STAT 150: Stochastic Processes

Fall 2017

Homework 3

Solutions by

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Exercises 3.4.1

Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let $T = \min\{n \geqslant 0 : X_n = 3\}$, $v_i = E(T|X_0 = i) \quad (i = 0, 1, 2)$

$\because$

$$\begin{cases} v_0 = 1 + 0.4v_0 + 0.3v_1 + 0.2v_2 \\ v_1 = 1 + 0.7v_1 + 0.2v_2 \\ v_2 = 1 + 0.9v_2 \end{cases}$$

$\therefore$

$$\begin{cases} v_0 = 10 \\ v_1 = 10 \\ v_2 = 10 \end{cases}$$

i.e. the mean time to reach state 3 starting from state 0 for the Markov chain is 10.

Exercises 3.4.4

A coin is tossed repeatedly until successive heads appear. Find the mean number of tosses required.
Hint: Let X_n be the cumulative number of successive heads. The state space is $0, 1, 2$ and the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let X_n be the cumulative number of successive heads, $T = \min\{n \geqslant 0 : X_n = 2\}$, $v_i = E(T|X_0 = i) \quad (i = 0, 1)$

$\because$

$$\begin{cases} v_0 = 1 + \frac{1}{2}v_0 + \frac{1}{2}v_1 \\ v_1 = 1 + \frac{1}{2}v_0 \end{cases}$$

$\therefore$

$$\begin{cases} v_0 = 6 \\ v_1 = 4 \end{cases}$$

i.e. the mean time to reach state 3 starting from state 0 for the Markov chain is 6.

Problems 3.5.1

As a special case of the successive maxima Markov chain whose transition probabilities are given in equation (3.34), consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0+a_1 & a_2 & a_3 \\ 0 & 0 & a_0+a_1+a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Starting in state 0, show that the mean time until absorption is $v_0 = \frac{1}{a_3}$.

Let $T = \min\{n \geqslant 0 : X_n = 3\}$, $v_i = E(T|X_0 = i) \quad (i = 0, 1, 2)$

$\because$

$$\begin{cases} v_0 = 1 + a_0 v_0 + a_1 v_1 + a_2 v_2 \\ v_1 = 1 + (a_0 + a_1) v_1 + a_2 v_2 \\ v_2 = 1 + (a_0 + a_1 + a_2) v_2 \end{cases}$$

$\therefore$

$$\begin{bmatrix} a_0 - 1 & a_1 & a_2 \\ 0 & a_0 + a_1 - 1 & a_2 \\ 0 & 0 & a_0 + a_1 + a_2 - 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$\therefore$

$$\begin{cases} v_0 = \dfrac{1}{1 - a_0 - a_1 - a_2} \\ v_1 = \dfrac{1}{1 - a_0 - a_1 - a_2} \\ v_2 = \dfrac{1}{1 - a_0 - a_1 - a_2} \end{cases}$$

$\because \quad a_0 + a_1 + a_2 + a_3 = 1$

$\therefore \quad v_0 = \frac{1}{a_3}$ i.e. the mean time to reach state 3 starting from state 0 for the Markov chain is $\frac{1}{a_3}$.

Problems 3.6.8

Consider the Markov chain $\{X_n\}$ whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \alpha & 0 & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$. Determine the mean time to reach state 3 starting from state 0. That is, find $E[T|X_0 = 0]$ where $T = \min\{n \geqslant 0; X_n = 3\}$.

