Appendices

# A The base model

At date 0, the household finances the purchase of a house with a mortgage loan. Given cash-on-hand at date 0 *M*0, which may come from previous savings or bequests, the household simultaneously chooses a house size (among a set of discrete house sizes) and a mortgage loan that meets the down-payment or Loan-to-Value requirement *D*1 ≤ (1 − *d*)*Q*0*H*1, where *d* is the proportional down-payment, *Q*0 is the unit-price of housing, and *H*1 is the chosen house size. An additional mortgage origination requirement observed in the literature is the Loan-to-Income ratio *D*1 ≤ *λP*1, which reflects initial mortgage affordability ([Campbell and Cocco](#_bookmark0) ([2015](#_bookmark0))). Coincidentally, the household leaves some cash-on-hand *M*1 to start next period with some liquidity.

V0(*M*0*, P*1) = max E*t*[V1(*M*1*, H*1*, D*1*, P*1)]

*D*1*,H*1

s.t.

*Q*0*H*1+ *M*1 = *M*0 + *D*1

*H*1 ∈ H = {*h*1*,* · · · *, hn*}

*D*1 ≤ (1 − *d*)*Q*0*H*1 (LTV)

*D*1 ≤ *λP*1 (LTI)

## Fixed-Rate Mortgage payments

(1)

When a household chooses a house size and mortgage loan, they also commit to a fixed payment amount and a loan duration of, for example, 30 years. To calculate the fixed rate mortgage payment (FRM*t*), we can start with the amount owed at the beginning of each period *Dt* and subtract a fixed payment, which is then multiplied by a fixed mortgage interest rate *RD*. The limiting condition that ensures repayment at time *T* is *DT* = 0, which leads to FRM*t* as shown below:

*Dt*

*Dt*+1 = (*Dt* − FRM*t*)*RD* = *DtRD* − FRM*tRD*

*Dt*+2 = (*Dt*+1 − FRM*t*)*RD* = (*DtRD* − FRM*tRD* − FRM*t*)*RD*

= *DtR*2 − FRM*tR*2 − FRM*tRD*

*D*

*D*

.

(2)

*DT* = (*DT* −1 − FRM*t*)*RD* = (*DT* −2*RD* − FRM*tRD* − FRM*t*)*RD*

= *DtRT* −*t* − FRM*t*(*RT* −*t* + *RT* −*t*−1 + · · · + *RD*) = *DtRT* −*t* − FRM*t*(*S*)

*D*

*D*

*D*

*D*

where *S* = *RD* (*RT −t*−1) . Setting *D*

*D*

= 0 as a repayment requirement, the fixed rate

*RD*−1

mortgage payment is

FRM

*T*

*t*(*Dt*) =

*RT* −*t*−1(*RD* − 1)

*RT* −*t* − 1

*D*

*D*

*Dt* (3)

The household is allowed to pay more than its required mortgage payment, which in this model leads to a decrease in the future minimum required payment FRM*t* to maintain the mortgage duration. If the household continually pays more than the minimum payment, FRM*t* becomes 0 when the mortgage debt is fully paid off. This flexibility allows for the fixed rate mortgage payment to depend only on current mortgage debt *Dt* and time to mortgage maturity, which is tracked by the age of the household *t*.

## The Investor’s problem in the presence of housing risk

* + 1. *Investing before Retirement*

A working household begins period 30 ≤ *t* ≤ 60 with cash-on-hand *Mt*, housing size *Ht*, mortgage debt *Dt*, and permanent income *Pt*. It must then choose a level of consumption *Ct*, mortgage payment *It*, and savings *At* of which a fraction *ςt* is invested into a risky asset and the rest into a safe asset.

The mortgage payment must be at least the fixed rate mortgage payment FRM*t*(*Dt*) in order to ensure payoff within an expected maturity of *J* = 30 years.

Although the model does not allow for explicit choices over housing adjustments, the household must nevertheless pay maintenance and upkeep costs due to housing depreciation at a cost of *δQ*0*Ht*. Additionally, the household must also exogenously

expand or contract their housing size given a life-cycle path of housing size Γ*H* . This

*t*+1

parameter is intended to represent the observed path of the housing size component of portfolio compositions over the life-cycle.

