Stochastic Processes

Fall 2017

•

Week 3

•

Solutions by

Jinhong Du

15338039

2.5

Suppose that $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t) + N_2(t), t \geqslant 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geqslant 0\}$ is $\frac{\lambda_1}{\lambda_1+\lambda_2}$, independently of the time of the event.

Let $\{N_1(t), t \geqslant 0\}, \{N_2(t), t \geqslant 0\}$ denote the independent Poisson precesses with rates λ_1 and λ_2 respetively. Then
(1) for any time points $t_0 = 0 < t_1 < \cdots < t_n$,

$$N_1(t_1) - N_1(t_0)\ , N_1(t_2) - N_1(t_1), \cdots,\ N_1(t_n) - N_1(t_{n-1})$$

are independent and

$$N_2(t_1) - N_2(t_0)\ , N_2(t_2) - N_2(t_1), \cdots,\ N_2(t_n) - N_2(t_{n-1})$$

are independent;
(2) for $s \geqslant 0$ and $t > 0$, $N_1(s+t) - N_1(s) \sim Poisson(\lambda_1 t)$ and $N_1(s+t) - N_1(s) \sim Poisson(\lambda_2 t)$;
(3) $N_1(0) = N_2(0) = 0$;
and $N(t) = N_1(t) + N_2(t)$.
(1$*$)
$\because$ $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent
$\therefore$ for any time points $t_0 = 0 < t_1 < \cdots < t_n$,

$$\begin{aligned}
N(t_1) - N(t_0) &= N_1(t_1) - N_1(t_0) + N_2(t_1) - N_2(t_0) \\
N(t_2) - N(t_1) &= N_1(t_2) - N_1(t_1) + N_2(t_2) - N_2(t_1) \\
&\vdots \\
N(t_n) - N(t_{n-1}) &= N_1(t_n) - N_1(t_{n-1}) + N_2(t_n) - N_2(t_{n-1})
\end{aligned}$$

are independent
(2$*$) for $s \geqslant 0$ and $t > 0$,
$\because$ $\forall\ k \in \mathbb{N}$,

$$\begin{aligned}
Pr\{N(s+t) - N(s) = k\} &= Pr\{N_1(s+t) - N_1(s) + N_2(s+t) - N_2(s) = k\} \\
&= \sum_{i=0}^{k} Pr\{N_1(s+t) - N_1(s) = i, N_2(s+t) - N_2(s) = k-i\} \\
&= \sum_{i=0}^{k} Pr\{N_1(s+t) - N_1(s) = i\} Pr\{N_2(s+t) - N_2(s) = k-i\} \\
&= \sum_{i=0}^{k} \frac{(\lambda_1 t)^i e^{-\lambda_1 t}}{i!} \frac{(\lambda_2 t)^{k-i} e^{-\lambda_2 t}}{(k-i)!} \\
&= \frac{t^k e^{-(\lambda_1+\lambda_2)t}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} \\
&= \frac{(\lambda_1+\lambda_2)^k t^k}{k!} e^{-(\lambda_1+\lambda_2)t}
\end{aligned}$$

$\therefore$ $N(s+t) - N(s) \sim Poisson((\lambda_1+\lambda_2)t)$
(3$*$)

$$N(0) = N_1(0) + N_2(0) = 0$$

Solution (cont.)

Therefore $\{N(t), t \geqslant 0\}$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

$$\begin{aligned}
\mathbb{P}\{N_1(t) = 1 | N(t) = 1\} &= \frac{\mathbb{P}\{N_1(t) = 1, N_2(t) = 0\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\lambda_1 t e^{-\lambda_2 t} e^{-\lambda_2 t}}{(\lambda_1 + \lambda_2) t e^{-(\lambda_1 + \lambda_2) t}} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

2.10

Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop then it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.

(a) Compute the expected time from when you arrive at the bus stop until you reach home.

