STAT 30900 : MATHEMATICAL COMPUTATIONS I

Fall 2019

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Homework 1

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Solutions by

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For Problem **5**, use any program you like but present your codes and results in a way that is comprehensible to someone who is unfamiliar with that program (e.g. comment your codes appropriately).

1

Let $\mathbf{x} \in \mathbb{C}^n$ and $A \in \mathbb{C}^{m\times n}$. We write $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ and $\|A\|_2 = \sup_{\|\mathbf{x}\|_2=1}\|A\mathbf{x}\|_2$ for the vector 2-norm and matrix 2-norm respectively.

(a) Show that there is no ambiguity in the notation, i.e., if $A \in \mathbb{C}^{n\times 1} = \mathbb{C}^n$, then $\|A\|_2$ is the same whether we regard it as the vector or matrix 2-norm. What if $A \in \mathbb{C}^{1\times n}$?

Proof. (1) $A \in \mathbb{C}^{n\times 1} = \mathbb{C}^n$.
For the vector 2−norm, if $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^\top \in \mathbb{C}^n$, then $\|A\|_2 = \sqrt{A^*A} = \sqrt{\sum_{i=1}^n \overline{a_i}a_i}$.
For the matrix 2−norm,

$$\begin{aligned}
\|A\|_2 &= \sup_{x\in\mathbb{C},\|x\|_2=1} \|Ax\|_2 \\
&= \sup_{x\in\mathbb{C},\sqrt{\overline{x}x}=1} \sqrt{(Ax)^*Ax} \\
&= \sup_{x\in\mathbb{C},\overline{x}x=1} \sqrt{\overline{x}xA^*A} \\
&= \sqrt{A^*A}
\end{aligned}$$

Therefore, $\|A\|_2$ is the same whether we regard it as the vector or matrix 2−norm.
(2) $A \in \mathbb{C}^{1\times n}$.
For the matrix 2−norm,

$$\begin{aligned}
\|A\|_2 &= \sup_{\mathbf{x}\in\mathbb{C}^n,\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 \\
&= \sup_{\mathbf{x}\in\mathbb{C}^n,\sqrt{\overline{\mathbf{x}}\mathbf{x}}=1} \sqrt{(A\mathbf{x})^*A\mathbf{x}} \\
&= \sup_{\mathbf{x}\in\mathbb{C}^n,\overline{\mathbf{x}}\mathbf{x}=1} |A\mathbf{x}|
\end{aligned}$$

From Cauchy-Schwarz inequality,

$$\langle A, \mathbf{x}\rangle \le \|A^\top\|_2\|\mathbf{x}\|_2,$$

where the norm on the right side is the vector 2−norm and the equality holds when $\mathbf{x} = cA^\top$ for all $c \neq 0$, thus

$$\|A\|_2 = \|A^\top\|_2,$$

i.e., if $A \in \mathbb{C}^{1\times n}$, then matrix norm $\|A\|_2$ is the same as the vector norm $\|A^\top\|_2$. $\square$

(b) Show that the vector 2-norm is unitarily invariant, i.e.,

$$\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

for all unitary matrices $U \in \mathbb{C}^{n\times n}$.

Proof. $\forall\ \mathbf{x} \in \mathbb{C}^n$, $\forall\ U \in \mathbb{C}^{n\times n}$ such that $U^*U = I_n$, we have

$$\begin{aligned}
\|U\mathbf{x}\|_2 &= \sqrt{(U\mathbf{x})^*U\mathbf{x}} \\
&= \sqrt{\mathbf{x}^*U^*U\mathbf{x}} \\
&= \sqrt{\mathbf{x}^*\mathbf{x}} \\
&= \|\mathbf{x}\|_2.
\end{aligned}$$

□

(c) *Bonus*: Show that no other vector p-norm is unitarily invariant, $1 \le p \le \infty$, $p \ne 2$.

Proof. $\forall\ p \in [1,\infty) \setminus \{2\}$, take $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^\top$, take the unitary matrix $U = \begin{bmatrix} u_{ij} \end{bmatrix}$ such that the first column $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}$,

$$\begin{aligned}
\|U\mathbf{x}\|_p^p &= \sum_{i=1}^n \left(\sum_{j=1}^n u_{ij}x_j \right)^p \\
&= \sum_{i=1}^n n^{-\frac{p}{2}} \\
&= n^{1-\frac{p}{2}}, \\
\|\mathbf{x}\|_p^p &= \sum_{i=1}^n |x_i|^p \\
&= 1.
\end{aligned}$$

Since $p \ne 2$, we have that $1 - \frac{p}{2} \ne 0$ and $\|U\mathbf{x}\|_p \ne \|\mathbf{x}\|_p$ for $n > 1$.
For $p = \infty$,

$$\begin{aligned}
\|U\mathbf{x}\|_\infty &= \max_i \left\{ \sum_{j=1}^n u_{ij}x_j \right\} \\
&= n^{-\frac{1}{2}}, \\
\|\mathbf{x}\|_p^p &= \sum_{i=1}^n |x_i|^p \\
&= 1,
\end{aligned}$$

so $\|U\mathbf{x}\|_\infty \ne \|\mathbf{x}\|_\infty$ for $n > 1$. □

(d) Show that the matrix 2-norm is unitarily invariant, i.e.,

$$\|UAV\|_2 = \|A\|_2$$

for all unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$.