V*t*(*Mt, Ht, Dt, Pt*) = max

*At,It,ςt*

s.t.

u(*Ct, Ht*) + *β* E*t*[V*t*+1(*Mt*+1*, Ht*+1*, Dt*+1*, Pt*+1)]

*At* = *Mt* − *Ct* − *It At* ≥ 0

*It* ≥ FRM*t* + [*Ht*+1 − (1 − *δ*)*Ht*]*Q*0 *Pt*+1 = Γ*t*+1*Pt*

*Yt*+1 = *θt*+1*Pt*+1

*Mt*+1 = *Yt*+1 + *At*(*ςt***R***t*+1 + (1 − *ςt*)R)

(4)

*Ht*+1 = Γ*H Ht*

*t*+1

*Dt*+1 = *RD*(*Dt* − *It*)

* + 1. *Investing after Retirement*

Households deterministically retire at age *t* = 60, and begin to experience exogenous risk of housing liquidation. By this time, the household has finished paying their mortgage debt, and so we lose the *Dt* state variable. If the household is exogenously forced to liquidate their house, they receive their home equity at the beginning of next period and become renters until death. The problem of a household which faces housing risk is

V*H* (*Mt, Ht, Pt*) = max u(*Ct, Ht*) + *βst* E*t*[V*H*

(*Mt*+1*, Ht*+1*, Pt*+1)]

*t Ct,At,ςt*

+ *β*(1 − *st*) E*t*[V*R*

*t*+1

*R*

(*M*

*t*+1

*t*+1

*, Pt*+1)]

s.t.

*At* = *Mt* − *Ct* − *It, At* ≥ 0

*It* ≥ [*Ht*+1 − (1 − *δ*)*Ht*]*Q*0 *Pt*+1 = Γ*t*+1*Pt*

*Yt*+1 = *θt*+1*Pt*+1

*Mt*+1 = *Yt*+1 + *At*(*ςt***R***t*+1 + (1 − *ςt*)R)

(5)

*R*

*M*

*t*+1

= *Mt*+1 + *Qt*+1*Ht*+1

*Ht*+1 = Γ*H Ht*

*t*+1

The renter’s problem then becomes a simple portfolio problem where the household additionally pays for rental housing.

V*R*(*Mt, Pt*) = max u(*Ct, HR*) + *β* E*t*[V*R*

(*Mt*+1*, Pt*+1)]

*t C ,HR,A ,ς t*

*t*+1

*t t t t*

s.t.

*At* = *Mt* − *Ct* − *HR, Ct, HR, At* ≥ 0

*t*

*t*

(6)

*Mt*+1 = *Yt*+1 + *At*(*ςt***R***t*+1 + (1 − *ςt*)R)

*Pt*+1 = Γ*t*+1*Pt*

*Yt*+1 = *θt*+1*Pt*+1

# Normalization

A useful strategy to facilitate finding the solution of these types of problems is normal- ization by permanent income. Throughout this section we assume

u(*C, H*) =

(*C*1−*αHα*)1−*ρ*

1 − *ρ*

(7)

which is a Cobb-Douglas function nested inside a CRRA utility function. The model parameter *α* determines the relative preference between non-durable consumption and housing size. In a simple rental housing problem, it also determines directly the rental housing share of total expenditures, as we’ll see below.

## Rental Housing in the Utility

A renter pays for rental housing every period until death. Rental housing enters the utility as a non-durable expenditure and, like consumption, it has no impact on the continuation value. Using the utility function described above, and representing non-

= *α*

durable total expenditures as *X* = *C* + *H*, utility maximization implies that *H*

*C*

1−*α* ,

*C* = (1 − *α*)*X*, and *H* = *αX*. Substituting these results into the utility function, it

becomes:

u(*C, H*) =

((1 − *α*)1−*αααX*)1−*ρ*

1 − *ρ*

*X*1−*ρ*

= *A*

˜

1 − *ρ*

(8)

Thus, we can represent the problem of total expenditures between consumption and

rental housing as u(*C, H*) =

coefficient.

*A*˜u(*X*), where u(·) is a CRRA utility function with *ρ*

## The Retired Renter’s last period of life

In the last period of life, a renter has no need to save and thus has to decide to spend all of his cash-on-hand (*MT* ) between non-durable consumption and rental housing.

V*R*(*MT , PT* ) = max u(*CT , HR*)

*T C ,HR T*

*T*

*T*

s.t.

