TTIC 31250
An Introduction to the Theory of Machine Learning

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Lecture 2: Online learning

Mistake-bound model:
- Basic results, relation to PAC, halving algorithm
- Connections to information theory

Combining "expert advice":
- (Randomized) Weighted Majority algorithm
- Regret-bounds, connections to game theory

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Recap from last time

- Last time: PAC model and Occam's razor.
  - If data set has $m \geq (1/\varepsilon)[s \ln(2) + \ln(1/\delta)]$ examples, then whp any consistent hypothesis with size(h) < s has err(h) < ε.
  - Equivalently, size(h) $\leq (\varepsilon m - \ln(1/\delta))/\ln(2)$ suffices.
  - "compression $\Rightarrow$ learning"
- Occam bounds $\Rightarrow$ any class is learnable in PAC model if computation time is no object.

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Online learning

- What if we don't want to make assumption that data is coming from some fixed distribution? Or any assumptions at all?
- Can no longer talk about past performance predicting future results.
- Can we hope to say anything interesting??

Idea: mistake bounds & regret bounds.

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Mistake-bound model

- View learning as a sequence of stages.
- In each stage, algorithm is given *x*, asked to predict *f(x)*, and then is told correct value.
- Make no assumptions about order of examples.
- Goal is to bound total number of mistakes.

Alg A learns class C with mistake bound M if A makes $\leq$ M mistakes on any sequence of examples consistent with some f $\in$ C.

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Mistake-bound model

Alg A learns class C with mistake bound M if A makes $\leq$ M mistakes on any sequence of examples consistent with some f $\in$C.

- Note: can no longer talk about "how much data do I need to converge?" Maybe see same examples over again and learn nothing new. But that's OK if don't make mistakes either…
- Try to bound in terms of size of examples n and complexity of target s.
- C is **learnable** in MB model if exists alg with mistake bound and running time per stage poly(n,s).

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Simple example: disjunctions

- Suppose features are boolean: $X = \{0,1\}^n$.
- Target is an OR function, like $x_3 \vee x_9 \vee x_{12}$.
- Can we find an on-line strategy that makes at most n mistakes?
- Sure.
  - Start with $h(x) = x_1 \vee x_2 \vee \ldots \vee x_n$
  - Invariant: {features in h} $\supseteq$ {features in f}
  - Mistake on negative: discard features in h set to 1 in x. Maintains invariant & decreases |h| by 1.
  - No mistakes on positives. So at most n mistakes total.

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Simple example: disjunctions

- Algorithm makes at most n mistakes.
- No deterministic alg can do better:

  1 0 0 0 0 0 0 + or – ?
  0 1 0 0 0 0 0 + or – ?
  0 0 1 0 0 0 0 + or – ?
  0 0 0 1 0 0 0 + or – ?
  ...

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MB model properties

An alg A is "conservative" if it only changes its state when it makes a mistake.

Claim: if C is learnable with mistake-bound M, then it is learnable by a conservative alg.

Why?

- Take generic alg A. Create new conservative A' by running A, but rewinding state if no mistake is made.
- Still $\leq$ M mistakes because A still sees a legal sequence of examples.

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MB learnable ⇒ PAC learnable

Say alg A learns C with mistake-bound M.

Transformation 1:

- Run (conservative) A until it produces a hyp h that survives $\geq (1/\varepsilon)\ln(M/\delta)$ examples.
- Pr(fooled by any given h) $\leq \delta/M$.
- Pr(fooled ever) $\leq \delta$.

  Uses at most $(M/\varepsilon)\ln(M/\delta)$ examples total.

- Fancier method gets $O(\varepsilon^{-1}[M + \ln(1/\delta)])$

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One more example...

- Say we view each example as an integer between 0 and 2^n-1.
- C = {[0,a] : a < 2^n}. (device fails if it gets too hot)
- In PAC model we could just pick any consistent hypothesis. Does this work in MB model?
- What would work?

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What can we do with unbounded computation time?

- "Halving algorithm": take majority vote over all consistent h ∈ C. Makes at most lg(|C|) mistakes.
- What if we had a "prior" p over fns in C? $\frac{1}{2^k} \geq p_f$
  - Weight the vote according to p. Make at most $\lg(1/p_f)$ mistakes, where f is target fn.
- What if f was really chosen according to p?
  - Expected number of mistakes $\leq \Sigma_h[p_h \lg(1/p_h)]$ = entropy of distribution p.

