STAT 150: Stochastic Processes

Fall 2017

Homework 4

Solutions by

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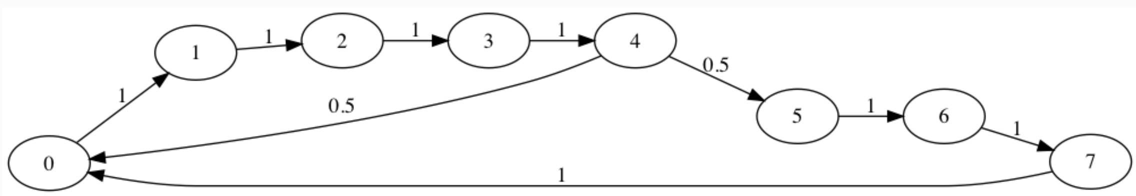
PK Exercises 4.3.1

A Markov chain has a transition probability matrix

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Find the equivalence classes. For which integers $n = 1, 2, \cdots, 20$, is it true that $P_{00}^{(n)} > 0$? What is the period of the Markov chain?

Hint: One need not compute the actual probabilities. See Section 4.3.1.



$\because \quad \forall\ i, j \in S = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $\exists$ a path from i to j

$\therefore \quad i \to j\ \forall\ i, , j \in S$

$\therefore \quad i \leftrightarrow j$, i.e. the Markov chain is irreducible, i.e. there is only an equivalence class $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

The chain can only return 0 after entering state 4 or 7. Begining at state 0, the chain returns to 0 moves at least 5 steps from state 4 and 8 steps from state 7.

Thus

$$\{n \geqslant 1 : P_{00}^{(n)} > 0\} = \{5, 8, 10, 13, 15, 16, 18, 20, \cdots, \} = \{5m + 8n : \forall\ m, n \in \mathbb{N}^{+}, mn \neq 0\}$$

$\because \quad$ the period of the state 0 is

$$d(0) = gcd(5, 8) = 1$$

and the states of the Markov chain are intercommunicative.

$\therefore \quad$ the period of the Markov chain is 1

PK Problem 4.3.1

A two-state Markov chain has the transition probability matrix

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}
\begin{array}{c}
\begin{array}{cc} 0 & 1 \end{array} \\
\begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}
\end{array}
$$

(a) Determine the first return distribution

$$f_{00}^{(n)} = Pr\{X_1 \neq 0, \cdots, X_{n-1} \neq 0, X_n = 0 | X_0 = 0\}$$

$$\begin{aligned}
f_{00}^{(0)} &= 0 \\
f_{00}^{(1)} &= Pr\{X_1 = 0 | X_0 = 0\} \\
&= 1 - a \\
f_{00}^{(n)} &= Pr\{X_n = 0, X_{n-1} \neq 0, \cdots, X_2 \neq 0, X_1 \neq 0 | X_0 = 0\} \\
&= Pr\{X_n = 0, X_{n-1} = 1, \cdots, X_2 = 1, X_1 = 1 | X_0 = 0\} \\
&= Pr\{X_n = 0 | X_{n-1} = 1\} Pr\{X_{n-1} = 1 | X_{n-2} = 1\} \cdots Pr\{X_1 = 1 | X_0 = 0\} \\
&= b(1-b)^{n-2} a \\
&= ab(1-b)^{n-2} \qquad\qquad (n \geqslant 2)
\end{aligned}$$