*MT* = *CT* + *Ht*

(9)

Again, utility maximization implies that *C* = (1 − *α*)*M* and *H* = *αM* . Substituting into the previous equation we obtain:

*M* 1−*ρ*

V*R*(*MT , PT* ) = *A*˜ *T*

*T*

(10)

1 − *ρ*

where *A*˜ = ((1 − *α*)1−*ααα*)1−*ρ*. We can now normalize by permanent income *P* such

*t*

that lowercase variables are *xt* = *Xt/Pt*. Substituting *PT* = *PT* −1Γ*T* , the expression becomes

*m*1−*ρ m*1−*ρ*

V*R*(*Mt, PT* ) = *P* 1−*ρA*˜ *T* = *P* 1−*ρ*Γ1−*ρA*˜ *T*

*T T*

(11)

1 − *ρ*

*T* −1 *T*

1 − *ρ*

If we define a normalized equation as v*R*(*mT*

*T*

*m*1*−ρ*

) = *T*

1−*ρ*

then the original problem can

be rewritten as

V*R*(*MT , PT* ) = *P* 1−*ρ*Γ1−*ρA*˜v*R*(*mT* ) (12)

*T T* −1 *T T*

## The Retired Renter’s second-to-last period

We can now consider the second-to-last period, although the same analysis will apply recursively to any period with a non-zero continuation value. Normalizing as above, where *xt* = *Xt/Pt*, we obtain

V (*MT* −1*, PT* −1) = max

*R*

u(*CT* −1*, HT* −1) + *β* E*t*[V*R*(*MT , PT* )]

*T* −1

*CT −*1

*,HR*

*T −*1

*T*

*,ςT −*1

= max

*A*˜u(*xT* −1*PT* −1) + *β* E*t*[*P* 1−*ρ*Γ1−*ρA*˜v*R*(*mT* )]

*xT −*1*,ςT −*1

*T* −1 *T T*

(13)

= *A*˜*P* 1−*ρ*  max u(*xT* −1) + *β* E*t*[Γ1−*ρ*v*R*(*mT* )]1

*T* −1

*xT −*1*,ςT −*1

*T*

*T*

We can again define the normalized value function as v*R*

*T* −1

= V*R*

*T* −1

*/*(*A*˜*P* 1−*ρ*) and

*T* −1

re-write the problem as

v (*mT* −1) = max

*R*

u(*xT* −1) + *β* E*t*[Γ1−*ρ*v*R*(*mT* )] (14)

*T* −1

*T T*

*xT −*1*,ςT −*1

The generalized renter’s problem with non-zero continuation value then simplifies to a simple consumption and portfolio choice problem with additionally defined control variables as presented below.

v*R*(*mt*) = max u(*xt*) + *β* E*t*[Γ1−*ρ*v*R*

(*mt*+1)]

*t*

s.t.

*xt,ςt*

*t*+1

*t*+1

*at* = *mt* − *xt, xt, at* ≥ 0

*ct* = (1 − *α*)*xt, ht* = *αxt*

*mt*+1 = *θt*+1 + *at*(*ςt***R***t*+1 + (1 − *ςt*)R)*/*Γ*t*+1

## The Retired Homeowner’s last period of life

(15)

We assume homeowners deterministically become renters before their last period of life.

## The Retired Homeowner’s second-to-last period of life

In their second-to-last period of life, a homeowner will become a renter in the next period with certainty. Note that by this stage of life the household has no outstanding mortgage debt. Their problem is

V (*MT* −1*, HT* −1*, PT* −1) = max

*H*

u(*CT* −1*, HT* −1) + *β* E*t*[V*R*(*M R, PT* )]

*T* −1

*T T*

*CT −*1*,AT −*1*,ςT −*1

s.t.

*CT* −1 + *AT* −1 = *MT* −1 − [Γ*H*

*T* −1

− (1 − *δ*)]*Q*0*HT* −1*, CT* −1*, AT* −1 ≥ 0

(16)

*M R* = *YT* + *AT* −1(*ςT* −1*RT* + (1 − *ςT* −1)R) + *QT* Γ*H*

*HT* −1

*T*

*PT* = Γ*T PT* −1

*YT* = *θT PT*

*T* −1

A similar normalization procedure as above yields the following equivalent problem

v (*mT* −1*, hT* −1) = max

*H*

u(*cT* −1*, hT* −1) + *A*˜*β* E[Γ1−*ρ*v*R*(*mR*)]

*T* −1

*cT −*1*,aT −*1*,ςT −*1

s.t.

*T T T*

(17)

*cT* −1 + *aT* −1 = *mT* −1 − [Γ*H*

*T* −1

− (1 − *δ*)]*Q*0*hT* −1*, cT* −1*, aT* −1 ≥ 0

*mR* = *θT* + *aT* −1(*ςT* −1*RT* + (1 − *ςT* −1)R)*/*Γ*T* + *QT* Γ*H*

*hT* −1*/*Γ*T*

*T*

## The Retired Homeowner’s general problem

*T* −1

A retired homeowner has no mortgage debt or payments, but does experience house liquidation and house price risk. With probability 1 − *st*, the homeowner will be forced to sell their house and become a renter next period, in which case they obtain the value of their home as a liquid asset.

