MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 9

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Solutions by

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Question 1

(a) Show that the Hermite polynomial $H_n(x)$ satisfies

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (x - ik)^n \mathrm{d}k.$$

$\because$

$$e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}} e^{-ikx} \mathrm{d}k$$

$\therefore$

$$\begin{aligned} e^{-x^2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}} e^{-ik\sqrt{2}x} \mathrm{d}k \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} e^{-2ikx} \mathrm{d}k \end{aligned}$$

$\therefore$

$$\begin{aligned} De^{-x^2} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} D^n e^{-2ikx} \mathrm{d}k \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (-2ik)^n e^{-2ikx} \mathrm{d}k \end{aligned}$$

$\therefore$

$$\begin{aligned} H_n(x) &= \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (2ik)^n e^{-2ikx} \mathrm{d}k \\ &= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k+ix)^2} (ik)^n \mathrm{d}k \\ &= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} [i(k - ix)]^n \mathrm{d}k \\ &= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (x - ik)^n \mathrm{d}k \end{aligned}$$

(b) Show that

$$|(x + ik)^n| \leqslant 2^n (|x|^n + |k|^n).$$

$\because \quad \forall\, n, j \in \mathbb{N}, n \geqslant j,$

$$\begin{aligned} |x^j k^{n-j}| &\leqslant \begin{cases} k^n & , x \leqslant 1 \\ \max\{|x|^n, |k|^n\} & , x > 1 \end{cases} \\ &\leqslant |x|^n + |k|^n \end{aligned}$$

$\therefore$

$$\begin{aligned} |(x + ik)^n| &= \left| \sum_{j=0}^{n} \binom{n}{j} x^k (ik)^{n-j} \right| \\ &\leqslant \sum_{j=0}^{n} \binom{n}{j} |x^k (ik)^{n-j}| \end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
&= \sum_{j=0}^{n} \binom{n}{j} |x^j k^{n-j}| \\
&\leqslant \sum_{j=0}^{n} \binom{n}{j} (|x|^n + |k|^n) \\
&\leqslant 2^n (|x|^n + |k|^n)
\end{aligned}
$$

(c) Use Stirling's approximation $n! \approx \left(\frac{n}{e}\right)^n$ to show

$$\frac{|h_n(x)|}{\|h_n\|} \leqslant 2^n e^{-\frac{x^2}{2}} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} e^{-\frac{x^2}{2}} |x|^n.$$

$$
\begin{aligned}
\frac{|h_n(x)|}{\|h_n\|} &= \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{n}{2}} \sqrt{n!}} |h_n(x)| \\
&\leqslant \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{3}{4}} (n!)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{-k^2} (|x|^n + |k|^n) \mathrm{d}k \\
&= \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{3}{4}} (n!)^{\frac{3}{2}}} \int_{0}^{\infty} e^{-t} t^{\frac{n-1}{2}} \mathrm{d}t + \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{3}{4}} (n!)^{\frac{3}{2}}} \sqrt{\pi} |x|^n \\
&= \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{3}{4}} (n!)^{\frac{3}{2}}} \Gamma\left(\frac{n+1}{2}\right) + \frac{2^{\frac{n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}} (n!)^{\frac{3}{2}}} |x|^n \\
&\leqslant \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{\pi^{\frac{3}{4}} (n!)^{\frac{3}{2}}} \cdot \frac{n!}{2^{\frac{n}{2}}} + \frac{2^{\frac{3n}{2}} e^{-\frac{x^2}{2}}}{(n!)^{\frac{1}{2}}} |x|^n \\
&\leqslant 2^n e^{-\frac{x^2}{2}} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} e^{-\frac{x^2}{2}} |x|^n
\end{aligned}
$$

(d) Show that

$$\frac{|h_n(x)|}{\|h_n\|} \leqslant 2 \left(\frac{16e\pi}{e^{2\pi}}\right)^{\frac{n}{2}}$$

for $|x| \geqslant \sqrt{2\pi n}$.

$\because$ when $x \geqslant \sqrt{2\pi n}$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[2^n e^{-\frac{x^2}{2}} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} e^{-\frac{x^2}{2}} x^n\right] &= -2^n x e^{-\frac{x^2}{2}} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} (-x^{n+1} + n x^{n-1}) e^{-\frac{x^2}{2}} \\
&= -2^n x e^{-\frac{x^2}{2}} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} x^{n-1} (n - x^2) e^{-\frac{x^2}{2}} \\
&< 0
\end{aligned}
$$

Solution (cont.)

