Problem Set 02 Solutions 118.F17.1

Q1. $f_j(x) = e^{-jx}$ $\quad j \geq 1$

$$\langle f_j, f_k \rangle = \int_0^\infty e^{-(j+k)x}\,dx = \frac{1}{j+k}$$

$$\boxed{\varphi_1} = \frac{f_1}{\|f_1\|} = \frac{f_1}{\sqrt{\langle f_1, f_1 \rangle}} = \boxed{\sqrt{2}\, f_1}$$

$$\varphi_2 = \frac{f_2 - \langle f_2, \varphi_1 \rangle \varphi_1}{\| \quad " \quad \|}$$

$$= \frac{f_2 - \langle f_2, \sqrt{2} f_1 \rangle \sqrt{2} f_1}{\| \quad " \quad \|}$$

$$= \frac{f_2 - \frac{1}{3} \cdot 2 f_1}{\| \quad " \quad \|}$$

$$= \frac{f_2 - \frac{2}{3} f_1}{\sqrt{\langle f_2, f_2 \rangle - 2 \cdot \frac{2}{3} \cdot \langle f_2, f_1 \rangle + \frac{4}{9} \langle f_1, f_1 \rangle}}$$

$$\boxed{\varphi_2 = 6 f_2 - 4 f_1}$$ (check!)

$$\varphi_3 = \frac{f_3 - \langle f_3, \varphi_1 \rangle \varphi_1 - \langle f_3, \varphi_2 \rangle \varphi_2}{\| \qquad '' \qquad \|}$$

$$= \frac{10 f_3 - 12 f_2 + 3 f_1}{\| \qquad '' \qquad \|}$$

$$\boxed{\varphi_3 = \sqrt{6}\,(10 f_3 - 12 f_2 + 3 f_1)}$$

Q2. $Pf(x) = \langle f, \varphi_1 \rangle \varphi_1 + \langle f, \varphi_2 \rangle \varphi_2(x) + \langle f, \varphi_3 \rangle \varphi_3(x)$

(a) $\langle f, f_j \rangle = \int_0^\infty x e^{-x/2} e^{-jx} dx$

$$= \int_0^\infty x \frac{d}{dx} \frac{e^{-(j+1/2)x}}{-(j+1/2)} dx$$

$$= - \int_0^\infty \frac{e^{-(j+1/2)x}}{-(j+1/2)} dx$$

$$\boxed{\langle f, f_j \rangle = \frac{1}{(j+1/2)^2}} = \frac{4}{(2j+1)^2}$$

$$\langle f, \varphi_1 \rangle = \sqrt{2} \langle f, f_1 \rangle = \sqrt{2} \frac{1}{(3/2)^2}$$

$$= \sqrt{2} \frac{4}{9}$$

$$\langle f, \varphi_2 \rangle = 6 \langle f, f_2 \rangle - 4 \langle f, f_1 \rangle$$

$$= 6 \cdot \frac{4}{5^2} - 4 \cdot \frac{4}{3^2}$$

$$= \frac{-184}{225}$$

$$\langle f, \varphi_3 \rangle = \sqrt{6} \langle f, 10 f_3 - 12 f_2 + 3 f_1 \rangle$$

$$= \sqrt{6} \left(10 \cdot \frac{4}{49} - 12 \cdot \frac{4}{25} + 3 \frac{4}{9} \right)$$

$$= \sqrt{6} \cdot \frac{844}{3675}$$

So

$$\boxed{\begin{aligned} Pf(x) &= \sqrt{2}\, \frac{4}{9} \cdot \varphi_1(x) \\ &\quad - \frac{184}{225} \varphi_2(x) \\ &\quad - \sqrt{6}\, \frac{844}{3675} \varphi_3(x) \end{aligned}}$$

