(1 - z^2\right)^{q-1} dz.$$

Let $z^2 = t. \Rightarrow 2z\,dz = dt. \;\; \Rightarrow dz = \frac{1}{2} t^{-\frac{1}{2}} dt.$

$z = 0 \;\Rightarrow t = 0, \;\; z = 1 \;\Rightarrow t = 1.$

Thus

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\,d\theta = \frac{1}{2}\int_0^1 t^{p-\frac{1}{2}}(1-t)^{q-1} t^{-\frac{1}{2}} dt = \frac{1}{2}\int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{1}{2}\mathrm{B}(p, q). \tag{6}$$

mploying equations (5) and (6) in equation (4) yields

$$\Gamma(p)\Gamma(q) = 4\left[\frac{1}{2}\Gamma(p+q)\right]\left[\frac{1}{2}\mathrm{B}(p, q)\right] = \Gamma(p+q)\mathrm{B}(p, q) \qquad \text{or}$$

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Note:

i. $\mathrm{B}(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}.$

ii. Beta function can also be defined by the integral

$\frac{1}{5}$.

$\frac{3\pi}{256}$.

$\frac{\pi}{2}$.

$\frac{5\pi}{8}$, obtained from $16B\left(\frac{3}{2}, \frac{7}{2}\right)$.

5. $B(4,1) = \frac{1}{4}$.

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,$$

we let $t = \frac{x}{1+x}$.

$$\Rightarrow dt = \frac{dx}{(1+x)^2}.$$

Also $x = \frac{t}{1-t}$. $\quad t=0 \Rightarrow x=0;$ $\quad t=1 \Rightarrow x=\infty$.

$$1-t = \frac{1}{1+x}.$$

$$\Rightarrow B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p-1}} \frac{1}{(1+x)^{q-1}} \frac{dx}{(1+x)^2} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx.$$

$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx.$$

nples

uate

1. $B\left(\frac{1}{2}, 2\right)$.
2. $B\left(3, \frac{1}{4}\right)$.
3. $B\left(\frac{3}{4}, \frac{1}{4}\right)$.

olutions:

1. $B\left(\frac{1}{2}, 2\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{1}{2}+2\right)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4}{3}\Gamma(2) = \frac{4}{3}.$
2. $B\left(3, \frac{1}{4}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(3+\frac{1}{4}\right)} = \frac{\Gamma(3)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{13}{4}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma(3)}{\frac{9}{4}\frac{5}{4}\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{64}{45}\Gamma(3) = \frac{64}{45}.2! = \frac{128}{45}.$
3. Since $B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$, we obtain, using $\alpha = \frac{3}{4}$,
   $B\left(\frac{3}{4}, \frac{1}{4}\right) = \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin\frac{3\pi}{4}} = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\pi.$

xercises

valuate

1. $B(1, 5)$.
2. $B\left(\frac{7}{2}, \frac{5}{2}\right)$.
3. $B\left(\frac{3}{2}, \frac{1}{2}\right)$.
4. $\int_0^2 x^{\frac{1}{2}}(2-x)^{\frac{5}{2}}\,dx$.
5. $\int_0^\infty \frac{t^3}{(1+t)^5}\,dt$.

Bessel's Equation and Bessel Functions

he differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{1}$$

where ν is a constant, is called Bessel's equation of order ν. In solving equation (1), we assume that $\nu \geq 0$. The solutions of Bessel's equation, called Bessel functions, are important in mathematical physics.

ince $x = 0$ is a regular singular point of equation (1), we assume a series solution of (1) in the form

$y = \sum_{n=0}^\infty c_n x^{n+r},\ c_0 \neq 0.$

his is differentiated twice with respect to x to obtain

$y' = \sum_{n=0}^\infty (n+r)c_n x^{n+r-1},\ \ y'' = \sum_{n=0}^\infty (n+r)(n+r-1)c_n x^{n+r-2}.$

mploying these in the given equation yields

$\sum_{n=0}^\infty (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^\infty (n+r)c_n x^{n+r} + \sum_{n=0}^\infty c_n x^{n+r+2} - \sum_{n=0}^\infty \nu^2 c_n x^{n+r} = 0.$

$\Rightarrow \sum_{n=0}^\infty \left[(n+r)^2 - \nu^2\right]c_n x^{n+r} + \sum_{n=0}^\infty c_n x^{n+r+2} = 0.$

xtracting x^r and replacing n with $n-2$ in the last summation, we get

$x^r\left\{\sum_{n=0}^\infty \left[(n+r)^2 - \nu^2\right]c_n - \sum_{n=2}^\infty c_{n-2}\right\}x^n = 0.$

$x^r \neq 0$, extracting the values of $n = 0$ and $n = 1$ from the first summation, and summing from $n = 2$, we
ain

$$-v^2]c_0 + [(1+r)^2 - v^2]c_1 x + \sum_{n=2}^{\infty}\{[(n+r)^2 - v^2]c_n + c_{n-2}\}x^n = 0. \quad (2)$$

he indicial equation is $r^2 - v^2 = 0$, since $c_0 \neq 0$, with roots $r_1 = v$, $r_2 = -v$.

Ve now obtain one solution corresponding to the indicial root $r_1 = v$.

$= v$: We have $[(1+v)^2 - v^2]c_1 = 0$, or $(1+2v)c_1 = 0$. Since $v \geq 0$, it follows that $1 + 2v \neq 0$, which implies hat $c_1 = 0$. From equation (2), the recursion formula is

$$(n+v)^2 - v^2]c_n + c_{n-2} = 0, \qquad \text{or} \quad n(n+2v)c_n + c_{n-2} = 0, \quad n = 2, 3, \ldots, \quad \text{or}$$

$$_n = -\frac{1}{n(n+2v)}c_{n-2}, \ n = 2, 3, \ldots. \quad (3)$$

vhich yields

$_3 = c_5 = c_7 = \cdots = 0$, since $c_1 = 0$.

ince n in equation (3) is an even number for non-trivial results, we have

$$_{2n} = -\frac{1}{2^2 n(n+v)}c_{2n-2}, \ n = 1, 2, \ldots, \quad (4)$$

vhich gives

$$_2 = -\frac{1}{2^2(1+v)}c_0, \ c_4 = -\frac{1}{2^2.2(2+v)}c_2 = \frac{(-1)^2}{2^4.2(1+v)(2+v)}c_0, \ c_6 = -\frac{1}{2^2.3(3+v)}c_4 = \frac{(-1)^3}{2^6.2.3(1+v)(2+v)(3+v)}c_0, \ldots,$$

$$_{2n} = \frac{(-1)^n}{2^{2n}n!(1+v)(2+v)\ldots(n+v)}c_0, \ n = 1, 2, \ldots. \quad (5)$$

n the theory of Bessel functions, it is customary to denote the constant c_0 by

$$_0 = \frac{1}{2^v\Gamma(1+v)}, \quad (6)$$

vhere Γ is gamma function. Thus, equation (5) becomes

$$_{2n} = \frac{(-1)^n}{2^{2n+v}n!(1+v)(2+v)\ldots(n+v)\Gamma(1+v)}c_0, \ n = 0, 1, 2, \ldots.$$

ut

$$(1+v+n) = (n+v)\Gamma(n+v) = \cdots = (n+v)(n+v-1)\ldots(1+v)\Gamma(1+v),$$

nd so

$$_{2n} = \frac{(-1)^n}{2^{2n+v}n!\Gamma(1+v+n)}, \ n = 0, 1, 2, \ldots.$$

lence, one solution of equation (1) is

$$_1 = \sum_{n=0}^{\infty} c_{2n}x^{2n+v} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}.$$

Ve have that one solution of the Bessel's equation of order v is Bessel function of the first kind of order v, and is lenoted by

$$_v(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}. \quad (7)$$

n the case of the indicial root $r = -v$, we have the second solution y_2 to be Bessel function of the first kind of order $-v$, denoted by

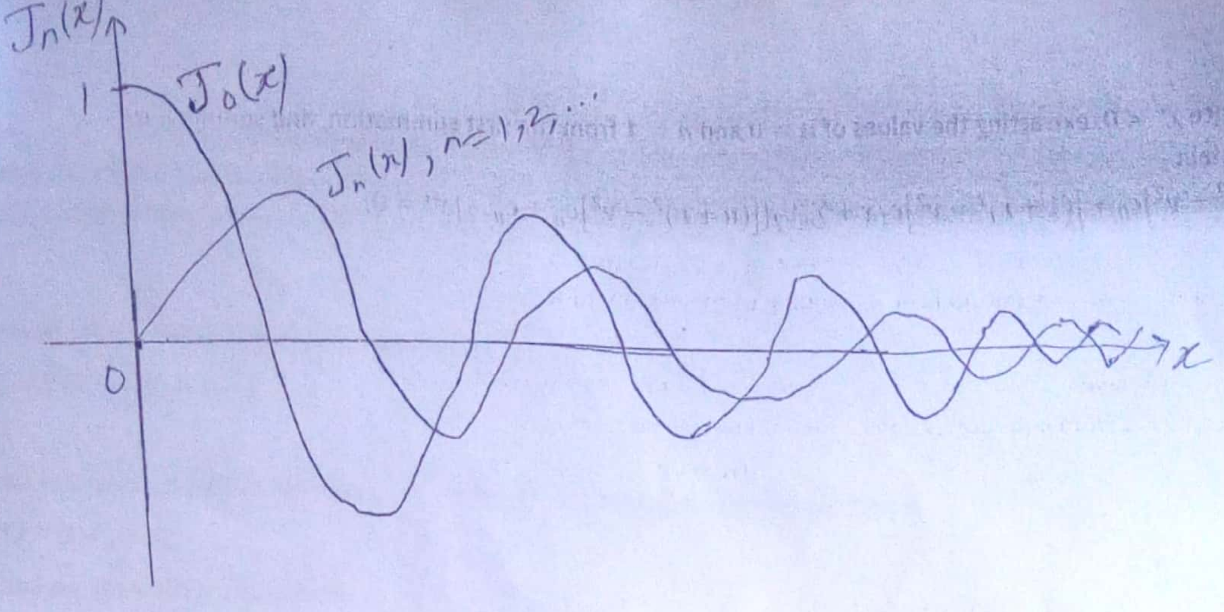
$$_{-v}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-v+n)}\left(\frac{x}{2}\right)^{2n-v}, \quad (8)$$

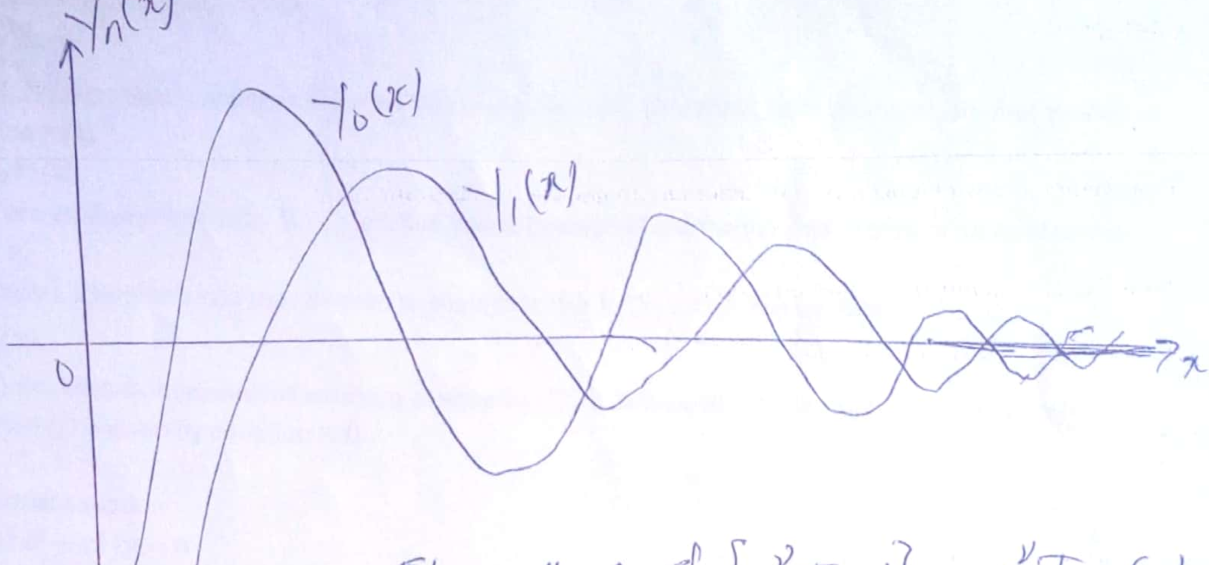
vhere $r_1 - r_2 = 2v$ is assumed not to be an integer.

herefore, if $r_1 - r_2 = 2v$ is not an integer, the general solution of Bessel's equation of order v is

$$= a_1 J_v(x) + a_2 J_{-v}(x), \quad (9)$$

ince $J_v(x)$ and $J_{-v}(x)$ are linearly independent solutions of equation (1), where a_1 and a_2 are arbitrary constants.





Show that $\frac{d}{dx}\left[x^{\nu}J_{\nu}(x)\right] = x^{\nu}J_{\nu-1}(x)$.

Since $J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}$,

t follows that

$$\frac{d}{dx}\left[x^{\nu}J_{\nu}(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)\,2^{2n+\nu}}\,x^{2(n+\nu)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{2(n+\nu)\,(-1)^n}{n!\,\Gamma(1+\nu+n)\,2^{2n+\nu}}\,x^{2(n+\nu)-1}$$

$$= x^{\nu}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}, \quad \text{or}$$

$$\frac{d}{dx}\left[x^{\nu}J_{\nu}(x)\right] = x^{\nu}J_{\nu-1}(x),$$

here $\Gamma(1+\nu+n) = (\nu+n)\,\Gamma(\nu+n)$ has been used.

ever, if $r_1 - r_2 = 2\nu$ is a positive integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly dependent, and there may exist a second tion of equation (1). It can be shown that if $r_1 - r_2 = 2\nu$ is a positive integer, and ν is half of an odd number, $x)$ and $J_{-\nu}(x)$ are linearly independent. Thus, equation (9) holds true for all $\nu \neq$ integer.

xample

ind the general solution of the differential equation

$$^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0.$$

olution: This is Bessel's equation of order $\nu = \frac{1}{3}$, since $\nu^2 = \frac{1}{9}$. Hence, its general solution is

$$= a_1 J_{\frac{1}{3}}(x) + a_2 J_{-\frac{1}{3}}(x),$$

vhere a_1 and a_2 are arbitrary constants.

Bessel Function of the Second Kind

f ν is not an integer, the function defined by

$$_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \qquad (10)$$

ind the function $J_\nu(x)$ are linearly independent solutions of equation (1). Therefore, for $\nu \neq$ integer, another general olution of equation (1) is

$$= a_1 J_\nu(x) + a_2 Y_\nu(x), \qquad (11)$$

vhere a_1 and a_2 are arbitrary constants. $Y_\nu(x)$ is called Bessel function of the second kind of order ν, or Neumann's unction of order ν.

ls $\nu \to m$ (an integer), L'Hôpital's rule may be used to show that $\lim_{\nu \to m} Y_\nu(x)$ exists, and we have

$$_m(x) = \lim_{\nu \to m} Y_\nu(x).$$

$_m(x)$ and $Y_m(x)$ are linearly independent solutions of equation (1). It follows that for any value of ν, the general olution of equation (1) is given by equation (11).

Jote: The differential equation

$$^2y'' + xy' + (\lambda^2 x^2 - \nu^2)y = 0$$

s called the parametric Bessel equation, since λ is a parameter. Its general solution is

$$= a_1 J_\nu(\lambda x) + a_2 Y_\nu(\lambda x),$$

vhere a_1 and a_2 are arbitrary constants. (12)

xample

ind the general solution of the parametric Bessel equation

$$x^2y'' + xy' + 9x^2y = 0.$$

olution: Here $\nu = 0$, $\lambda = 3$, since $\lambda^2 = 9$. Thus, the general solution is

$$= a_1 J_0(3x) + a_2 Y_0(3x),$$

vhere a_1 and a_2 are arbitrary constants.

$$\frac{d}{dx}\left[x^{\nu}J_\nu(x)\right] = x^{\nu}J_\nu'(x) + \nu x^{\nu-1}J_\nu(x) = x^{\nu}J_{\nu-1}(x). \quad (a)$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = x^{-\nu}J_\nu'(x) - \nu x^{-\nu-1}J_\nu(x) = -x^{-\nu}J_{\nu+1}(x) \quad (b)$$

Multiply (a) by $x^{-\nu}$ and multiply (b) by x^{ν} to obtain

$$J_\nu'(x) + \nu x^{-1}J_\nu(x) = J_{\nu-1}(x). \quad (c)$$

$$J_\nu'(x) - \nu x^{-1}J_\nu(x) = -J_{\nu+1}(x). \quad (d)$$

(c) plus (d) gives

$$2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x). \quad (e)$$

(c) minus (d) gives

$$\frac{2\nu}{x}J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x). \quad (f)$$

(e) and (f) are the recurrence relations listed as ii. and iv.

Show that $\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x)$.

Replacing ν by $-\nu$ in $\frac{d}{dx}\left[x^{\nu}J_\nu(x)\right] = x^{\nu}J_{\nu-1}(x)$, we get

$$\frac{d}{dx}\left[x^{-\nu}J_{-\nu}(x)\right] = x^{-\nu}J_{-\nu-1}(x).$$

Since $J_{-\nu}(x) = (-1)^{\nu}J_\nu(x)$, we have

$$\frac{d}{dx}\left[x^{-\nu}(-1)^{\nu}J_\nu(x)\right] = x^{-\nu}J_{-(\nu+1)}(x) = (-1)^{\nu+1}x^{-\nu}J_{\nu+1}(x).$$

$$\Rightarrow (-1)^{\nu}\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = (-1)^{\nu+1}x^{-\nu}J_{\nu+1}(x).$$

$$\Rightarrow \frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x).$$

Properties of Bessel Functions

me of the properties of Bessel functions are:

i. $\frac{d}{dx}[x^{\nu}J_{\nu}(x)] = x^{\nu}J_{\nu-1}(x)$.

ii. $\frac{d}{dx}[x^{-\nu}J_{\nu}(x)] = -x^{-\nu}J_{\nu+1}(x)$.

iii. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x)$.

iv. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x)$.

v. $J_{-m}(x) = (-1)^m J_m(x)$.

vi. $J_m(-x) = (-1)^m J_m(x)$.

vii. $J_m(0) = 0,\ m = 1, 2, \dots$

viii. $J_0(0) = 1$.

ix. $\lim_{x\to 0} Y_m(x) = -\infty$.

