TOPIC Edgeworth Expansions. Let X_1 be a random variable with characteristic function ψ_1, mean $E(X_1) = 0$, variance $E(X_1^2) = 1$, and $E(|X|^{r+2}) < \infty$ for some integer $r \geq 0$. Then X_1 has cumulants

$$\kappa_0 = 0 = \kappa_1,\ \kappa_2 = 1,\ \kappa_3,\ \ldots,\ \kappa_{r+2}$$

and by Taylor's theorem the cumulant generating function $K_1(\tau) := \log(\psi_1(\tau))$ satisfies

$$\begin{aligned} K_1(\tau) &= \sum_{j=0}^{r+2} \frac{\kappa_j (i\tau)^j}{j!} + o(\tau^{r+2}) \\ &\approx -\frac{\tau^2}{2} + \frac{\kappa_3 (i\tau)^3}{3!} + \frac{\kappa_4 (i\tau)^4}{4!} + \cdots + \frac{\kappa_{r+2} (i\tau)^{r+2}}{(r+2)!} \end{aligned}$$

as $\tau \to 0$. Now let X_n, $n \in \mathbb{N}$, be independent random variables, each distributed like X_1. Put

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{and} \quad S_n^* = S_n/\sqrt{n}\,. \tag{1}$$

S_n^* has cumulant generating function

$$K_n(t) = nK_1(t/\sqrt{n}\,) \approx -t^2/2 + \beta_n(t)$$

where

$$\beta_n(t) = \frac{\kappa_3 (it)^3}{3!} \frac{1}{\sqrt{n}} + \frac{\kappa_4 (it)^4}{4!} \frac{1}{n} + \cdots + \frac{\kappa_{r+2} (it)^{r+2}}{(r+2)!} \frac{1}{n^{r/2}} \tag{2}$$

and characteristic function

$$\begin{aligned} \psi_n(t) &= \big(\psi_1(t/\sqrt{n}\,)\big)^n = e^{K_n(t)} \\ &\approx e^{-t^2/2} e^{\beta_n(t)} \approx e^{-t^2/2} \Big(1 + \frac{\beta_n(t)}{1!} + \frac{\beta_n^2(t)}{2!} + \cdots + \frac{\beta_n^r(t)}{r!}\Big) \\ &\approx e^{-t^2/2} \Big(\sum_{j=0}^{r} \frac{Q_j(it)}{n^{j/2}}\Big) := \hat{\psi}_n(t), \end{aligned} \tag{3}$$

$\hat{\psi}_n(t) = e^{-t^2/2} \sum_{j=0}^{r} Q_j(it)/n^{j/2}$.

where

$$\begin{aligned} Q_0(z) &= 1, \\ Q_1(z) &= \frac{\kappa_3}{3!} z^3, \\ Q_2(z) &= \frac{\kappa_4}{4!} z^4 + \frac{\kappa_3^2}{2(3!)^2} z^6 = \frac{\kappa_4}{4!} z^4 + \frac{10\kappa_3^2}{6!} z^6, \\ Q_3(z) &= \frac{\kappa_5}{5!} z^5 + \frac{35\kappa_4\kappa_3}{7!} z^7 + \frac{280\kappa_3^3}{9!} z^9, \\ Q_4(z) &= \frac{\kappa_6}{6!} z^6 + \frac{35\kappa_4^2 + 56\kappa_5\kappa_3}{8!} z^8 + \frac{2100\kappa_4\kappa_3^2}{10!} z^{10} + \frac{15400\kappa_3^4}{12!} z^{12}, \end{aligned}$$

and so on; for each j, $Q_j(z)$ is a polynomial in z of degree $3j$ whose coefficients depend on $\kappa_3, \ldots, \kappa_{j+2}$.

If ψ_n is integrable, then S_n^* has a continuous bounded density f_n given by the inversion formula

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi_n(t)\, dt \tag{4}$$

$$\begin{aligned} &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{\psi}_n(t)\, dt \\ &= \sum_{j=0}^{r} \Big(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-t^2/2} Q_j(it)\, dt\Big) \frac{1}{n^{j/2}} := \hat{f}_n(x). \end{aligned} \tag{5}$$

For each $k \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-t^2/2} (it)^k\, dt &= (-1)^k \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d^k}{dx^k} e^{-itx} e^{-t^2/2}\, dt \\ &= (-1)^k \frac{d^k}{dx^k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-t^2/2}\, dt = (-1)^k \frac{d^k}{dx^k} \phi(x) \end{aligned}$$

$(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}(it)^k\,dt = (-1)^k\phi^{(k)}(x)$.

where

$$\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$$

is the standard normal density. We have

$$\begin{aligned}\phi'(x) &= \phi(x)(-x),\\ \phi''(x) &= \phi(x)(-x)^2 + \phi(x)(-1) = \phi(x)(x^2-1),\\ \phi'''(x) &= \phi(x)(x^2-1)(-x) + \phi(x)(2x) = \phi(x)(-x^3+3x),\end{aligned}$$

and so on. Thus

$$(-1)^k\phi^{(k)}(x) = \phi(x)H_k(x) \tag{6}$$

where

$$\begin{aligned}H_0(x) &= 1,\\ H_1(x) &= x,\\ H_2(x) &= x^2-1,\\ H_3(x) &= x^3-3x,\end{aligned}$$

and so on. For each $k \in \mathbb{N}$, $H_k(x)$ is a polynomial in x of degree k. The H_k's are called ***Hermite polynomials***; they are easily calculated from the recursion relations

$$\begin{aligned}H_k(x) &= xH_{k-1}(x) - H_{k-1}'(x), &&\text{for } k \ge 1, &(7)\\ &= xH_{k-1}(x) - (k-1)H_{k-2}(x), &&\text{for } k \ge 2, &(8)\end{aligned}$$

