STAT 150: Stochastic Processes

Fall 2017

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Homework 8

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Solutions by

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PK Problems 6.6.1

Let $Y_n = 0, 1, \cdots$ be a discrete time Markov chain with transition probabilities $\mathbf{P} = \|P_{ij}\|$, and let $\{N(t); t \geqslant 0\}$ be an independent Poisson process of rate λ. Argue that the compound process

$$X(t) = Y_{N(t)}, \qquad t \geqslant 0,$$

is a Markov chain in continuous time and determine its infinitesimal parameters.

$\because$ $\forall\ n \in \mathbb{N}^+,\ i_0, i_1, \cdots, i_n \in \mathbb{N},\ 0 \leqslant t_0 < t_1 < \cdots < t_n,$

$$\begin{aligned}
&\mathbb{P}(X(t_n) = i_n | X(t_0) = i_0, X(t_1) = i_1, \cdots, X(t_{n-1}) = i_{n-1}) \\
=&\mathbb{P}(Y_{N(t_n)} = i_n | Y_{N(t_0)} = i_0, Y_{N(t_1)} = i_1, \cdots, Y_{N(t_{n-1})} = i_{n-1}) \\
=&\sum_{k_0, \cdots, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_0} = i_0, \cdots, Y_{k_{n-1}} = i_{n-1}) \mathbb{P}(N(t_0) = k_0, \cdots, N(t_n) = k_n) \\
=&\sum_{k_0, \cdots, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_{n-1}} = i_{n-1}) \mathbb{P}(N(t_0) = k_0, \cdots, N(t_n) = k_n) \\
=&\sum_{k_0, \cdots, k_n} \mathbb{P}(Y_{k_n} = i_n, N(t_0) = k_0, \cdots, N(t_n) = k_n | Y_{k_{n-1}} = i_{n-1}) \\
=&\sum_{k_0, \cdots, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_{n-1}} = i_{n-1}) && \text{(independent increments)} \\
&\quad \cdot \mathbb{P}(N(t_0) - N(0) = k_0, \cdots, N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) \\
=&\sum_{k_{n-1}, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_{n-1}} = i_{n-1}) \mathbb{P}(N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) && \text{(independent increments)} \\
&\quad \cdot \left[\sum_{k_0, \cdots, k_{n-1}} \mathbb{P}(N(t_0) - N(0) = k_0) \cdots \mathbb{P}(N(t_{n-1}) - N(t_{n-2}) = k_{n-1} - k_{n-2}) \right] \\
=&\sum_{k_{n-1}, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_{n-1}} = i_{n-1}) \mathbb{P}(N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) \\
&\quad \cdot \mathbb{P}(N(t_0) = k_0, \cdots, N(t_{n-1}) = k_{n-1}) \\
=&\sum_{k_{n-1}, k_n} \mathbb{P}(Y_{k_n} = i_n | Y_{k_{n-1}} = i_{n-1}) \mathbb{P}(N(t_{n-1}) = k_{n-1}, N(t_n) = k_n) \\
=&\sum_{k_{n-1}, k_n} \mathbb{P}(Y_{N(t_n)} = i_n | Y_{N(t_{n-1})} = i_{n-1}, N(t_{n-1}) = k_{n-1}, N(t_n) = k_n) \\
&\quad \cdot \mathbb{P}(N(t_{n-1}) = k_{n-1}, N(t_n) = k_n) \\
=&\mathbb{P}(Y_{N(t_n)} = i_n | Y_{N(t_{n-1})} = i_{n-1}) \\
=&\mathbb{P}(X(t_n) = i_n | X(t_{n-1}) = i_{n-1})
\end{aligned}$$

$\therefore$ $X(t)$ is a Markov chain in continuous time

To determine the infinitesimal parameters, $\forall\ i, j \in \mathbb{N},\ s \geqslant 0,\ t > 0,$

(1) $i \neq j$

Solution (cont.)

