

STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2019
LECTURE 5

1. SOLVING LINEAR SYSTEMS

- assuming that we want to solve a linear system $A\mathbf{x} = \mathbf{b}$ that is known to be consistent, i.e. $\mathbf{b} \in \text{im}(A)$
- first we form the matrix-vector product $\mathbf{c} = U^*\mathbf{b}$
- let $\mathbf{y} = V^*\mathbf{x}$
- then $A\mathbf{x} = \mathbf{b}$ becomes $\Sigma\mathbf{y} = \mathbf{c}$
- since $A\mathbf{x} = \mathbf{b}$ is consistent, so

$$\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_n \end{bmatrix}$$

is consistent, which means that we must have $c_{r+1} = \dots = c_m = 0$, i.e.

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- now let

$$\mathbf{y} = \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix}$$

where y_{r+1}, \dots, y_n are free parameters

- all solutions of $A\mathbf{x} = \mathbf{b}$ are given by

$$\mathbf{x} = V\mathbf{y}$$

for some choices of y_{r+1}, \dots, y_n

- to see this, observe that

$$A\mathbf{x} = U\Sigma V^*V\mathbf{y} = U\Sigma\mathbf{y} = U \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix} = U \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = UU^*\mathbf{b} = \mathbf{b}$$

2. INVERTING NONSINGULAR MATRICES

- note that only square matrices could have inverses, when we say A is nonsingular or invertible, the fact that A is square is implied
- by the way, the set of all nonsingular matrices in $\mathbb{C}^{n \times n}$ forms group under matrix multiplication
- it is called **the general linear group** and denoted

$$\text{GL}(n) := \{A \in \mathbb{C}^{n \times n} : \det(A) \neq 0\}$$

- if $A \in \mathbb{C}^{n \times n}$ is nonsingular, then $\text{rank}(A) = n$ and so $\sigma_1, \dots, \sigma_n$ are all non-zero
- so

$$A^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^*$$

where

$$\Sigma^{-1} = \text{diag}(\sigma_1^{-1}, \dots, \sigma_n^{-1})$$

3. COMPUTING MATRIX 2-NORM AND F -NORM

- recall the definition of the matrix 2-norm,

$$\|A\|_2 = \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$$

which is also called the spectral norm

- we examine the expression

$$\|A\mathbf{x}\|_2^2 = \mathbf{x}^* A^* A \mathbf{x}$$

- the matrix A^*A is Hermitian and positive semidefinite, i.e. $\mathbf{x}^*(A^*A)\mathbf{x} \geq 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$
- exercise: show that if a matrix M is Hermitian positive semidefinite, then its EVD and SVD coincide
- as such, A^*A has SVD given by

$$A^*A = V\Sigma V^*$$

where V is a unitary matrix whose columns are the eigenvectors of A^*A , and Σ is a diagonal matrix of the form

$$\begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_n^2 & \\ & & & 0 \end{bmatrix}$$

where each σ_i^2 is nonnegative and an eigenvalue of A^*A

- these eigenvalues can be ordered such that

$$\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_r^2 > 0, \quad \sigma_{r+1}^2 = \dots = \sigma_n^2 = 0,$$

where $r = \text{rank}(A)$

- let $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$ and let $\mathbf{w} = V^* \mathbf{x}$, then we obtain

$$\begin{aligned} \frac{\|A\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} &= \frac{\mathbf{x}^* A^* A \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \\ &= \frac{\mathbf{x}^* V \Sigma V^* \mathbf{x}}{\mathbf{x}^* V V^* \mathbf{x}} \\ &= \frac{\mathbf{w}^* \Sigma \mathbf{w}}{\mathbf{w}^* \mathbf{w}} \\ &= \frac{\sum_{i=1}^n \sigma_i^2 |w_i|^2}{\sum_{i=1}^n |w_i|^2} \\ &\leq \sigma_1^2 \end{aligned}$$

- exercise: show that if a_1, \dots, a_n are nonnegative numbers, then $LHS \leq \frac{\max\{a_1, \dots, a_n\} (x_1 + \dots + x_n)}{x_1 + \dots + x_n}$

$$\max_{x_i \geq 0} \frac{a_1 x_1 + \dots + a_n x_n}{x_1 + \dots + x_n} = \max(a_1, \dots, a_n)$$

where the maximum is taken over $x_1, \dots, x_n \in [0, \infty)$ not all 0.