Proof. $\forall\, U \in \mathbb{C}^{m\times m},\ V \in \mathbb{C}^{n\times n}$ such that $U^*U = I_m$, $V^*V = I_n$,

$$\begin{aligned}
\|UAV\|_2 &= \sup_{\|\mathbf{x}\|_2=1} \|UAV\mathbf{x}\|_2 \\
&\xlongequal{\mathbf{y}=\mathbf{V}^*\mathbf{x}} \sup_{\|\mathbf{y}\|_2=1} \|UA\mathbf{y}\|_2 \\
&= \sup_{\|\mathbf{y}\|_2=1} \|U^*UA\mathbf{y}\|_2 \\
&= \sup_{\|\mathbf{y}\|_2=1} \|A\mathbf{y}\|_2 \\
&= \|A\|_2
\end{aligned}$$

where the second equality ($\|\mathbf{y}\|_2 = \|V^*\mathbf{x}\|_2 = \|\mathbf{x}\|_2 = 1$) and third equality ($\|UA\mathbf{y}\|_2 = \|U^*UA\mathbf{y}\|_2$) comes from unitarily invaraint property of the vector $2-$norm. □

(e) Show that the Frobenius norm is unitarily invariant, i.e.,

$$\|UAV\|_F = \|A\|_F$$

for all unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$. (*Hint*: First show that $\|A\|_F^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(AA^*)$).

Proof. Since

$$\begin{aligned}
\|A\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2 \\
\operatorname{tr}(A^*A) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{a_{ji}}a_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ji}|^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2 \\
\operatorname{tr}(AA^*) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{a_{ij}}a_{ij} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2,
\end{aligned}$$

we have $\|A\|_F^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(AA^*)$ and

$$\begin{aligned}
\|UAV\|_F^2 &= \operatorname{tr}(UAV) \\
&= \operatorname{tr}(V^*A^*U^*UAV) \\
&= \operatorname{tr}(V^*A^*AV) \\
&= \operatorname{tr}(VV^*A^*A) \\
&= \operatorname{tr}(A^*A) \\
&= \|A\|_F^2.
\end{aligned}$$

Thus, $\|UAV\|_F = \|A\|_F$. □

(f) Let $U \in \mathbb{C}^{n\times n}$. Show that the following are equivalent statements:

(i) $\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$;

(ii) $(U\mathbf{x})^*U\mathbf{y} = \mathbf{x}^*\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$;

(iii) U is unitary.

Proof. (i) $\implies$ (ii)
$\forall\, \mathbf{x} \in \mathbb{C}^n$, $\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$, so

$$\mathbf{x}^*U^*U\mathbf{x} = \mathbf{x}^*\mathbf{x},$$

Let $\mathbf{x} = \mathbf{e}_i$ where $\mathbf{e}_i$ is the unit n-vector whose ith element is 1 and other elements are 0. Then $\mathbf{e}_i^*U^*U\mathbf{e}_i = 1$, i.e. $(U^*U)_{ii} = 1$.
Let $\mathbf{e}'_j$ be the unit n-vector whose jth element is the unit imaginary number i. Let $\mathbf{x} = \mathbf{e}_i + \mathbf{e}'_j$ for $i < j$, then $(\mathbf{e}_i + \mathbf{e}'_j)^*U^*U(\mathbf{e}_i + \mathbf{e}'_j) = 2$, i.e. $(U^*U)_{ij} - (U^*U)_{ji} = 0$. So $(U^*U)_{ij} = (U^*U)_{ji} = 0$ for $i < j$. Therefore, $U^*U = I_n$ and $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $(U\mathbf{x})^*U\mathbf{y} = \mathbf{x}U^*U\mathbf{y} = \mathbf{x}^*\mathbf{y}$.
(ii) $\implies$ (iii)
$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $(U\mathbf{x})^*(U\mathbf{y}) = \mathbf{x}^*\mathbf{y}$, so

$$\mathbf{x}^*U^*U\mathbf{y} = \mathbf{x}^*\mathbf{y}.$$

For $i, j \in \{1, 2, \dots, n\}$, take $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$. Then

$$\mathbf{e}_i^*U^*U\mathbf{e}_j = (U^*U)_{ij} = \delta_{ij},$$

where $\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$. So $U^*U = I_n$, i.e., U is unitary.
(iii) $\implies$ (i)
Since U is unitary, from the unitarily invaraint property of the vector $2-$norm, $\forall\, \mathbf{x} \in \mathbb{C}^n$,

$$\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2.$$

□

Let $A \in \mathbb{C}^{n\times n}$. Let $\|\cdot\|$ be an operator norm of the form

$$\|A\| = \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|A\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha} \tag{2.1}$$

for some vector norm $\|\cdot\|_\alpha : \mathbb{C}^n \to [0,\infty)$. Show that if $\|A\| < 1$, then $I - A$ is nonsingular and furthermore,

$$\frac{1}{1+\|A\|} \leq \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

Proof. If $I - A$ is singular, then it has at least one eigenvalue 0, i.e., A has at least one eigenvalue $\lambda_i = 1$. Let $\mathbf{v}_i \in \mathbb{C}^n$ be the corresponding eigenvector of the eigenvalue $\lambda_i = 1$.
Since

$$\begin{aligned}\frac{\|A\mathbf{v}_i\|_\alpha}{\|\mathbf{v}_i\|_\alpha} &= \frac{\|\lambda_i\mathbf{v}_i\|_\alpha}{\|\mathbf{v}_i\|_\alpha}\\ &= \frac{|\lambda_i|\|\mathbf{v}_i\|_\alpha}{\|\mathbf{v}_i\|_\alpha}\\ &= |\lambda_i|\\ &= 1,\end{aligned}$$

from the definition of $\|A\|$, we have

$$\|A\| = \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|A\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha} \geq 1,$$

which is a contradiction. Therefore, $I - A$ is nonsingular.
Next we first prove the fact that this operator norm is submultiplicative, i.e., $\forall\ A, B \in \mathbb{C}^{n\times n}$,

$$\begin{aligned}\|AB\| &= \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|AB\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha}\\ &\leq \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|A\|\|B\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha}\\ &= \|A\|\|B\|\end{aligned}$$

where the inequality comes from the consistency property of the operator norm.
Since

$$\begin{aligned}\|(I-A)(I-A)^{-1}\| &= \|I\| = \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|I\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha} = 1\\ \|(I-A)(I-A)^{-1}\| &\leq \|I-A\| \cdot \|(I-A)^{-1}\|,\end{aligned}$$

we have

$$\|I-A\| \cdot \|(I-A)^{-1}\| \geq 1 \tag{1}$$

Also, since $\|A\| < 1$,

$$\begin{aligned}\|I-A\| &= \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|(I-A)\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha}\\ &\leq \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|I\mathbf{v}\|_\alpha + \|A\mathbf{v}\|_\alpha}{\|\mathbf{v}\|_\alpha}\\ &= 1 + \|A\|, \qquad (2)\end{aligned}$$