V*H* (*Mt, Ht, Pt*) = max u(*Ct, Ht*) + *βst* E*t*[V*H*

(*Mt*+1*, Ht*+1*, Pt*+1)]

*t Ct,At,ςt*

+ *β*(1 − *st*) E*t*[V*R*

*t*+1

*R*

(*M*

*t*+1

*t*+1

*, Pt*+1)]

s.t.

*Ct* + *At* = *Mt* − [*Ht*+1 − (1 − *δ*)*Ht*]*Q*0*, Ct, At* ≥ 0

*Mt*+1 = *Yt*+1 + *At*(*ςt***R***t*+1 + (1 − *ςt*)R)

(18)

*R*

*M*

*t*+1

= *Mt*+1 + *Qt*+1*Ht*+1

*Ht*+1 = Γ*H Ht*

*t*+1

*Pt*+1 = Γ*t*+1*Pt*

*Yt*+1 = *θt*+1*Pt*+1

The normalized version of their problem is

v*H* (*mt, ht*) = max u(*ct, ht*) + *β* E*t* Γ1−*ρ* (*st*v*H*

(*mt*+1*, ht*+1) + (1 − *st*)*A*˜v*R*

(*mR*

)

*t ct,at,ςt*

s.t.

*t*+1

*t*+1

*t*+1

*t*+1

*ct* + *at* = *mt* − [Γ*H* − (1 − *δ*)]*Q*0*ht, ct, at* ≥ 0

*t*+1

*mt*+1 = *θt*+1 + *at*(*ςt***R***t*+1 + (1 − *ςt*)R)*/*Γ*t*+1*,* 0 ≤ *ςt* ≤ 1

*ht*+1 = Γ*H*

*t*+1

*ht/*Γ*t*+1

*R*

*m*

*t*+1

= *mt*+1 + *Qt*+1*ht*+1

(19)

## The Working Homeowner that has mortgage debt

* 1. The borrowing constraint

Every household arrives at the last period of the model as a retired renter. Because they will die with certainty by the end of the period, there is a strict no-borrowing constraint imposed for these households (*aT* ≥ 0). This implies that the minimum allowable level of market resources is also *mR > mR* = 0. If the household arrived to the last period

*T T*

with a negative level of market resources, it would have to consume a negative amount, yielding −∞ utility.

This strict borrowing constraint in the last period leads to a self imposed borrowing constraint in the second to last period, as the precautionary savings motive induces households to meet a minimum allowable level of market resources next period in order to avoid −∞ utility.

For the retired renter household, that means

*mR* = *θT* + *aT* −1(*ςT* −1**R***T* + (1 − *ςT* −1)R)*/*Γ*T > mR* = 0*.* (20)

*T*

*T*

Because renting households have no collateral (a home), we can assume an artificial no-borrowing constraint. The minimum allowable level of market resources next period is met as long as *θT >* 0, which is true by construction. The no-borrowing constraint, in turn, induces a minimum allowable level of market resources in the second to last period

*R T* −1

as *m*

= 0. Recursively, we can continue to assume a no-borrowing constraint for

every period imposed on the retired renter household, which implies *aR* = 0 and *mR* = 0

*t t*

for all *t* during retirement.

The retired homeowner in their second to last period will become a renter with certainty by the next period, at which point they receive the cash value of their liquidated house.

*mR* = *mT* + *QT hT > mR* = 0 (21)

*T T*

which implies a natural minimum allowable level of market resources for homeowners

of

*mT* (*hT* ) = −*QT hT .* (22)

The last period market resources for homeowners is ruled by the transition equation

*mT* = *θT* + *aT* −1(*ςT* −1**R***T* + (1 − *ςT* −1)R)*/*Γ*T > mT .* (23)

Given a household’s decision on asset level *aT* −1, the lowest possible realization of next period market resources occurs when the household receives the lowest possible return, along with the lowest realizations of income shocks next period. For each *aT* −1, then, there is an upper bound on the risky share *ςT* −1 such that the minimum allowable level of future market resources is met.

