MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 7

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Solutions by

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Question 1

Suppose you can only afford to evaluate 11 terms of either side of the Poisson Sum Formula

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} e^{-\frac{(x-2\pi k)^2}{4t}} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-tk^2} e^{ikx}.$$

(a) Find δ such that the error in the right-hand side (truncated after 11 terms) is smaller than 10^{-14} for $t \geqslant \delta$ and $|x| \leqslant \pi$.

Let

$$\begin{aligned}\left| \frac{1}{2\pi} \sum_{|k|>5} e^{-tk^2} e^{ikx} \right| &\leqslant \frac{1}{2\pi} \sum_{|k|>5} e^{-tk^2} \\ &< \frac{1}{\pi} \sum_{k>5} e^{-6tk} \\ &= \frac{1}{\pi} \cdot \frac{e^{-36t}}{1-e^{-6t}} \leqslant 10^{-14}\end{aligned}$$

$\because \quad \forall\, t > 0$,

$$\left(\frac{e^{-36t}}{1-e^{-6t}}\right)' = \frac{-36e^{-30t} + 30e^{-42t}}{(1-e^{-6t})^2} < 0$$

and

$$\lim_{t\to 0+} \frac{e^{-36t}}{1-e^{-6t}} = +\infty$$

$$\lim_{t\to +\infty} \frac{e^{-36t}}{1-e^{-6t}} = 0$$

we have when $t = 1$,

$$\pi \frac{e^{-36}}{1-e^{-6}} < 10^{-14}$$

So choosing $\delta \geqslant 1$, and $t \geqslant \delta$ the error in the RHS is smaller than 10^{-14}

(b) Find $\Delta > \delta$ such that $\sqrt{4\pi t}$ times the error in the left hand side (truncated after 11 terms) is smaller than 10^{-14} for $0 < t \leqslant \Delta$ and $|x| \leqslant \pi$.

Let

$$\begin{aligned}\left| \sum_{|k|>5} e^{-\frac{(x-2\pi k)^2}{4t}} \right| &\leqslant \sum_{k>5} e^{-\frac{(-\pi-2\pi k)^2}{4t}} \\ &\leqslant e^{-\frac{\pi^2}{4t}} \sum_{|k|>5} e^{-\frac{(24\pi^2+24\pi)k}{t}} \\ &= 2e^{-\frac{\pi^2}{4t}} \cdot \frac{e^{-\frac{144(\pi^2+\pi)}{t}}}{1-e^{-6\frac{4\pi^2+4\pi}{t}}} \\ &< 2\frac{e^{-\frac{\pi^2}{4t}-\frac{144(\pi^2+\pi)}{t}}}{1-e^{-24\frac{\pi^2+\pi}{t}}} \leqslant 10^{-14}\end{aligned}$$

Solution (cont.)

$\because \quad \forall\, t > 0,$

$$\left(\frac{e^{-\frac{\pi^2}{4t}-\frac{144(\pi^2+\pi)}{t}}}{1-e^{-24\frac{\pi^2+\pi}{t}}}\right)' > 0$$

and

$$\lim_{t\to 0+}\frac{e^{-\frac{\pi^2}{4t}-\frac{144(\pi^2+\pi)}{t}}}{1-e^{-24\frac{\pi^2+\pi}{t}}} = 0$$

we have when $t = 1$,

$$\frac{e^{-\frac{\pi^2}{4t}-\frac{144(\pi^2+\pi)}{t}}}{1-e^{-24\frac{\pi^2+\pi}{t}}} < 10^{-14}$$

So choosing $\Delta \leqslant 1$, and $0 < t \leqslant \Delta$ the error in the LHS times $\sqrt{4\pi t}$ is smaller than 10^{-14}

(c) Invent an efficient strategy for evaluating $K(x,t)$ accurately for any $t > 0$ and $|x| \leqslant \pi$.

