TOPIC. Characteristic functions, cont'd. This lecture develops an inversion formula for recovering the density of a smooth random variable X from its characteristic function, and uses that formula to establish the fact that, in general, the characteristic function of X uniquely characterizes the distribution of X. We begin by discussing the characteristic function of sums and products of random variables.

Sums and products. $\mathbb{C}$-valued random variables $Z_1 = U_1 + iV_1$, ..., $Z_n = U_n + iV_n$, all defined on a common probability space, are said to be ***independent*** if the pairs (U_k, V_k) for $k = 1, \ldots, n$ are independent.

Theorem 1. *If $Z_1, \ldots, Z_n$ are independent $\mathbb{C}$-valued integrable random variables, then $\prod_{k=1}^n Z_k$ is integrable and*

$$E(\textstyle\prod_{k=1}^n Z_k) = \prod_{k=1}^n E(Z_k). \tag{1}$$

Proof The case $n = 2$ follows from the identity $(U_1+iV_1)(U_2+iV_2) = U_1U_2 - V_1V_2 + i(U_1V_2 + U_2V_1)$ and the analogue of (1) for integrable real-valued random variables. The general case follows by induction on n. ∎

Suppose X and Y are independent real random variables with distributions μ and ν respectively. The distribution of the sum $S := X + Y$ is called the ***convolution*** of μ and ν, denoted $\boldsymbol{\mu \star \nu}$, and is given by the formula

$$\begin{aligned}(\mu\star\nu)(B) &= P[S \in B] \\ &= \int_{-\infty}^{\infty} P[X+Y \in B \mid X = x]\,\mu(dx) = \int_{-\infty}^{\infty} P[x+Y \in B]\,\mu(dx) \\ &= \int_{-\infty}^{\infty} \nu(B-x)\,\mu(dx) = \int_{-\infty}^{\infty} \mu(B-y)\,\nu(dy) \end{aligned} \tag{2}$$

for (Borel) subsets B of $\mathbb{R}$; here $B - x := \{\, b - x : b \in B\,\}$. When X and Y have densities f and g respectively, this calculation can be pushed further (see Exercise 1) to show that S has density $f \star g$ (called

the convolution of f and g) given by

$$(f\star g)(s) = \int_{-\infty}^{\infty} f(x)g(s-x)\,dx = \int_{-\infty}^{\infty} f(s-y)g(y)\,dy. \tag{3}$$

Since the addition of random variables is associative — $(X+Y)+Z = X + (Y + Z)$ — so is the convolution of probability measures — $(\mu\star\nu)\star\rho = \mu\star(\nu\star\rho)$. In general, the convolution $\mu_1\star\cdots\star\mu_n$ of several measures is difficult to compute; however, the characteristic function of $\mu_1 \star\cdots\star \mu_n$ is easily obtained from the characteristic functions of the individual μ_k's:

Theorem 2. *If $X_1, \ldots, X_n$ are independent real random variables, then $S = \sum_{k=1}^n X_k$ has characteristic function $\phi_S = \prod_{k=1}^n \phi_{X_k}$.*

Proof Since e^{itS} is the product of the independent complex-valued integrable random variables e^{itX_k} for $k = 1, \ldots, n$, Theorem 1 implies that $E(e^{itS}) = \prod_{k=1}^n E(e^{itX_k})$. ∎

Example 1. (A) Suppose X and Y are independent and each is uniformly distributed over $[-1/2, 1/2]$. Using (3) it is easy to check that $S = X+Y$ has the so-called triangular distribution, with density $f_S(s) = (1 - |s|)^+$ (see Exercise 1). Since

$$\phi_X(t) = \int_{-1/2}^{1/2} \cos(tx)\,dx + i\int_{-1/2}^{1/2} \sin(tx)\,dx = \frac{\sin(t/2)}{t/2}$$

(this verifies line 10 of Table 12.1) and since $\phi_Y = \phi_X$, we have $\phi_S(t) = \sin^2(t/2)/(t^2/4) = 2(1-\cos(t))/t^2$; this gives line 11 of the Table.