(see Exercise 4). Upon writing $Q_j(z)$ in the form $\sum_k c_{jk}z^k$, we find

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}Q_j(it)\,dt &= \sum\nolimits_k c_{jk}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}(it)^k\,dt\\ &= \Big(\sum\nolimits_k c_{jk}H_k(x)\Big)\phi(x) := P_j(x)\phi(x) \qquad (9)\end{aligned}$$

with

$$\begin{aligned}P_0(x) &= 1,\\ P_1(x) &= \frac{\kappa_3}{3!}H_3(x),\\ P_2(x) &= \frac{\kappa_4}{4!}H_4(x) + \frac{10\kappa_3^2}{6!}H_6(x),\end{aligned}$$

and so on; for each $j \in \mathbb{N}$, $P_j(x)$ is obtained from $Q_j(z)$ by replacing z^k by $H_k(x)$ for each k. Thus it appears that

$$\begin{aligned}f_n(x) \approx \hat{f}_n(x) &= \sum\nolimits_{j=0}^{r}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}Q_j(it)\,dt\Big)\frac{1}{n^{j/2}}\\ &= \phi(x)\Big[P_0(x) + P_1(x)\frac{1}{\sqrt{n}} + P_2(x)\frac{1}{n} + \cdots + P_r(x)\frac{1}{n^{r/2}}\Big]\\ &:= e_{n,r}(x). \qquad (10)\end{aligned}$$

$e_{n,r}$ is called the ***r^{th}-order Edgeworth approximation to the density*** f_n of S_n^*. Note that $e_{n,0}$ is the standard normal density, so the 0^{th}-order Edgeworth approximation is just the usual simple normal approximation. The higher order approximations can be viewed as corrected, or adjusted, normal approximations that take into account the fact that the number n of summands is not infinite. The following theorem is concerned with the accuracy of $e_{n,r}$.

Theorem 1. *Let $X_1, X_2, \ldots$ be iid rvs with $E(X_1) = 0$, $E(X_1^2) = 1$, and $E(|X|^{r+2}) < \infty$ for some integer $r \ge 0$. Suppose also that for some $\nu \in \mathbb{N}$, the characteristic function ψ_1 of X_1 satisfies*

$$\int_{-\infty}^{\infty} |\psi_1(\tau)|^\nu\,d\tau < \infty. \tag{11}$$

Let $e_{n,r}$ be defined by (10). Then for $n \ge \nu$, S_n^* has a continuous bounded density f_n such that

$$\epsilon_{n,r}^* := \sup\{\,(1+|x|^{2+r})\,|f_n(x) - e_{n,r}(x)| : x \in \mathbb{R}\,\} = o(1/n^{r/2}) \tag{12}$$

as $n \to \infty$.

One nice feature of the Edgeworth approximation to the density f_n of S_n^* is that it leads to a simple approximation to the distribution of S_n^*, as follows. Suppose (12) holds. Then for each Borel subset B of $\mathbb{R}$,

$$\begin{aligned} \Big|P[\,S_n^*\in B\,]-\int_B e_{n,r}(x)\,dx\Big| &= \Big|\int_B \big(f_n(x)-e_{n,r}(x)\big)\,dx\Big| \\ &\le \int_B \frac{(1+|x|^{r+2})|f_n(x)-e_{n,r}(x)|}{1+|x|^{r+2}}\,dx \\ &\le \epsilon_{n,r}^*\int_{-\infty}^{\infty}\frac{1}{1+|x|^{r+2}}\,dx = o\big(1/n^{r/2}\big). \end{aligned}$$

as $n\to\infty$. In particular, the distribution function $F_n(x)=P[\,S_n^*\le x\,]$ of S_n^* satisfies

$$\sup_{x\in\mathbb{R}}|F_n(x)-E_{n,r}(x)| = o(1/n^{r/2}), \tag{13}$$

where

$$E_{n,r}(x)=\int_{-\infty}^x e_{n,r}(\xi)\,d\xi = \sum\nolimits_{j=0}^r\Big(\int_{-\infty}^x\phi(\xi)P_j(\xi)\,d\xi\Big)\,\frac{1}{n^{j/2}}. \tag{14a}$$

Recall that $P_0(x)=1$ and that for $j\ge 1$, $P_j=\sum_k c_{j,k}H_k$ is a linear combination of H_k's with $k\ge 1$. For such k's,

$$\begin{aligned} \int_{-\infty}^x \phi(\xi)H_k(\xi)\,d\xi &= (-1)^k\int_{-\infty}^x \phi^{(k)}(\xi)\,d\xi = (-1)^k\phi^{(k-1)}(\xi)\Big|_{-\infty}^x \\ &= -\phi(\xi)H_{k-1}(\xi)\Big|_{-\infty}^x = -\phi(x)H_{k-1}(x). \end{aligned}$$

It follows that the so-called ***r^{th}-order Edgeworth approximation to the distribution function*** F_n of S_n^* is simply

$$E_{n,r}(x)=\Phi(x)-\phi(x)\Big[\sum\nolimits_{j=1}^r P_j^*(x)\frac{1}{n^{j/2}}\Big], \tag{14b}$$

where Φ is the distribution function of the standard normal distribution and

$$\begin{aligned} P_1^*(x) &= \frac{\kappa_3}{3!}H_2(x), \\ P_2^*(x) &= \frac{\kappa_4}{4!}H_3(x)+\frac{10\kappa_3^2}{6!}H_5(x), \end{aligned}$$

and so on. In general, P_j^* is obtained from P_j by replacing H_k by H_{k-1} for each k. (13) says that the approximation $F_n(x)\approx E_{n,r}(x)$ is accurate to $o(1/n^{r/2})$, uniformly for $x\in\mathbb{R}$.

