ATLANTIS PRESS
Advances in Economics, Business and Management Research, volume 203
Proceedings of the 2021 3rd International Conference on Economic Management and Cultural Industry (ICEMCI 2021)

Comparison of the Applicability of Markowitz Model and Index Model Under 5 Real-World Constraints for Diverse Investors

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ABSTRACT
A notable trend observed within our research is that, despite the variations in actual values, both models have agreed on the stocks that are given the most extreme weights, either positive or negative. While the rest of the stocks, whose weights are closer than 0 and are used for nuanced balancing to arrive at the optimum, are allocated with significantly different weights as the models' underlying computations of standard deviation differ. Under the conditions of C1 to C3, Markowitz Model consistently performs better in predicting minimum variance portfolio, while Index Model is particularly apt at forecasting maximum Sharpe ratio portfolio. Such discrepancies, however, disappear under C4 and C5, where the Index Model claims greater potency in predicting both portfolios. However, their relative advantages over each other are never so significant under any constraint to demonstrate a consequential superiority hence arriving at an assertive conclusion. Nevertheless, our results may still provide an insight into the portfolio construction for diversified customers with multifarious optimization goals under eclectic circumstances.

Keywords: *Investment Management, Optimal Portfolio, Markowitz Model, Index Model, Investment Constraints.*

1. INTRODUCTION

The development of the finance industry has enabled the emergence of kinds of investment selections, such as the investment in mutual funds or index funds, and different investments have distinct constraints due to either meeting investors' preferences or adherence to regulations. Markowitz Model and Index Model have always been used in determining the optimal portfolios of stocks chosen and have been widely compared by scholars over the past years. Still, the comparisons have always based their conclusions on a general case rather than on considerations about the effects of practical variable constraints in the real world. In the real investment world, however, constraints applied by regulations or customers could influence the return and volatility of portfolios constructed and could influence the applicability of the Markowitz Model and Index model.

With the foundation of diversification, in 1952, an economist, Harry Markowitz, pioneered the Modern Portfolio Theory, also known as the Mean Variance portfolio theory [1]. The theory specifies a method an investor can use to construct a portfolio that gives the maximum return for a specified risk. Otherwise stated, the theory explains how an investor can construct a portfolio that yields the minimum risk for a specified return. The application of this theory first involves defining the properties of the portfolios available to the investor, that is, defining the opportunity set. Once this is done, the next aspect of considering is how the investor chooses one portfolio out of all the feasible portfolios in the opportunity set.

Various assumptions are usually made. It is assumed that the investor is risk-averse, non- satiated, and based his decision purely on expected return and variance. In addition, it is also assumed that there are no taxes or transaction costs and that the time-horizon is fixed. The following measures are therefore considered for portfolio optimization when using Markowitz's modern portfolio theory.

a) Mean/Expected portfolio return.

It refers to the weighted average of individual securities forming the portfolio.

b) Portfolio risk

As a result of uncertainty relating to future events, the risk is brought about, and it needs to be accounted for. Portfolio risk is measured by simply computing the standard deviation of the portfolio. Standard deviation is the square root of the variance. The portfolio variance can be defined as the measure of how the combined real returns of a collection of securities making up a portfolio vary from time to time. This statistic uses the standard deviations of every single asset in the portfolio and the correlations of every security duo in the portfolio.

The lower the covariance between the securities, the lower the overall variance of the portfolio; hence the lower the standard deviation. This means that the variance of a portfolio can be reduced by investing in securities whose returns are uncorrelated, that is, by diversification.

c) The optimal portfolio

In economics, it is assumed that the higher the risk, the higher the expected level of return, or the lower the risk, the lower the expected level of return. According to Markowitz's Mean-Variance theory, it's possible to construct an optimal or efficient portfolio that balances between risk and return.

When a plot of risk against expected return is made, the optimal portfolios' points are known as the "efficient frontier". Portfolios that lie outside the efficient frontier, that is, above or below the efficient frontier, are said to be sub-optimal.

Between 1962 and 1964, the Capital Asset Pricing Model (CAPM), built based on Markowitz's work, was introduced by William Sharpe, John Lintner, and Jan Mossin, which has been used as a base for investment tools and theories today [2]. In addition to the assumptions of the Modern Portfolio Theory, the model assumes that investors have the same single-period horizon, all investors can lend or borrow unlimited amounts at the same risk-free rate, and that the market for risky assets is perfect/ efficient.