(b) $Pf(x) = \int_0^\infty f(y) \overline{\varphi_1(y)}\, dy\, \varphi_1(x)$

$+ \int_0^\infty f(y) \overline{\varphi_2}(y)\, dy\, \varphi_2(x)$

$+ \int_0^\infty f(y) \overline{\varphi_3}(y)\, dy\, \varphi_3(x) =$

$$= \int_0^\infty K(x,y) f(y)\, dy$$

if

$$\boxed{K(x,y) = \sum_{j=1}^{3} \varphi_j(x) \overline{\varphi_j(y)}}$$

Alternate solutions via QR

1. Let $f_j(x) = e^{-jx}$ for $j = 1, 2, 3$.

Then

$$\langle f_i, f_j \rangle = \int_0^\infty e^{-(i+j)x}\,dx = \frac{1}{i+j}$$

so the Gram matrix is given by

$$G = \left[\frac{1}{i+j}\right] = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} = F^* F$$

where $F = [f_1 | f_2 | f_3] = QR = [\varphi_1 | \varphi_2 | \varphi_3] R$,

so

$$G = F^* F = (QR)^* QR = R^* Q^* Q R = R^* R.$$

Cholesky factorization of G gives

$$R = \begin{bmatrix} 0.70711 & 0.47140 & 0.35355 \\ 0 & 0.16667 & 0.20000 \\ 0 & 0 & 0.04082 \end{bmatrix},$$

2. First let's compute the vector

$$F^* f = \begin{bmatrix} \langle f_1, f \rangle \\ \langle f_2, f \rangle \\ \langle f_3, f \rangle \end{bmatrix}$$

of inner products between f and f_j:

$$\langle f_j, f \rangle = \int_0^\infty x e^{-x/2} e^{-jx} \, dx =$$

$$= \int_0^\infty x \frac{d}{dx} \frac{e^{-(j+1/2)x}}{-(j+1/2)} \, dx$$

$$= x \frac{e^{-(j+1/2)x}}{-(j+1/2)} \Big|_0^\infty + \int_0^\infty \frac{e^{-(j+1/2)x}}{+(j+1/2)} \, dx$$

$$= \frac{1}{(j+1/2)^2}.$$

Hence $F^* f = \begin{bmatrix} 4/9 \\ 4/25 \\ 4/49 \end{bmatrix}$.

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Now the projection of f onto range(Q) is given by

$$g = QQ^* f \qquad \text{where } Q = FR^{-1}, \quad Q^* = R^{-*} F^*$$

so

$$= F R^{-1} R^{-*} F^* f$$

$$= F\,(R^* R)^{-1} F^* f$$

$$= F(F^* F)^{-1} F^* f$$

$$= F G^{-1} F^* f.$$

Hence $G^{-1} F^* f$ is the vector of coefficients of the ON projection of F onto R(F), <u>in the basis</u> $\{f_1, f_2, f_3\}$: from Matlab Octave

$$G^{-1}F^*f = \begin{bmatrix} 72 & -240 & 180 \\ -240 & 900 & -720 \\ 180 & -720 & 600 \end{bmatrix} \begin{bmatrix} 4/9 \\ 4/25 \\ 4/49 \end{bmatrix}$$

$$= \frac{16}{9 \cdot 25 \cdot 49} \begin{bmatrix} 5715 \\ -14775 \\ 9495 \end{bmatrix}.$$

Hence the projection of f is given by

$$Pf = \frac{16}{9 \cdot 25 \cdot 49} \Big[5715\, e^{-x} - 14775\, e^{-2x} + 9495\, e^{-3x} \Big].$$

$$= 8.29\, e^{-x} - 21.4\, e^{-2x} + 13.8\, e^{-3x}$$

to three digit.

so

$$R^{-1} = \begin{bmatrix} 1.41421 & -4 & 7.34847 \\ 0 & 6 & -29.39388 \\ 0 & 0 & 24.49490 \end{bmatrix}$$

and

$$Q = FR^{-1}.$$

Squaring each entry gives

$$R^{-1} = \begin{bmatrix} \sqrt{2} & -4 & \sqrt{54} \\ 0 & 6 & -\sqrt{864} \\ 0 & 0 & \sqrt{600} \end{bmatrix}$$

so an ON basis for span $\{e^{-x}, e^{-2x}, e^{-3x}\}$ is

$$\varphi_1(x) = \sqrt{2}e^{-x}$$

$$\varphi_2(x) = -4e^{-x} + 6e^{-2x}$$

$$\varphi_3(x) = \sqrt{54}\,e^{-x} - \sqrt{864}\,e^{-2x} + \sqrt{600}\,e^{-3x}.$$