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What can we do with unbounded computation time?

- "Halving algorithm": take majority vote over all consistent h ∈ C. Makes at most lg(|C|) mistakes.
- What if C has functions of different sizes?
- For any (prefix-free) representation, can make at most 1 mistake per bit of target.
  - Think of writing random 0s and 1s until hit a legal hypothesis or no longer a prefix of one.
  - $p_f = \Pr(reach\ f) = 1/2^{size(f)}$
  - $\lg(1/p_f) = size(f)$.

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Is halving alg optimal?

- Not necessarily
- Can think of MB model as 2-player game between alg and adversary.
  - Adversary picks x to split C into $C_-(x)$ and $C_+(x)$. [fns that label x as – or + respectively]
  - Alg gets to pick one to throw out.
  - Game ends when all fns left are equivalent.
  - Adversary wants to make game last as long as possible.
- OPT(C) = MB when both play optimally.

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Is halving alg optimal?

- Halving algorithm: throw out larger set.
- Optimal algorithm: throw out set with larger mistake bound.

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What if there is no perfect function?

Think of as $h \in C$ as "experts" giving advice to you. Want to do nearly as well as best of them in hindsight.

These are called "regret bounds".
➢Show that our algorithm does nearly as well as best predictor in some class.

We'll look at a strategy whose running time is $O(|C|)$. So, only computationally efficient when C is small.

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Using "expert" advice

Say we want to predict the stock market.

- We solicit n "experts" for their advice. (Will the market go up or down?)
- We then want to use their advice somehow to make our prediction. E.g.,

Expt 1	Expt 2	Expt 3	neighbor's dog	truth
down	up	up	up	up
down	up	up	down	down
…	…	…	…	…

Can we do nearly as well as best in hindsight?

["expert" = someone with an opinion. Not necessarily someone who knows anything.]
[note: would be trivial in PAC (i.i.d.) setting]

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Using "expert" advice

If one expert is perfect, can get $\leq \lg(n)$ mistakes with halving alg.

But what if none is perfect? Can we do nearly as well as the best one in hindsight?

Strategy #1:

- Iterated halving algorithm. Same as before, but once we've crossed off all the experts, restart from the beginning.
- Makes at most $\lg(n)[OPT+1]$ mistakes, where OPT is #mistakes of the best expert in hindsight.

Seems wasteful. Constantly forgetting what we've "learned". Can we do better?

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Weighted Majority Algorithm

Intuition: Making a mistake doesn't completely disqualify an expert. So, instead of crossing off, just lower its weight.

Weighted Majority Alg:

- Start with all experts having weight 1.
- Predict based on weighted majority vote.
- Penalize mistakes by cutting weight in half.

Weights:	1	1	1	1		
Predictions:	U	U	U	D	We predict: U	Truth: D
Weights:	½	½	½	1		

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Analysis: do nearly as well as best expert in hindsight

- M = # mistakes we've made so far.
- m = # mistakes best expert has made so far.
- W = total weight (starts at n).
- After each mistake, W drops by at least 25%. So, after M mistakes, W is at most $n(3/4)^M$.
- Weight of best expert is $(1/2)^m$. So,

$$(1/2)^m \leq n(3/4)^M$$
$$(4/3)^M \leq n2^m$$
$$M \leq 2.4(m + \lg n)$$

constant ratio

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Randomized Weighted Majority

2.4(m + lg n) not so good if the best expert makes a mistake 20% of the time. Can we do better? Yes.

- Instead of taking majority vote, use weights as probabilities. (e.g., if 70% on up, 30% on down, then pick 70:30) Idea: smooth out the worst case.
- Also, generalize ½ to 1- ε.

Solves to: $M \leq \frac{-m\ln(1-\varepsilon)+\ln(n)}{\varepsilon} \approx (1+\varepsilon/2)m + \frac{1}{\varepsilon}\ln(n)$

M = expected #mistakes

$M \leq 1.39m + 2\ln n \quad \leftarrow \varepsilon = 1/2$

$M \leq 1.15m + 4\ln n \quad \leftarrow \varepsilon = 1/4$

$M \leq 1.07m + 8\ln n \quad \leftarrow \varepsilon = 1/8$

unlike most worst-case bounds, numbers are pretty good.

