MATH 118:
Fourier Analysis and Wavelets

Fall 2017

Problem Set 8

Solutions by

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Question 1

Show that $K = D^2 - x^2$ is a symmetric operator on $L^2(R)$: for nice smooth functions $f, g \in L^2(R)$ we have

$$\int_{-\infty}^{\infty} f(x)Kg(x)^*\mathrm{d}x =< f, Kg >=< Kf, g > .$$

$$\begin{aligned}
< f, Kg > &= \int_{-\infty}^{\infty} f(x)Kg(x)^*\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x)(D^2 - x^2)g(x)^*\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x)D^2g(x)\mathrm{d}x - \int_{-\infty}^{\infty} x^2f(x)g(x)\mathrm{d}x \\
&= f(x)g'(x)\Big|_{-\infty}^{\infty} - f'(x)g''(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} D^2f(x)g(x)\mathrm{d}x - \int_{-\infty}^{\infty} x^2f(x)g(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} Kf(x)g(x)^*\mathrm{d}x \\
&=< Kf, g >
\end{aligned}$$

Question 2

Show that

$$\|h_n\|^2 = \frac{\sqrt{\pi}}{n!}2^n.$$

(**Hint:** Square the expansion

$$\sum_{n=0}^{\infty} y^n h_n(x) = e^{\frac{x^2}{2}} e^{-(x-y)^2}$$

and integrate.)

The taylor series for $e^{-(x-y)^2}$ is

$$e^{-(x-y)^2} = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} D^n e^{-x^2} = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} y^n h_n(x)$$

$\therefore$

$$\sum_{n=0}^{\infty} y^n h_n(x) = e^{\frac{x^2}{2}} e^{-(x-y)^2}$$

$\therefore$

$$\left(\sum_{n=0}^{\infty} y^n h_n(x)\right)^2 = e^{x^2} e^{-2(x-y)^2}$$

i.e.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} y^m h_{m-n}(x)h_n(x) = e^{x^2} e^{-2(x-y)^2}$$

Solution (cont.)

$\because$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{m} y^m h_{m-n}(x) h_n(x) \mathrm{d}x &= \int_{-\infty}^{\infty} e^{x^2} e^{-2(x-y)^2} \mathrm{d}x \\
\sum_{m=0}^{\infty} \sum_{n=0}^{m} y^m \int_{-\infty}^{\infty} h_{m-n}(x) h_n(x) \mathrm{d}x &= 2\sqrt{\pi} e^{2y^2} \\
\sum_{n=0}^{\infty} y^{2n} \|h_n\|_2^2 &= \sum_{n=0}^{\infty} \sqrt{\pi} \frac{(2y^2)^n}{n!} \\
\sum_{n=0}^{\infty} \|h_n\|_2^2 y^{2n} &= \sum_{n=0}^{\infty} \sqrt{\pi} \frac{2^n}{n!} y^{2n}
\end{aligned}
$$

$\therefore$

$$
\|h_n\|_2^2 = \frac{\sqrt{\pi}}{n!} 2^n
$$

Question 3

(a) Show that the Hermite polynomials of degree less than or equal to n form a basis for the vector space of all polynomials of degree less than or equal to n.

$$
H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}
$$

$\because \quad \forall\, m \in \mathbb{N}$,

$$
\begin{aligned}
H_m'(x) &= (-1)^m 2x e^{x^2} D^m e^{-x^2} + (-1)^m e^{x^2} D^{m+1} e^{-x^2} \\
&= (-1)^m e^{x^2} (2x D^m e^{-x^2} + D^{m+1} e^{-x^2}) \\
&= (-1)^{m-1} e^{x^2} D^{m-1} e^{-x^2} \\
&= 2m H_{m-1}(x)
\end{aligned}
$$