(B) Suppose X and Z are independent real random variables, with $Z \sim N(0,1)$. Then $X_\sigma := X+\sigma Z$ has characteristic function $\phi_{X_\sigma}(t) = \phi_X(t)\phi_{\sigma Z}(t) = \phi_X(t)e^{-\sigma^2t^2/2}$. We're interested in X_σ because its distribution is very smooth (see Lemma 1 and Exercise 3) for any σ, and is almost the same as the distribution of X when σ is small. •

Theorem 3. *Let X and Y be independent real random variables with characteristic functions ϕ and ψ respectively. The product XY has characteristic function*

$$E\big(e^{itXY}\big) = E\big(\phi(tY)\big) = E\big(\psi(tX)\big). \tag{4}$$

Proof The condition expectation of e^{itXY} given that $X = x$ equals the unconditional expectation of e^{itxY}, namely, $E(e^{itxY}) = \psi(tx)$. Letting μ denote the distribution of X we thus have $E(e^{itXY}) = \int_{-\infty}^{\infty} E(e^{itXY} \mid X = x)\,\mu(dx) = \int_{-\infty}^{\infty} \psi(tx)\,\mu(dx) = E(\psi(tX))$. ∎

An inversion formula. This section shows how some probability densities on $\mathbb{R}$ can be recovered from their characteristic functions by means of a so-called inversion formula. We begin with a special case, which will be used later on in the proof of the general one.

Lemma 1. *Let X be a real random variable with characteristic function ϕ. Let $Z \sim N(0,1)$ be independent of X. For each $\sigma > 0$, the random variable $X_\sigma := X + \sigma Z$ has density f_σ (with respect to Lebesgue measure) given by*

$$f_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,e^{-\sigma^2t^2/2}\,dt \tag{5}$$

for $x \in \mathbb{R}$.

Proof For $\xi \in \mathbb{R}$, the random variable $X_\xi := X - \xi$ has characteristic function $\phi_{X_\xi}(y) = \phi(y)e^{-i\xi y}$. Taking $X = X_\xi$ and $t = 1$ in (4) gives

$$E\big(\phi(Y)e^{-i\xi Y}\big) = E\big(\psi(X-\xi)\big) \tag{6}$$

for any random variable Y with characteristic function ψ. Applying this to $Y \sim N(0, 1/\sigma^2)$ with density $f_Y(y) = e^{-\sigma^2y^2/2}/\sqrt{2\pi/\sigma^2}$ and characteristic function $\psi(t) = e^{-t^2/(2\sigma^2)}$, we get

$$\int_{-\infty}^{\infty}\phi(y)\,e^{-i\xi y}\,\frac{\sigma}{\sqrt{2\pi}}e^{-\sigma^2y^2/2}\,dy = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/(2\sigma^2)}\mu(dx),$$

(5): $X_\sigma = X + \sigma Z$ has density $f_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,e^{-\sigma^2t^2/2}\,dt$.

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where μ is the distribution of X. Rearranging terms gives

$$\int_{-\infty}^{\infty}\frac{e^{-(\xi-x)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}\mu(dx) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y}\phi(y)\,e^{-\sigma^2y^2/2}\,dy. \tag{7}$$

To interpret the left-hand side, note that conditional on $X = x$, one has $X_\sigma = x + \sigma Z \sim N(x, \sigma^2)$, with density

$$g_x(\xi) = \frac{e^{-(\xi-x)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}$$

at ξ. The left-hand side of (7), to wit $\int g_x(\xi)\,\mu(dx)$, is thus the unconditional density of X_σ, evaluated at ξ. This proves (5). ∎

Theorem 4 (The inversion formula for densities). *Let X be a real random variable whose characteristic function ϕ is integrable over $\mathbb{R}$, so $\int_{-\infty}^{\infty}|\phi(t)|\,dt < \infty$. Then X has a bounded continuous density f on $\mathbb{R}$ given by*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt. \tag{8}$$

Proof We need to show that

(A) the integral in (8) exists,
(B) f is bounded,
(C) f is continuous,
(D) $f(x)$ is real and nonnegative, and
(E) $P[a < X \le b] = \int_a^b f(x)\,dx$ for all $-\infty < a < b < \infty$.