Note:

a. m is an integer.

b. Properties iii. and iv. are recurrence relations.

c. Property vi. shows that $J_m(x)$ is an even function if m is an even integer, and an odd function if m is an odd integer.

Exercises

Find the general solution of each of the given differential equations.

1. $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$.
2. $16x^2y'' + 16xy' + (16x^2 - 1)y = 0$.
3. $x^2y'' + xy' + (9x^2 - 4)y = 0$.
4. $\frac{d}{dx}(xy') + \left(x - \frac{16}{x}\right)y = 0$.

Legendre's Equation and Legendre Polynomials

The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \qquad (1)$$

where n is a non-negative integer, is called Legendre's equation. It arises in numerous physical problems, particularly in boundary-value problems for spheres.

Since $x = 0$ is an ordinary point of equation (1), we assume a solution of (1) in the form

$y = \sum_{k=0}^{\infty} c_k x^k$,

which implies

$y' = \sum_{k=1}^{\infty} kc_k x^{k-1}$, $y'' = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$.

Employing these in the given differential equation yields

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} 2kc_k x^k + \sum_{k=0}^{\infty} n(n+1)c_k x^k = 0.$$

Replacing k with $k+2$ in the first term, we obtain

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} 2kc_k x^k + \sum_{k=0}^{\infty} n(n+1)c_k x^k = 0.$$

racting the value of $k = 0$ from the first and last series, extracting $k = 1$ from the first, third, and last terms, and mming from $k = 2$ yields

$$2!\,c_2 + n(n+1)c_0 + \{3!\,c_3 + [n(n+1) - 2]c_1\}x + \sum_{k=2}^{\infty}\{(k+2)(k+1)c_{k+2} + [n(n+1) - k(k+1)]c_k\}\,x^k = 0.$$

This implies

$$2!\,c_2 + n(n+1)c_0 = 0.$$

$$3!\,c_3 + [n(n+1) - 2]c_1 = 0, \text{ or } 3!\,c_3 + (n-1)(n+2)c_1 = 0,$$

and the recurrence relation is

$$(k+2)(k+1)c_{k+2} + [n(n+1) - k(k+1)]c_k = 0,\ \ k = 2, 3, \ldots. \qquad \text{or}$$

$$(k+2)(k+1)c_{k+2} + (n-k)(n+k+1)c_k = 0,\ \ k = 2, 3, \ldots.$$

We, therefore, get

$$c_2 = -\frac{n(n+1)}{2!}c_0;$$

$$c_3 = -\frac{(n-1)(n+2)}{3!}c_1;$$

$$c_{k+2} = -\frac{(n-k)(n+k+1)}{(k+2)(k+1)}c_k,\ \ k = 2, 3, \ldots. \qquad (2)$$

From equation (2), we have

$$c_4 = -\frac{(n-2)(n+3)}{4.3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0;$$

$$c_5 = -\frac{(n-3)(n+4)}{5.4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1;$$

$$c_6 = -\frac{(n-4)(n+5)}{6.5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0;$$

$$c_7 = -\frac{(n-5)(n+6)}{7.6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1; \ldots.$$

We, therefore, have two linearly independent series solutions

$$y_1 = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]; \qquad (3)$$

$$y_2 = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]. \qquad (4)$$

eries (3) and (4) converge for $|x| < 1$.

When n is an even integer, series (3) terminates at x^n, and series (4) is an infinite series. If n is an odd integer, series 3) is an infinite series, whereas series (4) terminates at x^n. It follows that, since n is a non-negative integer, a solution of Legendre's equation is an nth degree polynomial. A constant multiple of this polynomial solution that takes the value 1 at $x = 1$ is called a Legendre polynomial, denoted by $P_n(x)$.

Since a constant multiple of a solution of Legendre's equation is also a solution, it is customary to choose specific values for c_0 and c_1 according as n is an even and an odd positive integer, respectively. We select

$$c_0 = \begin{cases} 1, & n = 0 \\ (-1)^{\frac{n}{2}}\dfrac{1.3\ldots\ldots(n-1)}{2.4\ldots\ldots n}, & n = 2, 4, 6, \ldots \end{cases}$$

and

$$c_1 = \begin{cases} 1, & n = 1 \\ (-1)^{\frac{n-1}{2}}\dfrac{1.3\ldots\ldots n}{2.4\ldots\ldots(n-1)}, & n = 3, 5, 7, \ldots \end{cases}$$

mploying the above values of c_0 and c_1 in series (3) and (4), respectively, we get the Legendre polynomials

$P_0(x) = 1;\quad P_2(x) = -\frac{1}{2}\left(1 - \frac{2.3}{2!}x^2\right);\quad P_4(x) = \frac{1.3}{2.4}\left(1 - \frac{4.5}{2!}x^2 + \frac{2.4.5.7}{4!}x^4\right);$ ... from series (3) and

$P_1(x) = x;\quad P_3(x) = -\frac{1.3}{2}\left(x - \frac{2.5}{3!}x^3\right);\quad P_5(x) = \frac{1.3.5}{2.4}\left(x - \frac{4.7}{3!}x^3 + \frac{2.4.7.9}{5!}x^5\right);$ ... from series (4).

hus

$P_0(x) = 1;$

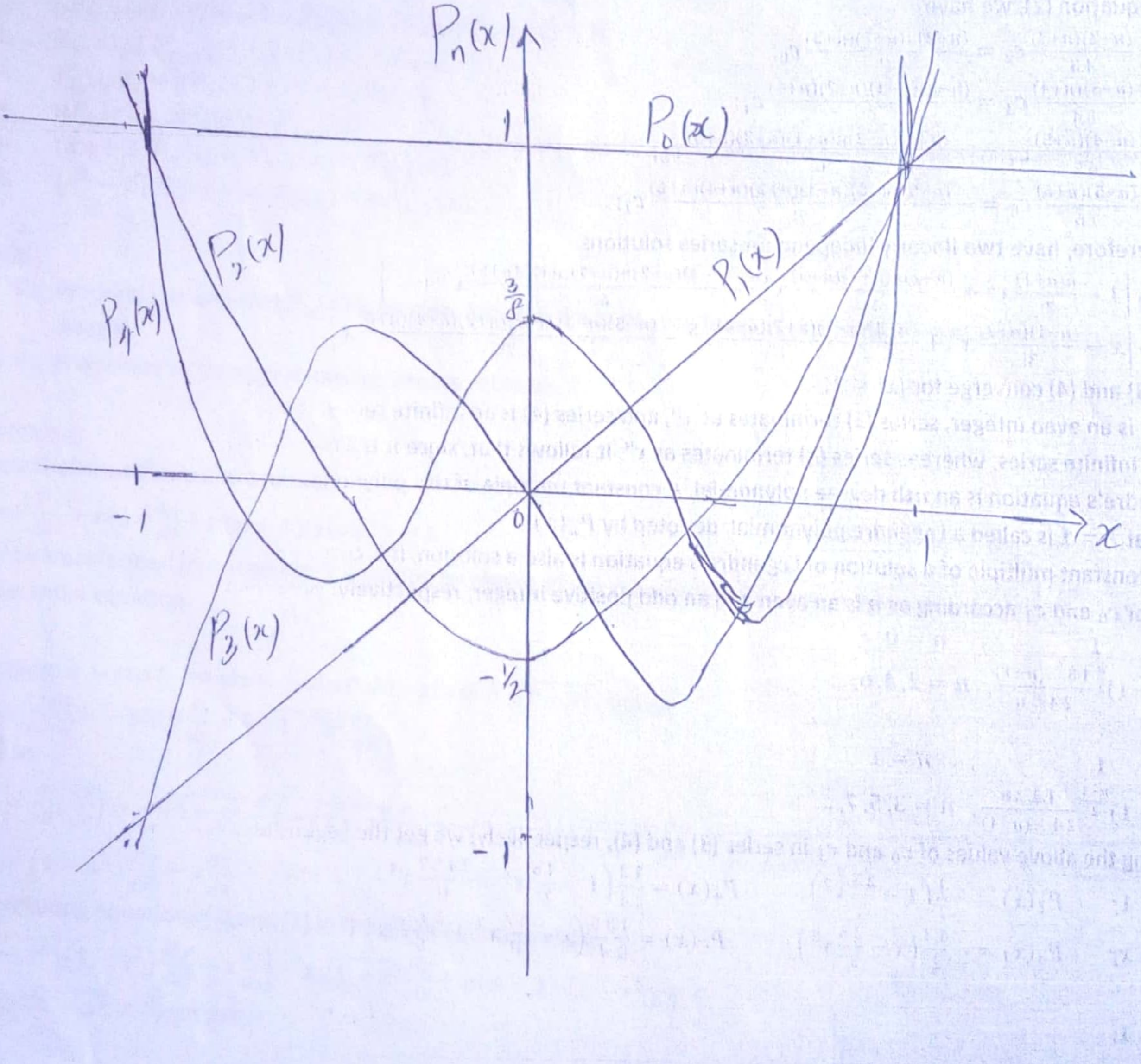
$P_1(x) = x;$

$$) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right).$$

$$x) = \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)$$

$$(x) = \frac{1}{128}\left(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35\right).$$

$$_9(x) = \frac{1}{128}\left(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x\right).$$



$x) = \frac{1}{2}(3x^2 - 1)$; $P_3(x) = \frac{1}{2}(5x^3 - 3x)$;

$(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$; $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$;

are Legendre polynomials.

he Legendre polynomials can also be generated by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

Properties of Legendre Polynomials

ome properties of Legendre polynomials are

i. $P_n(-x) = (-1)^n P_n(x)$.
ii. $P_n(1) = 1$.
iii. $P_n(-1) = (-1)^n$.
iv. $P_n(0) = 0,\ n = 1, 3, 5, \ldots$
v. $P'_n(0) = 0,\ n = 0, 2, 4, \ldots$
vi. $nP_{n-1}(x) - (2n+1)xP_n(x) + (n+1)P_{n+1}(x) = 0$.
vii. $P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$.
viii. $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$.
ix. $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$.
x. $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$.
xi. $(x^2 - 1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$.

Note:

a. Property i. shows that $P_n(x)$ is an even function if n is an even integer, and an odd function if n is an odd integer.

b. Properties vi. through xi. are recurrence relations.

xample 1

how that the differential equation

$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$

an be transformed into Legendre's equation by means of the substitution $x = \cos\theta$. Write down a solution of this ifferential equation.

olution: $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$; $\sin\theta = \sqrt{1-x^2}$, and so

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{dy}{dx} = -\sqrt{1-x^2}\frac{dy}{dx},$$

nd so (1)

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = -\sqrt{1-x^2}\frac{d}{dx}\left(-\sqrt{1-x^2}\frac{dy}{dx}\right) = \sqrt{1-x^2}\left(\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}\right), \quad \text{or}$$

$$\frac{d^2y}{d\theta^2} = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}.$$

ubstituting equations (1) and (2) in the given equation yields (2)

$$\sqrt{1-x^2}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] - x\sqrt{1-x^2}\frac{dy}{dx} + n(n+1)\sqrt{1-x^2}\,y = 0.$$

ince $\sqrt{1-x^2} \neq 0$, we get

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$

ich is Legendre's equation.

solution of this equation is $y = P_n(x) = P_n(\cos\theta)$, where n is a non-negative integer.

Example 2

Compute the Legendre polynomial corresponding to the Legendre's equation

$(1-x^2)y'' - 2xy' + 12y = 0.$ 3.

Solution: Here $n(n+1) = 12$, or $n^2 + n - 12 = 0$, or $(n-3)(n+4) = 0$, or $n = 3, n = -4$.

Since n is a non-negative integer, we have that $n = 3$. Thus, the corresponding Legendre polynomial is

$P_3(x) = \frac{1}{2}(5x^3 - 3x).$

Exercises

Compute the Legendre polynomial corresponding to each Legendre's equation.

1. $(1-x^2)y'' - 2xy' + 2y = 0.$
2. $(1-x^2)y'' - 2xy' + 6y = 0.$
3. $(1-x^2)y'' - 2xy' + 30y = 0.$
4. The Legendre polynomials are also be generated by Rodrigues' formula

   $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$

   Verify the results for $n = 0, 1, 2, 3$.
5. Using the text, compute the Legendre polynomials $P_6(x), P_7(x), P_8(x)$, and $P_9(x)$.

Hypergeometric Equation and Function

A differential equation of the form

$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0,$ (1)

where a, b, c are constants, is called Gauss' hypergeometric equation. Hypergeometric equation (1) is important because many second-order linear differential equations are reducible to it, and also because many important special functions are closely related to its solutions. We assume that c is not an integer.

Since $x = 0$ is a regular singular point of equation (1), we assume a series solution of (1) in the form

$y = \sum_{n=0}^{\infty} d_n x^{n+r},\ d_0 \neq 0.$

This is differentiated twice with respect to x to obtain

$y' = \sum_{n=0}^{\infty}(n+r)d_n x^{n+r-1},\quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)d_n x^{n+r-2}.$

Employing these in the given equation yields

$\sum_{n=0}^{\infty}(n+r)(n+r-1)d_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)d_n x^{n+r} + \sum_{n=0}^{\infty}c(n+r)d_n x^{n+r-1} - \sum_{n=0}^{\infty}(a+b+1)(n+r)d_n x^{n+r} - \sum_{n=0}^{\infty}abd_n x^{n+r} = 0.$

$\Rightarrow \sum_{n=0}^{\infty}(n+r)(c+n+r-1)d_n x^{n+r-1} - \sum_{n=0}^{\infty}[(n+r)(a+b+n+r)+ab]d_n x^{n+r} = 0.$

Extracting x^r and replacing n with $n+1$ in the first summation, we get

$x^r\{\sum_{n=-1}^{\infty}(n+r+1)(c+n+r)d_{n+1} - \sum_{n=0}^{\infty}(n+r+a)(n+r+b)d_n\}x^n = 0.$

Since $x^r \neq 0$, extracting the values of $n = -1$ from the first summation, and summing from $n = 0$, we obtain

$(r+c-1)d_0x^{-1} + \sum_{n=0}^{\infty}\{(n+r+1)(c+n+r)d_{n+1} - (n+r+a)(n+r+b)d_n\}x^n = 0.$ (2)

The indicial equation is $r(r+c-1) = 0$, since $d_0 \neq 0$, with roots $r_1 = 0$, $r_2 = 1-c$.

$$F\left(\frac{k}{2}, \frac{k+1}{2}, \frac{1}{2}; x^2\right) = (1+x)^{-k} + (1-x)^{-k}.$$

$$^{k}F\left(\frac{k}{2}, \frac{k+1}{2}, k+1; x\right) = \left(1+\sqrt{1-x}\right)^{-k}.$$

$\neq$ integer, it follows that the indicial roots do not differ by an integer. We now obtain one solution corresponding to the indicial root $r = 0$.

$r = 0$: From equation (2), the recurrence relation is

$(n+1)(c+n)d_{n+1} - (n+a)(n+b)d_n = 0, \quad n = 0, 1, 2, \ldots,$ or

$d_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)} d_n, \; n = 0, 1, 2, \ldots,$

which gives

$d_1 = \frac{ab}{c} d_0;$

$d_2 = \frac{(a+1)(b+1)}{2(c+1)} d_1 = \frac{a(a+1)b(b+1)}{2!c(c+1)} d_0;$

$d_3 = \frac{(a+2)(b+2)}{3(c+2)} d_2 = \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} d_0, \ldots.$

Generally, we get

$$d_n = \frac{a(a+1)(a+2)\ldots(a+n-1)b(b+1)(b+2)\ldots(b+n-1)}{n!c(c+1)(c+2)\ldots(c+n-1)} d_0, \; n = 1, 2, \ldots. \tag{3}$$

Setting $d_0 = 1$, we have one series solution of equation (1) to be

$$y_1 = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} x^3 + \cdots, \tag{4}$$

which converges for $|x| < 1$. Equation (4) is called the hypergeometric series, and is denoted by $F(a, b, c; x)$. From equation (3), we obtain

$$F(a, b, c; x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\ldots(a+n-1)b(b+1)(b+2)\ldots(b+n-1)}{n!c(c+1)(c+2)\ldots(c+n-1)} x^n. \tag{5}$$

$F(a, b, c; x)$ is called the hypergeometric function. $F(a, b, c; x) = F(b, a, c; x)$, which shows that $F(a, b, c; x)$ is symmetric with respect to a and b. Many functions can be obtained from the hypergeometric function for different values of the constants a, b, c. For example,

i. $F(a, b, c; 0) = 1.$
ii. $\lim\limits_{a \to \infty} F\left(a, b, b; \frac{x}{a}\right) = e^x.$
iii. $F(-a, b, b; -x) = (1+x)^a.$
iv. $F(k+1, -k, 1; x)$ is a polynomial of degree k, where k is a non-negative integer.
v. $xF(1, 1, 2; -x) = \ln(1+x).$
vi. $xF\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) = \tan^{-1} x.$
vii. $xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2\right) = \sin^{-1} x.$
viii. $2xF\left(\frac{1}{2}, 1, \frac{3}{2}; x^2\right) = \ln\left(\frac{1+x}{1-x}\right).$

A special case of equation iii. is

$F(1, b, b; x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots.$

Two properties of the hypergeometric function are

I. $\frac{d}{dx}[F(a, b, c; x)] = \frac{ab}{c} F(a+1, b+1, c+1; x).$
II. $\frac{d^2}{dx^2}[F(a, b, c; x)] = \frac{a(a+1)b(b+1)}{c(c+1)} F(a+2, b+2, c+2; x).$

Note: The second series solution of the hypergeometric equation (1) corresponding to the indicial root $r = 1 - c$ may be found to be

$$y_2 = x^{1-c}\left[1 + \frac{(a-c+1)(b-c+1)}{1!(-c+2)} x + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!(-c+2)(-c+3)} x^2 + \cdots\right], \tag{6}$$

coefficients of the Fourier-Legendre series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

of $f(x)$, defined on the interval $-1 < x < 1$, are given by

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx, \quad n = 0, 1, 2, \cdots$$

If $f(x) = P_n(x)$, then $c_n = 1 \ \forall\, n$, and so

$$\frac{2n+1}{2} \int_{-1}^{1} P_n^2(x)\, dx = 1,$$

or

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \cdots$$

e $c \neq 2, 3, 4, \ldots$, or

$= x^{1-c}F(a-c+1, b-c+1, -c+2; x)$. (7)

us, the general solution of the hypergeometric equation is

$= A_1F(a,b,c;x) + A_2x^{1-c}F(a-c+1, b-c+1, -c+2; x)$,

rovided $c \neq$ integer, where A_1 and A_2 are arbitrary constants and

$F(a-c+1, b-c+1, -c+2; x) = 1 + \sum_{n=1}^{\infty} \frac{(a-c+1)(a-c+2)\ldots(a-c+n)(b-c+1)(b-c+2)\ldots(b-c+n)}{n!(-c+2)(-c+3)\ldots(-c+n+1)} x^n$.

xample 1

ompute the hypergeometric series corresponding to the hypergeometric equation

$x(1-x)y'' + \left(\frac{3}{4} - 4x\right)y' - 2y = 0$.