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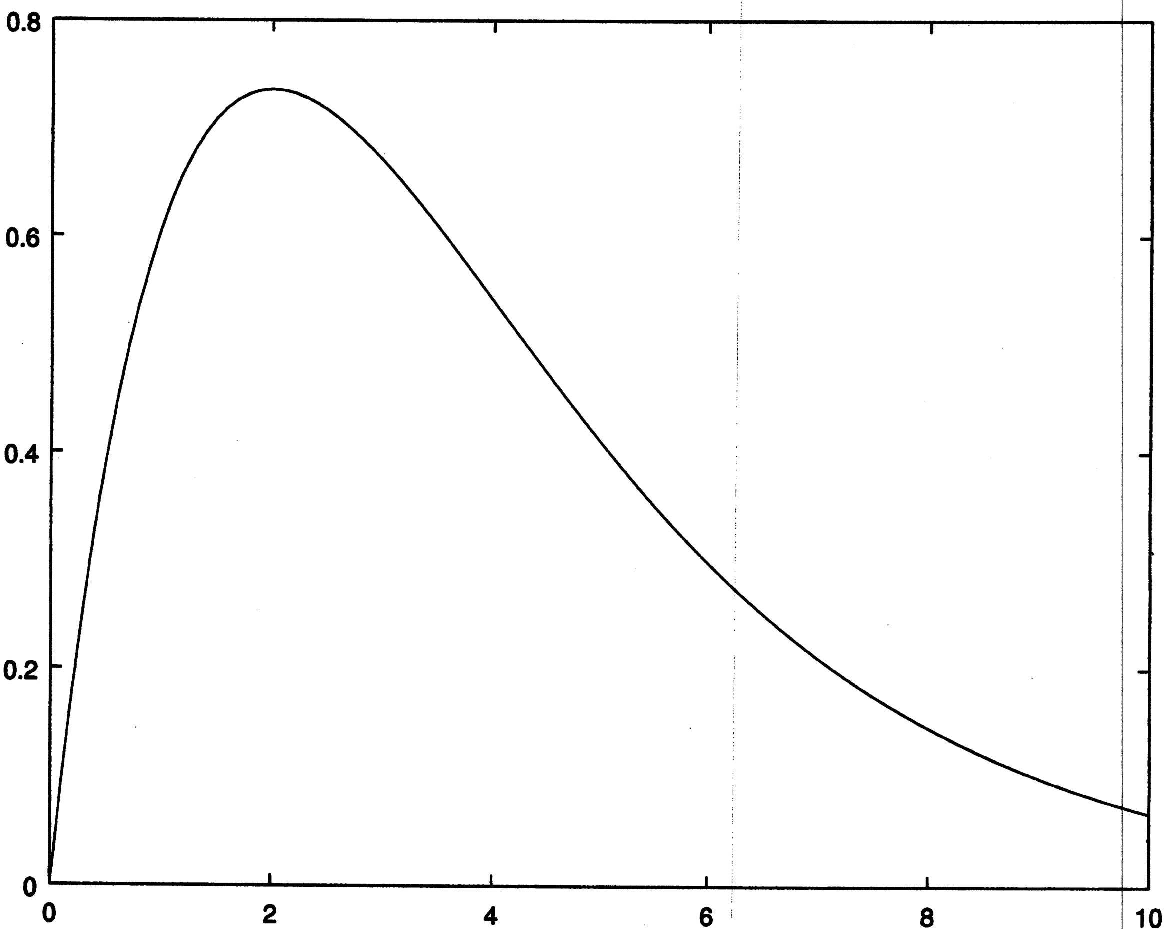

