STAT 30400 : Distribution Theory

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Homework 5

Solutions by

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1. (15 pts) Let W_1, W_2 and W_3 be three mutually independent exponential random variables with parameter $\lambda > 0$, and let J be a Bernoulli random variable, independent of the Ws, with parameter θ in (0,1). Set

$$X = (1-\theta)W_1 + JW_3,$$
$$Y = (1-\theta)W_2 + JW_3.$$

(a) Find the joint distribution function of X and Y.

Let $f_W(w) = \lambda e^{-\lambda w}\mathbb{1}_{\{w\geq 0\}}$ and $F_W(w) = [1-e^{-\lambda w}]\mathbb{1}_{\{w\geq 0\}}$ be the density function and distribution function of W_1, W_2 and W_3. Let $Z = JW_3$, since J and W_3 are independent,

$$\begin{aligned}
f_Z(z) &= \begin{cases} f_W(z)\mathbb{P}(J=1) & ,\ z=0 \\ \mathbb{P}(J=0) & ,\ z>0 \\ 0 & ,\ z<0 \end{cases} \\
&= \begin{cases} \theta f_W(z) & ,\ z=0 \\ 1-\theta & ,\ z>0 \\ 0 & ,\ z<0 \end{cases}
\end{aligned}$$

Then for $0 \leq x < y$,

$$\begin{aligned}
F_{(X,Y)}(x,y) &= \mathbb{P}(X\leq x, Y\leq y) \\
&= \mathbb{P}(X\leq x, Y\leq y|J=0)\mathbb{P}(J=0) + \mathbb{P}(X\leq x, Y\leq y|J=1)\mathbb{P}(J=1) \\
&= (1-\theta)\mathbb{P}\left(W_1 \leq \frac{x}{1-\theta}, W_2 \leq \frac{y}{1-\theta}\right) \\
&\quad + \int_0^\infty \theta\mathbb{P}\left(W_1 \leq \frac{x-z}{1-\theta}, W_2 \leq \frac{y-z}{1-\theta}, W_3 = z\right)\mathrm{d}z \\
&= (1-\theta)F\left(\frac{x}{1-\theta}\right)F_W\left(\frac{y}{1-\theta}\right) + \int_0^x \theta F\left(\frac{x-z}{1-\theta}\right)F\left(\frac{y-z}{1-\theta}\right)f_W(z)\mathrm{d}z
\end{aligned}$$

While

$$\begin{aligned}
&(1-\theta)F\left(\frac{x}{1-\theta}\right)F_W\left(\frac{y}{1-\theta}\right) \\
=&(1-\theta)(1-e^{-\frac{\lambda(x)}{1-\theta}})(1-e^{-\frac{\lambda(y)}{1-\theta}}) \\
&\int_0^x \theta F\left(\frac{x-z}{1-\theta}\right)F\left(\frac{y-z}{1-\theta}\right)f_W(z)\mathrm{d}z \\
=&\int_0^x \theta(1-e^{-\frac{\lambda(x-z)}{1-\theta}})(1-e^{-\frac{\lambda(y-z)}{1-\theta}})\lambda e^{-\lambda z}\mathrm{d}z \\
=&\theta\lambda\int_0^x e^{-\lambda z} - (e^{-\frac{\lambda}{1-\theta}x} + e^{-\frac{\lambda}{1-\theta}y})e^{\frac{\theta\lambda}{1-\theta}z} + e^{-\frac{\lambda}{1-\theta}(x+y)}e^{\frac{1+\theta}{1-\theta}z}\mathrm{d}z \\
=&\theta - \theta e^{-\lambda x} - (1-\theta)(e^{-\lambda x} + e^{-\frac{\lambda}{1-\theta}(y-\theta x)}) + (1-\theta)(e^{-\frac{\lambda}{1-\theta}x} + e^{-\frac{\lambda}{1-\theta}y}) \\
&+ \frac{\theta(1-\theta)}{1+\theta}e^{-\frac{\lambda}{1-\theta}(y-\theta x)} - \frac{\theta(1-\theta)}{1+\theta}e^{-\frac{\lambda}{1-\theta}(y+x)} \\
=&\theta - e^{-\lambda x} + (1-\theta)(e^{-\frac{\lambda}{1-\theta}x} + e^{-\frac{\lambda}{1-\theta}y}) - \frac{1-\theta}{1+\theta}e^{-\frac{\lambda}{1-\theta}(y-\theta x)} - \frac{\theta(1-\theta)}{1+\theta}e^{-\frac{\lambda}{1-\theta}(y+x)}.
\end{aligned}$$