Vrite down the general solution.

olution: Here $c = \frac{3}{4},\ a+b+1 = 4,\ ab = 2$. Solving $a+b = 3,\ ab = 2$, we get $a = 1,\ b = 2$, or $a = 2,\ b = 1$.

Ising $a = 1,\ b = 2, c = \frac{3}{4}$, we get the solution

$F\left(1, 2, \frac{3}{4}; x\right) = 1 + \sum_{n=1}^{\infty} \frac{1.2.3\ldots n.2.3.4\ldots(n+1)}{n!\frac{3}{4}\frac{7}{4}\frac{11}{4}\ldots(n-\frac{1}{4})} x^n = 1 + \sum_{n=1}^{\infty} \frac{n!(n+1)!4^n}{n!3.7.11\ldots(4n-1)} x^n = 1 + \sum_{n=1}^{\infty} \frac{(n+1)!4^n}{3.7.11\ldots(4n-1)} x^n$.

he general solution of the hypergeometric equation is given by

$y = A_1F(a,b,c;x) + A_2x^{1-c}F(a-c+1, b-c+1, -c+2; x)$,

vith $a = 1,\ b = 2, c = \frac{3}{4}$, or

$y = A_1F\left(1, 2, \frac{3}{4}; x\right) + A_2x^{\frac{1}{4}}F\left(\frac{5}{4}, \frac{9}{4}, \frac{5}{4}; x\right)$,

vhere A_1 and A_2 are arbitrary constants.

xample 2

how that any differential equation of the form

$x-A)(x-B)y'' + (Cx+D)y' + Ey = 0$,

vhere A, B, C, D, and E are constants and $A \neq B$ can be transformed into Gauss' hypergeometric equation

$(1-t)\ddot{y} + [c-(a+b+1)t]\dot{y} - aby = 0$

y means of the transformation $x = A + (B-A)t$, where $\dot{y} = \frac{dy}{dt}$.

olution: $x = A + (B-A)t \Rightarrow dx = (B-A)dt$, and so

$y' = \frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{B-A}\frac{dy}{dt} = \frac{1}{B-A}\dot{y}$;

$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{B-A}\frac{d}{dt}\left(\frac{1}{B-A}\frac{dy}{dt}\right) = \frac{1}{(B-A)^2}\frac{d^2y}{dt^2} = \frac{1}{(B-A)^2}\ddot{y}$.

$-A = (B-A)t,\ x - B = (A-B)(1-t) = -(B-A)(1-t)$, and so

$x-A)(x-B) = -(B-A)^2t(1-t)$. Also

$x + D = AC + D + C(B-A)t$.

ubstituting these into the given equation yields

$-(B-A)^2t(1-t).\frac{1}{(B-A)^2}\ddot{y} + [AC + D + C(B-A)t].\frac{1}{B-A}\dot{y} + Ey = 0$, or

$(1-t)\ddot{y} + \left[\frac{AC+D}{A-B} - Ct\right]\dot{y} - Ey = 0$,

hich is the hypergeometric equation, provided

$= A_1 F(1, 1, \frac{1}{2}; x) + A_2 x^{\frac{1}{2}} F(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; x).$

$y = A_1 F(1, \frac{1}{2}, -\frac{1}{2}; x) + A_2 x^{\frac{3}{2}} F(\frac{5}{2}, 2, \frac{5}{2}; x)$

$y = A_1 F(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x) + A_2 \sqrt{x}\, F(1, 1, \frac{3}{2}; x).$

$y = A_1 F(1, -\frac{1}{3}, \frac{1}{3}; x) + A_2 x^{\frac{2}{3}} F(\frac{5}{3}, \frac{1}{3}, \frac{5}{3}; x).$

$t(1-t)\ddot{y} + (\frac{1}{2} - t)\dot{y} - 2y = 0$, where $\dot{y} = \frac{dy}{dt}$

$t(1-t)\ddot{y} + 2\dot{y} + 6y = 0.$

$x(1-x)y'' + \left[\frac{Ct_1 + D}{t_1 - t_2} - Cx\right]y' - Ey$, where $y' = \frac{dy}{dx}$.

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$\frac{C+D}{A-B}$, $a+b+1=C,\ ab=E.$

ercises

or the following hypergeometric equations, compute the corresponding hypergeometric series, and find the general olution of each equation.

1. $x(1-x)y''+\left(\frac{1}{2}-3x\right)y'-y=0.$
2. $2x(1-x)y''-(1+5x)y'-y=0.$
3. $4x(x-1)y''+2(4x-1)y'+y=0.$
4. $3x(1-x)y''+(1-5x)y'+y=0.$

educe each given differential equation to hypergeometric equation by means of the transformation given in xample 2. above and obtain the corresponding hypergeometric series and its general solution in the new variable.

5. $(x-1)(x+2)y''+\left(x+\frac{1}{2}\right)y'+2y=0.$
6. $\left(x^2-\frac{1}{4}\right)y''+2y'-6y=0.$
7. Consider the differential equation

   $(t^2+At+B)\ddot{y}+(Ct+D)\dot{y}+Ey=0,$

   where A, B, C, D, E are constants, $\dot{y}=\frac{dy}{dt}$, and t^2+At+B has distinct zeros t_1 and t_2. Show that by introducing the new independent variable

   $x=\frac{t-t_1}{t_2-t_1},$

   the given equation becomes the hypergeometric equation, where the parameters are related by

   $Ct_1+D=-c(t_2-t_1),\ C=a+b+1,\ E=ab.$

Sturm-Liouville Problems

here is an important class of boundary-value problems (BVPs) in mathematical physics in which the differential quation is of the form

$p(x)y']'+[q(x)+\lambda r(x)]y=0,$ (1)

vhere $p(x)>0,\ q(x)$, and $r(x)>0$ are real and continuous functions on some interval $a\le x\le b$, and λ is a real arameter, satisfying boundary conditions of the form

$_1y(a)+\alpha_2y'(a)=0,\ \ \beta_1y(b)+\beta_2y'(b)=0.$ (2)

lere α_1 and α_2 are given real constants not both zero; so are β_1 and β_2.

quation (1) is called a Sturm-Liouville equation, and BVP (1)-(2) is called a Sturm-Liouville problem.

hese problems occur in various engineering applications, for instance, in connection with vibrations of strings and nembranes, and in heat conduction. Bessel's and Legendre's equations are examples of Sturm-Liouville equation, as ollows:

i. Bessel's equation

   $$x^2y''+xy'+(x^2-\nu^2)y=0\ \Rightarrow xy''+y'+\left(x-\frac{\nu^2}{x}\right)y=0\ \Rightarrow (xy')'+\left(x-\frac{\nu^2}{x}\right)y=0.$$

   Here $p(x)=x,\ q(x)=x,\ r(x)=\frac{1}{x},\ \lambda=-\nu^2.$

ii. Legendre's equation

   $$(1-x^2)y''-2xy'+n(n+1)y=0\ \Rightarrow [(1-x^2)y']'+n(n+1)y=0.$$

Here $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, $\lambda = n(n+1)$, with $-1 < x < 1$.

BVP (1)-(2) is always satisfied by the trivial solution $y(x) = 0$ for all x in the given interval. However, the sirable issue is to determine the values of λ for which the BVP (1)-(2) has a non-trivial solution. These values of λ re called the eigenvalues of the problem, and the corresponding non-trivial solutions $y(x)$ are called the igenfunctions of the problem.

xamples

ind the eigenvalues and eigenfunctions of the Sturm-Liouville problems.

1. $y'' + \lambda y = 0$, for $0 \le x \le L$, $y(0) = y'(L) = 0$.
2. $y'' + (-4 + \lambda)y = 0$, for $0 \le x \le 1$, $y'(0) = y'(1) = 0$.
3. $(x^{-1}y')' + (\lambda + 1)x^{-3}y = 0$, for $1 \le x \le e$, $y(1) = y(e) = 0$.

olutions:

1. The auxiliary equation of the differential equation $y'' + \lambda y = 0$ is $m^2 + \lambda = 0$, or $m = \pm\sqrt{-\lambda}$. We consider the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

   Case I: $\lambda < 0$

   Let $\lambda = -k^2$, where $k > 0$. The differential equation becomes $y'' - k^2 y = 0$, with the general solution
   $y(x) = c_1 e^{kx} + c_2 e^{-kx}$, and so
   $y'(x) = c_1 k e^{kx} - c_2 k e^{-kx}$,
   where c_1 and c_2 are arbitrary constants.
   $\Rightarrow y(0) = c_1 + c_2 = 0. \Rightarrow c_2 = -c_1$;
   $y'(L) = k(c_1 e^{kL} - c_2 e^{-kL}) = 0. \Rightarrow c_1 k(e^{kL} + e^{-kL}) = 0.$
   Since $k(e^{kL} + e^{-kL}) \neq 0$, it follows that $c_1 = 0$, and so $c_2 = 0$.
   Hence, we obtain the trivial solution $y(x) = 0$.

   Case II: $\lambda = 0$

   The differential equation becomes $y'' = 0$, whose general solution is
   $y(x) = c_3 x + c_4$, and so
   $y'(x) = c_3$,
   where c_3 and c_4 are arbitrary constants.
   $\Rightarrow y(0) = c_4 = 0$;
   $y'(L) = c_3 = 0$.
   We, therefore, get the trivial solution $y(x) = 0$.

   Case III: $\lambda > 0$

   Let $\lambda = k^2$, where $k > 0$. The differential equation becomes $y'' + k^2 y = 0$, with the general solution
   $y(x) = c_5 \cos kx + c_6 \sin kx$, and so
   $y'(x) = -c_5 k \sin kx + c_6 k \cos kx$,
   where c_5 and c_6 are arbitrary constants.
   $\Rightarrow y(0) = c_5 = 0$;
   $y'(L) = c_6 k \cos kL = 0$.
   For non-trivial solution, it follows that $c_6 \neq 0$. Since $k > 0$, we have that $\cos kL = 0$, and so $kL = (2n+1)\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$. Therefore, the eigenvalues of the problem are
   $\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2$, $n = 0, 1, 2, \ldots$,

and the corresponding eigenfunctions are
$y_n(x) = \sin\frac{(2n+1)\pi}{2L}x, \; n = 0, 1, 2, \ldots,$
with $c_5 = 0, \; c_6 = 1.$

2. The auxiliary equation of the differential equation $y'' + (-4 + \lambda)y = 0$ is $m^2 - 4 + \lambda = 0$, or $m = \pm\sqrt{4 - \lambda}$.
We consider the cases $4 - \lambda < 0, \; 4 - \lambda = 0, \; 4 - \lambda > 0.$

Case I: $4 - \lambda < 0$

Let $4 - \lambda = -k^2$, where $k > 0$. The differential equation becomes $y'' + k^2y = 0$, with the general solution
$y(x) = c_1 \cos kx + c_2 \sin kx.$ (1)
$\Rightarrow \; y'(x) = -c_1 k \sin kx + c_2 k \cos kx,$
where c_1 and c_2 are arbitrary constants.
$\Rightarrow \; y'(0) = c_2 k = 0. \;\; \Rightarrow \;\; c_2 = 0$ since $k > 0$;
$y'(1) = -c_1 k \sin k = 0.$
Since $c_1 \neq 0$ and $k > 0$, we have that $\sin k = 0. \Rightarrow k = n\pi, \; n = 1, 2, \ldots.$
Hence, $\lambda = 4 + k^2$ gives the eigenvalues
$\lambda_n = 4 + n^2\pi^2, \; n = 1, 2, \ldots,$ (2)
and the corresponding eigenfunctions, from equation (1), are
$y_n(x) = \cos n\pi x, \; n = 1, 2, \ldots,$ (3)
with $c_1 = 1, \; c_2 = 0.$

Case II: $4 - \lambda = 0$

The differential equation becomes $y'' = 0$, whose general solution is
$y(x) = c_3 x + c_4.$
$\Rightarrow \; y'(x) = c_3,$ (4)
where c_3 and c_4 are arbitrary constants.
$\Rightarrow \; y'(0) = y'(1) = c_3 = 0.$
Thus, the eigenvalue is
$\lambda = 4,$ (5)
and the corresponding eigenfunction is obtained from equation (4) with $c_3 = 0, \; c_4 = 1$ as
$y(x) = 1.$ (6)

Case III: $4 - \lambda > 0$

Let $4 - \lambda = k^2$, where $k > 0$. The differential equation becomes $y'' - k^2y = 0$, with the general solution
$y(x) = c_5 e^{kx} + c_6 e^{-kx}$, and so
$y'(x) = c_5 k e^{kx} - c_6 k e^{-kx},$
where c_5 and c_6 are arbitrary constants.
$\Rightarrow \; y'(0) = k(c_5 - c_6) = 0. \;\; \Rightarrow c_5 = c_6$, since $k > 0$;
$y'(1) = c_5 k(e^k - e^{-k}) = 0.$
Since $e^k \neq e^{-k}$, we get $c_5 = 0$ and so $c_6 = 0$. Therefore, we obtain a trivial solution $y(x) = 0$.
It will be observed that if we set $n = 0$ in equations (2) and (3), we obtain equations (5) and (6), respectively.
Hence, the eigenvalues and eigenfunctions, respectively, for the given Sturm-Liouville problem are
$\lambda_n = 4 + n^2\pi^2, \; n = 0, 1, 2, \ldots,$
and the corresponding eigenfunctions are
$y_n(x) = \cos n\pi x, \; n = 0, 1, 2, \ldots.$

The equation $(x^{-1}y')' + (\lambda+1)x^{-3}y = 0$ may be written as
$x^{-1}y'' - x^{-2}y' + (\lambda+1)x^{-3}y = 0$, or by multiplying through by x^3, we get
$$x^2y'' - xy' + (\lambda+1)y = 0, \tag{1}$$
which is a Cauchy-Euler equation. The form of the solution of equation (1) is $y = x^m$, and so the auxiliary equation of equation (1) is $m(m-1) - m + \lambda + 1 = 0$, or $m^2 - 2m + \lambda + 1 = 0$, with roots $m = 1 \pm \sqrt{-\lambda}$.
We consider the cases $\lambda < 0,\ \lambda = 0,\ \lambda > 0$.

<u>Case I: $\lambda < 0$</u>

Let $\lambda = -k^2$, where $k > 0$. The differential equation becomes $x^2y'' - xy' + (1-k^2)y = 0$, with the general solution
$y(x) = c_1x^{1+k} + c_2x^{1-k}$,
where c_1 and c_2 are arbitrary constants.
$\Rightarrow\ y(1) = c_1 + c_2 = 0. \ \Rightarrow\ c_2 = -c_1;$
$y(e) = c_1e^{1+k} + c_2e^{1-k} = 0 \ \Rightarrow\ c_1k(e^{1+k} - e^{1-k}) = 0.$
Since $e^{1+k} - e^{1-k} \neq 0$, it follows that $c_1 = 0$, and so $c_2 = 0$.
Hence, we obtain the trivial solution $y(x) = 0$.

<u>Case II: $\lambda = 0$</u>

The differential equation becomes $x^2y'' - xy' + y = 0$, whose general solution is
$y(x) = c_3x + c_4x\ln x$,
where c_3 and c_4 are arbitrary constants.
$\Rightarrow\ y(1) = c_3 = 0;$
$y(e) = c_4e\ln e = c_4e = 0. \ \Rightarrow\ c_4 = 0$, since $e \neq 0$.
We, therefore, get the trivial solution $y(x) = 0$.

<u>Case III: $\lambda > 0$</u>

Let $\lambda = k^2$, where $k > 0$. The differential equation becomes $x^2y'' - xy' + (k^2+1)y = 0$, with the general solution
$y(x) = x[c_5\cos(k\ln x) + c_6\sin(k\ln x)]$,
where c_5 and c_6 are arbitrary constants.
$\Rightarrow\ y(1) = c_5 = 0;$
$y(e) = c_6e\sin k = 0.$
For non-trivial solution, it follows that $c_6 \neq 0$. Since $e \neq 0$, we have that $\sin k = 0$, and so $k = n\pi,\ n = 1, 2, \ldots$, and so the eigenvalues of the problem are
$\lambda_n = n^2\pi^2,\ \ n = 1, 2, \ldots,$
and the corresponding eigenfunctions are
$y_n(x) = x\sin(n\pi\ln x),\ \ n = 1, 2, \ldots,$
with $c_5 = 0,\ c_6 = 1,\ k = n\pi$.