(b) Verify equation (4.16) when $i = 0$,

$$P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}, \qquad n \geqslant 1.$$

$$\begin{aligned}
P &= I + \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} \\
&= I + A \\
A^2 &= \begin{pmatrix} a(a+b) & -a(a+b) \\ -b(a+b) & b(a+b) \end{pmatrix} \\
A^3 &= \begin{pmatrix} -a(a+b)^2 & a(a+b)^2 \\ b(a+b)^2 & -b(a+b)^2 \end{pmatrix} \\
&\vdots \\
A^n &= \begin{pmatrix} a(-1)^n(a+b)^{n-1} & -a(-1)^n(a+b)^{n-1} \\ -b(-1)^n(a+b)^{n-1} & b(-1)^n(a+b)^{n-1} \end{pmatrix} \\
P^n &= (I + A)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} A^i \\
&= \begin{pmatrix} \frac{a}{a+b} \sum_{i=0}^{n} \binom{n}{i} (-a-b)^n + \frac{b}{a+b} & -\frac{a}{a+b} \sum_{i=0}^{n} \binom{n}{i} (-a-b)^n - \frac{b}{a+b} \\ -\frac{b}{a+b} \sum_{i=0}^{n} \binom{n}{i} (-a-b)^n - \frac{a}{a+b} & \frac{b}{a+b} \sum_{i=0}^{n} \binom{n}{i} (-a-b)^n + \frac{a}{a+b} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{a+b}(1-a-b)^n + \frac{b}{a+b} & -\frac{a}{a+b}(1-a-b)^n - \frac{b}{a+b} \\ -\frac{b}{a+b}(1-a-b)^n - \frac{a}{a+b} & \frac{b}{a+b}(1-a-b)^n + \frac{a}{a+b} \end{pmatrix}
\end{aligned}$$

Solution (cont.)

When $n = 1$,

$$\begin{aligned}
\sum_{k=0}^{1} f_{00}^{(k)} P_{00}^{(1-k)} &= 1 - a \\
&= \frac{a}{a+b}(1-a-b) + \frac{b}{a+b} \\
&= P_{00}^{(1)}
\end{aligned}$$

When $n \geqslant 2$,

$$\begin{aligned}
\sum_{k=0}^{n} f_{00}^{(k)} P_{00}^{(n-k)} &= (1-a)\left[\frac{a}{a+b}(1-a-b)^{n-1} + \frac{b}{a+b}\right] \\
&\quad + \sum_{k=2}^{n} ab(1-b)^{k-2}\left[\frac{a}{a+b}(1-a-b)^{n-k} + \frac{b}{a+b}\right] \\
&= \frac{a(1-a)(1-a-b)^{n-1} + (1-a)b}{a+b} + \frac{ab^2}{a+b}\sum_{k=2}^{n}(1-b)^{k-2} \\
&\quad - \frac{a^2b(1-a-b)^{n-2}}{a+b}\sum_{k=2}^{n}\left(\frac{1-b}{1-a-b}\right)^{k-2} \\
&= +\frac{a^2b}{a+b}\sum_{k=2}^{n}(1-b)^{k-2} \\
&= \frac{a(1-a)(1-a-b)^{n-1} + (1-a)b}{a+b} + \frac{ab}{a+b}[1-(1-b)^{n-1}] \\
&\quad + \frac{ab}{a+b}[(1-a-b)^{n-1} + (1-b)^{n-1}] \\
&= \frac{a}{a+b}(1-a-b)^n + \frac{b}{a+b} \\
&= P_{00}^{(n)}
\end{aligned}$$

PK Problem 4.3.3

Recall the first return distribution(Section 4.3.3),

$$f_{ii}^{(n)} = Pr\{X_1 \neq i, X_2 \neq i, \cdots, X_{n-1} \neq i, X_n = i | X_0 = i\} \qquad \text{for } n = 1, 2, \cdots$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (4.16), determine $f_{00}^{(n)}$, $n = 1, 2, 3, 4, 5$, for the Markov chain whose transition probability matrix is

$$\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}
\end{array}.$$

$$P^2 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$$

$$P^4 = \begin{pmatrix} \frac{3}{8} & \frac{1}{16} & \frac{1}{8} & \frac{7}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{8} & \frac{9}{16} \end{pmatrix}$$

$$P^5 = \begin{pmatrix} \frac{7}{32} & \frac{3}{16} & \frac{1}{16} & \frac{7}{32} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{4} & \frac{7}{16} \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{8} & \frac{9}{16} \end{pmatrix}$$

$$P_{00}^{(0)} = 1$$

$$f_{00}^{(0)} = 0$$

$\because$

$$\begin{aligned} P_{00}^{(1)} &= \sum_{k=0}^{1} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(1)} \end{aligned}$$

