MATH 118:
Fourier Analysis and Wavelets

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Problem Set 6

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Solutions by

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Question 1

Prove the Weierstrass Approximation Theorem: Every continuous function $f : [-1,1] \to \infty$ can be uniformly approximated by polynomials. I.e. given $\epsilon > 0$, there exists a degree $n \geqslant 0$ and a degree-n polynomial $p(x) = p_0 + p_1 x + \cdots + p_n x^n$ such that

$$|f(x) - p(x)| \leqslant \epsilon \qquad |x| \leqslant 1.$$

(a) Define

$$g(t) = f(\cos t) \qquad \text{for } t \leqslant \pi$$

Show that g is even, periodic and continuous for $|t| \leqslant \pi$.

Proof.
(1)
$\because$

$$g(-t) = f(\cos(-t)) = f(\cos t) = g(t)$$

$\therefore$ g is even
(2)
$\because$

$$g(t + 2\pi) = f(\cos(t + 2\pi)) = f(\cos t) = g(t)$$

$\therefore$ g is periodic
(3)
$\because$ $f(x)$ is continuous
$\therefore$ $\forall\ x_0 \in [-1,1],\ \forall\ \epsilon > 0,\ \exists\ \delta > 0$, s.t. $\forall\ x \in [-1,1]$, $|x - x_0| < \delta$,

$$|f(x_0) - f(x)| < \epsilon$$

$\because$ $\cos t$ is continuous
$\therefore$ $\forall\ t_0 \in \mathbb{R}, x_0 = \cos t$, for the given δ, $\exists\ \gamma > 0$, s.t. $\forall\ t \in \mathbb{R}$, $|t - t_0| < \gamma$,

$$|\cos t_0 - \cos t| < \delta$$

$\therefore$

$$|f(\cos t_0) - f(\cos t)| < \epsilon$$

i.e. $g(t)$ is continuous $\square$

(b) Find a sequence of even trigonometric polynomials

$$q_n(t) = \sum_{|k| \leqslant n} q_{nk} \cos(kt)$$

converging uniformly to g as $n \to \infty$.

$\because$ $g(t), \cos t$ is even and $\sin t$ is odd in $[-\pi, \pi]$

Solution (cont.)

$\therefore$

$$\begin{aligned}\hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(t)e^{-ikt}\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(t)\cos(kt)\mathrm{d}t\end{aligned}$$

$\therefore$ the fourier expansion of g is

$$g(t) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\hat{g}(k)e^{ikt}$$

Let

$$\begin{aligned}G_n(t) &= \frac{1}{\sqrt{2\pi}}\sum_{k=-n}^{n}\hat{g}(k)e^{ikt} \\ &= \frac{1}{\sqrt{2\pi}}\hat{g}(0) + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{n}\hat{g}(k)\cos(kt)\end{aligned}$$

$\because$ from Fejer's Theorem, $\frac{1}{n}\sum_{k=0}^{n-1}G_k(t) \to g(t)$ uniformly and

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}G_k(t) &= \frac{1}{n+1}\sum_{k=0}^{n}\left[\frac{1}{\sqrt{2\pi}}\hat{g}(0) + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{n}\hat{g}(k)\cos(kt)\right] \\ &= \frac{1}{\sqrt{2\pi}}\hat{g}(0) + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{n}\frac{n+1-k}{n+1}\hat{g}(k)\cos(kt)\end{aligned}$$

$\therefore$

$$q_n(t) = \frac{1}{n+1}\sum_{k=0}^{n-1}G_k(t)$$

where

$$q_{nk} = \begin{cases}\frac{1}{\sqrt{2\pi}}\hat{g}(0) & , k = 0 \\ \sqrt{\frac{2}{\pi}}\frac{n+1-k}{n+1}\hat{g}(k) & , k \neq 0\end{cases}$$

(c) Prove by induction that

$$T_n(x) = \cos(nt)$$

is a polynomial in the variable $x = \cos t$.

Proof.
When $n = 1$,

$$T_1(x) = \cos t = x$$

Solution (cont.)

When $n = 2$,

$$\begin{aligned} T_2(x) &= \cos(2t) \\ &= 2\cos^2 t - 1 \\ &= 2x^2 - 1 \end{aligned}$$

When $n \geqslant 3$

$$\begin{aligned} T_n(x) &= \cos(nt) \\ &= \cos[(n-1)t + t] \\ &= \cos[(n-1)t]\cos t - \sin[(n-1)t]\sin t \\ &= xT_{n-1}(x) - \sin[(n-2)t + t]\sin t \\ &= xT_{n-1}(x) - \sin[(n-2)t]\cos t \sin t - \cos[(n-2)t]\sin^2 t \\ &= xT_{n-1}(x) - \sin[(n-2)t]\cos t \sin t - \cos[(n-2)t](1 - \cos^2 t) \\ &= xT_{n-1}(x) - T_{n-2}(x) - \sin[(n-2)t]\cos t \sin t + \cos[(n-2)t]\cos^2 t \\ &= xT_{n-1}(x) - T_{n-2}(x) + \cos[(n-1)t]\cos t \\ &= 2xT_{n-1}(x) - T_{n-2}(x) \end{aligned}$$

Because $T_1(x), T_2(x)$ are both polynomials of x, $T_3(x)$ is aslo a polynomial. The same as $n \geqslant 3$. □

(d) Prove the Weierstrass Approximation Theorem.