Solution (cont.)

from (1) and (2), we have

$$\frac{1}{1+\|A\|} \leq \|(I-A)^{-1}\|.$$

Furthermore, for $\|A\| < 1$,

$$\begin{aligned}
\|(I-A)^{-1}\| &= \left\|\sum_{i=0}^{\infty} A^i\right\| \\
&\leq \sum_{i=0}^{\infty} \|A\|^i \\
&= \frac{1}{1-\|A\|}
\end{aligned}$$

where the inequality comes from the triangle-inequality and the submultiplicativity property of the operator norm.

Therefore,

$$\frac{1}{1+\|A\|} \leq \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

□

3

Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$, and $A = \mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$.

(a) Show that

$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \tag{3.2}$$

and that

$$\|A\|_\infty \leq \|\mathbf{x}\|_\infty\|\mathbf{y}\|_1.$$

What can you say about $\|A\|_1$?

Proof. Since

$$\begin{aligned}
\|A\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n} |\overline{y}_j x_i|^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} |y_j|^2|x_i|^2, \\
\|A\|_2 &= \sup_{\substack{\mathbf{z}\in\mathbb{C}^n \\ \|\mathbf{z}\|_2=1}} \|A\mathbf{z}\|_2 \\
&= \sup_{\substack{\mathbf{z}\in\mathbb{C}^n \\ \|\mathbf{z}\|_2=1}} \sqrt{\mathbf{z}^*\mathbf{y}\mathbf{x}^*\mathbf{x}\mathbf{y}^*\mathbf{z}} \\
&= \|\mathbf{x}\|_2 \cdot \sup_{\substack{\mathbf{z}\in\mathbb{C}^n \\ \|\mathbf{z}\|_2=1}} |\mathbf{z}^*\mathbf{y}| \\
&= \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2,
\end{aligned}$$

where the last equality comes from that the equality of the Cauthy-Schwarz inequality $|\mathbf{z}^*\mathbf{y}| \leq \|\mathbf{z}\|_2\|\mathbf{y}\|_2$ holds when $\mathbf{z} = \frac{1}{\|\mathbf{y}\|_2}\mathbf{y}$ if $\mathbf{y} \neq \mathbf{0}$,

$$\begin{aligned}
\|\mathbf{x}\|_2^2 \cdot \|\mathbf{y}\|_2^2 &= \left(\sum_{i=1}^{m} |x_i|^2\right)\left(\sum_{j=1}^{n} |y_j|^2\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} |x_i|^2|y_i|^2,
\end{aligned}$$

we have

$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2.$$

Solution (cont.)

Also,

$$\begin{aligned}
\|A\|_\infty &= \max_i \left\{ \sum_{j=1}^n |x_i \overline{y}_j| \right\} \\
&\le \max_i \{|x_i|\} \cdot \sum_{j=1}^n |y_j| \\
&= \|\mathbf{x}\|_\infty \|\mathbf{y}\|_1, \\
\|A\|_1 &= \max_j \left\{ \sum_{i=1}^n |x_i \overline{y}_j| \right\} \\
&\le \sum_{i=1}^n |x_i| \cdot \max_j \{|y_j|\} \\
&= \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty.
\end{aligned}$$

□

(b) Let $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{C}^m$ be linearly independent and $\mathbf{y}_1, \ldots, \mathbf{y}_r \in \mathbb{C}^n$ be linearly independent. Let

$$A = \mathbf{x}_1\mathbf{y}_1^* + \cdots + \mathbf{x}_r\mathbf{y}_r^*.$$

Show that $\operatorname{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.

Proof. Notice that $A = \sum_{i=1}^r \mathbf{x}_i\mathbf{y}_i^* = XY^*$, where $X = \begin{bmatrix} \mathbf{x}_1 & \cdots \mathbf{x}_r \end{bmatrix} \in \mathbb{R}^{m\times r}$ and $Y = \begin{bmatrix} \mathbf{y}_1 & \cdots \mathbf{y}_r \end{bmatrix} \in \mathbb{R}^{n\times r}$. Since $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{C}^m$ are linearly independent, X has full column rank. Analogously, Y has full column rank.

Since X has full column rank, if $\mathbf{0} = X\mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^r$, then $\mathbf{w} = \mathbf{0}$. $\forall\, \mathbf{z} \in \mathbb{R}^n$, $A\mathbf{z} = \mathbf{0}$, then $Y^*\mathbf{z} = \mathbf{0}$, i.e., $\mathbf{z} \in \ker(Y^*)$. So $\ker(XY^*) \subseteq \ker(Y^*)$.

From HW0 1(a), we have $\ker(A) = \ker(XY^*) \supseteq \ker(Y^*)$. So $\ker(A) = \ker(Y^*)$ and $\operatorname{nullity}(A) = \operatorname{nullity}(Y^*)$. By rank-nullity theorem, $\operatorname{rank}(A) = n - \operatorname{nullity}(A) = n - \operatorname{nullity}(Y^*) = \operatorname{rank}(Y^*) = r$.

This is not necessarily true if we drop either of the linear independence conditions. For example, let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $A = \sum_{i=1}^2 \mathbf{x}_i\mathbf{y}_i^* = \mathbf{x}_1\mathbf{y}_1^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Obviously, $\operatorname{rank}(A) = 1$. Similiarly, if we switch $\mathbf{x}_i$ and $\mathbf{y}_i$, i.e., $A = \sum_{i=1}^2 \mathbf{y}_i\mathbf{x}_i^* = \mathbf{y}_1\mathbf{x}_1^* = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which also has $\operatorname{rank}(A) = 1$.