(A) $f(x)$ exists since $|e^{-itx}\phi(t)| = |\phi(t)|$ and $\int_{-\infty}^{\infty}|\phi(t)|\,dt < \infty$.

(B) f is bounded since $2\pi|f(x)| = \left|\int e^{-itx}\phi(t)\,dt\right| \le \int|\phi(t)|\,dt < \infty$.

X has cf ϕ with $\int_{-\infty}^{\infty} |\phi(t)|\,dt < \infty$. $\quad f(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$.
(C) f is continuous. (D) $f(x)$ is real and nonnegative.

(C) We need to show that $x_n \to x \in \mathbb{R}$ entails $f(x_n) \to f(x)$, i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx_n}\phi(t)\,dt \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt.$$

This follows from the DCT, since

$$\begin{aligned}
&g_n(t) := e^{-itx_n}\phi(t) \to e^{-itx}\phi(t) \text{ for all } t \in \mathbb{R},\\
&|g_n(t)| \le D(t) := |\phi(t)| \text{ for all } t \in \mathbb{R} \text{ and all } n \in \mathbb{N}, \text{ and}\\
&\textstyle\int_{-\infty}^{\infty} D(t)\,dt < \infty.
\end{aligned}$$

(D) Let Z be a standard normal random variable independent of X and let $\sigma > 0$. Lemma 1 asserts that $X_\sigma := X + \sigma Z$ has density

$$f_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)e^{-\sigma^2t^2/2}\,dt.$$

Note that

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |f(x) - f_\sigma(x)| &= \sup_{x\in\mathbb{R}} \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)(1 - e^{-\sigma^2t^2/2})\,dt\right|\\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi(t)|\,\big(1 - e^{-\sigma^2t^2/2}\big)\,dt.
\end{aligned}$$

The DCT implies that the right-hand side here tends to 0 as $\sigma \to 0$, since

$$\begin{aligned}
&g_\sigma(t) := |\phi(t)|(1 - e^{-\sigma^2t^2/2}) \to 0 \text{ for all } t \in \mathbb{R},\\
&|g_\sigma(t)| \le D(t) := |\phi(t)| \text{ for all } t \in \mathbb{R} \text{ and all } n \in \mathbb{N}, \text{ and}\\
&\textstyle\int_{-\infty}^{\infty} D(t)\,dt < \infty.
\end{aligned}$$

This proves that $f_\sigma(x)$ tends uniformly to $f(x)$ as $\sigma \to 0$; in particular, since $f_\sigma(x)$ is real and nonnegative, so is $f(x)$.

X has cf ϕ with $\int_{-\infty}^{\infty} |\phi(t)|\,dt < \infty$. $\quad f(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$.
(D*): the density f_σ of $X_\sigma = X + \sigma Z$ converges uniformly to f

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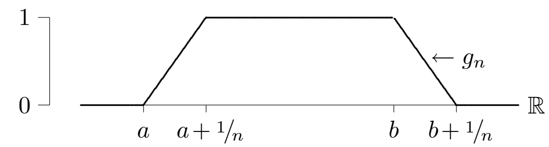
(E) Let $-\infty < a < b < \infty$ be given. We need to show that

$$P[a < X \le b] = \int_a^b f(x)\,dx. \tag{9}$$

To this end, let $n \in \mathbb{N}$ and let $g_n\colon \mathbb{R} \to \mathbb{R}$ be the function graphed below:



g_n is piecewise linear
$g_n(a) = 0 = g_n(b + 1/n)$
$g_n(a + 1/n) = 1 = g_n(b)$
$g_n(-\infty) = 0 = g_n(+\infty)$

As in the proof of (D), let $X_\sigma = X + \sigma Z$ with $\sigma > 0$ and $Z \sim N(0,1)$ independently of X. Since X_σ has density f_σ, we have

$$E\big(g_n(X_\sigma)\big) := \int_\Omega g_n\big(X_\sigma(\omega)\big)\,P(d\omega) = \int_{-\infty}^{\infty} g_n(x)f_\sigma(x)\,dx. \tag{10}$$