Solution (cont.)

so, for $0 \le x < y$,

$$F_{(X,Y)}(x,y) = 1 - e^{-\lambda x} - \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y-\theta x)} + \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y+x)}.$$

Since X and Y are symmetry in the sense that $F(x,y) = F(y,x)$, we have

$$F_{(X,Y)}(x,y) = \begin{cases} 1 - e^{-\lambda x} - \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y-\theta x)} + \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y+x)} & , 0 \le x < y \\ 1 - e^{-\lambda y} - \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(x-\theta y)} + \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y+x)} & , x \ge y \ge 0 \\ 0 & , x < 0 \text{ or } y < 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-\lambda \min\{x,y\}} - \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(x+y-(1-\theta)\min\{x,y\})} + \frac{1-\theta}{1+\theta} e^{-\frac{\lambda}{1-\theta}(y+x)} & , x \ge 0, y \ge 0 \\ 0 & , x < 0 \text{ or } y < 0 \end{cases}$$

(b) Find the marginal densities of X and Y.

For $x \ge 0$,

$$\begin{aligned} F_X(x) &= \lim_{y\to\infty} F_{(X,Y)}(x,y) \\ &= \begin{cases} 1 - e^{-\lambda x} & , x \ge 0 \\ 0 & , x < 0 \end{cases} \end{aligned}$$

Analogously, $F_Y(y) = (1 - e^{-\lambda y})\mathbb{1}_{y \ge 0}$.

(c) Find the copula, C_θ, associated with (X,Y). Show that C_θ is absolutely continuous in θ, and find C_0 and C_1.

Since for $u, v \in (0,1)$,

$$F_Y^{-1}(u) = F_X^{-1}(u) = -\frac{1}{\lambda}\log(1-u)$$

and $F_X^{-1}(u) \le F_Y^{-1}(v)$ for $u \le v$, we have for $u < v$

$$\begin{aligned} C_\theta(u,v) &= F_{(X,Y)}(F_X^{-1}(u), F_Y^{-1}(v)) \\ &= u - \frac{1-\theta}{1+\theta}(1-u)^{\frac{\theta}{1-\theta}}(1-v)^{\frac{1}{1-\theta}} + \frac{1-\theta}{1+\theta}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} C_\theta(u,v) &= \min\{u,v\} - \frac{1-\theta}{1+\theta}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}}(1-\min\{u,v\})^{-1} + \frac{1-\theta}{1+\theta}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}} \\ &= \min\{u,v\} - \frac{1-\theta}{1+\theta}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}}\frac{\min\{u,v\}}{1-\min\{u,v\}}, \\ \frac{\mathrm{d}C_\theta}{\mathrm{d}\theta} &= \frac{2}{(1+\theta^2)}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}}\frac{\min\{u,v\}}{1-\min\{u,v\}} \\ &\quad - \frac{1}{(1+\theta)(1-\theta)}(1-u)^{\frac{1}{1-\theta}}(1-v)^{\frac{1}{1-\theta}}\ln[(1-u)(1-v)]\frac{\min\{u,v\}}{1-\min\{u,v\}}, \end{aligned}$$

Solution (cont.)

