STAT 30400 : Distribution Theory

Fall 2019

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Homework 8

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Solutions by

Jinhong Du

12243476

1. (10 pts) (The inversion formula for lattice distributions). Let X be a random variable that takes values in the lattice $\{a + kh; k \in \mathbb{Z}\}$ (a and h are real numbers), with characteristic function ϕ. Show that for any x on the lattice,

$$\mathbb{P}(X = x) = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \phi(t) \mathrm{d}t.$$

Proof.

$$\begin{aligned}\phi(t) &= \mathbb{E}e^{itX} \\ &= \sum_{k\in\mathbb{Z}} e^{it(a+kh)} \mathbb{P}(X = a + kh)\end{aligned}$$

Let $x = a + mh$ for some $m \in \mathbb{Z}$.
If $k = m$, then $\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{it(k-m)h}\mathrm{d}t = \frac{2\pi}{h}$. If $k \neq m$, then

$$\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{it(k-m)h}\mathrm{d}t = \frac{1}{i(k-m)h}(e^{i(k-m)\pi} - e^{-i(k-m)\pi}) = 0.$$

So

$$\begin{aligned}&\frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \left[\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} e^{it(a+kh)} \mathbb{P}(X = a + kh) \mathrm{d}t \right] \\ =&\frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \left[\mathbb{P}(X = a + kh) \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{it(k-m)h} \mathrm{d}t \right] \\ =&\mathbb{P}(X = a + mh) \\ =&\mathbb{P}(X = x) < \infty\end{aligned}$$

By Fubini theorem, we have

$$\begin{aligned}&\frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \phi(t) \mathrm{d}t \\ =&\frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \left[\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} e^{it(a+kh)} \mathbb{P}(X = a + kh) \mathrm{d}t \right] \\ =&\mathbb{P}(X = x)\end{aligned}$$

□

2. (15 pts) We say that X_n converges in probability to X, written $X_n \xrightarrow{\mathbb{P}} X$, if, for any positive real number ϵ, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0$.

(a) Suppose that we have two sequences of random variables such that $X_n \xrightarrow{D} X$ and $Y_n - X_n \xrightarrow{\mathbb{P}} 0$. Show that $Y_n \xrightarrow{D} X$.

Proof. For $x \in \mathcal{C}_{F_X}$ and $\epsilon > 0$, since

$$\begin{aligned}\mathbb{P}(Y_n \leq x) &\leq \mathbb{P}(X_n \leq x + \epsilon, Y_n - X_n \leq -\epsilon) \\ &\leq \mathbb{P}(X_n \leq x + \epsilon) + \mathbb{P}(Y_n - X_n \leq -\epsilon) \\ &\leq \mathbb{P}(X_n \leq x + \epsilon) + \mathbb{P}(|Y_n - X_n| \geq \epsilon),\end{aligned}$$

we have

$$\limsup_{n\to\infty} \mathbb{P}(Y_n \leq x) \leq \mathbb{P}(X \leq x + \epsilon),$$

since $X_n \xrightarrow{D} X$ and $Y_n - X_n \xrightarrow{\mathbb{P}} 0$. Let $\epsilon \to 0^+$, since F_X is right-continuous, we have $\limsup_{n\to\infty} \mathbb{P}(Y_n \leq x) \leq \mathbb{P}(X \leq x)$.

Similarly, since

$$\begin{aligned}\mathbb{P}(X_n \leq x - \epsilon) &\leq \mathbb{P}(Y_n \leq x, X_n - Y_n \leq -\epsilon) \\ &\leq \mathbb{P}(Y_n \leq x) + \mathbb{P}(X_n - Y_n \leq -\epsilon) \\ &\leq \mathbb{P}(Y_n \leq x) + \mathbb{P}(|Y_n - X_n| \geq \epsilon),\end{aligned}$$

we have

$$\liminf_{n\to\infty} \mathbb{P}(Y_n \leq x) \geq \mathbb{P}(X \leq x - \epsilon).$$

Since $\mathbb{R} \setminus \mathcal{C}_{F_X}$ is a countable set, we can only choose a sequence of ϵ_k, such that $x - \epsilon_k \in \mathcal{C}_{F_X}$ and $\epsilon_k \to 0^+$. Then we have $\liminf_{n\to\infty} \mathbb{P}(Y_n \leq x) \geq \mathbb{P}(X \leq x)$.

