Life-Cycle Modeling is Ready for Prime Time

Christopher Carroll[*]
Professor of Economics,
Krieger School of Arts & Sciences,
Johns Hopkins University

Alan Lujan[†]
Program Coordinator & Lecturer,
Advanced Academic Programs,
Johns Hopkins University

Matthew N. White[‡]
Primary Developer,
Econ-ARK

Thursday 4th December, 2025

Abstract

The "life-cycle model" of optimal saving for retirement is familiar to anyone who has taken an intermediate economics class. When hiring a financial advisor, such people probably think the advisor's job is to tailor optimal life-cycle model choices to their particular circumstances. But academics and advisors know that the advice about both saving and portfolio choice provided by standard academic life-cycle models is deeply problematic. In particular, most such models imply that retirees should plan to run their wealth down to nearly zero and then live pension-check to pension-check in their old age. This paper makes the case that recent developments in the economics literature have finally given us the tools and insights required to construct rigorous life-cycle models whose advice is sensible. We provide an example of a simple model that can solve a number of problems by putting wealth in the utility function, which captures the intuitive desire for liquidity and security that standard models often overlook.

JEL Classification: XXX, XXX

Keywords: *words words words*

[*]ccarroll@llorracc.org
[†]alujan@jhu.edu
[‡]mnwhite@gmail.com

1 Introduction

Modigliani and Brumberg (1954) were the first to propose trying to understand consumer financial choices as optimal responses to the paths of income and of spending needs over the lifetime. An enormous academic literature has followed their pioneering work (see the literature appendix for an expansive summary), but it has proven difficult to build rational optimizing models that give sensible advice about both life-cycle saving choices and investment decisions. That is, the models yield implausible answers to questions about how much wealth should be retained later in life, and how much of one's retirement savings should be invested in the stock market. Indeed, the subtitle of a recent paper by Daga, Smart, and Pakula (2023) captures the current state of affairs nicely: "Why Practitioners Have Not Adopted the Lifecyle Model – Yet"; see also the literature review in De Nardi, French, and Jones (2016).

In this paper, we argue that the elements are already available to construct a model that "practitioners can adopt." All that is needed is to combine the relevant academic contributions with some wisdom from practitioners' own experience of advising clients, particularly their understanding that clients often resist aggressive drawdowns of wealth late in life. Our paper's central contribution is to provide a small open-source computational model that incorporates some of the features that make it possible to build rigorous optimizing life cycle models whose advice is not obviously wrong.

We begin with a (very) brief literature synopsis, then present a set of models that illustrate the difficulty of matching empirical facts with the life-cycle approach. We end with our proposed solution, which involves a small twist to the old idea (Carroll (2000)) of incorporating wealth directly in the utility function.

2 Recent Developments in the Literature

2.1 Computation, Complexity, and Uncertainty

Consumers face many uncertainties: about their own income, stock returns, interest rates, health expenditures, mortality, and much more. Further complications arise because of liquidity constraints and other financial frictions.

The incorporation of realistic descriptions of these complexities makes computation of the mathematically optimal solution to the consumer's problem is surprisingly difficult. Each new aspect of risk introduced to the model increases the burden of computing expectations of the future, while each added dimension of agent choice or personal circumstances expands the problem size (literally) exponentially. Modelers thus face a difficult tradeoff in designing their framework to be sufficiently realistic to serve as a plausible representation of the problem actual people face, while maintaining a degree of tractability to stay within the bounds of computational feasibility. The remarkable advance of computational power has now finally made it possible to compute a credible answer to the question, "What saving and portfolio choices are truly mathematically optimal?" in a context in which the key complexities are properly represented.

2.2 Survey Data on Expectations and Preferences

Another academic development has been a new openness to the idea that people's beliefs and preferences can be probed by *asking them* about their beliefs and preferences. In the context of motivations for saving, this leads us to take seriously the answers to a survey question about their 'most important' reason for saving that respondents to the Federal Reserve's Survey of Consumer Finances (SCF) have been asked for many years. Among retirees, one answer dominates the rest: "Liquidity / The Future." (See discussion below.)

A natural interpretation of the importance consumers put on "liquidity" is that precautionary saving motives matter for many households, highlighting the need for the aforementioned computational advancements. The traditional approach to constructing such models has been for economists to try to measure the necessary inputs (income uncertainty, e.g.), and to assume that agents' beliefs incorporate whatever it is that the economist has measured. However, the newly collected survey data on consumer expectations show that the beliefs that many people *actually* hold differ substantially from what economists postulate they "should" believe based on empirical measurements. Moreover, there is now considerable evidence that the decisions people make reflect their actual beliefs rather than whatever it is that economists think they *should* believe.[1]

Some recent work suggests that taking beliefs data into account could resolve many of the problems that have long beset the life cycle modeling literature. For example, Velásquez-Giraldo (2023) shows that even college-educated people systematically have held beliefs about stock market returns that are pessimistic compared to what the market has historically delivered. He argues that this explains why people have been less eager to invest in stocks than would be predicted by models calibrated with economists' more optimistic expectations, and that the portfolio investment behavior of college-educated people over most of their lives is reasonably consistent with *subjectively* rational decision-making (i.e. given their beliefs).

[1] A particularly troubling possibility is raised by the work of Gabaix et al. (2010), who point out that at least some elderly decision-makers (say, those with dementia) may be beyond the "age of reason."

Concretely, many people think that investment in stocks is a lousy deal, yielding a low return with high risk, so it's no mystery why such people do not invest.[2]

It is not astonishing to discover that many people hold beliefs that differ from those of experts, especially on subjects whose mastery requires considerable domain-specific education, like the returns and risk of stock investments. Indeed, the existence of a large industry offering financial advice is *prima facie* evidence that many people are not confident that they understand everything necessary to make good financial choices on their own.

Financial advice, however, is fraught with potential conflicts of interest. That is one reason that justifying such advice with an explicit and transparent modeling framework is so attractive. If the advice is consistent with the model, and the model can be checked both for mathematical correctness and conceptual soundness (by outside experts), then it is reasonable for a client hiring an advisor to trust the advice.

2.3 Model Specification and Estimation

In Section 3, we provide a formal description of the mathematical and computational structure of our optimizing models, beginning with the Life Cycle Portfolio model, which calculates optimal saving and optimal portfolio shares over the life cycle. In Section 4, we report that the model implies a rapid drawdown of wealth after retirement that is simply not observed in empirical observations, renewing attention to a longstanding problem with life cycle models (see, e.g., Hurd (1987), Heimer, Myrseth, and Schoenle (2019)).[3] We call this the "drawdown failure," which has been the subject of a large literature with both U.S. evidence (see, e.g., Hurd (1989), De Nardi, French, and Jones (2016), Kopecky and Koreshkova (2014), Mortenson, Schramm, and Whitten (2019), Poterba, Venti, and Wise (2018)) and international evidence (see, e.g., Christensen, Kallestrup-Lamb, and Kennan (2022), Ventura and Horioka (2020)).

We next modify the model by adding a bequest motive, because the literature has extensively explored whether such a motive could explain the drawdown failure (again see

[2] If the modeler is willing to assert that consumers have mistaken beliefs that cause them to make suboptimal choices, the advice the model gives will differ from the pattern of measured behavior. This could justify the model in recommending, for example, greater investment in risky assets than consumers tend to choose on their own. We compromise by adopting a believed equity premium of 0.03, which is lower than the historical average. Even lower beliefs would reduce the estimated risk aversion coefficient.

[3] Some impressive recent work by Ameriks, Caplin, and coauthors (Ameriks et al. (2011), Ameriks et al. (2020)) has argued that concerns about the possibility of extremely large medical costs (e.g. nursing home or other long term care) may be behind the drawdown failure for some people (see De Nardi, French, and Jones (2016)) for a survey). It would be interesting to see how our results might change if we were to include such shocks, but the inexorable logic behind the model's prediction of a rapid drawdown of wealth should still apply as the threat of mortality grows with age. Furthermore, the drawdown failure is also present in other countries whose social insurance systems are much more generous than the U.S., so it seems unlikely that uninsurable medical risk is a sufficient explanation.

De Nardi, French, and Jones (2016)). In Section 4 we find that in order for the bequest motive to explain the drawdown failure, the strength of that motive has to be extremely and implausibly strong— so much so that even early in their working lives, the primary motivation for saving is the accumulation of a bequest (rather than, say, to sustain one's own consumption in retirement).

This leads us into more speculative territory. If what consumers care most about is to hold wealth for "Liquidity / The Future," but that wealth is not explainable by precautionary saving against measurable shocks, a potential interpretation is that consumers value the ownership of wealth *in and of itself*. This interpretation has the advantage of being consistent with consumers' stated preferences for liquidity. After fleshing out this idea a bit, we propose a final model that makes wealth a direct input to the utility function in a different way than in the existing literature.

The main quantitative result of this paper is to show that this final model does a much better job of jointly matching the wealth profile and portfolio choices than does the Life Cycle Portfolio model. While it does not match the drawdown failure as well as the Warm Glow Bequest (WGB) model does, the WGB model fits the data only when we allow an implausibly intense bequest motive.

Our ability to judge that such a strong bequest motive is implausible rests on another methodological shift: Economists have become much more receptive to the use of "softer" data like surveys on household expectations and beliefs. This allows us to appeal to survey evidence in which respondents are asked directly about their motivations for saving; providing for a bequest turns out to rank very low on the list of consumers' expressed priorities.

But the broad purpose of our exercise is not to defend any particular modeling setup; instead, it is to call attention to the fact that modeling and conceptual tools (including the idea that softer data like surveys should be taken seriously) have advanced to the point where it is finally possible to construct rational optimizing models of life cycle financial choice that can serve as a credible justification for normative advice.

3 Models

The academic literature on life-cycle modeling is vast, and we cannot hope to do it justice (even in the broader sampling in the literature appendix). But the intrinsic nature of papers in any academic literature is to focus narrowly on one specific question at a time. Here, our goal is to examine the "big picture" question of what elements are needed to craft a model that can provide credible advice to retirees about spending and portfolio choices, while remaining reasonably consistent with the relevant well-established facts from the academic literature, as well as one new kind of further evidence that we view as vital: the experience of financial advisors themselves in interactions with their clients. This practitioner perspective helps ground the optimization problem in the realities of client preferences. We have been told,[4] for example, that a good way to get fired as a financial advisor is to recommend the LCP model's conclusion that it is optimal for retirees to plan to run their wealth down to zero and then live pension-check to pension-check.

