Robust Predictions in Dynamic Policy Games

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2. General Model
3. Example (1) : New Keynesian Model
4. Example (2): Sovereign Debt
5. Sunspots
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Introduction

- **Robust Predictions Literature**: look for common properties across all equilibria
- In static games, equilibrium outcomes $\leftrightarrow$ equilibrium strategies
- In dynamic games, strategies are not observable:

$$\text{data} = \text{eqm. path of some "feasible" equilibrium} \tag{1}$$

- "feasible" = set of equilibria being considered
- "Eqm path" = in the support of some equilibrium (potentially with sunspots)
- Robust predictions in Dynamic Settings: Obtain testable moment constraints on observables, if (1)

Introduction

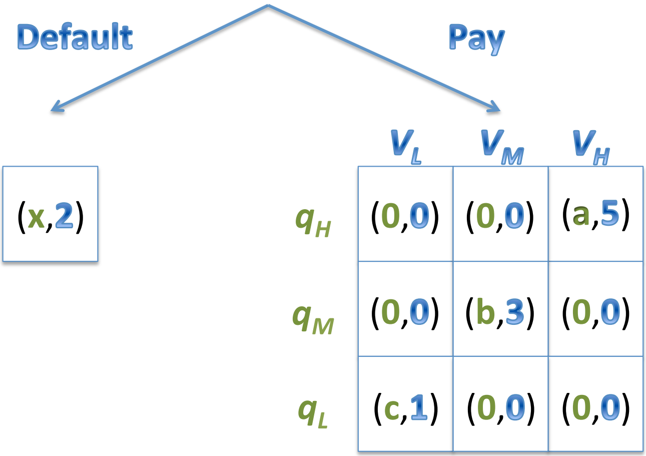
- How? **Revealed Expectations (or information) approach**
  - Assume we have been in an equilibrium up to $t-1$
  - If on path, agents maximized lifetime continuation utility at time t
  - This gives implied lower bounds on expected lifetime utility
  - Can be used to infer beliefs about future behavior of other agents.

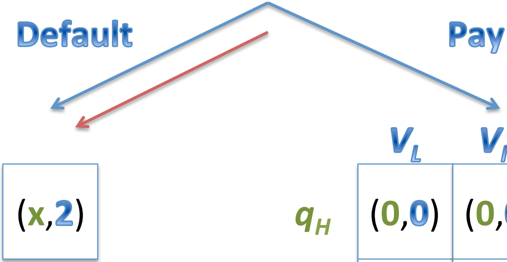
Introduction

- Today: Proof of concept: Application to NK and (mostly) Sovereign debt with endogenous default (Eaton & Gersovitz 83)
- In EG, best equilibrium exhibits
  - No "Confidence" Crises: self-fulfilling default (Cole & Kehoe, etc)
  - Prices are forward looking.
  - Too low default risk without sunspots.
- Goals: By looking at all equilibrium paths (with and without sunspots) we will obtain:
  - Bounds on debt prices moments (expectation, variance) conditional on the observed history
  - Bounds on crisis probabilities

Roadmap

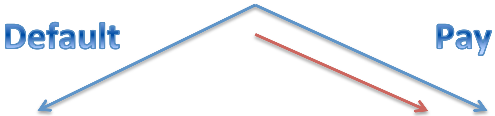
- 1 Period example
- General Model
- Example: Sovereign Debt
- Results on General Model


Default
Pay
(x,2)
V_L
V_M
V_H
q_H
(0,0)
(0,0)
(a,5)
q_M
(0,0)
(b,3)
(0,0)
q_L
(c,1)
(0,0)
(0,0)



$$\sigma_1 = \left(\left(Default, V_L\right), q_L\right)$$

	V_L	V_M	V_H
q_H	(0,0)	(0,0)	(a,5)
q_M	(0,0)	(b,3)	(0,0)
q_L	(c,1)	(0,0)	(0,0)

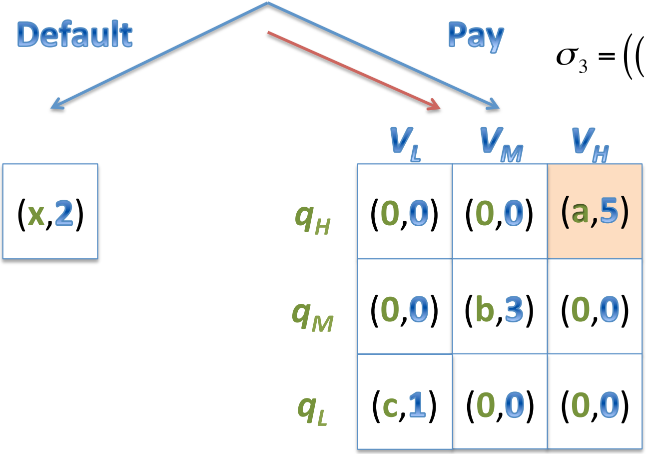


$$\sigma_2 = \left(\left(Pay, V_M \right), q_M \right)$$

(x,2)

	V_L	V_M	V_H
q_H	(0,0)	(0,0)	(a,5)
q_M	(0,0)	(b,3)	(0,0)
q_L	(c,1)	(0,0)	(0,0)

$$\sigma_3 = \left(\left(Pay, V_H \right), q_H \right)$$



Default **Pay**

(x,2)

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q_H	(0,0)	(0,0)	(a,5)
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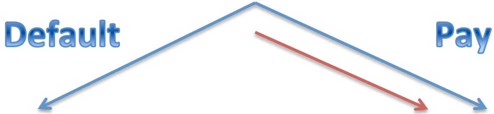

(x,2)

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Default

Pay

$(x, \boxed{4})$

	V_L	V_M	V_H
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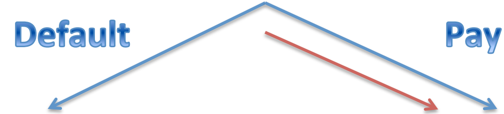

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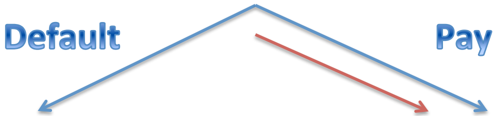