Index tracking is a form of passive portfolio management. Rather than trying to surpass a predetermined index as in active portfolio management, a portfolio manager can instead try to gain returns as close as possible to that of a theoretical portfolio such as the S&P 500 index. It is assumed that financial markets are not fully efficient in active portfolio management, whereas, in passive portfolio management, the assumption is that markets are efficient.

Therefore, we see that index tacking is also in line with the capital asset pricing model as a passive strategy. Index funds and mutual funds have grown tremendously since the 1980s, with the theoretical justification being provided by the Modern Portfolio Theory [3].

Therefore, index-tracking can be defined as an investment approach whose major aim is keeping the portfolio return as close as possible to a target index without purchasing all the index components [4]. The investor is interested in finding the portfolio that reduces the tracking error between the returns of the selected portfolio and the benchmark index as much as possible. The risk measure of how closely the portfolio follows the benchmark index return is referred to as the "tracking error" [5]. Such a portfolio, which an investor can use to track the index, is known as the "index portfolio" [6].

A realistic formulation of the index-tracking problem should include the number of assets in the portfolio, the restrictions on the position of each asset, the size of the transaction costs, and finally, the liquidity and exposure constraints [7]. However, it has been observed that the main source of tracking error is the transaction costs which is increased by the liquidity and exposure constraints [8], avoids the discussion on transaction costs as well as other costs under the condition that the number of assets in the portfolio is small relative to the number of assets in the market index. A high tracking error means that the index is not being followed accurately by the portfolio [6].

Index tacking strategies continue to be developed as a solution to the index-tracking problem. For instance, heuristic optimization techniques whose major advantage is that the objective function and the constraints of the index-tracking model are barely subjected to any restrictions. Some forms of heuristic optimizations algorithms have been proven to be efficient for index-tracking, like the threshold accepting (TA) algorithm, which Dueck and Winker and Gilli and Kellezi applied [9]. Other interesting techniques used are the co-integration approach [10] and the stochastic approach.

The structure of the paper is organized as follows: Section 2 introduces the firm stocks and data; Section 3 describes the method to build permissible portfolios, explaining the full Markowitz Model, the Index Model, and the five optimization additional constraints; Section 4 shows the result analysis, obtaining the optimal weights of both the Markowitz Model and the Index model and comparing them separately under five additional constraints; at last, we present our conclusion, reflect the flaws of our paper, and plan the further research.

2. DATA

To gain as approximate outcomes under real investment situations as possible, we have selected 20-year daily stock prices of 10 companies from 4 different industries, AMZN, AAPL, CTXS, JPM, BRK/A, PGR, UPS, FDX, JBHT, LSTR. We have also used SPX 500 index and 1-month federal funds rate data as market index data and risk-free rate return in our two models to link our simulation more closely with real-life diversified investment products. (Table 1). Stocks are chosen from 4 different industries, limiting the possibility that covariance of stocks is so high that the optimal portfolios constructed by the Markowitz Model will be too distorted to be compared with that by the Index Model.

To compare the portfolios constructed by the Markowitz Model and Index Model under different constraints, we are trying to diminish the effects of other factors that might misdirect the results. When dealing with daily returns, we have found a quite significant deviation from the Gaussian distribution. Therefore, we have turned to use monthly return data in lieu of the daily return for minimising the non-Gaussian effect. When taking SPX index return data as an example, the normal distribution of monthly return (Fig. 1) is more approximate to the Gaussian distribution than that of the daily return (Fig. 14).