For purposes like 401(k) or other pension plan design, the optimization problem should be constrained to be one that satisfies the legal obligations employers have to their employees. For example, the employer's contract is with the employee, not with the employee's heirs. The employer's duty is to craft a plan that is expected to permit the employee to have adequate resources for their own expenditures during retirement. These considerations constrain the advisor from including a bequest motive in the plan's default optimization objective.[5]

3.1 The Baseline Academic Model

We begin by describing the optimal consumption-saving problem over the life cycle for a consumer, focusing on the dynamics of their income while temporarily setting aside how returns to saving work. After we have finished describing the basic life-cycle model, we will augment it to add optimal portfolio choice between a safe asset and a risky asset (like the stock market) with a higher expected rate of return.

3.1.1 Basic Life-Cycle Consumption-Saving

In each year (indexed by t), a consumer's flow of utility depends on how much they consume from their available resources. We assume that the utility function is of the standard Constant Relative Risk Aversion (CRRA) form:

$$\mathrm{u}(\mathbf{c}) = \frac{\mathbf{c}^{1-\rho}}{1-\rho}. \tag{1}$$

The consumer is smart enough to realize that preserving some resources for the future is a good idea: they do not consume all of their wealth immediately.

At the time they choose how much to consume, the consumer has total market resources of $\mathbf{m}_t$, representing their previously owned resources (bank balances) and current income flow $\mathbf{y}_t$. Whatever the agent does not consume constitutes assets $\mathbf{a}_t$, which accrue interest by

[4] Personal communication, James Tzitzouris with Christopher Carroll, 2024-05-15.

[5] One way to accommodate this requirement would be to limit the empirical sample used to estimate the model to childless people. This might not be feasible with public datasets like the SCF because the sample sizes might be too small; but with administrative data of the kind available to the IRS it should be possible.

factor $\mathcal{R}_{t+1}$ between period t and period $t+1$. We follow most of the literature and assume that the consumer faces a hard liquidity (or borrowing) constraint: they cannot end any period with negative assets. These assumptions are expressed as:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t, && \text{remaining assets are market resources less consumption} && (2)\\
\mathbf{a}_t &\geq 0, && \text{consumer cannot borrow} && (3)\\
\mathbf{b}_{t+1} &= \mathcal{R}_{t+1}\mathbf{a}_t, && \text{bank balances are assets after yielding interest} && (4)\\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{y}_{t+1}. && \text{future market resources are bank balances plus income} && (5)\\
& && && (6)
\end{aligned}$$

One of the fundamental discoveries of the past 40 years or so is the extent to which optimal choice is profoundly altered by the presence of uncertainty. Friedman (1957) proposed a simple formulation of the labor income process that remains an excellent description of annual income shocks even today. According to Friedman, there are two components to income. The "permanent" component is roughly what the consumer would expect to earn in a "normal" year (say, their annual salary), and a "transitory" component reflects events like unemployment spells or lottery winnings, which make a given year's realized income deviate from its expected value. From a modeling perspective, the upshot is that a consumer's financial circumstances can be fully captured with two variables. First, the consumer's permanent income level $\mathbf{p}_t$ is the non-capital income they would normally expect to receive. Second, total market resources $\mathbf{m}_t$ represent the sum of financial assets and current income: the pool of resources that can be immediately spent, or "money" in the colloquial sense of, "how much money does grandma have?". The transitory component of income need not be tracked at all: as soon as this uncertainty is resolved, its information is fully incorporated by market resources $\mathbf{m}_t$.

A consumer's "value" of having a given amount of market resources $\mathbf{m}_t$ right now, and of knowing their current permanent income level to be $\mathbf{p}_t$, is the sum of consumption utility they will experience from today onward into the indefinite future, assuming that they make optimal choices in all periods. Potential future utility flows matter only to the extent that the agent expects to survive to that period, and might be further discounted due to placing more weight on the present than the future.

Mathematically, the consumer's objective is to maximize expected present discounted utility from consumption over a life cycle that ends no later than some terminal period T:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\{\mathbf{c}\}_t^T} \mathrm{u}(\mathbf{c}_t) + \mathbb{E}_t\left[\sum_{n=1}^{T-t} \mathcal{L}_t^{t+n}\beta^n \mathrm{u}(\mathbf{c}_{t+n})\right]. \tag{7}$$

$\mathcal{L}_t^{t+n}$: probability to $\mathcal{L}$ive until age $t+n$ given you are alive at age t

- $\mathcal{L}_{120}^{121} = 0.0$ says that a 120 year old has zero probability of living to 121
- $\mathcal{L}_{80}^{90} = 0.3$ says that an 80 year old has a 30 percent chance of reaching 90

To capture the predictable patterns that non-capital income follows over the life cycle (i.e. rising with age and experience, and falling at retirement to the level of any regular pension payments), we define a sequence to characterize the Modiglianian life cycle:

Γ_t : typical life cycle permanent income growth factor by age

The typical life cycle pattern is altered, in any particular consumer's case, by "permanent income shocks" that we represent with the variable ψ. At any given age, permanent income growth can deviate from the average experience of others of the same age in either a positive direction ($\psi > 1$ would correspond to an unexpected promotion or a switch to a higher-paying job) or a negative direction ($\psi < 1$ might be the result of a failure to be promoted or a change to a lower paying job).

This gives us the following description of the dynamics of income:

$$\begin{aligned} \mathbf{p}_{t+1} &= \mathbf{p}_t \Gamma_{t+1} \psi_{t+1}, && (8) \\ \mathbf{y}_{t+1} &= \mathbf{p}_{t+1} \theta_{t+1}, && (9) \\ \mathbb{E}_t[\mathbf{p}_{t+1}] &= \mathbf{p}_t \Gamma_{t+1}, && (10) \end{aligned}$$

where the third line follows because the expected value of the permanent shock is $\mathbb{E}_t[\psi] = 1$.

The transitory component θ of income has two modes. In unemployment spells, the consumer earns no income; we assume that such spells occur with probability $\wp$ each period. Otherwise, the consumer receives a transitory income shock $\xi > 0$ from some (mean one) distribution, rescaled to preserve the unit mean of θ:[6]

$$\xi_t = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \xi_t/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{11}$$

It is conventional to assume that shocks to permanent income and to the transitory income

[6] It is straightforward to extend the model to allow for a more realistic treatment of unemployment, for example by taking account of the existence of an unemployment insurance system; such an adjustment does not change the substantive conclusions we are interested in.

of the employed are (mean one) lognormally distributed:

$$\log \psi_t \sim \mathcal{N}(-\sigma^2_{[\psi,t]}/2, \sigma^2_{[\psi,t]}), \tag{12}$$

$$\log \xi_t \sim \mathcal{N}(-\sigma^2_{[\xi,t]}/2, \sigma^2_{[\xi,t]}). \tag{13}$$

3.1.2 Reducing Perceived Model Complexity

Following the standard Bellman representation, if the agent assumes that they *will* act optimally in all future periods, then their future expectations of discounted utility flows in (7) are their value function one period ahead. Defining $\mathcal{L}_t \equiv \mathcal{L}_t^{t+1}$ to reduce notation, this is:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} \ \mathrm{u}(\mathbf{c}_t) + \mathcal{L}_t \beta \mathbb{E}_t \left[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1}) \right]. \tag{14}$$

Under the assumptions we have made about the structure of the utility function (homotheticity), budget constraint (linearity and geometric returns), and income process (permanent and transitory shocks) it is possible to recast the problem entirely in terms of *ratios* of the model variables to permanent income $\mathbf{p}$. So, for example, italic $c = \mathbf{c}/\mathbf{p}$ is the ratio of the (boldface) level of consumption to the level of permanent income $\mathbf{p}$ (see Carroll (2023) for the math). This normalization is crucial because it reduces the complexity of the problem; instead of tracking both resources and income levels, we only need to track their ratio. The solution is thus homothetic with respect to permanent income level: $\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t \mathrm{c}_t(m_t)$. Normalizing the Bellman value function by $\mathbf{p}_t$ requires dividing by $\mathbf{p}^{1-\rho}$, yielding:

$$\mathrm{v}_t(m_t) \equiv \mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t)/\mathbf{p}_t^{1-\rho} = \max_{\mathrm{c}_t} \ \mathrm{u}(c_t) + \mathcal{L}_t \beta \mathbb{E}_t \left[(\psi_{t+1}\Gamma_{t+1})^{1-\rho} \mathrm{v}_{t+1}(m_{t+1}) \right]. \tag{15}$$

The consumer's problem can be further simplified by realizing that it boils down to a "now versus later" problem. All the consumer needs to know about the future is summarized by the value they will expect as a consequence of ending the current period with a certain ratio of assets to permanent income, $a = \mathbf{a}/\mathbf{p}$. Moreover, we combine several of the multiplicative terms into a single factor for the realized intertemporal discount factor $\boldsymbol{\beta}_{t+1}$. We represent the value of *ending* the period with assets of a_t using the Gothic variant of the letter v:

$$\mathfrak{v}_t(a_t) \equiv \mathbb{E}_t[\boldsymbol{\beta}_{t+1} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a_{t+1} + \theta_{t+1})], \quad \boldsymbol{\beta}_{t+1} \equiv \beta(\psi_{t+1}\Gamma_{t+1})^{1-\rho}. \tag{16}$$

With this definition, the period t problem can be summarized in Bellman form as simply:

$$\mathrm{v}_t(m_t) = \max_{c_t} \{ \mathrm{u}(c_t) + \mathcal{L}_{t+1} \mathfrak{v}_t(a_t) \} \quad \text{s.t.} \ \ a_t = m_t - c_t \geq 0. \tag{17}$$

Table 1: Summary of Life-Cycle Model Notation

Object	Meaning
$\mathcal{L}_{t+1} \equiv \mathcal{L}_t^{t+1}$	probability that a person alive at age t survives to age $t+1$
m	market resources normalized by permanent income
c	consumption normalized by permanent income
a	end-of-period assets normalizedby permanent income
$\mathrm{v}_t(m)$	normalized value function when consumption decision is made
$\mathfrak{v}_t(a)$	normalized value function when portfolio decision is made

Because a_t measures available market resources that are unspent, this formulation makes it crystal clear that the consumer faces a tradeoff between the utility of consumption today and the expected value of preserving assets for the future.

