STAT 30100 : Mathematical Statistics-1

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Homework 9

Solutions by

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STAT 30100, Homework 9

1. (Casella and Berger Problem 7.22) This exercise will prove the assertions in Example 7.2.16, and more. Let $X_1, \ldots, X_n$ be a random sample from a $\mathcal{N}(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $\mathcal{N}(\mu, \tau^2)$. Here we assume that σ^2, μ, and τ^2 are all known.

   (a) Find the joint pdf of $\overline{X}$ and θ.

   Since $\theta \sim \mathcal{N}(\mu, \tau^2)$ and $\overline{X}|\theta \sim \mathcal{N}(\theta, \frac{\sigma^2}{n})$, we have

$$f(\overline{x}, \theta) = f(\overline{x}|\theta)\pi(\theta) = \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} e^{-\frac{n(\overline{x}-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{(\theta-\mu)^2}{2\tau^2}} = \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{n(\overline{x}-\theta)^2}{2\sigma^2} - \frac{(\theta-\mu)^2}{2\tau^2}}.$$

   (b) Show that $m(\overline{x}|\sigma^2, \mu, \tau^2)$, the marginal distribution of $\overline{X}$, is $\mathcal{N}(\mu, \frac{\sigma^2}{n} + \tau^2)$.

   Proof. Since

$$\begin{aligned}
f(\overline{x}, \theta) &= \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{n}{2\sigma^2}\overline{x}^2 + \frac{n\theta}{\sigma^2}\overline{x} - \frac{n\theta^2}{2\sigma^2} - \frac{\theta^2}{2\tau^2} + \frac{\mu\theta}{\tau^2} - \frac{\mu^2}{2\tau^2}} \\
&= \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\theta^2 + \left(\frac{n\overline{x}}{\sigma^2} + \frac{\mu}{\tau^2}\right)\theta} e^{-\frac{n}{2\sigma^2}\overline{x}^2 - \frac{\mu^2}{2\tau^2}} \\
&= \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\left(\theta - \frac{\frac{n\overline{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)^2 + \frac{1}{2}\frac{\left(\frac{n\overline{x}}{\sigma^2} + \frac{\mu}{\tau^2}\right)^2}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}} e^{-\frac{n}{2\sigma^2}\overline{x}^2 - \frac{\mu^2}{2\tau^2}} \\
&= \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{\frac{\sigma^2}{n} + \tau^2}{2\frac{\sigma^2}{n}\tau^2}\left(\theta - \frac{\tau^2\overline{x}}{\tau^2 + \frac{\sigma^2}{n}} - \frac{\frac{\sigma^2}{n}\mu}{\tau^2 + \frac{\sigma^2}{n}}\right)^2 + \frac{1}{2}\frac{\left(\frac{n\overline{x}}{\sigma^2} + \frac{\mu}{\tau^2}\right)^2}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}} e^{-\frac{n}{2\sigma^2}\overline{x}^2 - \frac{\mu^2}{2\tau^2}} \\
m(\overline{x}|\sigma^2, \mu, \tau^2) &= \int_{\mathbb{R}} f(\overline{x}, \theta) \mathrm{d}\theta \\
&= \frac{\sqrt{n}}{\sqrt{2\pi\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\sigma^2\tau^2}} e^{-\frac{1}{2}\left(1 - \frac{n}{\sigma^2\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)}\right)\overline{x}^2 + \left(\frac{\frac{n\mu}{\sigma^2\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)\overline{x} - \frac{1}{2}\left(\frac{\frac{\mu^2}{\tau^4}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} + \frac{\mu^2}{2\tau^2}\right)} \\
&= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{n} + \tau^2\right)}} e^{-\frac{1}{2\left(\frac{\sigma^2}{n} + \tau^2\right)}(\overline{x}-\mu)^2},
\end{aligned}$$

   i.e., $\overline{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n} + \tau^2)$. □