$$
\begin{aligned}
&\mathbb{P}(X(t+s)=j|X(s)=i)\\
=&\mathbb{P}(Y_{N(t+s)}=j|Y_{N(s)}=i)\\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{P}(Y_{N(t+s)}=j|Y_{N(s)}=i,N(s)=m,N(t+s)=n)\mathbb{P}(N(s)=m,N(t+s)=n)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathbb{P}(Y_n=j|Y_m=i)\mathbb{P}(N(s)=m,N(t+s)=n)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathbb{P}(Y_n=j|Y_m=i)\mathbb{P}(N(t+s)-N(s)=n-m|N(s)=m)\mathbb{P}(N(s)=m)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathbb{P}(Y_n=j|Y_m=i)\mathbb{P}(N(t)=n-m)\mathbb{P}(N(s)=m)
\end{aligned}
$$

$\because$ as $t\to 0+$,

$$
\mathbb{P}(N(t)=n-m)=\begin{cases}1-t\lambda+o(t) & ,n-m=0\\ t\lambda+o(t) & ,n-m=1\\ o(t) & ,n-m\geqslant 2\end{cases}
$$

when $n-m=0$ and $i\neq j$,

$$
\mathbb{P}(Y_n=j|Y_m=i)=0
$$

$\therefore$ as $t\to 0+$,

$$
\begin{aligned}
\mathbb{P}(X(t+s)=j|X(s)=i)=&\mathbb{P}(Y_1=j|Y_0=i)[t\lambda+o(t)][1-t\lambda+o(t)]+o(t)\\
=&P_{ij}t\lambda+o(t)\\
q_{ij}=&\lim_{t\to 0}\frac{\mathbb{P}(X(t+s)=j|X(s)=i)-0}{t}\\
=&P_{ij}\lambda
\end{aligned}
$$

(2) $i=j$: as $t\to 0+$,

$$
\begin{aligned}
\mathbb{P}(X(t+s)=i|X(s)=i)=&1-\sum_{\substack{j=1\\ j\neq i}}^{\infty}\mathbb{P}(X(t+s)=j|X(s)=i)\\
=&1-\sum_{\substack{j=1\\ j\neq i}}^{\infty}P_{ij}t\lambda+o(t)\\
q_{ii}=&\lim_{t\to 0}\frac{\mathbb{P}(X(t+s)=i|X(s)=i)-1}{t}\\
=&-\sum_{\substack{j=1\\ j\neq i}}^{\infty}P_{ij}\lambda
\end{aligned}
$$

Solution (cont.)

Therefore, the infinitesimal matrix is

$$A = \begin{pmatrix} -\sum\limits_{\substack{j=1\\ j\neq 1}}^{\infty} P_{1j}\lambda & P_{12}\lambda & P_{13}\lambda & \cdots \\ P_{21}\lambda & -\sum\limits_{\substack{j=1\\ j\neq 2}}^{\infty} P_{2j}\lambda & P_{23}\lambda & \cdots \\ P_{31}\lambda & P_{32}\lambda & -\sum\limits_{\substack{j=1\\ j\neq 3}}^{\infty} P_{3j}\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

PK Problems 6.6.3

Let $X_1(t), X_2(t), \cdots, X_N(t)$ be independent two-state Markov chains having the same infinitesimal matrix

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \end{array}.$$

Determine the infinitesimal matrix for the Markov chain $Z(t) = X_1(t) + \cdots + X_N(t)$.

The state space of $Z(t)$ is $S = \{0, 1, \cdots, N\}$.