□

(c) Given any $0 \neq A \in \mathbb{C}^{m\times n}$, show that

$$\operatorname{rank}(A) = \min\left\{r \in \mathbb{N} : A = \sum\nolimits_{i=1}^{r} \mathbf{x}_i \mathbf{y}_i^*\right\}.$$

In other words, the rank of a matrix is the smallest r so that it may be expressed as a sum of r rank-1 matrices.

Proof. For $A \in \mathbb{C}^{m\times n}$ and $A \neq 0$, suppose that $\operatorname{rank}(A) = r_0$.
(1) Let $\mathbf{x}_1, \dots, \mathbf{x}_{r_0}$ be the basis of $\operatorname{im}(A)$, $X = \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_{r_0}\end{bmatrix}$. Since the ith column $\mathbf{a}_i$ of A can be expressed as a linear combination of them, i.e., $\exists\ \mathbf{b}_i \in \mathbb{C}^{r_0}$, s.t. $\mathbf{a}_i = X\mathbf{b}_j$, i.e. $A = XB = \sum_{i=1}^{r_0} \mathbf{x}_i \beta_i^*$ where $B = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_m\end{bmatrix} = \begin{bmatrix}\beta_1^* \\ \vdots \\ \beta_{r_0}^*\end{bmatrix}$. From (3), we know that $\operatorname{span}\{\beta_1, \dots, \beta_{r_0}\}$ is just the row space of A. Since $\operatorname{rank}(A) = r_0$, $\beta_1, \dots, \beta_{r_0}$ are linear independent. Then A can be expressed as the sum of r_0 rank-1 matrix.
(2) We just need to consider the case $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ are linearly independent and $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$ are linearly independent. Otherwise, if either one group of vectors is not, the basis of the span containing less than r vectors can be used to expressed A, which gives a smaller r.
By the definition of the basis of the column space and the row space, $r = r_0$ is the smallest number such that each column of A can be expressed as a linear combination of r linear independent column vectors, and each row of A can be expressed as a linear combination of r linear independent row vectors at the same time. Therefore,

$$\operatorname{rank}(A) = r_0 = \min\left\{r \in \mathbb{N} : A = \sum_{i=1}^{r} \mathbf{x}_i \mathbf{y}_i^*\right\}.$$

Otherwise, if there exists $r_1 < r_0$, linear independent $\mathbf{x}_1, \dots, \mathbf{x}_{r_1}$ and linear independent $\mathbf{y}_1, \dots, \mathbf{y}_{r_1}$ such that $A = \sum_{i=1}^{r_1} \mathbf{x}_i \mathbf{y}_i^*$, then by the result of (c), $\operatorname{rank}(A) = r_1 < r_0$, contradiction. □

(d) Show the following generalization of (3.2),

$$\|A\|_F \leq \sqrt{\operatorname{rank}(A)}\,\|A\|_2.$$

Note that $\nu\operatorname{rank}(A) = \|A\|_F^2/\|A\|_2^2$ is one of the three notions of numerical ranks that we discussed. It is often used as a continuous surrogate for matrix rank.

Proof. Let $r = \operatorname{rank}(A)$ and the singular value decomposition of A to be $A = U\Sigma V^*$ where $U \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ are unitary matrices and $\Sigma = \in \mathbb{R}^{m\times n}$ is a martix with non-negative real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0 = \cdots = 0$ on the diagonal. Then

$$\begin{aligned}
\|A\|_F^2 &= \operatorname{tr}(A^*A) \\
&= \operatorname{tr}(V\Sigma^*U^*U\Sigma V^*) \\
&= \operatorname{tr}(V\Sigma^*\Sigma V^*) \\
&= \operatorname{tr}(\Sigma^*U^*U\Sigma V^*V) \\
&= \operatorname{tr}(\Sigma^*U^*U\Sigma) \\
&= \sum_{i=1}^{r} \sigma_i^2.
\end{aligned}$$

Solution (cont.)

and

$$
\begin{aligned}
\|A\|_2^2 &= \sup_{\substack{\mathbf{x}\in\mathbb{C}^n \\ \|\mathbf{x}\|_2=1}} \|A\mathbf{x}\|_2^2 \\
&= \sup_{\substack{\mathbf{x}\in\mathbb{C}^n \\ \|\mathbf{x}\|_2=1}} \mathbf{x}^* V\Sigma^* U^* U\Sigma V^* \mathbf{x} \\
&= \sup_{\substack{\mathbf{x}\in\mathbb{C}^n \\ \|\mathbf{x}\|_2=1}} \mathbf{x}^* V\Sigma^* \Sigma V^* \mathbf{x} \\
&\overset{\mathbf{y}=V^*\mathbf{x}}{=\!=\!=} \sup_{\substack{\mathbf{x}\in\mathbb{C}^n \\ \|\mathbf{y}\|_2=1}} \mathbf{y}^* \Sigma^* \Sigma \mathbf{y} \\
&= \sup_{\substack{\mathbf{x}\in\mathbb{C}^n \\ \|\mathbf{y}\|_2=1}} \sum_{i=1}^r \sigma_i^2 |y_i|^2 \\
&= \sigma_1^2,
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|A\|_F &= \sqrt{\sum_{i=1}^r \sigma_i^2} \\
&\leq \sqrt{r\sigma_1^2} \\
&= \sqrt{\mathrm{rank}(A)}\|A\|_2
\end{aligned}
$$

□

(e) Show that with the nuclear norm we get instead

$$
\|A\|_* \leq \mathrm{rank}(A)\|A\|_2. \tag{3.3}
$$

In other words we could also use $\|A\|_*/\|A\|_2$ as a continuous surrogate for matrix rank.