Example 1. Suppose $X_n=Y_n-1$ where the Y_n's are iid exponential random variables with mean 1. The X_n's are iid with mean 0 and variance 1. Since X_1 has characteristic function

$$\psi_1(t)=\frac{e^{-it}}{1-it},$$

the integrability condition (11) of Theorem 1 is satisfied with $\nu=2$, and X_1 has cumulants

$$\kappa_r=(r-1)!$$

for $r\ge 3$. The Edgeworth expansions for the density f_n and cdf F_n of S_n^* are thus

$$\begin{aligned} f_n(x) &\sim \phi(x)\Big[1+\frac{H_3(x)}{3}\frac{1}{\sqrt{n}}+\Big(\frac{H_4(x)}{4}+\frac{H_6(x)}{18}\Big)\frac{1}{n}+\cdots\Big] \\ F_n(x) &\sim \Phi(x)-\phi(x)\Big[\frac{H_2(x)}{3}\frac{1}{\sqrt{n}}+\Big(\frac{H_3(x)}{4}+\frac{H_5(x)}{18}\Big)\frac{1}{n}+\cdots\Big]. \end{aligned}$$

Since $T_n=Y_1+\cdots+Y_n$ has a standard Gamma distribution with shape parameter n, it is possible to write f_n and F_n in simple form and study the accuracy of their Edgeworth approximations. Here we consider f_n; the corresponding study of F_n left to Exercise 14. Since $T_n\sim\text{Gamma}(n)$ has density $g_n(t)=t^{n-1}e^{-t}I_{(0,\infty)}(t)/\Gamma(n)$, the density f_n of $S_n^* = (T_n - n)/\sqrt{n}$ is given by

$$f_n(x) = \frac{(n+\sqrt{n}x)^{n-1}\, e^{-n-\sqrt{n}x}\sqrt{n}}{\Gamma(n)} I_{(-\sqrt{n},\infty)}(x). \tag{15}$$

Figure 1 exhibits the Edgeworth approximations $e_{n,r}$ to f_n for $n = 4$. The top panel plots $f_n(x)$ and $e_{n,r}(x)$ for $r = 0, \ldots, 4$ versus x. The bottom panel plots the corresponding errors $e_{n,r}(x) - f_n(x)$. Figure 2 shows how the maximum absolute error

$$\epsilon_{n,r} := \sup\{|e_{n,r}(x) - f_n(x)| : x \in \mathbb{R}\}$$

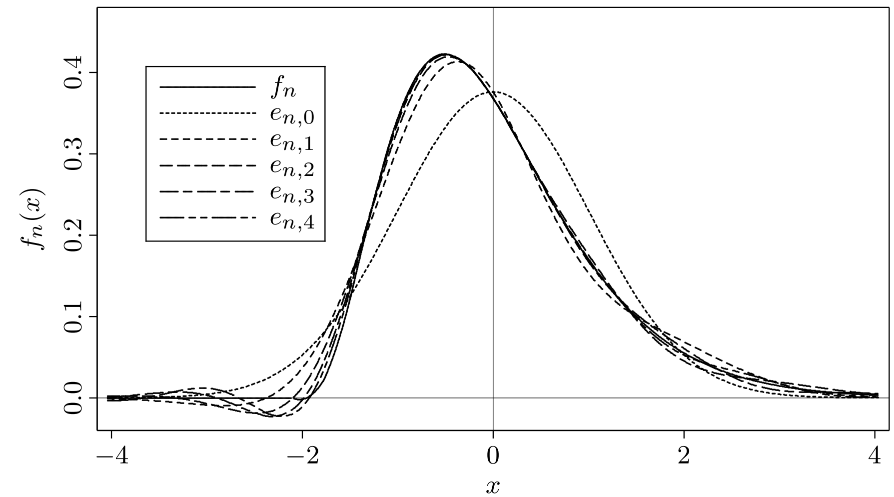
varies with n, again for $r = 0, \ldots, 4$. (The quantities $\epsilon^*_{n,r}$ in (12) can be studied in a similar manner; that's left to an exercise.) The features observed in these plots are typical of Edgeworth expansions in general. There are various things to note.

- The Edgeworth approximations to the density f_n may not themselves be probability densities, since they can take negative values. (They do however integrate to 1; see Exercise 7.)
- For a fixed r, the accuracy of the r^{th}-order approximation tends to increase with n, and the higher order approximations are asymptotically more accurate than the lower ones. However, for a fixed n, increasing the order r may worsen the approximation.
- The Edgeworth approximations tend to be most accurate near $x = 0$, the mean of S_n^*. This is especially true of the higher order approximations.
- As the top panel in Figure 2 shows, for each fixed r a plot of $\log_{10}(\epsilon_{n,r})$ versus $\log_{10}(n)$ is very nearly linear. To understand why, note that, at least for large n,

$$\begin{aligned}
&f_n(x) \approx e_{n,r+1}(x) = e_{n,r}(x) + \phi(x)P_{r+1}(x)/n^{(r+1)/2}\\
&\Longrightarrow \epsilon_{n,r} \approx \gamma_r/n^{(r+1)/2}\\
&\Longrightarrow \log_{10}(\epsilon_{n,r}) \approx \alpha_r - \big((r+1)/2\big)\log_{10}(n),
\end{aligned}$$

Figure 1

The standardized Gamma density f_n and its Edgeworth approximations $e_{n,r}$ for $n = 4$ and $r = 0, 1, \ldots, 4$.



Error in $e_{n,r}$ for $n = 4$ and $r = 0, 1, \ldots, 4$.