Let $v_i = E(T|X_0 = i) \quad (i = 0, 1, 2)$,

$\because$

$$\begin{cases} v_0 = 1 + \alpha v_0 + \beta v_2 \\ v_1 = 1 + \alpha v_0 \\ v_2 = 1 + \alpha v_0 + \beta v_1 \end{cases}$$

$$\alpha + \beta = 1$$

$\therefore$

$$\begin{bmatrix} 1-\alpha & 0 & -1+\alpha \\ -\alpha & 1 & 0 \\ -\alpha & -1+\alpha & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$

$$\begin{cases} v_0 = \dfrac{1}{1-\alpha} + \dfrac{1}{(1-\alpha)^2} + \dfrac{1}{(1-\alpha)^3} = \dfrac{1-(1-\alpha)^3}{\alpha(1-\alpha)^3} \\ v_1 = \dfrac{1}{(1-\alpha)^3} \\ v_2 = \dfrac{1}{(1-\alpha)^2} + \dfrac{1}{(1-\alpha)^3} \end{cases}$$

$\therefore \quad v_0 = \frac{1-(1-\alpha)^3}{\alpha(1-\alpha)^3}$ i.e. the mean time to reach state 3 starting from state 0 for the Markov chain is $\frac{1-(1-\alpha)^3}{\alpha(1-\alpha)^3}$.

Q1

In this question, we aim to show whether a simple random walk in $\mathbb{Z}^d$ is recurrent or not, for $d \in \mathbb{Z}^+$.
Let $\{X_n\}$ be a simple random walk with state space $S = \mathbb{Z}^d$ starting from $X_0 = \vec{0}$. Then by denition of simple random walk, $\mathbb{P}(X_{n+1} - X_n = \omega I_k) = \dfrac{1}{2d}$, for all $\omega \in \{+1, -1\}$ and $I_k \in \mathbb{Z}^d$ with only its k-th component to be 1 while all the other(s) to be 0.

1. Let T_k be the time that X returns to 0 for the k-th time, or ∞ when it only returns to 0 less than k times. Let $p = \mathbb{P}(T_1 < \infty)$. Prove that

$$p = \mathbb{P}(T_2 < \infty | T_1 = k) = \mathbb{P}(T_2 < \infty | T_1 < \infty), \qquad \forall k \in \mathbb{Z}^+;$$

$\because \quad$ given $X_0 = \vec{0}$,

$$\begin{aligned} \mathbb{P}(T_2 < \infty | T_1 = k) &= \frac{\mathbb{P}(T_1 = k, T_2 < \infty)}{\mathbb{P}(T_1 = k)} \\ &= \sum_{n=1}^{\infty} \frac{\mathbb{P}(X_{k+n} = \vec{0}, X_{k+n-1} \neq \vec{0}, \cdots, X_{k+1} \neq \vec{0}, X_k = \vec{0}, X_{k-1} \neq \vec{0}, \cdots, X_1 \neq \vec{0} | X_0 = \vec{0})}{\mathbb{P}(X_k = \vec{0}, X_{k-1} \neq \vec{0}, \cdots, X_1 \neq \vec{0} | X_0 = \vec{0})} \end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \mathbb{P}(X_{k+n} = \vec{0}, X_{k+n-1} \neq \vec{0}, \cdots, X_{k+1} \neq \vec{0} | X_k = \vec{0}, X_{k-1} \neq \vec{0}, \cdots, X_1 \neq \vec{0}, X_0 = \vec{0}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_{k+n} = \vec{0}, X_{k+n-1} \neq \vec{0}, \cdots, X_{k+1} \neq \vec{0} | X_k = \vec{0}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n = \vec{0}, X_{n-1} \neq \vec{0}, \cdots, X_1 \neq \vec{0} | X_0 = \vec{0}) \\
&= \mathbb{P}(T_1 < \infty)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(T_2 < \infty | T_1 < \infty) &= \frac{\mathbb{P}(T_1 < \infty, T_2 < \infty)}{\mathbb{P}(T_1 < \infty)} \\
&= \frac{1}{p} \sum_{k=1}^{\infty} \mathbb{P}(T_1 = k, T_2 < \infty) \\
&= \frac{1}{p} \sum_{k=1}^{\infty} \mathbb{P}(T_2 < \infty | T_1 = k) P(T_1 = k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(T_1 = k) \\
&= \mathbb{P}(T_1 < \infty)
\end{aligned}
$$