3.1.3 The Life Cycle Portfolio ('LCP') Model

We are now ready to add portfolio choice to the problem and discuss how the interest factor $\mathcal{R}_{t+1}$ is determined. Suppose the consumer can invest their assets in a risk-free asset with return factor R, and in a risky asset with returns distributed as $\log \mathbf{R}_{t+1} \sim \mathcal{N}(\mathsf{r}+\varphi-\sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2)$, where $\mathsf{r} = \log(\mathsf{R})$. That is, we make the standard assumption that returns are lognormally distributed with an expected equity premium of φ. The portfolio return the consumer earns will depend on the *share* ς_t of their assets that they invest in the risky asset:

$$\mathcal{R}_{t+1} = \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma \tag{18}$$

Now the consumer makes *two* choices in each period t: how much to consume c_t and the share ς_t of his assets to put into the risky asset. These choices are made simultaneously, but they can be thought of as being made sequentially, one immediately after the other: first consumption (conditioned on market resources m_t) and then the risky asset share (conditioned on assets a_t after consumption). The consumer makes the optimal choice of portfolio share, so we redefine the "Gothic" value function as:

$$\mathfrak{v}_t(a_t) = \max_{\varsigma_t} \quad \mathbb{E}_t \left[\boldsymbol{\beta}_{t+1} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_{t+1}) \right]. \tag{19}$$

A split second before choosing the risky share, the consumer's objective in the consumption stage of the problem is exactly the same as the Bellman form above, but with the redefined continuation value that accounts for optimal portfolio choice.

3.1.4 Calibration

Many of the parameters of the basic life-cycle consumption-saving model can be calibrated from well measured empirical data. For example, we use standard calibrations of both of expected income growth and income risk (the age trajectories of Γ_t, $\sigma^2_{[\psi,t]}$ and $\sigma^2_{[\theta,t]}$) during the working life based on Cagetti (2003). Survival probabilities by age are taken directly from actuarial mortality tables published by the Social Security Administration. We set the unemployment probability to a fairly standard value of $\wp = 5\%$, and model agents begin their "lifetime" at age 25 with no wealth.

We assume that annual returns on the risk free asset are $\mathsf{R} = 1.01$, and that the equity premium is $\varphi = 0.03$. These values are somewhat low compare to the economics literature (in which $\mathsf{R} = 1.02$ and $\varphi = 0.06$ are common), but from personal correspondence with financial planners, our understanding is that the profession tries to be conservative about projections of their clients' portfolio returns.

While we do not believe that values of $\beta < 1$ are "ethically indefensible" as claimed in Ramsey (1928), we nevertheless set the "pure" rate of time preference to $\beta = 1$, so that agents care exactly as much about their future self as their present self (conditional on surviving into the future). This also serves to constrain the model's ability to fit the data by declining to include an additional "free" parameter in the estimations.

Beyond those basic assumptions, we calibrate the model to include uncertainty after retirement. Specifically, we assume that there are expenditure shocks in retirement that can reduce *disposable* income net of exogenous expenditures by maintaining transitory income shocks at their age-64 variance (while zeroing out permanent income shocks); see also Fulford and Low (2024) for a recent estimation of expense shocks. This assumption acts as a kind of shorthand for other modeling assumptions that focus on modeling the dynamics of medical and nursing home expenses in old age, e.g. Ameriks et al. (2011) and De Nardi, French, and Jones (2010).

Even if old age medical expenses were incorporated into the model more realistically, this feature *cannot* be the primary explanation for the drawdown failure, as the phenomenon has been documented even in countries with much more robust public health insurance than the United States, including Sweden (Ljunge, Lockwood, and Manoli (2013)), Norway (Hattrem (2022)), Australia (Aus (2021)), Canada (Hamilton (2001)), Japan (Niimi and Horioka (2019)), and the United Kingdom (Crawford (2018)). In principle, the presence of such shocks provides a precautionary motive to draw down wealth more slowly. However, our estimation results show that even when we include this calibration of expense shocks, the model still predicts much more drawdown of wealth than the data show.

3.2 Alternative Preferences

Our specification of preferences in the LCP model is the standard assumption of time-separable Constant Relative Risk Aversion utility with Exponential Discounting. This is the workhorse tool for intertemporal choice models because it has a number of convenient mathematical properties and its implications satisfy some plausible economic desiderata (cf. Kimball (1993)). However, mathematical convenience provides no guarantee that the utility specification is *right* in the sense of giving a proper representation of what people's preferences really are. Economists have explored a number of modifications to these standard preferences in an attempt to make their models' predictions match various facts.

3.2.1 Habit Formation and Epstein-Zin Preferences

One intuitive idea is that people care not only about the current level of their consumption but also about how it compares to their past levels of consumption (they have "habit formation" in their preferences). For example, Carroll, Overland, and Weil (2000) show that a model with multiplicative habits can provide an explanation for otherwise puzzling patterns in the relationship between saving and growth across countries, and Michaelides (2002) examines the implications of such a model for life-cycle choices.

A second common modification to preferences considered by a substantial literature is the use of Epstein and Zin (1991) preferences, which allow agents to be extremely risk averse with respect to financial risk at a point of time, while simultaneously allowing the agents to be quite willing to tolerate changes in consumption over time. Such preferences have been proposed as a way to solve various puzzles related to the relatively high rate of return that equity investments have exhibited over time.

Both of these formulations are motivated mostly by the goal of matching macroeconomic data. In our view, however, they are both difficult to defend given some facts we can robustly observe in microeconomic data. The parameter values required to match macroeconomic puzzles often imply counterfactual behavior at the household level. In particular, both models would imply that households would be extremely eager to buy insurance to smooth away almost any risk to their microeconomic circumstances. While people do typically have insurance against large risks (fire insurance for the home, auto insurance for the car), the parameter values in these models required to match the macroeconomic facts would justify consumers in spending a large fraction of their income on insurance of all kinds. One particular example stands out: Households with either strong habits or a high Epstein-Zin instantaneous coefficient of relative risk aversion would be extremely eager to buy private unemployment insurance to supplement the default UI system provided by the state. Such private UI is available – and at prices such that a large fraction of households would be eager to buy it if they had such preferences – yet the fraction of households who participate in the private UI market is vanishingly small.

Our attention in this paper is therefore directed toward other modifications in preferences that seem to be plausible in both microeconomic and macroeconomic data.

3.2.2 The "Warm Glow" Bequest Motive

The LCP model sketched above assumes that the only reason to hold wealth is to spend it later, which means that eventually an age must come at which the wealth starts being spent down. As the literature has demonstrated, and as we will confirm below using data from the SCF from 1995 to 2022, the path of the median wealth ratio after retirement does not look anything like what that model predicts.

Of course, the model can make no sense at all of the behavior of the very rich. Bill Gates, for example, has chosen to allocate a large portion of his lifetime wealth to the Bill and Melinda Gates foundation rather than spending it on himself; and even with the relative pittance that remains to him today ($153 billion, per Business Insider) he would need to spend about $22 million per day to *avoid getting richer*.[7] In fact, he has pledged to give away nearly all of his wealth before he dies.

But the model also fails for people of much more modest means— the "drawdown failure" articulated above. From a mathematical point of view, it is clear that some other motive for holding onto wealth must be added to the framework if it is to explain the broad facts, never mind Bill Gates. A natural candidate is a bequest motive: the idea that people take pleasure in the thought of leaving something to their heirs.

This idea can be incorporated by adding another term to the sources of utility: the value the consumer places on the bequest, which we will denote as $e(a)$, the utility they experience from the thought of leaving an *e*state. The flow of utility that the consumer receives includes both their utility from consumption *and* the pleasure they take from the thought that, if they pass away before next period (which happens with probability $1 - \mathcal{L}$), their assets will pass to their heirs.

The consumer's new value function is therefore:

$$\mathrm{v}_t(m_t) = \max_{c_t} \overbrace{\mathrm{u}(c_t)}^{\text{present}} + \overbrace{\underbrace{\mathcal{L}_{t+1}\mathfrak{v}(a_t)}_{\text{live}} + \underbrace{(1 - \mathcal{L}_{t+1})e(a_t)}_{\text{die}}}^{\text{future}}. \tag{20}$$

[7]As of 2024-05-15, the Fed Funds rate is 5.3 percent at an annual rate. $153b $\times$ 0.053/365 days $\approx$ $22 million. At the current inflation rate of 3.4 percent, he would only have to spend a little over $8 million a day to run down his real wealth – assuming the Fed Funds rate is the highest rate of return he can earn.

The literature has commonly used a "warm glow from bequests" motive of the form:

$$e(a) = \alpha \frac{(a + \underline{a})^{1-\rho}}{1-\rho} \tag{21}$$

where the ρ coefficient is the same as in the utility function for consumption (see, e.g., De Nardi (2004)), and the α coefficient controls the importance of the bequest motive relative to the utility from consumption. To aid in both estimating and interpreting these parameters, we use an alternative parameterization that preserves the form of the bequest motive, similar to Love, Palumbo, and Smith (2009):

$$\gamma \equiv \alpha^{-1/\rho}, \quad \kappa \equiv \underline{a} \cdot \gamma \iff \alpha \equiv \gamma^{-\rho}, \quad \underline{a} \equiv \kappa/\gamma. \tag{22}$$

Rather than scaling warm glow utility with α, we instead can interpret γ and κ as describing the "consumption function at death": this person acts *as if* just after they die, they will consume one last time according to $c = \gamma a + \kappa$. Thus γ can be labeled as the "bequest MPC" and κ is the "bequest intercept".[8]

According to the evidence from historian Fredrick Cople Jaher (1980)'s chronicle of the behavior of the richest Americans since the Revolution, the bequest motive seems unlikely to be an important motivation, at least according to their own words. Jaher presents a feast of quotations articulating a host of motivations for extreme wealth accumulation; but among their many explanations of their behavior, almost none of the tycoons under study mention anything resembling the bequest motive as formulated in the standard academic life cycle literature. Andrew Carnegie was most explicit: "I would rather leave my son a curse than the almighty dollar."[9]

As mentioned above, the Fed has for many years asked respondents a question about their motivations for saving.[10] While respondents' answers are fairly heterogeneous, the SCF has a suggested aggregation of the many different answers into categories that correspond approximately to some of the motivations that the academic literature has considered. The category that best matches the "bequest" motivation is "family," which includes "to help the kids out" and "to leave an estate" but also includes saving for "weddings and other ceremonies"

[8]There is another way to re-parameterize the bequest motive that provides a third interpretation: a characterization of the optimal consumption function when the agent is *guaranteed* to die at the end of the period. This function is defined by two values: the level of market resources m below which this person would consume everything (i.e. the bequest motive does not bind) and the constant MPC above that level as they allocate a constant fraction of additional resources to their bequest. The "terminal MPC" from that parameterization is very closely related to the "bequest MPC" γ that we have estimated, and the "terminal kink point" is identical to our estimated "bequest intercept" κ.

