Macroeconomics III - Problem Set 1

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1 Literature Review: Mian, Straub and Sufi (2021)

What is the relationship between savings and wealth inequality? Mian, Straub and Sufi (2021) offer valuable answers using data from the United States. This paper is part of a wider literature about what is known as a "global savings glut", the idea that excess savings have contributed to global low-interest rate environments (pre-2020), which increased public debt and depressed output levels. The authors contribution is not only to ask *who* holds this excess in savings and in which *assets*, but also *who is financing whom* by undertaking what they call an "unveiling of the financial sector".

Their main finding is a "saving glut of the rich": The figure below shows the scaled saving rate of the top 1%, next 9%, and bottom 90%. Whether when looking at the income (panels a[1] and b[2]) or wealth (panel c[3]) distribution, it is shown that the top 1% have increased savings since the 1980s, while the bottom 90% have dissaved considerably. Savings were more stable for the "middle" 9%. The authors find that excess saving by the rich has not led to an increase in domestic investment, but has instead financed higher public and household debt levels, whosej' the main driver was the rise (decline) in accumulation of financial assets by the top 1% (bottom 90%). By calculating household net debt positions across the wealth distribution, they find that the top 1% were just as important in financing US government debt than foreign capital inflows *combined*.

We have selected this paper because we believe that it addresses important questions about how savings affect wealth accumulation and wealth distribution. If excess saving by the rich is driven by claims on public and private debt, low interest rates and higher borrowing will make it more difficult for the bottom 90% to accumulate wealth, raising questions about persistent inequalities and the cost of elevated public debt in the long run.

[1] from the DINA (Distributional National Account microfiles)
[2] from the CBO (Congressional Budget Office)
[3] from the DFA (Distributional Financial Account)

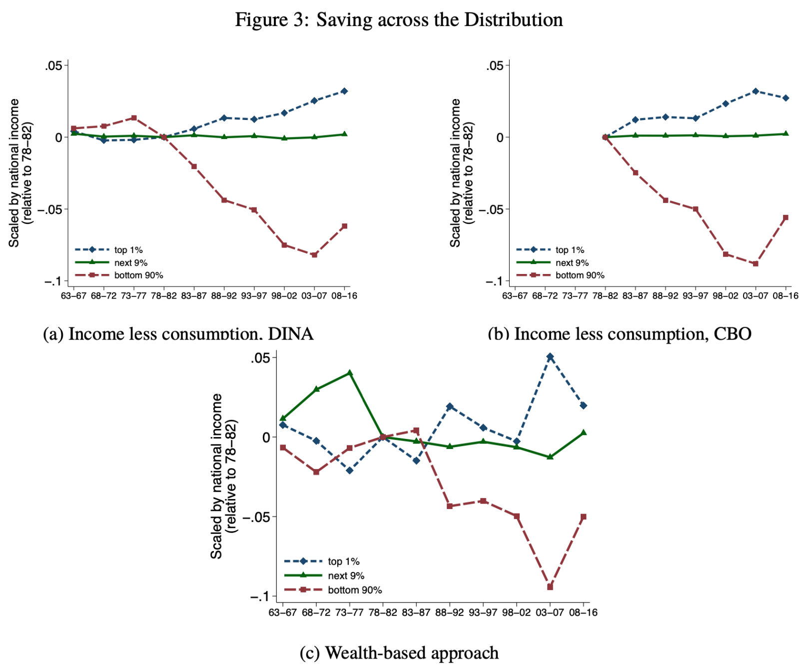

Figure 1: Mian, Straub and Sufi (2021), page 18.

2 Problem 1

Consider an economy where the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital. The consumer's utility function is:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where $\beta \in (0,1)$. The consumer is also endowed with k_0 units of capital in the first period. The feasible allocations satisfy:

$$c_t + k_{t+1} \leq \theta k_t^\alpha \quad \forall t \geq 0$$

where $0 < \alpha < 1$ and $\theta > 0$. We also have the following constraints:

$$c_t, k_t \geq 0$$

1. Write the problem in recursive form. Specify the Bellman equation, the state variable(s), the control variable(s), and the feasible set(s).