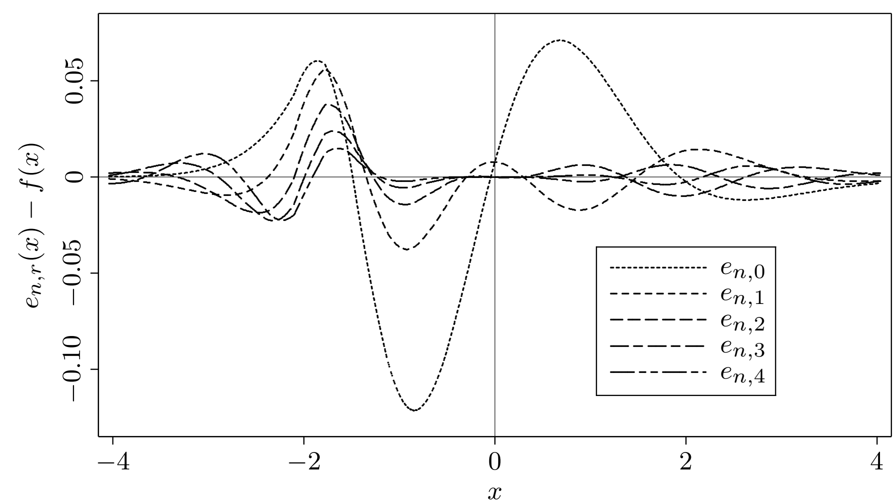
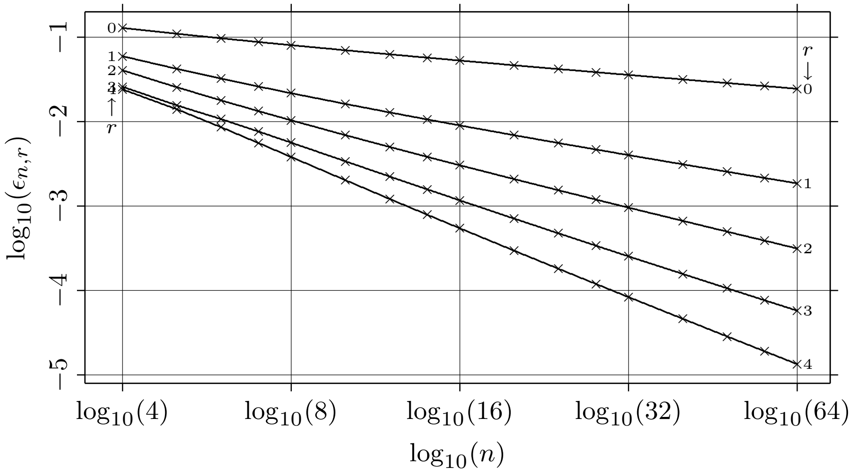
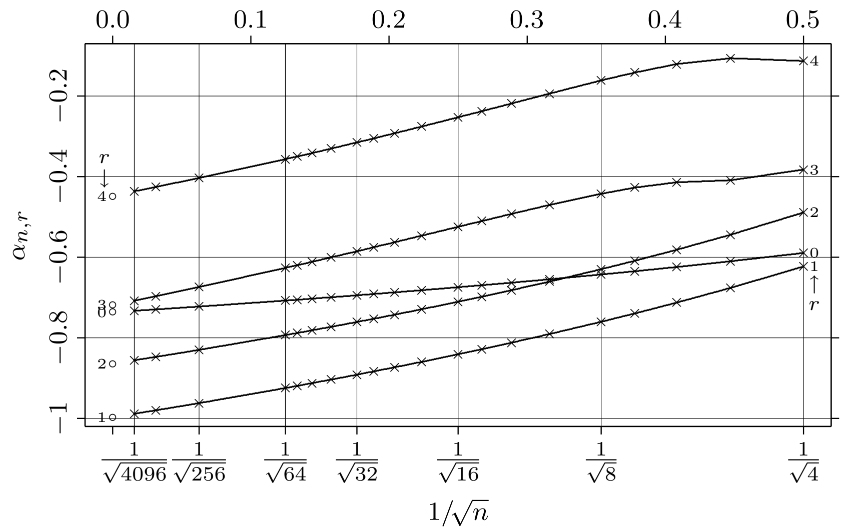


Figure 2

Maximum absolute error $\epsilon_{n,r} = \log_{10}(\sup\{|e_{n,r}(x) - f_n(x)| : x \in \mathbb{R}\}$ in the Edgeworth approximation $e_{n,r}$ to the standardized Gamma density f_n.



Exact intercepts $\alpha_{n,r} = \log_{10}(\epsilon_{n,r}) + ((r+1)/2)\log_{10}(n)$



$\log_{10}(\epsilon_{n,r}) \approx \alpha_r - \big((r+1)/2\big)\log_{10}(n)$

where

$$\gamma_r := \sup_{x\in\mathbb{R}} |\phi(x)P_{r+1}(x)| \quad \text{and} \quad \alpha_r = \log_{10}(\gamma_r). \tag{16}$$

Thus the error point $(\log_{10}(n), \log_{10}(\epsilon_{n,r}))$ almost lies on the straight line with intercept α_r and slope $-(r+1)/2$.

Of course, that error point lies exactly on the straight line with intercept

$$\alpha_{n,r} := \log_{10}(\epsilon_{n,r}) + ((r+1)/2)\log_{10}(n) \tag{17}$$

and slope $-(r+1)/2$. We'll call $\alpha_{n,r}$ the n^{th} exact intercept, and α_r the corresponding asymptotic intercept. The bottom panel in Figure 2 carries the analysis in this example a step further, by using $\times$'s to plot the exact intercept $\alpha_{n,r}$ versus $1/\sqrt{n}$. For comparison, the asymptotic intercept α_r is plotted as a $\circ$ above $1/\sqrt{\infty} = 0$. The plot reveals that for each r, $\alpha_{n,r}$ is nearly linear in $1/\sqrt{n}$. Thus it appears that a relationship of the form

$$\log_{10}(\epsilon_{n,r}) \approx \alpha_r + \beta_r/\sqrt{n} - ((r+1)/2)\log_{10}(n) \tag{18}$$

can be used to estimate $\epsilon_{n,r}$ very accurately, for all $n \geq 8$. The figure also reveals that the simpler relation

$$\log_{10}(\epsilon_{n,r}) \approx \alpha_r - ((r+1)/2)\log_{10}(n),$$

(which amounts to pretending that the difference $f_n(x) - e_{n,r}(x)$ is given by the next term in the Edgeworth expansion) underestimates $e_{n,r}$, but by no more than a factor of 2.5 for all $n \geq 4$ and $0 \leq r \leq 4$.•

A relationship like (18) always holds under some mild additional conditions; see Exercise 12. The adequacy of the relationship can be assessed in any particular case by using some technique (such as numerical quadrature applied to the inversion formula (4)) to compute f_n for some small to moderate values of n.