$\therefore \quad i, j \in \mathbb{N},\ i < j$,

$$
\begin{aligned}
< H_i(x), H_j(x) > &= \int_{-\infty}^{\infty} H_i(x) (-1)^j e^{x^2} \left(D^j e^{-x^2} \right) e^{-x^2} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} H_i(x) (-1)^j \left(D^j e^{-x^2} \right) \mathrm{d}x \\
&= (-1)^j H_i(x) D^j e^{-x^2} \Big|_{-\infty}^{\infty} - (-1)^j \int_{-\infty}^{\infty} H_i'(x) D^{j-1} e^{-x^2} \mathrm{d}x \\
&= (-1)^{j-1} \int_{-\infty}^{\infty} H_i'(x) D^{j-1} e^{-x^2} \mathrm{d}x \\
&= 2i < H_{i-1}(x), H_{j-1}(x) >
\end{aligned}
$$

Solution (cont.)

$$
\begin{aligned}
&= 2^i i! < H_0(x), H_{j-i}(x) > \\
&= 2^i i! \int_{-\infty}^{\infty} (-1)^{j-i} D^{j-i} e^{-x^2} \mathrm{d}x \\
&= \begin{cases} 2^i i! (-1)^{j-i} D_{j-i+1} e^{-x^2} \Big|_{-\infty}^{\infty} & , i < j \\ 2^i i! & , i = j \end{cases} \\
&= \begin{cases} 0 & , i < j \\ 2^i i! & , i = j \end{cases}
\end{aligned}
$$

i.e. $H_0(x), H_1(x), \cdots, H_n(x)$ form the basis for the vector space of all polynomials of degree less than or equal to n.

(b) Calculate the first three Hermite polynomials and use them to compute

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \mathrm{d}x.$$

$$
\begin{aligned}
H_0(x) &= (-1)^0 e^{x^2} D^0 e^{-x^2} \\
&= 1 \\
H_1(x) &= (-1)^1 e^{x^2} D^1 e^{-x^2} \\
&= 2x \\
H_2(x) &= (-1)^2 e^{x^2} D^2 e^{-x^2} \\
&= 4x^2 - 2
\end{aligned}
$$

$\because$

$$
\begin{aligned}
< H_0(x), H_0(x) > &= \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x \\
&= \sqrt{\pi} \\
< H_0(x), H_2(x) > &= 0
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-x^2} \mathrm{d}x &= \frac{1}{4} < H_2(x) + 2H_0(x), H_0(x) > \\
&= \frac{1}{2} < H_0(x), H_0(x) > \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

(c) Show that

$$\int_0^x e^{-s^2} H_n(s)\mathrm{d}s = C_n - e^{-x^2} H_{n-1}(x)$$

for some constant C_n, whenever $n \geqslant 1$.

$$\begin{aligned}\int_0^x e^{-s^2} H_n(s)\mathrm{d}s &= (-1)^n \int_0^x D^n e^{-s^2}\mathrm{d}s \\ &= (-1)^n D^{n-1} e^{-s^2}\Big|_0^x \\ &= (-1)^n D^{n-1} e^{-x^2} - (-1)^n \left(D^{n-1} e^{-s^2}\right)\Big|_{s=0} \\ &= C_n - e^{-x^2} H_{n-1}(x)\end{aligned}$$

where $C_n = (-1)^{n+1}\left(D^{n-1}e^{-s^2}\right)\Big|_{s=0}$.

(d) Show that the indefinite integral

$$\int_0^x P(s)e^{-s^2}\mathrm{d}s$$

can be evaluated explicitly whenever P is polynomial with

$$\int_{-\infty}^{\infty} P(s)H_0(s)e^{-s^2}\mathrm{d}s = 0.$$

$\because$ P is polynomial with $\int_{-\infty}^{\infty} P(s)H_0(s)e^{-s^2}\mathrm{d}s = 0$ and suppose its degree is n

$\therefore$

$$P(x) = \sum_{i=1}^{n} p_i H_i(x)$$

$\because$ $\forall\, n \in \mathbb{N}^+$,

$$\int_0^x e^{-s^2} H_n(s)\mathrm{d}s = C_n - e^{-x^2} H_{n-1}(x)$$

$\therefore$

$$\begin{aligned}\int_0^x P(s)e^{-s^2}\mathrm{d}s &= \sum_{i=1}^{n} p_i \int_0^x H_n(s)e^{-s^2}\mathrm{d}s \\ &= \sum_{i=1}^{n} p_i \left(C_i - e^{-x^2} H_{i-1}(x)\right)\end{aligned}$$