Take limits in (10) as $\sigma \downarrow 0$. In the middle term $g_n(X_\sigma(\omega))$ converges to $g_n(X(\omega))$ for each sample point ω; since the convergence is dominated by 1 and $E(1) = 1 < \infty$, the DCT implies that $E(g_n(X_\sigma)) \to E(g_n(X))$. On the right-hand side, $g_n(x)f_\sigma(x)$ converges pointwise to $g_n(x)f(x)$, with the convergence dominated by $(\int_{-\infty}^{\infty}|\phi(t)|\,dt)I_{(a,b+1]}$; since $\int_{-\infty}^{\infty} I_{(a,b+1]}(x)\,dx < \infty$, another application of the DCT shows that the right-hand side converges to $\int_{-\infty}^{\infty} g_n(x)f(x)\,dx$. The upshot is

$$E\big(g_n(X)\big) = \int_{-\infty}^{\infty} g_n(x)f(x)\,dx. \tag{11}$$

Now take limits in (11) as $n \to \infty$. Since $g_n \to g := I_{(a,b]}$ pointwise and boundedly, two more applications of the DCT (give the details!) yield

$$E\big(g(X)\big) = \int_{-\infty}^{\infty} g(x)f(x)\,dx.$$

This is the same as (9). ∎

X has cf ϕ with $\int_{-\infty}^{\infty} |\phi(t)|\,dt < \infty$. $\qquad f(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt.$

Example 2. (A) Suppose X has the standard exponential distribution, with density $f(x) = e^{-x} I_{[0,\infty)}(x)$ and characteristic function $\phi(t) = 1/(1-it)$. ϕ is not integrable because $|\phi(t)| \sim 1/|t|$ as $|t| \to \infty$. This is consistent with Theorem 4 because X doesn't admit a continuous density.

(B) Consider the two-sided exponential distribution, with density $f(x) = e^{-|x|}/2$ for $-\infty < x < \infty$. f has characteristic function

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \frac{1}{2}\Big[\int_{-\infty}^{0} e^{itx}e^{x}\,dx + \int_{0}^{\infty} e^{itx}e^{-x}\,dx\Big]\\ &= \frac{1}{2}\Big[\frac{1}{1+it} + \frac{1}{1-it}\Big] = \frac{1}{1+t^2}.\end{aligned}$$

This ϕ is integrable, so Theorem 4 implies that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{1}{1+t^2}\,dt = \frac{1}{2}e^{-|x|}.$$

Thus the standard Cauchy distribution, with density $1/(\pi(1+x^2))$ for $-\infty < x < \infty$, has characteristic function $e^{-|t|}$. This gives line 9 of Table 12.1. •

The uniqueness theorem. Here we establish the important fact that a probability measure on $\mathbb{R}$ is uniquely determined by its characteristic function.

Theorem 5 (The uniqueness theorem for characteristic functions). *Let X be a real random variable with distribution function F and characteristic function ϕ. Similarly, let Y have distribution function G and characteristic function ψ. If $\phi(t) = \psi(t)$ for all $t \in \mathbb{R}$, then $F(x) = G(x)$ for all $x \in \mathbb{R}$.*

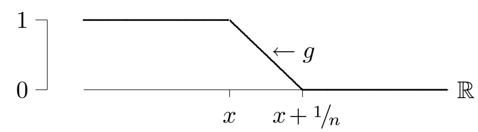
Proof Let $Z \sim N(0,1)$ independently of both X and Y. Set $X_\sigma = X + \sigma Z$ and $Y_\sigma = Y + \sigma Z$ for $\sigma > 0$. Since X_σ and Y_σ have the same integrable characteristic function, to wit $\phi(t)e^{-\sigma^2t^2/2} = \psi(t)e^{-\sigma^2t^2/2}$, Theorem 4 implies that they have same density, and hence that

$$E\big(g(X_\sigma)\big) = E\big(g(Y_\sigma)\big) \tag{12}$$

for any continuous bounded function $g\colon \mathbb{R} \to \mathbb{R}$. Letting $\sigma \to 0$ in (12) gives (how?)