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Example - Sunspots

- Public Correlating device $\zeta \sim U[0,1]$ after no-default.
- Extensive form Correlated Equilibrium (Forges 86)
- Myerson 94, Gul and Pearce 96
- **Outcomes:** distributions π over (V, q)
- IC is now

$$\mathbb{E}_{V,q}\{u_G(V,q)\} \geq u(\text{default})$$

Simplex

Example - Sunspots

- Let $\alpha = \max \Pr(q = q_L)$ across all equilibria.
- It is given by relaxing IC constraint as much as possible.

$$\alpha u_G(V_L, q_L) + (1-\alpha) u_G(V_H, q_H) = u_G(\text{default}) \iff$$

$$\alpha = \frac{u_G(V_H, q_H) - u_G(\text{default})}{u_G(V_H, q_H) - u_G(V_L, q_L)}$$

- If $u(\text{default}) = 2 \implies \alpha = 3/4$
- If $u(\text{default}) = 4 \implies \alpha = 1/4$
- Same bounds obtained if private signals $(\zeta_1, \zeta_2) \sim F$

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General Model: Dynamic Policy Game:

1. Enter period t with (endogenous) $b_t \in B$
2. Nature draws $y_t \sim F(\cdot \mid y_{t-1}, b_t)$. **Assumption:** y_t non-atomic.
3. Long lived player chooses next period state b_{t+1} and control d_t:

$$(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$$

4. Myopic agents set $q_t \in Q \subseteq \mathbb{R}^k$ according to:

$$q_t = \mathbb{E}_t \left\{ \sum_{\tau=t+1}^{\infty} \delta^{\tau-t-1} M(b_\tau, y_\tau, d_\tau, b_{\tau+1}) \right\}$$

- Long lived agent preferences:

$$U = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(b_t, y_t, d_t, b_{t+1}, q_t) \right\}$$

Examples

- We can also work with contemporaneous expectations (e.g. Barro Gordon)
- Mode is sequential: also works with simultaneous moves
- **Example 1:** Sovereign Debt models
  - $y_t = GDP$, d_t (default decision) $\in \{0,1\}$ and $b_t \geq 0$ bond issues
  - $q_t = \mathbb{E}_t \left[\frac{1}{1+r^*} (1 - d_{t+1}) \right] \quad (\delta = 0)$
  - $u = u(c_t) = u[d_t y_t + (1 - d_t)(y_t - b_t + q_t b_{t+1})]$
- **Example 2:** New Keynesian Model
  - $d_t = \pi_t =$ inflation
  - $q_t := \pi_t^e = \mathbb{E}_t (\pi_{t+1}) \quad (\delta = 0)$
  - **Phillips curve:** $\pi_t = \kappa x_t + \beta \pi_t^e + \varepsilon_t$
  - $u = -\left[\frac{1}{2} x_t^2 + \frac{1}{2} \chi (\pi_t - \pi_t^*)^2 \right]$, shocks $y_t = (\varepsilon_t, \pi_t^*)$

Subgame Perfection

- $h_t = (y_t, d_t, b_{t+1}, q_t)$
- A history $h^t = (h_0, h_1, ..., h_{t-1})$.
  - Long lived player (l) plays in histories $h = (h^t, y_t) \in \mathscr{H}_l$
  - Myopic agent (m) plays in histories $h = (h^t, y_t, d_t, b_t) \in \mathscr{H}_m$
- Strategies:
  - (l): $\sigma_l(h^t, y_t) = \left(d_t^{\sigma_l}, b_{t+1}^{\sigma_l}\right) \in \Gamma(b_t, y_t)$
  - (m): $q_m(h^t, y_t, d_t, b_{t+1}) \in \mathbb{R}^k$
- Continuation utility:

$$V(\sigma_g, q_m \mid h \in \mathscr{H}_l) := \mathbb{E}\left\{\sum_{\tau=0}^{\infty} \beta^{\tau} u\left(h^{t+\tau}\right) \mid \sigma_l, q_m\right\}$$

Subgame Perfection

- A Subgame Perfect Equilibrium (SPE) is a profile $\sigma = (\sigma_l, q_m)$ such that:
    - $V(\sigma_l, q_m \mid h^t, y_t) \geq V(\sigma_l', q_m \mid h^t, y_t)$ for all (h^t, y_t), $\sigma_l \in \Sigma_l$
    - $q_m(\cdot)$ is consistent with σ at all $h = (h^t, y_t, d_t, b_{t+1}) \in \mathscr{H}_m$:

$$q_m(h) = \mathbb{E}_t \left\{ \sum_{\tau=t+1} \delta^{\tau-t-1} M(b_\tau, y_\tau, d_\tau, b_{\tau+1}) \mid \sigma_l, q_m, h \right\}$$

Equilibrium Consistency

- We provide a recursive characterization of equilibrium consistent histories:

  Step 1: Suppose h^t is in the path of some equilibrium profile $\hat{\sigma}$

  Step 2: Characterize equilibrium constraints (across $\hat{\sigma}$) over period t observables: er

  $$(d_t, b_{t+1}, q_t) \text{ given } y_t$$

  Step 3: Show this can be done without knowing the specific $\hat{\sigma}$, only h^t. This characterizes the $h^{t+1} = (h^t, d_t, b_{t+1}, q_t)$ that are equilibrium consistent.

