STAT 150: Stochastic Processes

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Homework 6

Solutions by

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PK Problems 3.8.2

Let $Z = \sum_{n=0}^{\infty} X_n$ be the total family size in branching process whose offspring distribution has a mean $\mu = \mathbb{E}\xi < 1$. Assuming that $X_0 = 1$, show that $\mathbb{E}Z = \frac{1}{1-\mu}$.

$\because$

$$X_{n+1} = \xi_1^{(n)} + \xi_2^{(n)} + \cdots + \xi_{X_n}^{(n)}$$

$\therefore$

$$\mathbb{E}(X_{n+1}|X_n = x_n) = \mu x_n$$

$\therefore$ Given $X_0 = 1$,

$$\begin{aligned}\mathbb{E}X_n &= \mathbb{E}[\mathbb{E}(X_{n+1}|X_n)] \\ &= \mathbb{E}(\mu X_n) \\ &= \mu \mathbb{E}X_n \\ &= \mu^n X_0 \\ &= \mu^n\end{aligned}$$

$\therefore$ Given $X_0 = 1$,

$$\begin{aligned}\mathbb{E}(Z) &= \sum_{n=0}^{\infty} \mathbb{E}X_n \\ &= \sum_{n=0}^{\infty} \mu^n \\ &= \frac{1}{1-\mu}\end{aligned}$$

PK Problems 3.9.2

One-fourth of the married couples in a far-off society have exactly three children. The other three-fourths of couples continue to have children until the first boy and then cease childbearing. Assume that each child is equally likely to be a boy or girl. What is the probability that the male line of descent of a particular husband will eventually die out?

Let $\xi \in \{0, 1, 2, 3\}$ denotes the number of male children,

$$\begin{aligned}\mathbb{P}(\xi = 0) &= \frac{1}{4}\binom{3}{0}\left(\frac{1}{2}\right)^3 \\ &= \frac{1}{2^5}\end{aligned}$$

Solution (cont.)

$$\begin{aligned}
\mathbb{P}(\xi = 1) &= \frac{1}{4}\binom{3}{1}\left(\frac{1}{2}\right)^3 + \frac{3}{4} \\
&= \frac{27}{2^5} \\
\mathbb{P}(\xi = 2) &= \frac{1}{4}\binom{3}{2}\left(\frac{1}{2}\right)^3 \\
&= \frac{3}{2^5} \\
\mathbb{P}(\xi = 3) &= \frac{1}{4}\binom{3}{3}\left(\frac{1}{2}\right)^3 \\
&= \frac{1}{2^5}
\end{aligned}$$

$\therefore$ the generating function of ξ is

$$G_\xi(s) = \frac{1}{2^5} + \frac{27}{2^5}s^1 + \frac{3}{2^5}s^2 + \frac{1}{2^5}s^3$$

Let

$$G(s) = s$$

we have

$$s^3 + 3s^2 - 5s + 1 = 0$$

Therefore,

$$\begin{cases} s_1 = -2 - \sqrt{5} \\ s_2 = -2 + \sqrt{5} \\ s_3 = 1 \end{cases}$$

The smallest positive solution is $s_2 = -2+\sqrt{5}$, i.e., the probability that the male line of descent of a particular husband will eventually die out.

PK Problems 3.9.8

Consider a branching process whose offspring follow the geometric distribution $p_k = (1-c)c^k$ for $k = 0, 1, \cdots$ where $0 < c < 1$. Determine the probability of eventually extinction.

Let $\xi \in \mathbb{N}$ be the number of offspring. The generating function of ξ is

$$\begin{aligned}
G_\xi(s) &= \sum_{k=0}^{\infty} p_k s^k \\
&= \sum_{k=0}^{\infty} (1-c)c^k s^k \\
&= \frac{1-c}{1-cs}
\end{aligned}$$

Solution (cont.)

Let

$$s = G_\xi(s)$$

we have

$$(cs - 1 + c)(s - 1) = 0$$

therefore $s = \dfrac{1-c}{c}$ or $s = 1 \qquad (0 < c < 1)$

When $\dfrac{1-c}{c} < 1$, i.e. $c > \dfrac{1}{2}$, the smallest solution is $\dfrac{1-c}{c}$. The probability of eventually extinction is $\dfrac{1-c}{c}$.

When $\dfrac{1-c}{c} \geqslant 1$, i.e. $c \leqslant \dfrac{1}{2}$, the smallest solution is 1. The probability of eventually extinction is 1.