   The Bellman equation is:

   $$V(k_t) = \max_{c_t, k_{t+1}} [\log(c_t) + \beta V(k_{t+1})]$$

   subject to:

   $$c_t + k_{t+1} \leq \theta k_t^\alpha$$

   We can also write it as:

   $$V(k) = \max_{c, k'} [\log(c) + \beta V(k')]$$

   subject to:

   $$c + k' \leq \theta k^\alpha$$

   Thus:

   $$V(k) = \max_{0 \leq k' \leq \theta k^\alpha} [\log(\theta k^\alpha - k') + \beta V(k')]$$

   subject to:

   $$c + k' \leq \theta k^\alpha$$

   State variable: k
   Control variables: c, k'
   Feasible set: $k' \in \Gamma(k) \quad \text{where} \quad \Gamma(k) = [0, \theta k^\alpha]$

2. Euler equation:

(a) Derive the F.O.C.

From the Bellman equation:

$$V(k) = \max_{k'} [\log(\theta k^\alpha - k') + \beta V(k')]$$

The first-order condition is obtained by differentiating with respect to k':

$$-\frac{1}{\theta k^\alpha - k'} + \beta V'(k') = 0$$

Thus, the F.O.C. is:

$$\frac{1}{\theta k^\alpha - k'} = \beta V'(k')$$

(b) Derive the Envelope condition.

The envelope condition is obtained by differentiating the Bellman equation with respect to k:

$$V'(k) = \frac{\alpha\theta k^{\alpha-1}}{\theta k^\alpha - k'}$$

Thus we have:

$$V'(k') = \frac{\alpha\theta k'^{\alpha-1}}{\theta k'^\alpha - k''}$$

(c) Find the Euler equation.

Substitute $V'(k')$ from the envelope condition into the first-order condition to get the Euler equation:

$$\frac{1}{\theta k^\alpha - k'} = \beta \frac{\alpha\theta k'^{\alpha-1}}{\theta k'^\alpha - k''}$$

That we can rewrite as:

$$\frac{1}{c} = \beta \frac{\alpha\theta k'^{\alpha-1}}{c'}$$

3. **Value Function Iteration**:

(a) Let's make the initial guess that the value function has the following form: $V_0(x) = 0 \ \forall x \in \Gamma$, where Γ is the feasible set. Find the next guess V_1 such that $V_1 = TV_0$. Hint: to get V_1, you need to choose the value of k' that maximizes V_1 given V_0. This choice might not require taking the FOC.

Given the initial guess $V_0(x) = 0$, the next iteration V_1 is computed as:

$$V_1(k) = \max_{k'} \log(\theta k^\alpha - k')$$

The maximizing choice of k' is such that k' equals 0, leading to:

$$V_1(k) = \log(\theta k^\alpha) = \log(\theta) + \alpha \log(k)$$

(b) Find the next guess V_2 such that $V_2 = TV_1$.

Next, we compute V_2 as:

$$V_2(k) = \max_{k'} [\log(\theta k^\alpha - k') + \beta \log(\theta k'^\alpha)]$$

We solve for k' to maximize this expression:

$$\begin{aligned}
\frac{\mathrm{d}V_2(k)}{\mathrm{d}k'} &= 0 \\
\frac{1}{\theta k^\alpha - k'} &= \frac{\alpha\beta}{k'} \\
k' &= \frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta} \\
g_k(k) &= \frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta}
\end{aligned}$$