- We can add an extra restriction over equilibria (e.g. Less severe punishments)

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Equilibrium Values Correspondence

- Given a seed value $y_{-} = y$ and initial state $b_0 = b$, define

$$\mathscr{E}(y,b) := \left\{ (q,v) : \exists \sigma \in \mathbf{SPE}(y_{-},b) \begin{cases} v = V(\sigma \mid h_0 = (y_{-1} = y, b)) \\ q_0 = \mathbb{E}_0 \left\{ \sum_t \delta^t M_t \mid h_0, \sigma_g(h_0) \right\} \end{cases} \right\}$$

- Results also work with Self-Generating restriction $\mathscr{D}(y,b) \subset \mathscr{E}(y,b)$ (e.g. restricting punishments)

Main Objects

- Given $\mathscr{E}(y,b)$, the two main objects we actually need are

$$\underline{U}(b,y) := \max_{(d,b')\in\Gamma(y,b)} \left\{ \min_{(q,v)\in\mathscr{E}(y,b')} u\left(b,y,d,b',q\right) + \beta v \right\}$$

and

$$\overline{v}(y,b,q) = \max\{v : (q,v) \in \mathscr{E}(y,b)\}$$

Properties of $\overline{v}$:

- If $u(\cdot)$ concave in q and $\mathscr{E}$ is convex valued $\Longrightarrow \overline{v}$ concave in q
- $\overline{v}(\cdot)$ is the fixed point of a β−contraction, which depends on fundamentals, and on knowing $\underline{U}(y,b)$ Contraction

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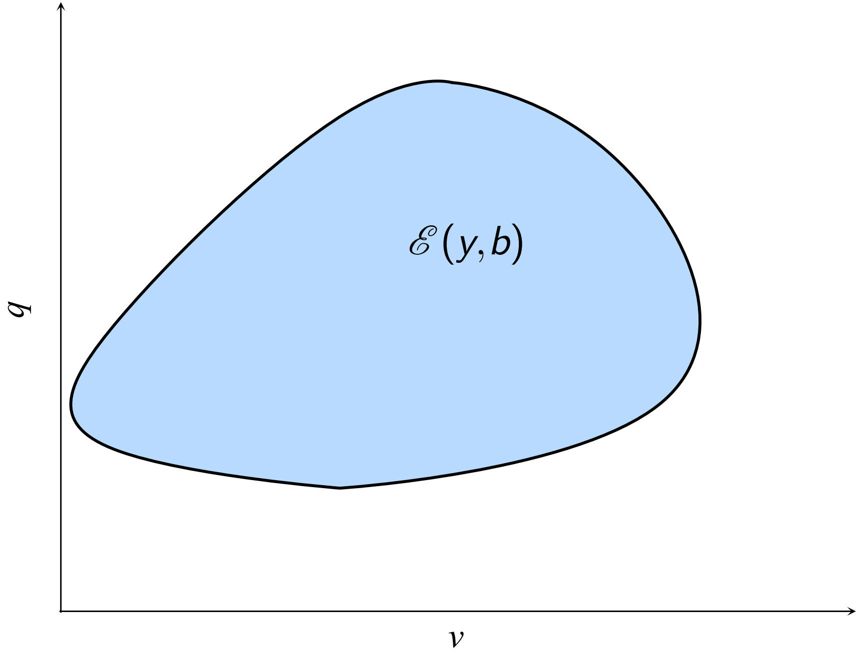
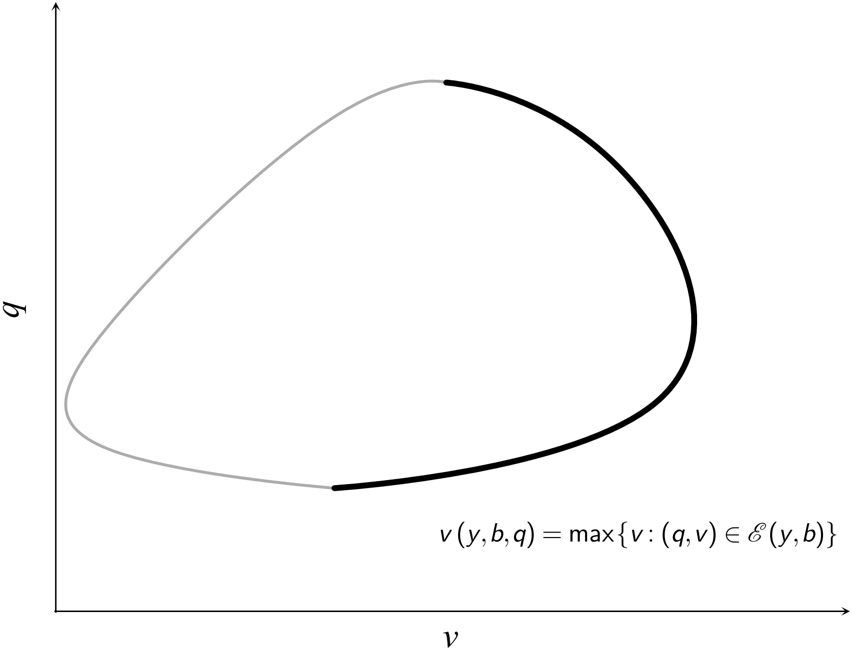

$\mathscr{E}(y,b)$
q
v



$$v(y,b,q)=\max\{v:(q,v)\in\mathscr{E}(y,b)\}$$

First Result - No Sunspots

Proposition.

If h^t is equilibrium consistent, then $h^{t+1} = (h^t, y_t, d_t, b_{t+1}, q_t)$ is eqm consistent $\iff$

$$u(b_t, y_t, d_t, b_{t+1}, q_t) + \beta \overline{v}(y_t, b_{t+1}, q_t) \geq \underline{U}(b_t, y_t) \tag{2}$$

- **Amnesia:** Only current period (and past state) is relevant
- Set of eqm consistent prices $q_t \in \mathbf{Q}(b_t, y_t, d_t, b_{t+1}) \iff (2)$
- If $u(\cdot)$ is concave in q and $\mathscr{E}$ is convex-valued $\Longrightarrow$

$$\Delta_t = u(h_t) + \beta \overline{v}(y_t, b_{t+1}, q_t) - \underline{U}(b_t, y_t)$$

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New Keynesian Model (Very Preliminary)

- Adapted from Waki et al (2018), with public information.
- Shocks are $z_t = \pi_t^*$ ($\varepsilon_t = 0$) with

$$z_t \sim_{i.i.d} f(z) \text{ with } \begin{cases} \mu_z = \mathbb{E}(z) \\ \sigma_z^2 = \mathbb{V}(z) \end{cases}$$