$\because$ $X_1(t), X_2(t), \cdots, X_N(t)$ are independent

$\therefore$ $\forall\ t \geqslant 0, h > 0,\ k, l \in \mathbb{N}^+$,

$$\begin{aligned} & \mathbb{P}(Z(t+h) = l | Z(t) = k) \\ =& \mathbb{P}\left(\sum_{i=1}^{N} X_i(t+h) = l \,\middle|\, \sum_{i=1}^{N} X_i(t) = k \right) \\ =& \sum_{\substack{l_1+\cdots+l_N=l \\ 0\leqslant l_1,\cdots,l_N\leqslant 1}} \sum_{\substack{k_1+\cdots+k_N=k \\ 0\leqslant k_1,\cdots,k_N\leqslant 1}} \prod_{i=1}^{N} \mathbb{P}\left(X_i(t+h) - X_i(t) = l_i | X_i(t) - X_i(0) = k_i\right) \end{aligned}$$

$\because$ $\forall\ i \in S$,

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 0) = 1 - \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 1) = 1 - \mu h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 0) = \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 1) = \mu h + o(h)$$

Solution (cont.)

$\therefore \quad \forall\ n \in S,\ k+n \in S$, and $n \geqslant 2$,

$$\begin{aligned}
&\mathbb{P}(Z(t+h)=k+n|Z(t)=k)\\
=&\mathbb{P}(Z(t+h)=k+n, \text{at least exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h)=X_{i_2}(t+h)=1, X_{i_1}(t)=X_{i_2}(t)=0|Z(t)=k)\\
\leqslant&\mathbb{P}(\text{exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h)=X_{i_2}(t+h)=1, X_{i_1}(t)=X_{i_2}(t)=0|Z(t)=k)\\
=&\binom{N}{2}[\lambda h+o(h)]^2\\
=&o(h)
\end{aligned}$$

$\forall\ n \in S,\ k-n \in S$, and $n \geqslant 2$,

$$\begin{aligned}
&\mathbb{P}(Z(t+h)=k-n|Z(t)=k)\\
=&\mathbb{P}(Z(t+h)=k-n, \text{at least exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h)=X_{i_2}(t+h)=0, X_{i_1}(t)=X_{i_2}(t)=1|Z(t)=k)\\
\leqslant&\mathbb{P}(\text{exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h)=X_{i_2}(t+h)=0, X_{i_1}(t)=X_{i_2}(t)=1|Z(t)=k)\\
=&\binom{N}{2}[\mu h+o(h)]^2\\
=&o(h)
\end{aligned}$$

I.e., only when for at most one $l_0 \in S$, X_{l_0} changes its state after $(t, t+h]$ and other $X_l(l \neq l_0)$ remain the same states, i.e. $|j-i| \leqslant 1$, the probability $\mathbb{P}(Z(t+h)=j|Z(t)=i)$ won't be $o(h)$ as $h \to 0$.

$$\begin{aligned}
\mathbb{P}(Z(t+h)=k+1|Z(t)=k) &= \binom{N-k}{1}[\lambda h+o(h)]\prod_{i=1}^{k}[1-\mu h+o(h)]\prod_{m=1}^{N-k-1}[1-\lambda h+o(h)]\\
&= (N-k)\lambda h+o(h)\\
\mathbb{P}(Z(t+h)=k-1|Z(t)=k) &= \binom{k}{1}[\mu h+o(h)]\prod_{i=1}^{k-1}[1-\mu h+o(h)]\prod_{m=1}^{N-k}[1-\lambda h+o(h)]\\
&= k\mu h+o(h)\\
\mathbb{P}(Z(t+h)=k|Z(t)=k) &= 1-\sum_{\substack{j=0\\ j\neq k}}^{N}\mathbb{P}(Z(t+h)=j|Z(t)=k)\\
&= 1-(N-k)\lambda h-k\mu h+o(h)
\end{aligned}$$

$\therefore \quad$ the infinitesimal matrix for $Z(t)$ is

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \end{array}\\
\begin{array}{c} 0\\ 1\\ 2\\ 3\\ \vdots\\ N-1\\ N \end{array} &
\begin{pmatrix}
-N\lambda & N\lambda & 0 & 0 & \cdots & 0 & 0\\
\mu & -(N-1)\lambda-\mu & (N-1)\lambda & 0 & \cdots & 0 & 0\\
0 & 2\mu & -(N-2)\lambda-2\mu & (N-2)\lambda & \cdots & 0 & 0\\
0 & 0 & 3\mu & -(N-3)\lambda-3\mu & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -\lambda-(N-1)\mu & \lambda\\
0 & 0 & 0 & 0 & \cdots & N\mu & -N\mu
\end{pmatrix}
\end{array}$$

GS 6.9.1

Let $\lambda\mu > 0$ and let X be a Markov chain on $\{1, 2\}$ with generator

$$\mathbf{G} = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.$$

(a) Write down the forward equations and solve them for the transition probabilities $p_{ij}(t), i, j = 1, 2$.