Let $a=(1-u)(1-v)\in(0,1)$, $x=\frac{1}{1-\theta}\in(1,\infty)$, $f(x)=xa^x$ for $x\in(1,\infty)$. $f'(x)=a^x(1+x\ln a)$. If $\ln a\geq -1$, then $f'(x)\leq 0$ and $f(x)\leq a$; If $\ln a<-1$, then $f'(x)<0$ for $x<-\frac{1}{\ln a}$ and $f'(x)<0$ for $x>-\frac{1}{\ln a}$, so $f(x)\leq -\frac{1}{\ln a}a^{-\frac{1}{\ln a}}$. Therefore,

$$\begin{aligned}\left|\frac{\mathrm{d}C_\theta}{\mathrm{d}\theta}\right| &\leq 2(1-u)(1-v)\frac{\min\{u,v\}}{1-\min\{u,v\}}+\frac{1}{1-\theta}[(1-u)(1-v)]^{\frac{1}{1-\theta}}\frac{\min\{u,v\}}{1-\min\{u,v\}}\\ &=\begin{cases}2a\frac{\min\{u,v\}}{1-\min\{u,v\}}+a\frac{\min\{u,v\}}{1-\min\{u,v\}} & \text{, if } \ln a\geq -1\\ 2a\frac{\min\{u,v\}}{1-\min\{u,v\}}-\frac{1}{\ln a}a^{-\frac{1}{\ln a}}\frac{\min\{u,v\}}{1-\min\{u,v\}} & \text{, if } \ln a<-1\end{cases}\end{aligned}$$

i.e., C_θ is Lipschitz continuous with respect to θ. Therefore, C_θ is absolutely continuous in θ.

$$\begin{aligned}C_0(u,v)&=\lim_{\theta\to 0^+}C_\theta(u,v)=\min\{u,v\}-(1-u)(1-v)(1-\min\{u,v\})^{-1}+(1-u)(1-v)\\ C_1(u,v)&=\lim_{\theta\to 1^-}C_\theta(u,v)=\min\{u,v\}\end{aligned}$$

2. (10 pts) Let X have a $N(\mu, 1)$ distribution and let $Y = X^2$.

(a) Find the density $f_Y(y)$ of Y. (This is known as *noncentral chi-square*)

Let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$ be the density function of X. Let $\mathcal{X}_1 = (-\infty, 0)$, $\mathcal{X}_2 = [0, \infty)$ and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, then in $\mathcal{X}_i$, X and Y have one-to-one relationship g_i.

$$
\begin{aligned}
f_Y(y) &= \sum_{\substack{x \in \mathcal{X}_i \\ x = g_i^{-1}(y)}} f_X(g_i^{-1}(x)) \left| \frac{\mathrm{d} g_i^{-1}(y)}{\mathrm{d}y} \right| \mathbb{1}_{y>0} \\
&= [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \left| \frac{1}{\sqrt{|y|}} \right| \mathbb{1}_{y>0} \\
&= \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y}-\mu)^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{y}-\mu)^2} \right] \frac{1}{\sqrt{y}} \mathbb{1}_{y>0} \\
&= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} (e^{\mu\sqrt{y}} + e^{-\mu\sqrt{y}}) \mathbb{1}_{y>0}
\end{aligned}
$$

(b) Show that we can write:

$$
f_Y(y) = \sum_{k=0}^{\infty} \mathbb{P}(R = k) f_{2k+1}(y),
$$

where R is distributed Poisson$(\mu^2/2)$ and f_m is the χ^2_m density. Give an interpretation of this formula.

Proof.

$$
\begin{aligned}
\mathbb{P}(R = k) &= \frac{\mu^{2k}}{2^k k!} e^{-\frac{\mu^2}{2}} \\
f_m(x) &= \frac{1}{2^{\frac{m}{2}} \Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x>0}
\end{aligned}
$$

we have

$$
\sum_{k=0}^{\infty} \mathbb{P}(R = k) f_{2k+1}(y) = y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} \sum_{k=0}^{\infty} \frac{\mu^{2k} y^k}{2^{\frac{2k+1}{2}} k!\ \Gamma(\frac{2k+1}{2})}
$$