- Phillips Curve is:

$$\pi_t = \kappa x_t + \beta \pi_t^e$$

New Keynesian Model (Very Preliminary)

- So, in the notation of the general theorem:
  - $y_t = z_t$
  - $d_t = \pi_t$
  - $q_t = \pi^e_{t+1}$
- Utility function for MA is

$$u\left(z_t, \pi_t, \pi^e_{t+1}\right) = -\frac{1}{2}x_t^2 - \frac{\chi}{2}\left(\pi_t - z_t\right)^2$$

which is concave in $\left(\pi_t, \pi^e_{t+1}\right)$

$\overline{v}(\pi^e)$

- Since there is no state variable and z_t is i.i.d $\Longrightarrow \overline{v} = \overline{v}(\pi^e)$
- Waki et al (2018) show that $\overline{v}(\pi^e)$ satisfies the fixed point equation:

$$\overline{v}(\pi^e) = \max_{\pi(\cdot),\pi^e_+(\cdot)} \mathbb{E}\left\{u\left(z,\pi(z),\pi^e_+(z)\right) + \beta\overline{v}\left(\pi^e_+(z)\right)\right\}$$

subject to

$$\mathbb{E}[\pi(z)] = \pi^e$$

which is the special case of our general fixed point equation.

- We show $\overline{v}(\pi^e)$ is a concave quadratic function.

Punishment - Stationary Equilibrium

- The stationary eqm value is

$$\underline{U}(b,y) = \underline{U} = -\frac{\chi \mathbb{E}_z \left[\chi \kappa^2 \left(z - \frac{\beta \chi \kappa^2}{1+\chi\kappa^2-\beta} \mu_z \right)^2 + \left(z - \frac{\beta}{1+\chi\kappa^2-\beta} \mu_z \right)^2 \right]}{2\left(1+\chi\kappa^2\right)^2 (1-\beta)}$$

- When $\mu_z = 0$ this simplifies to:

$$\underline{U} = -\frac{\chi \sigma_z^2}{2\left(1+\chi\kappa^2\right)(1-\beta)}$$

Equilibrium Consistency

If $h^t = (z_s, \pi_s, \pi_s^e)_{s \leq t-1}$ is equilibrium consistent, then $h^{t+1} = \left(h^t, z_t, \pi_t, \pi_{t+1}^e\right)$ is eqm consistent iff

$$-\frac{1}{2\kappa^2}\left(\pi_t - \beta\pi_{t+1}^e\right)^2 - \frac{\chi}{2}\left(\pi_t - z_t\right)^2 + \beta\overline{v}\left(\pi_{t+1}^e\right) \geq \underline{V}$$

- Given z_t, this translates into

$$\left(\pi_t, \pi_{t+1}^e\right) \in E\left(z_t\right)$$

  the region inside an Ellipsis
- Given (z_t, π_t) this can also be projected as

$$\pi_{t+1}^e \in \left[\underline{\pi}\left(z_t, \pi_t\right), \overline{\pi}\left(z_t, \pi_t\right)\right]$$

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Equilibrium Consistency

If $h^t = (z_s, \pi_s, \pi_s^e)_{s \leq t-1}$ is equilibrium consistent, then $h^{t+1} = \left(h^t, z_t, \pi_t, \pi_{t+1}^e\right)$ is eqm consistent iff

$$-\frac{1}{2\kappa^2}\left(\pi_t - \beta \pi_{t+1}^e\right)^2 - \frac{\chi}{2}\left(\pi_t - z_t\right)^2 + \beta \overline{v}\left(\pi_{t+1}^e\right) \geq \underline{V}$$

- Given z_t, this translates into

$$\left(\pi_t, \pi_{t+1}^e\right) \in E\left(z_t\right)$$

  the region inside an Ellipsis
- Given (z_t, π_t) this can also be projected as

$$\pi_{t+1}^e \in \left[\underline{\pi}\left(z_t, \pi_t\right), \overline{\pi}\left(z_t, \pi_t\right)\right]$$

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2. General Model

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6. Eqm Consistency with Sunspots - General Model

Setting: Eaton & Gersovitz (1981)

- Income $y \sim_{i.i.d} f(y)$ (density) over interval $Y = [\underline{y}, \overline{y}]$
- Continuum of households

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

- Benevolent Government
- Resource constraint

$$c_t = \underbrace{d_t y_t}_{\text{default}} + (1 - d_t) \left(\underbrace{y_t - b_t + q_t b_{t+1}}_{\text{no default}} \right)$$

where

- $d_t \in \{0, 1\}$ = default decision
- b_t = debt ≥ 0

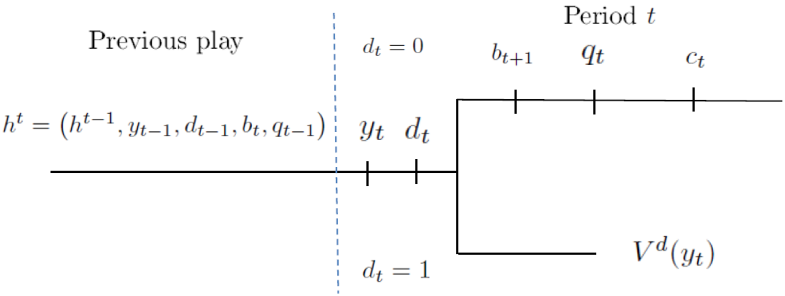
- No commitment

Setting

- Risk neutral competitive lenders maximize profits
- No Arbitrage:

$$q_t = \frac{1 - \Pr(d_{t+1} = 1)}{1 + r^*}$$

Setting



Previous play
Period t
$d_t = 0$
b_{t+1}
q_t
c_t
$h^t = (h^{t-1}, y_{t-1}, d_{t-1}, b_t, q_{t-1})$
y_t
d_t
$V^d(y_t)$
$d_t = 1$