Therefore,

$$\lim_{n\to\infty} \mathbb{P}(Y_n \leq x) = \mathbb{P}(X \leq x),$$

i.e., $Y_n \xrightarrow{D} X$. $\square$

(b) Show that if $X_n \xrightarrow{\mathbb{P}} X$, then $X_n \xrightarrow{D} X$.

Proof. For $x \in \mathcal{C}_{F_X}$ and $\epsilon > 0$, since

$$\begin{aligned}\mathbb{P}(X_n \leq x) &\leq \mathbb{P}(X \leq x+\epsilon, X_n - X \leq -\epsilon)\\ &\leq \mathbb{P}(X \leq x+\epsilon) + \mathbb{P}(X_n - X \leq -\epsilon)\\ &\leq \mathbb{P}(X \leq x+\epsilon) + \mathbb{P}(|X - X_n| \geq \epsilon),\end{aligned}$$

we have

$$\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x) \geq -\mathbb{P}(x < X \leq x+\epsilon) - \mathbb{P}(|X_n - X| \geq \epsilon).$$

Then

$$\liminf_{n\to\infty}[\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x)] \geq -\mathbb{P}(x < X \leq x+\epsilon).$$

Let $\epsilon \to 0^+$, we have $\liminf\limits_{n\to\infty}[\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x)] \geq 0$.
Also,

$$\begin{aligned}\mathbb{P}(X \leq x-\epsilon) &\leq \mathbb{P}(X_n \leq x, X - X_n \leq -\epsilon)\\ &\leq \mathbb{P}(X_n \leq x) + \mathbb{P}(X - X_n \leq -\epsilon)\\ &\leq \mathbb{P}(X_n \leq x) + \mathbb{P}(|X - X_n| \geq \epsilon),\end{aligned}$$

so

$$\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x) \leq \mathbb{P}(x-\epsilon < X \leq x) + \mathbb{P}(|X_n - X| \geq \epsilon).$$

Then

$$\limsup_{n\to\infty}[\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x)] \leq \mathbb{P}(x-\epsilon < X \leq x).$$

Let $\epsilon \to 0^+$, we have $\limsup\limits_{n\to\infty}[\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x)] \leq 0$.
Therefore,

$$\lim_{n\to\infty}[\mathbb{P}(X \leq x) - \mathbb{P}(X_n \leq x)] = 0,$$

i.e., $\lim\limits_{n\to\infty} F_n(x) = F(x)$ and $X_n \xrightarrow{D} X$.

$\square$

3. (15 pts)

(a) Let $X_1, X_2, \dots$ be independent random variables, having a standard exponential distribution. Let $M_n = \max(X_1, \dots, X_n)$. Show that as $n \to \infty$, $M_n - \log(n) \xrightarrow{D} Y$, where Y is a random variable having a double exponential distribution.

Proof. Let $T_n = M_n - \log n$, then $\forall\, x > 0$,

$$
\begin{aligned}
F_{X_1}(x) &= 1 - e^{-x} \\
F_{M_n}(x) &= [F_{X_1}(x)]^n = [1 - e^{-x}]^n
\end{aligned}
$$

Since $\forall\, x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x + \log n > 0$. So $\forall\, x \in \mathbb{R}$,

$$
\begin{aligned}
F_{T_n}(x) &= F_{M_n}(x + \log n) \\
&= \left(1 - \frac{e^{-x}}{n}\right)^n \\
&\to e^{-e^{-x}}, \qquad n \to \infty
\end{aligned}
$$

which means that $M_n - \log(n) \xrightarrow{D} Y$, where Y is a random variable having a double exponential distribution. $\square$

(b) Let k be a positive integer and for $n \geq k$ let $M_n^{(k)}$ be the k-th largest of $X_1, \dots, X_n$. Find numbers a_n and b_n such that $(M_n^{(k)} - a_n)/b_n$ converges in distribution to a nondegenerate random variable Y_k and give the distribution function of Y_k.