$f(x) = xe^{-x/2}$ on $[0,10]$

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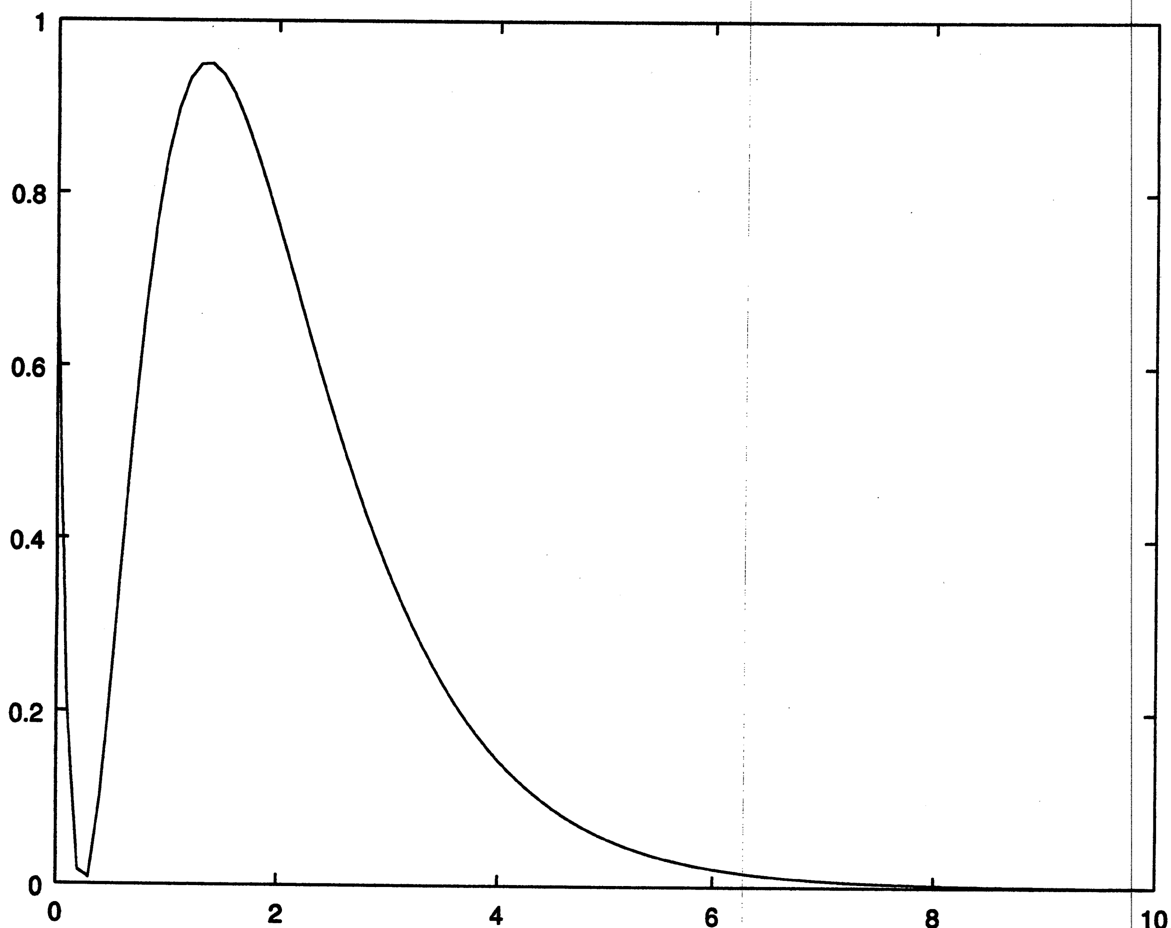


Pf(x) on [0,10]

3.

