Question 1 Use Gram-Schmidt orthogonalization to find an orthonormal basis for the span of $\{e^{-x}, e^{-2x}, e^{-3x}\}$ in $L^2(0,\infty)$ with inner product

$$< f, g >= \int_0^\infty f(x)\bar{g}(x)\ dx.$$

Question 2 (a) Find the orthogonal projection $Pf(x)$ of

$$f(x) = xe^{-x/2}$$

onto the subspace of Question 1.

(b) Express P in the form of an integral operator

$$Pf(x) = \int_0^\infty K(x,y)f(y)\ dy$$

and find the kernel $K(x,y)$.

Question 3 Let D be the unit disk in C,

$$L^2(D) = \{f : D \to C | \int\int_D |f(x,y)|^2\ dx\ dy < \infty\},$$

and

$$< f, g >= \int\int_D f(x,y)\bar{g}(x,y)\ dx\ dy.$$

(a) Show that

$$\varphi_n(x,y) = (x + iy)^n$$

for $n \in N$ is an orthogonal set in $L^2(D)$.

(b) Normalize them.

(c) Project

$$f(x,y) = \sqrt{x + iy}$$

onto the span of $\{\varphi_0, \ldots, \varphi_N\}$.

Question 4 Find a sequence $f_n \in L^2(0,1)$ such that $f_n \to 0$ in $L^2(0,1)$ but not uniformly on $[0,1]$.

Question 5 Let

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = \operatorname{sign}(x)$$
$$\varphi_2(x) = \varphi_1(2x-1)$$
$$\varphi_3(x) = \varphi_1(2x+1).$$

(a) Sketch φ_j for $0 \le j \le 3$.
(b) Show that these functions are orthogonal in $L^2(-1,1)$.
(c) Normalize them.
(d) Compute the orthogonal projection Pf of $f(x) = x$ onto the span of $\{\varphi_j | 0 \le j \le 3$.
(e) Express P in the form of an integral operator

$$Pf(x) = \int_{-1}^{1} K(x,y) f(y) \, \mathrm{d}y$$

(f) Sketch the kernel $K(x,y)$.

Question 6 Suppose $f \in L^2(0,1)$ is differentiable and f is orthogonal to $g(x) = \mathrm{e}^x + 1 - \mathrm{e}$.
(a) Show that f' is orthogonal to $G(x) = \mathrm{e}^x - 1 - (\mathrm{e}-1)x$.
(b) Explain why.