$$E\big(g(X)\big) = E\big(g(Y)\big) \tag{13}$$

Taking g to be the function with graph



(14)

in (13) and letting $n \to \infty$ gives (how?)

$$E\big(I_{(-\infty,x]}(X)\big) = E\big(I_{(-\infty,x]}(Y)\big);$$

but this is the same as $F(x) = G(x)$. (There is an inversion formula for cdfs implicit in this argument; see Exercise 6.) ∎

Example 3. According to the uniqueness theorem, a random variable X is normally distributed with mean μ and variance σ^2 if and only if $E(e^{itX}) = \exp(i\mu t - \sigma^2t^2/2)$ for all real t. It follows easily from this that the sum of two independent normally distributed random variables is itself normally distributed. In other words, the family $\{\, N(\mu,\sigma^2) : \mu \in \mathbb{R}, \sigma^2 \geq 0 \,\}$ of normal distributions on $\mathbb{R}$ is closed under convolution. •

Example 4. Suppose $X_1, X_2, \ldots, X_n$ are iid standard Cauchy random variables. By Example 2, each X_i has characteristic function $\phi(t) = e^{-|t|}$. The sum $S_n = X_1 + \cdots + X_n$ thus has characteristic function $\phi_{S_n}(t) = e^{-n|t|}$, and the sample average $\bar{X}_n = S_n/n$ has characteristic function $\phi_{\bar{X}_n}(t) = \phi_{S_n}(t/n) = e^{-|t|}$. Thus $\bar{X}_n$ is itself standard Cauchy. What does this say about using the sample mean to estimate the center of a distribution? •

Theorem 6 (The uniqueness theorem for MGFs). *Let X and Y be real random variables with respective real moment generating functions M and N and distribution functions F and G. If $M(u) = N(u) < \infty$ for all u in some nonempty open interval, then $F(x) = G(x)$ for all $x \in \mathbb{R}$.*

Proof Call the interval (a,b) and put $\mathcal{D} = \{\, z \in \mathbb{C} : a < \Re(z) < b \,\}$. We will consider two cases: $0 \in (a,b)$, and $0 \notin (a,b)$.

• *Case 1:* $0 \in (a,b)$. In this case the imaginary axis $I := \{\, it : t \in \mathbb{R} \,\}$ is contained in $\mathcal{D}$. The complex generating functions G_X and G_Y of X and Y exist and are differentiable on $\mathcal{D}$ and agree on $\{\, u + i0 : u \in (a,b) \,\}$ by assumption. By Theorem 13.4, G_X and G_Y agree on all of $\mathcal{D}$, and in particular on I. In other words, $E(e^{itX}) = G_X(it) = G_Y(it) = E(e^{itY})$ for all $t \in \mathbb{R}$. Thus X and Y have the same characteristic function, and hence the same distribution.

• *Case 2:* $0 \notin (a,b)$. We treat this by exponential tilting, as follows. For simplicity of exposition, suppose that X and Y have densities f and g respectively. Put $\theta = (a+b)/2$ and set

$$f_\theta(x) = e^{\theta x} f(x)/M(\theta) \quad \text{and} \quad g_\theta(x) = e^{\theta x} g(x)/N(\theta). \tag{15}$$

f_θ and g_θ are probability densities; the corresponding MGFs are

$$M_\theta(u) = M(u+\theta)/M(\theta) \quad \text{and} \quad N_\theta(u) = N(u+\theta)/N(\theta).$$

M_θ and N_θ coincide and are finite on the interval $(a-\theta, b-\theta)$. Since this interval contains 0, the preceding argument shows that the distributions with densities f_θ and g_θ coincide. It follows from (15) that the distributions with densities f and g coincide. ∎

Theorem 7 (The uniqueness theorem for moments). *Let X and Y be real random variables with respective distribution functions F and G. If (i) X and Y each have (finite) moments of all orders,*

(ii) $E(X^k) = E(Y^k) := \alpha_k$ for all $k \in \mathbb{N}$, and (iii) the radius R of convergence of the power series $\sum_{k=1}^\infty \alpha_k u^k/k!$ is nonzero, then $F(x) = G(x)$ for all $x \in \mathbb{R}$.

