An Attempt at the Replication of S. Rao Aiyagari and Ellen R. McGrattan's The Optimum Quantity of Debt

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Abstract
This paper uses a model of a large number of infinitely lived households whose saving behavior is influenced by precautionary saving motives and borrowing constraints to find that the welfare gains to being at the optimum quantity of debt rather than the US level are small. This model incorporates a different role for government debt than is found in standard models and captures different cost-benefit trade-offs.

Keywords Government debt, precautionary saving, borrowing constraints

JEL codes E6, H6

Powered by Econ-ARK

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Original Papper:  ScienceDirect (Paywall)
   GitHub:        Replication Project
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1 Introduction

1.1 Goal

The goal of this project is to replicate the main results from Aiyagari and McGrattan (1998) using the tools provided by HARK. I use the `ConsMarkovModel` along with the `ConsIndShockModel` and extend it accommodate the model developed by Aiyagari and McGrattan. Ultimately, I would like to completely replicate the results shown in Figure 3.

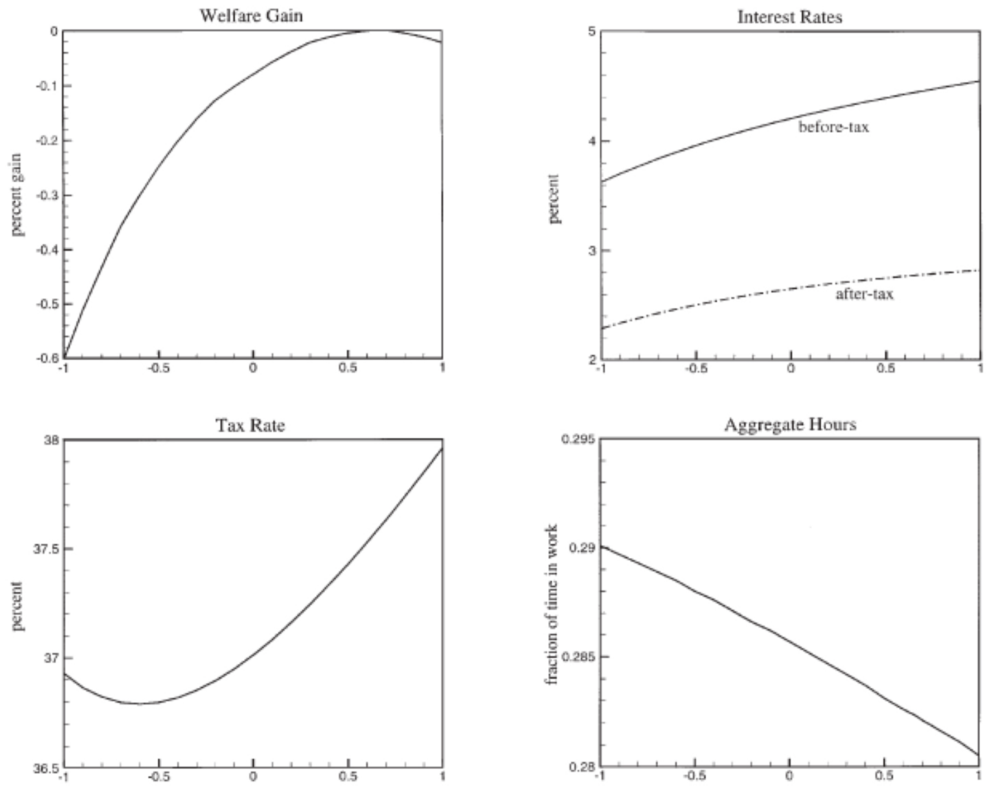

Figure 1 Welfare gain, interest rates, tax rate and aggregate hours versus debt/GDP ratio (x-axis) for the benchmark economy

Given my current limitations and lack of familiarity with HARK, I have decided to forgo replicating the model and instead use the basic model and produce the optimal consumption and labor function The main body of this paper will provide a summary of the original paper and go through some of the results produced by their simulation. Subsequently, the Appendix will show what we have managed to producce using HARK along with a brief explanation of the method we used to extend the tools currently available in HARK. For a more thorough elaboration of this process, I have made the annotated code that produces the replication publicly available on our GitHub page.