Given tolerance $\epsilon > 0$, from (a) we have $\forall\, n \in \mathbb{N}$, $\exists\, \delta > 0$, s.t. $\forall\, t \geqslant \delta$, $|x| \leqslant \pi$,

$$\left|\frac{1}{2\pi}\sum_{|k|>n} e^{-tk^2}e^{ikx}\right| < \epsilon$$

From (b) we have $\exists\, \Delta > \delta$, s.t. $\forall\, 0 < t \leqslant \Delta$ and $|x| \leqslant \pi$,

$$\sqrt{4\pi t}\left|\sum_{|k|>n} e^{-\frac{(x-2\pi k)^2}{4t}}\right| < \epsilon$$

Therefore we can evaluate $K(x,t)$ accurately for any $t > 0$ and $|x| \leqslant \pi$

Question 2

(a) Use the Poisson Sum Formula to prove the Euler-Maclaurin summation formula

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)\mathrm{d}x - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) - \cdots$$

for a smooth function f. (**Hint**: extend f to be even.)

Extend f to be even

$$F(x) = \begin{cases} f(-x) & x < 0 \\ f(x) & x \geqslant 0 \end{cases}$$

$\because$

$$\sum_{k\in\mathbb{Z}} f(x+kT) = \frac{\sqrt{2\pi}}{T}\sum_{k=-\infty}^{\infty}\hat{f}\left(\frac{2\pi k}{T}\right)e^{\frac{2\pi ikx}{T}}$$

Solution (cont.)

$\therefore$ let $x = 0$ and $T = 1$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k)$$

Suppose that $\forall\, n \in \mathbb{N}$, $f^{(n)}(x) \to 0 \qquad (x \to \infty)$

$\because$

$$\sum_{n\in\mathbb{Z}} f(n) = 2\sum_{n=1}^{\infty} f(n) + f(0)$$

$\therefore$

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n) &= \frac{1}{2}f(0) + \sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) \\
&= \frac{1}{2}f(0) + \frac{1}{2} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int_0^{\infty} f(x)\cos(2\pi i k x)\mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{f(x)\sin(2\pi i k x)}{2\pi i k}\bigg|_0^{\infty} - \int_0^{\infty} \frac{f'(x)\sin(2\pi i k x)}{2\pi i k}\mathrm{d}x \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int_0^{\infty} \frac{f'(x)\sin(2\pi i k x)}{2\pi i k}\mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{f'(x)\cos(2\pi i k x)}{(2\pi i k)^2}\bigg|_0^{\infty} - \int_0^{\infty} \frac{f^{(2)}(x)\cos(2\pi i k x)}{(2\pi i k)^2}\mathrm{d}x \right] \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{f'(0)}{(2\pi i k)^2} - \int_0^{\infty} \frac{f^{(2)}(x)\cos(2\pi i k x)}{(2\pi i k)^2}\mathrm{d}x \right] \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \frac{2f'(0)}{4\pi^2}\cdot\frac{\pi^2}{6} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int_0^{\infty} \frac{f^{(2)}(x)\cos(2\pi i k x)}{(2\pi i k)^2}\mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \frac{1}{12}f'(0) - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int_0^{\infty} \frac{f^{(2)}(x)\cos(2\pi i k x)}{(2\pi i k)^2}\mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \frac{1}{12}f'(0) + \frac{2f'''(0)}{(2\pi i)^4}\cdot\frac{\pi^4}{90} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int_0^{\infty} \frac{f^{(4)}(x)\cos(2\pi i k x)}{(2\pi i k)^4}\mathrm{d}x \\
&= \frac{1}{2}f(0) + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) - \cdots
\end{aligned}$$

(b) Find formulas for the rest of the coefficients B_{2k} in

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)\mathrm{d}x - \sum_{1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0)$$

by applying the formula to a suitable test function like $f(x) = e^{-tx}$.