Since

$$
\Gamma(\frac{2k+1}{2}) = \frac{2k-1}{2} \cdot \frac{2k-3}{2} \cdots \frac{1}{2} \Gamma(\frac{1}{2}) = \frac{(2k-1)!!}{2^k} \sqrt{\pi},
$$

and

$$
(2k)! = (2k)!! \cdot (2k-1)!! = 2^k \cdot k! \cdot (2k-1)!,
$$

we have

$$
\begin{aligned}
y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} \sum_{k=0}^{\infty} \frac{\mu^{2k} y^k}{2^{\frac{2k+1}{2}} k!\ \Gamma(\frac{2k+1}{2})} &= y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} \sum_{k=0}^{\infty} \frac{\mu^{2k} y^k}{\sqrt{2\pi}(2k)!} \\
&= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} \sum_{k=0}^{\infty} \frac{\mu^{2k} y^k}{(2k)!} \\
&= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} \left[\sum_{k=0}^{\infty} \frac{(\mu\sqrt{y})^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\mu\sqrt{y})^k}{k!} \right] \\
&= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}(y+\mu^2)} (e^{\mu\sqrt{y}} + e^{-\mu\sqrt{y}}) \\
&= f_Y(y)
\end{aligned}
$$

□

3. (10 pts) Let X and Y be independent random variable with distribution function F and representing function R. Let $\Delta = \mathbb{E}(|X-Y|)$ be the *Ginis mean difference*. Show that,

$$\Delta = 2\int_{-\infty}^{\infty} F(x)[1-F(x)]\mathrm{d}x = 2\int_0^1 (2u-1)R(u)\mathrm{d}u.$$

[Hint: for the first equality show that,

$$\Delta = 2\mathbb{E}\left(\int_{-\infty}^{\infty} \mathbb{1}_{X\le t<Y}\mathrm{d}t\right),$$

and reverse the order of integration; justify the reversal. For the second equality, give separate arguments depending on whether X is integrable or not. When X is integrable argue that

$$\Delta = 2\iint_{(u,v):u<v} [R(u)-R(v)]\mathrm{d}u\mathrm{d}v.$$

Proof. Let $X, Y \overset{iid}{\sim} F$. For all $x, y \in \mathbb{R}$, if $x < y$, then $y - x = \int_{\mathbb{R}} \mathbb{1}_{x\le t<y}\mathrm{d}t$; if $x \ge y$, then $x - y = \int_{\mathbb{R}} \mathbb{1}_{y\le t<x}\mathrm{d}t$. So

$$\begin{aligned}\Delta &= \mathbb{E}|X-Y| \\ &= \mathbb{E}\left(\int_{\mathbb{R}} \mathbb{1}_{X\le t<Y}\mathrm{d}t + \int_{\mathbb{R}} \mathbb{1}_{Y\le t<X}\mathrm{d}t\right) \\ &= 2\mathbb{E}\left(\int_{-\infty}^{\infty} \mathbb{1}_{X\le t<Y}\mathrm{d}t\right).\end{aligned}$$

Since the integrated function is non-negative, by Fubini Theorem we have

$$\begin{aligned}\Delta &= 2\int_{\mathbb{R}} \mathbb{E}\mathbb{1}_{X\le t<Y}\mathrm{d}t \\ &= 2\int_{\mathbb{R}} \mathbb{P}(X\le t<Y)\mathrm{d}t \\ &= 2\int_{\mathbb{R}} \mathbb{P}(X\le t)\mathbb{P}(Y>t)\mathrm{d}t \\ &= 2\int_{\mathbb{R}} F(t)[1-F(t)]\mathrm{d}t.\end{aligned}$$

Let $U, V \overset{iid}{\sim} Uniform(0,1)$, such that $X = R(U), Y = R(V)$.
(1) If X is not integrable, then $\mathbb{E}|Y| = \mathbb{E}|X| = \mathbb{E}|R(U)| = \int_0^1 |R(u)|\mathrm{d}u = \infty$