*ςT* −1

(R − **R***T*

) *<* R (*mT* − *θT* )Γ*T*

*a*

−

(24)

which implies a natural risky share constraint of

)

*T* −1

*ςT* −1(*aT* −1*, hT* −1) =

1

R − **R***T*

(*mT* (Γ*H*

R −

(

*T* −

1*hT* −1*/*Γ*T* ) − *θT* )Γ*T*

*aT* −1

(25)

The natural borrowing constraint, then, occurs when the implied natural risky share constraint is *ςT* −1(*aT* −1*, hT* −1) = 0, which is

*aT* −1(*hT* −1) = (*mT* (Γ*H* 1*hT* −1*/*Γ*T* ) − *θT* )Γ*T /*R (26)

*T* −

The minimum allowable level of market resources constraint is once again derived from the precautionary motive *cT* −1 *>* 0, which results in

*mT* −1(*hT* −1) = *aT* −1(*hT* −1) + [Γ*H* 1 − (1 − *δ*)]*Q*0*hT* −1*.* (27)

*T* −

An important issue arises in the third-to-last period. Households who will become renters next period only have to meet the minimum allowable level of market resources

for rental households next period *mR*

*T* −

1 = 0 or equivalently *mT* −1(*hT* −1) = −*Q*

*T* −1

*hT* −1.

Households who will remain homeowners instead have to meet the minimum allowable level of market resources for home-owning households next period (determined above). However, unlike in the second-to-last period, households in the third-to-last period do not know what type they will be in the next period, so they must meet both constraints, and thus must meet the stricter constraint. The effective minimum allowable level of market resources for the second to-last-period *t* = *T* − 1 and recursively for periods

before it is:

*m*∗*t* (*ht*) = max{*at*(*ht*) + [Γ*H* − (1 − *δ*)]*Q*0*ht,* −*Q ht*} (28)

*t*

*t*

and the general natural borrowing constraint for period *t* = *T* − 2 and recursively for every period before it is

*at*(*ht*) = (*m*∗*t*+1(Γ*H ht/*Γ*t*+1) − *θt*+1)Γ*t*+1*/*R*.* (29)

*t*

# Backsolving the problem

The overall return on the consumer’s portfolio is

*t*+1(*ςt*) = R(1 − *ςt*) + **R***t*+1*ςt*

= R + (**R***t*+1 − R)*ςt*

The first order condition with respect to *ct* is

(30)

u′ (*ct, ht*) = *β* E*t*  Γ−*ρ*  *t*+1 (*st∂m*v*H* (*mt*+1*, ht*+1) + (1 − *st*)*A*˜*∂m*v*R*

1

*t*+1

*t*+1

*t*+1

(*mR*

) (31)

The first order condition with respect to *ςt* is

*t*+1

*t*+1

*t*+1

0 = *β* E*t* Γ−*ρ* (**R***t*+1 − R) (*st∂m*v*H*

*t*+1

*t*+1

(*mt*+1*, ht*+1) + (1 − *st*)*A*˜*∂m*v*R*

(*mR*

) *at* (32)

A useful function to define is

v*t*(*at, ςt*) = *β* E*t* Γ1−*ρ* (*st*v*H*

(*mt*+1*, ht*+1) + (1 − *st*)*A*˜v*R*

(*mR*

) (33)

with first order conditions

*t*+1

*t*+1

*t*+1

*t*+1

v*a* = *β* E*t*  Γ−*ρ*  *t*+1 (*st∂m*v*H* (*mt*+1*, ht*+1) + (1 − *st*)*A*˜*∂m*v*R*

*t*

*t*+1

*t*+1

*t*+1

(*mR*

) (34)

v*ς* = *β* E*t*  Γ−*ρ* (**R***t*+1 − R) (*st∂m*v*H* (*mt*+1*, ht*+1) + (1 − *st*)*A*˜*∂m*v*R* (*mR* ) *a*(3*t* 5)

*t*+1

*t*

*t*+1

*t*+1

*t*+1

*t*+1

which implies first order conditions for the problem

u′1(*ct, ht*) = v*a*(*mt* − *ct, ςt*) (36)

*t*

0 = v*ς* (*at, ςt*) (37)

*t*

We can define the problem

which leads to solution

v˜*t*(*at*) = max v*t*(*at, ςt*)

*ςt*

s.t.