GS 5.4 4(a)

Let Z_n be the size of the nth generation of a branching process, and assume $Z_0 = 1$. Find an expression for the generating function G_n of Z_n, in the case when Z_1 has generating function given by $G(s) = 1 - \alpha(1-s)^\beta$, $0 < \alpha, \beta < 1$.

$$\begin{aligned}
G_2(s) &= 1 - \alpha(1 - G_1(s))^\beta \\
&= 1 - \alpha(1 - (1 - \alpha(1-s)^\beta))^\beta \\
&= 1 - \alpha^{1+\beta}(1-s)^{\beta^2} \\
G_3(s) &= 1 - \alpha(1 - G_2(s))^\beta \\
&= 1 - \alpha(1 - (1 - \alpha^{\beta+1}(1-s)^{\beta^2}))^\beta \\
&= 1 - \alpha^{1+\beta+\beta^2}(1-s)^{\beta^3}
\end{aligned}$$

Suppose that for $n \in \mathbb{N}$ the following equation holds,

$$G_n(s) = 1 - \alpha^{1+\beta+\cdots+\beta^{n-1}}(1-s)^{n\beta}$$

then for $n+1$,

$$\begin{aligned}
G_{n+1}(s) &= 1 - \alpha(1 - G_n(s))^\beta \\
&= 1 - \alpha(1 - (1 - \alpha^{1+\beta+\cdots+\beta^{n-1}}(1-s)^{\beta^n}))^\beta \\
&= 1 - \alpha^{1+\beta+\cdots+\beta^n}(1-s)^{\beta^{n+1}}
\end{aligned}$$

$\therefore$

$$G_n(s) = 1 - \alpha^{1+\beta+\cdots+\beta^{n-1}}(1-s)^{\beta^n}$$

GS 5.4 5

Branching with immigration Each generation of a branching process (with a single progenitor) is argumented by a random number of immigrants who are indistingushable from the other members of the population. Suppose that the numbers of immigrants in different generations are independent of each other and the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function G_n of the size of the nth generation satisfies $G_{n+1}(s) = G_n(G(s))H(s)$, where $G(s)$ is the probability generating function of a typical family of offspring.

Let Z_n denotes the size of the nth generation and I_n denotes the numbers of immigrants in nth generation. Suppose that $G_n(s)$ is the generationg function of Z_n and $H_{I_n}(s) = H(s)$ is the generating function of I_n. Suppose that in nth generation,

$$Z_{n+1} = X_{n+1,1} + X_{n+1,2} + \cdots + X_{n+1,Z_n} + I_{n+1}$$

where $X_{n+1,i}$ is the number of indeviduals in generation $n+1$ who are descendants of the ith individuals in nth generation.
Let

$$Y_{n+1} = X_{n+1,1} + X_{n+1,2} + \cdots + X_{n+1,Z_n}$$

we have

$$G_{Y_{n+1}}(s) = G_n(G(s))$$

$\because$ I_{n+1} is independent with $X_{n+1,1}, X_{n+1,2}, \cdots, X_{n+1,Z_n}$, i.e. I_{n+1} and Y_{n+1} are independent
$\therefore$

$$\begin{aligned} G_{n+1}(s) &= G_{Y_{n+1}}(s)H(s) \\ &= G_n(G(s))H(s) \end{aligned}$$

Question 1

For a branching process, let Z_n denote the number of individuals in generation n. Explain why $(Z_n : n \geqslant 0)$ is a Markov process. Show that, in this Markov process, every state $n \in \mathbb{N}$ with $n > 0$ is transient.

Let X_{ni} denotes the the number of direct successors of member i in period $n-1$.
$\because$ X_{ni} $(\forall\ n \in \mathbb{N},\ i \in \{1, 2, \cdots, Z_n\})$ are independent identically distributed
$\therefore$

$$\begin{aligned} &\mathbb{P}(Z_n = i_n | Z_{n-1} = i_{n-1}, \cdots, Z_0 = i_0) \\ =&\mathbb{P}(X_{n1} + \cdots + X_{n,Z_{n-1}} = i_n | Z_{n-1} = i_{n-1}, \cdots, Z_0 = i_0) \\ =&\mathbb{P}(X_{n1} + \cdots + X_{n,i_{n-1}} = i_n | Z_{n-1} = i_{n-1}) \\ =&\mathbb{P}(Z_n = i_n | Z_{n-1} = i_{n-1}) \end{aligned}$$

$\therefore$ $\{Z_n\}$ is a Markov chain

Solution (cont.)

