Stochastic Processes

Fall 2017

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Week 13

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Solutions by

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4.5

For states $i, j, k, k \neq j$, let

$$P_{ij/k}^n = \mathbb{P}\{X_n = j, X_l \neq k, l = 1, \ldots, n-1 | X_0 = i\}.$$

(a) Explain in words what $P_{ij/k}^n$ represents.

> $P_{ij/k}^n$ represents that starting at state i, the chain visits state j at time n without visiting state k during time 1 and $n-1$.

(b) Prove that, for $i \neq j$, $P_{ij}^n = \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k}$.

> $\because$ for $i \neq j$,
>
> $$P_{ij/k}^0 = \mathbb{P}\{X_0 = j | X_0 = i\} = 0$$
>
> $\therefore$
>
> $$\begin{aligned} P_{ii}^n &= \mathbb{P}\{X_n = i | X_0 = i\} \\ &= \sum_{k=0}^{n-1} \mathbb{P}\{X_n = i, X_{n-1} \neq i, \cdots, X_{k+1} \neq i, X_k = i | X_0 = i\} \\ &= \sum_{k=0}^{n-1} \mathbb{P}\{X_n = i, X_{n-1} \neq i, \cdots, X_{k+1} \neq i | X_k = i, X_0 = i\} \mathbb{P}\{X_k = i | X_0 = i\} \\ &= \sum_{k=0}^{n-1} \mathbb{P}\{X_n = i, X_{n-1} \neq i, \cdots, X_{k+1} \neq i | X_k = i\} \mathbb{P}\{X_k = i | X_0 = i\} \qquad \text{(Strong Markov Property)} \\ &= \sum_{k=0}^{n-1} P_{ij/i}^{n-k} P_{ii}^k \\ &= \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k} \end{aligned}$$

4.9

For a Markov chain prove that

$$\mathbb{P}\{X_k = i_k | X_j = i_j, \text{ for all } j \neq k\} = \mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\}.$$

$$\begin{aligned}
&\mathbb{P}\{X_k = i_k | X_j = i_j, \text{ for all } j \neq k\}\\
=&\frac{\mathbb{P}\{X_k = i_k, X_j = i_j, \text{ for all } j \neq k\}}{\mathbb{P}\{X_j = i_j, \text{ for all } j \neq k\}}\\
=&\frac{\mathbb{P}\{X_j = i_j, \text{ for all } j > k | X_j = i_j, \text{ for all } j \leqslant k\}\mathbb{P}\{X_j = i_j, \text{ for all } j \leqslant k\}}{\mathbb{P}\{X_j = i_j, \text{ for all } j > k | X_j = i_j, \text{ for all } j < k\}\mathbb{P}\{X_j = i_j, \text{for all } j < k\}}\\
=&\frac{\mathbb{P}\{X_j = i_j, \text{ for all } j > k | X_k = i_k\}\mathbb{P}\{X_j = i_j, \text{ for all } j \leqslant k\}}{\mathbb{P}\{X_j = i_j, \text{ for all } j > k | X_{k-1} = i_{k-1}\}\mathbb{P}\{X_j = i_j, \text{for all } j < k\}}\\
=&\frac{\mathbb{P}\{X_j = i_j, \text{ for all } j > k+1 | X_{k+1} = i_{k+1}, X_k = i_k\}\mathbb{P}\{X_{k+1} = i_{k+1} | X_k = i_k\}}{\mathbb{P}\{X_j = i_j, \text{ for all } j > k+1 | X_{k+1} = i_{k+1}, X_{k-1} = i_{k-1}\}\mathbb{P}\{X_{k+1} = i_{k+1} | X_{k-1} = i_{k-1}\}}\\
&\cdot \mathbb{P}\{X_k = i_k | X_j = i_j, \text{ for all } j < k\}\\
=&\frac{\mathbb{P}\{X_j = i_j, \text{ for all } j > k+1 | X_{k+1} = i_{k+1}\}\mathbb{P}\{X_{k+1} = i_{k+1} | X_k = i_k\}}{\mathbb{P}\{X_j = i_j, \text{ for all } j > k+1 | X_{k+1} = i_{k+1}\}\mathbb{P}\{X_{k+1} = i_{k+1} | X_{k-1} = i_{k-1}\}}\\
&\cdot \mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}\}\\
=&\frac{\mathbb{P}\{X_{k+1} = i_{k+1} | X_k = i_k\}\mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}\}}{\mathbb{P}\{X_{k+1} = i_{k+1} | X_{k-1} = i_{k-1}\}}\\
=&\frac{\mathbb{P}\{X_{k+1} = i_{k+1} | X_k = i_k, X_{k-1} = i_{k-1}\}\mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}\}}{\mathbb{P}\{X_{k+1} = i_{k+1} | X_{k-1} = i_{k-1}\}}\\
=&\frac{\mathbb{P}\{X_{k+1} = i_{k+1}, X_k = i_k | X_{k-1} = i_{k-1}\}}{\mathbb{P}\{X_{k+1} = i_{k+1} | X_{k-1} = i_{k-1}\}}\\
=&\mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\}
\end{aligned}$$