Proof. Let $U_1, \dots, U_n \overset{iid}{\sim} Uniform(0,1)$ and $U_{(1)} < \cdots < U_{(n)}$ be the order statistics.
Since $F(x) = 1 - e^{-x}$ and $F^{-1}(y) = -\ln(1-y)$, we have

$$
\begin{aligned}
M_n^{(k)} &= F^{-1}(U_{(n-k+1)}) \\
&= -\log(1 - U_{(n-k+1)}) \\
&\overset{D}{=\!=} -\log(U_{(k)}) \\
&\overset{D}{=\!=} -\log\left(\frac{Z_1 + \cdots + Z_k}{Z_1 + \cdots + Z_{n+1}}\right) \qquad Z_i \overset{iid}{\sim} Exp(1) \\
&= \log\left(1 + \frac{Z_{k+1} + \cdots + Z_n}{Z_1 + \cdots + Z_k}\right) \\
&\overset{\substack{W_k = Z_1 + \cdots + Z_k \\ D_{n,k} = \frac{Z_{k+1} + \cdots + Z_{n+1} - (n-k+1)}{\sqrt{n-k+1}}}}{=\!=\!=\!=\!=\!=} \log\left(1 + \frac{\sqrt{n-k+1}}{W_k}(D_{n,k} + \sqrt{n-k+1})\right) \\
&= \log\left(1 + \frac{n-k+1}{W_k}\right) + \log\left(1 + \frac{\frac{\sqrt{n-k+1}D_{n,k}}{W_k}}{1 + \frac{n-k+1}{W_k}}\right) \\
&= \log\left(\frac{1}{n-k+1} + \frac{1}{W_k} + \log\left(1 + \frac{\sqrt{n-k+1}D_{n,k}}{W_k + (n-k+1)}\right)\right) + \log(n-k+1)
\end{aligned}
$$

Solution (cont.)

Then

$$M_n^{(k)} - \log(n-k+1) \to \log\left(\frac{1}{W_k}\right) + \log 1 \qquad n \to \infty$$
$$= \log(\frac{1}{W_k})$$

So $a_n = -\log(\frac{1}{W_k})$, $b_n = 1$ and $Y_k = \log(\frac{1}{\mathrm{Gamma}(k,1)})$. $\square$

4. (10 pts) Let $X_{\alpha,\beta}$ be a random variable having a Beta distribution with parameters α and β. Show that for a fixed α and as $\beta \to \infty$, then $\beta X_{\alpha,\beta}$ converges strongly to X distributed Gamma$(\alpha, 1)$.

Proof. For $x \in (0, 1)$,

$$f_{X_{\alpha,\beta}}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Let f be the density of $X \sim \text{Gamma}(\alpha, 1)$. Then for $x > 0$,

$$\begin{aligned}
f_{\beta X_{\alpha,\beta}}(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x}{\beta}\right)^{\alpha-1} \left(1-\frac{x}{\beta}\right)^{\beta-1} \frac{1}{\beta} \\
\lim_{\beta\to\infty} f_{\beta X_{\alpha,\beta}}(x) &= \lim_{\beta\to\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x}{\beta}\right)^{\alpha-1} \left(1-\frac{x}{\beta}\right)^{\beta-1} \frac{1}{\beta} \\
&= \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \lim_{\beta\to\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\beta^{\alpha}} \left(1-\frac{x}{\beta}\right)^{\beta-1} \\
&= \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \lim_{\beta\to\infty} \left(1-\frac{x}{\beta}\right)^{\beta-1} \\
&= \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}.
\end{aligned}$$

For $x \leq 0$, it holds naturally that $\lim_{\beta\to\infty} f_{\beta X_{\alpha,\beta}}(x) = 0 = f(0)$. Therefore, f_β, the density of $\beta X_{\alpha,\beta}$ converges to f, the density of Gamma$(\alpha, 1)$.

□