$\therefore$

$$
p = \mathbb{P}(T_2 < \infty | T_1 = k) = \mathbb{P}(T_2 < \infty | T_1 < \infty) \qquad \forall\ k \in \mathbb{Z}^+
$$

2. Let $V = \max\{k : T_k < \infty\}$, where we adopt the convention that $\max \varnothing = 0$. Find the distribution of V in terms of p;

It's easy to know that if $T_n < \infty$, then $T_{n-1} < \infty, \cdots, T_1 < \infty$
Similar to 1, we have

$$
p = \mathbb{P}(T_{n+1} < \infty | T_n = k) = \mathbb{P}(T_{n+1} < \infty | T_n < \infty)
$$

because

$$
\begin{aligned}
\mathbb{P}(T_{n+1} < \infty | T_n = k) &= \sum_{n=1}^{\infty} \mathbb{P}(X_{k+n} = \vec{0}, X_{k+n-1} \neq \vec{0}, \cdots, X_{k+1} \neq \vec{0} | X_k = \vec{0}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n = \vec{0}, X_{n-1} \neq \vec{0}, \cdots, X_1 \neq \vec{0} | X_0 = \vec{0}) \\
&= \mathbb{P}(T_1 < \infty) \\
&= p
\end{aligned}
$$

Solution (cont.)

$$
\begin{aligned}
\mathbb{P}(T_{n+1} < \infty | T_n < \infty) &= \frac{\mathbb{P}(T_n < \infty, T_{n+1} < \infty)}{\mathbb{P}(T_n < \infty)} \\
&= \frac{\sum\limits_{k=1}^{\infty} \mathbb{P}(T_n = k, T_{n+1} < \infty)}{\mathbb{P}(T_n < \infty)} \\
&= \frac{\sum\limits_{k=1}^{\infty} \mathbb{P}(T_{n+1} < \infty | T_n = k)\mathbb{P}(T_n = k)}{\mathbb{P}(T_n < \infty)} \\
&= \frac{p\sum\limits_{k=1}^{\infty} \mathbb{P}(T_n = k)}{\mathbb{P}(T_n < \infty)} \\
&= \frac{p\mathbb{P}(T_n < \infty)}{\mathbb{P}(T_n < \infty)} \\
&= p
\end{aligned}
$$

$\therefore \quad \forall\ k \in \mathbb{N}^+$,

$$
\begin{aligned}
\mathbb{P}(V = k) &= \mathbb{P}(T_1 < \infty, \cdots, T_k < \infty, T_{k+1} = \infty) \\
&= \mathbb{P}(T_1 < \infty, \cdots, T_k < \infty) - \mathbb{P}(T_1 < \infty, \cdots, T_k < \infty, T_{k+1} < \infty) \\
&= \mathbb{P}(T_k < \infty | T_{k-1} < \infty) \cdots \mathbb{P}(T_2 < \infty | T_1 < \infty)\mathbb{P}(T_1 < \infty) \\
&\quad - \mathbb{P}(T_{k+1} < \infty | T_k < \infty)\mathbb{P}(T_k < \infty | T_{k-1} < \infty) \cdots \mathbb{P}(T_2 < \infty | T_1 < \infty)\mathbb{P}(T_1 < \infty) \\
&= p^k - p^{k+1} \\
&= p^k(1-p)
\end{aligned}
$$

$\because$

$$\mathbb{P}(V = 0) = \mathbb{P}(T_1 = \infty) = 1 - p$$

$$\mathbb{P}(V = k) = (1-p)p^k, \qquad k \in \mathbb{N}$$

3. Recall $p_{00}^{(k)} = \mathbb{P}(X_k = \vec{0})$. Show that

$$p_{00}^{(2n)} = \sum_{l_1 + l_2 + \cdots + l_d = n} \binom{2n}{n} \binom{n}{l_1 l_2 \cdots l_d}^2 \cdot \frac{1}{(2d)^{2n}};$$

Here,

$$\binom{n}{l_1 l_2 \cdots l_d} = \frac{n!}{l_1! l_2! \cdots l_d!}$$

is the multinomial coefficient that denotes the number of ways of colouring n labelled objects in d colours, with l_i in colour i.

