Stochastic Processes

Fall 2017

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Week 11

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Solutions by

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3.14

Let $A(t)$ and $Y(t)$ denote the age and excess at t of a renewal process. Fill in the missing terms:

(a) $A(t) > x \longleftrightarrow$ 0 events in the interval ____?

> $A(t) > x \longleftrightarrow$ 0 events in the interval $[t - x, t]$ for $0 < x < t$.

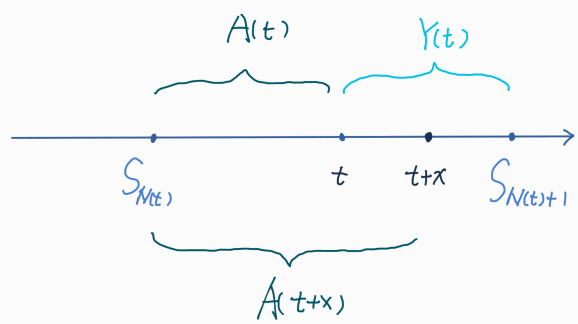
(b) $Y(t) > x \longleftrightarrow$ 0 events in the interval ____?

> $Y(t) > x \longleftrightarrow$ 0 events in the interval $(t, t + x]$.

(c) $\mathbb{P}\{Y(t) > x\} = \mathbb{P}\{A(\_\_\_\_) > \_\_\_\_\}$.

For $x, t > 0$,

$$\begin{aligned}\mathbb{P}\{Y(t) > x\} &= \mathbb{P}\{A(t + x) \geqslant x\} \\ &= \mathbb{P}\{A(t + x) > x\}\end{aligned}$$



(d) Compute the joint distribution of $A(t)$ and $Y(t)$ for a Poisson process.

When $\{N(t) : t \geqslant 0\}$ is a Poisson process, $A(t)$ and $Y(t)$ are independent because of the memoryless property. And for $x \geqslant 0$,

$$\begin{aligned}\mathbb{P}\{Y(t) \leqslant x\} &= 1 - e^{-\lambda x} \\ \mathbb{P}\{A(t) \leqslant x\} &= 1 - \mathbb{P}\{A(t) > x\} \\ &= \begin{cases} 0 & , x < 0 \\ 1 - e^{-\lambda x} & , 0 \leqslant x \leqslant t \\ 1 & , x > t \end{cases}\end{aligned}$$

Solution (cont.)

$\therefore$

$$
\begin{aligned}
F_{A(t),Y(t)}(x,y) &= F_{A(t)}(x)F_{Y(t)}(y) \\
&= \begin{cases} 0 & x<0 \text{ or } y<0 \\ (1-e^{-\lambda x})(1-e^{-\lambda y}) & ,0\leqslant x\leqslant t, y\geqslant 0 \\ 1-e^{-\lambda y} & ,x>t, y\geqslant 0 \end{cases}
\end{aligned}
$$

3.15

Let $A(t)$ and $Y(t)$ denote respectively the age and excess at t. Find:

(a) $\mathbb{P}\{Y(t)>x|A(t)=s\}$.

$$
\begin{aligned}
\mathbb{P}\{Y(t)>x|A(t)=s\} &= \mathbb{P}\{S_{N(t)+1}-t>x|t-S_{N(t)}=s\} \\
&= \mathbb{P}\{S_{N(t)+1}>t+x|S_{N(t)}=t-s\} \\
&= \mathbb{P}\{X_{N(t)+1}>s+x|S_{N(t)}=t-s\} \\
&= \sum_{n=0}^{\infty}\mathbb{P}\{X_{N(t)+1}>s+x|S_{N(t)}=t-s, N(t)=n\}\mathbb{P}\{N(t)=n|S_{N(t)}=t-s\} \\
&= \sum_{n=0}^{\infty}\mathbb{P}\{X_{n+1}>s+x|S_n=t-s, X_{n+1}>s\}\mathbb{P}\{N(t)=n|S_{N(t)}=t-s\} \\
&= \sum_{n=0}^{\infty}\mathbb{P}\{X_{n+1}>s+x|X_{n+1}>s\}\mathbb{P}\{N(t)=n|S_{N(t)}=t-s\} \\
&= \mathbb{P}\{X_1>s+x|X_1>s\} \\
&= \frac{\overline{F}(s+x)}{\overline{F}(s)}
\end{aligned}
$$

(b) $\mathbb{P}\left\{Y(t)>x|A\left(t+\frac{x}{2}\right)=s\right\}$.