Proof. Let $r = \mathrm{rank}(A)$ and $A = U\Sigma V^*$ be the singular value decomposition as defined in (e).

$$
\begin{aligned}
\|A\|_* &= \sum_{i=1}^r \sigma_i \\
&\leq r\sigma_1 \\
&= \mathrm{rank}(A)\|A\|_2.
\end{aligned}
$$

□

Recall that in the lectures, we mentioned that (i) there are matrix norms that are not submultiplicative and an example is the Hölder ∞-norm; (ii) we may always construct a norm that approximates the spectral radius of a given matrix A as closely as we want.

(a) Let $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ be a norm, defined for all $m, n \in \mathbb{N}$. Show that there always exists a $c > 0$ such that the constant multiple $\|\cdot\|_c := c\|\cdot\|$ defines a submultiplicative norm, i.e.,

$$\|AB\|_c \le \|A\|_c \|B\|_c$$

for any $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ (even if $\|\cdot\|$ does not have this property). Find the constant c for the Hölder ∞-norm.

Proof. Suppose that $\forall\ c > 0$, there exists $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, such that $\|AB\|_c > \|A\|_c\|B\|_c$. Then

$$c\|AB\| > c^2\|A\|\|B\|.$$

(1) If $\|A\| = 0$, then $A = 0$ since $\|\cdot\|$ is a norm. Then $\|AB\| = 0$, which means $c\|AB\| = c^2\|A\|\|B\|$. Similiarly, if $\|B\| = 0$, then $c\|AB\| = c^2\|A\|\|B\|$.
(2) If $\|A\|, \|B\| > 0$, let $c \to +\infty$, we have $\lim\limits_{c\to+\infty} c\|AB\| \le \lim\limits_{c\to+\infty} c^2\|A\|\|B\|$, since c^2 has higher order than c. So there exists $c_0 > 0$, s.t. $c_0\|AB\| \le c_0^2\|A\|\|B\|$. Contradiction.
Therefore, there always exists a $c > 0$ such that $\|AB\|_c \le \|A\|_c\|B\|_c$ for any $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$.
Next we prove that $\|\cdot\|_c$ is also a norm. It is easy to check that, for all $m, n \in \mathbb{N}$, $\forall\ A, B \in \mathbb{C}^{m\times n}$,

i $\|A\|_c = c\|A\| \ge 0$;

ii $\|A\|_c = 0$ iff $\|A\| = 0$ iff $A = 0$;

iii $\|\alpha A\|_c = c\|\alpha A\| = c|\alpha|\|A\| = |\alpha|\|A\|_c$;

iv $\|A + B\|_c = c\|A + B\| \le c(\|A\| + \|B\|) = \|A\|_c + \|B\|_c$.

Therefore, there always exists a $c > 0$ such that $\|\cdot\|_c$ defines a submultiplicative norm.
For $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}$, let $c = n$, then

$$\begin{aligned} n\|AB\|_{H,\infty} &= n \max_{i,j} \left| \sum_{k=1}^{n} a_{ik}b_{kj} \right| \\ &\le n^2 \max_{i,j} |a_{ij}| \cdot \max_{i,j} |b_{ij}| \\ &= (n\|A\|_{H,\infty}) \cdot (n\|B\|_{H,\infty}) \end{aligned}$$

□

(b) Let $J \in \mathbb{C}^{n\times n}$ be in Jordan form, i.e.,

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

where each block J_r, for $r = 1, \dots, k$, has the form

$$J_r = \begin{bmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}.$$

Let $\varepsilon > 0$ and $D_\varepsilon = \mathrm{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})$. Verify that

$$D_\varepsilon^{-1} J D_\varepsilon = \begin{bmatrix} J_{1,\varepsilon} & & \\ & \ddots & \\ & & J_{k,\varepsilon} \end{bmatrix}$$

where $J_{r,\varepsilon}$ is the matrix you obtain by replacing the 1's on the superdiagonal of J_r by ε's,

$$J_{r,\varepsilon} = \begin{bmatrix} \lambda_r & \varepsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda_r \end{bmatrix}$$

Proof. Consider a single Jordan block $J_r \in \mathbb{R}^{n_r \times n_r}$ where $n_r \in \mathbb{N}$. For any $\epsilon, \epsilon_0 > 0$, let $D_{r,\epsilon} = \epsilon_0 \cdot \mathrm{diag}(1, \epsilon, \dots, \epsilon^{n_r - 1})$. Then

$$\begin{aligned} D_{r,\epsilon}^{-1} J_r D_{r,\epsilon} &= \left[\frac{1}{\epsilon_0} \mathrm{diag}(1, \frac{1}{\epsilon}, \dots, \frac{1}{\epsilon^{n_r-1}})\right] J_r [\epsilon_0 \, \mathrm{diag}(1, \epsilon, \dots, \epsilon^{n_r-1})] \\ &= \begin{bmatrix} \lambda_r & 1 & & & \\ & \frac{\lambda_r}{\epsilon} & \frac{1}{\epsilon} & & \\ & & \ddots & \ddots & \\ & & & \frac{\lambda_r}{\epsilon^{n_r-1}} & \frac{1}{\epsilon^{n_r-1}} \end{bmatrix} \mathrm{diag}(1, \epsilon, \dots, \epsilon^{n_r-1}) \\ &= \begin{bmatrix} \lambda_r & \epsilon & & & \\ & \lambda_r & \epsilon & & \\ & & \ddots & \ddots & \\ & & & \lambda_r & \epsilon \end{bmatrix} \\ &\triangleq J_{r,\epsilon}. \end{aligned}$$

Next we split D_ϵ as $D = \begin{bmatrix} D_{1,\epsilon} & & & \\ & D_{2,\epsilon} & & \\ & & \ddots & \\ & & & D_{k,\epsilon} \end{bmatrix}$ where $D_{r,\epsilon} = \epsilon^{\sum_{i=1}^{r-1} n_r} \cdot \mathrm{diag}(1, \epsilon, \dots, \epsilon^{n_r - 1}) \in \mathbb{C}^{n_r \times n_r}$,