Gov. Best and Worst Equilibria - i.i.d

- Best equilibrium: "markov" in $(b_t, \mathbf{1}\{d_\tau = 0 \text{ for some } \tau \leq t\})$

$$\begin{cases} d_t = \mathbf{d}^*(b_t, y_t) \\ b_{t+1} = \mathbf{b}^*(b_t, y_t) \\ q_t = \overline{\mathbf{q}}(b_{t+1}) \\ V(\sigma \mid h^t) = \begin{cases} \overline{\mathbb{V}}(b_t) & \text{on path} \\ \underline{\mathbb{V}}^{aut} & \text{off path} \end{cases} \end{cases}$$

- Worst equilibrium: Autarky $\implies \underline{\mathbf{q}}(b_t) = 0$

Some Definitions

- Define

$$\underline{U}(y) = u(y) + \beta \underline{\mathbb{V}}^{aut}.$$

$$V^{nd}\left(b, y, b'\right) := u\left(y - b + \overline{\mathbf{q}}\left(b'\right) b'\right) + \beta \overline{\mathbb{V}}\left(b'\right)$$

$$\overline{V}^{nd}(b, y) = \max_{b' \geq 0} V^{nd}\left(b, y, b'\right)$$

Properties of $\overline{v}(b, q)$

1. $\exists ! \gamma = \gamma(b, y)$ such that

$$\overline{v}(b,q) = \int \left\{ \delta(y) \underline{U}(y) + [1 - \delta(y)] \overline{V}^{nd}(b,y) \right\} f(y)\, dy$$

where

$$\delta(y) = 0 \iff \overline{V}^{nd}(b,y) \geq \underline{U}(y) + \gamma$$

and

$$q = \frac{1}{1+r} \left(1 - \int \delta(y) f(y)\, dy \right)$$

2. It is
   - Non-increasing in b, increasing in q
   - concave in q
   - also a function of y if y_t is Markov

Equilibrium Consistency - No Sunspots

- Suppose h^t is an Equilibrium Consistent History
- Using Proposition, we know then that history $(d_t = 0, b_{t+1}, q_t)$ is also eqm consistent $\iff$

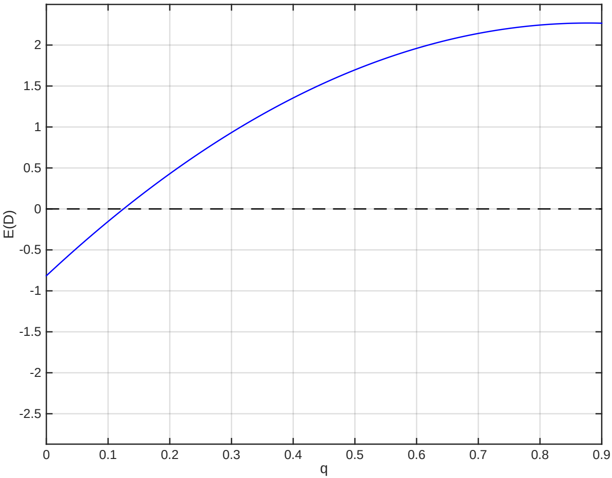
$$\Delta_t(q) := u(y_t - b_t + q_t b_{t+1}) + \beta \overline{v}(b_{t+1}, q_t) - \underline{U}(y_t) \geq 0$$

  which is concave and increasing.
- We want to find

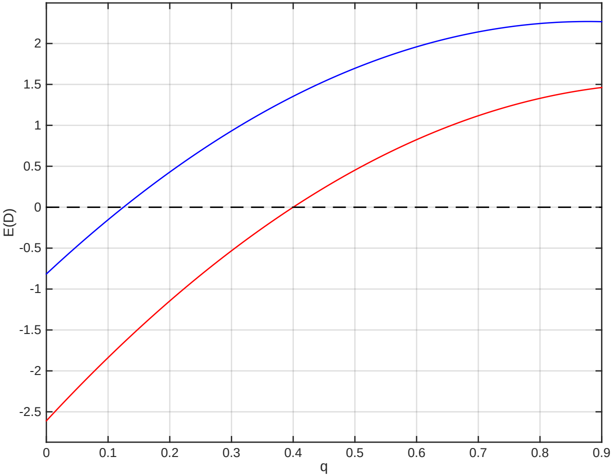
$$[\underline{\mathbf{q}}, \overline{\mathbf{q}}] = \text{ interval of eqm. consistent prices}$$

  Easy to see that $\overline{\mathbf{q}} = \overline{\mathbf{q}}(b')$ (price of "markov" eqm).

Find $\underline{\mathbf{q}}(b, y, b')$



Find $\underline{\mathbf{q}}(b, y, b'), \uparrow b$



Eqm. Consistent Prices

Corollary 2: Comparative Statics.

$\underline{\mathbf{q}}(b_t, y_t, b_{t+1})$ is increasing in the opportunity cost of not defaulting at t:

- Increasing in b_t
- Decreasing in y_t (only i.i.d)
- $\exists y : \underline{\mathbf{q}}(b_t, y, b_{t+1}) = \overline{\mathbf{q}}(b_{t+1})$ (if indifferent to defaulting under best eqm)

- "Reputation" effect

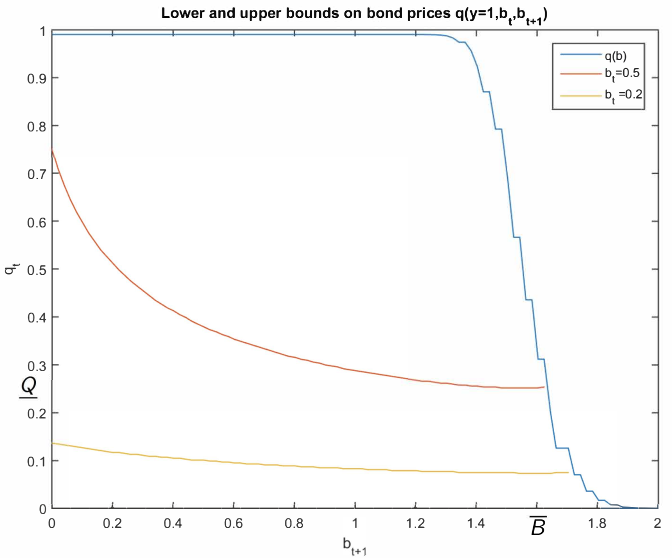
Numerical Example

- **Example:** Arellano's (2008) calibration
  - y_t follows Log-normal AR(1)
  - $u =$ CES with $\sigma = 2$
  - $\underline{U}(y_t)$ has an extra cost of default (subtracting from income) and punishment lasts for T periods, with $T \sim \text{Geo}(\alpha)$