0 ≤ *ςt* ≤ 1

(38)

(*c*1−*αhα*)−*ρ*(1 − *α*)*c*−*αhα* = v`˜*a*(*mt* − *ct*) (39)

*t*

*t*

*t*

*t*

*t*

we can solve for consumption function as

( \ 1

*c*

=

v`˜*a*(*m*

− *ct*)

*−ρ*(1*−α*)*−α*

(40)

*t*

*t*

*t*

(1 − *α*)*hα*(1−*ρ*)

Similarly, as before, the Envelope condition is

(*c*1−*αhα*)−*ρ*(1 − *α*)*c*−*αhα* = *∂m*v*H* (*mt, ht*) (41)

*t*

*t*

*t*

*t*

*t*

# Portfolio Choice after retirement

## The solution of risky share

Let **R***t*+1 be log-normally distributed such that log **R***t*+1 = **r***t*+1 ∼ N (**r***, σ*2), then it is

**r**

true that

E [**R**

*t*

*t*+1

] = *e***r**+*σ*2*/*2 and **Var** [**R**

] = (*eσ*2 − 1)*e*2**r**+*σ*2

(42)

Thus, if we want to produce a log-normal distribution with mean *µR* and variance *σ*2 ,

*R*

**r**

*t*

*t*+1

**r**

**r**

we can use a normal distribution with

( *µ*2

*R*

\ and 2

( *σ*2 )

(43)

**r** = log

*R*

*µ*2

+ *σ*2

*σ***r** = log

1 + 2 *.*

*R*

*R*

*µ*

According to Campbell and Viceira, the optimal share of stocks in financial wealth for an agent that is not facing income uncertainty is

*R*

)

2

(**r** + *σ /*2 − r

*α* = **r**

*γσ*2

**r**

1 + *Ht*

*Wt*

(44)

Using the above relations, we know that

log *µR* (

*σ*

(

*Ht* )

*α*(*µ , σ*

*R*

) =

*γ* log

*µ*

*R*

R

*R*

2

1 + 2

*R*

1 + *.*

*Wt*

(45)

## Exogenous Risky Share

Given the solution of portfolio choice after retirement, if we want to target a particular risky share *α* (for example, one that fits the observed data on portfolio choice after retirement), we can back out the agent’s beliefs on the risky asset return that would rationalize such an *α*. Assuming we know an agent’s financial wealth (human and non- human), we can fix *µ*∗*R* = *µR* to find an ex-ante belief on the variance of the risky distribution that rationalizes an exogenous risky share *α* as

(*σR*∗∗)2 =

*W* +*H*

*γαW*

*R*

(( )∗*µ*

R

− 1\

(*µ*∗*R*)2 (46)

Fixing *σR*∗ and finding a *µ*∗*R*∗(*σR*∗ ) that rationalizes an exogenous risky share *α* has no

analytical solution, although a numerical solution might exist under some conditions.

Of course, if instead we fix **r**∗ = **r**(*µR, σR*) or *σ***r**∗

rationalized beliefs as

= *σ***r**(*µR, σR*), we can obtain

∗∗ *αWtγ*(*σ***r**∗)2 ∗ 2

and

**r** = *W* + *H* + r − (*σ***r** ) */*2 (47)

(*σ***r**∗∗)2 = (**r** − r)

(1 +

*H* ) (*αγ* −

*W* + *H* −1

2*W*

)

*.* (48)

It’s important to note that these beliefs will result in different risky distribution parameters than those that pegged the log-normal distribution parameter.

*W*

# The Portfolio Choice Problem for Rental Households

Households that do not own and instead rent their homes have to decide how much to consume, how much to spend on rent, and how much to save. Their normalized problem can be stated as:

w*t*(*mt*) = max

{*at,ht,ςt*}

u(*ct, ht*) + *β* E*t* Γ1−*ρ*w*t*+1(*mt*+1)]

s.t.

*t*+1

*at* = *mt* − *ct* − *ht*

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt*

*mt*+1 = *at* *t*+1(*ςt*)*/*Γ*t*+1 + *θt*+1

(49)

Consider the problem of a consumer that has *xt* to spend on consumption and housing.

Their problem is

u(*x*) = max u(*c, h*)

{*c,h*}

s.t.

*x* = *c* + *h*

(50)

Given the functional form of utility we are using (CRRA with paramter *ρ*), the well known solution to this simple problem is *c*∗ = (1 − *α*)*x* and *h*∗ = *αx*. Restating the problem in terms of *x*, we obtain:

(*c*1−*αhα*)1−*ρ x*1−*ρ*

u(*x*) = u(*c*∗*, h*∗) =

∗ ∗

1 − *ρ*

= *χ*

1 − *ρ*

(51)

where *χ* = ((1 − *α*)1−*ααα*)1−*ρ*. Because both consumption and housing are non- durable in the case of a rental household, the consumer can first decide how much to spend on both goods (*xt*) and then decide how much to spend on each of the goods without changing the problem. A further step to simplify the problem is to use iterated expectations to split up the problem into subperiods. We can define

w *t*(*bt*+1) = E*t* Γ1−*ρ*w*t*+1(*mt*+1)]

*t*+1

where

*mt*+1 = *bt*+1*/*Γ*t*+1 + *θt*+1

Now, we can rewrite our original problem as

w*t*(*mt*) = max u(*xt*) + *β* E*t* w*t*(*bt*+1)

]

{*at,ςt*}

(52)

s.t.