(a) $\langle \varphi_n, \varphi_m \rangle = \iint (x+iy)^n (x-iy)^m$

$$= \int_0^{2\pi} e^{i(n-m)\theta} \int_0^1 r^{n+m+1} \, dr \, d\theta$$

$$= \frac{1}{n+m+2} \cdot 2\pi \cdot \delta_{nm},$$

so $\{\varphi_n\}$ is an Ogonal set. Thus

(b) $q_j(x) = \sqrt{\dfrac{2j+2}{2\pi}}\,\varphi_j = \boxed{\sqrt{\dfrac{j+1}{\pi}}\,\varphi_j(x)}$

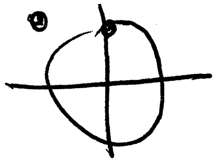
is an ONal set in $L^2(D)$.

(c) $\langle f, q_j \rangle = \sqrt{\dfrac{j+1}{\pi}} \displaystyle\int_0^{2\pi} e^{-ij\theta} e^{i\theta/2} \int_0^1 r^{j+3/2} \, dr \, d\theta$

$$= \sqrt{\frac{j+1}{\pi}} \; \frac{e^{i(1/2-j)2\pi} - 1}{i(1/2-j)} \; \frac{1}{j+5/2}$$

$$e^{i(1/2-j)2\pi} = e^{i\pi/2} = i \text{ so}$$

$$\langle f, \varphi_j \rangle = \sqrt{\frac{j+1}{\pi}} \; \frac{i-1}{i\,(1/2 - j)} \; \frac{1}{j+5/2}$$



$$= \underbrace{+i(i-1)}_{= -i-1} \; \frac{\sqrt{j+1}}{\sqrt{\pi}\,(j-1/2)(j+5/2)}$$

Hence the projection of $\sqrt{x+iy}$ onto $\operatorname{span}\{\varphi_0, \varphi_1, \dots, \varphi_N\}$ is

$$\sum_{j=0}^{N} \langle f, \varphi_j \rangle \varphi_j = \sum_{j=0}^{N} \frac{-i-1}{\sqrt{\pi}} \; \frac{j+1}{(j-1/2)(j+5/2)} \; \varphi_j(x)$$

$$\boxed{= \frac{-i-1}{\sqrt{\pi}} \sum_{j=0}^{N} \frac{(j+1)\,(x+iy)^j}{(j-1/2)\,(j+5/2)}}$$

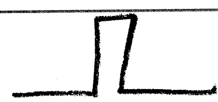
4. Many obvious possibilities. E.g.

$$f_n(x) = x^n$$

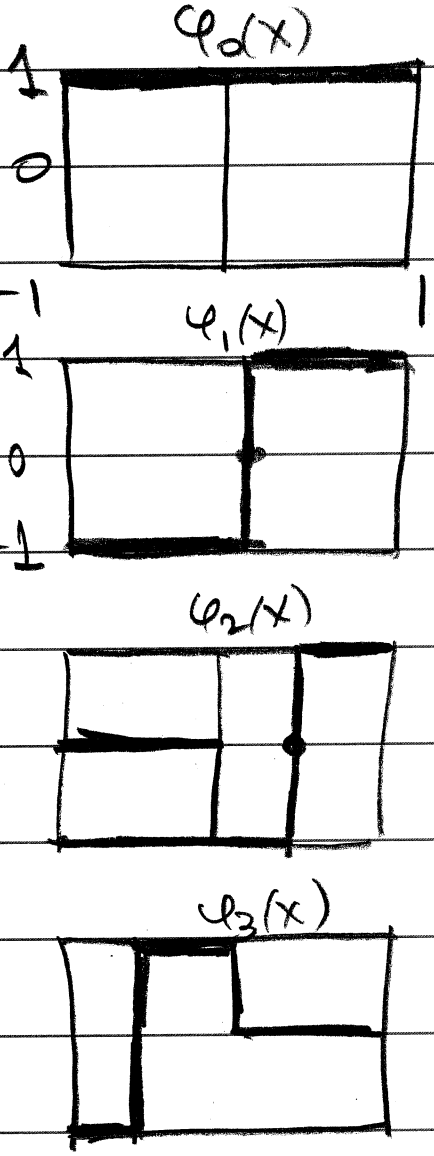
$$\| f_n \|^2 = \int_0^1 x^{2n}\, dx = \frac{1}{2n+1} \to 0$$

but

$$f_n(1) = 1 \quad \text{for all } n.$$

Or 

5.(a)



(b) Clearly $\langle \varphi_0, \varphi_j \rangle = 0$ by even-odd symmetry for $j \geq 1$. Similarly $\langle \varphi_1, \varphi_j \rangle = 0$ for $j \geq 2$. And $\langle \varphi_2, \varphi_3 \rangle = 0$ since their supports are disjoint.