Solution (cont.)

then

$$D_\epsilon^-1JD_\epsilon = \begin{bmatrix} D_{1,\epsilon} & & & \\ & D_{2,\epsilon} & & \\ & & \ddots & \\ & & & D_{k,\epsilon} \end{bmatrix}^{-1} \cdot \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \cdot \begin{bmatrix} D_{1,\epsilon} & & & \\ & D_{2,\epsilon} & & \\ & & \ddots & \\ & & & D_{k,\epsilon} \end{bmatrix}$$
$$= \begin{bmatrix} D_{1,\epsilon}^{-1}J_1D_{1,\epsilon} & & & \\ & D_{2,\epsilon}^{-1}J_2D_{2,\epsilon} & & \\ & & \ddots & \\ & & & D_{k,\epsilon}^{-1}J_kD_{k,\epsilon} \end{bmatrix}$$
$$= \begin{bmatrix} J_{1,\epsilon} & & & \\ & J_{2,\epsilon} & & \\ & & \ddots & \\ & & & J_{k,\epsilon} \end{bmatrix}$$

□

(c) Show that

$$\|D_\varepsilon^{-1}JD_\varepsilon\|_\infty \le \rho(J) + \varepsilon.$$

Proof. Since J is an upper-triangle matrix, its diagonal entries are its eigenvalues. So $\rho(J) = \max_{1\le i\le k} |\lambda_r|$. Since each row of $D_\epsilon^{-1}JD_\epsilon$ has at most two non-zero entries and the norm $\|\cdot\|_\infty$ equals to the maximum sum of absolute values in a row, we have

$$\begin{aligned} \|D_\epsilon^{-1}JD_\epsilon\|_\infty &\le \max_{1\le i\le k}(|\lambda_r| + |\epsilon|) \\ &= \max_{1\le i\le k}|\lambda_r| + \epsilon \\ &= \rho(J) + \epsilon \end{aligned}$$

□

(d) Hence, or otherwise, show that for any given $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, there exists an operator norm $\|\cdot\|$ of the form (2.1) with the property that

$$\rho(A) \le \|A\| \le \rho(A) + \varepsilon.$$

(*Hint*: Transform A into Jordan form).

Proof. $\forall\ A \in \mathbb{C}^{n\times n}$, $\exists\ X \in \mathbb{C}^{n\times n}$, s.t. $A = XJX^{-1}$ where $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$ is of Jordan canonical

Solution (cont.)

form. $\forall\ \epsilon > 0$, let $D_\epsilon = \text{diag}(1, \epsilon, \dots, \epsilon^{n-1})$, define an operator norm as

$$\begin{aligned}\|M\| &= \max_{\mathbf{0}\neq\mathbf{v}\in\mathbb{C}^n} \frac{\|D_\epsilon^{-1}X^{-1}M\mathbf{v}\|_\infty}{\|D_\epsilon^{-1}X^{-1}\mathbf{v}\|_\infty} \\ &\overset{\mathbf{y}=D_\epsilon^{-1}X^{-1}\mathbf{v}}{=\!=\!=\!=\!=\!=} \max_{\mathbf{0}\neq\mathbf{y}\in\mathbb{C}^n} \frac{\|D_\epsilon^{-1}X^{-1}MXD_\epsilon\mathbf{y}\|_\infty}{\|\mathbf{y}\|_\infty} \\ &= \|D_\epsilon^{-1}X^{-1}MXD_\epsilon\|_\infty,\end{aligned}$$

then $\|A\| = \|D_\epsilon^{-1}JD_\epsilon\|_\infty \leq \rho(J) + \epsilon = \rho(A) + \epsilon$, here $\rho(J) = \rho(A)$ since the eigenvalues of A and J are the same.

Also, if $\mathbf{v}_1$ is the eigenvector of $D_\epsilon^{-1}JD_\epsilon$ corresponding to the eigenvalue λ_1 with largest absolute value, then λ_1 is also the eigenvalue of A that has largest absolute value since the eigenvalues of A, J and $D_\epsilon^{-1}JD_\epsilon$ are the same. So

$$\begin{aligned}\|A\| &= \|D_\epsilon^{-1}JD_\epsilon\|_\infty \\ &\geq \frac{\|D_\epsilon^{-1}JD_\epsilon\mathbf{v}\|_\infty}{\|\mathbf{v}\|_\infty} \\ &= \frac{\|\lambda_1\mathbf{v}\|_\infty}{\|\mathbf{v}\|_\infty} \\ &= |\lambda_1| \\ &= \rho(A).\end{aligned}$$

Therefore, for the operator norm $\|A\| = \|D_\epsilon^{-1}JD_\epsilon\|_\infty$, $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$.

$\square$

5

Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} n+1-\max(i,j) & i \leq j+1, \\ 0 & i > j+1. \end{cases}$$

This is an example of an *upper Hessenberg* matrix: it is upper triangular except that the entries on the subdiagonal $a_{j+1,j}$ may also be non-zero. For $n = 12$ and $n = 25$, do the following:

(a) Compute $\|A\|_\infty$ and $\|A\|_1$.

```
upper_Hessenberg_matrix <- function(n){
    A <- matrix(0, n, n)
    for (i in 1:n) {
        for (j in (i-1):n) {
            A[i,j] <- n+1-max(i,j)
        }
    }
    return(A)
}

A <- upper_Hessenberg_matrix(12)
A_norm_infity <- max(rowSums(abs(A)))
A_norm_1 <- max(colSums(abs(A)))
cat('For n = 12, the infinity norm of A is ',  A_norm_infity,
    'and the 1 norm of A is ', A_norm_1)

## For n = 12, the infinity norm of A is  78 and the 1 norm of A is  48

A <- upper_Hessenberg_matrix(25)
A_norm_infity <- max(rowSums(abs(A)))
A_norm_1 <- max(colSums(abs(A)))
cat('For n = 25, the infinity norm of A is ',  A_norm_infity,
    'and the 1 norm of A is ', A_norm_1)

## For n = 25, the infinity norm of A is  325 and the 1 norm of A is  181
```

(b) Compute $\rho(A)$ and $\|A\|_2$. You may use any built-in functions of your program.

```
A <- upper_Hessenberg_matrix(12)
rho_A <- max(abs(eigen(A)$values))
A_norm_2 <- norm(A, '2')
cat('For n = 12, the spectral radius of A is ',  rho_A,
    'and the 2 norm of A is ', A_norm_2)
```

Solution (cont.)