[9]To the astonishment of skeptics, he gave away more than 90 percent of his wealth before he died.

[10]See Aladangady et al. (2023), and specifically the material starting at line 848 in the documentation.

Table 2: Most Important Reason for Saving, 1995-2022 SCF Waves

Reason	Proportion	Explanation
"Family"	0.06	bequests; weddings, etc
"Retirement"	0.27	
"Liquidity / The Future"	0.40	
"Purchases"	0.13	cars, vacation homes, etc
"Cannot save"	0.06	
Other	0.08	

and "to have children / a family."

An ambitious agenda would be to tabulate the answers to this survey question for people at different ages and then to construct a model that would imply the same age pattern of motivations as the data. For example, one might find that for people who have just entered the labor market (say, the 26-30 age group) the survey responses showed that saving for "retirement" was not a priority, while saving for "purchases" and "liquidity" were important. In order for a model to be credible, its implications would need to comport with the survey data. Our aim here is to take a first step in that direction, by constructing a model that is at least consistent with the responses of retirees.

Table 2 presents the responses to this question for college-educated households older than age 70 from the 1995 to the 2022 waves of the SCF. If bequests were a primary motivation for saving for most (college-educated) people, it would be surprising for them to mention this motivation so rarely. The "Family" motivation, which includes bequests, accounts for only 6% of responses. Given these (and other) objections to the bequest motive, as well as the problems of the model without a bequest motive, it is natural to consider alternative modifications to the framework.

3.3 Wealth in the Utility Function

Explaining the motivation to save is one of those places where economists' new openness to the idea of taking seriously what people say about their motivations has bite. While it is reasonable to be skeptical about taking quotations from Jaher (1980) at face value, Carroll (2000) argues that essentially all of the motivations articulated (wealth brings power; wealth allows philanthropy; wealth is a way of "keeping score"; and more) can be captured in a mathematical formulation in which wealth enters the utility function directly.

The most general way that economists have for incorporating people's motivations into models of behavior is simply to assume that the decision-maker directly values something—in this case, wealth. The question is how best to incorporate the item in the utility function to study any particular question. Carroll (2000), for example, proposed a utility function specifically designed to capture saving behavior as wealth approached infinity, and accomplishing that goal required some mathematical structure that delivered the desired results but was unwieldy (and not obviously necessary for explaining the behavior of the bottom 99 percent, whose wealth does not approach infinity).[11]

3.3.1 Money in the Utility Function

There is a literature in macroeconomics, pioneered by Miguel Sidrauski (1967), that has long included "money" (in the monetary sense) in the utility function of the representative agent in one form or another. A well-known paper by Rotemberg (1984) proposed a specific utility function designed to capture the stability of the ratio of money to GDP, and the model was estimated on U.S. data in Poterba and Rotemberg (1986).

The structure of their utility function is

$$\mathrm{u}(c, \ell) = \frac{\left(c^{1-\delta}\ell^{\delta}\right)^{1-\rho}}{1-\rho} \tag{23}$$

where ℓ captures the the ℓiquidity services provided by money-holding.

To be clear, the aim of that literature was to explain the holding of ℓ defined as dollar cash holdings, to study questions like the "velocity" of money and the role of money supply and money demand in determining interest rates– *not* to explain saving behavior of individual households.

3.3.2 Wealth In the Utility Function: Cobb-Douglas Form

But for the question of how to incorporate wealth in the utility function, Tzitzouris (2024) proposed a mathematically identical formulation in which assets a takes the place of ℓ in the Rotemberg-Poterba utility function.[12] The Cobb-Douglas functional form is commonly used in other contexts, but does not seem to have been previously explored as a formulation for how to put a direct wealth-holding motive in the utility function.

The upshot is that if we credit the proposition that the ownership of wealth yields utility, then there is good precedent for the functional form of Tzitzouris (2024). Henceforth we will call this the Tzitzouris-Rotemberg-Poterba or "TRP" utility function. This functional form

[11] Specifically, a separable utility-from-wealth function was added to the maximizer's objective and with a coefficient of relative risk aversion smaller than that for the utility from consumption.

[12] The question of whether a or m should be in the utility function is of little importance; here we prefer a because assets after consumption are immune to considerations of whether the time period is a year, a quarter, or a month.

is attractive because it maintains the homotheticity required for normalization; moreover, the marginal utility gained from holding additional wealth is greater when consumption is higher, and vice-versa, capturing a nuanced view of financial well-being. It is a relatively simple matter to solve the revised problem with wealth in the utility function using the TRP utility specification. The revised utility and value functions of the problem are:

$$\mathrm{u}_t(c_t, a_t) = \frac{\left(c_t^{1-\delta} a_t^{\delta}\right)^{1-\rho}}{1-\rho}, \tag{24}$$

$$\mathrm{v}_t(m_t) = \max_{c_t} \ \mathrm{u}(c_t, a_t) + \mathcal{L}_{t+1}\mathfrak{v}_t(a_t). \tag{25}$$

We are open to the possibility that wealth in the utility function is a reduced form for other motivations— indeed, that was the thesis of Carroll (2000). In particular, the fact that in our SCF table above, "Liquidity / The Future" is the most popular answer among retirees for the most important reason to save might signal that the forms of uncertainty that we can measure – like the Ameriks et al. (2020) calculations about nursing home expense risks – constitute only a fraction of the risks retirees might worry about. Maintaining a buffer stock of wealth to protect oneself against "unknown unknowns" is plainly reasonable, but also nearly impossible to calibrate in a quantitative model. That is, we would need to have an accurate representation of people's beliefs about the magnitude, frequency, and persistence of "unknown unknowns". Indeed, even if you knew those answers, they would be, at best, "known unknowns." Directly incorporating wealth into the utility function may serve as a tractable proxy for this profound uncertainty aversion.

3.4 Comparisons to Other Models Familiar to Practitioners

Gordon and Joseph (2014) asserted that financial planning practitioners mostly used rules of thumb and heuristics to provide their advice. That paper aimed to introduce the key concepts of formal life-cycle modeling to the audience of practitioners. Since its publication, there appears to have been considerable movement in the direction advocated by its authors. A number of leading financial institutions have made available partial descriptions of proprietary life cycle models that they are developing. Here we describe only models that have been published in peer-reviewed journals and whose detailed characteristics are knowable.

Daga, Smart, and Pakula (2023) looks at the post-retirement period and normalizes all variables by retirement income, but it does not incorporate risk to permanent income, nor does it attempt to reconcile post-retirement with pre-retirement behavior. The model uses an additive utility of bequest with no shifter term, which has the implication that even income-poor households have a powerful bequest motive. The multigoal framework additionally disaggregates consumption expenses into different categories, each with the same CRRA coefficient. This reduces the effect of the diminishing marginal utility of consumption, such that the sum of utilities of consumption is greater than the utility of the sums of consumption. Moreover, each goal has an allocation coefficient (relative importance) that is set subjectively by the researchers.

O'Hara and Daverman (2015) incorporate an additively separable utilty of bequest like the one described above, and like our models it incorporates both permanent and transitory shocks to income. However, the model uses Epstein-Zin preferences (see our above objections to this feature). Their model also assumes that there is no uncertainty (except for mortality) in the post-retirement period, because that assumption has the convenient implication that the post-retirement period is extremely simple; the portfolio share, in particular, should remain constant at the infinite-horizon Merton-Samuelson solution.

Lanski et al. (2022) uses a life-cycle consumption and portfolio choice model without any consideration of bequests like the one discussed above (although other aspects of the model are simpler than our specification).

There have also been publications that partake of the mathematical flavor of life cycle modeling but do not use the intertemporal choice framework. Idzorek (2023) uses a static model of portfolio optimization, where the objective is to maximize the mean-variance trade off of a portfolio position with additional features such as non-pecuniary benefits. Because this model is not dynamic, it cannot examine questions like the role of increasing health and mortality risk with age, the degree of uncertainty in expenses, or any other question whose answer depends on changes in the consumer's circumstances as they age.

4 Estimation

4.1 Indirect Inference Described

Even if one knew all the parameters of the model (consumers' coefficient of relative risk aversion, etc), solving an optimization problem that includes the many real-world complications described above (especially those due to uncertainty) is such a formidable problem that it only became possible about 25 years ago (when solving such models took days).

But of course we do not know the best values to choose for unobservable parameters like relative risk aversion. The increasingly standard approach to this problem is the method of "indirect inference." Essentially, this means specifying the structure of your model except for the values of parameters that you cannot measure well (like time preference and risk aversion), and asking a numerical search algorithm to seek the values of those parameters that lets the model fit the data as well as it is capable of doing. This requires solving the model perhaps thousands of times, which is why indirect inference has only begun to come into its own recently, as computer speeds have become fast enough to tackle the problem.

4.2 Indirect Inference Implemented: Method of Simulated Moments

Because of this paper's focus on the drawdown failure, we are particularly interested in finding the optimal post-retirement choices for both the rate of spending and for portfolio allocation between safe and risky assets. The "method of simulated moments" finds the parameters that make the model's simulated moments (statistics), like the median wealth and the median portfolio share, match the corresponding empirical facts as closely as possible.

Consider an empirical moment q_i where $i \in \{1, ..., N\}$ and the corresponding simulated moment $\hat{q}_i(\theta)$, where θ is the vector of parameters that we are interested in estimating. By solving and simulating our structural model with different θ parameters, we can calculate the simulated moments $\hat{q}_i(\theta)$ for each parameter set. The method of simulated moments then consists of searching for the parameter set θ that minimizes the distance between the simulated versus empirical moments. This is done by minimizing the objective function:

$$\min_{\theta} \sum_{i=1}^{N} \left(\omega_i[q_i - \hat{q}_i(\theta)]\right)^2 \tag{26}$$

Here, ω_i is the weight of each moment in the objective function, representing the relative importance of each moment in the estimation process. For example, we might be more interested in matching the median wealth than the median portfolio share, and thus assign a higher weight to the former.