(c) $\langle \varphi_0, \varphi_0 \rangle = \int_{-1}^{1} 1^2 \, dx = 2$ so

$\varphi_0(x) = \frac{1}{\sqrt{2}}$ is normalized.

$\langle \varphi_1, \varphi_1 \rangle = 2$

$\langle \varphi_2, \varphi_2 \rangle = 1 = \langle \varphi_3, \varphi_3 \rangle$

so

$\varphi_0(x) = \frac{1}{\sqrt{2}} \qquad \varphi_1(x) = \frac{1}{\sqrt{2}} \operatorname{sign}(x)$

$$\varphi_2(x) = \begin{cases} \operatorname{sign}(2x-1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_3(x) = \begin{cases} \operatorname{sign}(2x+1) & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

are normalized.

(d) $\langle x, \varphi_0 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x \, dx = 0$

$\langle x, \varphi_1 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} |x| \, dx = \frac{2}{\sqrt{2}} \int_0^1 x \, dx = \frac{1}{\sqrt{2}}$

$\langle x, \varphi_2 \rangle = \int_0^{1/2} -x \, dx + \int_{1/2}^{1} x \, dx$

$= -\frac{1}{2} x^2 \Big|_0^{1/2} + \frac{1}{2} x^2 \Big|_{1/2}^{1}$

$= \frac{1}{4}$

$\langle x, \varphi_3 \rangle = -\int_{-1}^{-1/2} x \, dx + \int_{-1/2}^{0} x \, dx$

$= -\frac{1}{2} x^2 \Big|_{-1}^{-1/2} + \frac{1}{2} x^2 \Big|_{-1/2}^{0}$

$= \frac{1}{4}$

So

$Pf(x) = \frac{1}{\sqrt{2}} \varphi_1(x) + \frac{1}{4} \varphi_2(x) + \frac{1}{4} \varphi_3(x)$

normalized! $\longrightarrow \frac{1}{\sqrt{2}} \text{sign}(x)$

$$Pf(x) = \begin{cases} -\frac{1}{\sqrt{2}} - \frac{1}{4} & -1 \le x \le -1/2 \\ -\frac{1}{\sqrt{2}} + \frac{1}{4} & -1/2 \le x \le 0 \\ \frac{1}{\sqrt{2}} - \frac{1}{4} & 0 \le x \le 1/2 \\ \frac{1}{\sqrt{2}} + \frac{1}{4} & 1/2 \le x \le 1 \end{cases}$$