Let $p_i = \mathbb{P}(\text{number of offspring of a family } = i)$

(1) $p_1 = 1$, the number of members in every generation remains the same. So it will not be transient for every state $n \in \mathbb{N}^+$.

(2) $p_0 > 0$, then state 0 is the absorbing state and $p_1 \leqslant 1 - p_0 < 1$

$\because \quad \forall\, m, n \in \mathbb{N},\ n > 0,\ \mathbb{P}(Z_{m+1} = 0 | Z_m = n) \geqslant p_0^n > 0$

$\therefore$

$$\mathbb{P}(Z_i \neq n,\ \forall\, i \in \mathbb{N},\ i \geqslant m | Z_m = n) \geqslant \mathbb{P}(Z_{m+1} = 0 | Z_m = n) > 0$$

$\therefore \quad$ every state $n \in \mathbb{N}$ with $n > 0$ is transient

(3) $p_0 = 0$ and $p_1 < 1$

$\because$

$$\mathbb{P}(\text{number of offspring of a family} \geqslant 2) = 1 - p_1 > 0$$

$$\mathbb{P}\{Z_i \neq n \text{ for some } i \in \mathbb{N}^+ | Z_m = n\} \geqslant (1 - p_1)^n > 0$$

$\therefore \quad$ every state $n \in \mathbb{N}$ with $n > 0$ is transient

Question 2

In **GS Q5** (Branching with immigration), suppose that $G'(1) < 1$ and $H'(1) < \infty$. Show that

(a) The number of members has a stationary distribution π;

Let I_n denotes the number of immigration in generation n. $I_1, I_2, \cdots$ are independent identical distributed with probability distribution $h(n)$.

$\because$

$$0 \leqslant G(1) - \mathbb{P}(Z_1 = 0) \leqslant G'(1) < 1$$

$\therefore$

$$p_0 = \mathbb{P}(Z_1 = 0) > 0$$

(1) If $p_0 = 1$, then $Z_n = I_n$, i.e. then stationary distribution is given by

$$\pi = \begin{pmatrix} h(0) & h(1) & \cdots \end{pmatrix}$$

(2) If $\mathbb{P}(I_1 = 0) = 1$ and $0 < p_0 < 1$, then from question 1 we have state 0 is the absorbing state and state $n(\forall\, n \in \mathbb{N}^+)$ are transient. So the stationary distribution is given by

$$\pi = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}$$

(3) If $\mathbb{P}(I_1 = 0) < 1$ and $0 < p_0 < 1$, then $\exists\, M_1, M_2 \in \mathbb{N}^+$, s.t. $\mathbb{P}(I_1 = M_1) > 0$, $\mathbb{P}(Z_1 = M_2) > 0$. Let $G_p = \{n \in \mathbb{N} : \mathbb{P}(Z_1 = n) > 0\}$, $H_p = \{n \in \mathbb{N} : h(n) > 0\}$ and $S_p = \{h + \sum_{i=1}^{n} g_i : \forall\, h \in H_p,\ \forall\, n \in \mathbb{N}, \forall\, g_i \in G_p\}$, then $|S_p| = \aleph_0$. We know that $Z_{n+1} \in S_p$ since $Z_{n+1} = H_{n+1} + X_{n+1,1} + \cdots + X_{n+1,Z_n}$.

Irreducibility

Solution (cont.)

$\because \quad \forall\ i,j \in S_p,\ \exists\ n_2 \in \mathbb{N}^+$, s.t. $n_0 = M_1 M_2^{n_2} > \max\{i,j\}$ and $n_0 \in S_p$,

$$\begin{aligned}\mathbb{P}(Z_n = n_0 | Z_0 = i) &\geqslant p_0^i h(M_1) \mathbb{P}(Z_1 = M_2)^{M_1 + M_1 M_2 + \cdots + M_1 M_2^{n_2 - 1}} \\ &= p_0^i h(M_1) \mathbb{P}(Z_1 = M_2)^{\frac{M_1(1-M_2^{n_2})}{1-M_2}} > 0\end{aligned}$$

i.e. the 0th generation die out and M_1 immigrations appear, then no more immigration and every individual produces M_2 offsprings until n_2th generation.

$\because \quad$ suppose that $j = h_j + \sum\limits_{k=1}^{m_j} g_{jk}$,

$$\mathbb{P}(Z_1 = j | Z_0 = n_0) \geqslant p_0^{n_0 - j} h(h_j) \mathbb{P}(Z_1 = g_{jk})^{m_j} > 0$$

i.e. in 0th generation, $n_0 - j$ individuals have no offspring, the others produce $g_{j1}, \cdots, g_{jm_j}$ offsprings respectively and $h(h_j)$ immigrants appear in the 1th generation.