- it follows that

$$\frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \sigma_1$$

for all nonzero \mathbf{x}

- since V is a unitary matrix, it follows that there exists an \mathbf{x} such that

$$\mathbf{w} = V^* \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1$$

- in which case

$$\mathbf{x}^* A^* A \mathbf{x} = \mathbf{e}_1^* \Sigma \mathbf{e}_1 = \sigma_1^2$$

- in fact, this vector \mathbf{x} is the eigenvector of $A^* A$ corresponding to the eigenvalue σ_1^2
- we conclude that

$$\|A\|_2 = \sigma_1$$

- note that we have also shown that

$$\|A\|_2 = \sqrt{\rho(A^* A)}$$

since the eigenvalues of $A^* A$ are simply the squares of the singular values of A by Lemma ??

- another way to arrive at this same conclusion is to use the fact in Homework 1 that the 2-norm of a vector is invariant under multiplication by a unitary matrix, i.e. if $Q^* Q = I$, then $\|\mathbf{x}\|_2 = \|Q\mathbf{x}\|_2$, from which it follows that

$$\|A\|_2 = \|U \Sigma V^*\|_2 = \|\Sigma\|_2 = \sigma_1$$

- by the same problem in Homework 1, the Frobenius norm is also unitarily invariant
- this yields an expression in terms of singular values

$$\|A\|_F = \|U \Sigma V^*\|_F = \|\Sigma\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$$

where $r = \text{rank}(A)$

4. COMPUTING OTHER MATRIX NORMS

- the Schatten and Ky Fan norms can be expressed in terms of singular values
- this is cheating a bit, because we will in fact *define* them in terms of singular values
- for any $p \in [1, \infty]$, the **Schatten p -norm** of $A \in \mathbb{C}^{m \times n}$ is

$$\|A\|_{\sigma,p} := \left[\sum_{i=1}^{\min(m,n)} \sigma_i(A)^p \right]^{1/p}$$

and for $p = \infty$, we have

$$\|A\|_{\sigma,\infty} = \max\{\sigma_1(A), \dots, \sigma_{\min(m,n)}(A)\}$$

- of course, the sum will stop at $r = \text{rank}(A)$ since $\sigma_{r+1}(A) = \dots = \sigma_{\min(m,n)}(A) = 0$
- as usual, we also have

$$\lim_{p \rightarrow \infty} \|A\|_{\sigma,p} = \|A\|_{\sigma,\infty} \quad (4.1)$$

for all $A \in \mathbb{C}^{m \times n}$

- for the special values of $p = 1, 2, \infty$, we have
 - $p = 1$: nuclear norm

$$\|A\|_{\sigma,1} = \sigma_1(A) + \dots + \sigma_{\min(m,n)}(A) = \|A\|_*$$

- $p = 2$: Frobenius norm

$$\|A\|_{\sigma,2} = \sqrt{\sigma_1(A)^2 + \dots + \sigma_{\min(m,n)}(A)^2} = \|A\|_F$$

- $p = \infty$: spectral norm

$$\|A\|_{\sigma,\infty} = \max\{\sigma_1(A), \dots, \sigma_{\min(m,n)}(A)\} = \sigma_1(A) = \|A\|_2$$

- in fact one may also define ‘Schatten p -norm’ for values of $p \in [0, 1)$ by dropping the root $1/p$ outside the sum

$$\|A\|_{\sigma,p} := \sum_{i=1}^{\min(m,n)} \sigma_i(A)^p$$

- these will not be norms in the usual sense of the word because they do not satisfy the triangle inequality but they satisfy an analogue of (4.1)

$$\lim_{p \rightarrow 0} \|A\|_{\sigma,p} = \text{rank}(A)$$

- we may regard matrix rank as the ‘Schatten 0-norm’ since

$$\|A\|_{\sigma,0} = \sigma_1(A)^0 + \dots + \sigma_{\min(m,n)}(A)^0 = \text{rank}(A)$$

provided we define $0^0 := 0$

- ‘Schatten p -norm’ for $p < 1$ are useful in the same way ‘ ℓ^p -norms’ are useful for $p < 1$, namely, as continuous surrogates of matrix rank (vector ‘ ℓ^p -norms’ for $p < 1$ are often used as continuous surrogates as the ℓ^0 -norm, i.e. a count of the number of nonzero entries), which is a discontinuous function on $\mathbb{C}^{m \times n}$
- the nuclear norm, i.e. Schatten 1-norm, being convex in addition to continuous, is the most popular surrogate for matrix rank
- for any $p \in [1, \infty)$ and $k \in \mathbb{N}$ the **Ky Fan (p, k) -norm** of $A \in \mathbb{C}^{m \times n}$ is