Proof of Theorem 1. For simplicity, we will prove a weaker version of (12), namely,

$$\epsilon_{n,r} := \sup\{\, |f_n(x) - e_{n,r}(x)| : x \in \mathbb{R}\,\} = o(1/n^{r/2}) \tag{19}$$

as $n \to \infty$; (12) itself can be proved in a similar manner, using ideas from Exercise 16.22. The heart of the proof of (19) is the following theorem about the behavior of the characteristic function ψ_n of S_n^*; note that this theorem doesn't require the integrability assumption (11).

Theorem 2. *Let X_1 be a random variable with mean $E(X_1) = 0$, variance $E(X_1^2) = 1$, $E(|X|^{r+2}) < \infty$, and cumulants $\kappa_3, \ldots, \kappa_{r+2}$. Put $\psi_n(t) = \big(\psi_1(t/\sqrt{n}\,)\big)^n$ where ψ_1 is the characteristic function of X_1, and put*

$$\hat{\psi}_n^\circ(t) = e^{-t^2/2}\Big(1 + \frac{\beta_n(t)}{1!} + \frac{\beta_n^2(t)}{2!} + \cdots + \frac{\beta_n^r(t)}{r!}\Big) \tag{20}$$

where

$$\beta_n(t) = \frac{\kappa_3(it)^3}{3!}\frac{1}{\sqrt{n}} + \frac{\kappa_4(it)^4}{4!}\frac{1}{n} + \cdots + \frac{\kappa_{r+2}(it)^{r+2}}{(r+2)!}\frac{1}{n^{r/2}}. \tag{21}$$

Then for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\hat{\psi}_n^\circ(t) - \psi_n(t)| \le \Big(\frac{\epsilon|t|^{r+2}}{n^{r/2}} + \frac{(C|t|^3)^{r+1}}{(r+1)!\, n^{(r+1)/2}}\Big)e^{-t^2/4} \tag{22}$$

for all t and n such that $|t|/\sqrt{n} \le \delta$; here δ depends just on ϵ, the distribution of X_1, and r, while C is a constant depending just on κ_3, $\ldots$, κ_{r+2}.

Proof X_1 has cumulant generating function

$$\log\big(\psi_1(\tau)\big) = -\tau^2/2 + \alpha_1(\tau) := -\tau^2/2 + \beta_1(\tau) + \rho_1(\tau)$$

where

$$\beta_1(\tau) = \frac{\kappa_3(i\tau)^3}{3!} + \frac{\kappa_4(i\tau)^4}{4!} + \cdots + \frac{\kappa_{r+2}(i\tau)^{r+2}}{(r+2)!}$$

and

$$\rho_1(\tau) = o(\tau^{r+2}) \ \text{ as } \tau \to 0.$$

$$\log\big(\psi_1(\tau)\big) = -\tau^2/2 + \alpha_1(\tau) := -\tau^2/2 + \beta_1(\tau) + \rho_1(\tau)$$

Accordingly, S_n^* has cumulant generating function

$$\begin{aligned}\log\big(\psi_n(t)\big) &= \log\big(\psi_1^n(t/\sqrt{n}\,)\big) = n\log\big(\psi_1(t/\sqrt{n}\,)\big)\\ &= -t^2/2 + \alpha_n(t) = -t^2/2 + \beta_n(t) + \rho_n(t)\end{aligned}$$

with $\alpha_n(t) = n\alpha_1(t/\sqrt{n}\,)$, $\beta_n(t) = n\beta_1(t/\sqrt{n}\,)$, and $\rho_n(t) = n\rho_1(t/\sqrt{n}\,)$. Note that $\beta_n(t)$ here is the same as in (21). The characteristic function $\psi_n(t)$ of S_n^* and its approximation $\hat{\psi}_n^\circ(t)$ are respectively

$$\psi_n(t) = e^{-t^2/2}e^{\alpha_n(t)} \quad \text{and} \quad \hat{\psi}_n^\circ(t) = e^{-t^2/2}\Big(\sum\nolimits_{j=0}^{r} \frac{\beta_n^j(t)}{j!}\Big). \tag{23}$$

Exercise 20 asserts that

$$\Big|e^\alpha - \sum\nolimits_{j=0}^{r}\frac{\beta^j}{j!}\Big| \le \Big(|\beta - \alpha| + \frac{|\beta|^{r+1}}{(r+1)!}\Big)\exp\big(\max(|\alpha|, |\beta|)\big) \tag{24}$$

for any complex numbers α and β. We are going to deduce (22) by applying (24) with $\alpha = \alpha_n(t)$ and $\beta = \beta_n(t)$.

Let $\epsilon > 0$ be given. Since $\rho_1(\tau) = o(\tau^{r+2})$ as $\tau \to 0$, there exists a number δ_1, depending just on ϵ, the distribution of X_1, and r, such that $|\rho_1(\tau)| \le \epsilon|\tau|^{r+2}$ for $|\tau| \le \delta_1$. This implies

$$|\beta_n(t) - \alpha_n(t)| = |\rho_n(t)| = n|\rho_1(t/\sqrt{n}\,)| \le \frac{\epsilon|t|^{r+2}}{n^{r/2}} \tag{25}$$

provided $|t|/\sqrt{n} \le \delta_1$.