The forward equations

$$P'(t) = P(t)G$$

i.e.

$$\begin{cases} p'_{11}(t) = -\mu p_{11}(t) + \lambda p_{12}(t) \\ p'_{12}(t) = \mu p_{11}(t) - \lambda p_{12}(t) \\ p'_{21}(t) = -\mu p_{21}(t) + \lambda p_{22}(t) \\ p'_{22}(t) = \mu p_{21}(t) - \lambda p_{22}(t) \end{cases}$$

with initial condition

$$P(0) = I$$

and constraints

$$\begin{cases} p_{11}(t) + p_{12}(t) = 1 \\ p_{21}(t) + p_{22}(t) = 1 \end{cases}$$

$\therefore$

$$\begin{cases} p'_{11}(t) = \lambda - (\mu + \lambda)p_{11}(t) \\ p_{12}(t) = 1 - p_{11}(t) \\ p_{21}(t) = 1 - p_{22}(t) \\ p'_{22}(t) = \mu - (\mu + \lambda)p_{22}(t) \end{cases}$$

Solve and get

$$P(t) = \begin{pmatrix} \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t} & \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t} \\ \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t} & \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t} \end{pmatrix}$$

(b) Calculate $\mathbf{G}^n$ and hence find $\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{G}^n$. Compare your answer with that to part (a).

$\because$

$$\begin{aligned} \mathbf{G}^2 &= \begin{pmatrix} \mu^2 + \mu\lambda & -(\mu^2 + \mu\lambda) \\ -(\mu\lambda\lambda^2) & \mu\lambda\lambda^2 \end{pmatrix} \\ &= -(\mu + \lambda)\mathbf{G} \end{aligned}$$

Solution (cont.)

$\therefore \quad \forall\, n \in \mathbb{N}, n \geqslant 2$

$$
\begin{aligned}
\mathbf{G}^n &= \mathbf{G}^2 \cdot \mathbf{G}^{n-2} \\
&= -(\mu+\lambda)\mathbf{G} \cdot \mathbf{G}^{n-2} \\
&= -(\mu+\lambda)\mathbf{G}^{n-1} \\
&= \cdots \\
&= (-1)^{n-1}(\mu+\lambda)^{n-1}\mathbf{G}
\end{aligned}
$$

$\because \quad \mu\lambda > 0$

$\therefore \quad \mu + \lambda \neq 0$

$\therefore$

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{G}^n &= \mathbf{I} - \frac{1}{\mu+\lambda}\sum_{n=1}^{\infty}\frac{(-1)^n(\mu+\lambda)^n t^n}{n!}\mathbf{G} \\
&= \mathbf{I} - \frac{1}{\mu+\lambda}[e^{-(\mu+\lambda)t} - 1]\mathbf{G} \\
&= \mathbf{I} + \frac{1}{\mu+\lambda}\mathbf{G} - \frac{1}{\mu+\lambda}e^{-(\mu+\lambda)t}\mathbf{G} \\
&= \begin{pmatrix} \frac{\lambda}{\mu+\lambda} & \frac{\mu}{\mu+\lambda} \\ \frac{\lambda}{\mu+\lambda} & \frac{\mu}{\mu+\lambda} \end{pmatrix} + \begin{pmatrix} \frac{\mu}{\mu+\lambda} & -\frac{\mu}{\mu+\lambda} \\ -\frac{\lambda}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda} \end{pmatrix} e^{-(\mu+\lambda)t} \\
&= \begin{pmatrix} \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t} & \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t} \\ \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t} & \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t} \end{pmatrix}
\end{aligned}
$$

Therefore,

$$
P(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{G}^n
$$

(c) Solve the equation $\pi\mathbf{G} = 0$ in order to find the stationary distribution. Verify that $p_{ij}(t) \to \pi_j$ as $t \to \infty$.