For our exercise, we are interested in matching the median wealth to income ratios throughout the life cycle, as well as the share of wealth held in equities as given by S&P's target date fund (TDF) glidepath. As noted in Aboagye et al. (2024), this age-dependent portfolio allocation is used by many individuals who make "default" decisions via their (former) employer's TDF plan. Because the SCF has relatively few observations of older households, we aggregate the data into 5-year age bins to smooth out the noise from the small samples. Starting at age 25, we calculate the median wealth-to-income ratio as follows: Wealth is defined as the sum of all assets and liabilities, including financial assets, housing, vehicles, and debt. For income, we use the sum of all wages, salaries, Social Security, and retirement income, excluding capital gains and other non-recurring income. We then calculate the wealth to income ratio of every household in the age bin and remove households with an income of zero. The median wealth-to-income ratio is calculated from the remaining households.

Because the SCF data is increasingly sparse at older ages, the raw empirical moments show a "zig-zag" pattern above age 75 due to the small sample size, even with 5-year-wide bins.

In our structural model, we assume retirement occurs automatically at exactly age 65, whereas in the data we observe retirement at different ages, but predominantly between ages 60 and 70. Therefore, we omit moments for ages 60 to 69, where our (relatively parsimonious) model is known to deviate most strongly from empirical reality, but keep the data for ages 70 and above to capture the behavior of retirees.

Considering the moments we have chosen, it is clear that there is an imbalance: There are more wealth to income moments than portfolio share moments (12 vs 5), and the portfolio share moments lie between 0 and 1, whereas the wealth to income ratios can be much larger. To account for this, we set the weights to normalize the wealth to income ratios by the highest ratio in the data, making them all lie between 0 and 1, and adjust the weights for the portfolio share moments by a factor of 12/5, so that the two sets are about equally weighted in the estimation process, despite the difference in scale and number of moments.

Having chosen the moments we are interested in matching and their respective weights, we can now proceed to a discussion of estimating the parameters of our various models. We use the Econ-ARK project's `HARK` package to solve and simulate the models, and `optimagic` (Gabler (2022), formerly `estimagic`) to perform the estimation process. The estimation process is computationally expensive, requiring the solving and simulation of the model given a parameter set many times.[13]

4.3 Indirect Inference Results

Table 3 shows the results of our estimation exercise, and the fit of the three estimated models is plotted below, in Figure 1 and Figure 3.

For the standard Life-Cycle Portfolio (LCP) model, we estimate a CRRA coefficient of over 9, which lines up with the literature finding that portfolio choice models require a very high ρ in order to prevent agents from holding excessively risky portfolios (often 100% equities). In general, "typical" values for the CRRA coefficient (from experimental evidence and other contexts) are considered to be between 1 and 5. The "criterion" column of Table 3 lists the minimum value that the objective function achieves for each model– the

[13]Because our moments require simulation, our moment generating functions $\hat{q}_i(\theta)$ have no analytical derivatives with respect to the parameters, so we must rely on numeric differentiation and optimization algorithms to find the optimal parameter set. We use the `tranquilo` algorithm (Gabler et al. (2024)), which stands for TrustRegion Adaptive Noise robust QuadratIc or Linear approximation Optimizer. The `tranquilo` optimizer has many attractive features, such as being able to evaluate the function in parallel and estimate even noisy objective functions with many parameters, as well as being especially designed for least squares problems, such as the MSM.

Table 3: Estimated Parameter Values

Model	Criterion	CRRA (ρ)	Wealth Share (δ)	Bequest MPC (γ)	Bequest Intercept (κ)
Life-Cycle Portfolio	1.049	9.195			
Warm-Glow LC Portfolio	0.104	4.594		0.112	0.321
TRP / WIUF LC Portfolio	0.306	5.136	0.248		

smallest weighted squared distance between simulated and empirical moments. The LCP model performs poorly by this measure, as illustrated in the figures. LCP consumers want to quickly run down their wealth at older ages, as the probability of death increases with age and they know that they "can't take it with them." To try to match the empirical wealth trend (red dashed line in Figure 1), which holds steady at a high wealth-to-income ratio at older ages, the LCP model (solid blue line) exceeds observed wealth accumulation through the working life. Even then, the wealth drawdown is so rapid that the best the LCP model can achieve is to overshoot wealth significantly before age 65, and then vastly undershoot it in retirement.

As discussed above, there are multiple model features that can ameliorate or eliminate the wealth drawdown problem, beginning with a simple bequest motive. The Warm-Glow Portfolio model (orange lines on figures) estimates a much more realistic CRRA coefficient of $\rho \approx 4.6$. With a strong bequest motive, the Warm-Glow model is able to match the high levels of wealth observed deep into retirement. That is, these consumers do not quickly draw down their assets because they take great pleasure in passing their estate on to their heirs. Under our parameterization, we estimate that consumers act *as if* they will experience a final "consumption at death" of $c = 0.11a + 0.32$; this implies that if they were faced with guaranteed, imminent death (at some very old age), consumers would allocate to their heirs 89% of any resources in excess of one-third of their permanent income– most of their wealth.

This strong bequest motive at the very end of life propagates backward to more reasonable ages (albeit not as directly interpretable), and ultimately it applies for essentially *everyone*.[14] Indeed, the Warm-Glow model predicts that saving behavior *when young* is strongly motivated by the bequest motive. In Figure 2, we reproduce the median wealth moment fit for the Warm-Glow Portfolio model from Figure 1 and add a new model moment plot for agents whose bequest motive has been turned off. That is, we maintain the *same* ρ as estimated,

[14]In contrast, the literature has generally found (e.g. De Nardi (2004)) that the bequest motive comes into play only for relatively wealthy households, and is mostly inoperative around median wealth. This discrepancy might arise because of the simplified approach we have used here, matching *only* the median wealth-to-income ratio by age, rather than wealth levels conditional on income.

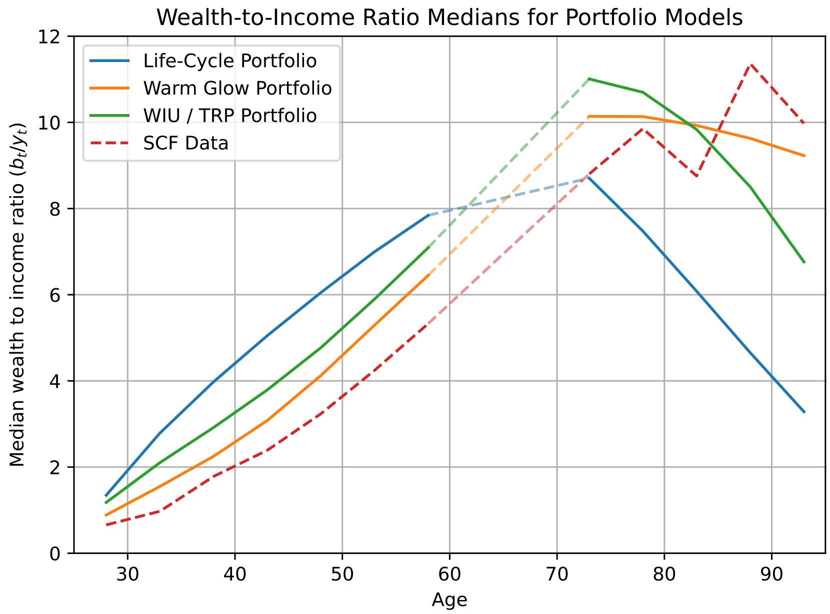

Figure 1: Median Wealth to Income Ratio for different portfolio models. Red dashed line indicates median wealth-to-income ratios for college-educated households in the SCF.

but set $\kappa = 1000$ so that marginally allocating resources to bequests has (essentially) no effect on expected utility– the agents are already almost satiated in this dimension.

Unsurprisingly, the drawdown failure returns in Figure 2 because retirees have no incentive to retain wealth as the likelihood of mortality rises. However, agents *also* drastically reduce their saving behavior in their working years and don't build nearly as large a nest egg for retirement. This is what we mean by an "implausibly strong" bequest motive: it drives wealth accumulation even in middle age, not just among the elderly. Recall from the discussion of Aladangady et al. (2023) and Jaher (1980) that very few older people ascribe their wealth-holding behavior to a bequest motive, and yet the Warm-Glow model has the saving choices of *40 year olds* driven by the urge to bequeath. This suggests that while the bequest model can match the drawdown pattern in the data, it does so for the wrong reasons. Even if a model can *mechanically* reproduce observed data features or hit empirical targets, that does not make it "right" or "true," especially if its underlying logic is implausible and contradictory to qualitative evidence. And as discussed above, the bequest motive is inconsistent with an investment advisor's fiduciary duty *to the client*. We include the Warm-Glow model in our presentation not to advocate for it, but merely to demonstrate that there are *multiple ways* for life-cycle models to generate more realistic wealth trajectories in retirement.

Our preferred specification also has the agents value wealth itself as a motivation to retain assets later in life, but in a way that is more consistent with qualitative responses. The

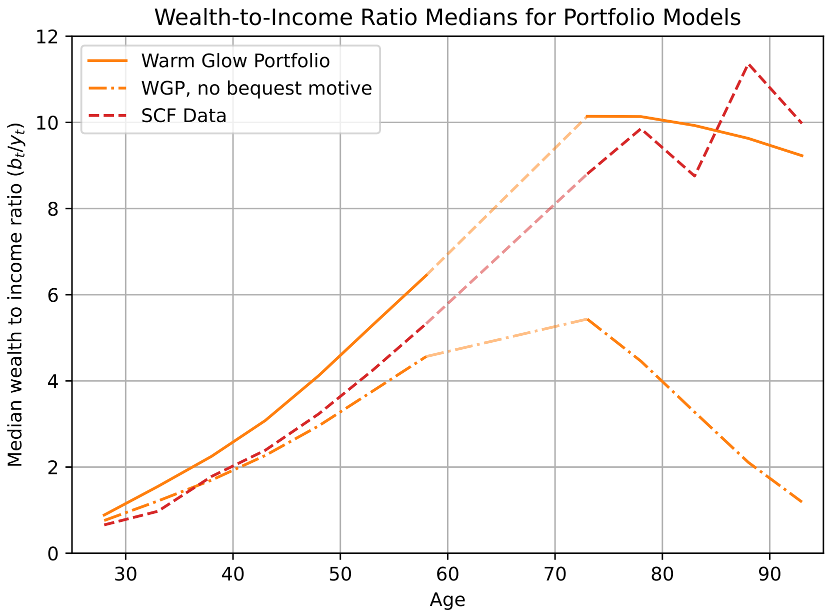

Figure 2: Median Wealth to Income Ratio in the estimated Warm-Glow Portfolio model (solid orange) vs with bequest motive turned off (dashed orange), as compared to college-educated households in the SCF (red). The bequest motive is (implausibly) a strong motivator of wealth accumulation among working age agents.