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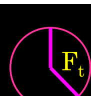
Analysis



- Say at time t we have fraction F_t of weight on experts that made mistake.
- So, we have probability F_t of making a mistake, and we remove an εF_t fraction of the total weight.
  - $W_{final} = n(1-\varepsilon F_1)(1 - \varepsilon F_2)\ldots$
  - $\ln(W_{final}) = \ln(n) + \sum_t [\ln(1 - \varepsilon F_t)] \leq \ln(n) - \varepsilon \sum_t F_t$ (using $\ln(1-x) < -x$)

  $= \ln(n) - \varepsilon M.$ ($\sum F_t$ = E[# mistakes])
- If best expert makes m mistakes, then $\ln(W_{final}) > \ln((1-\varepsilon)^m)$.
- Now solve: $\ln(n) - \varepsilon M > m \ln(1-\varepsilon)$.

$$M \leq \frac{-m\ln(1-\varepsilon)+\ln(n)}{\varepsilon} \approx (1+\varepsilon/2)m + \frac{1}{\varepsilon}\log(n)$$

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Summarizing

- E[# mistakes] $\leq (1+\varepsilon)$OPT + $\varepsilon^{-1}\log(n)$ = OPT + (εOPT + $\varepsilon^{-1}\log(n)$)

Assuming here that $OPT \geq \log(n)$

- If set $\varepsilon=(\log(n)/OPT)^{1/2}$ to balance the two terms out (or use guess-and-double), get bound of $M \leq OPT+2(OPT \cdot \log n)^{1/2} \leq OPT+2(T \log n)^{1/2}$
- Define average regret in T time steps as: (avg per-day cost of alg) – (avg per-day cost of best fixed expert in hindsight). Goes to 0 or better as T→ ∞ = "no-regret" algorithm].

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Extensions

- What if experts are actions? (rows in a matrix game, ways to drive to work,…)
- At each time t, each has a loss (cost) in {0,1}.
- Can still run the algorithm
  - Rather than viewing as "pick a prediction with prob proportional to its weight" ,
  - View as "pick an expert with probability proportional to its weight"
  - Alg pays expected cost $\vec{p_t} \cdot \vec{c_t} = F_t$.
- Same analysis applies.

Do nearly as well as best action in hindsight!

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Extensions

- What if losses (costs) in [0,1]?
- Just modify alg update rule: $w_i \leftarrow w_i(1 - \epsilon c_i)$.
- Fraction of wt removed from system is: $(\sum_i w_i \epsilon c_i)/(\sum_j w_j) = \epsilon \sum_i p_i c_i = \epsilon[our\ expected\ cost]$
- Analysis very similar to case of {0,1}.

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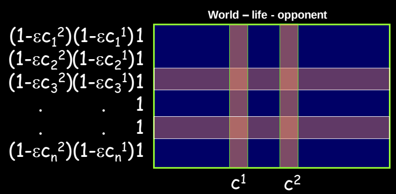
RWM (multiplicative weights alg)



scaling so costs in [0,1]

Guarantee: do nearly as well as fixed row in hindsight

Which implies doing nearly as well (or better) than minimax optimal

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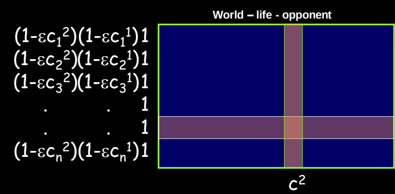
Connections to minimax optimality



scaling so costs in [0,1]

If play RWM against a best-response oracle, $\vec{p}$ will approach minimax optimality.

(If if didn't, wouldn't be getting promised guarantee)

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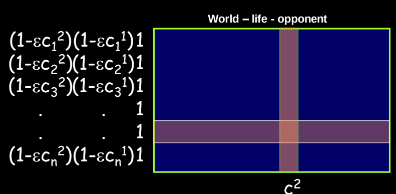
Connections to minimax optimality



scaling so costs in [0,1]

If play two RWM against each other, then empirical distributions must be near-minimax-optimal.

(Else, one or the other could & would take advantage)

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