Now consider $|\beta_n(t)|$. If $r > 0$, we argue as follows. Since $\beta_1(\tau)/\tau^3 \to i^3\kappa_3/3!$ as $\tau \to 0$, there exist a finite constant C and a number δ_2, each depending only on $\kappa_3, \ldots, \kappa_{r+2}$, such that $|\beta_1(\tau)| \le C|\tau|^3$ for $|\tau| \le \delta_2$. This implies

$$|\beta_n(t)| = |n\beta_1(t/\sqrt{n}\,)| \le C\frac{|t|^3}{\sqrt{n}} \tag{26}$$

for $|t|/\sqrt{n} \le \delta_2$. If $r = 0$, (26) holds trivially, since in that case $\beta_1(\tau) = 0 \Longrightarrow \beta_n(t) = 0$.

(20): $\hat{\psi}^\circ_n(t) = e^{-t^2/2}\Big(1 + \frac{\beta_n(t)}{1!} + \frac{\beta_n^2(t)}{2!} + \cdots + \frac{\beta_n^r(t)}{r!}\Big)$

Now consider $\gamma_n(t) := \max(|\alpha_n(t)|, |\beta_n(t)|)$. Since $\gamma_1(\tau) = o(\tau^2)$ as $\tau \to 0$, there exists a δ_3, depending just on the distribution of X_1, such that $\gamma_1(\tau) \le \tau^2/4$ for $|\tau| \le \delta_3$. This implies that

$$\gamma_n(t) = n\gamma_1(t/\sqrt{n}\,) \le t^2/4 \Longrightarrow e^{-t^2/2}e^{\gamma_n(t)} \le e^{-t^2/4} \tag{27}$$

for $|t|/\sqrt{n} \le \delta_3$.

To complete the proof, take $\delta = \min(\delta_1, \delta_2, \delta_3)$. (22) follows directly from (23)–(27). ∎

We turn now to the proof of (19). By the inversion theorem, X_1 has for $n \ge \nu$ the bounded continuous density

$$f_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_n(t)\,dt. \tag{28}$$

Let $\hat{\psi}^\circ_n$ be the approximation (20) to ψ_n and let $e^\circ_{n,r}$ be its inverse Fourier transform:

$$e^\circ_{n,r}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\hat{\psi}^\circ_n(t)\,dt. \tag{29}$$

Lemma 1. *Let f_n and $e^\circ_{n,r}$ be defined by* (28) *and* (29) *respectively. Under the conditions of Theorem 1, one has*

$$\epsilon^\circ_{n,r} := \sup\{\,|f_n(x) - e^\circ_{n,r}(x)| : x \in \mathbb{R}\,\} = o(1/n^{r/2}) \tag{30}$$

as $n \to \infty$.

Proof Since

$$\epsilon^\circ_{n,r} \le I_n := \frac{1}{2\pi}\int_{-\infty}^{\infty} |\psi_n(t) - \hat{\psi}^\circ_n(t)|\,dt$$

it suffices to show that

$$\lim_{n\to\infty} n^{r/2} I_n = 0. \tag{31}$$

For this, let $\epsilon > 0$ be given. Using Theorem 2, choose $\delta > 0$ such that

$$|\hat{\psi}^\circ_n(t) - \psi_n(t)| \le \Big(\frac{\epsilon|t|^{r+2}}{n^{r/2}} + \frac{(C|t|^3)^{r+1}}{(r+1)!\,n^{(r+1)/2}}\Big)e^{-t^2/4} \tag{32}$$

for all t and n such that $|t|/\sqrt{n} \le \delta$. Write

$$I_n = \textstyle\int_{-\infty}^{\infty} |\psi_n(t) - \hat{\psi}^\circ_n(t)|\,dt \le A_n + B_n + C_n$$

where

$$A_n := \int_{|t|\le\delta\sqrt{n}} |\psi_n(t) - \hat{\psi}^\circ_n(t)|\,dt,$$

$$B_n := \int_{|t|\ge\delta\sqrt{n}} |\psi_n(t)|\,dt, \quad \text{and} \quad C_n := \int_{|t|\ge\delta\sqrt{n}} |\hat{\psi}^\circ_n(t)|\,dt.$$

We are going to show that there exists a finite constant c (not depending on ϵ or on n) such that

$$n^{r/2}|A_n| \le c\epsilon + o(1), \;\; n^{r/2}B_n = o(1), \;\; \text{and} \;\; n^{r/2}C_n = o(1) \tag{33}$$

as $n \to \infty$. (31) will follow by taking limits, first as $n \to \infty$ and then as $\epsilon \to 0$. To establish (33), first note that by (32)

$$n^{r/2}A_n \le c\epsilon + O(1/\sqrt{n}\,) = c\epsilon + o(1) \tag{34}$$

where $c = \int_{-\infty}^{\infty} |t|^{r+2}e^{-t^2/4}\,dt < \infty$. By an argument used in the proof of Theorem 16.4,

$$n^{r/2}B_n \le n^{(r+1)/2}\,b(\delta)^{n-\nu}\int_{-\infty}^{\infty} |\psi_1(t)|^\nu\,dt = o(1), \tag{35}$$

since (11) implies that $b(\delta) := \sup\{\,|\psi_1(\tau) : |\tau| \ge \delta\,\} < 1$. Finally, since $\sigma := \sup\{\,|\hat{\psi}^\circ_n(t)|e^{t^2/4} : n \in \mathbb{N},\ t \in \mathbb{R}\,\} < \infty$, we have

$$n^{r/2}C_n \le \sigma\, n^{r/2}\int_{|t|\ge\delta\sqrt{n}} e^{-t^2/4}dt \le \sigma\, n^{r/2}\,\frac{2e^{-\delta^2 n/4}}{\delta\sqrt{n}} = o(1). \tag{36}$$

Together (34)–(36) give (33). ∎

(19): $\epsilon_{n,r} := \sup\{\, |f_n(x) - e_{n,r}(x)| : x \in \mathbb{R}\,\} = o(1/n^{r/2})$.