$$\begin{aligned}\Delta &= \mathbb{E}|X-Y| \\ &= \int_0^{\infty} \mathbb{P}(|X-Y|\ge t)\mathrm{d}t \\ &\ge \int_0^{\infty} \mathbb{P}(|X|\ge t+k, |Y\le k|)\mathrm{d}t \\ &= \int_0^{\infty} \mathbb{P}(|X|\ge t+k)\mathbb{P}(|Y\le k|)\mathrm{d}t \\ &= \mathbb{P}(|Y\le k|)\int_0^{\infty} \mathbb{P}(|X|\ge t+k)\mathrm{d}t \\ &= \mathbb{P}(|Y\le k|)\cdot\infty \\ &= \infty\end{aligned}$$

for sufficiently large k such that $\mathbb{P}(|Y\le k|) > 0$. Decompose $(0,1)$ as $(0,1) = \{u : R(u) \ge 0, u \ge$

Solution (cont.)

$\frac{1}{2}\} \cup \{u : R(u) < 0, u < \frac{1}{2}\} \cup \{u : R(u) > 0, u < \frac{1}{2}\} \cup \{u : R(u) < 0, u > \frac{1}{2}\}$, we have

$$0 \geq \int_{\{u:R(u)>0,u<\frac{1}{2}\}} (2u-1)R(u)\mathrm{d}u \geq \int_{\{u:R(u)>0,u<\frac{1}{2}\}} (2u-1)R(\frac{1}{2})\mathrm{d}u > -\infty$$
$$0 \geq \int_{\{u:R(u)<0,u>\frac{1}{2}\}} (2u-1)R(u)\mathrm{d}u \geq \int_{\{u:R(u)<0,u>\frac{1}{2}\}} (2u-1)R(\frac{1}{2})\mathrm{d}u > -\infty$$

Since X is not integrable, neither $R(u)$ is integrable and therefore $\int_0^1 R^+(u)\mathrm{d}u = \int_0^1 R^-(u)\mathrm{d}u = \infty$. Since $R^+(u)$ is non-negative and non-decreasing, there exists $u_0 < \infty$ such that $u_0 = \inf\{R(u) \geq 0,\ u \geq \frac{1}{2}\}$,

$$\begin{aligned}
\int_{\{u:R(u)\geq 0,u\geq\frac{1}{2}\}} (2u-1)R(u)\mathrm{d}u &= \int_{u_0}^1 (2u-1)R(u)\mathrm{d}u \\
&\geq (2u_0-1)\int_{u_0}^1 R(u)\mathrm{d}u \\
&= (2u_0-1)\left[\int_0^1 R^+(u)\mathrm{d}u - \int_{u_0}^1 R^+(u)\mathrm{d}u\right] \\
&\geq (2u_0-1)\int_0^1 R^+(u)\mathrm{d}u - (2u_0-1)R^+(u_0) \\
&= \infty.
\end{aligned}$$

Analogously, we also have $\int_{\{u:R(u)<0,u<\frac{1}{2}\}}(2u-1)R(u)\mathrm{d}u \geq \infty$. So $\int_0^1 (2u-1)R(u)\mathrm{d}u = \infty = \Delta$.

(2) If X is integrable, $\mathbb{E}|X| = \mathbb{E}|Y| = \int_0^1 |R(u)|\mathrm{d}u < \infty$, and $\iint_{(0,1)\times(0,1)} |R(u) - R(v)|\mathrm{d}u\mathrm{d}v \leq \iint_{(0,1)\times(0,1)} [|R(u)| + |R(v)|]\mathrm{d}u\mathrm{d}v \xlongequal{\text{Fubini}} \int_0^1 \left(\int_0^1 [|R(u)| + |R(v)|]\mathrm{d}u\right)\mathrm{d}v = \left(\int_0^1 |R(u)|\mathrm{d}u\right) + \left(\int_0^1 |R(v)|\mathrm{d}v\right) < \infty$.