Wealth-in-Utility-Function (WIUF) / TRP Portfolio model estimates a CRRA ρ coefficient of about 5.14 and a wealth share of utility δ coefficient of 0.25. This result is significant because the CRRA ρ coefficient required to match the wealth accumulation patterns is significantly lower, and thus more plausible, than that of the standard Life-Cycle Portfolio choice model, whose high CRRA ρ has long been a puzzle in the literature. In the LCP model, high risk aversion is the primary tool to prevent excessive investment in risky assets. In the WIUF model, however, the direct utility from wealth provides an additional motive to hold safe assets (to ensure a minimum level of wealth utility), reducing the need for extreme risk aversion to match the observed portfolio shares. As seen in Figure 1, the WIUF / TRP model (green line) does not need to overshoot wealth accumulation in early life by nearly as much as the basic LCP model, as agents want to retain assets in retirement to generate utility directly. While the WIUF model still predicts a faster drawdown in late retirement than observed in the SCF data, it represents a substantial improvement over the standard LCP model and avoids the behavioral implausibility required by the Warm-Glow model. Compared to the Warm-Glow model, the WIUF / TRP specification does predict more wealth accumulation early in life, but for more immediate reasons: young consumers value money and liquidity *now*, rather than saving at age 35 to leave a large bequest at age 85.

Moreover, because the CRRA parameter doesn't need to be so high, the WIUF model can

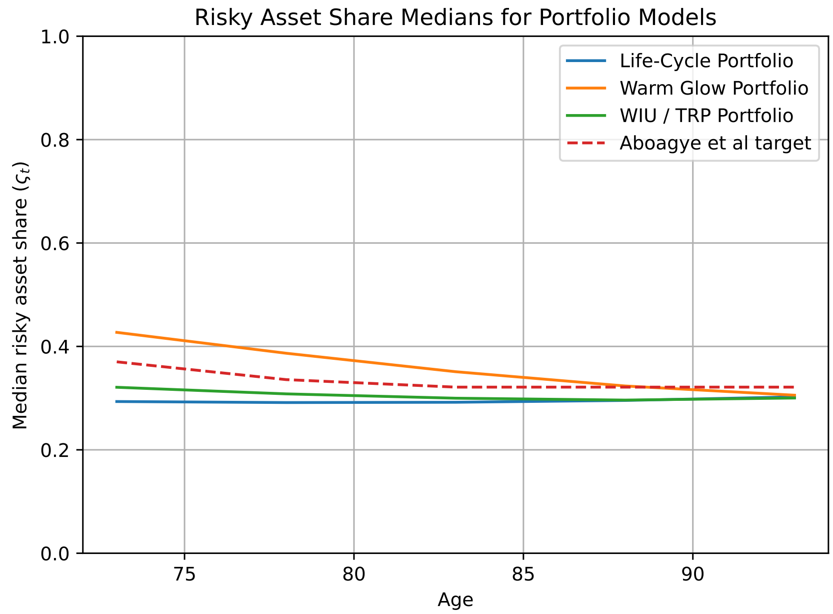

Figure 3: Median Portfolio Share for different portfolio models. The red line shows the target moments from Aboagye et al. (2024).

somewhat better match the target risky assets share moments (red dashed line in Figure 3). That paper presents the typical glidepath of target-date funds (TDFs) which provide a basis for much of commercial financial advice. While the whole life-cycle glidepath is provided in Aboagye et al. (2024), here we only target (and plot) those moments starting at age 70. The model fit with respect to risky asset share is comparable for the Warm-Glow model, generally matching the level and recommended shallow downward slope.

4.4 The Role of Subjective Beliefs

The results above depend on the standard modeling assumption that agents have rational expectations: they know and understand the true distribution of risks that they face. However, Velásquez-Giraldo (2023) reports that even college-educated people often have *subjective* beliefs about the distribution of stock returns that are pessimistic relative to observed historic returns– they believe stocks are considerably riskier given their mean return. Our estimates of the coefficient of relative risk aversion for the two models that could fit the wealth and portfolio shares reasonably well were around $\rho \approx 5$, the top of the "normal" range for risk aversion. If people *actually* believe that stocks are much riskier, then they would not need to be so risk averse for the model to fit the targets risky portfolio share around 30-40% in retirement. That is, for any level of risk aversion, agents that subjectively believe that stocks

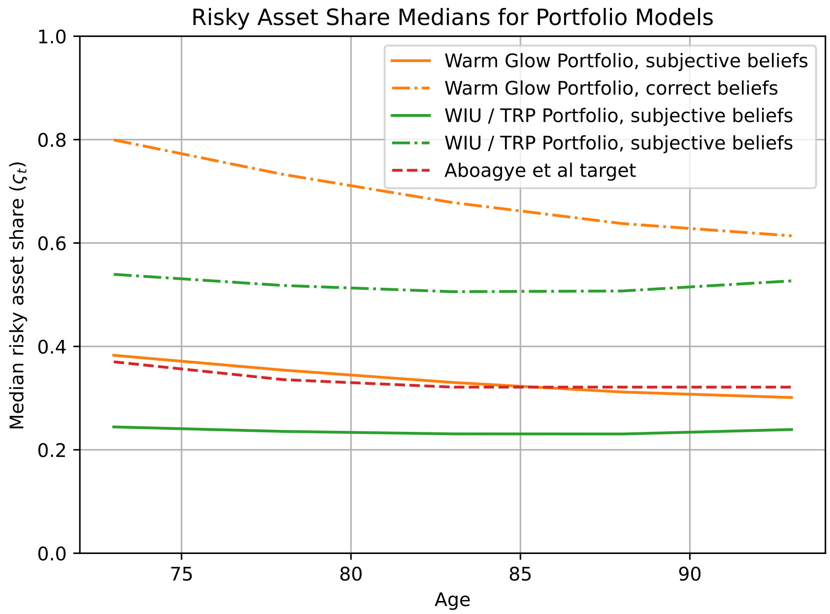

Figure 4: Median risky asset share under parameter estimates with incorrect subjective beliefs about stock returns. Solid lines show fitted portfolio shares if agents *believe* stocks are much riskier; dash-dot lines show portfolio shares with same estimated preferences but *correct* beliefs about stock returns. Red dashed line is target moments from Aboagye et al. (2024).

are riskier will want to hold a lower share of risky assets, so risk aversion must decrease to raise risky asset holdings back to the empirical target.

The analysis in Velásquez-Giraldo (2023) is quite rich and incorporates heterogeneous beliefs within and across demographic groups, but we need only perform a relatively straightforward exercise to illustrate our point. We repeat the estimation procedure from above, but have the agents *solve* their problem as if the equity premium were $\widetilde{\varphi} \approx 0.08\%$ and the standard deviation of log stock returns were $\widetilde{\sigma_{\mathbf{r}}} \approx 0.48$– higher return but with massively greater risk. When the agents live out the model in simulation, the "true" stock return distribution is used ($\varphi = 0.03$ and $\sigma_{\mathbf{r}} = 0.2$). To best fit the target moments, we estimate $\rho = 3.26$ for the Warm-Glow Portfolio model and $\rho = 2.84$ for the Wealth-in-Utility Portfolio model, right in the middle of the typical range of risk aversion.[15]

If agents have incorrect subjective beliefs, it's reasonable to ask what the model predicts they would do if they knew the *true* distribution of stock returns. That is, use the preference parameters estimated under subjective stock returns, but give the agents correct beliefs as in the original estimation. Risky asset shares for both sets of beliefs are plotted on Figure 4.

[15]The re-estimated Cobb-Douglas share for wealth δ and bequest motive parameters (κ, γ) are comparable to their original values. The basic Life-Cycle Portfolio model even more poorly fits the target moments at its estimated ρ.

Unsurprisingly, agents would want to hold a considerably greater share of their wealthy in risky assets if they learned that stock returns have much lower variance than they previously believed. The upshot is that if financial advisers want to use life-cycle models to provide normative advice to their clients for how "best" to allocate their wealth over time, and to inform those models with inferences about preferences from empirical observations, they must think carefully about the beliefs that generated the observed behavior. This suggests a crucial role for advisors not just in optimization, but also in educating clients about realistic expectations, as optimal advice depends heavily on the client's beliefs about risk and return.

5 Conclusion

To thoughtful academics, it has long been disturbing that the financial advice industry has paid so little attention to our hard work in constructing and solving impressively sophisticated dynamic stochastic optimization models of financial behavior. Those of us with a bit of humility have always suspected that the failure has been on our side: If all we could offer was models that produced risible advice like "everyone should spend down their wealth to zero and live pension-check to pension-check," while financial analysts' real world experience told them that such advice would get them fired, then it was reasonable to disregard the academic literature.

The thesis of this paper, however, is that a confluence of factors has now finally brought us to a point where state-of-the-art mathematical/computational life-cycle optimization models can provide advice that makes sense– so long as the model assumptions are also disciplined by survey data and the practical knowledge of financial advisors.

Much more remains to be done to improve the models further; for example, a question of great practical importance that is now just at the edge of possibility of being computationally solved is to calculate the implications of nonfinancial (principally, housing) wealth for optimal financial choice. Because homeownership is such a complex phenomenon, the academic literature is only now reaching the point at which it may be possible to answer questions like, "If I own a house, how should I modify my spending and portfolio plans to take that into account?" We do know the *direction* of the effect. Kimball (1993) shows that the addition of a new uncontrollable risk reduces the optimal choice of exposure to controllable risks like the stock market. But *by how much* one's stock exposure should be reduced because of house-price risk can only be answered by solving a quantitatively plausible model.

It would be a better world if financial advice could be justified as reflecting the mathematically optimal solution to a well-defined problem. Not only would academics have the satisfaction of knowing that they had finally come close to fulfilling the vision of Modligliani and Brumberg 70 years ago. Financial analysts could also sleep more soundly in the knowledge that the advice they were giving were what many people probably think it already is: The adaptation to the client's particular circumstances of the advice that is the best that can be delivered by the latest high-tech computational optimization tools. The time seems ripe for a much closer collaboration between academia and the financial industry in building this better world by combining computational rigor with practical insight.

References

(2021). "Wealth transfers and their economic effects." techreport, Productivity Commission, Canberra, AU.