(20): $\hat{\psi}_n^{\circ}(t) = e^{-t^2/2}\Big(1 + \dfrac{\beta_n(t)}{1!} + \dfrac{\beta_n^2(t)}{2!} + \cdots + \dfrac{\beta_n^r(t)}{r!}\Big)$

(2): $\beta_n(t) = \dfrac{\kappa_3(it)^3}{3!}\dfrac{1}{\sqrt{n}} + \dfrac{\kappa_4(it)^4}{4!}\dfrac{1}{n} + \cdots + \dfrac{\kappa_{r+2}(it)^{r+2}}{(r+2)!}\dfrac{1}{n^{r/2}}$

To complete the proof of (19), expand the term $\sum_{j=0}^{r} \beta_n^j(t)/j!$ in (20) in powers of $1/\sqrt{n}$ to get

$$\begin{aligned}\hat{\psi}_n^{\circ}(t) &= e^{-t^2/2}\Big[\sum_{j=0}^{r} \frac{Q_j(it)}{n^{j/2}} + \sum_{j=r+1}^{r^2} \frac{Q_{j,r}(it)}{n^{j/2}}\Big] \\ &= \hat{\psi}_n(t) + e^{-t^2/2}\Big[\sum_{j=r+1}^{r^2} \frac{Q_{j,r}(it)}{n^{j/2}}\Big]\end{aligned}$$

where Q_0, ..., Q_r are the polynomials appearing in (3), and $Q_{r+1,r}$, ..., $Q_{r^2,r}$ are some other polynomials whose coefficients depend only on κ_3, ..., κ_{r+2}. Thus

$$\begin{aligned}e_{n,r}^{\circ}(x) - e_{n,r}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\big(\hat{\psi}_n^{\circ}(t) - \hat{\psi}_n(t)\big)\,dt \\ &= \phi(x)\Big[\sum_{j=r+1}^{r^2} \frac{P_{j,r}(x)}{n^{j/2}}\Big]\end{aligned}$$

where $P_{j,r}(x)$ is the polynomial obtained from $Q_{j,r}(z) = \sum_k c_{j,r,k} z^k$ by replacing z^k by $H_k(x)$ for each k. It follows that

$$n^{r/2} \sup\{\, |e_{n,r}^{\circ}(x) - e_{n,r}(x)| : x \in \mathbb{R}\,\} = O(1/\sqrt{n}\,)$$

as $n \to \infty$. Together with (30), this gives (19), and hence completes the proof of (the weakened version of) Theorem 1. ∎

Exercise 1. (a) The polynomial Q_j appearing in (3) has one term for each of the additive partitions of the integer j; explain how the κ's and the power of z in the term depend on the partition. (b) How many terms are there in Q_5 and Q_6? ◇

The next six exercises deal with the Hermite polynomials defined by (6).

Exercise 2. Show that

$$\sum_{r=0}^{\infty} H_r(x)\frac{t^r}{r!} = e^{tx - t^2/2} \tag{37}$$

for all t and x. [Hint: expand $\phi(x-t)$ in a Taylor series about $t=0$; use the fact that $\phi(x-z)$ is a differentiable function of the complex variable z to show that the Taylor series converges for all t.] ◇

Exercise 3. Show that

$$H_r(x) = x^r - \frac{r^{[2]}}{2!!}x^{r-2} + \frac{r^{[4]}}{4!!}x^{r-4} - \cdots, \tag{38}$$

where $r^{[j]} = r \times (r-1) \times \cdots \times (r-j+1)$ and $(2j)!! = 2^j j!$. [Hint: multiply the Taylor expansions of e^{tx} and $e^{-t^2/2}$.] ◇

Exercise 4. Use (38) to show that

$$H_r'(x) = rH_{r-1}(x) \tag{39}$$

for $r \geq 1$. Deduce the recursion relation (8) from (7). Use (8) and `Maple` to compute $H_r(x)$ for $0 \leq r \leq 12$. ◇

Exercise 5. Show that

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{tx - t^2/2}\, e^{ux - u^2/2}\phi(x)\,dx \\ &\quad = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{t^r u^s}{r!s!}\int_{-\infty}^{\infty} H_r(x)H_s(x)\phi(x)\,dx,\end{aligned}$$

and deduce that

$$\int_{-\infty}^{\infty} H_r(x)H_s(x)\phi(x)\,dx = \begin{cases} r!, & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases} \tag{40}$$ ◇

Exercise 6. Show that

$$\int_{-\infty}^{\infty} e^{zt}H_r(t)\phi(t)\,dt = e^{z^2/2}z^r \tag{41}$$

for all $z \in \mathbb{C}$ and all $r \in \mathbb{N}$. [Hint: Integrate by parts, or use the Fourier inversion theorem to show that (41) holds for $z = -ix$ with $x \in \mathbb{R}$.] ◇

Exercise 7. Let $e_{n,r}$ be defined by (10). (a) Show that

$$\int_{-\infty}^{\infty} e_{n,r}(x)\,dx = 1. \tag{42}$$

(b) Show that for each j, the j^{th} derivative of $e_{n,r}$ has the form

$$e_{n,r}^{(j)}(x) = \phi(x)\Big[\sum\nolimits_{i=0}^{r} \frac{P_i^{[j]}(x)}{n^{j/2}}\Big] \tag{43}$$

for certain polynomials $P_i^{[j]}$; explain how these polynomials are related to the polynomials P_i in (10). (c) Show that the Fourier transform of $e_{n,r}$ is the function $\hat{\psi}_n$ defined by (3). ◇

Exercise 8. Let the polynomials P_j be defined by (9). Show that

$$P_2(0) = \frac{1}{8}\kappa_4 - \frac{5}{24}\kappa_3^2, \tag{44_2}$$

$$P_4(0) = -\frac{1}{48}\kappa_6 + \frac{35}{384}\kappa_4^2 + \frac{7}{48}\kappa_3\kappa_5 - \frac{35}{64}\kappa_3^2\kappa_4 + \frac{385}{1152}\kappa_3^4, \tag{44_4}$$

while $P_0(0) = 1$ and $P_1(0) = 0 = P_3(0)$. ◇

Exercise 9. For a real number $C > 0$, let $\mathcal{G}_{r,C}$ be the collection of (Borel measurable) functions $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |g(x)|/(1 + |x|^{r+2})\,dx \leq C.$$

