Problem Set 1: Bellman equations

Sciences Po - Macroeconomics 3 - Fall 2024
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The deadline submission is 22th, September 23:59. Send a unique pdf file.

Literature: Wealth distribution

Present and comment a result or a stylized fact about wealth distribution of an article of your choice. Please confirm the article with me by email (paloma.peligry@sciencespo.fr). Once it has been approved, register the paper on the Google Sheet.

Exercise 1: Recursive problem and Value Function Iteration

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t log(c_t)$$

Where $\beta \in (0, 1)$. The consumer is also endowed with k_0 units of capital in the first period. The feasible allocations satisfy:

$$c_t + k_{t+1} \leq \theta k_t^{\alpha}$$

Where $0 < \alpha < 1$ and $\theta > 0$. Notice we also have the following constraints:

$$c_t, k_t \geq 0$$

1. Write the problem in recursive form. Specify the Bellman equation, the state variables(s), the control variable(s) and the feasible set(s).
2. **Euler equation**
   (a) Derive the F.O.C.
   (b) Derive the Envelope condition.
   (c) Find the Euler equation.
3. **Value Function Iteration:**
   (a) Let's make the initial guess that the value function has the following form: $V_0(x) = 0$ $\forall x \in \Gamma$, where Γ is the feasible set. Find the next guess V_1 such that $V_1 = TV_0$.
   (b) Find the next guess V_2 such that $V_2 = TV_1$.
   (c) Compute the error term between the two iterations. Are we getting closer to the true solution V^*?
4. **Guess and verify**: Assume that the value function has the form $V(k) = a_1 + a_2 logk$. Solve for the analytical solution of the value function and recover the values of a_1 and a_2.

   Note: this question is independent of question 3.

5. Find the policy functions.
6. **Recover the sequence of consumption.** Assume that $k_0 = 1$ $\alpha = 0.5$, $\beta = 0.9$ and $\theta = 2$, find the values of c_0, c_1 and c_2.

Exercise 2: Housing problem

Houses are durable goods from which households derive some utility. To model the demand for houses, a simple shortcut consists in introducing houses in the utility function. The goal of this exercise is to be able to use data on house prices and interest rates to derive properties of the demand for houses.

Households thus derive utility from consumption and from having houses. The instantaneous utility function is $u(c_t, H_t)$ where H_t is the amount of housing. Households also have access to financial savings denoted b_t at period t, remunerated at a real rate r_t between period t and $t+1$. Houses depreciate at rate δ. The program of the households is

$$\max_{\{c_t, H_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$

$$c_t + b_t + P_t H_t = y_t + (1 + r_{t-1}) b_{t-1} + P_t (1 - \delta) H_{t-1}$$

1. What are the underlying assumptions about the transactions in the housing market?
2. State the **transversality conditions**.
3. Write the **Bellman equation** of the problem, with the value function denoted as $V(b, H)$ (be careful about the timing notation). What are the control variable(s)? What are the state variable(s)?
4. Find the two **Euler equations** (step by step).
5. Assume the utility function has the following form:

   $$u(c_t, H_t) = (c_t^\rho + H_t^\rho)^{\frac{1}{\rho}}$$

   Explain what is the economic meaning of the ρ coefficient. Using the two Euler equations, express c_t as a function of P_t, P_{t+1} and r_t. How can we get ρ from the data?

BONUS Exercise 3: Bellman equation

Consider a single agent problem where each period, w total output is produced and can be divided into consumption of a perishable good, c_t, and investment in a durable good, d_{xt}. The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, d_t, produces a flow of services that enters the utility function. The income w is exogenous. Thus, the problem faced by the household with initial stock d_0 is:

$$\max_{\{c_t, d_t, d_{xt}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u_1(c_t) + u_2(d_t)]$$

$$\text{s.t.} \quad c_t + d_{xt} \le w \quad \forall t$$

$$d_{t+1} \le (1 - \delta) d_t + d_{xt} \quad \forall t$$

$$d_t, c_t \ge 0 \quad \forall t$$

$$d_0 \text{ given}$$

where both u_1 and u_2 are strictly increasing and continuous.

1. State a condition on either u_1 or u_2 (or both) such that you can write an equivalent problem in the following form:

$$\max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1})$$
$$\text{s.t.} \quad d_{t+1} \in \Gamma(d_t) \quad \forall t$$

   where $\Gamma(d_t) \in \mathbb{R}^+$. What is F? What is the correspondence Γ (i.e., this is the set of possible values where the variable d_{t+1} can be chosen)?

   For the remaining questions, assume that both u_1 and u_2 satisfy the Inada conditions and are continuously differentiable.

2. Write the **Bellman equation** for this problem.

3. State the envelope condition and the F.O.C.

4. Find the **Euler equation** of this problem.

5. Show that there is a unique steady state value of the stock, d^*, such that if $d_0 = d^*$, then $d_t = d^* \quad \forall t$. Show that $d^* > 0$.