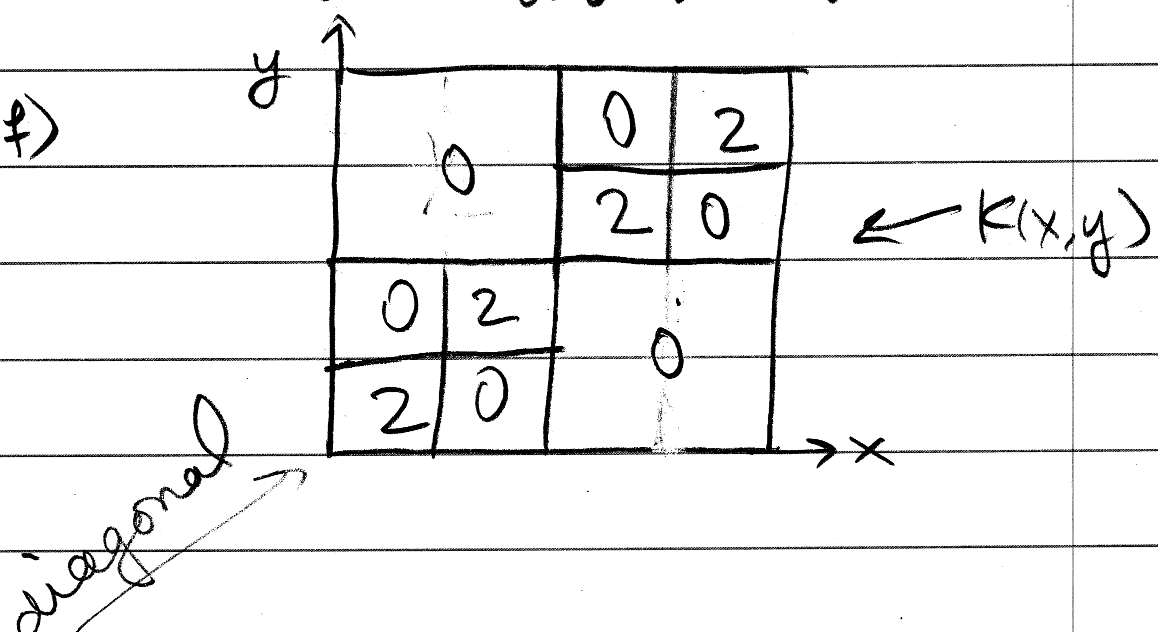
(e) $Pf(x) = \langle f, \varphi_0 \rangle \varphi_0(x) + \dots + \langle f, \varphi_3 \rangle \varphi_3(x)$

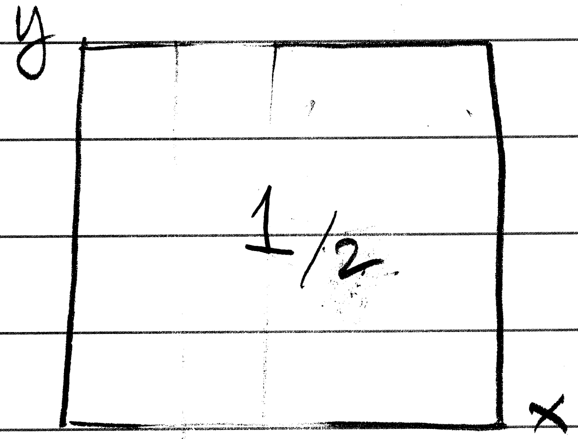
$$= \int_{-1}^{1} K(x,y) f(y)\, dy$$

where

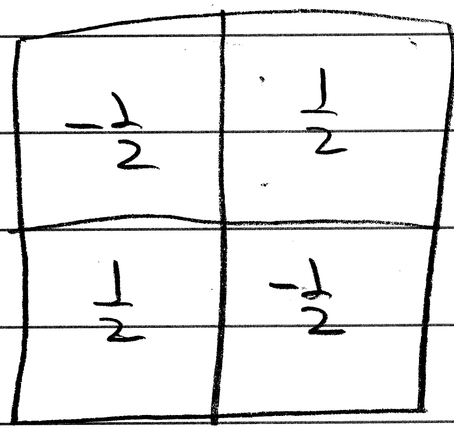
$$K(x,y) = \sum_{j=0}^{3} \varphi_j(x) \varphi_j(y)$$

(f)

