Question 1 (a) Compute an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = QR$$

(b) find the orthonormal and upper-triangular matrices Q and R.
(c) Compute the orthogonal projection P onto the range of A.

Question 2 Find a_0 and a_1 minimizing

$$F(a_0, a_1) = \int_0^1 |a_0 + a_1 x - \mathrm{e}^{-x}|^2 \, \mathrm{d}x.$$

Question 3 (a) Find an orthonormal basis for the 3-dimensional subspace of $L^2(-1, 1)$ spanned by $1, x$ and x^2.
(b) Interpret as a QR factorization.

Question 4 Let

$$H^1 = H^1(0, 1) = \{f \in L^2(0, 1) | f' \in L^2(0, 1)\}$$

with inner product

$$< f, g >= \int_0^1 f(x)g(x) + f'(x)g'(x) \, \mathrm{d}x.$$

(For simplicity assume all functions are real-valued.)
(a) Show that every $f \in H^1$ is continuous and bounded on (0,1).
(b) Let $g \in H^1$ and suppose also that g' and g'' are continuous except at some point $x_0 \in (0, 1)$. Show that

$$< f, g >= f(1)g'(1) + f(x_0)\left(g'(x_0^-) - g'(x_0^+)\right) - f(0)g'(0) + \int_0^1 f(x)\left(g(x) - g''(x)\right) \, \mathrm{d}x$$

for every $f \in H^1$.
(c) Find $g \in H^1$ such that

$$< f, g >= f(x_0)$$

for every $f \in H^1$.

Question 5 Given $n+1$ distinct points $-1 < x_0 < x_1 < \ldots < x_n < 1$, let P_n be the linear operator which takes $f \in H^1$ into the unique degree-n polynomial

$$p_n(x) = P_n f(x) = \sum_{j=0}^{n} L_j(x) f(x_j)$$

which interpolates the $n+1$ values $f(x_j)$. Here $L_j(x)$ are the degree-n polynomials satisfying

$$L_i(x_j) = \delta_{ij}.$$

(a) Show that P_n is a projection.
(b) Find the adjoint operator $P_n^* g$ for $g \in H^1$.
(c) Show that P_n is not an orthogonal projection.
(d) Find a basis $\{e_0, e_1, e_2, e_3\}$ for the range of P_3 which is orthogonal in the H^1 inner product.
(e) Find the orthogonal projection Q_3 onto the range of P_3. Express Q_3 as an integrodifferential operator

$$Q_3 f(x) = \int_0^1 K(x,y) f(y) + K'(x,y) f'(y) \, \mathrm{d}y$$

and compute the kernels K and K' in terms of $\{e_0, e_1, e_2, e_3\}$.
(f) Show that $q = Q_3 f$ minimizes the H^1 norm $\|q - f\|$ over q in the range of P_3.