4.11

If $f_{ii} < 1$ and $f_{jj} < 1$, show that

(a) $\sum\limits_{n=1}^{\infty} P_{ij}^n < \infty$;

Define the generaing function

$$P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^n s^n$$
$$F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^n s^n$$

where

$$P_{ij}^0 = \delta_{ij}$$
$$f_{ij}^0 = 0$$

Given $i, j \in S$, $\forall\ n \in \mathbb{N}$, define $A_n = \{X_n = j\}$, $B_n = \{n \in \mathbb{N}^+ : X_n = j, X_l \neq j, \text{ for } 1 \leqslant l < n\}$
$\because$

$$A_n = \bigcup_{k=0}^{n} B_k \cap A_n$$

$\therefore$

$$\begin{aligned}
\mathbb{P}\{A_n | X_0 = i\} &= \sum_{k=1}^{n} \mathbb{P}\{A_n \cap B_k | X_0 = i\} \\
&= \sum_{k=1}^{n} \mathbb{P}\{B_k | X_0 = i\} \mathbb{P}\{A_n | B_k, X_0 = i\} \\
&= \sum_{k=1}^{n} \mathbb{P}\{B_k | X_0 = i\} \mathbb{P}\{A_n | B_k\} \qquad \text{(Markov Property)}
\end{aligned}$$

$\therefore \quad \forall\ n \in \mathbb{N}^+$,

$$P_{ij}^n = \sum_{k=1}^{n} f_{ij}^k p_{jj}^{n-k}$$

$\therefore$

$$\begin{aligned}
P_{ii}(s) &= P_{ij}^0 + \sum_{n=1}^{\infty} P_{ij}^n \\
&= \delta_{ij} + F_{ij}(s) P_{jj}(s)
\end{aligned}$$

$\therefore \quad$ for $|s| < 1$,

$$P_{jj} = \frac{1}{1 - F_{jj}(s)}$$

$\because$

$$f_{jj} = \lim_{s \uparrow 1} F_{jj}(s) < 1$$

Solution (cont.)

$\therefore$

$$\begin{aligned}\sum_{n=0}^{\infty} P_{jj}^n &= \lim_{s\uparrow 1} P_{jj}(s) \\ &= \frac{1}{1-f_{jj}(s)} < \infty\end{aligned}$$

$\therefore \quad \forall\, i \neq j,$

$$\begin{aligned}\sum_{n=1}^{\infty} P_{ij}^n &= \sum_{n=0}^{\infty} P_{ij}^n \\ &= \lim_{s\uparrow 1} P_{ij}(s) \\ &= \lim_{s\uparrow 1} F_{ij}(s) \lim_{s\uparrow 1} P_{jj}(s) \\ &= f_{ij} \sum_{n=0}^{\infty} P_{jj}^n \\ &< \infty\end{aligned}$$

(b) $f_{ij} = \dfrac{\sum_{n=1}^{\infty} P_{ij}^n}{1+\sum_{n=1}^{\infty} P_{jj}^n}$.

From Abel's Theorem,

$$\begin{aligned}\sum_{n=0}^{\infty} P_{ij}^n &= \delta_{ij} + \lim_{s\uparrow 1} P_{ij}(s) \\ &= \delta_{ij} + \lim_{s\uparrow 1} F_{ij}(s) \lim_{s\uparrow 1} P_{jj}(s) \\ &= \delta_{ij} + f_{ij} \sum_{n=0}^{\infty} P_{jj}^n\end{aligned}$$

$\therefore \quad \forall\, i \neq j,$

$$\sum_{n=0}^{\infty} P_{ij}^n = f_{ij} \sum_{n=0}^{\infty} P_{jj}^n$$

i.e.,

$$\sum_{n=1}^{\infty} P_{ij}^n = f_{ij} \left(1 + \sum_{n=1}^{\infty} P_{jj}^n\right)$$

i.e.,

$$f_{ij} = \frac{\sum_{n=1}^{\infty} P_{ij}^n}{1+\sum_{n=1}^{\infty} P_{jj}^n}$$