$\therefore$

$$\begin{aligned}&\mathbb{P}(Z_n = j \text{ for some } n \in \mathbb{N}^+) | Z_0 = i) \\ \geqslant &\mathbb{P}(Z_n = n_0 | Z_0 = i) \mathbb{P}(Z_n = j \text{ for some } n \in \mathbb{N}^+ | Z_0 = n_0) > 0\end{aligned}$$

$\therefore \quad$ the chain in S_p is irreducible

Aperiodic

Let $I_{\min} = \min\{n \in \mathbb{N} : \mathbb{P}(I_1 = n) > 0\}$, then $I_{\min} \in S_p$.

$\because \quad$ starting from $I_{\min}$, to return $I_{\min}$, it can take 2 generations (the 0th generation dies out and $I_{\min}$ immigrants appear) or it can take 3 generations (the 0th generation don't die out, the 1th generation dies out and $I_{\min}$ immigrants appear in 3th generation)

$\therefore \quad$ the chain in S_p is aperiodic

Positive Recurrence

Suppose that the irreducible chian in S_p is not positive recurrent, then $\forall\ i \in S_p,\ j \in \mathbb{N}^+$, $p_{ij}(n) \to 0$ $(n \to \infty)$, since $\forall\ n \in \mathbb{N} \setminus S_p$, $p_{in} = 0$

$\therefore \quad \forall\ m \in \mathbb{N}^+$

$$\sum_{k=1}^{m} p_{ik}(n) \to 0 \quad (n \to \infty)$$

$\therefore$

$$\lim_{n \to \infty} \mathbb{E} Z_{n+1} \geqslant m + 1 \to \infty \quad (m \to \infty)$$

$\because$

$$G'_{n+1}(s) = G'_n(G(s)) G'(s) H(s) + G_n(G(s)) H'(s)$$

$\therefore$

$$\begin{aligned}\mathbb{E} Z_{n+1} &= G'_{n+1}(1) \\ &= G'_n(G(1)) G'(1) H(1) + G_n(G(1)) H'(1) \\ &= G'(1) \mathbb{E} Z_n + H'(1) \\ &= G'(1)^n \mathbb{E} Z_1 + H'(1)[1 + G'(1) + G'(1)^2 + \cdots + G'(1)^{n-1}]\end{aligned}$$

$$\lim_{n \to \infty} \mathbb{E} Z_{n+1} = \lim_{n \to \infty} G'(1)^n \mathbb{E} Z_1 + \frac{H'(1)}{1 - G'(1)} < \infty$$

There is a contradiction.

Solution (cont.)

$\therefore$ the chain in S_p is positive recurrent, i.e., the stationary distribution $\pi = \left(\pi_i\right)_{i \in S_p}$ exists in S_p. For $i \in \mathbb{N} \setminus S_p$, $\pi_i = 0$.
$\therefore$ the number of members has a stationary distribution π.

(b) Suppose the stationary distribution has generating function $J(s)$, then

$$J(s) = J(G(s))H(s)$$

(1) If $\mathbb{P}(I_1 = 0) = 1$, $J(s) = \mathbb{P}(Z_1 = 0)$, $H(s) = \mathbb{P}(I_1 = 0)$, then

$$J(s) = J(G(s))H(s)$$

(2) If $p_0 = 1$, $J(s) = H(s)$, $G(s) = \mathbb{P}(Z_1 = 0) = 1$, then

$$\begin{aligned} J(G(s))H(s) &= H(1)H(s) \\ &= H(s) \\ &= J(s) \end{aligned}$$

(3) If $\mathbb{P}(I_1 = 0) < 1$ and $p_0 < 1$, since the chain in S_p is irreducible, aperiodic and positive recurrent, the limiting disrtibution exists and equals to the unique stationary distribution.
$\because$ $\forall\, i \in S_p$,

$$\lim_{n \to \infty} \mathbb{P}(Z_n = i) = \pi_i$$

and $\forall\, j \in \mathbb{N} \setminus S_p$, $\forall\, n \in \mathbb{N}^+$,

$$\mathbb{P}(Z_n = j) = 0 = \pi_j$$

$\therefore$

$$\begin{aligned} J(s) &= \lim_{n \to \infty} G_{n+1}(s) \\ &= \lim_{n \to \infty} G_n(G(s))H(s) \\ &= J(G(s))H(s) \end{aligned}$$