Table 1. Stock Introduction (Source: Yahoo! Finance)

	Stock Code	Company Name	Sector
Market Index	SPX	-	-
Stock #1	AMZN	Amazon.com, Inc.	Consumer Cyclical
Stock #2	AAPL	Apple Inc.	Technology
Stock #3	CTXS	Citrix Systems, Inc.	Technology
Stock #4	JPM	JPMorgan Chase & Co.	Financial Services
Stock #5	BRK/A	Berkshire Hathaway Inc.	Financial Services
Stock #6	PGR	The Progressive Corporation	Financial Services
Stock #7	UPS	United Parcel Service, Inc.	Industrials
Stock #8	FDX	FedEx Corporation	Industrials
Stock #9	JBHT	J.B. Hunt Transport Services, Inc.	Industrials
Stock #10	LSTR	Landstar System, Inc.	Industrials
Risk-free rate	FEDL01	-	-

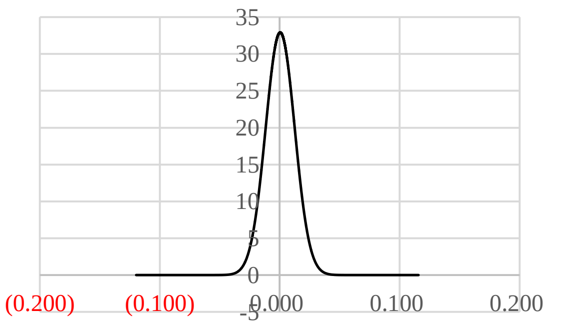

Fig 1. Normal Distribution of SPX Daily Return (%)

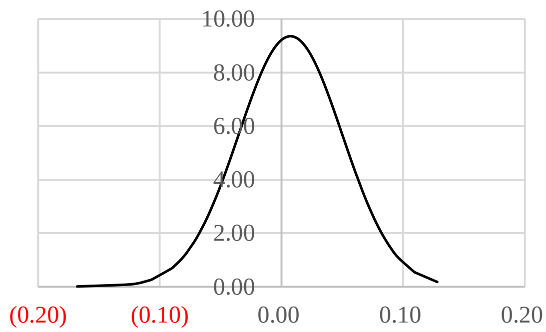

Fig 2. Normal Distribution of SPX Monthly Return (%)

3. MATH AND EQUATIONS

This project generalized the permissible portfolios of ten stocks using the full Markowitz Model and Index Model under five additional constraints. In the following, we will separately explain the materials, the two models, and the five constraints.

We mainly aimed to obtain the Minimal Risk or Variance Frontier, Efficient Frontier, and Minimal Return Frontier. As shown in section 2, after generating the monthly observation of the ten stocks, we calculated the excess return and residual returns. Then we used Excel to get the return, standard deviation, and the Sharpe ratio of the stocks. With the two models and the five constraints in mind, the main tool that we used to solve the optimization problems is the Excel Solver. For example, when we were trying to find the Minimal Risk or Variance Frontier, first, we set up the standard deviations as objectives. Then we selected the weights of the assets as the changing variables. Finally, we set dummy variables and typed in the constraints. We clicked the button of running Excel Solver and used the result to plot the Permissible Portfolios. On the one hand, we compared the five different constraints for each optimization problem (MM and IM model). On the other hand, we compared the two optimization problem solutions (MM and IM model) for each same constraint.

3.1. The Markowitz Model

The Markowitz Model attempts to maximise the portfolio's expected return using the approach of the following three parts. First, determine the Minimal Variance Frontier, which is the opportunity set. Second, identify the optimal risky portfolio, when the steepest capital allocation line is tangential to the opportunity set. And finally, choose the appropriate portfolio by mixing with the risk-free asset given risk-aversion. The formula is

$$min\sigma^2(r_p) \quad = \sum\sum x_i x_j \, Cov(r_i - r_j) \quad (1)$$

$$r_p = \sum x_i r_i \quad (2)$$

The full Markowitz Model requires huge amounts of estimates which embodies greater uncertainty, and it is also regarded with investors' behaviours.

3.2. The Index Model

The index model is an asset pricing model to measure both the risk and the return of a stock. It greatly simplifies the estimation of the covariance matrix and enhances the analysis of security risk premium. However, the Index Model is based on the assumption that return residuals are uncorrelated. That is, stocks are correlated only through the general performance of the macroeconomy. The formula is

$$\gamma_{it} - \gamma_f = \alpha_i + \beta(\gamma_{mt} - \gamma_f) + \epsilon_i \quad (3)$$

$$\gamma_i = E\ (\gamma_i) + m_i + e_i \quad (4)$$

a) r_{it} is return to stock i in period t

b) r_f is the risk-free rate (i.e., the interest rate on treasury bills)