Proof Let M and N be the MGFs of X and Y, respectively. By Theorem 12.5, $M(u) = \sum_{k=1}^\infty \alpha_k u^k/k! = N(u)$ for all u's in the nonempty open interval $(-R,R)$. Theorem 6 thus implies $F = G$. ∎

Example 5. (A) Suppose X is a random variable such that

$$\alpha_k := E(X^k) = \begin{cases} (k-1)\cdot(k-3)\cdots 5\cdot 3\cdot 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Then $X \sim N(0,1)$, because a standard normal random variable has these moments and the series $\sum_{k=1}^\infty \alpha_k u^k/k!$ has an infinite radius of convergence.

(B) It is possible for two random variables to have the same moments, but yet to have different distributions. For example, suppose $Z \sim N(0,1)$ and set

$$X = e^Z. \tag{16}$$

X is said to have a ***log-normal distribution***, even though it would be more accurate to say that the log of X is normally distributed. By the change of variables formula, X has density

$$f_X(x) = f_Z(z)\frac{dz}{dx} = \frac{1}{\sqrt{2\pi}}\, e^{-(\log(x))^2/2}\frac{1}{x} = \frac{1}{\sqrt{2\pi}}\frac{1}{x^{1+\log(x)/2}} \tag{17}$$

for $x > 0$, and 0 otherwise. The k^{th} moment of X is

$$E(X^k) = E(e^{kZ}) = e^{k^2/2};$$

this is finite, but increases so rapidly with k that the power series $\sum_{k=1}^\infty E(X^k)u^k/k!$ converges only for $u = 0$ (check this!). For real numbers α put

$$g_\alpha(x) = f_X(x)\big[1 + \sin\big(\alpha \log(x)\big)\big]$$

for $x > 0$, and $= 0$ otherwise. I am going to show that one can pick an $\alpha \neq 0$ such that

$$\int_{-\infty}^{\infty} x^k g_\alpha(x)\,dx = \int_{-\infty}^{\infty} x^k f_X(x)\,dx \tag{18}$$

for $k = 0, 1, \ldots$. For $k = 0$, (18) says that the nonnegative function g_α integrates to 1, and thus is a probability density. Let Y be a random variable with this density. Then by (18) we have $E(Y^k) = E(X^k)$ for $k = 1, 2, \ldots$, even though X and Y have different densities, and therefore different distributions.

It remains to show that there is a nonzero α such that (18) holds, or, equivalently, such that

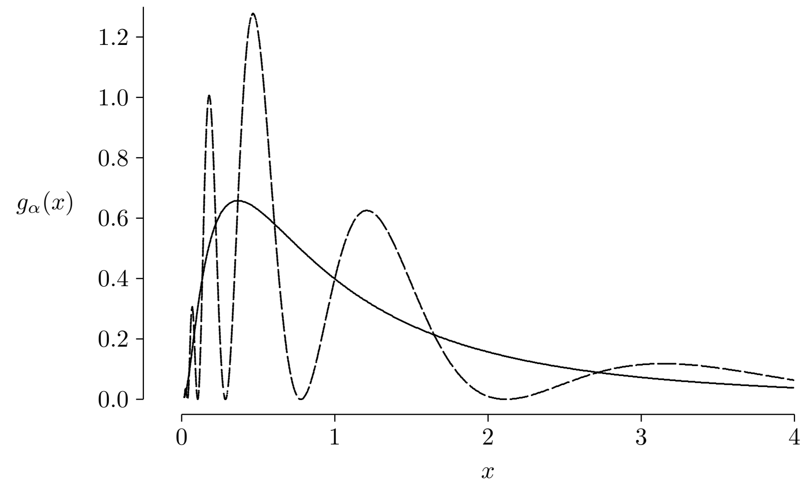
$$\nu_k := \int_0^\infty x^k f_X(x) \sin\big(\alpha \log(x)\big)\,dx = 0$$

for $k = 0, 1, 2, \ldots$. Letting $\Im(z)$ denote the imaginary part v of the complex number $z = u + iv$, we have

$$\begin{aligned}
\nu_k &= E\big[X^k \sin\big(\alpha(\log(X))\big)\big] = E\big(e^{kZ}\sin(\alpha Z)\big) \\
&= \Im\big[E\big(e^{kZ} e^{i\alpha Z}\big)\big] = \Im\big[E\big(e^{(k+i\alpha)Z}\big)\big] = \Im\big[e^{(k+i\alpha)^2/2}\big] \\
&= \Im\big[e^{(k^2-\alpha^2)/2} e^{ik\alpha}\big] = e^{(k^2-\alpha^2)/2}\,\sin(k\alpha).
\end{aligned}$$