<u>lote 1</u>: The form of the solution of the Cauchy-Euler equation
$$\iota_4x^4y^{(4)} + a_3x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0, \tag{1}$$
vhere $a_i,\ i = 1, 2, 3, 4$, are constants, is $y = x^m$, and so the auxiliary equation of equation (1) is
$\iota_4m(m-1)(m-2)(m-3) + a_3m(m-1)(m-2) + a_2m(m-1) + a_1m + a_0 = 0.$
f this quartic equation has <u>real and distinct</u> roots, $m_i,\ i = 1, 2, 3, 4$, then the general solution of equation (1) is
$' = c_1x^{m_1} + c_2x^{m_2} + c_3x^{m_3} + c_4x^{m_4}.$

$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \cdots$

$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + \cdots$

$f(x) = \frac{2}{5}P_1(x) - \frac{2}{5}P_3(x)$.

$f(x) = P_1(x) + 2P_2(x)$.

$f(x) = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$.

6. Show that the sketch is given by ~~f(x)~~

$$f(x) = \begin{cases} 1+x, & -1 < x < 0 \\ 1-x, & 0 \le x < 1 \end{cases},$$

and compute its Fourier-Legendre series.

. Show that the sketch is given by

$$f(x) = \begin{cases} 6(x+1), & -1 < x < -\frac{1}{2} \\ -6x, & -\frac{1}{2} \le x < \frac{1}{2} \\ -3, & \frac{1}{2} \le x < 1 \end{cases}.$$

Thereafter, obtain its Fourier-Legendre series.

he roots of the quartic equation are real and repeated, i.e. m_1 (4 times), then the general solution of equation

$= c_1x^{m_1} + c_2x^{m_1}\ln x + c_3x^{m_1}(\ln x)^2 + c_4x^{m_1}(\ln x)^3$.

he roots of the quartic equation are repeated complex conjugates, i.e. $m_1 = \alpha + i\beta$ (twice) and $m_2 = \alpha - i\beta$ (2 mes), then the general solution of equation (1) is

$= x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x) + c_3(\ln x)\cos(\beta\ln x) + c_4(\ln x)\sin(\beta\ln x)]$,

here c_i, $i = 1,2,3,4$, are arbitrary constants.

lote 2: Another method of solving the Cauchy-Euler equation is to let $x = e^t$, or $t = \ln x$, and so

$y' = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{e^t}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt}. \Rightarrow x\frac{dy}{dx} = \frac{dy}{dt}$, or $xy' = \dot{y}$.

$y'' = \frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{e^t}\frac{d}{dt}\left(\frac{1}{e^t}\frac{dy}{dt}\right) = \frac{1}{x}\left(\frac{1}{e^t}\frac{d^2y}{dt^2} - \frac{1}{e^t}\frac{dy}{dt}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right). \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$, or $x^2y'' = \ddot{y} - \dot{y}$.

ontinuing in this manner, we will obtain $x^3y''' = \dddot{y} - 3\ddot{y} + 2\dot{y}$.

rom the foregoing, we see that employing the substitution $x = e^t$ is tedious and impracticable when we have higher rder differential equation.

xercises

ind the eigenvalues and eigenfunctions of the Sturm-Liouville problems.

1. $y'' + \lambda y = 0$, for $0 \le x \le \pi$, $\quad y(0) = y(\pi) = 0$.
2. $y'' + \lambda y = 0$, for $-\frac{1}{2} \le x \le \frac{1}{2}$, $\quad y\left(-\frac{1}{2}\right) = y\left(\frac{1}{2}\right) = 0$.
3. $y'' + \lambda y = 0$, for $0 \le x \le 2\pi$, $\quad y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$.
4. $x^2y'' + 2xy' + \lambda y = 0$, for $1 \le x \le e^2$, $\quad y(1) = y(e^2) = 0$.
5. $y'' + (2+\lambda)y = 0$, for $0 \le x \le 1$, $\quad y(0) = y(1) = 0$.
6. $(xy')' + \lambda x^{-1}y = 0$, for $1 \le x \le e$, $\quad y(1) = y'(e) = 0$.
7. $(xy')' + \lambda x^{-1}y = 0$, for $e^{\frac{1}{2}} \le x \le e^{\frac{3}{2}}$, $\quad y\left(e^{\frac{1}{2}}\right) = y\left(e^{\frac{3}{2}}\right) = 0$.
8. $\left(e^{2x}y'\right)' + (\lambda+1)e^{2x}y = 0$, for $0 \le x \le \pi$, $\quad y(0) = y(\pi) = 0$.
9. Show that the eigenvalues of the Sturm-Liouville problem $y'' + \lambda y = 0$, $\quad y(0) = 0$, $y(1) + y'(1) = 0$ are obtained as solutions of the equation $\tan k = -k$, where $k = \sqrt{\lambda}$. Show graphically that this equation has infinitely many solutions $k = k_n$ and the eigenfunctions are $y_n = \sin k_n x \quad (k_n \ne 0)$.

Orthogonal Functions

efinition 1: Let $f(x)$ and $g(x)$ be two real-valued functions defined on an interval $a \le x \le b$. The inner product of he functions $f(x)$ and $g(x)$, denoted by (f,g), is defined by

$$(f,g) = \int_a^b f(x)g(x)\,dx, \tag{1}$$

rovided the integral exists.

he functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $a \le x \le b$ if $(f,g) = 0$. That is

$$(f,g) = \int_a^b f(x)g(x)\,dx = 0. \tag{2}$$

efinition 2: A set of real-valued functions $\{\phi_n(x)\}$, $n = 0,1,2,\ldots$, is said to be orthogonal on an interval $a \le x \le b$

$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \; m \neq n.$ (3)

non-negative square root of (ϕ_n, ϕ_n) is called the norm of $\phi_n(x)$, and is denoted by $\|\phi_n(x)\|$. Hence

$\|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)\, dx}.$ (4)

An orthogonal set $\{\phi_n(x)\}, \; n = 0, 1, 2, \ldots$, on an interval $a \leq x \leq b$ whose functions have norm 1 (i.e. $\|\phi_n(x)\| = 1$) satisfies the relation

$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$ (5)

where $m, n = 0, 1, 2, \ldots$. Such a set is called an orthonormal set of functions on the interval $a \leq x \leq b$.

Equation (5) may be written as $(\phi_m, \phi_n) = \delta_{mn}$, where

$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$ (6)

is called Kronecker delta, with $m, n = 0, 1, 2, \ldots$.

In light of the foregoing, it is obvious that an orthonormal set can be obtained from an orthogonal set by dividing each function by its norm.

<u>Definition 3</u>: A set of functions $\{\phi_n(x)\}, \; n = 0, 1, 2, \ldots$, is said to be an orthogonal set with respect to a weight function $w(x)$ on an interval $a \leq x \leq b$ if

$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0, \; m \neq n.$ (7)

In this case, the norm of $\phi_n(x)$ is defined as

$\|\phi_n(x)\| = \sqrt{\int_a^b w(x)\phi_n^2(x)\, dx},$ (8)

and if the norm $\|\phi_n(x)\| = 1$, the set is said to be orthonormal on that interval with respect to the weight function $w(x)$.

Series of Orthogonal Functions (Generalized Fourier Series)

Let $\{\phi_n(x)\}, \; n = 0, 1, 2, \ldots$, be an orthogonal set of functions on an interval $a \leq x \leq b$, and let $f(x)$ be a given function that can be represented in terms of the functions $\phi_n(x), \; n = 0, 1, 2, \ldots$ by a convergent series

$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x).$ (9)

This series is called a generalized Fourier series of $f(x)$, and the coefficients $c_n, \; n = 0, 1, 2, \ldots$ are called the Fourier constants of $f(x)$ with respect to that orthogonal set of functions.

Owing to orthogonality, the Fourier constants can be determined by multiplying equation (9) by $\phi_n(x)$ and integrating over the interval $a \leq x \leq b$ to obtain

$\int_a^b f(x)\phi_m(x)\, dx = c_0 \int_a^b \phi_0(x)\phi_m(x)\, dx + c_1 \int_a^b \phi_1(x)\phi_m(x)\, dx + \cdots + c_n \int_a^b \phi_n(x)\phi_m(x)\, dx + \cdots =$

$c_n \int_a^b \phi_n^2(x)\, dx = c_n \|\phi_n(x)\|^2, n = 0, 1, 2, \ldots,$

which is obtained when $m = n$.

Thus, the Fourier constants are given by

$c_n = \frac{1}{\|\phi_n(x)\|^2} \int_a^b f(x)\phi_n(x)\, dx = \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}, n = 0, 1, 2, \ldots.$ (10)

Using equation (10) in equation (9) gives

$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x),$ (11)

as a generalized Fourier series of the function $f(x)$.

$A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$s A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

er, if the set $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to a weight function $w(x)$ on an interval $a \le$, then multiplying equation (9) by $w(x)\phi_m(x)$ and integrating over the interval $a \le x \le b$ yields

$$\frac{1}{\|\phi_n(x)\|^2}\int_a^b w(x)f(x)\phi_n(x)\,dx, \tag{12}$$

ere $\|\phi_n(x)\|^2 = \int_a^b w(x)\phi_n^2(x)\,dx$.

this case, the generalized Fourier series of $f(x)$ is still given by equation (9), but the Fourier constants are given by quation (12).

xamples

how that the given functions are orthogonal on the indicated interval.

1. $f(x) = e^x,\ g(x) = xe^{-x} - e^{-x},\ 0 \le x \le 2.$
2. $f(x) = x,\ g(x) = \cos 2x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$

olutions:

1. $\int_0^2 f(x)g(x)\,dx = \int_0^2 e^x(xe^{-x} - e^{-x})\,dx = \int_0^2 (x-1)\,dx = \left[\frac{1}{2}x^2 - x\right]_0^2 = 0.$

   Thus, the given functions are orthogonal on the interval $0 \le x \le 2$.

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)g(x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos 2x\,dx = \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$

   Hence, the given functions are orthogonal on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

xamples

how that the given set of functions is orthogonal on the given interval and determine the corresponding rthonormal set.

3. $\sin x, \sin 2x, \sin 3x, \ldots, 0 \le x \le 2\pi.$
4. $\left\{1, \cos\frac{n\pi x}{L}\right\}, n = 1, 2, \ldots, 0 \le x \le L.$
5. $\{1, \cos mx, \sin nx\}, m, n = 1, 2, \ldots, -\pi \le x \le \pi.$

olutions:

3. For $\sin x, \sin 2x, \sin 3x, \ldots, 0 \le x \le 2\pi$, we let $\phi_n(x) = \sin nx$, $n = 1, 2, \ldots$, and so

   $(\phi_m, \phi_n) = \int_0^{2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_0^{2\pi}[\cos(m-n)x - \cos(m+n)x]\,dx = \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right]_0^{2\pi} = 0,\ m \ne n$ for $m, n = 1, 2, \ldots$.

   Thus, the given set of functions is orthogonal on the given interval. The norm of the function $\phi_n(x)$ is

   $\|\phi_n(x)\| = \sqrt{\int_0^{2\pi}\sin^2 nx\,dx} = \sqrt{\frac{1}{2}\int_0^{2\pi}(1 - \cos 2nx)dx} = \sqrt{\frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^{2\pi}} = \sqrt{\pi}, n = 1, 2, \ldots.$

   Hence, the corresponding orthonormal set on the interval $0 \le x \le 2\pi$ is

   $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots.$

4. For $\left\{1, \cos\frac{n\pi x}{L}\right\}$, $n = 1, 2, \ldots$, $0 \le x \le L$, we let $\phi_0(x) = 1$, $\phi_n(x) = \cos\frac{n\pi x}{L}$, $n = 1, 2, \ldots$, and so

   $(\phi_0, \phi_n) = \int_0^L \cos\frac{n\pi x}{L}\,dx = \frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L = 0, n \ne 0.$

   The square norm of $\phi_0(x)$ is

$\|\phi_0(x)\|^2 = \int_0^L dx = L. \Rightarrow \|\phi_0(x)\| = \sqrt{L}.$

$(\phi_m, \phi_n) = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left[\cos\frac{(m+n)\pi x}{L} + \cos\frac{(m-n)\pi x}{L}\right] dx = \frac{L}{2}\left[\frac{1}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L} + \frac{1}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\right]_0^L = 0, m \neq n$ for $m, n = 1, 2, \ldots.$

Thus, the given set of functions is orthogonal on the given interval. The square norm of the function $\phi_n(x)$ is

$\|\phi_n(x)\|^2 = \int_0^L \cos^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left(1+\cos\frac{2n\pi x}{L}\right) dx = \frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_0^L = \frac{L}{2}. \Rightarrow \|\phi_n(x)\| = \sqrt{\frac{L}{2}},\ n = 1, 2, \ldots.$

Hence, the corresponding orthonormal set on the interval $0 \le x \le L$ is

$\left\{\frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right\},\ n = 1, 2, \ldots.$

5. For $\{1, \cos mx, \sin nx\}$, $m, n = 1, 2, \ldots,\ -\pi \le x \le \pi$, we let $\phi_0(x) = 1$, $\phi_m(x) = \cos mx$, $\psi_n(x) = \sin nx$, $m, n = 1, 2, \ldots$, and so

   $(\phi_0, \phi_m) = \int_{-\pi}^{\pi} \cos mx\,dx = \frac{1}{m}\sin mx\Big|_{-\pi}^{\pi} = 0,\ m \neq 0.$

   $(\phi_0, \psi_n) = \int_{-\pi}^{\pi} \sin nx\,dx = -\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} = 0,\ n \neq 0.$

   The square norm of $\phi_0(x)$ is

   $\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi. \Rightarrow \|\phi_0(x)\| = \sqrt{2\pi}.$

   $(\phi_m, \psi_n) = \int_{-\pi}^{\pi} \cos mx \sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x - \sin(m-n)x]\,dx = \frac{1}{2}\left[-\frac{1}{(m+n)}\cos(m+n)x + \frac{1}{(m-n)}\cos(m-n)x\right]_{-\pi}^{\pi} = 0,\ m \neq n$ for $m, n = 1, 2, \ldots.$

   Thus, the given set of functions is orthogonal on the given interval. The square norm of the function $\phi_m(x)$ is

   $\|\phi_m(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2mx)\,dx = \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_{-\pi}^{\pi} = \pi. \Rightarrow \|\phi_m(x)\| = \sqrt{\pi},$ $m = 1, 2, \ldots.$

   $\|\psi_n(x)\|^2 = \int_{-\pi}^{\pi} \sin^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2nx)\,dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_{-\pi}^{\pi} = \pi. \Rightarrow \|\psi_n(x)\| = \sqrt{\pi},$ $n = 1, 2, \ldots.$

   Hence, the corresponding orthonormal set on the interval $-\pi \le x \le \pi$ is

   $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\sin nx\right\},\ m, n = 1, 2, \ldots.$

Example 6

Verify that the given functions are orthogonal on the given interval with respect to the weight function $w(x)$. Find the norm of each function.

$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ -1 \le x \le 1;\quad w(x) = (1-x^2)^{-\frac{1}{2}}.$

Solution:

$\int_{-1}^{1} w(x)T_0(x)T_1(x)\,dx = \int_{-1}^{1}\frac{x}{\sqrt{1-x^2}}\,dx.$

Let $x = \sin\theta. \Rightarrow dx = \cos\theta\,d\theta.$

$x = -1 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2};\ x = 1 \Rightarrow \theta = \frac{\pi}{2}$, and so

$\frac{x}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta}{\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, d\theta = -[\cos\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$

$\int_{-1}^{1} w(x)T_0(x)T_1(x)\, dx = 0.$

$\int_{-1} w(x)T_0(x)T_2(x)\, dx = \int_{-1}^{1} \frac{2x^2-1}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin^2\theta - 1)\, d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\, d\theta = -\frac{1}{2}[\sin 2\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$

$\Rightarrow \int_{-1}^{1} w(x)T_0(x)T_2(x)\, dx = 0.$

$\int_{-1}^{1} w(x)T_1(x)T_2(x)\, dx = \int_{-1}^{1} \frac{x(2x^2-1)}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin^3\theta - \sin\theta)\, d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos 2\theta\, d\theta =$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta - \sin 3\theta)\, d\theta = \frac{1}{2}\left[\frac{1}{3}\cos 3\theta - \cos\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$

$\Rightarrow \int_{-1}^{1} w(x)T_1(x)T_2(x)\, dx = 0.$

hus, the functions $T_0(x), T_1(x), T_2(x)$ are orthogonal on the given interval with respect to the given weight function $v(x)$.

he norms of these functions are obtained as follows:

$T_0(x)\|^2 = \int_{-1}^{1} w(x)T_0^2(x)\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi,$ and so

$T_0(x)\| = \sqrt{\pi}.$

$|T_1(x)\|^2 = \int_{-1}^{1} w(x)T_1^2(x)\, dx = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2\theta\cos\theta}{\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2\theta)\, d\theta =$

$\theta - \frac{1}{2}\sin 2\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2},$ and so

$|T_1(x)\| = \sqrt{\frac{\pi}{2}}.$

$|T_2(x)\|^2 = \int_{-1}^{1} w(x)T_2^2(x)\, dx = \int_{-1}^{1} \frac{(2x^2-1)^2}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\sin^2\theta-1)^2\cos\theta}{\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin^2\theta - 1)^2\, d\theta =$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 2\theta\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 4\theta)\, d\theta = \left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2},$ and so

$|T_2(x)\| = \sqrt{\frac{\pi}{2}}.$

xercises

how that the given functions are orthogonal on the indicated interval. Find the norm of each function.

1. $f(x) = x^3,\ g(x) = x^2 + 1,\ -1 \le x \le 1.$
2. $f(x) = \cos x,\ g(x) = \sin^2 x,\ 0 \le x \le \pi.$
3. $f(x) = e^x,\ g(x) = \sin x,\ \frac{\pi}{4} \le x \le \frac{5\pi}{4}.$

how that the given set of functions is orthogonal on the indicated interval and determine the corresponding rthonormal set.