Since R is a non-decreasing function, we have

$$\begin{aligned}
\Delta &= \mathbb{E}|R(U) - R(V)| \\
&= \iint_{(0,1)\times(0,1)} |R(u) - R(v)|\mathrm{d}u\mathrm{d}v \\
&= \iint_{0<u<v<1} [R(v) - R(u)]\mathrm{d}u\mathrm{d}v + \iint_{0<v\leq u<1} [R(u) - R(v)]\mathrm{d}u\mathrm{d}v \\
&= 2\iint_{0<u<v<1} [R(v) - R(u)]\mathrm{d}u\mathrm{d}v \\
&\xlongequal{\text{Fubini}} 2\int_0^1 \left(\int_u^1 [R(v) - R(u)]\mathrm{d}v\right)\mathrm{d}u \\
&= 2\int_0^1 \left[\left(\int_u^1 R(v)\mathrm{d}v\right) - (1-u)R(u)\right]\mathrm{d}u \\
&= 2\int_0^1 \left(\int_u^1 R(v)\mathrm{d}v\right)\mathrm{d}u - 2\int_0^1 (1-u)R(u)\mathrm{d}u \\
&\xlongequal{\text{Fubini}} 2\int_0^1 \left(\int_0^v R(v)\mathrm{d}u\right)\mathrm{d}v - 2\int_0^1 (1-u)R(u)\mathrm{d}u \\
&= 2\int_0^1 vR(v)\mathrm{d}v + 2\int_0^1 (u-1)R(u)\mathrm{d}u \\
&= 2\int_0^1 (2u-1)R(u)\mathrm{d}u
\end{aligned}$$

Solution (cont.)

A double integral can be done as an iterated integral (in either order) provided the integrand is non-negative, or the double integral is absolutely convergent. $\square$

4. (15 pts) Let X be an integrable random variable with standard deviation σ, mean deviation δ and mean difference Δ. Show that:

(a) $\Delta \le 2\delta$.

Proof. Suppose that X' and X are independent identical distributed with mean μ.

$$\begin{aligned}\Delta &= \mathbb{E}|X - X'| \\ &\le \mathbb{E}|X-\mu| + \mathbb{E}|\mu - X'| \\ &= 2\delta\end{aligned}$$

□

(b) $\delta \le \Delta$.

Proof. Suppose that X' and X are independent identical distributed. Since

$$\begin{aligned}\Delta &= \mathbb{E}|X - X'| \\ &= \mathbb{E}[\mathbb{E}(\ |X-X'|\ \big|X)] \\ &\ge \mathbb{E}[\ |\mathbb{E}(X-X')|\ \big|X] \\ &= \mathbb{E}|X - \mathbb{E}X'| \\ &= \delta\end{aligned}$$

□

(c) $\Delta \le 2\sigma/\sqrt{3}$. If $\sigma < \infty$, equality holds if and only if X has a uniform distribution. [Hint: Use Exercise 3 and Exercise 4 from Homework 3.]

Proof. Choose quantile function $Q(u)$ as the representing function $R(u)$. Let $\mu = \mathbb{E}X = \int_0^1 Q(u)\mathrm{d}u$, then

$$\int_0^1 (2u-1)\mu \mathrm{d}u = \frac{\mu}{2}(2u-1)^2\Big|_0^1 = 0.$$

So

$$\begin{aligned}\Delta &= 2\int_0^1 (2u-1)Q(u)\mathrm{d}u \\ &= 2\int_0^1 (2u-1)(Q(u)-\mu)\mathrm{d}u \\ &\le 2\left(\int_0^1 (2u-1)^2\mathrm{d}u\right)^{\frac{1}{2}}\left(\int_0^1 (Q(u)-\mu)^2\mathrm{d}u\right)^{\frac{1}{2}} \\ &= 2\sqrt{\frac{4}{3}u^3 - 2u^2 + u\ }\Bigg|_0^1 \cdot \sqrt{\mathbb{E}(X-\mu)^2} \\ &= \frac{2}{\sqrt{3}}\sigma,\end{aligned}$$

the inequality holds if and only if $(2u-1)^2 = (Q(u)-\mu)^2$ almost everywhere for u in $[0,1]$, i.e., $Q(u) = \mu + (2u-1)$ a.s. ($Q(u)$ is nondecreasing), which means $F(x) = \frac{x-\mu+1}{2}\mathbb{1}_{(\mu-1,\mu+1)}$, i.e. X has a uniform distribution. □