Problem Set 01 Solutions 

1. $\langle u,v\rangle = u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v \qquad u,v\in\mathbb{C}^2.$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u^* = [\bar{u}_1, \bar{u}_2].$

Note that our convention puts complex conjugation on second factor v, so ~~one~~ technically correct answer is: "this is not an inner product since complex conjugation on u destroys linearity."

Another technically correct answer is to require inner products that are complex-conjugated in u and linear in v. (This is actually more common in mathematics.)

Then we have a proof:

$\langle u, v \rangle = u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v$ is inner product:

(1) linear in $u$ ✓

(2) conjugate-symmetric in $(u, v)$

(3) positive:

$$\langle u, u \rangle = u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} u$$

$$= [\bar{u}_1 \ \bar{u}_2] \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= [\bar{u}_1 \ \bar{u}_2] \begin{bmatrix} 8u_1 - u_2 \\ -u_1 + 8u_2 \end{bmatrix}$$

$$= 8|u_1|^2 - \bar{u}_1 u_2 - u_1 \bar{u}_2 + 8|u_2|^2$$

$$\geq 7(|u_1|^2 + |u_2|^2) \geq 0$$

and $= 0$ iff $u = 0$, because

$|\bar{u}_1 u_2| \leq |u_1||u_2| \leq \frac{1}{2}|u_1|^2 + \frac{1}{2}|u_2|^2$.

2. linear, conjugate-symmetric obvious for all. Check positivity:

(a) $\langle p,p\rangle = \sum_{j=0}^{n} |p_j|^2$ ✓

(b) $\langle p,p\rangle = \int_0^{\pi} |p(x)|^2 dx = 0$

Can happen only if $p(x) \equiv 0$ on $0 \le x \le \pi$, by FTA means $p=0$.

(c) $\int_{-\infty}^{\infty} |p(x)|^2 dx = \infty$ for any nonzero polynomial. Not inner product!

(d) $\langle p,p\rangle = \int_{-\infty}^{\infty} e^{-|x|} |p(x)|^2 dx < \infty$

b/c exponential wins over polynomial.

$\langle p,p\rangle = 0 \Rightarrow p(x) \equiv 0 \Rightarrow p=0$.

| | $\langle 1,x\rangle$ | $\langle 1,x^2\rangle$ |
|---|---|---|
| a | 0 | 0 |
| b | $\pi^2/2$ | $\pi^3/3$ |
| c | "$\infty$" | "$\infty$" |
| d | 0 | 4 |

3. (a) Finite because of Cauchy-Schwarz:

$$\left| \sum_{n=0}^{N} u_n \bar{v}_n \right| \le \sqrt{\sum_{n=0}^{N} |u_n|^2} \sqrt{\sum_{n=0}^{N} |v_n|^2} < \infty$$

as $N \to \infty$ b/c $u, v \in \ell^2(\mathbb{N})$. Clearly linear & conjugate symmetric & positive.

(b) $$\langle u, v \rangle = \sum_{n=0}^{\infty} 2^{-n} 3^{-n} = \sum_{n=0}^{\infty} 6^{-n}$$

$$= \frac{1}{1 - 1/6} = \frac{6}{5}$$ by geom series.

$$\langle u, u \rangle = \frac{1}{1 - 1/4} = \frac{4}{3} \qquad \|u\| = \frac{2}{\sqrt{3}}$$

$$\langle v, v \rangle = \qquad = 9/8 \qquad \|v\| = \frac{3}{2\sqrt{2}}$$

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \frac{6/5}{2/\sqrt{3} \;\; 3/2\sqrt{2}} = \frac{6}{5} \cdot \frac{\sqrt{2}}{\sqrt{3}}$$

$$= 0.979796\ldots$$

$$\theta = 0.201358\ldots \text{ radians}$$

$$= 11.53696 \text{ degrees}$$

4. (a) $\|x \pm y\|^2 = \langle x \pm y, x \pm y \rangle$

$= \langle x, x \rangle \pm \langle y, x \rangle \pm \langle x, y \rangle + \langle y, y \rangle$

so

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \left\langle \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle$

$+ \left\langle \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle - 2 \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$

$= 2 - 2 \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$

$\therefore \langle x, y \rangle = \|x\| \, \|y\| \left( 1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right)$

giving another formula for

$$\cos\theta = 1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2$$

in terms of the distance between $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$.