Suppose that the stationary distribution is $\pi = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$, then

$$
\pi\mathbf{G} = 0
$$

is equivalent to

$$
\begin{cases}
-\pi_1\mu + \lambda\pi_2 = 0 \\
\pi_1\mu - \lambda\pi_2 = 0 \\
\pi_1 + \pi_2 = 1
\end{cases}
$$

Solving and get

$$
\pi = \begin{pmatrix} \frac{\lambda}{\mu+\lambda} & \frac{\mu}{\mu+\lambda} \end{pmatrix}
$$

Solution (cont.)

We have

$$\begin{aligned}
\lim_{t\to\infty} p_{11}(t) &= \lim_{t\to\infty}\left[\frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t}\right] = \frac{\lambda}{\mu+\lambda} = \pi_1 \\
\lim_{t\to\infty} p_{12}(t) &= \lim_{t\to\infty}\left[\frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t}\right] = \frac{\mu}{\mu+\lambda} = \pi_2 \\
\lim_{t\to\infty} p_{21}(t) &= \lim_{t\to\infty}\left[\frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}\right] = \frac{\lambda}{\mu+\lambda} = \pi_1 \\
\lim_{t\to\infty} p_{22}(t) &= \lim_{t\to\infty}\left[\frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}\right] = \frac{\mu}{\mu+\lambda} = \pi_2
\end{aligned}$$

since $\mu, \lambda > 0$.

GS 6.9.2

As a continuation of the previous exercise, find:

(a) $\mathbb{P}(X(t) = 2|X(0) = 1, X(3t) = 1)$,

$$\begin{aligned}
&\mathbb{P}(X(t) = 2|X(0) = 1, X(3t) = 1) \\
=&\frac{\mathbb{P}(X(t) = 2, X(3t) = 1|X(0) = 1)}{\mathbb{P}(X(3t) = 1|X(0) = 1)} \\
=&\frac{\mathbb{P}(X(3t) = 1|X(0) = 1, X(t) = 2)\mathbb{P}(X(t) = 2|X(0) = 1)}{\mathbb{P}(X(3t) = 1|X(0) = 1)} \\
=&\frac{\mathbb{P}(X(3t) = 1|X(t) = 2)\mathbb{P}(X(t) = 2|X(0) = 1)}{\mathbb{P}(X(3t) = 1|X(0) = 1)} \\
=&\frac{p_{21}(2t)p_{12}(t)}{p_{11}(3t)} \\
=&\frac{\left[\frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)2t}\right]\left[\frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t}\right]}{\frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)3t}} \\
=&\frac{\mu\lambda}{\mu+\lambda}\frac{(1+e^{-(\mu+\lambda)2t})(1+e^{-(\mu+\lambda)t})}{\lambda+\mu e^{-(\mu+\lambda)3t}}
\end{aligned}$$

(b) $\mathbb{P}(X(t) = 2|X(0) = 1, X(3t) = 1, X(4t) = 1)$.