$$\sum_{n=0}^{\infty} e^{-tn} = \frac{1}{2} + \int_0^{\infty} e^{-tx}\mathrm{d}x - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(-t)^{2k-1}$$

$$\frac{1}{1-e^{-t}} = \frac{1}{2} + \frac{1}{t} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(-t)^{2k-1}$$

$$t = \left[\frac{1}{2}t + 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}t^{2k}\right]\left(-\sum_{n=1}^{\infty} \frac{(-t)^n}{n!}\right)$$

$\therefore$ by comparing the power coefficients, we have

$$\begin{cases} 0 = 0 \\ 1 = 1 \\ 0 = \frac{1}{2} - \frac{1}{2} \\ 0 = \frac{B_2}{2} - \frac{1}{4} + \frac{1}{3} \\ 0 = \frac{1}{2} \cdot \frac{1}{(2k-1)!} - \sum\limits_{i=0}^{k-1} \frac{B_{2i}}{(2i)!} \cdot \frac{1}{(2k-2i)!} & ,k \in \mathbb{N}^+ \\ 0 = \frac{1}{2} \cdot \frac{1}{(2k)!} + \sum\limits_{i=0}^{k} \frac{B_{2i}}{(2i)!} \cdot \frac{1}{(2k+1-2i)!} & ,k \in \mathbb{N}^+ \end{cases}$$

here $B_0 = 1$

$\therefore$

$$\begin{cases} B_2 = \frac{1}{6} \\ B_4 = -\frac{1}{30} \\ B_6 = \frac{1}{42} \\ B_8 = -\frac{1}{30} \\ \qquad \vdots \end{cases}$$

Question 3

Fix $t > 0$ and let

$$G(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

(a) Compute $\hat{G}(k,t)$.

$$\begin{aligned}
\hat{G}(k,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x,t)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{2t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{2t}}\int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x+2itk)^2-tk^2}\mathrm{d}x \\
&\xlongequal{z=\frac{x+2itk}{2\sqrt{t}}} \frac{1}{\sqrt{2\pi e^{tk^2}}}\int_{-\infty}^{\infty} e^{-z^2}\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}}e^{-tk^2}
\end{aligned}$$

(b) Compute $\hat{G}(k,t)$ by a different method.

$$\begin{aligned}
\hat{G}(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x,t)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x,t)\sum_{n=0}^{\infty}\frac{(-ikx)^n}{n!}\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{t}}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{4t}}\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{t}}\sum_{m=0}^{\infty}\frac{(-ik)^{2m}}{(2m)!}\int_{-\infty}^{\infty} x^{2m} e^{-\frac{x^2}{4t}}\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{t}}\sum_{m=0}^{\infty}\frac{(4t)^{\frac{2m+1}{2}}k^{2m}}{(2m)!}\int_{-\infty}^{\infty} z^{m+\frac{1}{2}} e^{-z}\mathrm{d}z \\
&= \frac{1}{2\pi\sqrt{t}}\sum_{m=0}^{\infty}\frac{(4t)^{\frac{2m+1}{2}}k^{2m}}{(2m)!}\Gamma\left(m+\frac{1}{2}\right) \\
&= \frac{1}{2\pi\sqrt{t}}\sum_{m=0}^{\infty}\frac{(4t)^{\frac{2m+1}{2}}k^{2m}}{(2m)!}\frac{(2m-1)!}{2^{2m-1}(m-1)!}\sqrt{\pi} \\
&= \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{\infty}\frac{(-tk^2)^m}{m!} \\
&= \frac{1}{\sqrt{2\pi}}e^{-tk^2}
\end{aligned}$$

(c) Show that

$$G_t = G_{xx}$$

for $t > 0$.