Question 4

(a) Show that

$$-<Kf,f>=\int_{-\infty}^{\infty} f'(x)^2+x^2f(x)^2\mathrm{d}x=\sum_{n=0}^{\infty}(2n+1)\frac{<f,h_n>^2}{\|h_n\|^2}$$

for real-valued $f\in L^2(R)$.

$\because$ $f\in L^2(R)$

$\therefore$

$$\lim_{x\to\pm\infty} f(x)=0$$

$$\begin{aligned}
-<Kf,f> &= -\int_{-\infty}^{\infty}[(D^2-x^2)f(x)]f(x)\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} f''(x)f(x)\mathrm{d}x+\int_{-\infty}^{\infty} x^2f(x)^2\mathrm{d}x \\
&= -f'(x)f(x)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty} f'(x)^2\mathrm{d}x+\int_{-\infty}^{\infty} x^2f(x)^2\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f'(x)^2+x^2f(x)^2\mathrm{d}x
\end{aligned}$$

$\because$ $\left\{\varphi_n(x)=\dfrac{h_n(x)}{\|h_n(x)\|}\right\}$ is the orthonormal basis of L^2

$\therefore$

$$f=\sum_{n=0}^{\infty}<f,\varphi_n(x)>\varphi_n(x)$$

$\therefore$

$$\begin{aligned}
-<Kf,f> &= -<\sum_{n=0}^{\infty}<f,\varphi_n(x)>K\varphi_n(x),\sum_{n=0}^{\infty}<f,\varphi_n(x)>\varphi_n(x)> \\
&=<\sum_{n=0}^{\infty}<f,\varphi_n(x)>(2n+1)\varphi_n(x),\sum_{n=0}^{\infty}<f,\varphi_n(x)>\varphi_n(x)> \\
&= \sum_{n=0}^{\infty}(2n+1)<f,\varphi_n(x)>^2 \\
&= \sum_{n=0}^{\infty}(2n+1)\frac{<f,h_n>^2}{\|h_n\|_2^2}
\end{aligned}$$

$\therefore$

$$-<Kf,f>=\int_{-\infty}^{\infty} f'(x)^2+x^2f(x)^2\mathrm{d}x=\sum_{n=0}^{\infty}(2n+1)\frac{<f,h_n>^2}{\|h_n\|^2}$$

(b) Prove the weak Heisenberg inequality

$$\int_{-\infty}^{\infty} f'(x)^2+x^2f(x)^2\mathrm{d}x\geqslant\int_{-\infty}^{\infty} f(x)^2\mathrm{d}x$$

for such f.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)^2 \mathrm{d}x =& < f, f > \\
&=< \sum_{n=0}^{\infty} < f, \varphi_n(x) > \varphi_n(x), \sum_{n=0}^{\infty} < f, \varphi_n(x) > \varphi_n(x) > \\
&= \sum_{n=0}^{\infty} < f, \varphi_n(x) >^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{< f, h_n >^2}{\|h_n\|_2^2} \\
&\leqslant \sum_{n=0}^{\infty} (2n+1) \frac{< f, h_n >^2}{\|h_n\|_2^2} \\
&= \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \mathrm{d}x
\end{aligned}
$$

Question 5

Show that

$$
e^{2its} = e^{-t^2} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H_n(s)
$$

(**Hint:** Seek an expansion of the form

$$
e^{2its} = \sum_{n=0}^{\infty} f_n(t) H_n(s)
$$

and use orthogonality of the H_ns.)