2 The Two Models

2.1 Overview

The paper presents two models to present the central idea of the paper, namely a basic model along with a benchmark model. Both models are extensions of the model first introduced by Aiyagari (1994) with the following general features:

- Individual stochastic labor productivity risk without aggregate risks
- Perfectly competitive firms utilizing labor and capital
- Incomplete market with risk free assets and a borrowing constraint
- Precautionary savings

2.2 Basic Model

The Basic model extends the Aiyagari (1994) by adding these features:

- Government debt
- Exogenous and wasteful government consumption
- Lump sum taxes (has no inurance and incentive effects)
- Exogenous labor supplyIndividual stochastic labor productivity risk without aggregate risks

The household problem is defined by:

$$\max_{c_t, a_{t+1}} E\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\nu}}{1-\nu}\right]$$

s.t.

$$c_t + a_{t+1} \leq (1+r)a_t + w_t e_t - T_t$$
$$c_t \geq 0; a_t \geq 0; a_0, e_0 \text{ given}$$

Technology is defined by:

- Stochastic labor productivity, e_t, which is normalized by $E(e_t) = 1$
- Labor augmenting technological progress $z_t = z(1+g)^t$
- Growth adjustment: $Y_t = F(K_t, z_t N_t)$
- Capital depreciates at rate δ

The government budget is defined by:
$G_t + rB_t = B_{t+1} - B_t + T_t$

The asset market is then setup in the following environment:
$A_t = K_t + B_t \quad (A_t :$ per capita assets)

Along the balanced growth path, we have:

- constant r
- Y, K, T, B, A (variables in per capita terms) and w grow at rate g
- lower case / tilde hat letters denote variables that is normalized by output

We then transform the problem into the following form:
$\max_{\{\tilde{c}_t, \tilde{a}_{t+1}\}} \quad E\left[Y_0^{1-v} \sum_{t=0}^{\infty} \left[\beta(1+g)^{1-v}\right]^t \tilde{c}_t^{1-v}/(1-v) \mid \tilde{a}_0, e_0\right]$

subject to

$\tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1+r)\tilde{a}_t + \tilde{w}e_t - \tau$
$\tilde{c}_t \geq 0, \tilde{a}_t \geq 0, t \geq 0$

$$Y : per\ capita\ output \tag{1}$$
$$\beta : discount\ factor \tag{2}$$
$$g : rate\ of\ technical\ progress \tag{3}$$
$$v : relative\ risk\ aversion\ coefficient \tag{4}$$
$$\tilde{c} : output\ normalized\ per\ capita\ consumption \tag{5}$$
$$\tilde{a} : output\ normalized\ per\ capita\ asset\ held\ by\ consumers \tag{6}$$
$$\tilde{w} : output\ normalized\ per\ capita\ wage \tag{7}$$
$$e : individual\ labor\ productivity \tag{8}$$

We can then obtain the transformed government budget and asset market equations that are given as the following.

Government budget:
$\gamma + (r-g)b = \tau \quad (\gamma = G_t/Y_t)$

Asset Market:
$\bar{a} = k + b \quad (\bar{a} = A_t/Y_t)$

The competitive equilibrium is then a set of:

- Household policy function: $\alpha(\tilde{a}, e)$ (asset accumulation decision rule)
- Factor inputs L and K
- Factor prices w and r
- Government debt B
- Taxes T

Such that:

- The equilibrium distribution of household over the state space $\lambda(a, e)$ associated with $\alpha(\tilde{a}, e)$ and $\pi(e'|e)$ is stationary
- Given w, r and T, $\alpha(\tilde{a}, e)$ maximizes the houhsehold problem
- Given w and r: firms choose L and K
- Households savings supply equals demand by firms and government
- Households labor supply equals demand by firms
- Government budget is satisfied
- Goods market clears