Price Bounds

Bounds on Debt

Equilibrium Schedules



Debt issue constraints

- In figure, if $b_{t+1} > \overline{B}$ such that $\underline{q}\left(\overline{B}\right) = \overline{q}\left(\overline{B}\right) \Longrightarrow h^t$ is not eqm consistent.
- Maximum debt level also satisfies

$$\overline{B}\left(h^t, y_t\right) := \max\left\{b_{t+1} : V^{nd}\left(b_t, y_t, b_{t+1}\right) \geq \underline{U}\left(y_t\right)\right\}$$

so if $b_{t+1} > \overline{B} \Longrightarrow$ outside of any eqm path.

Debt issue constraints

- Other outcomes we can bound across all equilibria
  - Maximum debt/gdp ratio: $\overline{D}_t (y_t) = \overline{B}_{t+1}/y_t$
  - Lowest price: $\underline{Q}_t (y_t) := \overline{\mathbf{q}} (y_t, \overline{B}_{t+1})$ $(= \underline{\mathbf{q}})$

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Adding Sunspots (Public Correlation Devices)

- Agents in the economy observe
  - Publicly observed signals, uncorrelated to fundamentals (i.e. sunspots)
- Econometrician does not know
  a) what these signals are
  b) how they coordinate actions and expectations
- **Solution concept:** Extensive Form Correlated Equilibrium with public signals (Forges 1986)
- If $h \neq h^{\emptyset}$, equivalent to agents having privately observed, payoff irrelevant signals (Forges 1986)
- Two important, distinct cases...

Type I: Before $I's$ decision

- Timing:

1. $y_t \sim F(\cdot \mid y_{t-1}, b_t)$
2. ζ_t
3. $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$
4. q_t

Type II: After $I's$ decision

- Timing:

1. $y_t \sim F(\cdot \mid y_{t-1}, b_t)$
2. $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$
3. ζ_t
4. q_t

Type I Sunspots

- Suppose economy observes both y_t and a public randomizing device $\zeta_t \in S$ such that $\zeta_t \perp y_t$
- **Same results:** y_t serves itself as a correlating device if non-atomic
- In general, we can model $\hat{y}_t = (y_t, \zeta_t)$

Type II Sunspots - Sovereign Debt

- We focus on Type II sunspots (more general)
- That means that $q_t \mid h^t, y_t, d_t, b_{t+1}$ may still be random (since it may be a function of ζ_t).
- **Outcomes**: equilibrium consistent price distributions:

$$\mathscr{Q}(h) = \{Q \in \Delta([0, \overline{\mathbf{q}}(b_{t+1})]) \ : \ Q \text{ is an equilibrium price distribution}\}$$

- If government did not default, she expected:

$$\int_0^1 [u(y_t - b_t + b_{t+1} q_t(\zeta_t)) + \beta V_{t+1}(\zeta_t)] \, d\zeta_t \geq \underline{U}(y_t)$$

- Revealed Information (or beliefs)

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$$\int [u(y_t - b_t + b_{t+1} q_t) + \beta V_{t+1}] \, d\hat{Q}(V_{t+1}, q_t) \geq \underline{U}(z_t)$$

- Revealed Information (or beliefs)

Characterization

Proposition.

Suppose h^t is equilibrium consistent. Then $\left(d_t = 0, b_{t+1}, \hat{Q}\right)$ is eqm consistent $\iff \hat{Q} \in \Delta\left[0, \overline{\mathbf{q}}\left(b_{t+1}\right)\right]$ and

$$\mathbb{E}^{\hat{Q}}\left(\Delta\right) := \int \left\{u\left(y_t - b_t + \hat{q}_t b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \hat{q}_t\right)\right\} d\hat{Q}\left(\hat{q}_t\right) - \underline{U}\left(y_t\right) \geq 0 \tag{3}$$

and hence $\mathscr{Q}\left(h\right) = \mathscr{Q}\left(b_t, y_t, b_{t+1}\right)$

No Sunspots

$$u\left(y_t - b_t + \hat{q}_t b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \hat{q}_t\right) - \underline{U}\left(y_t\right) \geq 0$$

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Basic Properties of $\mathscr{Q}$

1. **Comparative Statics:** In the set order sense, $\mathscr{Q}(b_t, y_t, b_{t+1})$ is
   - Increasing in b_t
   - Decreasing in y_t (i.i.d)
2. **Expected prices:**

$$\bigcup_{Q \in \mathscr{Q}(h)} \mathbb{E}^Q(q_t) := \bigcup_{Q \in \mathscr{Q}(h)} \left\{ \int \hat{q}_t dQ(\hat{q}_t) \right\} = \left[\underline{\mathbf{q}}(b_t, y_t, b_{t+1}), \overline{\mathbf{q}}(b_{t+1}) \right]$$