$\therefore$ when $x = \sqrt{2\pi n}$, the RHS is maximized, i.e.

$$\begin{aligned}\frac{|h_n(x)|}{\|h_n\|} &\leqslant 2^n e^{-\pi n} + \left(\frac{8e}{n}\right)^{\frac{n}{2}} e^{-\pi n}(2\pi n)^{\frac{n}{2}} \\ &= 2^n e^{-\pi n} + (16e\pi)^{\frac{n}{2}} e^{-\pi n} \\ &\leqslant 2\left(\frac{16e\pi}{e^{2\pi}}\right)^{\frac{n}{2}}\end{aligned}$$

(e) Explain why scaled Hermite functions $h_0(cx)$, $h_1(cx)$,$\cdots$, $h_n(cx)$ might form a suitable basis for approximating functions $f \in L^2$ which are approximately band- and time-limited in the sense that

$$\int_{|x|>T} |f(x)|^2 \mathrm{d}x \leqslant \epsilon^2 \|f\|^2$$

and

$$\int_{|k|>K} |\hat{f}(k)|^2 \mathrm{d}k \leqslant \epsilon^2 \|\hat{f}\|^2$$

How should n and c relate to K and T?

From (d), for $|x| \geqslant \sqrt{2\pi n}$,

$$\frac{|h_n(x)|}{\|h_n\|} \leqslant 2\left(\frac{16e\pi}{e^{2\pi}}\right)^{\frac{n}{2}}$$

Given $T > 0$, $\exists\ c = \frac{T}{\sqrt{2\pi n}}$, s.t. $\forall\ i \in \mathbb{N}, 0 \leqslant i \leqslant n$, when $|x| \geqslant \frac{T}{c} \geqslant \sqrt{2\pi n}$,

$$\frac{|h_i(x)|}{\|h_i\|} \leqslant 2\left(\frac{16e\pi}{e^{2\pi}}\right)^{\frac{i}{2}}$$

We can find n such that LHS can be as small as possible.

$\because$

$$\widehat{h_n}(k) = (-1)^n h_n(x)$$

$$\widehat{h_n}(ck) = \frac{(-1)^n}{c} h_n\left(\frac{x}{c}\right)$$

$\therefore$ in the fourier space, the error term can be bounded as well

h_n is approximately time-limited to $(-\sqrt{2\pi n}, \sqrt{2\pi n})$ and so is its fourier transform. Hence $h_0, \cdots, h_n$ might be a useful basis for functions band-limited with $b = O(\sqrt{n})$ and time-limited with $a = O(\sqrt{n})$. If a is not the same as b, then we use c to scale $h_n(cx)$ to interval $[-a, a]$ and its fourier transform $\frac{(-i)^n}{c} h_n\left(\frac{x}{c}\right)$ to interval $[-b, b]$ after making sure $n = O(ab)$ is large enough.

$$c\sqrt{2\pi n} = \frac{\sqrt{2\pi n}}{c}$$

and

$$c\sqrt{2\pi n} \leqslant T$$

$$\frac{\sqrt{2\pi n}}{c} \leqslant K$$

Question 2

(a) Show that

$$FDf(k) = \hat{f}'(k) = ik\hat{f}(k) = ikFf(k)$$

and

$$F(xf)(k) = \widehat{xf}(k) = i\hat{f}'(k) = iDFf(k).$$

$$\begin{aligned}
FDf(x) &= Ff'(x) \\
&= \hat{f}'(k) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} f(x)e^{-ikx}\Big|_{-\infty}^{\infty} - \frac{-ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \\
&= ik\hat{f}(k) \\
&= ikFf(k) \\
F(xf)(k) &= \widehat{xf}(k) \\
&= i(\widehat{-ixf})(k) \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-ix)e^{-ikx}\mathrm{d}x \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)D_k e^{-ikx}\mathrm{d}x \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-ikx}\mathrm{d}x \\
&= i\hat{f}'(k) \\
&= iDFf(k)
\end{aligned}$$

(b) Show that the differential operator

$$D_{ab}f(x) = [(a^2 - x^2)f'(x)]' - b^2x^2f(x)$$

satisfies

$$FD_{ab} = D_{ba}F.$$

$$
\begin{aligned}
FD_{ab}f(k) &= F\{[(a^2-k^2)f'(k)]' - b^2k^2f(k)\} \\
&= F[-2kf'(k) + (a^2-k^2)f''(k) - b^2k^2f(k)] \\
&= -2F[xf'](k) + a^2FD^2f(k) - F[x^2f''(x)](k) - b^2F[x^2f(x)](k) \\
&= -2iDFDf(k) + a^2(ik)^2Ff(k) - (i)^2D^2Ff''(k) - b^2(i)^2D^2Ff(k) \\
&= -2i(ik)DFf(k) - a^2k^2Ff(k) + (ik)^2D^2Ff(k) + b^2D^2Ff(k) \\
&= 2kDFf(k) - a^2k^2Ff(k) - k^2D^2Ff(k) + b^2D^2Ff(k) \\
&= [(b^2-(ik)^2)DFf(k)]' - a^2k^2Ff(k) \\
&= D_{ba}Ff(k)
\end{aligned}
$$