We plug $g_k(k)$ in $V_2(k)$:

$$V_2(k) = \log(\theta k^\alpha - g_k(k) + \beta \log(\theta) + \beta\alpha \log(g_k(k))$$

$$\begin{aligned}
V_2(k) &= \log(\theta k^\alpha - \frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta}) + \beta \log(\theta) + \beta\alpha \log(\frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta}) \\
V_2(k) &= log(\theta k^\alpha(1 - \frac{-\alpha\beta}{1+\alpha\beta}) + \beta \log(\theta) + \alpha\beta \log(\frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta}) \\
V_2(k) &= log(\frac{\theta k^\alpha}{1+\alpha\beta}) + \beta \log(\theta) + \alpha\beta \log(\frac{\alpha\beta\theta k^\alpha}{1+\alpha\beta}) \\
V_2(k) &= \log(\theta k^\alpha) - log(1+\alpha\beta) + \beta log(\theta) + \alpha\beta log(\alpha\beta\theta k^\alpha) - \alpha\beta log(1+\alpha\beta)
\end{aligned}$$

(c) **BONUS**: Compute the error term between the two iterations. Are we getting closer to the true solution V^*?

The error term is the difference between successive iterations, which should decrease as we approach the true value function.

$$V_1(k) - V_0(k) = \log(\theta) + \alpha \log(k)$$

$$V_2(k) - V_1(k) = \log(\theta k^\alpha) - log(1 + \alpha\beta) + \beta log(\theta) + \alpha\beta log(\alpha\beta\theta k^\alpha) - \alpha\beta log(1 + \alpha\beta) - (\log(\theta) + \alpha \log(k))$$

$$V_2(k) - V_1(k) = \log(\theta) + \alpha \log(k) - log(1 + \alpha\beta) + \beta log(\theta) + \alpha\beta log(\alpha\beta\theta k^\alpha) - \alpha\beta log(1 + \alpha\beta) - \log(\theta) - \alpha \log(k)$$

$$V_2(k) - V_1(k) = -log(1 + \alpha\beta) + \beta log(\theta) + \alpha\beta log(\alpha\beta\theta k^\alpha) - \alpha\beta log(1 + \alpha\beta)$$

$$V_2(k) - V_1(k) = -log(1 + \alpha\beta) + \beta log(\theta) + \alpha\beta log(\tfrac{\alpha\beta\theta k^\alpha}{1+\alpha\beta})$$

At first sight, it is not clear whether the error term gets smaller, but if we plug these expressions into Desmos, we notice that there is indeed a convergence. Thus error term is getting smaller so we are getting closer to the true solution V*.

4. Guess and verify: Assume that the value function has the form $V(k) = a_1 + a_2 \log k$. Solve for the analytical solution of the value function and recover the values of a_1 and a_2.

Substitute V(k) into the Bellman equation with V(k') = $a_1 + a_2 \log(k')$:

$$V(k) = \max_{k'} [\log(\theta k^\alpha - k') + \beta(a_1 + a_2 \log(k'))]$$

We solve for the optimal value of k':

$$\frac{\mathrm{d}V(k)}{\mathrm{d}k'} = 0$$
$$\frac{1}{\theta k^\alpha - k'} = \frac{\beta a_2}{k'}$$
$$k' = \frac{\beta a_2}{1 + \beta a_2}\theta k^\alpha$$

We set the initial guess equal to the Bellman Equation evaluated at the optimal value of k':

$$a_1 + a_2 \log(k) = \log(\theta k^\alpha - \tfrac{\beta a_2}{1+\beta a_2}\theta k^\alpha) + \beta(a_1 + a_2 \log(\theta k^\alpha \tfrac{\beta a_2}{1+\beta a_2}))$$

$$a_1 + a_2 \log(k) = \log(\tfrac{\theta k^\alpha}{1+\beta a_2}) + \beta(a_1 + a_2 \log(\theta k^\alpha \tfrac{\beta a_2}{1+\beta a_2}))$$

$$a_1 + a_2 \log(k) = \beta a_1 - \log(1+\beta a_2) + \beta a_2(\log(\beta a_2) - \log(1+\beta a_2)) + (1+\beta a_2)\log(\theta) + (1+\beta a_2)\alpha \log(k)$$