2.3 How interest rate (r) is determined

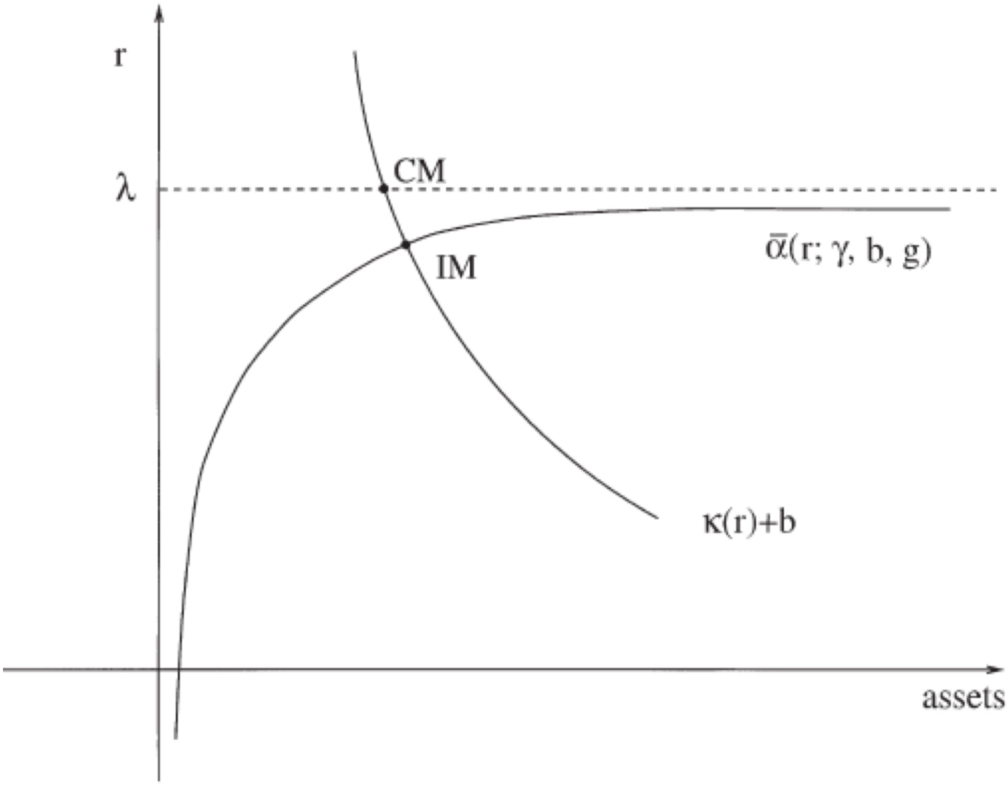

Figure 2 Interest rate determination

Where:

- $\lambda \equiv \frac{(1+g)^{\nu}}{\beta} - 1$ (Complete Market Asset Demand)
- Asset demand: $\bar{\alpha}(r; \gamma, b, g)$
- Asset supply: $\kappa(r) + b$ where k is a function of r

2.4 Welfare Function

The welfare function used in the paper is a utilitarian welfare function of the following form:

$$\Omega = \iint V(a, e) \mathrm{d}H(a, e)$$

- V: optimal value function
- H: steady state distribution of assets and productivities
- Ω expresses welfare changes in percentage of consumption

2.5 Benchmark Model

The stationary steady state model is defined by:

$$\max_{\tilde{c}_t, l_t, \tilde{a}_{t+1}} E\left[(Y_0)^{\eta(1-\mu)} \sum_{t=0}^{\infty} \left[\beta(1+g)^{\eta(1-\mu)}\right]^t \frac{\left(\tilde{c}_t^{\eta}\big|_t^{1-\eta}\right)^{1-\mu}}{1-\mu}\right]$$

s.t.

$$\tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1 + (1-\tau_y)\, r)\, \tilde{a}_t + (1-\tau_y)\, w_t e_t \,(1 - I_t) + \chi$$

$$\tilde{c}_t \geq 0; \tilde{a}_t \geq 0; 1 \geq l_t \geq 0; \tilde{a}_0, e_0, Y_0$$

The parameters used in the benchmark model was obtained from:

- Production function: Cobb Douglas (with capital share θ)
- Labor productivity process:
  - Assumed to be AR(1)
  - Approximated as seven state Markov Chain, Tauchen [1986]
  - From Aiyagari [1994]: $\rho = 0.6, \sigma = 0.3$
- Government policies and parameters:
  - $\gamma = 21.7\%$
  - $\chi = 8.2\%$
  - $b = 66\%$ (of GDP)
  - $g = 1.85\%, \delta = 0.075, \theta = 0.3$
- $\rho, \sigma, \mu, \beta, \eta$ determines precautionary savings motive