c) r_{mt} is the return to the market portfolio in period t

d) α_i is the stock's alpha or abnormal return

e) β_i is the stocks' beta or responsiveness to the market return

f) ϵ_{it} is the residual (random) return

3.3. Additional Five Constraints

The first optimization constraint is designed to simulate Regulation T by FINRA, allowing broker-dealers to allow their customers to have positions, 50% or more of which are funded by the customer's account equity. The second optimization constraint is designed to simulate some arbitrary "box" constraints on weights, which the client may provide. Third, a "free" problem, without any additional optimization constraints, illustrates how the area of permissible portfolios in general and the efficient frontier, particularly, looks like if you have no constraints. The fourth additional optimization constraint is designed to simulate the typical limitations existing in the U.S. mutual fund industry: a U.S. open-ended mutual fund is not allowed to have any short positions. Lastly, we would like to see if the inclusion of the broad index into our portfolio has a positive or negative effect.

4. RESULT ANALYSIS

In this section, we predict the optimal weights for both Markowitz Model and Index Model, on each individual stock under different constraints that resolve three optimization problems: portfolios constructed to give the minimum expected variance (risk), maximum expected Sharpe ratio (efficiency), and maximum expected excess return. Based on the estimates obtained from following the procedures described in Section 3, we can apply two models with weights on individual stock i as independent variables, hence arrive at the specific combinations of stock wights, which allows the optimization problems to be met within different constraints. Table 2 presents the results derived from fundamental manipulations of our raw data that constitute the underlying parameters for our models. These parameters themselves nevertheless shed light on the weight composition, as we shall see that alpha plays a paramount role in predicting maximum return portfolios.

Table 2. Stock Parameters

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
Annualized Average Return	7.542%	33.796%	34.016%	15.636%	11.865%	9.002%	15.406%	9.850%	12.952%	22.525%	17.387%
StDev	14.850%	41.410%	34.452%	41.514%	29.013%	16.243%	21.059%	21.437%	26.690%	30.654%	23.924%
Beta	1.000	1.351	1.257	1.221	1.361	0.572	0.712	0.830	1.104	1.076	0.798
Alpha	0.000%	23.604%	24.536%	6.430%	1.601%	4.687%	10.036%	3.592%	4.628%	14.407%	11.372%
Residual StDev	0.000%	36.223%	28.957%	37.348%	20.818%	13.844%	18.212%	17.543%	21.066%	26.158%	20.786%

4.1 Constraint 1 (C1) - The 'Free' Constraint

We begin the data analysis with this constraint since it involves none of the extraneous limitations. Hence it is apt to serve as a benchmark for the purpose of comparison, to which we may frequently refer in the latter analysis.

Table 3&4 demonstrates the weight estimates that result in portfolios satisfying minimum variance (risk) and maximum Sharpe ratio, correspondingly. Note that the optimising weights which maximise expected excess return are generally unobtainable. This is presumably due to the nonexistent limit on investors' positions under C1: the manager could theoretically construct a portfolio that unboundedly expands the wights on these stocks to long the promising stocks, and conversely short the ones that are looked down - as long as the intrinsic requirement, that the weights placed on all stocks sum up to one, is satisfied. Such a condition is uncontrovertibly unpractical. It nevertheless explores one possibility that one should not ignore: once certain restraints are placed to leverage level, as we shall see in the cases of C2 and C3, the weights that optimize this maximisation problem may appear pertinent to investors' interest.

The estimation results of the Markowitz Model show that a portfolio can be constructed to generate the global minimum variance that is as low as 12.24%, given the investor is highly risk-averse and seeks the safest, hence least turbulent, risky portfolio regardless of the premium. The maximum Sharpe ratio portfolio, on the other hand, reveals a carefully balanced relationship between risk and return so that the portfolio constructed is most efficient, with the Sharpe ratio being equal to 1.5385, that 1.5385% of return is paid as compensation for one percent more risk you bear at the point. Not surprisingly, the latter embodies a much higher risk of standard deviation being 32.35% - nearly three times of the former's. In comparison, it forecasts an even larger gain in remuneration: the expected excess return is 49.61%, almost seven times the former's 7.15%. The disproportionately greater gain in return underscores the portfolio's efficiency again: the percentage increase in return exceeds the extra risk investor must undertake. Hence a more rational customer with less risk-aversion may prefer the latter. A closer examination of the weights reveals the underlying rationale: SPX, as a nearly well-diversified index asset, possesses the lowest standard deviation of 14.85%, according to Table 1. To obtain the lowest overall risk of the whole portfolio, greater weight is placed on SPX till the point where $w_{SPX}\sigma_{SPX}$ catches up with the rest so that it is no longer attractive, which eventually leads us to the point where SPX accounts for 72.24% of the total portfolio. Once we renounce the objective of minimising risk and seek to maximise efficiency, SPX is abnegated and shorted heavily with a weight of -2.3752 since it has the lowest alpha being equal to 0, and to leave rooms for other assets such as BRK/A, whose weight surges from 0.3621 to 0.9159.