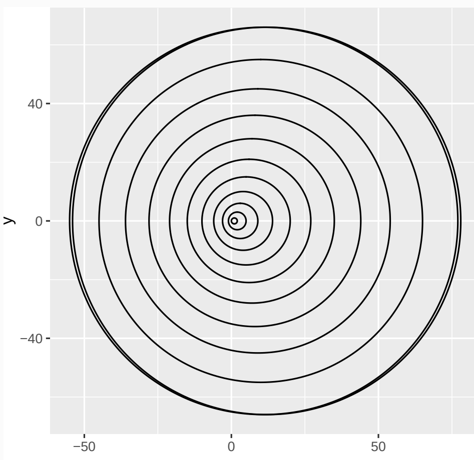
```
## For n = 12, the spectral radius of A is  32.22889 and the 2 norm of A is  47.73602
```

```
A <- upper_Hessenberg_matrix(25)
rho_A <- max(abs(eigen(A)$values))
A_norm_2 <- norm(A, '2')
cat('For n = 25, the spectral radius of A is ',  rho_A,
    'and the 2 norm of A is ', A_norm_2)
```

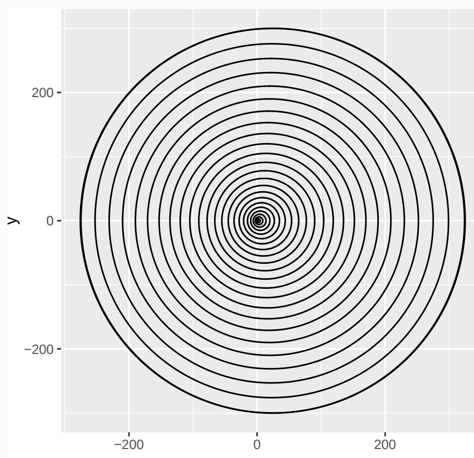
```
## For n = 25, the spectral radius of A is  77.98369 and the 2 norm of A is  180.7546
```

(c) Using Gerschgorin's theorem, describe the domain that contains all of the eigenvalues.

Let $G_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}$, where $r_i = \sum_{j \neq i} |a_{ij}|$. As the Gerschgorin's Theorem states, the sepctrum of A satisfies $\lambda(A) \subseteq \cup_{i=1}^{n} G_i$.
The diagonal entries are all real, and the boundaries of disks are shown as below.



$n = 12$



$n = 25$

Code:

```
library(ggplot2)
library(ggforce)
A <- upper_Hessenberg_matrix(12)
cat('r_ii = ', rowSums(abs(A)) - diag(abs(A)))
```

```
## r_ii =  66 66 55 45 36 28 21 15 10 6 3 1
```

```
cat('a_ii = ', diag(A))
```

```
## a_ii =  12 11 10 9 8 7 6 5 4 3 2 1
```

```
df <- data.frame(x= Re(diag(A)), y= Im(diag(A)), r = rowSums(abs(A)) - diag(abs(A)))
ggplot() +
  geom_circle(aes(x0 = x, y0 = y, r = r), data = df) +
  coord_fixed()
```

Solution (cont.)

```
A <- upper_Hessenberg_matrix(25)
cat('r_ii = ', rowSums(abs(A)) - diag(abs(A)))
```

```
## r_ii =  300 300 276 253 231 210 190 171 153 136 120 105 91 78 66 55 45 36 28 21 15
## 10 6 3 1
```

```
cat('a_ii = ', diag(A))
```

```
## a_ii =  25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
```

```
df <- data.frame(x= Re(diag(A)), y= Im(diag(A)), r = rowSums(abs(A)) - diag(abs(A)))
ggplot() +
  geom_circle(aes(x0 = x, y0 = y, r = r), data = df) +
  coord_fixed()
```

(d) Compute all of the eigenvalues and singular values of A. How many of the eigenvalues are real and how many are complex? You may use any built-in functions of your program.

```
A <- upper_Hessenberg_matrix(12)
eigen(A)$values
```

```
##  [1] 32.22889150 20.19898865 12.31107740  6.96153309  3.51185595
##  [6]  1.55398871  0.64350532  0.28474972  0.14364652  0.08122763
## [11]  0.04950747  0.03102804
```

```
cat('For n = 12, ', sum(Im(eigen(A)$values)==0), ' eigenvalues are real, ',
    sum(Im(eigen(A)$values)!=0), ' eigenvalues are complex.')
```

```
## For n = 12,  12  eigenvalues are real,  0  eigenvalues are complex.
```

```
cat('Singular values:', svd(A)$d)
```

```
## Singular values: 47.73602 18.02815 10.79075 7.622463 5.898032 4.826981 4.000138
## 3.211231 2.423965 1.638687 0.8693957 1.11863e-08
```

```
A <- upper_Hessenberg_matrix(25)
eigen(A)$values
```

```
##  [1] 77.9836861+0.0000000i 60.5984151+0.0000000i 47.7776517+0.0000000i
##  [4] 37.5667120+0.0000000i 29.2021313+0.0000000i 22.2855770+0.0000000i
##  [7] 16.5771913+0.0000000i 11.9192521+0.0000000i  8.2006342+0.0000000i
## [10]  5.3359397+0.0000000i  3.2478955+0.0000000i  1.8456428+0.0000000i
## [13]  0.9999471+0.0000000i  0.5385177+0.0000000i  0.3175409+0.2204004i
## [16]  0.3175409-0.2204004i  0.3819660+0.0000007i  0.3819660-0.0000007i
```