3 Results

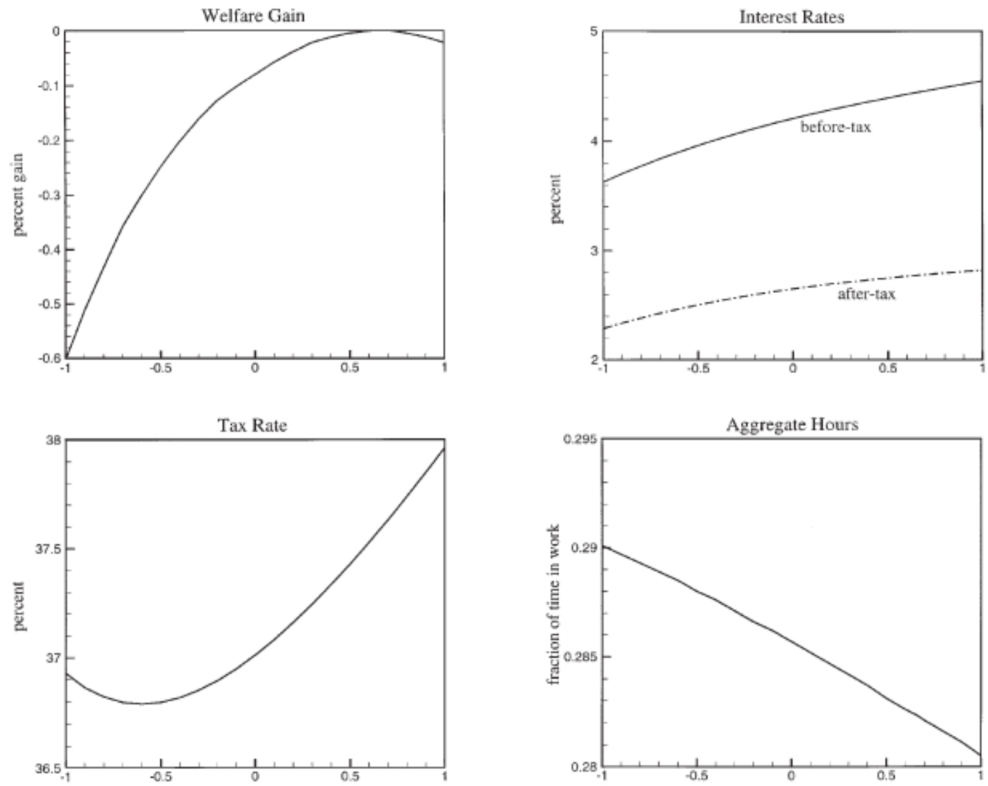

Figure 3 Welfare gain, interest rates, tax rate and aggregate hours versus debt/GDP ratio (x-axis) for the benchmark economy

As can be seen from the figure, the authors found that public debt is welfare improving only if taxes are costly. Furthermore, the optimal level of debt is either indeterminate or set by initial conditions. This can be explained by three points:

1. Enhances household consumption smoothing (+)

2. Requires costly taxation (-)

3. Crowds out productive capital and increases interest rate (-)

4 Conclusion

The optimum quantity of debt found in the model is equal to the average level in the post-war US economy. Nevertheless, the welfare function is very flat. Deviating from the optimal level impacts welfare by a very small level. For some pertrubations in the parameter, there is significant change in the quantity of debt even though the welfare effects remain insignificant.

Appendix

A Notes on the Replication Attempt

My original intention in replicating this paper was to alter the `ConsMarkovModel` in order to fit the Aiyagari and McGrattan paper. That attempt did not succeed and the remnants is left in the folder aptly named Failed Attempt. In that failed attempt, my plan was to create three main classes which are:

- McGariConsumerSolution, which is the usual consumer solution with the addition of the labor supply function
- McGariSolver, whcih solves a single period of the consumption-labor-saving problem with a stochastic transtion between discreet states
- McGariConsumerType, which represents the agent in the consumption-labor-saving model.

Given the failed attempt, I have instead tried to use the `ConsMarkovModel` and fit it using parameters and techniques used by Aiyagari and Mcgrattan in their paper. Other than the three graphs produced below, I have also written a code which attempts to find the steady state solution to the paper's problem, which I run using the SteadyState code although it has failed to converge.