4. $\sin x,\ \sin 3x,\ \sin 5x, \ldots,\ 0 \le x \le \frac{\pi}{2}.$
5. $1,\ \cos \pi x,\ \cos 2\pi x,\ \cos 3\pi x, \ldots,\ -1 \le x \le 1.$
6. $\cos x,\ \cos 2x,\ \cos 3x, \ldots,\ 0 \le x \le \frac{\pi}{2}.$
7. $P_0(x),\ P_1(x),\ P_2(x), -1 \le x \le 1,$

$_0 = \sqrt{2}$, $b_0 = 0$, $b_1 = \sqrt{\frac{\ }{2}}$, $c_0 = \frac{\ }{2}\sqrt{\frac{\ }{2}}$,

$c_1 = 0$, $c_2 = -\frac{3}{2}\sqrt{\frac{5}{2}}$.

$\|H_0(x)\| = \pi^{1/4}$, $\|H_1(x)\| = \sqrt{2}\,\pi^{1/4}$, $\|H_2(x)\| = 4\pi^{1/4}$.

. $\|L_n(x)\| = 1$, $n = 0, 1, 2$.

1. $\|U_n(x)\| = \sqrt{\frac{\pi}{2}}$, $n = 0, 1, 2$.

where $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)$ are Legendre polynomials.

8. Determine the constants $a_0, b_0, b_1, c_0, c_1, c_2$ so that the functions $g_0(x) = a_0, g_1(x) = b_0 + b_1x, g_2(x) = c_0 + c_1x + c_2x^2$ form an orthonormal set on the interval $-1 \le x \le 1$. Compare the result with that of Problem 7.

erify that the given functions are orthogonal with respect to the given weight function $w(x)$ on the given interval. valuate the norm of each function.

9. $H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ -\infty < x < \infty;\ w(x) = e^{-x^2}$.
10. $L_0(x) = 1,\ L_1(x) = 1 - x,\ L_2(x) = 1 - 2x + \frac{1}{2}x^2,\ 0 \le x < \infty;\ w(x) = e^{-x}$.
11. $U_0(x) = 1,\ U_1(x) = 2x,\ U_2(x) = 4x^2 - 1,\ -1 \le x \le 1;\ w(x) = (1 - x^2)^{\frac{1}{2}}$.
12. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on the interval $a \le x \le b$ such that $\phi_0(x) = 1$ and $\phi_1(x) = x$. Show that $\int_a^b (\alpha x + \beta)\,\phi_n(x)dx = 0$ for $n = 2, 3, \dots$ and any constants α and β.
13. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on the interval $a \le x \le b$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2,\ m \ne n$.

Orthogonality of Bessel Functions

he orthogonality of the Bessel functions $J_n(x),\ n = 0, 1, 2, \dots$, is very important in engineering applications, for xample, in connection with vibrations of circular membranes.

he Bessel function $y = J_\nu(ax)$ was obtained by solving the parametric Bessel equation

$^2y'' + xy' + (a^2x^2 - \nu^2)y = 0,$

hich may be put in the form

$(xy')' + (a^2x^2 - \nu^2)y = 0,$ or

$$xy')' + \left(a^2x - \frac{\nu^2}{x}\right)y = 0. \qquad (1)$$

imilarly, $y = J_\nu(bx)$ is a solution of

$$xy')' + \left(b^2x - \frac{\nu^2}{x}\right)y = 0. \qquad (2)$$

rom equations (1) and (2), we get

$$xJ_\nu'(ax)]' + \left(a^2x - \frac{\nu^2}{x}\right)J_\nu(ax) = 0. \qquad (3)$$

$$xJ_\nu'(bx)]' + \left(b^2x - \frac{\nu^2}{x}\right)J_\nu(bx) = 0. \qquad (4)$$

ince the product $J_\nu(ax)J_\nu(bx)$ is of interest, we multiply equation (3) by $J_\nu(bx)$ and equation (4) by $J_\nu(ax)$, and ubtract to get

$$_\nu(bx)[xJ_\nu'(ax)]' - J_\nu(ax)[xJ_\nu'(bx)]' + (a^2 - b^2)xJ_\nu(ax)J_\nu(bx) = 0,$$

hich may be rewritten as

$$[xJ_\nu(bx)J_\nu'(ax) - xJ_\nu(ax)J_\nu'(bx)] + (a^2 - b^2)xJ_\nu(ax)J_\nu(bx) = 0. \qquad (5)$$

tegrating equation (5) over a given interval $0 \le x \le R$, we get

$$J_\nu(bx)J_\nu'(ax) - xJ_\nu(ax)J_\nu'(bx)]_0^R + (a^2 - b^2)\int_0^R xJ_\nu(ax)J_\nu(bx)\,dx = 0,$$

hich gives

$$J_\nu(bR)J_\nu'(aR) - J_\nu(aR)J_\nu'(bR)] + (a^2 - b^2)\int_0^R xJ_\nu(ax)J_\nu(bx)\,dx = 0. \qquad (6)$$

ice aR and bR are zeros of $J_\nu(x)$, it follows that at $x = R, J_\nu(aR) = J_\nu(bR) = 0$. Hence, equation (6) becomes

$$- b^2) \int_0^R xJ_\nu(ax)J_\nu(bx)\,dx = 0. \tag{7}$$

$a \neq b$, the integral in equation (7) is zero, and so

$$\int_0^R xJ_\nu(ax)J_\nu(bx)\,dx = 0, \;\; a \neq b, \tag{8}$$

which proves that the set of functions $J_\nu(a_n x)$, $n = 1, 2, \ldots$, is orthogonal on the interval $0 \leq x \leq R$ with respect to he weight function x, where $a_n R$, $n = 1, 2, \ldots$, are zeros of $J_\nu(x)$.

f $a = b$, the integral in equation (7) is not zero, and can be shown to be

$$\|J_\nu(ax)\|^2 = \int_0^R xJ_\nu^2(ax)\,dx = \frac{1}{2}R^2 J_{\nu+1}^2(aR). \tag{9}$$

Note: aR is a zero of $J_\nu(x)$ and not of $J_{\nu+1}(x)$.

Simple Set of Polynomials

A set of polynomials $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, is called a simple set if $f_n(x)$ is of degree n. The set contains polynomials of degrees $0, 1, 2, \ldots, n, \ldots$. An example of a simple set of polynomials is $1, x, x^2, x^3$.

An important property of simple sets is that if $g_m(x)$ is any polynomial of degree m and $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, is a simple set of polynomials, then there exist constants c_k such that

$$g_m(x) = \sum_{k=0}^{m} c_k f_k(x),$$

which is assumed to be a convergent series.

Orthogonal Polynomials

Theorem: If $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, is a simple set of polynomials, a necessary and sufficient condition that $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, be orthogonal with respect to the weight function $w(x)$ over an interval $a \leq x \leq b$ is that

$$\int_a^b w(x)x^k f_n(x)\,dx \begin{cases} = 0, & k = 0, 1, 2, \ldots, n-1 \\ \neq 0, & k = n \end{cases}$$

Zeros of Orthogonal Polynomials

Theorem: If $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, is a simple set of real polynomials orthogonal with respect to the weight function $w(x)$ over an interval $a \leq x \leq b$, and if $w(x) > 0$ over the interval $a < x < b$, then the zeros of $f_n(x)$, $n = 0, 1, 2, \ldots$, are distinct and all lie in the open interval $a < x < b$.

Orthogonality of Legendre Polynomials

The orthogonality of Legendre polynomials is useful in mathematical physics, especially in sphere problems.

The Legendre polynomials $y = P_n(x)$, $n = 0, 1. 2, \ldots$, were obtained by solving the Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

which may be written as

$$\left[(1 - x^2)y'\right]' + n(n+1)y = 0. \tag{1}$$

$P_n(x)$, $n = 0, 1. 2, \ldots$, form a simple set of polynomials for which we now obtain an orthogonality property.

Since $y = P_n(x)$, $n = 0, 1. 2, \ldots$, are solutions of equation (1), we have

$$\left[(1 - x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0. \tag{2}$$

f(x) is a function defined on the interval
$x < R$ and is given by the convergent series

$$f(x) = \sum_{n=1}^{\infty} c_n J_\nu(a_n x),$$

en the coefficients

$$c_n = \frac{2}{R^2 J_{\nu+1}^2(a_n R)} \int_0^R x f(x) J_\nu(a_n x)\,dx, \quad n=1,2,\dots$$

r $f(x) = J_\nu(a_n x)$, it follows that $c_n = 1$. Hence

$$\frac{2}{R^2 J_{\nu+1}^2(a_n R)} \int_0^R x J_\nu^2(a_n x)\,dx = 1,$$

r

$$\int_0^R x J_\nu^2(a_n x)\,dx = \frac{1}{2} R^2 J_{\nu+1}^2(a_n R).$$

larly, $y = P_m(x),\ m = 0, 1.2, \ldots$, are solutions of

$$-x^2)P_m'(x)\big]' + m(m+1)P_m(x) = 0. \tag{3}$$

nce the product $P_m(x)P_n(x)$ is of interest, we multiply equation (2) by $P_m(x)$ and equation (3) by $P_n(x)$, and ubtract to get

$$_m(x)\big[(1-x^2)P_n'(x)\big]' - P_n(x)\big[(1-x^2)P_m'(x)\big]' + [n(n+1) - m(m+1)]P_m(x)P_n(x) = 0,$$

hich may be rewritten as

$$\frac{l}{x}\big[(1-x^2)P_n(x)P_m'(x) - (1-x^2)P_m(x)P_n'(x)\big] + (n-m)(n+m+1)P_m(x)P_n(x) = 0. \tag{4}$$

ntegrating equation (4) over the interval $a \le x \le b$, we get

$$(1-x^2)P_n(x)P_m'(x) - (1-x^2)P_m(x)P_n'(x)\big]_a^b + (n-m)(n+m+1)\int_a^b P_m(x)P_n(x)\,dx = 0. \tag{5}$$

ny a and b may be chosen. Since $1 - x^2 = 0$ gives $x = \pm 1$, we select $a = -1, b = 1$, and so equation (5) becomes

$$n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0. \tag{6}$$

f $m \ne n$ and since m and n are non-negative integers, we get $n - m \ne 0$ and $n + m + 1 \ne 0$. Hence, the integral in quation (6) is zero, and so

$$_{-1}^{1} P_m(x)P_n(x)\,dx = 0,\ m \ne n, \tag{7}$$

hich proves the orthogonality of the Legendre polynomials over the interval $-1 \le x \le 1$.

f $m = n$, the integral in equation (6) is not zero, and can be shown to be

$$P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1},\ n = 0, 1, 2, \ldots. \tag{8}$$

Fourier Series

Ve assume that a function $f(x)$, defined on an interval $-L < x < L$, can be expanded in the trigonometric series

$$(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{1}$$

vhich is assumed to be convergent. We aim at determining the coefficients $a_0, a_n, b_n,\ n = 1, 2, \ldots$.

o determine a_0, we integrate both sides of equation (1) with respect to x from $-L$ to L to get

$$_{-L}^{L} f(x)\,dx = \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx\right) = a_0 L, \tag{2}$$

ince $\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx = 0, n = 1, 2, \ldots$. Equation (2) gives

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx. \tag{3}$$

$a_n,\ n = 1, 2, \ldots$ are determined by multiplying both sides of equation (1) by $\cos\frac{m\pi x}{L}$, and integrating over the interval $-L < x < L$:

$$_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right). \tag{4}$$

ut $\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx = 0, m = 1, 2, \ldots;$

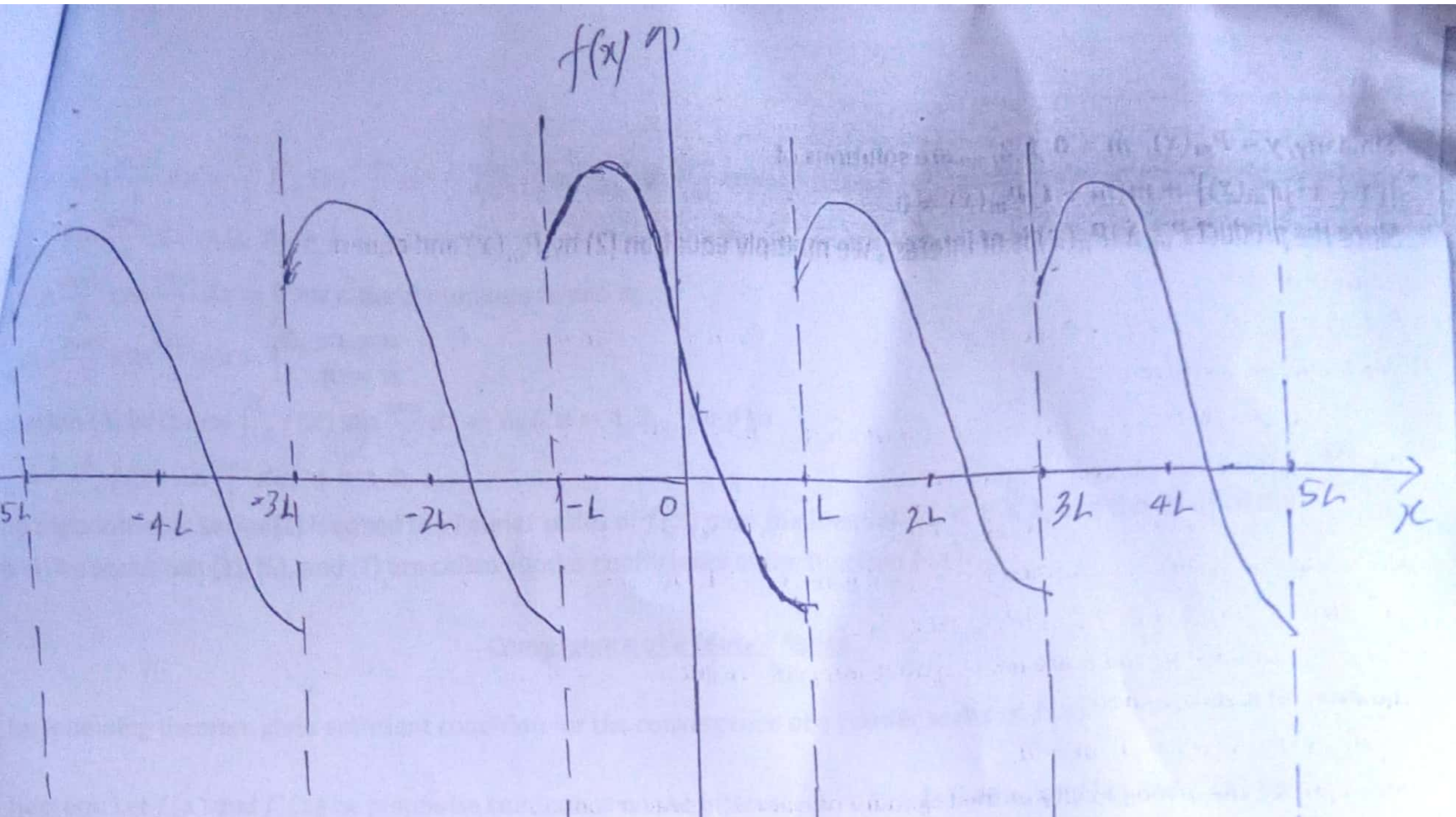
$$_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases}0, & m \ne n\\ L, & m = n\end{cases};$$

$_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$ for natural numbers m and n.

quation (4) becomes $\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = a_n L, n = 1, 2, \ldots$, and so

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, \ldots. \tag{5}$$

$b_n,\ n = 1, 2, \ldots$ are determined by multiplying both sides of equation (1) by $\sin\frac{m\pi x}{L}$, and integrating over the interval $-L < x < L$:



D	I
$x(L+x)$	$\sin\frac{n\pi x}{L}$
$L+2x$	$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$
2	$-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$
0	$\frac{L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}$

	D	I
$+$	$(L-x)^2$	$\sin\frac{n\pi x}{L}$
$-$	$-2(L-x)$	$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$
$+$	2	$-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$
	0	$\frac{L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}$

$(x)\sin\frac{m\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx\right).$ (6)

$\int_{-L}^{L}\sin\frac{m\pi x}{L}dx = 0,\ m = 1, 2, \ldots;$

$\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0$ for natural numbers m and n;

$\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases}0, & m \neq n\\ L, & m = n\end{cases}.$

quation (4) becomes $\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = a_n L, n = 1, 2, \ldots$, and so

$_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,\ n = 1, 2, \ldots.$ (7)

he trigonometric series (1) is called the Fourier series of $f(x)$ over the interval $-L < x < L$, and the coefficients iven by equations (3), (5), and (7) are called Fourier coefficients of the function $f(x)$.

Convergence of a Fourier Series

he following theorem gives sufficient condition for the convergence of a Fourier series of $f(x)$.

heorem: Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$; that is, let $f(x)$ and $f'(x)$ be ontinuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then ne Fourier series of $f(x)$ on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the ourier series will converge to the average

$\frac{(x_+)+f(x_-)}{2},$

rhere $f(x_+)$ and $f(x_-)$ denote the limit of $f(x)$ at x from the right and the left, respectively.

lote: For a point x on the interval $-L < x < L$ and $h > 0$, we have that

$(x_+) = \lim_{h\to 0} f(x+h),\ f(x_-) = \lim_{h\to 0} f(x-h).$

Periodic Extension

ince the Fourier series (1) of $f(x)$ over the interval $-L < x < L$ is periodic with period $2L$, it follows that it not only epresents $f(x)$ on the interval $-L < x < L$, but also gives the periodic extension of $f(x)$ outside this interval. The eriodicity of $f(x)$ with period $2L$ means $f(x + 2L) = f(x)$.

Vhen $f(x)$ is piecewise continuous, and the limit of $f(x)$ at $x = L$ from the left and at $x = -L$ from the right exist, ıen the series (1) will converge to the average

$\frac{(-L_+)+f(L_-)}{2}$

t the endpoints $x = \pm L$, and to this value extended periodically to $x = \pm 3L, \pm 5L, \pm 7L, \ldots$.

