Question 1 Prove the Weierstrass Approximation Theorem: Every continuous function $f : [-1,1] \to \infty$ can be uniformly approximated by polynomials. I.e. given $\epsilon > 0$, there exists a degree $n \geq 0$ and a degree-n polynomial $p(x) = p_0 + p_1 x + \cdots + p_n x^n$ such that

$$|f(x) - p(x)| \leq \epsilon \qquad \text{for} |x| \leq 1.$$

(a) Define

$$g(t) = f(\cos t) \qquad \text{for} |t| \leq \pi.$$

Show that g is even, periodic and continuous for $|t| \leq \pi$.

(b) Find a sequence of even trigonometric polynomials

$$q_n(t) = \sum_{|k| \leq n} q_{nk} \cos(kt)$$

converging uniformly to g as $n \to \infty$.

(c) Prove by induction that

$$T_n(x) = \cos(nt)$$

is a polynomial in the variable $x = \cos t$.

(d) Prove the Weierstrass Approximation Theorem.

Question 2 Solve the classical moment problem: is every continuous function $f : [-1,1] \to C$ uniquely determined by the sequence $\{m_0, m_1, \ldots\}$ of its moments

$$m_k = \int_{-1}^{1} x^k f(x) \mathrm{d}\, x?$$

Question 3 (a) Compute all the moments m_k over $[0, \infty)$

$$m_k = \int_0^\infty x^k f(x) \mathrm{d}\, x$$

for $f(x) = \exp(-x^{1/4}) \sin(x^{1/4})$.

(b) Discuss in view of your answer to Question 2.

Question 4 (a) Compute the coefficients $\hat{f}(k)$ of the Fourier sine series

$$\sum_{k=1}^{\infty} \hat{f}(k) \sin kx$$

over the interval $|x| \leq \pi$ for the odd function $f(x) = \frac{1}{2}\text{sign}(x)$.

(b) Find an explicit formula for the first critical point $\theta_N > 0$ of the partial sum error

$$g_N(x) = \sum_{k=1}^{N} \hat{f}(k) \sin kx - \frac{1}{2}.$$

(I.e. find the smallest positive solution θ_N of the equation $g'_N(\theta) = 0$.)

(c) Evaluate the limiting overshoot

$$\lim_{N \to \infty} g_N(\theta_N)$$

(d) Explain Gibbs' phenomenon quantitatively.