$$\|A\|_{\sigma,p,k} := \left[\sum_{i=1}^k \sigma_i(A)^p \right]^{1/p}$$

- clearly for any $A \in \mathbb{C}^{m \times n}$,

$$\|A\|_{\sigma,p,\infty} = \|A\|_{\sigma,p,\min(m,n)} = \|A\|_{\sigma,p}$$

and so Ky Fan norms generalize Schatten norms

5. COMPUTING MAGNITUDE OF DETERMINANT

- note that determinants are only defined for square matrices $A \in \mathbb{C}^{n \times n}$
- recall the formula

$$\det(A) = \prod_{i=1}^n \lambda_i(A) \quad (5.1)$$

where $\lambda_i(A) \in \mathbb{C}$ is the i th eigenvalue of A

- recall that every $n \times n$ matrix has exactly n eigenvalues counted with multiplicity
- by convention we usually order eigenvalues in decreasing order of magnitudes, i.e.

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \dots \geq |\lambda_n(A)|$$

- now if we have only the singular values of A we cannot get $\det(A)$ like we did with its eigenvalues in (5.1)
- however we can get the magnitude of the determinant

$$|\det(A)| = |\det(U\Sigma V^*)| = |\det(U) \det(\Sigma) \det(V)| = \det(\Sigma) = \det \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} = \prod_{i=1}^n \sigma_i(A)$$

since $|\det(U)| = |\det(V)| = 1$

- exercise: show that all eigenvalues of a unitary matrix U must have absolute value 1 and so $|\det(U)| = 1$ by (5.1)

6. COMPUTING PSEUDOINVERSE

- as we mentioned above, only square matrices $A \in \mathbb{C}^{n \times n}$ may have an inverse, i.e. $X \in \mathbb{C}^{n \times n}$ such that

$$AX = XA = I$$

$$AB = BA = I \\ B = BA A^{-1} = A^{-1}$$

- if such an X exists, we denote it by A^{-1}
- exercise: show that A^{-1} , if exist, must be unique
- can we define something that behaves like an inverse (we will say exactly what this means in the next lecture when we discuss minimum length least squares problem) for square matrices that are not invertible in the traditional sense?
- more generally, can we define some kind of inverse for rectangular matrices?
- such considerations lead us to the notion of pseudoinverse
- the most famous one is the **Moore–Penrose pseudoinverse**

Theorem 1 (Moore–Penrose). *For any $A \in \mathbb{C}^{m \times n}$, there exists a $X \in \mathbb{C}^{n \times m}$ satisfying*

- (i) $(AX)^* = AX$,
- (ii) $(XA)^* = XA$,
- (iii) $XAX = X$,
- (iv) $AXA = A$.

Furthermore the X satisfying these four conditions must be **unique** and is denoted by $X = A^\dagger$.

- other types of pseudoinverse, sometimes also called **generalized inverse**, may be defined by choosing a subset of these four properties
- the Moore–Penrose theorem is actually true over any field but for \mathbb{C} (and also \mathbb{R}) we can say a lot more

- in fact we can compute A^\dagger explicitly using the SVD
- easy fact: if $A \in \mathbb{C}^{n \times n}$ is invertible, then

$$A^{-1} = A^\dagger \quad (6.1)$$

- to show this, just see that all four properties in the theorem hold true if we plug in $X = A^{-1}$
- another easy fact: if $D = \text{diag}(d_1, \dots, d_{\min(m,n)}) \in \mathbb{C}^{m \times n}$ is diagonal in the sense that $d_{ij} = 0$ for all $i \neq j$, then

$$D^\dagger = \text{diag}(\delta_1, \dots, \delta_{\min(m,n)}) \in \mathbb{C}^{n \times m} \quad (6.2)$$

- where

$$\delta_i = \begin{cases} 1/d_i & \text{if } d_i \neq 0 \\ 0 & \text{if } d_i = 0 \end{cases}$$

- to show this, just see that all four properties in the theorem hold true if we plug in $X = \text{diag}(\delta_1, \dots, \delta_{\min(m,n)})$
- in general

$$(AB)^\dagger \neq B^\dagger A^\dagger, \quad AA^\dagger \neq I, \quad A^\dagger A \neq I$$

- we can get A^\dagger in terms of the SVD of $A \in \mathbb{C}^{m \times n}$: if $A = U\Sigma V^*$, then

$$A^\dagger = V\Sigma^\dagger U^* \quad (6.3)$$

where

$$\Sigma^\dagger = \begin{bmatrix} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \in \mathbb{C}^{n \times m}$$