Aboagye, K., Page, S., Schafer, L., and Tzitzouris, J. (2024). "Personalized target date funds." *The Journal of Portfolio Management*, 50(5).

Aladangady, A., Bricker, J., Chang, A. C., Goodman, S., Krimmel, J., Moore, K. B., Reber, S., Volz, A. H., and Windle, R. A. (2023). "Changes in U.S. Family Finances from 2019 to 2022: Evidence from the Survey of Consumer Finances." *Federal Reserve Bulletin*.

Ameriks, J., Caplin, A., Laufer, S., and Van Nieuwerburgh, S. (2011). "The joy of giving or assisted living? Using strategic surveys to separate public care aversion from bequest motives." *The journal of finance*, 66(2): 519–561.

Ameriks, J., Briggs, J., Caplin, A., Shapiro, M. D., and Tonetti, C. (2020). "Long-Term-Care Utility and Late-in-Life Saving." *Journal of Political Economy*, 128(6): 2375–2451.

Cagetti, M. (2003). "Wealth accumulation over the life cycle and precautionary savings." *Journal of business & economic statistics: a publication of the American Statistical Association*, 21(3): 339–353.

Carroll, C. D. (2000). *Why Do the Rich Save So Much?*, chap. 14. Harvard University Press.

Carroll, C. D. (2006). "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems." *Economics Letters*, 91(3): 312–320.

Carroll, C. D. (2023). "Theoretical Foundations of Buffer Stock Saving." *Revise and Resubmit, Quantitative Economics*.

Carroll, C. D., Overland, J., and Weil, D. N. (2000). "Saving and Growth with Habit Formation." *American Economic Review*, 90(3): 341–355.

Christensen, B. J., Kallestrup-Lamb, M., and Kennan, J. (2022). *Consumption and Saving after Retirement.* National Bureau of Economic Research.

Crawford, R. (2018). "The use of financial wealth in retirement." techreport, The Institute for Fiscal Studies.

Daga, A., Smart, T., and Pakula, D. C. (2023). "The Multi-Goal Framework: Why Practitioners Have Not Adopted the Lifecycle Model—Yet." *The Journal of Retirement*, 10(3): 71–94.

De Nardi, M. (2004). "Wealth inequality and intergenerational links." *The Review of Economic Studies*, 71(3): 743–768.

De Nardi, M., French, E., and Jones, J. B. (2010). "Why do the elderly save? The role of medical expenses." *The journal of political economy*, 118(1): 39–75.

De Nardi, M., French, E., and Jones, J. B. (2016). "Savings after retirement: A survey." *Annual Review of Economics*, 8(1): 177–204.

Epstein, L. G. and Zin, S. E. (1991). "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis." *Journal of Political Economy*, 99(2): 263–286.

Friedman, M. (1957). *The permanent income hypothesis*, pp. 20–37. Princeton University Press.

Fulford, S. and Low, D. (2024). "Expense Shocks Matter." *Working Paper.*

Gabaix, X., Agarwal, S., Laibson, D., and Driscoll, J. C. (2010). "The Age of Reason: Financial Decisions over the Life Cycle and Implications for Regulation." *Brookings Papers on Economic Activity: Fall 2009*, p. 51.

Gabler, J. (2022). "A Python Tool for the Estimation of Large Scale Scientific Models."

Gabler, J., Gsell, S., Mensinger, T., and Petrosyan, M. (2024). "Tranquilo: An Optimizer for the Method of Simulated Moments." techreport, University of Bonn.

Gordon, I. and Joseph, T. (2014). "Retirement Income Research: What Can We Learn from Economics?" *The Journal of Retirement*, 1(4): 118–128.

Hamilton, M. (2001). “The Financial Circumstances of Elderly Canadians and the Implications for the Design of Canada’s Retirement Income System.”

Hattrem, A. (2022). “Mindre inntekter, men storre formue for pensjonistene.” techreport, Statistics Norway.

Heimer, R. Z., Myrseth, K. O. R., and Schoenle, R. S. (2019). “YOLO: Mortality Beliefs and Household Finance Puzzles.” *The Journal of Finance*, 74(6): 2957–2996.

Hurd, M. D. (1987). “Savings of the elderly and desired bequests.” *The American Economic Review*, pp. 298–312.

Hurd, M. D. (1989). “Mortality Risk and Bequests.” *Econometrica*, 57(4): 779.

Idzorek, T. M. (2023). “Personalized Multiple Account Portfolio Optimization.” *Financial Analysts Journal*, 79(3): 155–170.

Jaher, F. C. (1980). *The Gilded Elite: American Multimillionaires, 1865 to the Present*. Croon Helm, London.

Kimball, M. S. (1993). “Standard Risk Aversion.” *Econometrica*, 61(3): 589–611.

Kopecky, K. A. and Koreshkova, T. (2014). “The Impact of Medical and Nursing Home Expenses on Savings.” *American Economic Journal: Macroeconomics*, 6(3): 29–72.

Lanski, I., Paramaguru, R., Phoa, W., Wang, Y., and Hammond, P. B. (2022). “Using a Life Cycle Model to Design a Target Date Glidepath.” *The Journal of Portfolio Management*, 48(4): 228–240.

Ljunge, M., Lockwood, L., and Manoli, D. (2013). “The Drawdown of Assets during Retirement: Evidence from Sweden.” *NBER Working Paper*, (NB13-03).

Love, D. A., Palumbo, M. G., and Smith, P. A. (2009). “The trajectory of wealth in retirement.” *Journal of Public Economics*, 93(1): 191–208.

Michaelides, A. (2002). “Buffer Stock Saving and Habit Formation.” *SSRN Electronic Journal*.

Modigliani, F. and Brumberg, R. (1954). “Utility analysis and the consumption function: An interpretation of cross-section data.” *Post-Keynesian Economics*, 1(338-436).

MORTENSON, J. A., SCHRAMM, H. R., AND WHITTEN, A. (2019). "The Effects of Required Minimum Distribution Rules On Withdrawals from Traditional IRAs." *National Tax Journal*, 72(3): 507–542.

NIIMI, Y. AND HORIOKA, C. Y. (2019). "The wealth decumulation behavior of the retired elderly in Japan: The relative importance of precautionary saving and bequest motives." *Journal of the Japanese and International Economies*, 51: 52–63.

O'HARA, M. AND DAVERMAN, T. (2015). "Reexamining "To vs. Through": What New Research Tells Us about an Old Debate." *The Journal of Retirement*, 2(4): 30–37.

POTERBA, J. AND ROTEMBERG, J. (1986). *Money in the Utility Function: An Empirical Implementation*. National Bureau of Economic Research.

POTERBA, J., VENTI, S., AND WISE, D. A. (2018). "Longitudinal determinants of end-of-life wealth inequality." *Journal of Public Economics*, 162: 78–88.

RAMSEY, F. P. (1928). "A Mathematical Theory of Saving." *The Economic Journal*, 38(152): 543–559.

ROTEMBERG, J. J. (1984). "A monetary equilibrium model with transactions costs." *The journal of political economy*, 92(1): 40–58.

SAMUELSON, P. A. (1969). "Lifetime Portfolio Selection By Dynamic Stochastic Programming." *The Review of Economics and Statistics*, 51(3): 239–246.

SIDRAUSKI, M. (1967). "Rational choice and patterns of growth in a monetary economy." *The American Economic Review*, pp. 534–544.

TZITZOURIS, J. A. (2024). "A novel model to measure utility from consumption and wealth." *SSRN Electronic Journal*.

VELÁSQUEZ-GIRALDO, M. (2023). "Life-Cycle Portfolio Choices and Heterogeneous Stock Market Expectations." *Working paper*.

VENTURA, L. AND HORIOKA, C. Y. (2020). "The wealth decumulation behavior of the retired elderly in Italy: the importance of bequest motives and precautionary saving." *Review of Economics of the Household*, 18(3): 575–597.

A Appendix: Solution Methods

In this appendix, we provide additional details for how the intertemporal optimization problems in our life-cycle models are solved. The method for finding consumption in the basic life-cycle and warm-glow bequest models is taken directly from Carroll (2006), and the method for finding optimal portfolio shares (in all models) uses standard techniques. The wealth-in-utility model has a novel solution method.

A.1 Optimal Portfolio Share

Each of our models has two control variables– consumption c_t and the risky asset share ς_t– that are chosen simultaneously, but we solve them as if they are chosen sequentially. The agent's state variable when they choose ς_t is thus their assets after buying consumption, $a_t = m_t - c_t$. The agent's problem at this point is:

$$\mathfrak{v}_t(a_t) = \max_{\varsigma \in [0,1]} \ \mathbb{E}_t \left[\boldsymbol{\beta}_{t+1} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_{t+1}) \right] \quad \text{s.t.} \quad \mathcal{R}_{t+1} = \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma. \tag{27}$$

Note that the survival probability $\mathcal{L}_{t+1}$ is *not* included in this problem because it is a constant scaling factor that cannot affect the optimal choice of ς_t. Likewise, the warm-glow bequest motive $e(a_t)$ is also irrelevant because it is constant with respect to ς.

We will not reproduce the proof here, but it can be shown that the first order condition is necessary and sufficient to characterize the optimal risky portfolio share as long as v_{t+1} is strictly concave (which can also be proven). Taking the derivative of the maximand with respect to ς and equating it to zero yields:

$$\mathbb{E}_t \left[\boldsymbol{\beta}_{t+1} (\mathbf{R}_{t+1} - \mathsf{R}) a_t \mathrm{v}'_{t+1}((\mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma) a_t + \theta_{t+1}) \right] = 0. \tag{28}$$

The left-hand side is monotone in ς and thus the FOC has at most one interior solution. The constraint that the consumer cannot short the risky asset ($\varsigma \geq 0$) is never binding as long as mean risky return exceeds the risk-free return. However, the constraint that they cannot short the risk-free asset to buy even more of the risky asset ($\varsigma \leq 1$) *does* sometimes bind. Our algorithm for find the optimal ς is as follows:

1. Fix an exogenous grid of a_t values from 0 to a large value (say, a wealth-to-income ratio of 100).

2. For each a in that grid, evaluate the left-hand side of the FOC above. If it is less than or equal to zero, then $\varsigma = 1$ for that gridpoint. Otherwise, continue.

3. Solve the FOC for ς using a standard numeric rootfinder (e.g., Brent's method), bounded above by 1 and below by the Samuelson risky asset share.