3. If Q' FOSDs Q and $Q \in \mathscr{Q} \Longrightarrow Q' \in \mathscr{Q}$ as well

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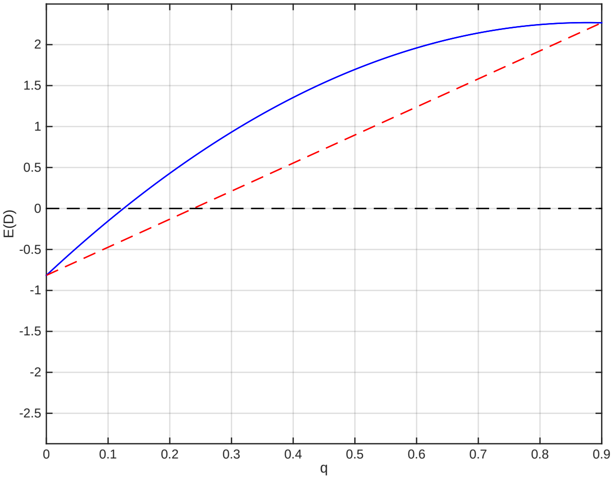
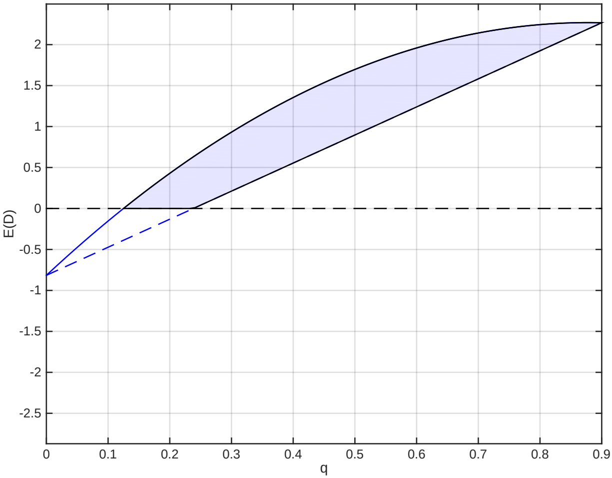
Moment Conditions



Moment Conditions



Bounds on Crisis Probabilities

- We want to find

$$p_0 = \max_{Q \in \mathcal{Q}(b_t, z_t, b_{t+1})} Q(\{\hat{q}_t = 0\})$$

  i.e. the maximum probability of a "confidence crisis"
- If $d_t = 0 \Longrightarrow p_0 < 1$ (wouldn't have payed b_t in the first place!!)
- Max. p_0 must make IC the least binding:

$$p_0 \Delta(0) + (1 - p_0)\Delta(\overline{\mathbf{q}}) = 0 \iff$$

$$p_0 = \frac{\Delta(\overline{\mathbf{q}})}{\Delta(\overline{\mathbf{q}}) - \Delta(0)} \in (0, 1)$$

Bounds on cdf's

- In general, we want to get the FOSD infimum cdf:

$$\underline{Q}\left(q \mid h^t\right) = \max_{Q \in \mathscr{Q}(b_t, z_t, b_{t+1})} Q\left(\{\hat{q} : \hat{q} \leq q\}\right)$$

Corollary

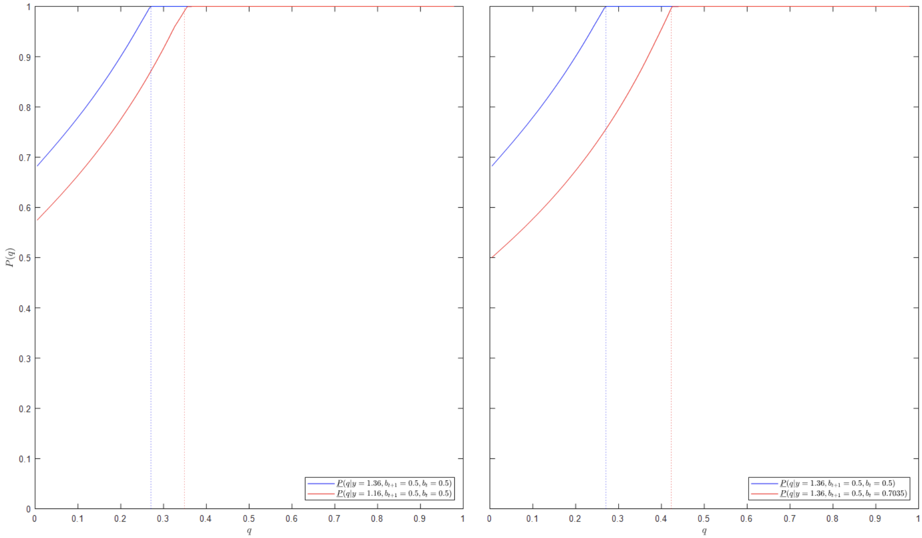
For every $q \in \left(0, \underline{\mathbf{q}}(\cdot)\right)$

$$\underline{Q}\left(q \mid h^t\right) = \frac{\Delta\left(\overline{\mathbf{q}}\right)}{\Delta\left(\overline{\mathbf{q}}\right) - \Delta\left(q\right)} < 1$$

and for $q \in \left[\underline{\mathbf{q}}(\cdot), \overline{\mathbf{q}}(\cdot)\right]$

$$\underline{Q}\left(q \mid h^t\right) = 1$$

Infimum Distribution $\underline{Q}(q)$



Bounds on Variance

Proposition.

Let $h = (h^t, y_t, d_t, b_{t+1})$ and suppose $(d_t, b_{t+1}, Q(\cdot))$ is eqm consistent,. Let $\mu_Q := \mathbb{E}^Q(q_t \mid h)$ and $\sigma_Q^2 = \mathbf{Var}^Q(q_t \mid h)$. Also,

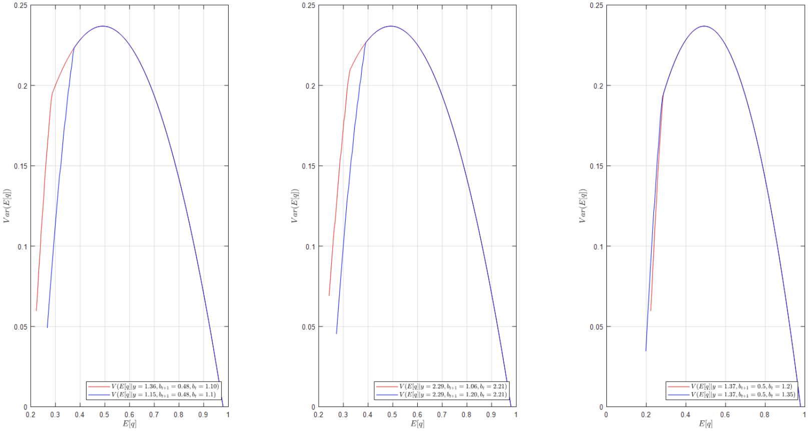
$$q^* = \left[1 - \underline{Q}(0)\right] \times \overline{\mathbf{q}}$$