Given $X_0 = \vec{0}$, the Markov chain goes to $X_{2n} = \vec{0}$. Then in every dimension k, the chain must go one direction I_k for l_k times and go to another direction $-I_k$ for the same times, i.e., l_k times.
We have

$$2l_1 + 2l_2 + \cdots + 2l_d = 2n$$

Solution (cont.)

i.e.

$$l_1 + l_2 + \cdots + l_d = n$$

In every step, the Markov chian may move in in 2 directions of d dimensions. Totally $2d$ choices, and in this specific case the number of moves to one choice should be $l_1, l_1, l_2, l_2, \cdots, l_d, l_d$ respectively. And the permutation is

$$\begin{aligned}\binom{2n}{l_1\ l_1\ l_2\ l_2 \cdots l_d\ l_d} &= \frac{(2n)!}{[(l_1)!]^2[(l_2)!]^2 \cdots [(l_d)!]^2} \\ &= \frac{(2n)!}{n!n!} \cdot \left(\frac{n!}{l_1!l_2! \cdots l_d!}\right)^2 \\ &= \binom{2n}{n}\binom{n}{l_1 l_2 \cdots l_d}^2\end{aligned}$$

$\because$

$$\mathbb{P}(X_{n+1} - X_n = I_k) = \mathbb{P}(X_{n+1} - X_n = -I_k) = \frac{1}{2d}$$

$\therefore$

$$p_{00}^{(2n)} = \sum_{l_1+l_2+\cdots+l_d=n} \binom{2n}{n}\binom{n}{l_1 l_2 \cdots l_d}^2 \frac{1}{(2d)^{2n}}$$

4. Let $G = \sum_{k=1}^{\infty} p_{00}^{(k)}$. Show that the following conditions are equivalent:

   (a) $G < \infty$;

   (b) $p < 1$;

   (c) $\mathbb{P}(V = 0) > 0$;

   (d) X_n does not return to the origin with positive probability.

$(a) \iff (b)$

$\because$

$$\mathbb{P}(T_k < \infty) = \mathbb{P}(T_k < \infty | T_{k-1} < \infty) \cdots \mathbb{P}(T_2 < \infty | T_1 < \infty)\mathbb{P}(T_1 < \infty) = p^k$$

$$\begin{aligned}G &= \sum_{k=1}^{\infty} p_{00}^{(k)} \\ &= \sum_{k=1}^{\infty} \mathbb{P}(X_k = \vec{0} | X_0 = \vec{0}) \\ &= \sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathbb{P}(T_i = k) \\ &= \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} \mathbb{P}(T_k = i) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(T_i < \infty) \\ &= \sum_{i=1}^{\infty} (1-p)p^k < \infty\end{aligned}$$

Solution (cont.)

$\therefore$

$$\begin{aligned} G < \infty &\iff \sum_{i=1}^{\infty}(1-p)p^k < \infty \\ &\iff p < 1 \end{aligned}$$

$(b) \iff (c)$

From 2 we have $\mathbb{P}(V = k) = (1-p)p^k \qquad (k \in \mathbb{N})$,

$\therefore$

$$p < 1 \iff \mathbb{P}(V = 0) = 1 - p > 0$$

$(c) \iff (d)$

$$\begin{aligned} \mathbb{P}(V = 0) > 0 &\iff \mathbb{P}(T_1 = \infty) > 0 \\ &\iff X_n \text{ does not return to the origin with positive probability} \end{aligned}$$