Let $S_1 = X_1$ denotes the waiting time until the 1st bus arrives and T denotes the time from when you arrive at the bus stop until you reach home. Then

$$T = \begin{cases} S_1 + R & , S_1 \leqslant s \\ s + W & , S_1 > s \end{cases}$$

$\because \quad S_1 \sim \Gamma(1, \lambda)$

$\therefore$

$$\begin{aligned}
\mathbb{E}T &= \int_0^s \lambda e^{-\lambda t}(t + R)\mathrm{d}t + \int_s^\infty \lambda e^{-\lambda t}(s + W)\mathrm{d}t \\
&= -s e^{-\lambda s} - \frac{1}{\lambda} e^{-\lambda s} + \frac{1}{\lambda} + R(1 - e^{-\lambda s}) + (s + W) e^{-\lambda s} \\
&= \frac{1 - e^{-\lambda s}}{\lambda} + (W - R) e^{-\lambda s} + R
\end{aligned}$$

(b) Show that if $W < \frac{1}{\lambda} + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > \frac{1}{\lambda} + R$ then it is minimized by letting $s = \infty$ (that is, you continue to wait for the bus), and when $W = \frac{1}{\lambda} + R$ all values of s give the same expected time.

Let $f(s) = \mathbb{E}T, \ (s \geqslant 0)$.

$\because$

$$\begin{aligned}
f'(s) &= e^{-\lambda s} - \lambda(W - R) e^{-\lambda s} \\
&= \lambda \left(\frac{1}{\lambda} + R - W \right) e^{-\lambda s}
\end{aligned}$$

Solution (cont.)

(1) $W < \frac{1}{\lambda} + R$

$\because \quad f'(s) > 0$

$\therefore \quad f(s)$ increases as s increases

$\therefore \quad f(s)$ is minimized when $s = 0$

(2) $W > \frac{1}{\lambda} + R$

$\because \quad f'(s) < 0$

$\therefore \quad f(s)$ decreases as s increases

$\therefore \quad f(s)$ is minimized when $s = \infty$

(3) $W = \frac{1}{\lambda} + R$

$\because \quad f'(s) \equiv 0$

$\therefore \quad f(s) \equiv c$, where c is a constant. I.e., all values of s given the same expected time.

(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.

From the memoryless property of the exponential distribution, if it was worth waiting some time $s_0 > 0$ for a bus, and the bus has not arrived at s_0, then resetting time suggests that it must be worth waiting another s_0 time units. Thus, if the optimal s is positive, it must be infinite.

2.12

Events, occurring according to a Poisson process with rate λ, are registered by a counter. However, each time an event is registered the counter becomes inoperative for the next b units of time and does not register any new events that might occur during that interval. Let $R(t)$ denote the number of events that occur by time t and are registered.

(a) Find the probability that the first k events are all registered.

$$
\begin{aligned}
\mathbb{P}\{R(t) = k, N(t) = k\} &= \mathbb{P}\left\{X_2 \geqslant b, \cdots, X_k \geqslant b\right\} \\
&= \prod_{n=2}^{k} \mathbb{P}\{X_n \geqslant b\} \\
&= \prod_{n=2}^{k} e^{-\lambda b} \\
&= e^{-\lambda(k-1)b}
\end{aligned}
$$

(b) For $t \geqslant (n-1)b$, find $\mathbb{P}\{R(t) \geqslant n\}$.

For $t \geqslant (n-1)b$, let $U(t)$ denotes the number of events that occur by time t and are not registered.
$\because$ the event that occurs at time s and is registered has pobability

$$P(s) = \begin{cases} 0 & \text{, the counter is inoperative at time } s \\ 1 & \text{, the counter is operative at time } s \end{cases}$$

$\therefore$ by Proposition 2.3.2, $R(t)$ and $U(t)$ are independent Poisson random variables having respective parameters λtp and $\lambda t(1-p)$, where

$$\begin{aligned} p &= \frac{1}{t}\int_0^t P(s)\mathrm{d}s \\ &= \frac{t-(n-1)b}{t} \end{aligned}$$

$\therefore$

$$\begin{aligned} \mathbb{P}\{R(t) \geqslant n\} &= \sum_{k=n}^{\infty} \mathbb{P}\{N(t-(n-1)b) = k\} \\ &= \sum_{k=n}^{\infty} \frac{[t-(n-1)b]^k \lambda^k}{k!} e^{-\lambda[t-(n-1)b]} \end{aligned}$$