Consequently we can make $\nu_k = 0$ for all k by taking $\alpha = \pi$, or 2π, or 3π, … . We have not only produced two distinct densities with the same moments, but in fact an infinite sequence $g_0 = f_X$, g_π, $g_{2\pi}$, $g_{3\pi}$, … of such densities! The plot on the next page exhibits g_0 and $g_{2\pi}$.•

Graphs of $g_\alpha(x) = [1 + \sin(\alpha \log(x))]/\big(\sqrt{2\pi}\, x^{1+\log(x)/2}\big)$ versus $x \in (0, 4)$, for $\alpha = 0$ and 2π. g_0 is the log-normal density. $g_{2\pi}$ has the same moments as g_0.



Exercise 1. (a) Suppose that μ and ν are probability measures on $\mathbb{R}$ with densities f and g respectively. Show that the convolution $\mu \star \nu$ has density $f \star g$ given by (3). (b) Verify the claim made in Example 1 (A), that the sum S of two independent random variables, each uniformly distributed over $[-1/2, 1/2]$, has density $f_S(s) = (1 - |s|)^+$ with respect to Lebesgue measure. [Hint for (a): Continue (2) by writing $\nu(B - x) = \int_B g(s - x)\,ds$ (why?) and then use Fubini's theorem to get $P[S \in B] = \int_B (f \star g)(s)\,ds$.] ◇

Exercise 2. Show that the function f_σ defined by (5) is infinitely differentiable. [Hint: replace the real argument x by $-iz$ with $z \in \mathbb{C}$ and show that the resulting integral is a linear combination of complex generating functions. ◇

Exercise 3. The function g_n in (10) may be replaced by $g := I_{(a,b]}$, to get $P[a < X_\sigma \le b] = \int_a^b f_\sigma(x)\,dx$. Show how (9) may be deduced from this. ◇

Exercise 4. Use line 11 of Table 12.1 and the inversion formula (8) for densities to get line 12 (for simplicity take $\alpha = 1$). Show that $f(x) := \big(1 - \cos(x)\big)/(\pi x^2)$ is in fact a probability density on $\mathbb{R}$ and deduce that $\int_0^\infty \big(1 - \cos(x)\big)/x^2\, dx = \pi/2$. ◇

Exercise 5 (*An inversion formula for point masses*). Let X be a random variable with characteristic function ϕ. Show that for each $x \in \mathbb{R}$,

$$P[\, X = x\,] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx}\, \phi(t)\, dt\,. \tag{19}$$

Deduce that if $\phi(t) \to 0$ as $|t| \to \infty$, then $P[\, X = x\,] = 0$ for all $x \in \mathbb{R}$. [Hint for (19): write $\phi(t)$ as $E(e^{itX})$, use Fubini's Theorem, and the DCT (with justifications). ◇

Exercise 6 (*An inversion formula for cdfs*). Let X be a random variable with distribution function F, density f, and characteristic function ϕ. (a) Suppose that ϕ is integrable. Use the inversion formula (8) for densities to show that

$$F(b) - F(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)\, dt \tag{20}$$

for $-\infty < a < b < \infty$. [Hint: $\big(F(b) - F(a)\big)/(b-a)$ is the density at b for the random variable $X + U$, where U is uniformly distributed over $[0, b-a]$ independently of X.] (b) Now drop the assumption that ϕ is integrable. Show that

$$F(b) - F(a) = \lim_{\sigma\downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi(t) e^{-\sigma^2 t^2/2}\, dt \tag{21}$$

provided that F is continuous at a and b. ◇

Exercise 7 (*The inversion formula for lattice distributions*). Let ϕ be the characteristic function of a random variable X which takes almost all its values in the lattice $L_{a,h} = \{\, a + kh : k = 0, \pm 1, \pm 2, \dots\,\}$. Show that

$$\frac{P[\, X = x\,]}{h} = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx} \phi(t)\, dt \tag{22}$$

for each $x \in L_{a,h}$. Give an analogous formula for $S_n = \sum_{i=1}^n X_i$, where $X_1, \dots, X_n$ are independent random variables, each distributed like X. [Hint: use Fubini's theorem.] ◇

