(1a) Find an orthonormal basis e_1, e_2 for the range of the matrix

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 7 & 3 \\ 1 & 8 & 3 \end{bmatrix} = [a_1|a_2|a_3]$$

(1b) Find the 3×3 matrix P which projects orthonormally onto the range of A.

(1c) Find the closest point y in the range of A to

$$b = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}.$$

(2a) Let $u(x,t)$ be the solution of the wave equation

$$u_t = u_x$$

which is 2π-periodic in x and satisfies the initial condition $u(x,0) = g(x)$ where $g \in L^2(-\pi,\pi)$. Find the complex Fourier coefficients $\hat{u}(k,t)$ in terms of $\hat{g}$.

(2b) Show that

$$\int_{-\pi}^{\pi} |u(x,t)|^2 dx = \int_{-\pi}^{\pi} |g(x)|^2 dx$$

for all $t \geq 0$.

(2c) Sum the Fourier series to express $u(x,t)$ directly in terms of g.

(2d) Show that u is 2π-periodic in t:

$$u(x, t + 2\pi) = u(x, t)$$

for all $t \geq 0$.

(3a) Compute the complex Fourier coefficients on the interval $-\pi < x < \pi$ of the function $f(x) = x(\pi^2 - x^2)$. (Hint: $f(x)\mathrm{e}^{-\mathrm{i}kx} = (\mathrm{i}D)(\pi^2 + D^2)\mathrm{e}^{-\mathrm{i}kx}$ where $D = d/dk$ is independent of x.)

(3a continued)

(3a continued)

(3b) Show that

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.$$

(3c) State a theorem justifying (3b) and verify its hypotheses on $f(x) = x(\pi^2 - x^2)$.