(c) Show that D_{ab} commutes with the orthogonal projection onto time-limited functions

$$P_af(t) = f(t)$$

for $|t| \leqslant a$ and

$$P_af(t) = 0$$

for $|t| > a$.

$$
\begin{aligned}
D_{ab}P_af(t) &= \begin{cases} D_{ab}f(t) & ,|t| \leqslant a \\ D_{ab}0, & ,|t| > a \end{cases} \\
&= \begin{cases} [(a^2-x^2)f'(x)]' - b^2x^2f(x) & ,|t| \leqslant a \\ 0, & ,|t| > a \end{cases} \\
&= P_aD_{ab}
\end{aligned}
$$

(d) Use (b) and (c) to show that D_{ab} commutes with the integral operator

$$S_{ab}f(t) = P_aQ_bP_af(t) = \frac{1}{\pi}\int_{-a}^{a} \frac{\sin[b(t-s)]}{t-s} f(s)\mathrm{d}s$$

where $Q_b = F^*P_bF$ is the orthogonal projection onto bandlimited functions.

$\because$

$$D_{ab}F = FD_{ab}$$

$\therefore$

$$F^*D_{ab} = D_{ab}F^*$$

Solution (cont.)

$$\begin{aligned} D_{ab}S_{ab}f(t) &= D_{ab}P_aQ_bP_af(t) \\ &= P_aD_{ab}Q_bP_af(t) \\ &= P_aD_{ab}F^*P_bFP_af(t) \\ &= P_aF^*D_{ab}P_bFP_af(t) \\ &= P_aF^*P_bD_{ab}FP_af(t) \\ &= P_aF^*P_bFD_{ab}P_af(t) \\ &= P_aQ_bP_aD_{ab}f(t) \\ &= S_{ab}D_{ab}f(t) \end{aligned}$$

(e) Explain why the eigenfunctions of D_{ab} might be useful in representing approximately time- and band-limited functions.

S_{ab} can be used to represent approximately time- and band-limited functions. But its eigenfunctions are hard to compute. Since $S_{ab}D_{ab} = D_{ab}S_{ab}$, the eigenfunctions of D_{ab} diagonalize S_{ab}. And therefore we can get the orthonormal basis of eigenfunctions of S_{ab}. So the approximate time- and band-limited functions can be presented in the eigenfunctions space of S_{ab}.

Question 3

(a) Use Fourier transform to find a bounded solution u of

$$u_{xx} + u_{tt} = 0$$

in the upper half plane $x \in \mathbb{R},\ t > 0$, with boundary conditions

$$u(x, 0) = g(x)$$

where $g \in L^2(\mathbb{R})$ is bounded and continuous.

$\because$

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(u_{tt} + u_{xx})e^{-ikx}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}D_t^2\int_{-\infty}^{\infty}u_{tt}e^{-ikx}\mathrm{d}x + \frac{1}{\sqrt{2\pi}}(u_xe^{-ikx} - ikue^{-ikx})\Big|_{\infty}^{\infty} + \frac{(ik)^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ue^{-ikx}\mathrm{d}x \\ &= \hat{u}_{tt} - k^2\hat{u} \end{aligned}$$

$\therefore$

$$\hat{u}_{tt} = k^2\hat{u}$$

Solution (cont.)

$\therefore$

$$\hat{u}(k,t) = Ae^{-t|k|} + Be^{t|k|}$$

To let $u(x,t) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{ikx} \mathrm{d}k$ be bounded, we have $B = 0$ and