Proof.

$\because$ from (b) we have that $q_n(t)(n \geqslant 0)$ are linear combination of $T_n(x)$ and from (c) $T_n(x)(n \in \mathbb{N})$ are polynomials in terms of $x = \cos t$

$\therefore$ $q_n(t)$ is degree-n polynomials in terms of $x = \cos t$, i.e. $q_n(t) = \sum\limits_{k=0}^{n} p_{nk} x^k = p_n(x)$, and $q_n(t) \to g(t) = f(\cos t)$ uniformly

$\therefore$ $\forall\, \epsilon > 0$, $\exists\, n \in \mathbb{N}$, s.t. $\forall\, t \in \mathbb{R}$, $|f(\cos t) - p_n(t)| < \epsilon$

i.e. $\forall\, x \in [-\pi, \pi]$,

$$|f(x) - p_n(x)| < \epsilon$$

□

Question 2

Solve the classical moment problem: is every continuous function $f : [1, 1] \to C$ uniquely determined by the sequence $\{m_0, m_1, \cdots\}$ of its moments

$$m_k = \int_{-1}^{1} x^k f(x) \mathrm{d}x?$$

$\because \quad f(x)$ is continuous on $[-1,1]$

$\therefore \quad \exists\, M > 0$, s.t. $\forall\, k \in \mathbb{N}, \forall\, x \in [-1,1]$,

$$|x^k f(x)| \leqslant |f(x)| \leqslant M$$

i.e.

$$m_k < \infty$$

Suppose that $f(x), g(x)$ are continuous on $[-1,1]$ and $m_{f,k} = m_{g,k}$, then $f, g \in L^2(-1,1)$, $\forall\, k \in \mathbb{N}$,

$$\int_{-1}^{1} x^k f(x)\mathrm{d}x = \int_{-1}^{1} x^k g(x)\mathrm{d}x$$

i.e.

$$\int_{-1}^{1} x^k [f(x) - g(x)]\mathrm{d}x = 0$$

From the Weierstrass Approximation Theorem, $\forall\, \epsilon > 0, \exists\, n \in \mathbb{R}$ and $p(x) = \sum\limits_{i=0}^{n} p_i x^i$ s.t. $|f(x) - g(x) - p(x)| < \epsilon$ $|x| \leqslant 1$ since $f(x) - g(x)$ is also continuous and $f - g \in L^2(-1,1)$.

$\because$

$$\sum_{i=0}^{n} \int_{-1}^{1} p_i x^k [f(x) - g(x)]\mathrm{d}x = 0$$

i.e.

$$\int_{-1}^{1} p(x)[f(x) - g(x)]\mathrm{d}x = 0$$

$\therefore$

$$\begin{aligned}
\int_{-1}^{1} [f(x) - g(x)]^2 \mathrm{d}x &= \int_{-1}^{1} [f(x) - g(x)][f(x) - g(x) - p(x) + p(x)]\mathrm{d}x \\
&= \int_{-1}^{1} [f(x) - g(x)][f(x) - g(x) - p(x)]\mathrm{d}x \\
&\leqslant \sqrt{\int_{-1}^{1} [f(x) - g(x)]^2 \mathrm{d}x} \sqrt{\int_{-1}^{1} [f(x) - g(x) - p(x)]^2 \mathrm{d}x} \\
&\leqslant \|f - g\| \epsilon
\end{aligned}$$

$\therefore$

$$\int_{-1}^{1} [f(x) - g(x)]^2 \mathrm{d}x = 0$$

i.e.

$$f(x) - g(x) = 0 \qquad (a.e.)$$

i.e.

$$f(x) = g(x) \qquad (a.e.)$$

$\because \quad f(x), g(x)$ are continuous

$\therefore$

$$f(x) \equiv g(x)$$

i.e. every contiunous function in $[-1,1]$ can be unique determined by the sequence of its moments.