$$
\begin{aligned}
&\mathbb{P}(X(t)=2|X(0)=1,X(3t)=1,X(4t)=1)\\
=&\frac{\mathbb{P}(X(t)=2,X(3t)=1,X(4t)=1|X(0)=1)}{\mathbb{P}(X(3t)=1,X(4t)=1|X(0)=1)}\\
=&\frac{\mathbb{P}(X(4t)=1|X(0)=1,X(t)=2,X(3t)=1)\mathbb{P}(X(3t)=1|X(0)=1,X(t)=2)\mathbb{P}(X(t)=2|X(0)=1)}{\mathbb{P}(X(4t)=1|X(3t)=1,X(0)=1)\mathbb{P}(X(3t)=1|X(0)=1)}\\
=&\frac{\mathbb{P}(X(4t)=1|X(3t)=1)\mathbb{P}(X(3t)=1|X(t)=2)\mathbb{P}(X(t)=2|X(0)=1)}{\mathbb{P}(X(4t)=1|X(3t)=1)\mathbb{P}(X(3t)=1|X(0)=1)}\\
=&\frac{p_{11}(t)p_{21}(2t)p_{12}(t)}{p_{11}(t)p_{11}(3t)}\\
=&\frac{p_{21}(2t)p_{12}(t)}{p_{11}(3t)}\\
=&\frac{\mu\lambda}{\mu+\lambda}\frac{(1+e^{-(\mu+\lambda)2t})(1+e^{-(\mu+\lambda)t})}{\lambda+\mu e^{-(\mu+\lambda)3t}}
\end{aligned}
$$

GS 6.9.4

Pasta property. Let $X=\{X(t):t\geqslant 0\}$ be a Markov chain having stationary distribution π. We may sample X at the times of a Poisson process: let N be a Poisson process with intensity λ, independent of X, and define $Y_n=X(T_n+)$, the value taken by X immediately after the epoch T_n of the nth arrival of N. Show that $Y=\{Y_n:n\geqslant 0\}$ is a discrete-time Markov chain with the same stationary distribution as X. (This exemplifies the 'Pasta' property: Poisson arrivals see time averages.)

The full assumption of the independence of N and X is not necessary for the conclusion. it suffices that $\{N(s):s\geqslant t\}$ be independent of $\{X(s):s\leqslant t\}$, a property known as 'lack of anticipation'. It is not even necessary that X be Markov; the Pasta property holds for many suitable ergodic processes.

Suppose the state space of X is S and its stationary distribution is $\pi=\left(\pi_i\right)_{i\in S}$. Then the state space of Y is also S.

$\because\quad N(t)$ is a Poisson process

$\therefore\quad T_1-T_0,T_2-T_1,\cdots$ are independent identically distributed

Solution (cont.)

(1) **Markov Property**: $\forall\ n \in \mathbb{N}^+,\ i_0, \cdots, i_n \in S$,

$$\begin{aligned}
&\mathbb{P}(Y_n = i_n | Y_{n-1} = i_{n-1}, \cdots, Y_0 = i_0) \\
=&\mathbb{P}(X(T_n+) = i_n | X(T_{n-1}+) = i_{n-1}, \cdots, X(T_0+) = i_0) \\
=&\int\limits_{t_1,\cdots,t_n \geqslant 0} \mathbb{P}(X(t_n+) = i_n | X(t_{n-1}+) = i_{n-1}, \cdots, X(0) = i_0) \\
&\quad \mathbb{P}(T_n - T_{n-1} = t_n - t_{n-1}, \cdots, T_1 - T_0 = t_1)\mathrm{d}t_1 \cdots \mathrm{d}t_n \\
=&\int\limits_{t_1,\cdots,t_n \geqslant 0} \mathbb{P}(X(t_n+) = i_n | X(t_{n-1}+) = i_{n-1})\mathbb{P}(T_n - T_{n-1} = t_n - t_{n-1}) \\
&\quad \cdots \mathbb{P}(T_1 - T_0 = t_1)\mathrm{d}t_1 \cdots \mathrm{d}t_n \\
=&\int\limits_{t_{n-1},t_n \geqslant 0} \mathbb{P}(X(t_n+) = i_n | X(t_{n-1}+) = i_{n-1})\mathbb{P}(T_n - T_{n-1} = t_n - t_{n-1}) \\
&\quad \cdot \left[\int\limits_{t_1,\cdots,t_{n-2} \geqslant 0} \mathbb{P}(T_{n-1} - T_{n-2} = t_{n-1} - t_{n-2}, \cdots, T_1 = t_1)\mathrm{d}t_1 \cdots \mathrm{d}t_{n-2}\right] \mathrm{d}t_{n-1}\mathrm{d}t_n \\
=&\int\limits_{t_{n-1},t_n \geqslant 0} \mathbb{P}(X(t_n+) = i_n | X(t_{n-1}+) = i_{n-1})\mathbb{P}(T_n - T_{n-1} = t_n - t_{n-1}) \\
&\quad \cdot \left[\int\limits_{t_1,\cdots,t_{n-2} \geqslant 0} \mathbb{P}(T_{n-1} = t_{n-1}, \cdots, T_1 = t_1)\mathrm{d}t_1 \cdots \mathrm{d}t_{n-2}\right] \mathrm{d}t_{n-1}\mathrm{d}t_n \\
=&\int\limits_{t_{n-1},t_n \geqslant 0} \mathbb{P}(X(t_n+) = i_n | X(t_{n-1}+) = i_{n-1})\mathbb{P}(T_n = t_n, T_{n-1} = t_{n-1})\mathrm{d}t_n\mathrm{d}t_{n-1} \\
=&\mathbb{P}(X(T_n+) = i_n | X(T_{n-1}+) = i_{n-1}) \\
=&\mathbb{P}(Y_n = i_n | Y_{n-1} = i_{n-1})
\end{aligned}$$