- to see this, just verify that (6.3) satisfies the four properties in Theorem 1
- we may also deduce from (6.3) that

$$A^\dagger = (A^* A)^{-1} A^*$$

if A has full column rank ($r = n$) and that

$$A^\dagger = A^* (A A^*)^{-1}$$

if A has full row rank ($r = m$)

7. SOLVING LEAST SQUARES PROBLEMS

- given $A \in \mathbb{C}^{m \times n}$ and $\mathbf{b} \in \mathbb{C}^m$, the least squares problem ask to find $\mathbf{x} \in \mathbb{C}^n$ so that

$$\|\mathbf{b} - A\mathbf{x}\|_2^2$$

is minimized

- note that \mathbf{x} minimizes $\|\mathbf{b} - A\mathbf{x}\|_2^2$ iff it minimizes $\|\mathbf{b} - A\mathbf{x}\|_2$, so whether we write the ‘squared’ or not doesn’t really matter
- using SVD of A and the unitary invariance of the vector 2-norm in Homework 2, we can simplify this minimization problem as follows

$$\begin{aligned} \|\mathbf{b} - A\mathbf{x}\|_2^2 &= \|\mathbf{b} - U\Sigma V^* \mathbf{x}\|_2^2 \\ &= \|U^* \mathbf{b} - \Sigma V^* \mathbf{x}\|_2^2 \\ &= \|\mathbf{c} - \Sigma \mathbf{y}\|_2^2 \\ &= (c_1 - \sigma_1 y_1)^2 + \dots + (c_r - \sigma_r y_r)^2 + c_{r+1}^2 + \dots + c_m^2 \end{aligned}$$

where $\mathbf{c} = U^* \mathbf{b}$ and $\mathbf{y} = V^* \mathbf{x}$

- so we see that in order to minimize $\|A\mathbf{x} - \mathbf{b}\|_2$, we must set $y_i = c_i/\sigma_i$ for $i = 1, \dots, r$
- the unknowns y_i , for $i = r+1, \dots, m$, can have any value, since they do not affect the value of $\|\mathbf{c} - \Sigma \mathbf{y}\|_2$
- so all the least squares solution are of the form

$$\mathbf{x} = V\mathbf{y} = V \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix}$$

where $y_{r+1}, \dots, y_n \in \mathbb{C}$ are arbitrary

- our analysis above also shows that the minimum value of $\|\mathbf{b} - A\mathbf{x}\|_2^2$ is given by

$$\min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2^2 = c_{r+1}^2 + \dots + c_m^2$$

8. SOLVING MINIMUM LENGTH LEAST SQUARES PROBLEMS

- one of the best-known applications of the SVD is that it can be used to obtain the solution to the problem

$$\|\mathbf{b} - A\mathbf{x}\|_2 = \min, \quad \|\mathbf{x}\|_2 = \min.$$

- alternatively, we can write

$$\min \left\{ \|\mathbf{x}\|_2 : \mathbf{x} \in \operatorname{argmin}_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\| \right\}$$

- or using the normal equations

$$\min \{ \|\mathbf{x}\|_2 : A^* A \mathbf{x} = A^* \mathbf{b} \}$$

- notation: $f : X \rightarrow \mathbb{R}$, for $S \subseteq X$, write

$$\operatorname{argmin}_{x \in S} f(x) \quad \text{or} \quad \operatorname{argmin} \{ f(x) : x \in S \}$$

for the *set* of minimizers, i.e.

$$\operatorname{argmin}_{x \in S} f(x) = \left\{ x_* \in S : f(x_*) = \min_{x \in S} f(x) \right\}$$

we often write (sloppily) $x_* = \operatorname{argmin} f(x)$ or $\operatorname{argmin} f(x) = x_*$ to mean that x_* is a minimizer of f over S even though the proper notation ought to have been $x_* \in \operatorname{argmin} f(x)$

- note that from the last part of the last lecture that if A does not have full rank, then there are infinitely many solutions to the least squares problem because the residual in does not depend on y_{r+1}, \dots, y_m and these are thus free parameters that we may freely choose:

$$\mathbf{x} = V\mathbf{y} = V \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix} \tag{8.1}$$

where $\mathbf{c} = U^* \mathbf{b}$

- we can claim that the unique solution with minimum 2-norm is given by setting $y_{r+1} = \dots = y_m = 0$, i.e.