The Index Model presents rather similar outlines: the predictions for minimum variance achievable being 12.43%, and the maximum Sharpe ratio being 1.5960. This, however, does not imply similar compositions for these portfolios: the maximum Sharpe portfolio, for instance, involves a much more negative position with a weight of -3.298 on SPX, whereas the wights on the other assets depart from and are greater than the ones predicted by Markowitz Model. Consequently, this produces a portfolio that embodies higher risk and premium, even though the eventual efficiency - the quotation of these two - remains alike.

In general, Markowitz Model plays a slightly better role in predicting the minimum achievable standard deviation, which is merely 0.19% less than Index Model's prediction;. At the same time, the latter presents a more favourable potential maximum efficiency, which is 0.0575 greater than the former's prediction.

Table 3. Construction Predictions for Markowitz Model Under C1

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.7224	-0.0235	-0.0385	-0.0104	-0.1847	0.3621	0.1391	0.0343	-0.1028	-0.0055	0.1075	7.15%	12.24%	0.5839
Max Sharpe	-2.3752	0.3706	0.6540	0.0042	0.1730	0.9159	0.6819	0.0127	-0.0869	0.3097	0.3402	49.61%	32.25%	1.5385

Table 4. Construction Predictions for Index Model Under C1

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.6579	-0.0414	-0.0474	-0.0244	-0.1287	0.3451	0.1342	0.0856	-0.0361	-0.0172	0.0724	6.46%	12.43%	0.5200
Max Sharpe	-3.2979	0.4302	0.6996	0.1102	0.0883	0.5848	0.7235	0.2791	0.2494	0.5035	0.6293	60.91%	38.16%	1.5960

4.2 Constraint 2 (C2) - The FINRA Constraint

This constraint considers a more realistic condition that investors are prohibited from overly leveraging their assets more than double their initial equity. One conspicuous point is that the weight compositions of minimum variance portfolios for both models are identical to those under C1, as they did not involve overly radical leverage. Hence they are not bounded and affected by this change in constraint.

The maximum Sharpe portfolio, on the other hand, is restrained by C2: a noticeable change is in the weights of SPX, that Markowitz Model suggests a dramatic reduction in the short position from the previous level of -2.3752 to -0.4845, and similarly, the Index Model rises its weight from -3.2979 to -0.4730. By preventing the portfolio from fully exploiting the benefits of leverage, ceteris paribus, C2 should construct a less efficient portfolio which is exactly what we observe: the maximum potential Sharpe ratio drops to 1.414 for Markowitz Model and to 1.430 for Index Model.

Unlike C1, the expected risk premium will not be able to increase unboundedly under the circumstance of C2 since the manager cannot arbitrarily place as much weight as possible on stocks. This allows us to arrive at the weight composition of the maximum expected return portfolio: the two models should propose the same weight composition due to the fact that they are only differentiated in the computation of standard deviation. The return optimization should produce identical conclusions with the same underlying inputs such as alpha and beta, which our results have confirmed. Despite possessing as large a premium as 47.23%, both models predict the potential risk exposure, not surprisingly, to be the largest among three: with values of 47.49% and 48.00%, respectively. Like the maximum Sharpe portfolio, it decides to short the more conservative stocks such as SPX, UPS, and FDX, but then invest heavily in technology companies such as AAPL to the extent where the weight accounts for 149.72% of the total portfolio. Interestingly, it still provides a Sharpe ratio of 0.984, which is slightly below 1, hence for customers who are close to risk-neutral, yet not seeking efficiency. However, simply prospective gains, such radical tactic may still be plausible.