Exercise 8. Use the inversion formula (22) to recover the probability mass function of the Binomial(n, p) distribution from its characteristic function $(pe^{it} + q)^n$. ◇

Exercise 9. Use characteristic functions to show that the following families of distributions in Table 1 are closed under convolution: Degenerate, Binomial (fixed p), Poisson, Negative binomial (fixed p), Gamma (fixed α), and Symmetric Cauchy. ◇

A random variable X is said to have an ***infinitely divisible distribution*** if for each positive integer n, there exist iid random variables $X_{n,1}, \dots, X_{n,n}$ whose sum $X_{n,1} + \cdots + X_{n,n}$ has the same distribution as X.

Exercise 10. Show that a Normal distribution and all but one (which one?) of the distributions in the preceding exercise is infinitely divisible. ◇

Exercise 11. (a) Suppose X and Y are independent random variables, with densities f and g respectively. Show that $Z := Y - X$ has density

$$h(z) := \int_{-\infty}^{\infty} f(x) g(x+z)\, dx. \tag{23}$$

(b) Suppose the random variables X and Y in part (a) are each Gamma$(r, 1)$, for a positive integer r. Use (23) to show that Z has density

$$h_r(z) := e^{-|z|} \sum_{j=0}^{r-1} \left[\frac{1}{\Gamma(r)} \frac{\Gamma(r+j)}{\Gamma(r-j)\, j!} \frac{1}{2^{r+j}} \right] |z|^{r-1-j}. \tag{24}$$

(c) Find the characteristic function of the unnormalized t-distribution with ν degrees of freedom, for an odd integer ν. ◇

Exercise 12. Let $\mathfrak{IT}_\alpha$ be the so-called inverse triangular distribution with parameter α, having characteristic function $\phi_\alpha(t) = (1 - |t|/\alpha)^+$ (see line 12 of Table 1). Show that the characteristic function $\phi(t) = \big((1-|t|)^+ + (1-|t|/2)^+\big)/2$ of the mixture $\mu = (\mathfrak{IT}_1 + \mathfrak{IT}_2)/2$ agrees with the characteristic function ψ of $\nu = \mathfrak{IT}_{4/3}$ on a nonempty open interval, even though $\mu \neq \nu$. Why doesn't this contradict Theorem 5? ◇

Let f be a continuous integrable function from the real line $\mathbb{R}$ to the complex plane $\mathbb{C}$. (Here "integrable" means $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.) The ***Fourier transform*** of f is the function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \tag{25}$$

for $-\infty < t < \infty$. If f is a probability density, then $\hat{f}$ is its characteristic function; the goal of the next two exercises is to extend some of the things we know about characteristic functions to Fourier transforms.

Exercise 13. Let $f\colon \mathbb{R} \to \mathbb{C}$ be continuous and integrable and let $\sigma > 0$. Show that

$$\int_{-\infty}^{\infty} f(x-z) \frac{e^{-z^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}(t) e^{-\sigma^2 t^2/2}\, dt \tag{26}$$

for all $x \in \mathbb{R}$. [Hint: by the inversion formula for densities, (26) is true when f is a probability density (i.e., real, positive, and integrable); deduce the general case (f complex and integrable) from the special one.] ◇

Exercise 14. Let $f\colon \mathbb{R} \to \mathbb{C}$ be continuous and integrable. Show that if $\hat{f}$ is integrable, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}(t)\, dt \tag{27}$$

for all $x \in \mathbb{R}$. [Hint: let σ tend to 0 in (26) — but beware, f need not be bounded.] ◇