$$\mathbf{x}_+ = V \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (8.2)$$

where $\mathbf{c} = U^* \mathbf{b}$

- this follows because

$$\|\mathbf{x}_+\|_2^2 = \left| \frac{c_1}{\sigma_1} \right|^2 + \dots + \left| \frac{c_r}{\sigma_r} \right|^2 \leq \left| \frac{c_1}{\sigma_1} \right|^2 + \dots + \left| \frac{c_r}{\sigma_r} \right|^2 + |y_{r+1}|^2 + \dots + |y_n|^2 = \|\mathbf{x}\|_2^2$$

whatever our choice of y_{r+1}, \dots, y_n in (8.1)

- we may also write (8.2) in terms of the Moore–Penrose pseudo inverse

$$\mathbf{x}_+ = V \Sigma^\dagger \mathbf{c} = V \Sigma^\dagger U^* \mathbf{b} = A^\dagger \mathbf{b}$$

since Σ^\dagger has precisely the right form

$$\Sigma^\dagger = \begin{bmatrix} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

9. FINDING BASES FOR FOUR FUNDAMENTAL SUBSPACES

- given $A \in \mathbb{C}^{m \times n}$ we may regard it as a linear operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $\mathbf{x} \mapsto A\mathbf{x}$
- there are four subspaces associated with A that we call the fundamental subspaces
 - $\ker(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}$
 - $\operatorname{im}(A) = \{\mathbf{y} \in \mathbb{C}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{C}^n\}$
 - $\ker(A^*) = \{\mathbf{y} \in \mathbb{C}^m : A^* \mathbf{x} = \mathbf{0}\}$
 - $\operatorname{im}(A^*) = \{\mathbf{x} \in \mathbb{C}^n : A^* \mathbf{y} = \mathbf{x} \text{ for some } \mathbf{y} \in \mathbb{C}^m\}$
- they are called the kernel, image, cokernel, and coimage respectively
 - the word null space is often used in place of kernel
 - the word range space is often used in place of image
- as we see from Homework 1, they are related by

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp$$

- furthermore they decompose the domain and codomain of A into orthogonal subspaces

$$\mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A) \quad \text{and} \quad \mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A)$$

- this decomposition is sometimes called **Fredholm alternative** and is very useful for studying linear systems $A\mathbf{x} = \mathbf{b}$ and least squares problems (cf. Homework 1)
- there is a variant called Farkas Lemma that applies to linear inequalities $A\mathbf{x} \leq \mathbf{b}$
- the full SVD of A allows us to simply read off orthonormal bases for the four fundamental subspaces, which will be useful if we want to compute projections
- let $\operatorname{rank}(A) = r$ and let $\mathbf{u}_1, \dots, \mathbf{u}_m$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the left and right singular vectors of A , indexed in the usual way in descending magnitude of their corresponding singular values

- $\ker(A) = \text{span}\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$
- $\text{im}(A) = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$
- $\ker(A^*) = \text{span}\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$
- $\text{im}(A^*) = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$
- this is easy to see, let us do $\text{im}(A)$ for example
 - first observe that an orthonormal basis for $\text{im}(\Sigma)$ is the first r standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_r \in \mathbb{C}^m$ since

$$\Sigma \mathbf{x} = \sigma_1 x_1 \mathbf{e}_1 + \dots + \sigma_r x_r \mathbf{e}_r$$

- now just observe that
$$\text{im}(A) = \text{im}(U\Sigma V^*) = \text{im}(U\Sigma) = \text{span}\{U\mathbf{e}_1, \dots, U\mathbf{e}_r\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$$
- there is a useful fact worth noting: for nonsingular matrices $S \in \text{GL}(m)$ and $T \in \text{GL}(n)$,

$$\text{im}(AT) = \text{im}(A) \quad \text{and} \quad \ker(SA) = \ker(A)$$

10. FINDING A BEST RANK- r APPROXIMATION

- given $A \in \mathbb{C}^{m \times n}$, we want to find $X \in \mathbb{C}^{m \times n}$ of rank not more than r so that $\|A - X\|$ is minimized
- in notations, we want

$$\min_{\text{rank}(X) \leq r} \|A - X\| \tag{10.1}$$

- such an X is called a best rank- r approximation to A or a rank- r projection of A
- if $r \geq \text{rank}(A)$, then clearly $X = A$ and the problem is trivial
- so we shall always assume that $r < \text{rank}(A)$
- we will see how to construct such an X explicitly when the norm $\|\cdot\|$ is unitarily invariant, i.e., satisfying

$$\|UXV\| = \|X\|$$

for all $X \in \mathbb{C}^{m \times n}$ whenever U and V are unitary matrices

- we will start with the classical case where $\|\cdot\|$ is the matrix 2-norm or spectral norm