We notice the first part of the second term is constant (a_1) and the second part is linear to $\log(k)$, thus we can identify:

$$a_2 = (1+\beta a_2)\alpha$$
$$a_2 = \frac{\alpha}{1-\beta\alpha}$$

We plug a_2 in the previous equation:

$$a_1 + \tfrac{\alpha}{1-\beta\alpha}\log(k) = \beta a_1 - \log(1+\beta\tfrac{\alpha}{1-\beta\alpha}) + \beta\tfrac{\alpha}{1-\beta\alpha}(\log(\beta\tfrac{\alpha}{1-\beta\alpha}) - \log(1+\beta\tfrac{\alpha}{1-\beta\alpha})) + (1+\beta\tfrac{\alpha}{1-\beta\alpha})\log(\theta) + (1+\beta\tfrac{\alpha}{1-\beta\alpha})\alpha\log(k)$$

$$a_1(1-\beta) = \log(1-\beta\alpha) + \tfrac{1}{1-\alpha\beta}(\alpha\beta\log(\alpha\beta) + \log(\theta))$$

$$a_1 = \tfrac{1}{(1-\beta)}(\log(1-\beta\alpha) + \tfrac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \tfrac{1}{1-\alpha\beta}\log(\theta))$$

5. Find the policy functions. We solve for $g_k(k)$ given a_2:

$$g_k(k) = k' = \frac{\beta a_2}{1+\beta a_2}\theta k^\alpha$$

$$g_k(k) = \frac{\beta\frac{\alpha}{1-\beta\alpha}}{1+\beta\frac{\alpha}{1-\beta\alpha}}\theta k^\alpha$$
$$g_k(k) = \beta\alpha\theta k^\alpha$$

And thus :

$$g_c(k) = c = \theta k^\alpha - g_k(k)$$
$$g_c(= \theta k^\alpha - \beta\alpha\theta k^\alpha$$
$$g_c(k) = \theta k^\alpha(1-\beta\alpha)$$

6. Recover the sequence of consumption. Assume $k_0 = 1$, $\alpha = 0.5$, $\beta = 0.9$, and $\theta = 2$. Find the values of c_0, c_1, and c_2.

$$c_0 = \theta \times k_0^\alpha \times (1 - \beta\alpha) = 2 \times 1^{0.5} \times (1 - 0.9 \times 0.5) = 1.1$$

$$k_1 = \beta\alpha\theta k_0^\alpha = 0.9 \times 0.5 \times 2 \times 1^{0.5} = 0.9$$

$$c_1 = \theta k_1^\alpha \times (1 - \beta\alpha) = 2 \times 0.9^{0.5} \times (1 - 0.9 \times 0.5) = 1.044$$

$$k_2 = \beta\alpha\theta k_1^\alpha = 0.9 \times 0.5 \times 2 \times 0.9^{0.5} = 0.854$$

$$c_2 = \theta k_2^\alpha \times (1 - \beta\alpha) = 2 \times 0.854^{0.5} \times (1 - 0.9 \times 0.5) = 1.017$$

3 Problem 2

We are given the problem:

$$\max_{\{c_t, H_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$

subject to:

$$c_t + b_t + P_t H_t = y_t + (1 + r_{t-1}) b_{t-1} + P_t (1 - \delta) H_{t-1}$$

1) Assumptions About Housing Market Transactions

- Perfect Information
- No transaction costs
- Perfect competition
- We also assume that the depreciation rate is between 0 and 1.

2) Transversality Condition

The transversality condition generally states that the marginal utility of assets must approach zero in the limit as $t \to \infty$. This is because otherwise, the agent would be better off reducing assets held tomorrow to consume more today, which cannot be an optimal solution.
In this example, we have two assets, hence:

$$\lim_{t \to \infty} \beta u'(c_t, H_t) \frac{P_t}{(1 - \delta) P_{t+1}} = 0$$

$$\lim_{t \to \infty} \beta u'(c_t, H_t) \frac{1}{(1 + r_t)} = 0$$

3) Bellman Equation

State variables must be known in the first period to the agent to solve the problem, whereas control variables are chosen to influence the future states.