*at* = *mt* − *xt*

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt*

*bt*+1 = *at t*+1(*ςt*)

which embeds the simple subproblem and our defined iterated expectation. We can rewrite the problem as

(53)

w*t*(*mt*) = max u(*mt at*) + *β* E*t* w*t* (*at*(R + (**R***t*+1 R)*ςt*)) (54)

− − ]

{*at,ςt*}

First order condition with respect to *a* provides the Euler equation

u′(*xt*) = *β* E*t* w′*t*(*bt*+1) *t*+1(*ςt*) (55) and the first order condition with respect to *ςt* is

]

*β* E*t* w′*t*(*bt*+1)*at*(**R***t*+1 − R) = 0 (56) The envelope condition is given by

]

w*t*′ (*mt*) = u′(*xt*) (57)

And finally,

w ′ (*bt*+1) = E*t* Γ1−*ρ*w′ (*mt*+1)*/*Γ*t*+1] = E*t* Γ−*ρ* w′ (*mt*+1)] (58)

*t*

*t*+1

*t*+1

*t*+1

*t*+1

# The portfolio problem of a homeowner with no mortgage

A homeowner with no mortgage debt is allowed to invest more on their house to increase its size (or they can let it depreciate). In doing so, they choose home investment, consumption, and savings. Their problem is summarized as follows:

v*t*(*mt, ht*−1) = max u(*ct, ht*) + *β* E*t* Γ1−*ρ* (1 − *ϑ*)v*t*+1(*mt*+1*, ht*+1) + *ϑ*w*t*+1(*m*w

) ]

*at,ςt,it*

s.t.

*t*+1

*t*+1

*ht* = (1 − *δ*)*ht*−1 + *it/***Q**0 *ht*+1 = *ht/*Γ*t*+1

*at* = *mt* − *ct* − *it*

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt*

*mt*+1 = *at* *t*+1(*ςt*)*/*Γ*t*+1 + *θt*+1

w

*m*

*t*+1

= *mt*+1 + **Q***t*+1*ht*+1

(59)

To facilitate the solution method, we can split the above problem into different subperiods.

In the first subperiod, the household arrives with cash on hand and their previous housing size. They then pick their current size by investing *it* where housing costs are

**Q**0. After investing, they are left with net cash on hand after housing costs, and a new housing size.

v*t*(*mt, ht* 1) = max v˜*t*(*nt, ht*)

−

*it*

s.t.

*nt* = *mt* − *it*

*ht* = (1 − *δ*)*ht*−1 + *it/***Q**0

(60)

In the second subperiod, the household arrives with net cash on hand and their current housing size. This subperiod is a standard portfolio choice problem, indexed by their house size. The agent must then choose a level of savings *at* and the proportion of their savings that will go into the risky asset *ςt* versus the safe asset (1 − *ςt*).

v˜*t*(*nt, ht*) = max u(*ct, ht*) + *β* E*t* v˜`*t*(*bt*+1*, ht*)

{*at,ςt*}

*at* = *nt* − *ct*

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt*

*bt*+1 = *at t*+1(*ςt*)

(61)

Finally in the last subperiod, the household’s uncertainty is realized. Simultaneously, they observe their permanent and transitory income shocks, whether they will become renters in the next period (function w*t*+1 with probability *ϑ*), and if they do become renters, the liquidation price of their house per unit of housing.

v`˜*t*(*bt*+1*, ht*) = E*t*  Γ1−*ρ*  (1 − *ϑ*)v*t*+1(*mt*+1*, ht*+1) + *ϑ*w*t*+1(*m*w

where

*t*+1

*t*+1

) ]

*ht*+1 = *ht/*Γ*t*+1

*mt*+1 = *bt*+1*/*Γ*t*+1 + *θt*+1

(62)

w

*m*

*t*+1

= *mt*+1 + *ht*+1**Q***t*+1

## First order conditions: Choosing home investment

The problem is

v*t*(*mt, ht*−1) = max v˜*t*(*mt* − *it,* (1 − *δ*)*ht*−1 + *it/***Q**0) (63)