Under C2, Markowitz Model presents a more auspicious minimum risk due to the unchanged weight construction to C1, and similarly Index Model forecasts a greater efficiency of Sharpe ratio by 0.0164. Interestingly, despite performing poorly in predicting maximum Sharpe ratio portfolio, Markowitz Model seems to be more optimistic in delivering the Sharpe ratio for maximum return portfolio, which is 0.0105 greater than the one of Index Models.

Table 5. Construction Predictions for Markowitz Model Under C2

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.7224	-0.0235	-0.0385	-0.0104	-0.1847	0.3621	0.1391	0.0343	-0.1028	-0.0055	0.1075	7.15%	12.24%	0.5839
Max Sharpe	-0.4845	0.1642	0.3003	-0.0003	-0.0003	0.4134	0.3296	-0.0001	-0.0146	0.1249	0.1674	26.43%	18.70%	1.4136
Max Return	-0.4972	0.0021	1.4972	-0.0001	-0.0003	-0.0007	-0.0002	-0.0005	-0.0003	0.0000	-0.0001	47.23%	47.49%	0.9944

Table 6. Construction Predictions for Index Model Under C2

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.6579	-0.0414	-0.0474	-0.0244	-0.1287	0.3451	0.1342	0.0856	-0.0361	-0.0172	0.0724	6.46%	12.43%	0.5200
Max Sharpe	-0.4730	0.1784	0.3052	0.0050	-0.0268	0.2244	0.3153	0.0117	0.0011	0.1920	0.2666	28.57%	19.98%	1.4300
Max Return	-0.3015	0.6406	0.8296	-0.0003	-0.0008	-0.1693	-0.0005	-0.0270	-0.0005	0.0291	0.0005	47.23%	48.00%	0.9839

4.3 Constraint 3 (C3) - Arbitrary' Box' Constraint

In analogous to C2, C3 places some degrees of limitation on customers' potential positions despite not being as strong. The weight threshold of 1 applied here is arbitrary to serve the simulation purpose: individual customers may encounter multifarious constraints specific to themselves, while C3 is to mimic such an incidence when the customers are prohibited from the position on any single asset that exceeds their account equity. Again, the constraint is fortunately not binding for a minimum risk portfolio for both models. Therefore C1-C3 share the same weight arrangements for this optimization problem.

Moving to the second optimization, however, we shall see the fact that C3 is less suppressive than C2: the maximum Sharpe ratio achievable of Markowitz Model is predicted to be 1.501 instead of 1.414 in C2, and 1.521 comparing to the previous 1.430 for Index Model. Both models achieve this by allowing the weight on SPX to decrease to -1 where the constraint is binding and consequently allocate more to the ones with previously positive weight (AMZN, AAPL, BRK/A, etc.), and conversely reduce the weights on the negative ones with small tweaks (JPM, UPS, etc.).

Finally, the weight composition for the maximum return portfolio again unambiguously demonstrates what is absent under C1. Observing from Table 7&8, both models predict a consistent arrangement where all weights have hit the constraint, either the upper or the lower bound. However, the five assets placed with the weight of -1 are the exact ones that obtain the smallest predicted alphas. Even though all the non-market premiums of these stocks are positive, the model would seek to long the ones that have the greatest alpha as much as possible. The Solver was following the same logic under C3 where it reports an error since unbounded positions can lead to infinite weights. Finally, sharing the same expected risk premium, both models predict a Sharps ratio around 1 as the standard deviations are computed differently: Markowitz Model suggests a higher value of 91.966% than Index Model does for its 83.059%.

To sum up, under the circumstance of C3, Markowitz Model maintains its advantage in predicting minimum risk, and Index Model is more suitable for maximum efficiency prediction, whose Sharpe ratio exceeds the former's by 0.0203. In contrast to C2, it is Index Model which predicts a greater efficiency for the maximum return portfolio, which is 0.1021 greater than the one suggested by Markowitz Model.