(c) Fix $x \in V \setminus A$. Since $x \notin A$, we know that

$$\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$$

for all $y \in A$. (Otherwise $\|y\| \frac{x}{\|x\|} \in A$ and since $A$ is a subspace $x \in A$.)

Hence

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| > 0$$

for all $y \in A$, so by (b) we have

$$\cos \vartheta = 1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 < 1$$

By the triangle inequality, we also have

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \left\| \frac{x}{\|x\|} \right\| + \left\| \frac{y}{\|y\|} \right\| = 2,$$

with equality only when $\frac{x}{\|x\|}$ and $\frac{-y}{\|y\|}$ are parallel (hence equal). Since $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$ we have

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| < 2$$

for all $y \in A$. Putting it together,

$$-1 < \cos\theta < 1$$

so there is a constant $\gamma < 1$ such that

$$|\cos\theta| \leq \gamma < 1$$

for all $y \in A$. Since

$$\langle x, y \rangle = \cos\theta \, \|x\| \, \|y\|,$$

we have a strengthened Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \gamma \, \|x\| \, \|y\|.$$

5. (a) Clearly linear and conjugate-symmetric. For positivity,

$\langle f, f \rangle = 0 \Rightarrow f(x) = f'(x) = 0$ almost everywhere. By Sobolev inequality $f$ is continuous so $f = 0$.

(b) $f^{(q)}(x) = (ia)^q f(x)$ so

$$\|f\|_p = \sqrt{\int_{-\pi}^{\pi} \left(1 + |ia|^2 + \dots + |ia|^{2p}\right) |f(x)|^2 dx}$$

$$= \sqrt{2\pi \left(1 + a^2 + a^4 + \dots + a^{2p}\right)}$$

$$\|f\|_p = \begin{cases} = \sqrt{2\pi} \sqrt{\dfrac{1 - a^{2p+2}}{1 - a^2}} & \text{for } |a| \neq 1. \\ = \sqrt{2\pi \cdot p} & \text{for } |a| = 1 \end{cases}$$

$$\langle f, g\rangle = \int_{-\pi}^{\pi} \left(1 + ia\cdot\overline{ib} + \dots + (ia)^p\overline{(ib)^p}\right)\cdot e^{i(a-b)x}\,dx$$

$$= \begin{cases} 0 & a \neq b \\ 2\pi\left(1+|a|^2+\dots+|a|^{2p}\right) & a = b \end{cases}$$

since $\int_{-\pi}^{\pi} e^{iax}\,dx = \left.\frac{e^{iax}}{ia}\right|_{-\pi}^{\pi} = 0$

for $a \in \mathbb{Z}$, $a \neq 0$.

Hence $\cos\theta = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases}$

and

$$\theta = \begin{cases} \pi/2 & a \neq b \\ 0 & a = b. \end{cases}$$

(c) $f(x) = 1$ for $x > 0$ so

$$\langle f^{(q)}, g\rangle = (-1)^q \langle f, g^{(q)}\rangle$$

$$= (-1)^q \int_{-\pi}^{\pi} f(x)\overline{g^{(q)}}(x)\,dx$$

$$= (-1)^q \int_{0}^{\pi} \overline{g^{(q)}}(x)\,dx$$

$$\langle f^{(q)}, g\rangle = (-1)^q \left[g^{(q-1)}(\pi) - g^{(q-1)}(0)\right]$$

(d) $$\langle f^{(q)}, g\rangle = (-1)^q \left[g^{(q-1)}(b) - g^{(q-1)}(a)\right]$$

(e) $$\langle f^{(q)}, g\rangle = (-1)^q \int_a^b Q(x)\, g^{(q)}(x)\, dx$$

$$= (-1)^q \left[ Q g^{(q-1)}\Big|_a^b - Q' g^{(q-2)}\Big|_a^b + Q'' g^{(q-3)} + \Big|_a^b \cdots \right.$$

$$\left. + (-1)^{q-1} Q^{(q-1)} g^{(1)}\Big|_a^b + (-1)^q \int_a^b Q^{(q)} g\, dx \right]$$

Of course if $q > n$ the last few terms vanish.