Then,

- If $\mu_Q \geq q^* \implies \sigma_Q^2 \leq \mu(\overline{\mathbf{q}} - \mu)$
- If $\mu < q^* \implies \sigma_Q^2 \leq \mu\left(\overline{\mathbf{q}} + q_\mu - \mu\right) - q_\mu \overline{\mathbf{q}}$ where q_μ is the unique solution to
$$\underline{Q}(q_\mu) q_\mu + \left[1 - \underline{Q}(q_\mu)\right] \times \overline{\mathbf{q}} = \mu$$

- This bounds are the maximum variances across all possible eqm consistent distributions

Mean-Variance Bounds



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General Model with Sunspots

- Let $\mathscr{E}^s(y,b)$ be the set of eqm values with sunspots, and $\mathscr{Q}^s(y,b)$ the eqm prices with sunspots.
- If **(a)** $\mathscr{E}(y,b)$ is convex and compact valued and **(b)** $u(\cdot)$ is concave in q, then

1. $\overline{v}(y,b,q)$ is concave in q, and hence so is $\Delta(q)$
2. $\mathscr{E}^s(y,b) = \mathscr{E}(y,b)$ and hence $\mathscr{Q}(y,b) = \mathscr{Q}^s(y,b)$ and is convex

General Model with Sunspots

- With sunspots, an outcome in period t is a policy (d_t, b_{t+1}) and a distribution $P \in \Delta\left(\mathbb{R}^k\right)$ over q
- Autonomous mechanism sends recommendations q with dist. P
- Revelation Principle in Info Design ("straightforward messages" in KG)

General Model with Sunspots

Proposition.

Suppose h^t is equilibrium consistent, and **(a)** and **(b)** hold. Then (d_t, b_{t+1}, P) is eqm consistent $\iff$ $\mathbf{Supp}(P) \subseteq \mathscr{Q}(y_t, b_{t+1})$ and

$$\int \left[u\left(b_t, y_t, d_t, b_{t+1}, \hat{q}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dP\left(\hat{q}\right) \geq \underline{U}\left(y_t, b_t\right)$$

- Because $\Delta\left(\cdot, q\right)$ is concave, then $\bigcup_P \mathbb{E}^P\left(q\right) = \mathbf{Q}\left(b_t, y_t, d_t, b_{t+1}\right)$

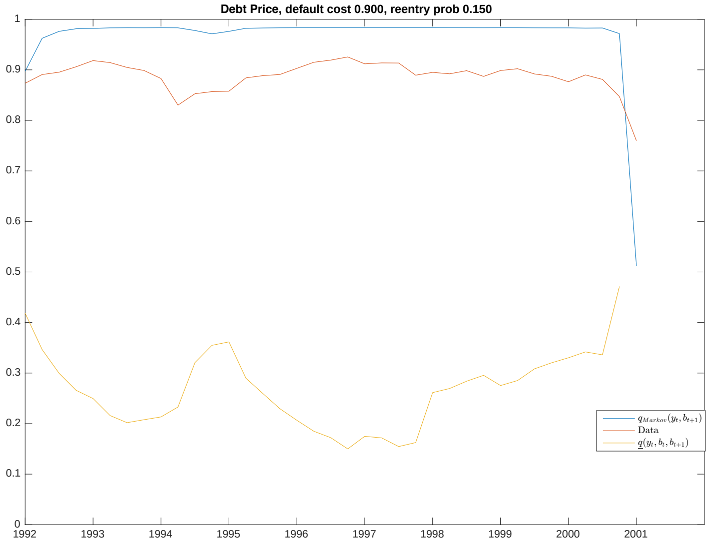
Limitations

- Without the concavity assumption on $u(\cdot)$, the proposition is still valid: however, $\underline{U}(\cdot)$ and $\overline{v}(\cdot)$ may be different from the non-sunspot case.
- Tighter constraints obtain only when long lived players deviate from spot optimum. When behavior is close to myopic behavior, almost no constraints can be obtained.
- No "learning": past behavior does not impose hard constraints on the future. E.g. the fact that a gov did not default in the past when she was tempted to do so, does not help impose tighter constraints on more than 1 period ahead.
- This is a consequence of the robustness analysis: we can get tighter constraints by imposing more assumptions over the set of equilibria.

Conclusions

- **Our approach:** General methodology of testing equilibrium conditions without selection.
- Translates conditions about continuation values into observables.
- Particularly useful for providing testable implications of equilibria with (unobserved) correlation devices.
- Sunspots = Information about strategies player about other agents).
- Next: How to econometrically test this properly? (Partial Identification, Moment Inequalities).

Price Intervals



Bond issue bound
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1992
1993
1994
1995
1996
1997
1998
1999
2000
2001
$\max b_{t+1}(b_t, y_t)$
Data

$\overline{v}(z,x,q)$ as a fixed point

- When $\delta = 0$, $\overline{v}(\cdot)$ is the unique fixed point of

$$\mathbb{T}(f)(y,b,q) =$$

$$\max_{(d,x',q',w)(\cdot)} \int \left\{ u\left(b,y',d(y'),b'(y'),q'(y')\right) + \beta w(y') \right\} dF\left(y' \mid y,b\right)$$

$$\text{s.t. } : \begin{cases} u\left[b,y',d(y'),b'(y'),q'(y')\right] + \beta w(y') \geq \underline{U}\left(b'(y'),y'\right) & \forall y' \\ q = \int M\left(b,y',d(y'),b'(y')\right) dF\left(y' \mid y,b\right) & \\ w(y') \leq f\left(y',b'(y'),q'(y')\right) & \forall y' \end{cases}$$

- Waki et al (2018) for New Keynesian model. When $\delta \in (0,1)$ need to add $\lim \delta^t q_t = 0$
- **Special case:** if $\delta = 0 \implies \underline{U}(y,b)$ value function of a Markov-perfect equilibrium. Go Back