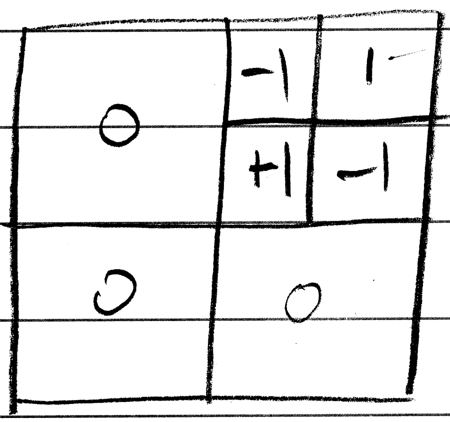




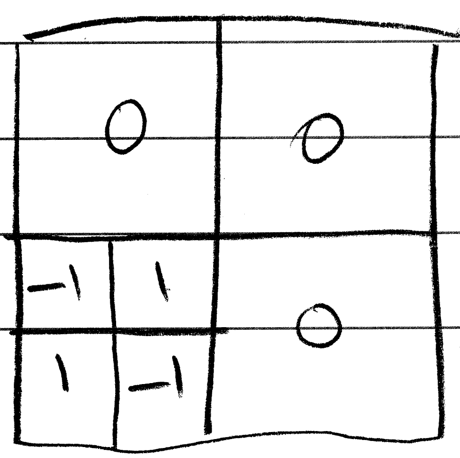
$$\varphi_0(x)\varphi_0(y) = \frac{1}{2}$$



$$\varphi_1(x)\varphi_1(y) = \frac{1}{2}\,\mathrm{sign}(x)\,\mathrm{sign}(y)$$



$$\varphi_2(x)\varphi_2(y) = \mathrm{sign}(2x-1)\,\mathrm{sign}(2y-1)$$



$$\varphi_3(x)\varphi_3(y) = \mathrm{sign}(2x+1)\,\mathrm{sign}(2y+1)$$

6. (a) $\langle f', G\rangle = \int_0^1 f'(x)\,\overline{G(x)}\,dx$

$= -\int_0^1 f(x)\,\overline{G'(x)}\,dx$ since $G(0) = G(1) = 0$

$= -\int_0^1 f(x)\,\overline{g(x)}\,dx$ since $G'(x) = g(x)$

$= -\langle f, g\rangle = 0$ since $f \perp g$.

(b) Differentiation is antisymmetric (modulo boundary values).

Question 1 Use Gram-Schmidt orthogonalization to find an orthonormal basis for the span of $\{e^{-x}, e^{-2x}, e^{-3x}\}$ in $L^2(0,\infty)$ with inner product

$$< f, g >= \int_0^\infty f(x)\bar{g}(x)\ dx.$$

Question 2 (a) Find the orthogonal projection $Pf(x)$ of

$$f(x) = xe^{-x/2}$$

onto the subspace of Question 1.

(b) Express P in the form of an integral operator

$$Pf(x) = \int_0^\infty K(x,y)f(y)\ dy$$

and find the kernel $K(x,y)$.

Question 3 Let D be the unit disk in C,

$$L^2(D) = \{f : D \to C | \int\int_D |f(x,y)|^2\ dx\ dy < \infty\},$$

and

$$< f, g >= \int\int_D f(x,y)\bar{g}(x,y)\ dx\ dy.$$

(a) Show that

$$\varphi_n(x,y) = (x + iy)^n$$

for $n \in N$ is an orthogonal set in $L^2(D)$.

(b) Normalize them.

(c) Project

$$f(x,y) = \sqrt{x + iy}$$

onto the span of $\{\varphi_0, \ldots, \varphi_N\}$.

Question 4 Find a sequence $f_n \in L^2(0,1)$ such that $f_n \to 0$ in $L^2(0,1)$ but not uniformly on $[0,1]$.

Question 5 Let

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = \operatorname{sign}(x)$$
$$\varphi_2(x) = \varphi_1(2x-1)$$
$$\varphi_3(x) = \varphi_1(2x+1).$$

(a) Sketch φ_j for $0 \le j \le 3$.
(b) Show that these functions are orthogonal in $L^2(-1,1)$.
(c) Normalize them.
(d) Compute the orthogonal projection Pf of $f(x) = x$ onto the span of $\{\varphi_j | 0 \le j \le 3$.
(e) Express P in the form of an integral operator

$$Pf(x) = \int_{-1}^{1} K(x,y) f(y)\, \mathrm{d}y$$

(f) Sketch the kernel $K(x,y)$.

Question 6 Suppose $f \in L^2(0,1)$ is differentiable and f is orthogonal to $g(x) = \mathrm{e}^x + 1 - \mathrm{e}$.
(a) Show that f' is orthogonal to $G(x) = \mathrm{e}^x - 1 - (\mathrm{e}-1)x$.
(b) Explain why.