$\because$

$$
e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}} e^{-ikx} \mathrm{d}k
$$

$\therefore$

$$
\begin{aligned}
e^{-x^2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}} e^{-ik\sqrt{2}x} \mathrm{d}k \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} e^{-2ikx} \mathrm{d}k
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
De^{-x^2} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} D^n e^{-2ikx} \mathrm{d}k \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (-2ik)^n e^{-2ikx} \mathrm{d}k
\end{aligned}
$$

$\therefore$

$$
H_n(x) = \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (2ik)^n e^{-ikx} \mathrm{d}k
$$

Solution (cont.)

The generating function of $\{H_n(s)\}$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{H_n(s)}{n!} t^n &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{e^{s^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2-2iks} (2ik)^n \mathrm{d}k \\
&= \frac{e^{s^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(2ikt)^n}{n!} e^{-k^2} e^{-2iks} \mathrm{d}k \\
&= \frac{e^{s^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} e^{ik(2s-2t)} \mathrm{d}k \\
&\xlongequal{e^{-k^2} \leftrightarrow \frac{1}{\sqrt{2}} e^{-\frac{x^2}{4}}} e^{s^2} e^{-\frac{(2s-2t)^2}{4}} \\
&= e^{-t^2+2st}
\end{aligned}$$

$\therefore$

$$e^{2ts} = e^{t^2} \sum_{n=0}^{\infty} \frac{H_n(s)}{n!} t^n$$

$\therefore$

$$e^{2its} = e^{-t^2} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H_n(s)$$

Question 6

Use Cramers inequality

$$|H_n(s)| \leqslant 1.09 \cdot 2^{\frac{n}{2}} \sqrt{n!} e^{\frac{s^2}{2}}$$

and Stirlings approximation to show that the error in N terms of the approximation in Question 5 is bounded by

$$\left| e^{2its} - \sum_{n=0}^{N-1} f_n(t) H_n(s) \right| \leqslant 10 \left(\frac{2e}{N} \right)^{\frac{N}{2}}$$

for $N > 10$, $|t| \leqslant 1$ and $|s| \leqslant 2$. How many terms are required to get 10-digit accuracy?

$$\begin{aligned}
\left| e^{2its} - \sum_{n=0}^{N-1} f_n(t) H_n(s) \right| &= \left| \sum_{n=N}^{\infty} \frac{(it)^n e^{-t^2}}{n!} H_n(s) \right| \\
&\leqslant \sum_{n=N}^{\infty} \frac{t^n e^{-t^2}}{n!} |H_n(s)| \\
&\leqslant 1.09 \cdot \sum_{n=N}^{\infty} \frac{t^n e^{-t^2}}{n!} 2^{\frac{n}{2}} \sqrt{n!} e^{\frac{s^2}{2}} \\
&\leqslant 1.09 \cdot \sum_{n=N}^{\infty} \frac{1}{\sqrt{n!}} 2^{\frac{n}{2}} e^2
\end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
&\approx 1.09 \cdot \sum_{n=N}^{\infty} \frac{1}{(2\pi n)^{\frac{1}{4}}} \left(\frac{2e}{n}\right)^{\frac{n}{2}} e^2 \\
&\leqslant \frac{1.09 \cdot e^2}{(20\pi)^{\frac{1}{4}}} \sum_{n=N}^{\infty} \left(\frac{2e}{N}\right)^{\frac{n}{2}} \\
&= \frac{1.09 \cdot e^2}{(20\pi)^{\frac{1}{4}}} \cdot \frac{\left(\frac{2e}{N}\right)^{\frac{N}{2}}}{1 - \left(\frac{2e}{N}\right)^{\frac{1}{2}}} \\
&= \frac{1.09 \cdot e^2}{(20\pi)^{\frac{1}{4}} \left[1 - \left(\frac{2e}{10}\right)^{\frac{1}{2}}\right]} \cdot \left(\frac{2e}{N}\right)^{\frac{N}{2}} \\
&\approx 10 \left(\frac{2e}{N}\right)^{\frac{N}{2}}
\end{aligned}
$$

Let

$$10 \left(\frac{2e}{N}\right)^{\frac{N}{2}} < 10^{-10}$$

we get

$$N \geqslant 30$$