B Replication Results

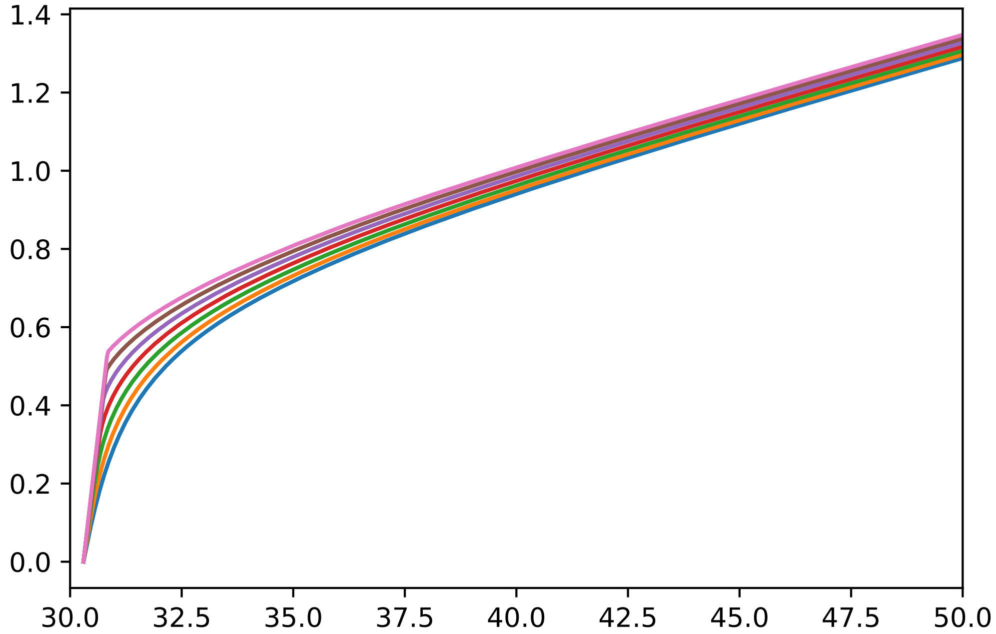

Figure 4 The Optimal Consumption Function

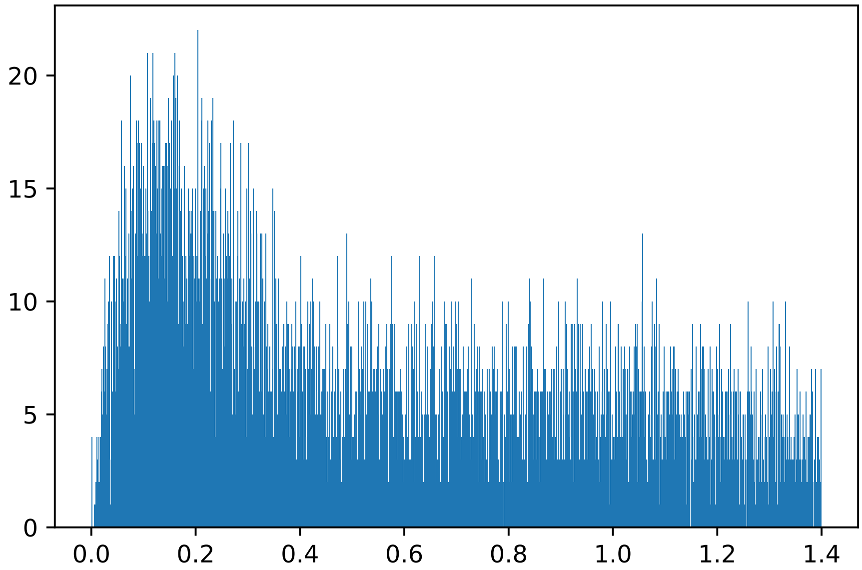

Figure 5 The Distribution of Asset

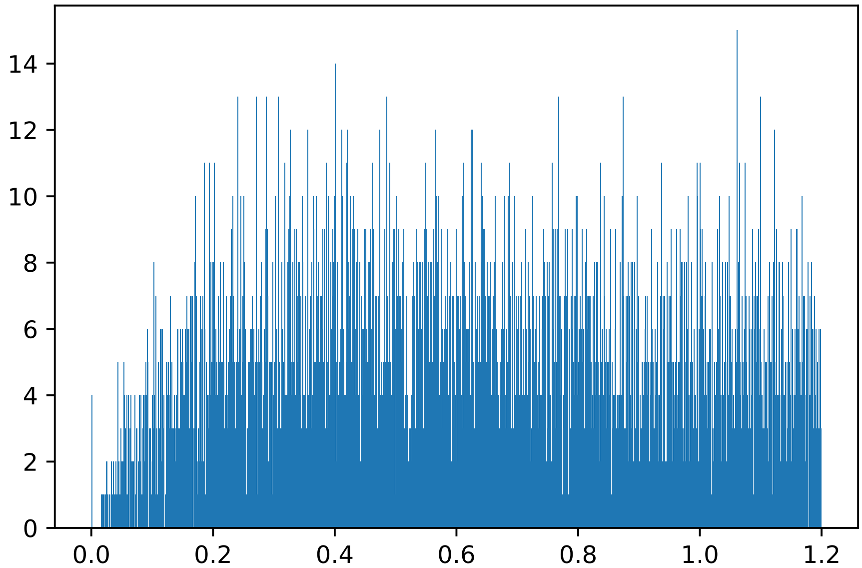

Figure 6 The Distribution of Wealth Level

References

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