Theorem 2 (Eckart–Young). *Let the SVD of A be*

$$A = \sum_{i=1}^{\text{rank}(A)} \sigma_i \mathbf{u}_i \mathbf{v}_i^*, \quad \sigma_1 \geq \dots \geq \sigma_r > 0.$$

Then for any $r \in \{1, \dots, \text{rank}(A) - 1\}$, a solution to (10.1) when $\|\cdot\| = \|\cdot\|_2$ is given by

$$X = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^*. \tag{10.2}$$

Furthermore, we have

$$\min_{\text{rank}(X) \leq r} \|A - X\|_2 = \sigma_{r+1}. \tag{10.3}$$

In matrix form, we have

$$X = U \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} V^*, \tag{10.4}$$

where $A = U\Sigma V^$ is the SVD of A .*

Proof. Suppose there is a $B \in \mathbb{C}^{m \times n}$ with $\text{rank}(B) \leq r$ and $\|A - B\|_2 < \sigma_{r+1}$. Then by the rank-nullity theorem

$$\text{rank}(B) + \dim(\ker(B)) = n$$

and so

$$\dim(\ker(B)) \geq n - r.$$

Let $\mathbf{w} \in \ker(B)$. Then $B\mathbf{w} = \mathbf{0}$ and so

$$\|A\mathbf{w}\|_2 = \|(A - B)\mathbf{w}\|_2 \leq \|A - B\|_2 \|\mathbf{w}\|_2 < \sigma_{r+1} \|\mathbf{w}\|_2. \quad (10.5)$$

Let $\mathbf{w} \in W := \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{r+1}\}$. Then $\mathbf{w} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_{r+1} \mathbf{v}_{r+1}$. Rewriting this in matrix form

$$\mathbf{w} = V_{r+1} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \end{bmatrix} = V_{r+1} \boldsymbol{\alpha}$$

where $V_{r+1} = [\mathbf{v}_1, \dots, \mathbf{v}_{r+1}] \in \mathbb{C}^{n \times r}$, i.e., the first $r + 1$ columns of V .

$$\begin{aligned} \|A\mathbf{w}\|_2^2 &= \|U\Sigma V^* V_{r+1} \boldsymbol{\alpha}\|_2^2 = \left\| \Sigma \begin{bmatrix} I_{r+1} \\ O \end{bmatrix} \boldsymbol{\alpha} \right\|_2^2 = \left\| \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{r+1} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \end{bmatrix} \right\|_2^2 \\ &= \sum_{i=1}^{r+1} \sigma_i^2 |\alpha_i|^2 \geq \sigma_{r+1}^2 \sum_{i=1}^{r+1} |\alpha_i|^2 = \sigma_{r+1}^2 \|\mathbf{w}\|_2^2. \end{aligned}$$

Hence if $\mathbf{w} \in W$, then

$$\|A\mathbf{w}\|_2 \geq \sigma_{r+1} \|\mathbf{w}\|_2. \quad (10.6)$$

But since $\dim(\ker(B)) \geq n - r$ and $\dim(W) = r + 1$, the two subspaces must intersect nontrivially, i.e., $\dim(\ker(B) \cap W) \geq 1$ and so there exists a non-zero vector $\mathbf{w} \in \ker(B) \cap W$. Such a vector would satisfy both (10.5) and (10.6), a contradiction. Hence our original assumption is false: There is no rank- r matrix B that could beat the bound in (10.3). On the other hand it is easy to verify that the choice of X in (10.4) satisfies (10.3):

$$\|A - X\|_2 = \left\| U \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \sigma_{r+1} & \\ & & & & \ddots \\ & & & & & \sigma_{\text{rank}(A)} \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{bmatrix} V^* \right\|_2 = \sigma_{r+1}.$$

□

- the generalization of Eckart–Young theorem to any arbitrary unitarily invariant norm is due to Mirsky and this theorem is sometimes also called the **Eckart–Young–Mirsky theorem**
- note that the general theorem only says that (10.2) is the best rank- r approximation of A , the value in (10.3) would in general be different

- for example if we use the Frobenius norm

$$\min_{\text{rank}(X) \leq r} \|A - X\|_F = \sqrt{\sigma_{r+1}^2 + \cdots + \sigma_{\text{rank}(A)}^2}.$$