$(d) \iff (a)$

From the definition of transient, we have

$$\begin{aligned} G < \infty &\iff \text{state } \vec{0} \text{ is transient} \\ &\iff \mathbb{P}(X_n = \vec{0} \text{ for some } n \geqslant 1 | X_0 = \vec{0}) < 1 \\ &\iff \mathbb{P}(X_n \text{ does not return to the origin} | X_0 = \vec{0}) \\ &\quad = 1 - \mathbb{P}(X_n = \vec{0} \text{ for some } n \geqslant 1 | X_0 = \vec{0}) > 0 \\ &\iff X_n \text{ does not return to the origin with positive probability} \end{aligned}$$

5. Use $Stirlings\ formula$ $m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$ to show that the 2-dimensional simple random walk is recurrent(i.e., the origin is a recurrent state for this Markov chain);

From 3 we have

$$\begin{aligned} p_{00}^{2n} &= \sum_{l_1+l_2=n} \binom{2n}{n} \binom{n}{l_1 l_2}^2 \frac{1}{4^{2n}} \\ &= \sum_{i=0}^{n} \binom{2n}{n} \binom{n}{i}^2 \frac{1}{4^{2n}} \\ &= \frac{\binom{2n}{n}^2}{4^{2n}} \\ &= \frac{1}{4^{2n}} \left(\frac{(2n)!}{n!n!} \right)^2 \\ &\approx \frac{1}{4^{2n}} \left(\frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n} 2\pi n} \right)^2 \\ &= \frac{1}{4^{2n}} \left(\frac{2^{2n}}{\sqrt{\pi n}} \right)^2 \end{aligned}$$

Solution (cont.)

$$= \frac{1}{\pi n}$$

and

$$p_{00}^{2n+1} = 0$$

because there is at least a odd number of moves in one dimension.

$\because$

$$\sum_{n=1}^{\infty} \frac{1}{\pi n} = \infty$$

$\therefore$

$$\sum_{n=1}^{\infty} p_{00}^{n} = \infty$$

i.e. the 2-dimensional simple random walk is recurrent

6. Use *Stirlings formula* and the bound (valid for any (l_1, l_2, l_3) that sum to $3n$)

$$\binom{3n}{l_1 l_2 l_3} \leqslant \binom{3n}{nnn}$$

to show that the 3-dimensional simple random walk is transient;

From 3 we have

$$\begin{aligned} p_{00}^{2n} &= \sum_{l_1+l_2+l_3=n} \binom{2n}{n} \binom{n}{l_1 l_2 l_3}^2 \frac{1}{6^{2n}} \\ &= \frac{1}{6^{2n}} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(2n)!}{[i!j!(n-i-j)!]^2} \\ &\leqslant \frac{1}{2^{2n}} \binom{2n}{n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \left(\frac{1}{3^n} \frac{n!}{i!j!(n-i-j)!} \right)^2 \end{aligned}$$

$\because$

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{1}{3^n} \frac{n!}{i!j!(n-i-j)!} = 1$$

$\therefore$

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} \left(\frac{1}{3^n} \frac{n!}{i!j!(n-i-j)!} \right)^2 \leqslant \max_{i,j} \frac{1}{3^n} \frac{n!}{i!j!(n-i-j)!}$$

$\therefore$

$$p_{00}^{2n} \leqslant \frac{1}{2^{2n}} \binom{2n}{n} \max_{i,j} \frac{1}{3^n} \frac{n!}{i!j!(n-i-j)!}$$

Solution (cont.)