**Question 1** Prove or disprove:

$$< u, v >= u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v$$

is an inner product on $C^2$. Check the properties.

**Question 2** Which of the following define an inner product on degree-$n$ polynomials

$$p(x) = p_0 + p_1 x + \cdots + p_n x^n?$$

(a)

$$< p, q >= \sum_{j=0}^{n} p_j \bar{q}_j$$

(b)

$$< p, q >= \int_0^{\pi} p(x)\bar{q}(x) \, \mathrm{d}x$$

(c)

$$< p, q >= \int_{-\infty}^{\infty} p(x)\bar{q}(x) \, \mathrm{d}x$$

(d)

$$< p, q >= \int_{-\infty}^{\infty} p(x)\bar{q}(x)\mathrm{e}^{-|x|} \, \mathrm{d}x$$

Justify your answers with proof or counterexample. Evaluate $< 1, x >$ and $< 1, x^2 >$ for each case.

**Question 3** (a) Prove or disprove:

$$< u, v >= \sum_{n=0}^{\infty} u_n \bar{v}_n$$

is an inner product on $l^2(N)$. Check the properties.

(b) For $u_n = 2^{-n}$ and $v_n = 3^{-n}$ compute $< u, v >$ and the angle between $u$ and $v$.

**Question 4** (a) Prove the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for any $x$ and $y$ vectors in a *real* inner product space with norm $\| \|$.

(b) Prove

$$< x, y >= \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right)$$

for nonzero vectors $x$ and $y$.

(c) Given a subspace $A$ of a real inner-product space $V$, and a vector $x \in V$ which is not in $A$, show that there is a constant $\gamma < 1$ such that

$$| < a, x > | \leq \gamma \|a\|\|x\|$$

for all $a \in A$.

**Question 5** For $p = 1, 2, \ldots$ define the Sobolev space $H^p = H^p(-\pi, \pi)$ by

$$H^p = \{g \in L^2 = L^2(-\pi,\pi) | g \text{ is } 2\pi\text{-periodic and } g', g'', \ldots, g^{(p)} \in L^2\},$$

with

$$< f, g >_p = \int_{-\pi}^{\pi} f(x)\bar{g}(x) + f'(x)\bar{g}'(x) + \cdots + f^{(p)}(x)\bar{g}^{(p)}(x)\, \mathrm{d}x.$$

For $p = 0$ we set $H^0 = L^2$ with the usual $L^2$ inner product $<,>$.

(a) Show that $< f, g >_p$ defines an inner product on $H^p$.

(b) Compute the norm $\|f\|_p = \sqrt{< f, f >_p}$ in $H^p$ of $f(x) = \mathrm{e}^{iax}$ and the angle in $H^p$ between $f$ and $g(x) = \mathrm{e}^{ibx}$ for $a, b \in Z$.

(c) For $f \in L^2$ define a generalized derivative $f^{(p)}$ by the requirement

$$< f^{(p)}, g >= (-1)^p < f, g^{(p)} >$$

for all $g \in H^p$. Let $f \in L^2$ be given by $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x < 0$. For $1 \leq q \leq p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g \in H^p$.

(d) Fix $-\pi < a < b < \pi$ and let $f \in L^2$ be given by $f(x) = 1$ for $a < x < b$ and $f(x) = 0$ otherwise. For $1 \leq q \leq p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g \in H^p$.

(e) Compute the generalized derivatives of $f(x) = Q(x)$ for $a < x < b$ and $f(x) = 0$ otherwise, where $Q$ is a degree-$n$ polynomial.