Table 7. Construction Predictions for Markowitz Model Under C3

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.7224	-0.0235	-0.0385	-0.0104	-0.1847	0.3621	0.1391	0.0343	-0.1028	-0.0055	0.1075	7.15%	12.24%	0.5839
Max Sharpe	-1.0000	0.2233	0.3976	-0.0120	-0.0050	0.6248	0.4601	-0.0311	-0.1056	0.2088	0.2391	33.18%	22.11%	1.5007
Max Return	-1.0000	1.0000	1.0000	1.0000	-1.0000	-1.0000	1.0000	-1.0000	-1.0000	1.0000	1.0000	87.55%	91.97%	0.9520

Table 8. Construction Predictions for Index Model Under C3

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.6579	-0.0414	-0.0474	-0.0244	-0.1287	0.3451	0.1342	0.0856	-0.0361	-0.0172	0.0724	6.46%	12.43%	0.5200
Max Sharpe	-1.0000	0.2185	0.3662	0.0309	-0.0879	0.3332	0.3998	0.0952	0.0562	0.2497	0.3383	34.07%	22.40%	1.5210
Max Return	-1.0000	1.0000	1.0000	1.0000	-1.0000	-1.0000	1.0000	-1.0000	-1.0000	1.0000	1.0000	87.55%	83.06%	1.0541

4.4 Constraint 4 (C4) - Typical U.S. Open-Ended Mutual Fund Constraint

Tables 9&10 present the outputs of the models adhering to the typical U.S. open-ended mutual fund regulation, which effectively prohibits both short and long positions. The models generate relatively similar predictions on both weight allocations and their optimising subject. BRK/A and SPX are given the highest weights for minimum risk portfolios while all the tech stocks are emptied. With a closer inspection at Table 2, one may discover the underlying rationale, since the Solver selected five assets with the least tribulation, despite their ostensibly high covariance: three out of the five industrials were chosen albeit our intuitive misgivings about high interdependency for the Markowitz Model. The estimation of standard deviation is rather close, with 12.99% for the Index Model while 13.09% for the other. Moving towards a maximum efficiency portfolio, both models seem to forsake SPX as a safeguard but embrace AMZN and AAPL to effectively promote efficiency. Markowitz Model reports a prospective Sharpe ratio of 1.2536, while the Index Model suggests 1.2766. As for maximum return portfolio, both models unsurprisingly choose to include the only AAPL due to its largest annualised average return, resulting in an expected portfolio return of -34.016% given the limited freedom of allocating the weighted sum of 1.

Within the fourth constraint, we may see that Index Model actually performs consistently better than Markowitz Model. Although the discrepancies are not astoundingly large: Index Model predicts a minimum variance achievable 0.1% less than Markowitz, while the maximum Sharpe ratio is 0.023 greater. Since the maximum return portfolios for both models consist of a single stock with identical weight, all the estimates for such portfolios are equal to the stock's intrinsic parameters within Table 2.

Table 9. Construction Predictions for Markowitz Model Under C4

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.3847	0.0000	0.0000	0.0000	0.0000	0.3849	0.1423	0.0072	0.0000	0.0000	0.0809	10.04%	13.09%	0.7669
Max Sharpe	0.0000	0.1296	0.2521	0.0000	0.0000	0.1926	0.2272	0.0000	0.0000	0.0881	0.1104	22.09%	17.62%	1.2536
Max Return	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	34.02%	34.45%	0.9873

Table 10. Construction Predictions for Index Model Under C4

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.3046	0.0000	0.0000	0.0000	0.0000	0.3765	0.1464	0.0934	0.0000	0.0000	0.0790	10.24%	12.99%	0.7883
Max Sharpe	0.0000	0.1546	0.2712	0.0000	0.0000	0.0326	0.2136	0.0000	0.0000	0.1398	0.1882	24.46%	19.16%	1.2766
Max Return	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	34.02%	34.45%	0.9873

4.5 Constraint 5 (C5) - Exclusion of the Index Asset

Tables 11&12 would illustrate the wight predictions if SPX was not to be considered as an option. Other than that, the constraint is equivalent to C3. The consequence is apparent: without SPX as a means for further diversification, the minimum risk achievable is higher in both models, with 13.39% for the Markowitz and 13.21% for the Index. Both have placed the largest weight on BRK/A, whose StDev is unquestionably the least among all stock, except for SPX. The second-largest weight is placed on PGR subsequently, whose StDev, as one may easily conjecture, is the second-lowest of 21.059%.