State variables: $b_{t-1}, H_{t-1}, y_t, P_t, r_{t-1}$.
Control variables: c_t, H_t, b_t.

The Bellman equation is:

$$V(b_{-1}, H_{-1}) = \max_{\{c, H\}_{t=0}^{\infty}} u(c, H) + \beta V(b, H)$$

Subject to:

$$c = y + (1 + r_{-1}) b_{-1} + P(1 - \delta) H_{-1} - b - PH$$

4) Euler Equations

FOCs must be taken w.r.t b and H. Taking the first-order condition with respect to b:

$$-u'_c(c, H) + \beta V'(b, H) = 0$$

This implies:

$$u'_c(c, H) = \beta V'(b, H)$$

For housing, H:

$$-Pu'_c(c, H) + u'_H + \beta V'(b, H) = 0$$

It follows that:

$$Pu'_c(c, H) - u'_H(c, H) = \beta V'(b, H)$$

By the Envelope theorem, we can now write:

$$V'_c(b_{-1}, H_{-1}) = (1 + r_{-1})u'_c(c, H)$$

$$V'_c(b, H) = (1 + r)u'_c(c', H')$$

Similarly:

$$V'_H(b_{-1}, H_{-1}) = P(1 - \delta)u'_c(c, H)$$

$$V'_H(b, H) = P'(1 - \delta)u'_c(c', H')$$

Now we can use these expressions and plug them into each other. **We obtain the following Euler equations:**

$$u'_c(c, H) = \beta(1 + r)u'_c(c', H') \tag{1}$$

$$\beta P'(1 - \delta)u'_c(c', H') = Pu'_c(c, H) - u'_H(c, H) \tag{2}$$

5) Risk Aversion

Let:

$$u(c_t, H_t) = (c_t^\rho + H_t^\rho)^{\frac{1}{\rho}}$$

Where ρ measures the risk aversion of the agent.

Mathematically, this follows from the concavity of the utility function by Jensen's inequality. ρ could be estimated in a lab experiment (microeconomics) or using data from financial and insurance markets. For examole, Cesarini *et al.* (2018) study data from Swedish lottery earnings to study substantial wealth shocks, and show a significant decrease in labor income following random assignment of lottery prizes.

By plugging the Euler equations into each other, obtain:

$$\beta P'(1-\delta)u'_c(c',H') = P\beta(1+r)u'_c(c',H') - u'_H(c,H)$$

Write:

$$u'_H(c,H) = \beta u'_c(c',H')\left[P(1+r) - P'(1-\delta)\right]$$

Now, we can use the expression for the utility function to obtain the respective derivatives, plug these into the above statement, and then solve for $\frac{c_t}{H_t}$.

$$u'_c(c,H) = \frac{1}{\rho}(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}}\rho c^{\rho-1}$$

$$u'_c(c,H) = (c^\rho + H^\rho)^{\frac{1-\rho}{\rho}} c^{\rho-1}$$

Similarly:

$$u'_H(c,H) = \frac{1}{\rho}(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}}\rho H^{\rho-1}$$

$$u'_H(c,H) = (c^\rho + H^\rho)^{\frac{1-\rho}{\rho}} H^{\rho-1}$$

Restating our combined Euler equation:

$$(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}} c^{\rho-1} = \beta(1+r)(c')^{\rho-1}(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}}$$

$$(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}} H^{\rho-1} = \beta(c')^{\rho-1}(c^\rho + H^\rho)^{\frac{1-\rho}{\rho}}\left[P(1+r) - P'(1-\delta)\right]$$

Finally, dividing both equations by each other and rearranging for $\frac{c_t}{H_t}$ yields:

$$\frac{c^{\rho-1}}{H^{\rho-1}} = \frac{1+r}{P(1+r) - P'(1-\delta)}$$

$$\frac{c_t}{H_t} = \left(\frac{1+r}{P(1+r) - P'(1-\delta)}\right)^{\frac{1}{\rho-1}} \tag{3}$$

4 Bonus Problem

We are given:

$$\max_{\{c_t,H_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \Big[\beta^t u_1(c_t) + u_2(d_t)\Big]$$

subject to:

$$c_t + dx_t \leq w \quad \forall t$$

$$d_{t+1} \leq (1-\delta)d_t + dx_t \quad \forall t$$

$$d_t,\ c_t \geq 0 \quad \forall t$$

where some d_0 is given.