$\because$

$$\begin{aligned}
\widehat{G_t}(k,t) &= -\frac{k^2}{\sqrt{2\pi}}e^{-tk^2} \\
\widehat{G_{xx}}(k,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G_{xx}(x,t)e^{-ikx}\mathrm{d}x \\
&= (-ik)^2\hat{G}(k,t) \\
&= -\frac{k^2}{\sqrt{2\pi}}e^{-tk^2}
\end{aligned}$$

$\therefore$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{G_t}(k,t)\mathrm{d}k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{G_{xx}}(k,t)\mathrm{d}k$$

i.e.

$$G_t = G_{xx}$$

(d) Let $f \in L^2(\mathbb{R})$ be continuous and bounded. Show that

$$\int_{-\infty}^{\infty} G(x-y,t)f(y)\mathrm{d}y \to f(x)$$

for every $x \in \mathbb{R}$ as $t \to 0$.

$$\int_{-\infty}^{\infty} G(x-y,t)f(y)\mathrm{d}y \xlongequal{z=\frac{x-y}{\sqrt{t}}} \int_{-\infty}^{\infty} -\sqrt{t}G(\sqrt{t}z,t)f(x-\sqrt{t}z)\mathrm{d}z$$

$\because$ f is bounded

$\therefore$ $\exists\ M > 0$, s.t. $\forall\ x \in \mathbb{R}$, $|f(x)| < M$

$\therefore$

$$\left|-\sqrt{t}G(\sqrt{t}z,t)f(x-\sqrt{t}z)\right| \leqslant \frac{M}{\sqrt{4\pi}}$$

$\because$ f is continuous

$\therefore$

$$\begin{aligned}
\lim_{t\to 0} -\sqrt{t}G(\sqrt{t}z,t)f(x-\sqrt{t}z) &= \lim_{t\to 0}\frac{e^{-z^2}}{\sqrt{4\pi}}f(x-\sqrt{t}z) \\
&= \frac{e^{-z^2}}{\sqrt{4\pi}}f(x)
\end{aligned}$$

$\therefore$ by Dominated Convegence Theorem, $\forall\ x \in \mathbb{R}$, as $t \to 0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} G(x-y,t)f(y)\mathrm{d}y &\to \int_{\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{4\pi}}f(x)\mathrm{d}z \\
&= f(x)\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{4\pi}}\mathrm{d}z \\
&= f(x)
\end{aligned}$$

(e) Solve the inhomogeneous initial-value problem

$$u_t = u_{xx} + \rho(x,t)$$

for $x \in \mathbb{R},\ t > 0$, subject to the initial condition

$$u(x,0) = 0.$$

$\because$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(u_t - u_{xx} - \rho(x,t))e^{-ikx}\mathrm{d}x = \hat{u}_t(k,t) + k^2\hat{u}(k,t) - \hat{\rho}(k,t) = 0$$

$\therefore$

$$\hat{u}_t(k,t) = -k^2\hat{u}(x,t) + \hat{\rho}(k,t)$$

$\therefore$

$$\begin{aligned}\hat{u}(k,t) &= e^{-k^2 t}\left(\int_0^t \hat{\rho}(k,y)e^{k^2 y}\mathrm{d}y + \hat{u}(k,0)\right)\\ &= e^{-k^2 t}\int_0^t \hat{\rho}(k,y)e^{k^2 y}\mathrm{d}y + e^{-k^2 t}\hat{u}(k,0)\end{aligned}$$

$\therefore$

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,t)e^{ikx}\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{-k^2 t}\int_0^t \hat{\rho}(k,y)e^{k^2 y}\mathrm{d}y\cdot e^{ikx} + e^{-k^2 t}\hat{u}(k,0)e^{ikx}\right]\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k^2 t}\int_0^t \hat{\rho}(k,y)e^{k^2 y}\mathrm{d}y\cdot e^{ikx}\mathrm{d}k + e^{-k^2 t}u(k,0)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^t \hat{\rho}(k,y)e^{k^2 y}e^{ikx-k^2 t}\mathrm{d}y\mathrm{d}k\end{aligned}$$

Question 4

(a) Find $t > 0$ such that the Gaussian $G(x,t)$ from Question 3 is an eigenfunction of the Fourier transform.