Compared to their previous level of 1.53-ish under C1, the maximum Sharpe ratios are unambiguously lower at the current level of 1.3261 and 1.3311. The weight compositions, however, are disparate between the two. When arranged in descending order, Markowitz Model places the largest weights on BRK/A, PGR, and AAPL, whereas Index Model focuses on AAPL, PGR, and LSTR.

With the absence of SPX, the comparative advantage of the Markowitz Model in predicting minimum variance portfolio seem to depart from what we observed in C1. As Index Model now allows an even less standard deviation of 13.21%, which is 0.18% less than the Markowitz.

Table 11. Construction Predictions for Markowitz Model Under C5

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.0000	0.0245	0.0419	0.0084	-0.0709	0.5631	0.2352	0.1141	-0.0805	0.0060	0.1582	13.20%	13.39%	0.9858
Max Sharpe	0.0000	0.1465	0.2673	-0.0344	-0.1554	0.3635	0.3199	-0.1217	-0.1321	0.1783	0.1681	23.89%	18.02%	1.3261

Table 12. Construction Predictions for Markowitz Model Under C5

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	RETURN	STDEV	SHARP
Min Var	0.0000	-0.0155	-0.0081	-0.0011	-0.0502	0.4799	0.2165	0.1782	0.0345	0.0282	0.1376	11.47%	13.21%	0.8685
Max Sharpe	0.0000	0.1856	0.3169	-0.0047	-0.2123	0.1167	0.2766	-0.0530	-0.0540	0.1857	0.2425	26.95%	20.24%	1.3311

5. CONCLUSION

In this paper, we mainly researched the influences of using different models (Markowitz Model and Index Model) and how the influences change under different constraints on the eventual optimal portfolios that involved 10 stocks, SPX index, and 1-month federal funds rate. Based on the Markowitz Model and Index Model and aiming at studying practical questions that could occur in real life, we proceeded with 20-year monthly return data of 10 stocks, SPX index, and 1-month federal funds rate, and then established efficient frontiers within 5 constraints and 2 models separately by excel solver. Finally, we compared the portfolios' returns, standard deviations, and sharp ratios to evaluate the influences of different models and under variable constraints. Some conclusions are proposed accordingly.

Based on our results analysis presented above, both the Markowitz and the Index models indicate that we should embrace AMZN, AAPL, CTXS, PGR, JBHT, and LSTR, which result in positive returns for our portfolio selection purposes. Both models give us similar Sharpe Ratios with similar standard deviations concerning both portfolios. Yet, the Markowitz model may be a better fit for our purposes with a lower standard deviation, which suggests lower variance.

One drawback with the Markowitz model is that the variance of a portfolio is not a complete measure of the risk taken by the investor. What is the Value at Risk for a given portfolio? That is impossible to answer if only the variance and mean and not the distribution are known. Hence the Markowitz model does not tell an investor which portfolio he/she can afford to buy if he/she is willing to take a certain risk to get bankrupted. Suppose an investor wants to use the Markowitz model to choose a suitable portfolio. In that case, it is probably a good idea to do some complementary calculations of the risk of extreme events using extreme value statistics.

In terms of the Index Model, it is important to note that risks in the stock market cannot be diversified away; such risks are termed systematic risk. Unlike the unsystematic risks that occur due to company-specific features, systematic risks result from general market factors. In other words, they are risks affecting the entire industry, and they cannot be solved or reduced through the diversification process. Therefore, portfolio managers must clearly understand the type of risk facing their stock. The understanding will help in enacting proper and effective measures. Additionally, the portfolio managers will be able to save their companies from undergoing unnecessary costs and losses. For instance, it will be irrational to use a diversification process to mitigate systematic risks. Instead, such risks are beyond the organization's control, and they can only be controlled at the government level. Besides, unsystematic risks can be managed through the collaboration of all market players through the formation of adequate policies. Therefore, readers of this current portfolio analysis need to interpret our results with active reflections and considerations.

The shortcomings of the paper are correlated with the flaws of the two models we used. The Markowitz Model requires a great number of estimates, which can cause considerable estimation errors. The index model has the precondition that the return residuals don't play a role. We assume the stocks are only correlated with the performance of the macroeconomy and neglecting the fact that firms also interact within the same or similar sectors. In the future, we will specifically figure out and summarize which model is more suitable under different circumstances.

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