(2) **Homogeneous**:

$\because\quad X$ is homogeneous

$\therefore$

$$\begin{aligned}
\mathbb{P}(Y_n = i_n | Y_{n-1} = i_{n-1}) &= \mathbb{P}(X(T_n+) = i_n | X(T_{n-1}+) = i_{n-1}) \\
&= \mathbb{P}(X((T_n - T_{n-1})+) = i_n | X(0+) = i_{n-1}) \\
&= \mathbb{P}(X((T_1 - T_0)+) = i_n | X(0+) = i_{n-1}) \\
&= \mathbb{P}(X(T_1+) = i_n | X(T_0+) = i_{n-1}) \\
&= \mathbb{P}(Y_1 = i_n | Y_0 = i_{n-1})
\end{aligned}$$

i.e. Y is homogeneous

(3) **Stationary Distribution**:

$\because\quad \pi$ is the stationary distribution of X

$\therefore\quad \forall\ j \in S,\ n \in \mathbb{N}^+$,

$$\pi_j = \sum_{i \in S} \pi_i \mathbb{P}(X_n = j | X_{n-1} = i)$$

Solution (cont.)

$\because$

$$
\begin{aligned}
\sum_{i\in S}\pi_i\mathbb{P}(Y_n=j|Y_{n-1}=i) &= \sum_{i\in S}\pi_i\mathbb{P}(Y_1=j|Y_0=i)\\
&= \sum_{i\in S}\pi_i\mathbb{P}(X(T_1+)=j|X(T_0+)=i)\\
&= \sum_{i\in S}\pi_i\int\limits_{t_1\geqslant 0}\mathbb{P}(X(t_1+)=j|X(0)=i)\mathbb{P}(T_1=t_1)\mathrm{d}t_1\\
&= \int\limits_{t_1\geqslant 0}\sum_{i\in S}\pi_i\mathbb{P}(X(t_1+)=j|X(0)=i)\mathbb{P}(T_1=t_1)\mathrm{d}t_1\\
&= \int\limits_{t_1\geqslant 0}\pi_j\mathbb{P}(T_1=t_1)\mathrm{d}t_1\\
&= \pi_j
\end{aligned}
$$

and π is a distribution.

$\therefore$ the stationary distribution of Y is the same as X

GS 6.9.12

Let Z be an irreducible discrete-time Markov chain on a countably infinite state space S, having transition matrix $\mathbf{H}=(h_{ij})$ satisfying $h_{ii}=0$ for all states i, and with stationary distribution ν. Construct a continuous-time process X on S for which Z is the imbedded chain, such that X has no stationary distribution.

Suppose that $v=\left(v_i\right)_{i\in S}$. Define the generator G of X by

$$
g_{ij}=\begin{cases} v_ih_{ij} & ,i\neq j\\ -v_i & ,i=j\end{cases}
$$

then the imbeded chain is Z with probability transition matrix $\mathbf{H}$.