$\therefore$

$$f_{00}^{(1)} = 0$$

$\because$

$$\begin{aligned} P_{00}^{(2)} &= \sum_{k=0}^{2} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(1)} P_{00}^{(1)} + f_{00}^{(2)} P_{00}^{(0)} \\ &= f_{00}^{(2)} \end{aligned}$$

$\therefore$

$$f_{00}^{(2)} = \frac{1}{4}$$

$\because$

$$\begin{aligned} P_{00}^{(3)} &= \sum_{k=0}^{3} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(1)} P_{00}^{(2)} + f_{00}^{(2)} P_{00}^{(1)} + f_{00}^{(3)} P_{00}^{(1)} \\ &= f_{00}^{(3)} \end{aligned}$$

Solution (cont.)

$\therefore$

$$f_{00}^{(3)} = \frac{1}{8}$$

$\because$

$$\begin{aligned} P_{00}^{(4)} &= \sum_{k=0}^{4} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(2)} P_{00}^{(2)} + f_{00}^{(3)} P_{00}^{(1)} + f_{00}^{(4)} P_{00}^{(0)} \\ &= \frac{1}{16} + f_{00}^{(4)} \end{aligned}$$

$\therefore$

$$f_{00}^{(4)} = \frac{5}{16}$$

$\because$

$$\begin{aligned} P_{00}^{(5)} &= \sum_{k=0}^{5} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(2)} P_{00}^{(3)} + f_{00}^{(3)} P_{00}^{(2)} + f_{00}^{(4)} P_{00}^{(1)} + f_{00}^{(5)} P_{00}^{(0)} \\ &= \frac{1}{32} + \frac{1}{32} + f_{00}^{(5)} \end{aligned}$$

$\therefore$

$$f_{00}^{(5)} = \frac{5}{32}$$

GS Section 6.4, Page 236, Q4

Show by example that chains which are not irreducible may have different stationary distributions.

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

given the initial distribution $\begin{cases} \mathbb{P}(X_0 = 0) = p \\ \mathbb{P}(X_0 = 1) = 1 - p \end{cases}$, where $0 \leqslant p \leqslant 1$.

Then

$$\pi = (p, 1-p) \qquad \forall\, p \in [0, 1]$$

since

$$p, 1-p \geqslant 0$$

Solution (cont.)

and

$$\pi P = (p, 1-p) = \pi.$$

It is not unique.

GS Page 236, Q7

Show that a random walk on the infinite binary tree is transient.

Let $S = \mathbb{N}$, $\{X_n\}$ be a Markov chain where $X_n = i$ denotes that the chain are at depth i at time n and $X_0 = 0$.

We have $p_{00} = \frac{1}{3}$, $p_{i,i+1} = \frac{2}{3} \quad (\forall\ i \geqslant 0)$ and $p_{i,i-1} = \frac{1}{3} \quad (\forall\ i \geqslant 1)$, then

$$P_{00}^{(0)} = 1$$

$\forall\ n \in \mathbb{N}^+$,

$$\begin{aligned} P_{00}^{(2n-1)} &= 0 \\ P_{00}^{(2n)} &= Pr(\text{move down for } n \text{ times and move up for } n \text{ times}) \\ &= \binom{2n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n \\ &\approx \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n}2\pi n}\frac{2^n}{3^{2n}} \\ &= \left(\frac{8}{9}\right)^n \frac{1}{\sqrt{\pi n}} \end{aligned}$$

$\because$

$$\begin{aligned} \sum_{n=0}^{\infty} P_{00}^{n} &= 1 + \sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n \frac{1}{\sqrt{\pi n}} \\ &\leqslant \sum_{n=0}^{\infty}\left(\frac{8}{9}\right)^n \\ &= 9 < \infty \end{aligned}$$