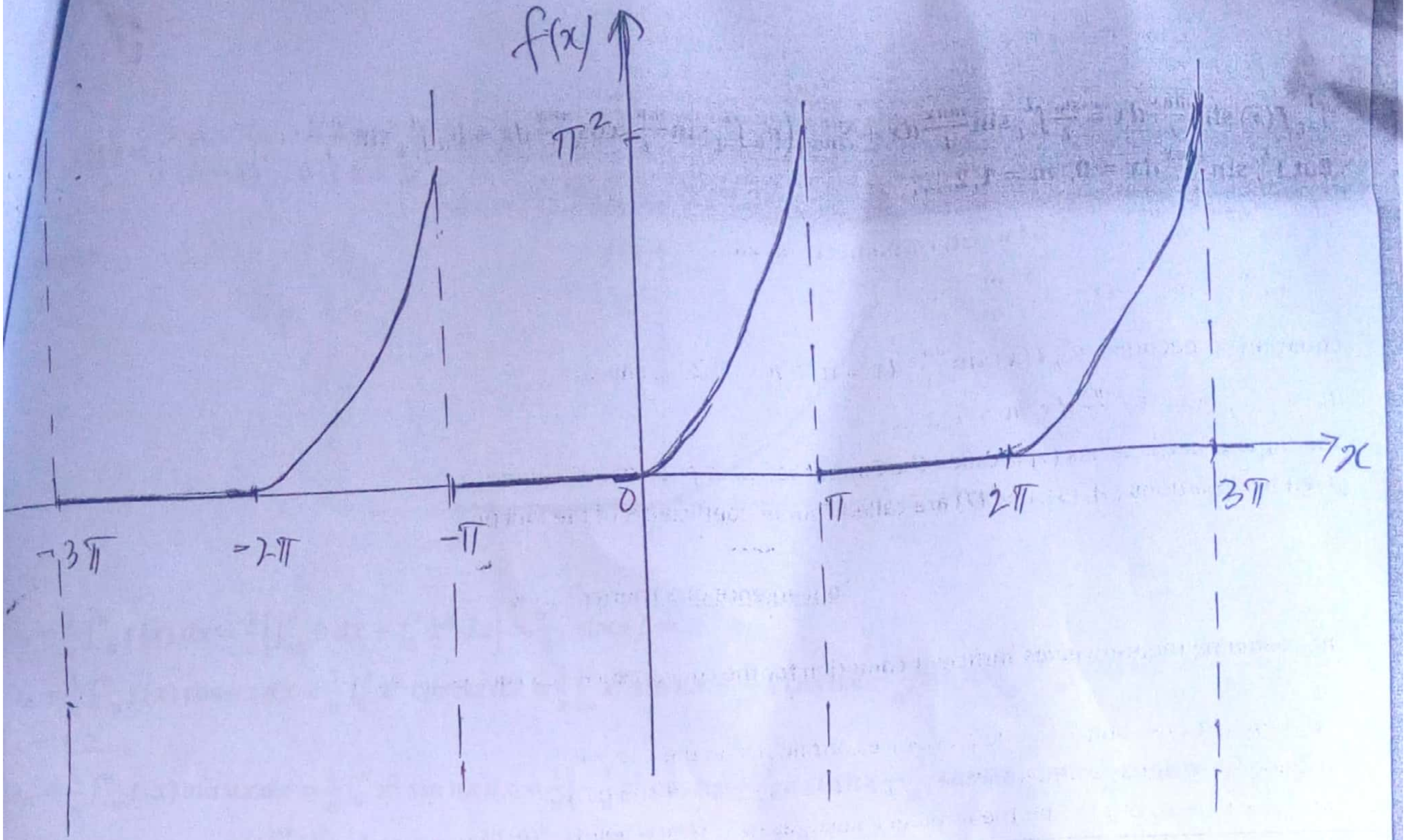
xample

ind the Fourier series of $f(x)$ over the indicated interval. Sketch the function that is the sum of the series obtained.

1. $f(x) = \begin{cases}0, & -\pi < x < 0\\ x^2, & 0 \le x < \pi\end{cases}.$

   Use the result to show that

   $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ and $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$



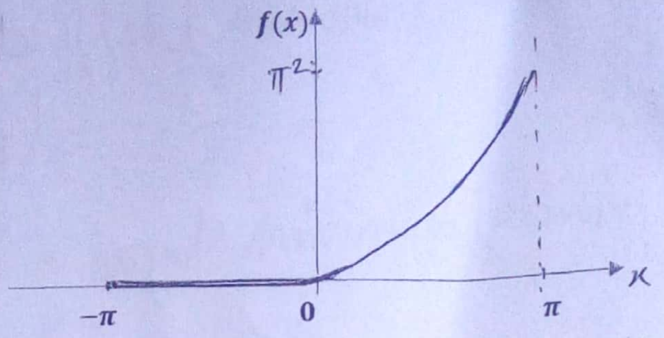
D	I
$x(L+x)$	$\cos\frac{n\pi x}{L}$
$L+2x$	$\frac{L}{n\pi}\sin\frac{n\pi x}{L}$
2	$-\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}$
0	$-\frac{L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}$

	D	I
+	$(L-x)^2$	$\cos\frac{n\pi x}{L}$
−	$-2(L-x)$	$\frac{L}{n\pi}\sin\frac{n\pi x}{L}$
+	2	$-\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}$
	0	$-\frac{L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}$

2. $f(x) = \begin{cases} x(L+x), & -L<x<0 \\ (L-x)^2, & 0 \le x < L \end{cases}$.

lutions:

1.



$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2\,dx\right] = \frac{\pi^2}{3}$, since $L=\pi$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[\frac{1}{n}x^2\sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{2}{n^2}\cos n\pi = \frac{2(-1)^n}{n^2}$,

$n = 1, 2, \ldots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\left[-\frac{1}{n}x^2\cos nx + \frac{2}{n^2}x\sin nx - \frac{2}{n^3}\cos nx\right]_0^{\pi} = -\frac{\pi}{n}\cos n\pi + \frac{2}{n^3}(\cos n\pi - 1) = \frac{2[(-1)^n-1]}{\pi n^3} - \frac{\pi}{n}(-1)^n = \frac{2[(-1)^n-1]}{\pi n^3} + \frac{\pi(-1)^{n+1}}{n}$, $n = 1, 2, \ldots$

hus, the Fourier series of $f(x)$ on the interval $-\pi < x < \pi$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$, or

$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{2[(-1)^n-1]}{\pi n^3} + \frac{\pi(-1)^{n+1}}{n}\right)\sin nx\right\}$. (1)

t the endpoint $x = \pi$, the series converges to the average

$\frac{f(-\pi_+)+f(\pi_-)}{2} = \frac{\pi^2}{2}$, (2)

ince $f(-\pi_+) = 0$ and $f(\pi_-) = \pi^2$. Substituting $x = \pi$ into series (1) yields

$f(\pi) = \frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{2[(-1)^n-1]}{\pi n^3} + \frac{\pi(-1)^{n+1}}{n}\right)\sin n\pi\right\} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos n\pi$.

$\Rightarrow \frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^{2n}}{n^2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$, or

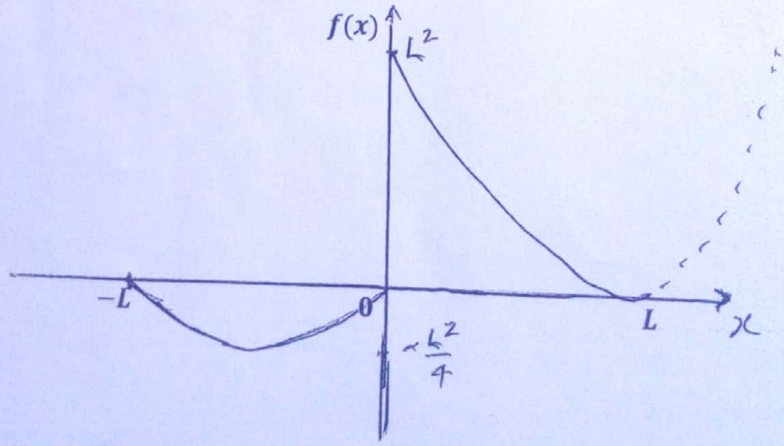
$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

t $x = 0$, the series converges to, from (1),

$f(0) = 0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$, or

$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$.

2.



$f(x) = x(L+x)$, $-L < x < 0$.

$f'(x) = L + 2x = 0$ for turning points $\Rightarrow x = -\frac{L}{2}$.

$f''(x) = 2 > 0$ (local minimum).

$f(-\frac{L}{2}) = -\frac{L}{2}(\frac{L}{2}) = -\frac{L^2}{4}$.

$\Rightarrow (-\frac{L}{2}, -\frac{L^2}{4})$ is a local minimum point.

Also, $f(-L) = f(0) = 0$.

$f(x) = (L-x)^2$, $0 \le x < L$.

$f'(x) = 2(x-L) = 0$ for turning points. $\Rightarrow x = L$.

$\Rightarrow f''(x) = 2 > 0$ (local minimum).

$f(L) = 0$.

$\Rightarrow (L, 0)$ is a local minimum point

Also, $f(0) = L^2$.

$= \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\left[\int_{-L}^{0} x(L+x)\,dx + \int_{0}^{L}(L-x)^2\,dx\right] = \frac{1}{L}\left[\frac{1}{2}x^2(L+x) - \frac{1}{6}x^3\right]_{-L}^{0} - \frac{1}{3L}\left[(L-x)^3\right]_0^L = \frac{L^2}{6}.$

$_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\left[\int_{-L}^{0} x(L+x)\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L}(L-x)^2\cos\frac{n\pi x}{L}\,dx\right] = \frac{1}{L}\left[\frac{L}{n\pi}x(L+x)\sin\frac{n\pi x}{L} + \right.$
$\left.\frac{L^2}{{}^2\pi^2}(L+2x)\cos\frac{n\pi x}{L} - \frac{2L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}\right]_{-L}^{0} + \frac{1}{L}\left[\frac{L}{n\pi}(L-x)^2\sin\frac{n\pi x}{L} - \frac{2L^2}{n^2\pi^2}(L-x)\cos\frac{n\pi x}{L} - \frac{2L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}\right]_0^L = \frac{L^2}{n^2\pi^2}(3 +$
$\text{os}\, n\pi) = \frac{L^2[3+(-1)^n]}{n^2\pi^2},\ n = 1, 2, \ldots.$

$_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\left[\int_{-L}^{0} x(L+x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L}(L-x)^2\sin\frac{n\pi x}{L}\,dx\right] = \frac{1}{L}\left[-\frac{L}{n\pi}x(L+x)\cos\frac{n\pi x}{L} + \right.$
$\left.\frac{L^2}{{}^2\pi^2}(L+2x)\sin\frac{n\pi x}{L} - \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right]_{-L}^{0} + \frac{1}{L}\left[-\frac{L}{n\pi}(L-x)^2\cos\frac{n\pi x}{L} - \frac{2L^2}{n^2\pi^2}(L-x)\sin\frac{n\pi x}{L} + \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right]_0^L = \frac{L^2}{n\pi},\ n =$
$, 2, \ldots.$

hus, the Fourier series of $f(x)$ on the interval $-\pi < x < \pi$ is

$(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$ or

$(x) = \frac{L^2}{12} + \frac{L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{[3+(-1)^n]\cos\frac{n\pi x}{L} + n\pi\sin\frac{n\pi x}{L}\right\}.$

xercises

ketch the function $f(x)$, and obtain its Fourier series over the indicated interval.

1. $f(x) = x + \pi,\ -\pi < x < \pi.$
   Use the result to show that
   $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$
2. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}.$
   Use the result to show that
   $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots.$
3. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}.$
4. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}.$
5. $f(x) = \pi - 2|x|, -\pi < x < \pi.$
6. $f(x) = x^2,\ -L < x < L.$
   Use the result to show that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$
7. $f(x) = \begin{cases} 0, & -L < x < 0 \\ (L-x)^2, & 0 \le x < L \end{cases}.$
8. $f(x) = x^3,\ -L < x < L.$
9. $f(x) = e^x,\ -\pi < x < \pi.$
10. $f(x) = x^4,\ -L < x < L.$

$$f(x) = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{[(-1)^n + 1]}{1-n^2} \cos nx.$$

$$f(x) = -\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \left[-\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2} + \frac{3}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}\right]$$

$$\therefore f(x) = \frac{3}{8} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left\{2\left[\cos\frac{n\pi}{2} - 1\right]\cos\frac{n\pi x}{2} + \left[2\sin\frac{n\pi}{2} + n\pi(-1)^{n+1}\right]\sin\frac{n\pi x}{2}\right\}$$

$$\therefore f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^2} \cos nx.$$

$$\therefore f(x) = \frac{L^2}{3} + \frac{4L^2}{3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{L}.$$

$$\therefore f(x) = \frac{L^2}{6} + \frac{L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{n^3}\left\{2n\pi\cos\frac{n\pi x}{L} + \left[n^2\pi^2 - 2 + 2(-1)^n\right]\sin\frac{n\pi x}{L}\right\}$$

$$\therefore f(x) = \frac{2L^3}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2\pi^2-6)}{n^3} \sin\frac{n\pi x}{L}.$$

$$\therefore f(x) = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\right].$$

$$\therefore f(x) = \frac{L^4}{5} + \frac{8L^4}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n(n^2\pi^2-6)}{n^4} \cos\frac{n\pi x}{L}.$$

Fourier Cosine and Sine Series

[F]unction $y = f(x)$ is said to be even if $f(-x) = f(x)$, and its graph is symmetric with respect to the y-axis. A [fu]nction $y = f(x)$ is said to be odd if $f(-x) = -f(x)$, and its graph is symmetric with respect to the origin.

[T]he following are the properties of even and odd functions:

i. The product of two even functions is even.
ii. The product of two odd functions is even.
iii. The product of an even function and an odd function is odd.
iv. If $f(x)$ is an even function, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$.
v. If $f(x)$ is an odd function, then $\int_{-L}^{L} f(x)\,dx = 0$.

[L]et $f(x)$ be an even function defined on an interval $-L < x < L$. From the foregoing section, we get

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx.$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, \ldots,$ since $f(x)\cos\frac{n\pi x}{L}$ is an even function.

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0,\ n = 1, 2, \ldots,$ since $f(x)\sin\frac{n\pi x}{L}$ is an odd function.

[A]lso, if $f(x)$ is an odd function defined on an interval $-L < x < L$, we have that

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0.$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0,\ n = 1, 2, \ldots,$ since the function $f(x)\cos\frac{n\pi x}{L}$ is odd.

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, \ldots,$ since the function $f(x)\sin\frac{n\pi x}{L}$ is even.

[H]ence, the Fourier series of an even function $f(x)$ on an interval $-L < x < L$ is the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \tag{1}$$

[w]here

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \tag{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, \ldots. \tag{3}$$

[T]he Fourier series of an odd function $f(x)$ on an interval $-L < x < L$ is the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{4}$$

[w]here

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, \ldots. \tag{5}$$

[E]xamples

[P]rove that:

1. The product of an even function and an odd function is odd.
2. If $f(x)$ is an even function, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$.

[S]olutions:

1. Let $f(x)$ be an even function and let $g(x)$ be an odd function. Then $h(x)$, the product of $f(x)$ and $g(x)$, is $h(x) = f(x)g(x)$.
   But $f(-x) = f(x)$ and $g(-x) = -g(x)$, and so
   $h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x)$,
   which shows that $h(x)$ is an odd function.

2. Since $f(x)$ is an even function, then $f(-x) = f(x)$, and so

$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx = \int_{-L}^{0} f(-x)\,dx + \int_{0}^{L} f(x)\,dx.$

Let $u = -x$ in the first integral. $\Rightarrow du = -dx.$ $x = 0 \Rightarrow u = 0;$ $x = -L \Rightarrow u = L$, and so

$\int_{-L}^{L} f(x)\,dx = -\int_{L}^{0} f(u)\,du + \int_{0}^{L} f(x)\,dx = \int_{0}^{L} f(u)\,du + \int_{0}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx,$

since u is a dummy variable.

xamples

etermine whether the function is even, odd, or neither.

1. $f(x) = x\cos x.$
2. $f(x) = x^2 + x.$
3. $f(x) = \begin{cases} x+5, & -2 < x < 0 \\ -x+5, & 0 \le x < 2 \end{cases}.$

olutions:

3. $f(x) = x\cos x \Rightarrow f(-x) = -x\cos(-x) = -x\cos x \Rightarrow f(-x) = -f(x)$, and so $f(x)$ is odd.
4. $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 - x = x^2 - x. \Rightarrow f(-x) \ne f(x)$ and $f(-x) \ne -f(x)$, and so $f(x)$ is neither even nor odd.
5. $f(x) = \begin{cases} x+5, -2 < x < 0 \\ -x+5, \ 0 \le x < 2 \end{cases} \Rightarrow f(-x) = \begin{cases} -x+5, -2 < -x < 0 \text{ or } 0 < x < 2 \\ x+5, \ 0 \le -x < 2 \text{ or } -2 < x \le 0 \end{cases} = \begin{cases} x+5, -2 < x < 0 \\ -x+5, \ 0 \le x < 2 \end{cases}.$

$\Rightarrow f(-x) = f(x)$, and so $f(x)$ is an even function.

xamples

xpand the given function in an appropriate Fourier cosine or sine series.

6. $f(x) = \begin{cases} -1, -\pi < x < 0 \\ 1, \ 0 \le x < \pi \end{cases}.$
7. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}.$

olutions:

4. $f(x) = \begin{cases} -1, -\pi < x < 0 \\ 1, \ 0 \le x < \pi \end{cases} \Rightarrow f(-x) = \begin{cases} -1, -\pi < -x < 0 \text{ or } 0 < x < \pi \\ 1, \ 0 \le -x < \pi \text{ or } -\pi < x \le 0 \end{cases} = \begin{cases} 1, -\pi < x < 0 \\ -1, \ 0 \le x < \pi \end{cases}.$

$\Rightarrow f(-x) = -f(x)$, and so $f(x)$ is is an odd function on the interval $-\pi < x < \pi$.