When $0\leqslant\frac{x}{2}\leqslant s$, from 3.15 (a) we have

$$
\begin{aligned}
\mathbb{P}\left\{Y(t)>x\middle|A\left(t+\frac{x}{2}\right)=s\right\} &= \mathbb{P}\left\{Y(t)>x\middle|A(t)=s-\frac{x}{2}\right\} \\
&= \frac{\overline{F}\left(s+\frac{x}{2}\right)}{\overline{F}(s)}
\end{aligned}
$$

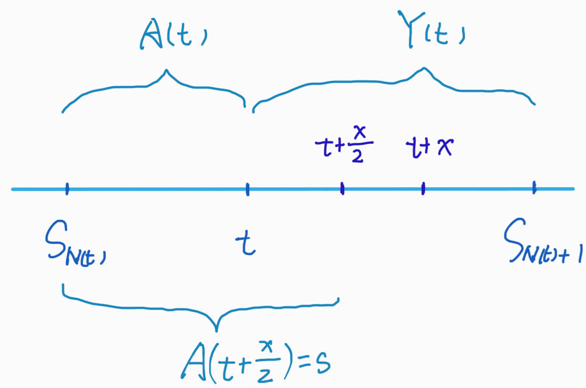
When $\frac{x}{2}>s$ or $x<0$,

$$\mathbb{P}\left\{Y(t)>x\middle|A\left(t+\frac{x}{2}\right)=s\right\}=0$$

since

$$\left\{A\left(t+\frac{x}{2}\right)=s\right\} \quad\Longleftrightarrow\quad \{0\leqslant Y(t)\leqslant s-\frac{x}{2}\}=\varnothing$$

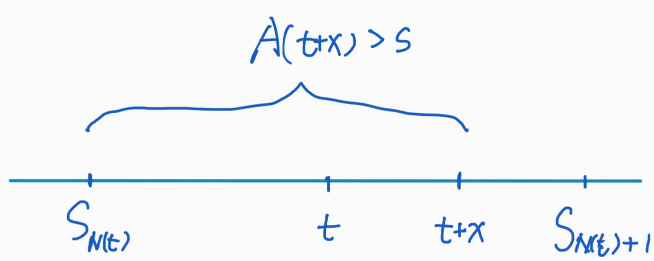
Solution (cont.)



(c) $\mathbb{P}\{Y(t) > x | A(t+x) > s\}$ for a Poisson process.

From independent increments of Poisson process, we have

$$\begin{aligned}
\mathbb{P}\{Y(t) > x | A(t+x) > s\} &= \mathbb{P}\{S_{N(t)+1} - t > x | S_{N(t+x)} - S_{N(t)} > s\} \\
&= \begin{cases} 1 & , s \geqslant x \text{ or } x < 0 \\ \mathbb{P}\{X_1 > x - s\} & , x > s \\ 1 - e^{-\lambda x} & , x \geqslant 0, s < 0 \end{cases} \\
&= \begin{cases} 1 & , s \geqslant x \geqslant 0 \text{ or } x < 0 \\ e^{-\lambda(x-s)} & , x > s \geqslant 0 \\ 1 - e^{-\lambda x} & , s < 0 \end{cases}
\end{aligned}$$



(d) $\mathbb{P}\{Y(t) > x, A(t) > y\}$.