$$\hat{u}(k,t) = Ae^{-t|k|}$$

$\because$

$$u(x,0) = g(x)$$

$\therefore$

$$\hat{u}(k,0) = \hat{g}(k)$$

$\therefore$

$$\hat{u}(k,t) = \hat{g}(k) e^{-t|k|}$$

$\therefore$

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(k) e^{-t|k|} e^{ikx} \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t|k|} e^{ikx} \int_{-\infty}^{\infty} g(y) e^{-iky} \mathrm{d}y \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} e^{-t|k|} e^{ik(x-y)} \mathrm{d}k \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left[\int_{0}^{\infty} e^{[-t+i(x-y)]k} \mathrm{d}k + \int_{-\infty}^{0} e^{[t+i(x-y)]k} \mathrm{d}k \right] \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left[\frac{e^{[-t+i(x-y)]k}}{i(x-y)-t} \Big|_{0}^{\infty} + \frac{e^{[t+i(x-y)]k}}{i(x-y)+t} \Big|_{-\infty}^{0} \right] \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left(\frac{1}{t-i(x-y)} + \frac{1}{t+i(x-y)} \right) \mathrm{d}y \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} g(y) \frac{t}{t^2+(x-y)^2} \mathrm{d}y \\
&\xlongequal{z=\frac{x-y}{t}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(x-tz)}{1+z^2} \mathrm{d}z
\end{aligned}$$

(b) Show that u attains its boundary values in the sense that

$$u(x,t) \to g(x)$$

as $t \to 0$.

$\because$ $g(x)$ is bounded and continuous

$\therefore$ $\exists\ M > 0$, s.t. $\forall\ t > 0$,

$$\left| \frac{g(x+tz)}{1+z^2} \right| \leqslant \frac{M}{1+z^2} \leqslant M$$

$\because$

$$\lim_{t \to 0} \frac{g(x+tz)}{1+z^2} = \frac{g(x)}{1+z^2}$$

Solution (cont.)

$\therefore$ from Dominated Convergence Theorem,

$$\begin{aligned}\lim_{t\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x+tz)}{1+z^2}\mathrm{d}z &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{1+z^2}\mathrm{d}z\\ &= \frac{g(x)}{\pi}\arctan z\Big|_{-\infty}^{\infty}\\ &= g(x)\end{aligned}$$

(c) Assume that $g' \in L^2(\mathbb{R})$ is also bounded and continuous. Argue directly from the Laplace equation that if the Dirichlet-Neumann operator Λ is defined by

$$u_t(x,t) \to \Lambda g(x).$$

as $t \to 0$, then Λ must satisfy

$$\Lambda^2 g(x) = -g''(x).$$

$\because$

$$\lim_{t\to 0} u_t(x,t) = \Lambda g(x)$$

$\therefore$

$$\lim_{t\to 0} u_{tt}(x,t) = \Lambda^2 g(x)$$

$\because$

$$u_{xx} + u_{tt} = 0$$

$\therefore$

$$\lim_{t\to 0} u_{xx}(x,t) = -\Lambda^2 g(x)$$

$\therefore$

$$\begin{aligned}\Lambda^2 g(x) &= \frac{\partial^2}{\partial x^2}\lim_{t\to 0} u(x,t)\\ &= -\frac{\partial^2}{\partial x^2} g(x)\\ &= -g''(x)\end{aligned}$$

(d) Find the kernel of the Hilbert transform operator H such that

$$\Lambda g = H(g')$$

$\because$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x-tz)}{1+z^2}\mathrm{d}z$$

Solution (cont.)

$\therefore$

$$
\begin{aligned}
u_t(x,t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{-z}{1+z^2} g'(x-tz)\mathrm{d}z \\
&\xlongequal{y=x-tz} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-y}{t^2+(x-y)^2} g'(y)\mathrm{d}y
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\Lambda g(x) &= \lim_{t\to 0} u_t(x,t) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x-y} g'(y)\mathrm{d}y \\
&= H(g'(x))
\end{aligned}
$$

Question 4

Solve the integral equation

$$
D^{-\frac{1}{2}}h(t) = \int_0^t \frac{1}{\sqrt{\pi(t-s)}} h(s)\mathrm{d}s = g(t)
$$

where g is a nice function with $g(0)=0$.
(**Hint:** Square $D^{-\frac{1}{2}}$.)

$\because$

$$
D^{-\frac{1}{2}}h(t) = \int_0^t \frac{1}{\sqrt{\pi(t-s)}} h(s)\mathrm{d}s = g(t)
$$

$\therefore$

$$
\begin{aligned}
D^{-1}h(t) &= D^{-\frac{1}{2}}g(t) \\
&= \int_0^t \frac{1}{\sqrt{\pi(t-s)}} g(s)\mathrm{d}s \\
&\xlongequal{x=t-s} -\int_0^t \frac{1}{\sqrt{\pi x}} g(t-x)\mathrm{d}x \\
h(t) &= -\frac{1}{\sqrt{\pi x}} g(t-x)\bigg|_{x=t} - \int_0^t \frac{1}{\sqrt{\pi x}} g'(t-x)\mathrm{d}x \\
&= -\frac{g(0)}{\sqrt{\pi t}} - \int_0^t \frac{1}{\sqrt{\pi x}} g'(t-x)\mathrm{d}x \\
&\xlongequal{s=t-x} \int_0^t \frac{1}{\sqrt{\pi(t-s)}} g'(s)\mathrm{d}s
\end{aligned}
$$