We, therefore, expect a Fourier sine series. Here $L = \pi$, and so

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{1-\cos n\pi}{n}\right] = \frac{2}{n\pi}[1-(-1)^n],\ n = 1, 2, \ldots.$

Thus, the required Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, or

$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]\sin nx.$

5. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases} \Rightarrow f(-x) = \begin{cases} 1, & -2 < -x < -1 \text{ or } 1 < x < 2 \\ x, & -1 \le -x < 0 \text{ or } 0 < x \le 1 \\ -x, & 0 \le -x < 1 \text{ or } -1 < x \le 0 \\ 1, & 1 \le -x < 2 \text{ or } -2 < x \le -1 \end{cases} = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases} = f(x),$

which implies that $f(x)$ is an even function on the interval $-2 < x < 2$.

D	I
x	$\cos\frac{n\pi x}{2}$
1	$\frac{2}{n\pi}\sin\frac{n\pi x}{2}$
0	$-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}$

7. Odd.

8. Even.

1. Neither.

0. Odd.

1. $f(x)=\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}.$

2. $f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2}\cos nx.$

3. $f(x)=12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx.$

4. $f(x)=\frac{2}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{1-n^2}\cos nx.$

5. $f(x)=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x.$

6. $f(x)=\frac{\sinh kL}{kL}+2kL\sinh kL\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2+k^2L^2}\cos\frac{n\pi x}{L}.$

Hence, we expect a Fourier cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, where

$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (1)\,dx = \left[\frac{1}{2}x^2\right]_0^1 + [x]_1^2 = \frac{3}{2}$, since $L = 2$,

$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \cos\frac{n\pi x}{2}\,dx = \left[\frac{2}{n\pi}x\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^1 +$

$\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_1^2 = \frac{4}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right], n = 1, 2, \ldots.$

Therefore, the required Fourier cosine series is

$f(x) = \frac{3}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi x}{2}.$

xercises

rove that:

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. If $f(x)$ is an even function, then $|f(x)|$, $f^2(x)$, and $f^3(x)$ are even functions.
4. If $f(x)$ is an odd function, then $|f(x)|$ and $f^2(x)$ are even functions.
5. If $g(x)$ is any function, defined for all x, then $p(x) = \frac{1}{2}[g(x) + g(-x)]$ is even and $q(x) = \frac{1}{2}[g(x) - g(-x)]$ is odd, and $g(x) = p(x) + q(x)$.
6. If $f(x)$ is an odd function, then $\int_{-L}^{L} f(x)\,dx = 0$.

etermine whether the function is even, odd or neither.

7. $f(x) = x|x|$.
8. $f(x) = |x^5|$.
9. $f(x) = x^3,\ 0 \le x \le 2$.
10. $f(x) = \begin{cases} x^2, & -1 < x \le 0 \\ -x^2, & 0 < x < 1 \end{cases}$.

xpand the given function in an appropriate Fourier cosine or sine series.

11. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 \le x < 1. \\ 1, & 1 \le x < 2 \end{cases}$
12. $f(x) = |x|,\ -\pi < x < \pi$.
13. $f(x) = x(\pi^2 - x^2),\ -\pi < x < \pi$.
14. $f(x) = |\sin x|,\ -\pi < x < \pi$.
15. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \le x < 1 \end{cases}$.
16. $f(x) = \cosh kx,\ -L < x < L$, where k is a constant.

Fourier-Bessel Series

set of orthogonal functions is defined as complete on a given interval if there is no other function orthogonal to all them in that interval. Bessel functions, therefore, form a complete orthogonal set on the interval $0 < x < R$. ence, we can expand functions in a Fourier-Bessel series, just as we did with Fourier series.

et $f(x)$ be a given function defined on the interval $0 < x < R$ that can be represented in terms of Bessel functions

$$x) = \sum_{n=1}^{\infty} c_n J_\nu(a_n x), \qquad (1)$$

n is assumed to be convergent. Equation (1) is called a Fourier-Bessel series of $f(x)$, and the coefficients c_n, $n =$ ... are to be determined, using the orthogonality property of Bessel functions.

Multiplying equation (1) by $xJ_v(a_m x)$ and integrate over the interval $0 \le x \le R$ gives

$$\int_0^R xf(x)J_v(a_m x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^R xJ_v(a_m x)J_v(a_n x)\,dx. \quad (2)$$

Only the integral corresponding to $m = n$ is non-zero, since

$$\int_0^R xJ_v(a_m x)J_v(a_n x)\,dx = 0,\ m \neq n.$$

Equation (2) becomes

$$\int_0^R xf(x)J_v(a_n x)\,dx = c_n \int_0^R xJ_v^2(a_n x)\,dx = \frac{c_n}{2} R^2 J_{v+1}^2(a_n R).$$

Thus, the coefficients of the Fourier-Bessel series (1) are given by

$$c_n = \frac{2}{R^2 J_{v+1}^2(a_n R)} \int_0^R xf(x)J_v(a_n x)\,dx,\ n = 1, 2, \dots. \quad (3)$$

Notes:

1. $a_n R,\ n = 1, 2, \dots$ are roots of $J_v(x)$ and not of $J_{v+1}(x)$.
2. Use will be made of the properties
   i. $\frac{d}{dt}[t^v J_v(t)] = t^v J_{v-1}(t) \Rightarrow \int t^v J_{v-1}(t)\,dt = t^v J_v(t)$,
   ii. $\frac{d}{dt}[t^{-v} J_v(t)] = -t^{-v} J_{v+1}(t) \Rightarrow \int t^{-v} J_{v+1}(t)\,dt = -t^{-v} J_v(t)$,

   in the evaluation of $\int t^k J_v(t)\,dt$. We use i. if $|k| > v$, and use ii. if $|k| \le v$.
3. We restrict ourselves to integral values of v, since this is a case of significant practical importance.

Examples

1. Develop the function $f(x) = 1$, defined on the interval $0 < x < R$ in a Fourier-Bessel series $f(x) = \sum_{n=1}^{\infty} c_n J_0(a_n x)$.
2. Expand the following function in a Fourier-Bessel series of order 1:
   $f(x) = \begin{cases} k, & 0 < x < b \\ 0, & b < x < 3 \end{cases}$.
3. Represent $f(x) = x$, defined on the interval $0 < x < 2$, by a Fourier-Bessel series of order 3.

Solutions:

1. The coefficients of the given Fourier-Bessel series are given by

   $c_n = \frac{2}{R^2 J_1^2(a_n R)} \int_0^R xJ_0(a_n x)\,dx,\ n = 1, 2, \dots$, since $f(x) = 1, v = 0$.

   Let $t = a_n x. \Rightarrow dt = a_n dx$.

   $x = 0 \Rightarrow t = 0,\ x = R \Rightarrow t = a_n R$, and so

   $c_n = \frac{2}{a_n^2 R^2 J_1^2(a_n R)} \int_0^{a_n R} tJ_0(t)\,dt,\ n = 1, 2, \dots$.

   $\int t^v J_{v-1}(t)\,dt = t^v J_v(t) \Rightarrow \int tJ_0(t)\,dt = tJ_1(t)$, since $v = 1$, and so

   $c_n = \frac{2[tJ_1(t)]_0^{a_n R}}{a_n^2 R^2 J_1^2(a_n R)} = \frac{2}{a_n R J_1(a_n R)},\ n = 1, 2, \dots$.

   Thus, the Fourier-Bessel series of $f(x)$ is

   $f(x) = \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(a_n x)}{a_n J_1(a_n R)}$,

   where $a_n R,\ n = 1, 2, \dots$ are the zeros of $J_0(x)$.

2. The required Fourier-Bessel series of order 1 of $f(x)$ over the interval $0 < x < 3$ is given by

$f(x) = \sum_{n=1}^{\infty} c_n J_1(a_n x)$, (1)

where

$c_n = \frac{2}{3^2 J_2^2(3a_n)} \int_0^3 x f(x) J_1(a_n x)\, dx = \frac{2}{9 J_2^2(3a_n)} \int_0^3 kx J_1(a_n x)\, dx,\ n = 1, 2, \ldots$, since $\nu = 1, R = 3$.

Let $t = a_n x. \Rightarrow dt = a_n dx.\ x = 0 \Rightarrow t = 0,\ x = b \Rightarrow t = a_n b$.

$\Rightarrow c_n = \frac{2k}{9a_n^2 J_2^2(3a_n)} \int_0^{a_n b} t J_1(t)\, dt,\ n = 1, 2, \ldots.$ (2)

But $\int t^{-\nu} J_{\nu+1}(t)\, dt = -t^{-\nu} J_\nu(t)$. (3)

$\Rightarrow \int J_1(t)\, dt = -J_0(t)$, where $\nu = 0$ in equation (3), and so

$\int_0^{a_n b} t J_1(t)\, dt = -[t J_0(t)]_0^{a_n b} + \int_0^{a_n b} J_0(t)\, dt = -a_n b J_0(a_n b) + \int_0^{a_n b} J_0(t)\, dt,$

which is employed in equation (2) to obtain

$c_n = \frac{2k}{9a_n^2 J_2^2(3a_n)} \left[-a_n b J_0(a_n b) + \int_0^{a_n b} J_0(t)\, dt\right],\ n = 1, 2, \ldots.$

Hence, the required Fourier-Bessel series is

$f(x) = \frac{2k}{9} \sum_{n=1}^{\infty} \frac{1}{a_n^2 J_2^2(3a_n)} \left[-a_n b J_0(a_n b) + \int_0^{a_n b} J_0(t)\, dt\right] J_1(a_n x),$

where $3a_n,\ n = 1, 2, \ldots$ are zeros of $J_1(x)$.

3. The required Fourier-Bessel series of order 1 of $f(x)$ over the interval $0 < x < 3$ is

$f(x) = \sum_{n=1}^{\infty} c_n J_3(a_n x)$, (1)

where

$c_n = \frac{2}{2^2 J_4^2(2a_n)} \int_0^2 x f(x) J_3(a_n x)\, dx = \frac{2}{4 J_4^2(2a_n)} \int_0^2 x^2 J_3(a_n x)\, dx,\ n = 1, 2, \ldots$, since $f(x) = x, \nu = 1, R = 2$.

Let $t = a_n x. \Rightarrow dt = a_n dx.$

$x = 0 \Rightarrow t = 0,\ x = b \Rightarrow t = a_n b.$

$\Rightarrow c_n = \frac{1}{2a_n^3 J_4^2(2a_n)} \int_0^{2a_n} t^2 J_3(t)\, dt,\ n = 1, 2, \ldots.$ (2)

But $\int t^{-\nu} J_{\nu+1}(t)\, dt = -t^{-\nu} J_\nu(t)$. (3)

$\Rightarrow \int t^{-2} J_3(t)\, dt = -t^{-2} J_2(t)$, where $\nu = 2$ in equation (3), and so

$\int t^2 J_3(t)\, dt = \int t^4 [t^{-2} J_3(t)]\, dt = -t^2 J_2(t) + 4 \int t J_2(t) dt$, using integration by parts.

But $\int t^{-1} J_2(t)\, dt = -t^{-1} J_1(t)$, from equation (3), where $\nu = 1$, and so

$\int t J_2(t)\, dt = \int t^2 [t^{-1} J_2(t)]\, dt = -t J_1(t) + 2 \int J_1(t) dt.$

$\Rightarrow \int t^2 J_3(t)\, dt = -t^2 J_2(t) + 4[-t J_1(t) + 2 \int J_1(t) dt] = -t^2 J_2(t) - 4t J_1(t) + 8 \int J_1(t)\, dt.$

But $\int J_1(t)\, dt = -J_0(t)$, from equation (3), where $\nu = 0$.

$\Rightarrow \int_0^{a_n b} t J_3(t)\, dt = -[8J_0(t) + 4t J_1(t) + t^2 J_2(t)]_0^{2a_n} = -[8J_0(2a_n) + 4(2a_n) J_1(2a_n) +$
$(2a_n)^2 J_2(2a_n) - 8J_0(0)] = -[8J_0(2a_n) + 8a_n J_1(2a_n) + 4a_n^2 J_2(2a_n) - 8]$, since $J_0(0) = 1$,

which is employed in equation (2) to obtain

$c_n = \frac{2[2 - 2J_0(2a_n) - 2a_n J_1(2a_n) - a_n^2 J_2(2a_n)]}{a_n^3 J_4^2(2a_n)},\ n = 1, 2, \ldots.$

Therefore, the required Fourier-Bessel series is

$f(x) = 2 \sum_{n=1}^{\infty} \frac{[2 - 2J_0(2a_n) - 2a_n J_1(2a_n) - a_n^2 J_2(2a_n)]}{a_n^3 J_4^2(2a_n)} J_3(a_n x),$

where $2a_n,\ n = 1, 2, \ldots$ are zeros of $J_3(x)$.

ises

the Fourier-Bessel series of $f(x)$ over the indicated interval in terms of Bessel functions of order 0.

1. $f(x) = x^2,\ 0 < x < 1.$
2. $f(x) = 1 - x^2,\ 0 < x < 1.$
3. $f(x) = R^2 - x^2,\ 0 < x < R.$
4. $f(x) = x^4,\ 0 < x < R.$

xpand $f(x)$ on the indicated interval in a Fourier-Bessel series of order 1.

5. $f(x) = 1,\ 0 < x < 3.$
6. $f(x) = x,\ 0 < x < 1.$
7. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}R \\ 0, & \frac{1}{2}R < x < R \end{cases}.$

epresent $f(x)$, defined on the indicated interval, in a Fourier-Bessel series of order 2.

8. $f(x) = \begin{cases} 0, & 0 < x < a \\ k, & a < x < R \end{cases}$, where k is a constant.
9. $f(x) = x^5,\ 0 < x < 4.$

Ind the Fourier-Bessel series of $f(x)$ over the indicated interval in terms of the Bessel function J_3.

10. $f(x) = x,\ 0 < x < R.$
11. $f(x) = x^5,\ 0 < x < 2.$

Fourier-Legendre Series

Ince the Legendre polynomials $P_n(x),\ n = 1, 2, \dots$ form a complete orthogonal set on the interval $-1 \le x \le 1$, we an expand functions in Fourier-Legendre series.

et $f(x)$ be a given function, defined on the interval $-1 < x < 1$, that can be represented in terms of Legendre olynomials by the series

$$(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{1}$$

hich is assumed to be convergent. Equation (1) is called a Fourier-Legendre series of the function $f(x)$. We desire to nd the coefficients $c_n,\ n = 0, 1, 2, \dots$. Use will be made of the orthogonality property of Legendre polynomials to nd $c_n,\ n = 0, 1, 2, \dots$.

/e multiply equation (1) by $P_m(x)$ and integrate with respect to x from -1 to 1 to get

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} c_n \int_{-1}^{1} P_m(x)P_n(x)\,dx = c_n \int_{-1}^{1} P_n^2(x)\,dx = \frac{2c_n}{2n+1},\quad n = 0, 1, 2, \dots,$$

hich is obtained when $m = n$, since all other integrals corresponding to $m \neq n$ vanish, owing to orthogonality.

ence, the coefficients $c_n, n = 0, 1, 2, \dots$, of the Fourier-Legendre series (1) are given by

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{2c_n}{2n+1},\quad n = 0, 1, 2, \dots, \text{ or}$$

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx,\quad n = 0, 1, 2, \dots. \tag{2}$$

ote:

i. It is not necessary for the function $f(x)$ to be continuous on the interval $-1 < x < 1$ for it to be expandable in a Fourier-Legendre series.

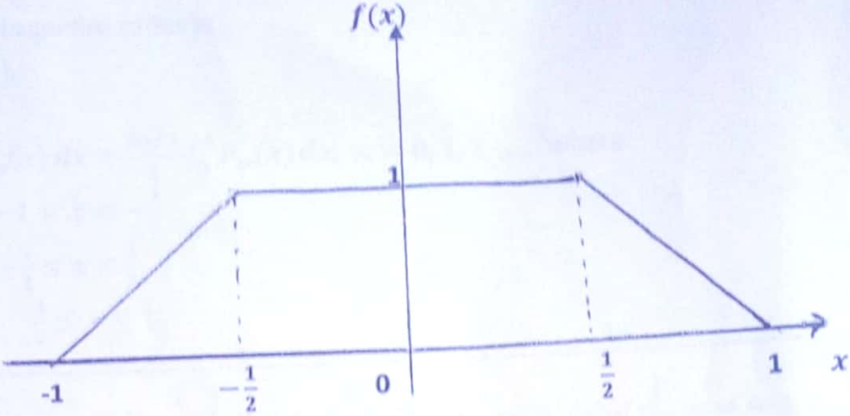
The Fourier-Legendre series converges to $f(x)$ anywhere on the interval $-1 < x < 1$ that $f(x)$ is continuous and converges to the mid-point of the jump at discontinuities.