Let

$$\hat{G}(k,t) = G(k,t)$$

i.e.

$$\frac{1}{\sqrt{2\pi}}e^{-tk^2} = \frac{e^{-\frac{k^2}{4t}}}{\sqrt{4\pi t}}$$

we have

$$\begin{cases} t = \dfrac{1}{4t} \\ \dfrac{1}{\sqrt{2\pi}} = \dfrac{1}{\sqrt{4\pi t}} \\ t > 0 \end{cases}$$

i.e.

$$t = \frac{1}{2}$$

(b) Let F be the $N \times N$ discrete Fourier transform matrix with elements

$$F_{jk} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i jk}{N}}$$

for $0 \leqslant j,\ k \leqslant N-1$. Apply the Poisson Sum Formula to $G(x,t)$ and choose parameters x and T to find a formula for an eigenvector $g \in \mathbb{C}^N$ and eigenvalue $\lambda \in \mathbb{C}$ of F.

(**Hint**: write the index of summation $k = p + qN$ and the sum over k as a double sum over $p = 0$ to $N-1$ and $q \in \mathbb{Z}$.)

$\because$

$$\sum_{k=-\infty}^{\infty} G(x+kT, t) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \hat{G}\left(\frac{2\pi k}{T}, t\right) e^{\frac{2\pi i k x}{T}}$$

$\therefore$

$$\sum_{k=-\infty}^{\infty} G(x+kT, t) = \frac{\sqrt{2\pi}}{T} \sum_{q=-\infty}^{\infty} \sum_{p=0}^{N-1} \hat{G}\left(\frac{2\pi(p+qN)}{T}, t\right) e^{\frac{2\pi i (p+qN) x}{T}}$$

let $t = \frac{1}{2}$,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{[x+kT]^2}{2}}}{\sqrt{2\pi}} &= \frac{\sqrt{2\pi}}{T} \sum_{q=-\infty}^{\infty} \sum_{p=0}^{N-1} \frac{e^{-\frac{1}{2}\left[\frac{2\pi(p+qN)}{T}\right]^2}}{\sqrt{2\pi}} e^{\frac{2\pi i (p+qN) x}{T}} \\ &= \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left[\frac{2\pi(p+qN)}{T}\right]^2}}{T} e^{\frac{2\pi i (p+qN) x}{T}} \end{aligned}$$

Let $T = \sqrt{2\pi N}$ and $x = j\sqrt{\frac{2\pi}{N}}$ $(\forall\ j \in \mathbb{N}, 0 \leqslant j \leqslant N-1)$,

$$\sum_{k=-\infty}^{\infty} \frac{e^{-\frac{\pi}{N}[j+(p+qN)N]^2}}{\sqrt{2\pi}} = \sum_{q=-\infty}^{\infty} \sum_{p=0}^{N-1} \frac{e^{-\frac{1}{2}\left[\frac{2\pi(p+qN)}{\sqrt{2\pi N}}\right]^2}}{\sqrt{2\pi N}} e^{\frac{2\pi i (p+qN) j}{N}}$$

$$\sum_{q=-\infty}^{\infty} e^{-\frac{\pi}{N}(j+qN)^2} = \sum_{p=0}^{N-1} \sum_{q=-\infty}^{\infty} e^{-\frac{\pi}{N}(p+qN)^2} \frac{e^{\frac{2\pi i jp}{N}}}{\sqrt{N}} \tag{1}$$

Suppose that the eigenvector $g = \begin{pmatrix} g_0 & g_1 & \cdots & g_N \end{pmatrix}^T$, then (1) becomes $\forall\ j \in \mathbb{N},\ 0 \leqslant j \leqslant N-1$,

$$g_j = \sum_{p=0}^{N-1} F_{jp} g_p$$

$\therefore$

$$g_p = \sum_{q=-\infty}^{\infty} e^{-\frac{\pi}{N}(p+qN)^2}$$

$$\lambda = 1$$