For $x, y > 0$,

$$
\begin{aligned}
&\mathbb{P}\{Y(t) > x, A(t) > y\} \\
=&\mathbb{P}\{S_{N(t)+1} > t + x, t - S_{N(t)} > y\} \\
=&\mathbb{P}\{X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y\} \\
=&\mathbb{P}\{X_1 > t + x, t > y | S_{N(t)} = 0\}\mathbb{P}\{S_{N(t)} = 0\} \\
&\quad + \int_0^t \mathbb{P}\{X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y | S_{N(t)} = s, X_{N(t)+1} > t - S_{N(t)}\} \mathrm{d}F_{S_{N(t)}}(s) \\
=&\mathbb{1}_{\{t>y\}}(y)\mathbb{P}\{X_1 > t + x | X_1 > t\}\mathbb{P}\{X_1 > t\} \\
&\quad + \int_0^t \mathbb{1}_{\{t-s>y\}}(s)\mathbb{P}\{X_{N(t)+1} > t - s + x | X_{N(t)+1} > t - s\} \mathrm{d}F_{S_{N(t)}}(s) \\
=&\mathbb{1}_{\{[0,t)\}}(y)\mathbb{P}\{X_1 > t + x | X_1 > t\}\mathbb{P}\{X_1 > t\} \\
&\quad + \int_0^t \mathbb{1}_{(-\infty, t-y)}(s)\mathbb{P}\{X_1 > t - s + x | X_1 > t - s\} \mathrm{d}F_{S_{N(t)}}(s) \\
=&\mathbb{1}_{\{[0,t)\}}(y)\overline{F}(t+x) + \int_0^{t-y} \overline{F}(t + x - s) \mathrm{d}m(s)
\end{aligned}
$$

For $x \leqslant 0, y > 0$,

$$
\begin{aligned}
\mathbb{P}\{Y(t) > x, A(t) > y\} &= \mathbb{P}\{A(t) > y\} \\
&= \mathbb{P}\{t - S_{N(t)} > y\} \\
&= \mathbb{P}\{S_{N(t)} < t - y\} \\
&= \begin{cases} 0 & , y > t \\ \overline{F}(t) + \int_0^{t-y} \overline{F}(t-s)\mathrm{d}m(s) & , y \leqslant t \end{cases}
\end{aligned}
$$

For $x > 0, y \leqslant 0$,

$$
\begin{aligned}
&\mathbb{P}\{Y(t) > x, A(t) > y\} \\
=&\mathbb{P}\{Y(t) > x\} \\
=&\mathbb{P}\{S_{N(t)+1} - t > x\} \\
=&\mathbb{P}\{S_{N(t)+1} > t + x\} \\
=&\mathbb{P}\{X_{N(t)+1} > t - S_{N(t)} + x\} \\
=&\mathbb{P}\{X_1 > t + x | S_{N(t)} = 0\}\mathbb{P}\{S_{N(t)} = 0\} \\
&\quad + \int_0^t \mathbb{P}\{X_{N(t)+1} > t - S_{N(t)} + x | S_{N(t)} = s, X_{N(t)+1} > t - S_{N(t)}\} \mathrm{d}F_{S_{N(t)}}(s) \\
=&\mathbb{P}\{X_1 > t + x | X_1 > t\}\mathbb{P}\{X_1 > t\} \\
&\quad + \int_0^t \mathbb{P}\{X_1 > t - s + x | X_1 > t - s\} \mathrm{d}F_{S_{N(t)}}(s) \\
=&\overline{F}(t+x) + \int_0^t \overline{F}(t + x - s)\mathrm{d}m(s)
\end{aligned}
$$

(e) If $\mu < \infty$, show that, with probability 1, $\dfrac{A(t)}{t} \to 0$ as $t \to \infty$.

$\because$

$$\frac{N(t)}{t} \xrightarrow{P} \frac{1}{\mu}$$

$$\frac{S_{N(t)}}{N(t)} \xrightarrow{P} \mu$$

$\therefore$ with probability one,

$$\begin{aligned}
\lim_{t\to\infty} \frac{A(t)}{t} &= \lim_{t\to\infty} \frac{t - S_{N(t)}}{t} \\
&= 1 - \lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} \lim_{t\to\infty} \frac{N(t)}{t} \\
&= 1 - \frac{\mu}{\mu} \\
&= 0
\end{aligned}$$