When $f(x)$ is a polynomial of degree m, its Fourier-Legendre series must terminate at $c_m P_m(x)$.

xamples

xpand in a Fourier-Legendre series, the functions given by

1. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$.
2. $f(x) = 5 - 2x, \ -1 < x < 1.$
3.



olutions:

1. The required Fourier-Legendre series is

$f(x) = \sum_{n=0}^{\infty} c_n P_n(x),$

where

$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx,\ n = 0, 1, 2, \dots,\ n = 0, 1, 2, \dots$, and so

$c_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 dx = \left[\frac{1}{2}x\right]_0^1 = \frac{1}{2};$

$c_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x dx = \frac{3}{2}\left[\frac{1}{2}x^2\right]_0^1 = \frac{3}{4};$

$c_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2 - 1)dx = \left[\frac{5}{4}(x^3 - x)\right]_0^1 = 0;$

$c_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3 - 3x)dx = \left[\frac{7}{4}\left(\frac{5}{4}x^4 - \frac{3}{2}x^2\right)\right]_0^1 = -\frac{7}{16};$

$c_4 = \frac{9}{2}\int_0^1 P_4(x)\,dx = \frac{9}{2}\int_0^1 \frac{1}{8}(35x^4 - 30x^2 + 3)dx = \left[\frac{9}{16}(7x^5 - 10x^3 + 3x)\right]_0^1 = 0;$

$c_5 = \frac{11}{2}\int_0^1 P_5(x)\,dx = \frac{11}{2}\int_0^1 \frac{1}{8}(63x^5 - 70x^3 + 15x)dx = \left[\frac{11}{16}\left(\frac{63}{6}x^6 - \frac{35}{2}x^4 + \frac{15}{2}x^2\right)\right]_0^1 = \frac{11}{32}, \dots.$

Thus, the required Fourier-Legendre series is

$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + c_4P_4(x) + c_5P_5(x) + \cdots$, or

$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots.$

2. The required Fourier-Legendre series is

$f(x) = \sum_{n=0}^{\infty} c_n P_n(x),$

where

$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{2n+1}{2}\int_{-1}^{1}(5-2x)P_n(x)\,dx,\ n = 0,1,2,\ldots,$ $n = 0,1,2,\ldots,$ and so

$c_0 = \frac{1}{2}\int_{-1}^{1}(5-2x)P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1}(5-2x)dx = \left[\frac{1}{2}\left(5x - x^2\right)\right]_0^1 = 5;$

$c_1 = \frac{3}{2}\int_{-1}^{1}(5-2x)P_1(x)\,dx = \frac{3}{2}\int_{-1}^{1}(5-2x)x\,dx = \frac{3}{2}\left[\frac{5}{2}x^2 - \frac{2}{3}x^3\right]_0^1 = -2.$

Since $f(x) = 5 - 2x$ is a polynomial of degree 1, it follows that $c_n = 0,\ n = 2,3,4,\ldots$

Hence, the required Fourier-Legendre series is

$f(x) = c_0P_0(x) + c_1P_1(x)$, or

$f(x) = 5P_0(x) - 2P_1(x).$

3. The required Fourier-Legendre series is

$f(x) = \sum_{n=0}^{\infty} c_nP_n(x),$

where

$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx,\ n = 0,1,2,\ldots,$ where

$$f(x) = \begin{cases} 2(1+x), & -1 < x < -\frac{1}{2} \\ 1, & -\frac{1}{2} \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x < 1 \end{cases}.$$

Therefore,

$c_n = \frac{2n+1}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)P_n(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} P_n(x)\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)P_n(x)\,dx\right],\ n = 0,1,2,\ldots,$ and so

$c_0 = \frac{1}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)P_0(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} P_0(x)\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)P_0(x)\,dx\right] = \frac{1}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} dx + \int_{\frac{1}{2}}^{1} 2(1-x)\,dx\right] = \frac{1}{2}\left\{\left[(1+x)^2\right]_{-1}^{-\frac{1}{2}} + [x]_{-\frac{1}{2}}^{\frac{1}{2}} - \left[(1-x)^2\right]_{\frac{1}{2}}^{1}\right\} = \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{1}{4}\right] = \frac{3}{4};$

$c_1 = \frac{3}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)P_1(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} P_1(x)\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)P_1(x)\,dx\right] = \frac{3}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)x\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)x\,dx\right] = \frac{3}{2}\left\{\left[\frac{1}{2}x^2 + \frac{1}{3}x^3\right]_{-1}^{-\frac{1}{2}} + \left[\frac{1}{2}x^2\right]_{-\frac{1}{2}}^{\frac{1}{2}} + 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{\frac{1}{2}}^{1}\right\} = \frac{3}{2}\left[-\frac{1}{6} + 0 + \frac{1}{6}\right] = 0;$

$c_2 = \frac{5}{2}\left[\int_{-1}^{-\frac{1}{2}} 2(1+x)P_2(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} P_2(x)\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)P_2(x)\,dx\right] = \frac{5}{2}\left[\int_{-1}^{-\frac{1}{2}}(1+x)\left(3x^2-1\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\left(3x^2-1\right)dx + \int_{\frac{1}{2}}^{1}(1-x)\left(3x^2-1\right)dx\right] = \frac{5}{2}\left[\int_{-1}^{-\frac{1}{2}}\left(3x^3 + 3x^2 - x - 1\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\left(3x^2-1\right)dx + \int_{\frac{1}{2}}^{1}\left(-3x^3 + 3x^2 + x - 1\right)dx\right] = \frac{5}{2}\left\{\left[\frac{3}{4}x^4 + x^3 - \frac{1}{2}x^2 - x\right]_{-1}^{-\frac{1}{2}} + \left[\frac{1}{2}\left(x^3 - x\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \left[-\frac{3}{4}x^4 + x^3 + \frac{1}{2}x^2 - x\right]_{\frac{1}{2}}^{1}\right\} = \frac{5}{2}\left[\frac{3}{64} - \frac{3}{8} + \frac{3}{64}\right] = -\frac{45}{64};\ \ldots$

Thus, the required Fourier-Legendre series is

$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + \cdots,$ or

$f(x) = \frac{3}{4}P_0(x) - \frac{45}{64}P_2(x) + \cdots.$

xercises

xpand the following functions in Fourier-Legendre series.

$\cancel{\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]}$

$\lambda_n = n^2; \quad y_n(x) = \sin nx, \quad n = 1, 2, \dots$

$\lambda_n = n^2\pi^2; \quad y_n(x) = \begin{cases} \sin n\pi x, & n \text{ even} \\ \cos n\pi x, & n \text{ odd} \end{cases}, \quad n = 1, 2, \dots$

$\lambda_n = n^2; \quad y_0(x) = 1, \quad y_n(x) = \cos nx, \sin nx, \quad n = 1, 2, \dots$

$\lambda_n = \frac{1}{4}(1 + n^2\pi^2); \quad y_n(x) = x^{-\frac{1}{2}} \sin\left(\frac{1}{2} n\pi \ln x\right), \quad n = 1, 2, \dots$

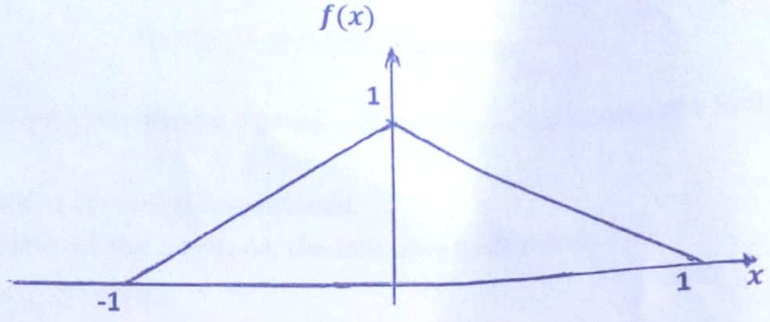
$\lambda_n = n^2\pi^2 - 2; \quad y_n(x) = \sin n\pi x, \quad n = 1, 2, \dots$

$\lambda_n = \left[\frac{(2n+1)\pi}{2}\right]^2; \quad y_n(x) = \sin\left[\frac{(2n+1)\pi}{2} \ln x\right], \quad n = 0, 1, 2, \dots$

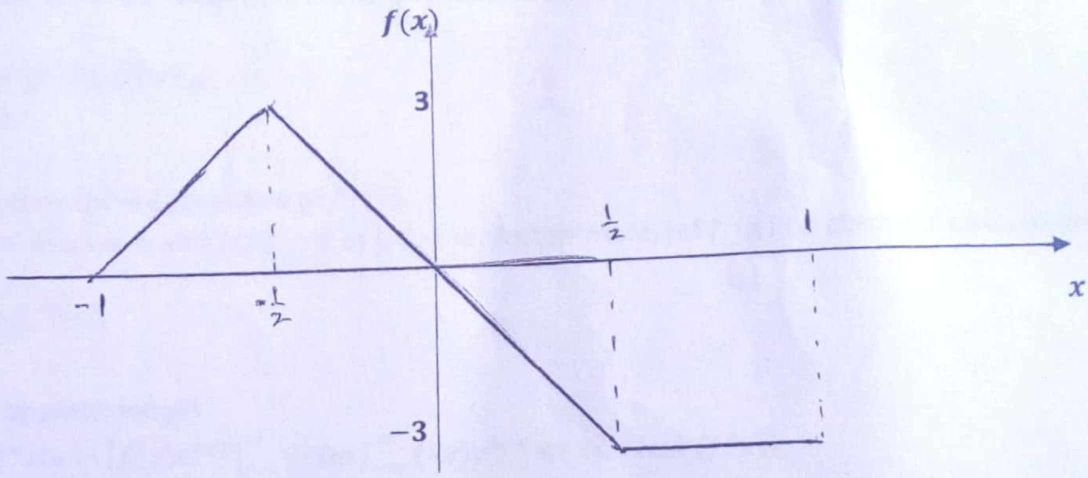
$\lambda_n = n^2\pi^2; \quad y_n(x) = \dfrac{\sin \frac{n\pi(1 - 2\ln x)}{2}}{\sin \frac{n\pi}{2}}, \quad n = 1, 2, \dots$

$\lambda_n = n^2; \quad y_n(x) = e^{-x} \sin nx, \quad n = 1, 2, \dots$

1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}$.
2. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$.
3. $f(x) = x - x^3$.
4. $f(x) = 3x^2 + x - 1$.
5. $f(x) = x^4$.
6.



7.



Fourier Transformation

The Fourier series representation of a function is useful for applications involving periodic functions or functions defined on a finite interval. Frequently, however, the functions of interest are non-periodic and defined for all values of the independent variable. In these situations, the Fourier transform is often employed.

The Fourier transform of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is given by the integral

$$\mathcal{F}\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx, \tag{1}$$

and

$$\mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-ix\omega}\, d\omega \tag{2}$$

is called the inverse Fourier transform of $F(\omega)$.

Note: The Fourier transform of $f(x)$ can also be defined as
$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx,$
and the inverse Fourier transform of $F(\omega)$ will then become
$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-ix\omega}\,d\omega.$
The process of obtaining the Fourier transform of a given function is called Fourier transformation. Similarly, for the inverse.

Existence of Fourier Transforms

The following two conditions are sufficient for the existence of the Fourier transform of a function $f(x)$ defined on the x-axis.

I. $f(x)$ is piecewise continuous on every finite interval.
II. $f(x)$ is absolutely integrable on the x-axis, i.e. the following limit exists:
$\lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx.$
When this limit exists, it is equal to $\int_{-\infty}^{\infty}|f(x)|\,dx.$

Theorem 1: (Linearity of the Fourier transformation)
The Fourier transformation is a linear operation, i.e. for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist, we have
$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\},$
for any constants a and b.

Theorem 2: (Fourier transform of the derivative of $f(x)$)
Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then
$\mathcal{F}\{f'(x)\} = -i\omega\mathcal{F}\{f(x)\}.$

Proof: Using integration by parts, we get
$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{i\omega x}\,dx = \left[f(x)e^{i\omega x}\right]_{-\infty}^{\infty} - i\omega\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = -i\omega\mathcal{F}\{f(x)\},$
since $f(x) \to 0$ as $|x| \to \infty$.

Note: By successive applications of integration by parts, we find that the Fourier transforms of higher derivatives of $f(x)$ are
$\mathcal{F}\{f''(x)\} = (-i\omega)^2\mathcal{F}\{f(x)\},$
$\mathcal{F}\{f'''(x)\} = (-i\omega)^3\mathcal{F}\{f(x)\}, \ldots,$
$\mathcal{F}\{f^{(n)}(x)\} = (-i\omega)^n\mathcal{F}\{f(x)\},$
provided $f'(x), f''(x), \ldots, f^{(n)}(x)$ each tends to zero as $|x| \to \infty$, respectively.

Convolution

The convolution $f * g$ of functions $f(x)$ and $g(x)$ is defined by

$$x) = (f * g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)\,d\tau = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)\,d\tau. \qquad (1)$$

he convolution is employed in obtaining the inverse of the product of two transform functions. If $F(\omega)$ and $G(\omega)$ are the respective Fourier transforms of $f(x)$ and $g(x)$, we have

$$\mathcal{F}^{-1}\{F(\omega)G(\omega)\} = f * g = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)\,d\tau.$$

Theorem 3: (Convolution theorem)

Let $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}.$$

Examples

Find the Fourier transform of the function $f(x)$.

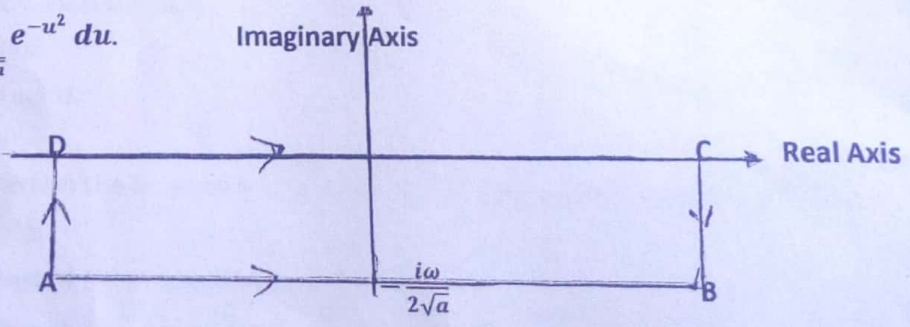
1. $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$.
2. $f(x) = \begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases}$.
3. $f(x) = e^{-ax^2}$, where $a > 0$.
4. $f(x) = xe^{-x^2}$.

Solutions:

1. The Fourier transform of $f(x)$ is
$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \int_0^a ke^{i\omega x}\,dx = \frac{k}{i\omega}\left[e^{i\omega x}\right]_0^a = \frac{k}{i\omega}\left(e^{i\omega a} - 1\right).$$

2. $$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \int_b^c e^{ax}e^{i\omega x}\,dx = \int_b^c e^{(a+i\omega)x}\,dx = \frac{1}{a+i\omega}\left[e^{(a+i\omega)x}\right]_b^c = \frac{1}{a+i\omega}\left[e^{(a+i\omega)c} - e^{(a+i\omega)b}\right].$$

3. $$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \int_{-\infty}^{\infty} e^{-ax^2}e^{i\omega x}\,dx = \int_{-\infty}^{\infty} e^{(-ax^2+i\omega x)}\,dx.$$
We complete the squares to obtain
$$-ax^2 + i\omega x = -a\left(x^2 - \frac{i\omega}{a}x\right) = -a\left[\left(x - \frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right] = -a\left(x - \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a},$$ and so
$$\mathcal{F}\left\{e^{-ax^2}\right\} = e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{i\omega}{2a}\right)^2}\,dx.$$
Let $u = \sqrt{a}\left(x - \frac{i\omega}{2a}\right) = \sqrt{a}x - \frac{i\omega}{2\sqrt{a}}$. $\Rightarrow du = \sqrt{a}dx$.
$x = \pm\infty \Rightarrow u = \pm\infty - \frac{i\omega}{2\sqrt{a}}$. Hence
$$\mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{4a}}\int_{-\infty-\frac{i\omega}{2\sqrt{a}}}^{\infty-\frac{i\omega}{2\sqrt{a}}} e^{-u^2}\,du.$$



The path of integration is A to B. This may be replaced by integration path D to C, since integration from A to D and that from C to B both t end to zero. This stems from the fact along these paths (A to D and C to B), $dx = du = 0$. Therefore,

$$\mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-u^2}\, du.$$

When we studied Gamma functions, we proved that $\int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$, and so $\int_{-\infty}^{\infty} e^{-u^2}\, du = 2\int_0^{\infty} e^{-u^2}\, du = \sqrt{\pi}$.

Thus,

$$\mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{4a}}.\sqrt{\pi} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}.$$

4. $\mathcal{F}\left\{xe^{-x^2}\right\} = \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\} = -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\} = -\frac{1}{2}(-i\omega)\mathcal{F}\left\{e^{-x^2}\right\} = \frac{i\omega}{2}\sqrt{\pi}e^{-\frac{\omega^2}{4}}$,

   since $\mathcal{F}\left\{e^{-ax^2}\right\} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$, where $a = 1$.

<u>Exercises</u>

Find the Fourier transforms of the following functions.

1. $f(x) = \begin{cases} e^{iax}, -b < x < b \\ 0, \ \text{otherwise} \end{cases}.$
2. $f(x) = \begin{cases} e^{-x}, \ x > 0 \\ 0, \ x < 0 \end{cases}.$
3. $f(x) = \begin{cases} x, & 0 < x < b \\ 2x - a, & b \le x < 2b. \\ 0, & \text{otherwise} \end{cases}$
4. Show that if the function $f(x)$ has a Fourier transform, so does $f(x-a)$, and $\mathcal{F}\{f(x-a)\} = e^{i\omega a}\mathcal{F}\{f(x)\}$.
5. Show that if $F(\omega)$ is the Fourier transform of the function $f(x)$, then $F(\omega - a)$ is the Fourier transform of the function $e^{-iax}f(x)$.

<u>Fourier Cosine and Sine Transforms</u>

If $f(x)$ is an even function, so that $f(-x) = f(x)$, then by symmetry,

$$\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx = 0. \qquad (1)$$

This is because $f(x)\sin\omega x$ is an odd function. Hence, the Fourier transform of $f(x)$ is

$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx = \int_{-\infty}^{\infty} f(x)\,(\cos\omega x + i\sin\omega x)\, dx,$$

which, on using equation (1), reduces to

$$\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx = 2\int_0^{\infty} f(x)\cos\omega x\, dx,$$

since $f(x)\cos\omega x$ is an even function.

Thus, if $f(x)$ is an even function defined on the interval $0 < x < \infty$, its Fourier cosine transform is defined as

$$\mathcal{F}_c\{f(x)\} = F(\omega) = \int_0^{\infty} f(x)\cos\omega x\, dx, \qquad (2)$$

and the inverse Fourier cosine transform of $F(\omega)$ is defined as

$$\mathcal{F}_c^{-1}\{F(\omega)\} = f(x) = \frac{2}{\pi}\int_0^{\infty} F(\omega)\cos x\omega\, d\omega. \qquad (3)$$

Similarly, if $f(x)$ is an odd function defined on the interval $0 < x < \infty$, its Fourier sine transform is defined as