   (c) Show that $\pi(\theta|\overline{x}, \sigma^2, \mu, \tau^2)$, the posterior distribution of θ, is normal with mean and variance given by

$$\mathbb{E}(\theta|\overline{x}) = \frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}}\overline{x} + \frac{\frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}}\mu, \qquad \mathrm{Var}(\theta|\overline{x}) = \frac{\frac{\sigma^2}{n}\tau^2}{\frac{\sigma^2}{n} + \tau^2}.$$

   Proof. From (b), we have $\theta|\overline{x} \sim \mathcal{N}\left(\frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}}\overline{x} + \frac{\frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}}\mu, \frac{\frac{\sigma^2}{n}\tau^2}{\frac{\sigma^2}{n} + \tau^2}\right)$. □

2. Suppose $X_1, \dots, X_n$ i.i.d. with density $f(x_i|\theta) = \theta \exp(-\theta x_i)$, where $\theta > 0$.

(a) Show that the gamma family of priors is conjugate for inference about θ given $X_1, \dots, X_n$.

Proof. $\theta \sim \Gamma(\alpha, \beta)$, for $\theta > 0$,

$$\begin{aligned}\pi(\theta|\boldsymbol{X} = \boldsymbol{x}) &\propto \pi(\theta) L(\theta; \boldsymbol{x}) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-\theta \sum_{i=1}^n x_i} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{x_i}\right)\theta}\end{aligned}$$

so $\theta|\boldsymbol{X} = \boldsymbol{x} \sim \Gamma\left(\alpha + n, \beta + \sum_{i=1}^n x_i\right)$, i.e., the gamma family of priors is conjugate for inference about θ given $X_1, \dots, X_n$. □

(b) Find the Jeffreys prior for this problem.

$$\begin{aligned}\frac{\partial l(\theta; \boldsymbol{x})}{\partial \theta} &= \frac{\partial}{\partial \theta}\left[\log\theta - \theta \sum_{i=1}^n x_i\right] = \frac{1}{\theta} - \sum_{i=1}^n x_i \\ \frac{\partial^2 l(\theta; \boldsymbol{x})}{\partial \theta^2} &= -\frac{1}{\theta^2} \\ \mathcal{I}_n(\theta) = -\mathbb{E}_\theta &\left[\frac{\partial^2 l(\theta; \boldsymbol{x})}{\partial \theta^2}\right] = \frac{1}{\theta^2}\end{aligned}$$

So the Jeffreys prior satisfies $\pi_J(\theta) \propto \frac{1}{\theta}$.

(c) Is the prior in part (b) proper or improper?

Since $\int_0^{+\infty} \frac{1}{\theta} \mathrm{d}\theta = \log(\theta)|_0^{+\infty} = \infty$, the prior in part (b) is improper.

(d) Find the posterior mean of θ using the prior in part (b).

Since

$$\begin{aligned}\pi_J(\theta|\boldsymbol{X} = \boldsymbol{x}) &\propto \pi_J(\theta) L(\theta; \boldsymbol{x}) \\ &= \theta^{n-1} e^{-\theta \sum_{i=1}^n x_i} \\ \int_0^{+\infty} \theta^{n-1} e^{-\theta \sum_{i=1}^n x_i} \mathrm{d}\theta &= \left(\sum_{i=1}^n x_i\right)^{-n} \int_0^{+\infty} t^{n-1} e^{-t} \mathrm{d}t \\ &= \frac{\Gamma(n)}{\left(\sum_{i=1}^n x_i\right)^n},\end{aligned}$$

we have

$$\pi_J(\theta|\boldsymbol{X} = \boldsymbol{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta \sum_{i=1}^n x_i}.$$

So the posterior mean is

$$\mathbb{E}(\theta|\boldsymbol{X} = \boldsymbol{x}) = \int_0^{+\infty} \frac{\left(\sum_{i=1}^n x_i\right)^n}{\Gamma(n)} \theta^n e^{-\theta \sum_{i=1}^n x_i} \mathrm{d}\theta = \frac{n}{\sum_{i=1}^n x_i}.$$