1. Conditions on u_1 and u_2

Both must be continuously differentiable and exhibit decreasing returns to scale, as well as being additively separable. F would be the function that expresses c_t as a function of d_t. d_{t+1} can be chosen from the correspondence that is governed by c_t, d_t and limited by w_t.

Thus we have:

$$c_t = w - d_t = w + (1-\delta)d_t + d_{t+1}$$

This expression relies on the market clearing condition ; incomes must be entirely spent at time t.
Thus we can write F the total utility function derived from the current stock of durables and future investments:

$$F(d_t, d_{t+1}) = u_1(w + (1-\delta)d_t + w + d_{t+1}) + u_2(d_t)$$

$$F(d_t, d_{t+1}) = u_1(d_t, d_{t+1}) + u_2(d_t)$$

The correspondence Γ reflects the feasible choices for the future stock of durables, considering depreciation and available resources:

$$\Gamma \in [0; (1-\delta)d_t + w]$$

2. Bellman Equation

The state variable is d_t. The control variables are c_t, d_{xt}, d'.

The Bellman equation is:

$$V(d) = \max_{\{d_t\}_{t=0}^{\infty}} (u_1(c) + u_2(d) + \beta V(d'))$$

subject to:

$$c_t + d_{t+1} \leq w + (1-\delta)d_t$$

3. Envelope Condition and FOC

Using the provided constraints, we can eliminate d_{x_t} and write c as a function of d and d':

$$r(d') = (1-\delta)d + w - c$$

$$c = (1-\delta)d + w - d'$$

The Bellman equation becomes:

$$V(d) = \max_{\{d_t\}_{t=0}^{\infty}} \left[u_1\Big((1-\delta)d + w - d'\Big) + u_2(d')\right] + \beta V(d')$$

The first-order condition with respect to d' is:

$$0 = -\frac{\delta u_1}{\delta c}\left[(1-\delta)d + w - d'\right] + \beta \frac{\delta V}{\delta d'}(d')$$

It follows that:

$$\beta \frac{\delta V}{\delta d'}(d') = \frac{\delta u_1}{\delta c}\left[(1-\delta)d + w - d'\right]$$

By the Envelope Theorem, we can write:

$$\frac{\delta V}{\delta d}(d) = \frac{\delta u_2}{\delta d} u_2(d)$$

$$\frac{\delta V}{\delta d'}(d') = \frac{\delta u_2}{\delta d} u_2(d')$$

4. Euler Equation

Finally, by combining the FOC and the Envelope condition, the Euler equation can be written as:

$$\frac{\delta u_1}{\delta c}\left[(1-\delta)d + w - d'\right] = \beta \frac{\delta u_2}{\delta d} u_2(d') \tag{4}$$

5. Steady-state $d*$

Let the steady-state value of d be d^*, such that if $d_0 = d^*$, then $d_t = d^*$ for all t.

The constraint:

$$d_{t+1} \leq (1-\delta)d_t + dx_t$$

becomes:

$$d^* = (1-\delta)d^* + d_x^*$$

Rearranging the equation:

$$d^* - (1 - \delta)d^* = d_x^*$$

$$\delta d^* = dx^*$$

$$d^* = \frac{d_x^*}{\delta}$$

Since $\delta > 0$ (the depreciation rate is positive), and as long as $dx^* > 0$, we have $d^* > 0$.

Moreover, the steady-state equation is linear, meaning the solution for d^* is unique.