Samuelson (1969) characterized the optimal risky asset share in the absence of labor income risk, which is also the lower limit of the risky asset share as a becomes arbitrarily large (and thus labor income risk becomes increasingly irrelevant). This value can thus be used both to bound the search for an interior solution to the FOC, and in the extrapolation of the risky share policy function above the top gridpoint. Specifically, we construct the policy function as a linear interpolant over the (a, ς) gridpoints with an exponential decay extrapolation to the Samuelson limit above the top gridpoint.

As discussed further below, marginal value $\mathrm{v}'_{t+1}(m_{t+1})$ in all of our models is simply equal to the marginal utility of consumption at that value of market resources. Hence we never need to represent the value function itself, as the consumption function in $t+1$ (and preference parameters) carries all information needed to make optimal choices in period t.

A.2 Optimal Consumption in Baseline and Warm-Glow Bequest Models

Stepping back within the period, we can now consider the choice of optimal consumption, taking as given that the agent will optimally allocate their assets between the risky and risk-free assets. For our first two models, we express the *continuation value* over a_t as:

$$\bar{\mathfrak{v}}_t(a_t) = \begin{cases} \mathcal{L}_{t+1}\mathfrak{v}_t(a_t) & \text{in baseline life-cycle model} \\ \mathcal{L}_{t+1}\mathfrak{v}_t(a_t) + (1-\mathcal{L}_{t+1})e(a_t) & \text{in warm-glow bequest model} \end{cases}. \tag{29}$$

The optimization problem in the consumption-choice step can then be written as simply:

$$\mathrm{v}_t(m_t) = \max_c \left[\frac{c^{1-\rho}}{1-\rho} + \bar{\mathfrak{v}}_t(m_t - c) \right]. \tag{30}$$

This has a first order condition that is necessary and sufficient to characterize optimal consumption:

$$c_t^{-\rho} - \bar{\mathfrak{v}}'_t(m_t - c_t) = 0 \Longrightarrow c_t = \bar{\mathfrak{v}}'_t(a_t)^{-1/\rho}. \tag{31}$$

The algorithm for solving for optimal consumption is thus:

1. Fix an exogenous grid of a_t values from 0 to a large value (say, a wealth-to-income ratio of 100); this can be the same or a different grid from above.

2. For each a in the grid, evaluate the right-hand side of (31), yielding the c_t consistent with this level of end-of-period assets.

3. Find the associated decision-time state by inverting the intraperiod budget: $m_t = a_t + c_t$.

4. Construct the consumption function as a linear interpolant over those (m_t, c_t) pairs, adding a point at $(0, 0)$ to incorporate the liquidity-constrained portion of the consumption function.

5. Construct the marginal value function $\mathrm{v}_t'(m_t)$ as the composition of the marginal utility function and the consumption function.

In the final step, we apply the standard envelope condition logic, which says that the marginal value of holding just a bit more market resources is equal to the marginal utility of consuming a tiny bit more (relative to the optimal level). This is obvious when the consumer is liquidity constrained: if they had a bit more cash-on-hand, they would consume that marginal dollar, yielding the marginal utility of consumption and *still* end the period with no assets. When the consumer is not constrained, the first order condition for optimal consumption means that they were indifferent to the allocation of the last bit of resources between consumption and saving. Hence the marginal value of a bit more market resources equals both the marginal continuation value *and* the marginal utility of consumption.

A.3 Optimal Consumption in the Wealth-in-Utility Model

With a standard CRRA utility function over consumption, marginal utility of consumption is very simple expression, and so the first order condition for optimal consumption was trivial to solve, as in (31). With the alternate preferences in the Wealth-in-Utility model, this is not the case. Recall that utility in this model involves a Cobb-Douglas aggregation of consumption c_t and assets a_t inside of a CRRA function:

$$\mathrm{u}(c, a) = \frac{\left(c^{1-\delta} a^{\delta}\right)^{1-\rho}}{1-\rho} = \frac{\left(c^{1-\delta} (m-c)^{\delta}\right)^{1-\rho}}{1-\rho}. \tag{32}$$

Fixing market resources m_t at some value of interest, the marginal return to utility from consumption is:

$$\frac{\mathrm{d}}{\mathrm{d}c}\mathrm{u}(c, m-c) = \left[(1-\delta)c^{-\delta}(m-c)^{\delta} - \delta c^{1-\delta}(m-c)^{\delta-1}\right] \cdot \left((m-c)^{\delta}c^{1-\delta}\right)^{-\rho} \tag{33}$$

$$= \left[(1-\delta)\left(\frac{c}{m-c}\right)^{-\delta} - \delta\left(\frac{c}{m-c}\right)^{1-\delta}\right] \cdot \left((m-c)^{\delta}c^{1-\delta}\right)^{-\rho} \tag{34}$$

$$= \left[(1-\delta)\left(\frac{c}{a}\right)^{-\delta} - \delta\left(\frac{c}{a}\right)^{1-\delta}\right] \cdot \left(aa^{\delta-1}c^{1-\delta}\right)^{-\rho} \tag{35}$$

$$= \left[(1-\delta)\chi^{-\delta} - \delta\chi^{1-\delta}\right] \cdot \left(a\chi^{1-\delta}\right)^{-\rho}, \quad \chi \equiv (c/a). \tag{36}$$

These algebraic manipulations will prove useful when we solve the first order condition momentarily. With wealth-in-utility preferences, the agent's optimal consumption problem is:

$$\mathrm{v}_t(m_t) = \max_{c}\left[\frac{\left(c^{1-\delta}(m_t-c)^{\delta}\right)^{1-\rho}}{1-\rho} + \bar{\mathfrak{v}}_t(m_t-c)\right]. \tag{37}$$

Consider the first order condition for optimality by taking the derivative of the maximand with respect to c and equating it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}c}\mathrm{u}(c_t, m_t-c_t) - \bar{\mathfrak{v}}_t'(m_t-c_t) = 0. \tag{38}$$

We can substitute the final form of the marginal return to consumption, move the second term to the right-hand side, and then rearrange slightly to get:

$$\left[(1-\delta)\chi_t^{-\delta} - \delta\chi_t^{1-\delta}\right] \cdot \left(a_t\chi_t^{1-\delta}\right)^{-\rho} = \bar{\mathfrak{v}}_t'(a_t) \Longrightarrow$$

$$\left[(1-\delta)\chi_t^{-\delta} - \delta\chi_t^{1-\delta}\right]^{-1/\rho} \cdot \chi_t^{1-\delta} = \underbrace{\bar{\mathfrak{v}}_t'(a_t)^{-1/\rho}/a_t}_{\equiv\ \omega_t}. \tag{39}$$

Note that the left-hand side of the rearranged FOC is monotonically increasing with respect to $\chi = c/a > 0$, starting from zero and growing without bound. Moreover, the RHS (which uses *only* information about the continuation value through a_t) must be strictly positive, as both marginal value and end-of-period assets are strictly positive (the consumer will never choose $a = 0$ with these preferences because it would yield infinitely negative utility). Hence the first order condition has a unique solution in χ_t for each a_t.

Unlike with basic CRRA utility, there is no closed form solution for (39) to recover χ_t from ω_t. While it would be possible to use a rootfinder to solve (39) for each value of ω_t as it comes up during the solution process, it is more efficient to pre-compute a function that accurately maps from ω to χ. We now describe a method for constructing such a function by working with the inverse relationship.

First, note that the expression in square brackets is not positive for all values of $\chi > 0$, with the bounding condition defined by:

$$(1-\delta)\chi^{-\delta} - \delta\chi^{1-\delta} > 0 \implies (1-\delta)\chi^{-\delta} > \delta\chi^{1-\delta} \implies \frac{1-\delta}{\delta} > \chi. \tag{40}$$

The expression in square brackets is raised to a negative power, hence we need only to consider values of $\chi \in (0, (1-\delta)/\delta)$. These χ values represent the *range* of the function that maps from ω to χ, and we want our constructed approximation to cover as much of that range as possible. To do so, we will use the logit transformation to map from an auxiliary variable $z \in \mathbb{R}$ to $\chi \in (0, (1-\delta)/\delta)$:

$$\chi = \frac{\exp(z)}{1+\exp(z)} \cdot \frac{1-\delta}{\delta}. \tag{41}$$

The domain of the function we want to approximate is $\omega > 0$ or $\omega \in \mathbb{R}_+$, so we can work with $\log(\omega)$ to ensure that we are working with strictly positive numbers. To approximate the mapping from ω to χ:

1. Fix a grid of z values centered around zero; we use a uniformly spaced grid with 301 gridpoints between ± 15.
2. For each z in the grid, calculate the corresponding χ using (41). With our z grid, this spans the inner 99.99994% of the feasible range of χ.
3. For each χ in that grid, calculate ω using (39), then compute $\log(\omega)$.
4. Construct a linear spline interpolant that maps from the vector of $\log(\omega)$ values to the grid of z, with linear extrapolation above and below; call this function $g(\cdot)$.
5. Define function $f(\cdot)$ as the composition of (41), g, and log.

By construction, f is an approximation of the mapping from ω to χ implicitly defined by (39), by successively applying the logarithm to ω, the interpolant from $\log(\omega)$ to z, and the logit transformation to recover χ. The approximation is extremely accurate, even on the extrapolated region on ω, because the underlying mapping from $\log(\omega)$ to z approaches linearity on both ends. Conveniently, the f function depends on only ρ and δ and can be constructed once for all periods of the life-cycle.

With the mapping $f : \omega \rightarrow \chi$ in hand, we can now describe our algorithm for solving for optimal consumption under wealth-in-utility preferences:

1. Fix an exogenous grid of a_t values from 0 to a large value (say, a wealth-to-income ratio of 100); this can be the same or a different grid from above.

2. For each a in the grid, evaluate $\bar{\mathfrak{v}}_t'(a_t)$, then raise it to the $-1/\rho$ power and divide by a, yielding a vector of ω values.

3. For each ω in that vector, compute $\chi = f(\omega)$, then multiply by a to recover optimal c.

4. Find the associated decision-time state by inverting the intraperiod budget: $m_t = a_t + c_t$.

5. Construct the consumption function as a linear interpolant over those (m_t, c_t) pairs, adding a point at $(0, 0)$ to incorporate the liquidity-constrained portion of the consumption function.

6. Construct the marginal value function $\mathrm{v}_t'(m_t)$ as the composition of the marginal utility function and the consumption function.

Solving the wealth-in-utility model is thus *barely* more computationally burdensome than solving the basic life-cycle model, despite the considerably more complex preferences.