Question 5

(a) Solve the initial-boundary value problem for the heat equation

$$u_t = u_{xx}$$

for $x > 0, t > 0$, with homogeneous initial conditions $u(x,0) = 0$ and boundary conditions $u(0,t) = g(t)$ where g is a nice function with $g(0) = 0$.
(**Hint:** Try $u(x,t) = \int_0^t K_{t-s}(x)h(s)ds$ and solve an integral equation for h.)

Suppose that for $x \geqslant 0$,

$$\begin{aligned} u(x,t) &= \int_0^t K_{t-s}(x)h(s)\mathrm{d}s \\ &= \int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}h(s)\mathrm{d}s \\ &= [G(x,\cdot) * h(\cdot)](t) \end{aligned}$$

Then $u(x,0) = 0$ and $u(x,t)$ can be defined on the whole plane by setting $u(x,t) = u(-x,t) \quad (x < 0)$. We have

$$\begin{aligned} \hat{u}(k,t) &= \sqrt{2\pi}\hat{G}(k,t)\hat{h}(k) \\ &= e^{-tk^2}\hat{h}(k) \\ \hat{u}_{xx}(k,t) &= -k^2\hat{u}(k,t) \\ \hat{u}_t(k,t) &= -k^2 e^{-tk^2}\hat{h}(k) \\ &= -k^2\hat{u}(k,t) \end{aligned}$$

$\therefore$

$$\hat{u}_t(k,t) = \hat{u}_{xx}(k,t)$$

i.e.

$$u_t = u_{xx}$$

Let

$$u(0,t) = g(t)$$

we have

$$\int_0^t \frac{1}{\sqrt{4\pi(t-s)}}h(s)\mathrm{d}s = g(t)$$

Similar to Question 4, we have

$$h(t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}g'(s)\mathrm{d}s$$

$\therefore$

$$u(x,t) = \int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\int_0^s \frac{1}{\sqrt{4\pi(s-y)}}g'(y)\mathrm{d}y\mathrm{d}s$$

(b) Assume that $g' \in L^2(\mathbb{R})$ is also bounded and continuous. Argue directly from the heat equation that if

$$u_x(x,t) \to \Lambda g(t)$$

as $x \to 0$, then the Dirichlet-Neumann operator Λ must satisfy

$$\Lambda^2 g(t) = g'(t).$$

$\because$

$$\lim_{x\to 0} u_x(x,t) = \Lambda g(t)$$

$\therefore$

$$\lim_{x\to 0} u_{xx}(x,t) = \Lambda^2 g(t)$$

$\because$

$$u_{xx} = u_t$$

$\therefore$

$$\lim_{x\to 0} u_t(x,t) = \Lambda^2 g(t)$$

$\therefore$

$$\begin{aligned}\Lambda^2 g(t) &= \frac{\partial}{\partial t}\lim_{x\to 0} u(x,t)\\ &= \frac{\partial}{\partial t} g(t)\\ &= g'(t)\end{aligned}$$

(c) Find the Dirichlet-Neumann operator Λ.

$\because$

$$g(t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} h(s)\mathrm{d}s$$

$$u_x(x,t) = \int_0^t \frac{-xe^{-\frac{x^2}{4(t-s)}}}{2(t-s)\sqrt{4\pi(t-s)}} \int_0^s \frac{1}{\sqrt{4\pi(s-y)}} g'(y)\mathrm{d}y\mathrm{d}s$$

$\because$ $x > 0$ and

$$\Lambda^2 g(t) = g'(t)$$

$\therefore$

$$\begin{aligned}\Lambda g(t) &= \lim_{x\to 0}\int_0^t \frac{-xe^{-\frac{x^2}{4(t-s)}}}{2(t-s)\sqrt{4\pi(t-s)}} h(s)\mathrm{d}s\\ &\xlongequal{z=\frac{x}{\sqrt{4(t-s)}}} \lim_{x\to 0}\frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4t}}}^{\infty} e^{-z^2} h\left(t-\frac{x^2}{4z^2}\right)\mathrm{d}z\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2} h(t)\mathrm{d}z\\ &= h(t)\\ &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} g(s)\mathrm{d}s\end{aligned}$$