$\because$ Z is irreducible and has stationary distribution v.

$\therefore$ $v_i>0(\forall\ i\in S)$. Otherwise, if $\exists\ j_0\in S$, s.t. $v_{j_0}=0$, then by irreducibility, $\forall\ j\in S$, $v_j=0$ and $\sum_{j\in S}v_i=0$ which is a contradiction. And we also have that Z is positive recurrent.

For any $\pi=\left(\pi_i\right)_{i\in S}$, s.t. $\pi G=0$, we have $\forall\ i,j\in S$,

$$
\begin{aligned}
\sum_{i\in S}\pi_ig_{ij}&=0\\
-\pi_jv_j+\sum_{\substack{i\in S\\ i\neq j}}\pi_iv_jh_{ij}&=0
\end{aligned}
$$

Solution (cont.)

Let $u = \Big(\pi_i v_i\Big)_{i \in S}$, then

$$u_j = \sum_{\substack{i \in S \\ i \neq j}} u_i h_{ij}$$

i.e.

$$u = uH$$

$\because$ v is the stationary distribution of positive recurrent and irreducible chain Z

$\therefore$ $x = v$ is the only solution such that $\sum\limits_{i \in S} x_i = 1$ to the equation

$$x = xH$$

$\therefore$

$$u = av$$

where a is a constant. I.e. $\forall\ i \in S$,

$$\pi_i v_i = a v_i$$

$\because$ $\forall\ i \in S$,

$$v_i > 0$$

$\therefore$ $\forall\ i \in S$,

$$\pi_i = a$$

$\because$ S is a countably infinite set

$\therefore$

$$\sum_{i \in S} \pi_i = \sum_{i \in S} a = 0 \quad \text{or} \quad \infty$$

which is not a distribution

$\because$ if π is the stationary distribution of X, then $\forall\ t \geqslant 0$, $\pi P(t) = \pi \Leftrightarrow \pi G = 0$ and the solution of $\pi G = 0$ won't be a distribution

$\therefore$ X has no stationary distribution

Question 1

Show that an irreducible continuous-time Markov chain X starting from $X(0) = i$ does not explode if the stationary distribution π exists and $\pi(i) > 0$.

When $S = \{i\}$, then the case is trivial. Below we only discuss when $|S| > 1$.
Let T_n denotes the time of the nth change in value of the chain X and set $T_0 = 0$ and define the jump chain of X by $Z_n = X(T_n+)$, i.e. the value of X immediately after its jumps. Suppose that the infinitesimal matrix for X is $\mathbf{G} = \Big(g_{ij}\Big)_{S \times S}$, then $g_{ii} \neq 0 (\forall\ i \in S)$ becasue of irreducibility otherwise the chain remains forever in

Solution (cont.)

state i once it has arrived there for the first time. Then $\{Z_n\}$ has transition probability $h_{ij} = \begin{cases} -\frac{g_{ij}}{g_{ii}} & ,i \neq j \\ 0 & ,i = j \end{cases}$.

Suppose that $\forall\ i, j \in S$, $p_{ij}(t) = \mathbb{P}(X(T) = i | X(0) = j)$.

$\because$ $\forall\ j \in S$,

$$\lim_{t\to\infty} p_{ji}(t) = \pi(i) > 0$$

$\therefore$ $\exists\ t_0 > 0$, s.t. $\forall\ t > t_0$,

$$p_{ii}(t) \geqslant \frac{\pi(i)}{2}$$

$\because$

$$g_{ii} > 0$$

and

$$\int_0^\infty p_{ii}(t)\mathrm{d}t \geqslant \int_{t_0}^\infty \frac{\pi(i)}{2}\mathrm{d}t = \infty$$

$\therefore$ from Theorem (27), state i is recurrent

$\because$ $X(0) = i$ and state i is a recurrent state for the jump chain Z

$\therefore$ from Theorem (24), the chain X does not explode