Show that if the hypotheses of Theorem 1 are satisfied, then

$$E\big(g(S_n^*)\big) - \int_{-\infty}^{\infty} g(x)e_{n,r}(x)\,dx \to 0$$

as $n \to \infty$, uniformly for $g \in \mathcal{G}_{r,C}$. What can be said about the collection $\mathcal{G}_{r,C}^*$ of functions g such that $|g(x)| \leq C(1 + |x|^{r+2})$ for all x?◇

Exercise 10. Suppose that the conditions of Theorem 1 hold with (11) strengthened to

$$\int_{-\infty}^{\infty} |\psi_1(\tau)|^{\nu}\,|\tau|^k\,d\tau < \infty \tag{45}$$

for a positive integer k. Show that the density f_n of S_n^* has k continuous bounded derivatives and

$$\sup\{\,|f_n^{(j)}(x) - e_{n,r}^{(j)}(x)| : x \in \mathbb{R}\,\} = o(1/n^{r/2}) \tag{46}$$

for $j = 1, \ldots, k$. ◇

Exercise 11. Consider the situation in Example 1. Use Figure 2 to answer the following questions. (a) What is the value of $\epsilon_{100,4}$? (b) For what n is $\epsilon_{n,2} \approx 10^{-5}$? ◇

Exercise 12. Suppose that the conditions of Theorem 1 strengthened by requiring $E(|X|^{r+4}) < \infty$. Let $\alpha_{n,r}$ and α_r be defined by (17) and (16) respectively, and let the polynomials P_j be defined by (9). Show that

$$\alpha_{n,r} = \alpha_r + \beta_r/\sqrt{n} + o(1/\sqrt{n}), \tag{47}$$

where $\beta_r = \big(1/\log_e(10)\big)P_{r+2}(\xi_r)/P_{r+1}(\xi_r)$, with ξ_r being the point that maximizes $|\phi(x)P_{r+1}(x)|$. [Hint: First show that $n^{(r+1)/2}\epsilon_{n,r} = \sup\{\,\phi(x)|P_{r+1}(x) + P_{r+2}(x)/\sqrt{n}\,| : x \in \mathbb{R}\,\} + o(1/\sqrt{n}\,)$.] ◇

Exercise 13. In the context of Example 1, use `Splus` or the equivalent to carry out a numerical study of the quantity $\epsilon^*_{n,r}$ in (12). Your analysis should include plots similar to Figures 1 and 2. $\diamond$

Exercise 14. In the context of Example 1, use `Splus` or the equivalent to carry out a numerical study of the accuracy of the Edgeworth approximations $E_{n,r}$ to the distribution function F_n of S_n^*. Your analysis should include plots similar to Figures 1 and 2. $\diamond$

Exercise 15. Use the Edgeworth expansion of $f_n(0)$ for the density f_n in (15) to deduce the following version of Stirling's famous approximation to $n!$:

$$n! = \sqrt{2\pi n}\, n^n\, e^{-n}\Big[1 + \frac{1}{12n} + O\Big(\frac{1}{n^2}\Big)\Big] \tag{48}$$

as $n \to \infty$. $\diamond$

Exercise 16. Sharpen (48) by showing that

$$n! = \sqrt{2\pi n}\, n^n\, e^{-n}\Big[1 + \frac{1}{12n} + \frac{1}{288n^2} + O\Big(\frac{1}{n^3}\Big)\Big] \tag{49}$$

as $n \to \infty$. Use may find it helpful to use `Maple` to do the algebra. $\diamond$

Exercise 17. Carry out a numerical study of the accuracy of the approximations (48) and (49) to $n!$, along with the simpler approximations $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ and $n! \approx \sqrt{2\pi(n+1/6)}\, n^n\, e^{-n}$. What do you conclude? $\diamond$

Exercise 18. Suppose $X_1, X_2, \ldots$ are iid Uniform$(0,1)$. (a) Write down explicit formulas for the r^{th}-order Edgeworth approximations to the density f_n of S_n^*, for $r = 0, 1, \ldots, 4$. (b) Study the accuracy of these approximations numerically. [There is a closed form expression for $f_n(x)$; alternatively $f_n(x)$ may be computed by using numerical integration to evaluate (4).] (c) Sometimes the standardized sum of 6 iid Uniform random variables is used as an approximate normal deviate. How good is this approximation? $\diamond$

Exercise 19. Suppose the conditions of Theorem 2 hold. Let $\hat{\psi}_n$ be defined by (3) and $e_{n,r}$ by (10). (a) Use inequality (22) to show that for each n, $\hat{\psi}_n(t) - \psi_n(t) = o(t^{r+2})$ as $t \to 0$. (b) Use part (a) to generalize (42) by showing that

$$E((S_n^*)^j) = \int_{-\infty}^{\infty} x^j\, e_{n,r}(x)\, dx \tag{50}$$

for $j = 0, 1, \ldots, r+2$. [Hint: think about the $(r+2)^{\text{nd}}$-order Taylor expansions of ψ_n and $\hat{\psi}_n$ about $t = 0$.] $\diamond$

Exercise 20. Prove (24). [Hint: Put $e_r(\beta) = \sum_{j=0}^{r} \beta^j/j!$. One has $\big|e^\alpha - e_\ell(\beta)\big| \le |e^\alpha - e^\beta| + \big|e^\beta - e_\ell(\beta)\big|$ by the triangle inequality. Now use the Taylor series expansions of e^α and e^β about 0, together with the identity $\alpha^j - \beta^j = (\alpha-\beta)(\alpha^{j-1} + \cdots + \beta^{j-1})$.] $\diamond$