Let $m = \left\lceil \frac{n}{3} \right\rceil$, i.e. $\frac{n}{3} \leqslant m < \frac{n}{3} + 1$, i.e. $n \leqslant 3m < n + 3$, then we have

$$\begin{aligned} p_{00}^{2n} &\leqslant \frac{1}{2^{2n}} \binom{2n}{n} \max_{i,j} \frac{1}{3^{3m-2}} \frac{(3m)!}{(i+3m-n)!j!(n-i-j)!} \\ &\leqslant \frac{1}{2^{2n}3^{3m-2}} \binom{2n}{n} \frac{(3m)!}{m!m!m!} \\ &\approx \frac{1}{2^{2n}3^{3m-2}} \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n} 2\pi n} \frac{\left(\frac{3m}{e}\right)^{3m} \sqrt{6\pi m}}{\left(\frac{m}{e}\right)^{3m} (2\pi m)^{\frac{3}{2}}} \\ &= \frac{9\sqrt{6}}{(2\pi m)^{\frac{3}{2}}} \\ &\leqslant \frac{9\sqrt{6}}{\left(\frac{2}{3}\pi n\right)^{\frac{3}{2}}} \end{aligned}$$

and

$$p_{00}^{2n+1} = 0$$

because there is at least a odd number of moves in one dimension.

$\because$

$$\sum_{n=1}^{\infty} p_{00}^{2n} = \sum_{k=1}^{\infty} \frac{9\sqrt{6}}{\left(\frac{2}{3}\pi 2k\right)^{\frac{3}{2}}} < \infty$$

$\therefore$ the 3-dimensional simple random walk is transient

7. What about $d \geqslant 4$? Provide a proof if you can.

Define $S_n = (X_n^{(1)}, X_n^{(2)}, X_n^{(3)})^T$ where $X_n^{(i)}$ is the i-th dimension of X_n and $S_0 = \vec{0}$.

$\because$ $\forall\, k \in \{1, 2, 3\}$,

$$\mathbb{P}(X_{n+1} - X_n = \omega I_k) = \frac{1}{2d}$$

$\therefore$ S_n is a 3-dimensional simple random walk

$\because$ we have proved in 6 that the 3-dimensional simple random walk is transient, i.e., S_n is transient which implies that $\exists\, n \in \mathbb{N}^+,\ s.t.$

$$T_1', \cdots, T_n' < \infty,\ T_{n+1}', T_{n+2}', \cdots = \infty$$

where T_k' is the time that S_n returns to 0 for the $k-th$ time

And we have

$$T_k \geqslant T_k'$$

$\therefore$

$$T_{n+1}, T_{n+2}, \cdots = \infty$$

i.e. X_n is transient

2

The distinct pair i, j of states of a Markov chain is called *symmetric* if

$$\mathbb{P}_j(T_j < T_i) = \mathbb{P}_i(T_i < T_j),$$

where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | X_0 = i)$ and $T_i = \min\{n \geqslant 1 : X_n = i\}$.
Given $X_0 = i$ and i, j symmetric and recurrent, find the expected number of visits to j before the chain revisits i.

Let

$$p = \mathbb{P}_j(T_j < T_i) = \mathbb{P}_i(T_i < T_j)$$

Let N_{ij} denotes the number of visits to j before the chain revisits i, then $X_0 = i, N_{ij} = n$ correspondes to the paths such that $i \to j$, then $j \to j$ for $n-1$ times then $j \to i$

$\because \quad \forall\, n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}_i(N_{ij} = n) &= \mathbb{P}_i(T_i > T_j)[\mathbb{P}_j(T_j < T_i)]^{n-1}\mathbb{P}_j(T_i < T_j) \\
&= [1 - \mathbb{P}_i(T_i < T_j)][\mathbb{P}_j(T_j < T_i)]^{n-1}[1 - \mathbb{P}_j(T_j < T_i)] \\
&= (1-p)p^{n-1}(1-p) \\
&= (1-p)^2 p^{n-1}
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\mathbb{E}(N_{ij} | X_0 = i) &= \sum_{n=0}^{\infty} n\mathbb{P}_i(N_{ij} = n) \\
&= \sum_{n=0}^{\infty} n(1-p)^2 p^{n-1} \\
&= (1-p)^2 \sum_{n=0}^{\infty} np^{n-1} \\
&= (1-p)^2 \frac{1}{(1-p)^2} \\
&= 1
\end{aligned}$$