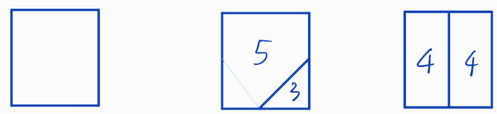
$\therefore \quad$ a random walk on the infinite binary tree is transient

GS Page 236, Q9

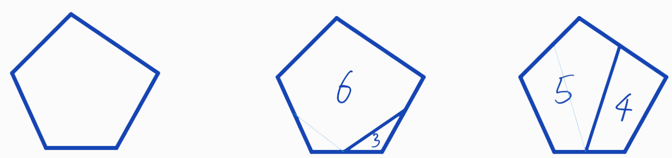
A random sequence of convex polygons is generated by picking two edges of the current polygon at random, joining their midpoints, and picking one of the two resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number of edges of the nth polygon thus constructed. Find $\mathbb{E}(X_n)$ in terms of X_0, and find the stationary distribution of the Markov chain X.

$\because$

i	0	1
$\mathbb{P}(X_{n+1}=i \mid X_n=0)$	$\frac{1}{2}$	$\frac{1}{2}$



i	0	1	2
$\mathbb{P}(X_{n+1}=i \mid X_n=1)$	$\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$



i	0	1	2	3
$\mathbb{P}(X_{n+1}=i \mid X_n=2)$	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

$$\vdots$$

$$\mathbb{P}(X_{n+1}=i \mid X_n=2k) = \frac{2}{2k+2}\times\frac{1}{2} = \frac{1}{2k+2}$$

$$\mathbb{P}(X_{n+1}=i \mid X_n=2k+1) = \begin{cases} \frac{2}{(2k+1)+2}\times\frac{1}{2} = \frac{1}{(2k+1)+2}, & i\neq k \\ \frac{1}{(2k+1)+2}, & i=k \end{cases}$$

$\therefore$ given $X_n = k (k \geqslant 0)$, $X_{n+1} \in A_k = \{0, 1, \cdots, k+1\}$ and $\forall\, i \in A_k$,

$$\mathbb{P}(X_{n+1} = i | X_n = k) = \frac{1}{k+2}$$

Solution (cont.)

$\therefore$ given X_0,

$$\begin{aligned}
\mathbb{E}X_n &= \mathbb{E}[\mathbb{E}(X_n|X_{n-1})] \\
&= \mathbb{E}\left(\sum_{i=0}^{X_{n-1}+1} \frac{i}{X_{n-1}+2}\right) \\
&= \frac{1}{2}\mathbb{E}(X_{n-1}) + \frac{1}{2} \\
&= \cdots \\
&= \sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^n}X_0 \\
&= 1 - \frac{1}{2^n} + \frac{1}{2^n}X_0
\end{aligned}$$

$\because$

$$\mathbf{P} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots
\end{pmatrix}$$

and

$$\pi = \pi P$$

suppose that $\pi = \begin{pmatrix}\pi_0 & \pi_1 & \cdots\end{pmatrix}$

$\therefore$

$$\begin{cases}
\pi_0 = \sum\limits_{i=0}^{\infty} \dfrac{1}{i+2}\pi_i \\
\pi_k = \sum\limits_{i=k-1}^{\infty} \dfrac{1}{i+2}\pi_i \quad , k \geqslant 1 \\
\sum\limits_{i=0}^{\infty} \pi_i = 1
\end{cases}$$

by subtracting π_{k-1} from π_k we get

$$\begin{cases}
\pi_k = \dfrac{1}{k!}\pi_0 \quad , k \in \mathbb{N}^+ \\
\sum\limits_{i=0}^{\infty} \pi_i = 1
\end{cases}$$

$\therefore$

$$\begin{cases}
\pi_0 = e^{-1} \\
\pi_k = \dfrac{1}{k!}e^{-1}
\end{cases}$$

i.e.

$$\pi \sim Poisson(1)$$