Question 3

(a) Compute all the moments m_k over $[0, \infty)$

$$m_k = \int_0^\infty x^k f(x) \mathrm{d}x$$

for $f(x) = e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}$.

$$\begin{aligned} m_k &= \int_0^\infty x^k f(x) \mathrm{d}x \\ &= \int_0^\infty x^k e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}} \mathrm{d}x \\ &\xlongequal{t=x^{\frac{1}{4}}} \int_0^\infty 4t^{4k+3} e^{-t} \sin t \mathrm{d}t \end{aligned}$$

$\because$

$$\begin{aligned} \int_0^\infty t^{4k+3} e^{(i-1)t} \mathrm{d}t &= \frac{1}{i-1} t^{4k+3} e^{(i-1)t} \Big|_0^\infty - \frac{4k+3}{i-1} \int_0^\infty t^{4k+2} e^{(i-1)t} \mathrm{d}t \\ &= \cdots \\ &= -\frac{(4k+3)!}{(i-1)^{4k+3}} \int_0^\infty e^{(i-1)t} \mathrm{d}t \\ &= \frac{(4k+3)!}{(i-1)^{4k+4}} \\ &= \frac{(-1)^{k+1}(4k+1)!}{4^{k+1}} + i \cdot 0 \\ \int_0^\infty t^{4k+3} e^{(i-1)t} \mathrm{d}t &= \int_0^\infty t^{4k+3} e^{-t} \cos t \mathrm{d}t + i \int_0^\infty t^{4k+3} e^{-t} \sin t \mathrm{d}t \end{aligned}$$

$\therefore$

$$m_k = \int_0^\infty 4t^{4k+3} e^{-t} \sin t \mathrm{d}t = 0$$

(b) Discuss in view of your answer to Question 2.

In Question 3 (a), the moments m_k cannot determine $f(x)$ uniquely since $m_k \equiv 0$ for both $f(x) = e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}$ and $f(x) = 0$.

It is because that in infinite interval, we cannot assure that $\exists\ p(x) = \sum_{i=0}^{n} p_i x^i$ s.t. $p(x)$ conveges in $f(x) - g(x)$ in the solution of Question 2. I.e., in infinite interval Weierstrass Approximation Theorem doesn't hold.

Question 4

(a) Compute the coefficients $f(k)$ of the Fourier sine series

$$\sum_{k=1}^{\infty} \hat{f}(k)\sin(kx)$$

over the interval $|x| \leqslant \pi$ for the odd function $f(x) = \frac{1}{2}sign(x)$.

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(kt)\mathrm{d}t - \frac{1}{\pi}\int_{-\pi}^{0} \sin(kt)\mathrm{d}t \\
&= \frac{1}{k\pi}\cos(kt)\Big|_0^{\pi} - \frac{1}{k\pi}\cos(kt)\Big|_{-\pi}^{0} \\
&= \frac{2[(-1)^k - 1]}{k\pi} \qquad k \in \mathbb{N}^+
\end{aligned}$$

(b) Find an explicit formula for the first critical point $\theta_N > 0$ of the partial sum error

$$g_N(x) = \sum_{k=1}^{N} \hat{f}(k)\sin(kx) - \frac{1}{2}.$$

(I.e. find the smallest positive solution θ_N of the equation $g_N'(\theta) = 0$.)

Let

$$\begin{aligned}
g_N'(x) &= \sum_{k=1}^{N} \hat{f}(k)k\cos(kx) \\
&= \sum_{k=1}^{N} \frac{2[(-1)^k - 1]}{\pi}\cos(kx) \\
&= -\frac{4}{\pi}\sum_{k=1}^{\lceil \frac{N}{2} \rceil} \cos[(2k-1)x] \\
&= 0
\end{aligned}$$

the first critical point satisfies $\forall\ i \in \mathbb{N},\ 1 \leqslant i \leqslant \lceil \frac{N}{2} \rceil$,

$$\cos\left[\left(2\left\lceil\frac{N}{2}\right\rceil - 2i - 1\right)x\right] = -\cos\left[(2i-1)\,x\right]$$

i.e.

$$\left\lceil\frac{N}{2}\right\rceil x - \frac{\pi}{2} = \frac{\pi}{2} - x$$

we get

$$\theta_N = \frac{\pi}{\lceil \frac{N}{2} \rceil + 1}$$

(c) Evaluate the limiting overshoot

$$\lim_{N\to\infty} g_N(\theta_N)$$

$$
\begin{aligned}
g_N(\theta_N) &= \sum_{k=1}^{N} \hat{f}(k)\sin(k\theta_N) - \frac{1}{2} \\
&= \sum_{k=1}^{N} \frac{2[(-1)^k - 1]}{k\pi} \sin(k\theta_N) - \frac{1}{2} \\
&= -\frac{4}{\pi} \sum_{k=1}^{\lceil \frac{N}{2} \rceil} \frac{\sin[(2k-1)\theta_N]}{2k-1} - \frac{1}{2} \\
&= -\frac{2}{\pi} \sum_{k=1}^{\lceil \frac{N}{2} \rceil} \frac{\sin[(2k-1)\theta_N]}{(2k-1)\theta_N}(2\theta_N) - \frac{1}{2} \\
&\to -\frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x - \frac{1}{2}
\end{aligned}
$$

(d) Explain Gibbs phenomenon quantitatively.

Gibbs phenomennon appears at the discontinuous point, i.e. the end points of the interval $[-\pi, \pi]$, $\lim\limits_{N\to\infty} \theta_N = \pi$ and the partial sum $\lim\limits_{N\to\infty} g_N(\theta_N) = -\frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\mathrm{d}x - \frac{1}{2}$. When N increases, the absolute value of partial sum near π will not decrease.