*it*

The first order condition with respect to *it* is

v˜*n*(*nt, ht*) = v˜*h*(*nt, ht*)*/***Q**0 (64)

*t t*

which equalizes the marginal benefit of additional net cash-on-hand (cash-on-hand net of home investment) with the marginal cost of a larger house. The envelope conditions are

v*m*(*mt, ht*−1) = v˜*n*(*nt, ht*)

*t t* (65)

v*h*(*mt, ht*−1) = v˜*h*(*nt, ht*)(1 − *δ*)

*t*

*t*

## First order conditions: Choosing consumption and portfolio investment

Once again, let’s reduce the problem to 1 line.

v˜*t*(*nt, ht*) = max u(*nt at, ht*) + *β* E*t* v`˜*t*(*at*(R + (**R***t*+1 R)*ςt*)*, ht*) (66)

− −

{*at,ςt*}

Notice that *ht* passes through this problem unaltered. Indeed, in this subproblem, the house size indexes the portfolio choice (and may affect marginal utility) but does not need further addressing beyond a simple portfolio choice model.

The first order condition with respect to *at* is

u*c*(*ct, ht*) = *β* E*t* v`˜*b*(*bt*+1*, ht*) *t*+1(*ςt*) (67)

*t*

The first order condition with respect to *ςt* is

*β* E*t* v`˜*b*(*bt*+1*, ht*)*at*(**R***t*+1 − R) = 0 (68) Finally, the envelope conditions are

*t*

v˜*n*(*nt, ht*) = u*c*(*ct, ht*)

*t*

v˜*h*(*n , h* ) = u*h*(*c , h* ) + *β* E

*t*

*t*

*t*

*t*

*t*

v`˜*h*(*b*

*, h* ) (69)

The second envelope condition is due to the nature of the *ht* pass-through.

*t*

*t*

*t*+1

*t*

## Envelope conditions: Uncertainty is realized

The last subperiod is harder to re-write in one line, but because there is no maximization it is straight forward to calculate the derivatives.

*t*+1

*t*+1

*t*+1

v`˜*b*(*bt*+1*, ht*) = E*t*  Γ−*ρ*

*t*

*t*+1

(1 − *ϑ*)v*m*

(*mt*+1*, ht*+1) + *ϑ*w*m*

(*m*w

) ]

v`˜*h*(*bt*+1*, ht*) = E*t*  Γ−*ρ*

*t*

*t*+1

(1 − *ϑ*)v*h*

(*mt*+1*, ht*+1) + *ϑ*w*m*

(*m*w

)**Q***t*+1 ]

# Solving the homeowner with mortgage problem

(70)

*t*+1

*t*+1

*t*+1

s.t.

*at,ςt,it*

*t*+1

*dt* = *dt*−1 + (1 − *δ*)*ht* − *it at* = *mt* − *ct* − *it*

v*t*(*mt, ht, dt*−1) = max u(*ct, ht*) + *β* E*t* Γ1−*ρ*v*t*+1(*mt*+1*, ht*+1*, dt*+1)]

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt mt*+1 = *at* *t*+1(*ςt*)*/*Γ*t*+1 + *θt*+1 *ht*+1 = *ht/*Γ*t*+1

(71)

w

*m*

*t*+1

= *mt*+1 + **Q***t*+1*ht*+1

Can also be split up into subparts

v*t*(*mt, ht, dt* 1) = max v˜*t*(*nt, ht, dt*)

−

*it*

*nt* = *mt* − *it*

*dt* = *dt*−1 + (1 − *δ*)*ht* − *it*

v˜*t*(*nt, ht, dt*) = max u(*ct, ht*) + *β* E*t* v˜`*t*(*bt*+1*, ht, dt*)

{*at,ςt*}

*at* = *nt* − *ct*

*t*+1(*ςt*) = R + (**R***t*+1 − R)*ςt*

*bt*+1 = *at t*+1(*ςt*)

(72)

(73)

v`˜*t*(*bt*+1*, ht, dt*) = E*t*  Γ1−*ρ*v*t*+1(*mt*+1*, ht*+1*, dt*+1)]

*t*+1

where

*ht*+1 = *ht/*Γ*t*+1 *dt*+1 = *dt*R*D/*Γ*t*+1

*mt*+1 = *bt*+1*/*Γ*t*+1 + *θt*+1

(74)

w

*m*

*t*+1

= *mt*+1 + *ht*+1**Q***t*+1

# References

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