Solution (cont.)

```
## [19]  0.1326689+0.3098789i  0.1326689-0.3098789i -0.0572034+0.2833596i
## [22] -0.0572034-0.2833596i -0.1934173+0.1658994i -0.1934173-0.1658994i
## [25] -0.2423038+0.0000000i
```

```r
cat('For n = 25, ', sum(Im(eigen(A)$values)==0), ' eigenvalues are real, ',
    sum(Im(eigen(A)$values)!=0), ' eigenvalues are complex.')
```

```
## For n = 25,  15  eigenvalues are real,  10  eigenvalues are complex.
```

```r
cat('Singular values:', svd(A)$d)
```

```
## Singular values: 180.7546 70.34427 43.29843 31.15827 24.31744 19.9658 16.98294
## 14.837 13.24611 12.04573 11.10841 10.29396 9.505691 8.719352 7.932963 7.146449
## 6.359787 5.572948 4.785905 3.998644 3.211219 2.423965 1.638687 0.8693957 1.9824e-16
```

6

You are not allowed to use the SVD for this problem, i.e., no arguments should depend on the SVD of A or A^*. Let W be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ below is called the *orthogonal complement* of W.

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n : \mathbf{v}^*\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

For any subspace $W \subseteq \mathbb{C}^n$, we write $P_W \in \mathbb{C}^{n\times n}$ for an orthogonal projection onto W.

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.

Proof. (1) Since $\forall\, \mathbf{w} \in W \oplus W^\perp$, $\exists\, \mathbf{u} \in W \subseteq \mathbb{C}^n, \mathbf{v} \in W^\perp \subseteq \mathbb{C}^n$, s.t. $\mathbf{w} = \mathbf{u} + \mathbf{v} \in \mathbb{C}$. Therefore, $W \oplus W^\perp \subseteq \mathbb{C}^n$. Next we prove that $W \oplus W^\perp \supseteq \mathbb{C}^n$.
Since W is a subspace of $\mathbb{C}^n$, let $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be the orthogonal basis of W where $r = \dim W \le n$. Also, there exists $\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_n\}$ such that $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ becomes the orthogonal basis of $\mathbb{C}^n$. Thus, $\forall\, j \in \{r+1, \dots, n\}$, $\forall\, i \in \{1, \dots, r\}$, $\mathbf{w}_i^*\mathbf{w}_j = 0$. Then $\forall\, \mathbf{w} \in W$, $\exists\, \alpha_1, \dots, \alpha_r \in \mathbb{C}$, s.t. $\mathbf{w} = \sum_{i=1}^r \alpha_i \mathbf{w}_i$,

$$\mathbf{w}_j^*\mathbf{w} = \sum_{i=1}^r \alpha_i \mathbf{w}_j^*\mathbf{w}_i = 0,$$

so $\mathbf{w}_j \in W^\perp$ for all $j \in \{r+1, \dots, n\}$.
$\forall\, \mathbf{w} \in \mathbb{C}^n$, $\exists\, \beta_1, \dots, \beta_n \in \mathbb{C}$, s.t. $\mathbf{w} = \sum_{i=1}^n \beta_i \mathbf{w}_i$. Let $\mathbf{u} = \sum_{i=1}^r \beta_i \mathbf{w}_i \in W$, $\mathbf{v} = \sum_{i=r+1}^n \beta_i \mathbf{w}_i \in W^\perp$, then $\mathbf{w} = \mathbf{u} + \mathbf{v} \in W \oplus W^\perp$. Therefore, $\mathbb{C}^n \subseteq W \oplus W^\perp$.
Therefore, $\mathbb{C}^n = W \oplus W^\perp$.
(2) Since $\operatorname{span}\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_n\} \subseteq W^\perp$, $\operatorname{span}\{\mathbf{w}_1, \dots, \mathbf{w}_r\} \nsubseteq W^\perp$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis of $\mathbb{C}^n$, we have $W^\perp = \operatorname{span}\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_n\}$ and $\dim(W^\perp) = n - r$. Similarly, we have $\dim((W^\perp)^\perp) = n - \dim(W^\perp) = r$.
Since $\forall\, \mathbf{w} \in W^\perp$, $\exists\, \theta_{r+1}, \dots, \theta_n \in \mathbb{C}$, s.t. $\mathbf{w} = \sum_{i=r+1}^n \theta_i \mathbf{w}_i$, then

$$\mathbf{w}_i^*\mathbf{w} = \sum_{i=r+1}^n \theta_i \mathbf{w}_i^*\mathbf{w}_j = 0,$$

i.e., $\mathbf{w}_i \in (W^\perp)^\perp$ for all $i \in \{1, \dots, r\}$. Then $W = \operatorname{span}\{\mathbf{w}_1, \dots, \mathbf{w}_r\} \subseteq (W^\perp)^\perp$. Since $\dim(W) = r = \dim((W^\perp)^\perp)$, we have that $(W^\perp)^\perp = W$. $\square$

(b) Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp.$$

Proof. (1) $\forall\, \mathbf{x} \in \operatorname{im}(A)^\perp$, we have that $\forall\, \mathbf{y} \in \operatorname{im}(A)$, $\mathbf{y}^*\mathbf{x} = 0$. Since $\mathbf{y} \in \operatorname{im}(A)$, there exists $\mathbf{z} \in \mathbb{C}^n$ such that $\mathbf{y} = A\mathbf{z}$. Then $\mathbf{z}^*A^*\mathbf{x} = 0$ for all $\mathbf{z} \in \mathbb{C}^n$, which implies $A^*\mathbf{x} = 0$. Therefore $\mathbf{x} \in \ker(A^*)$ and $\operatorname{im}(A)^\perp \subseteq \ker(A^*)$.

Solution (cont.)

Furthermore, since

$$\begin{aligned}\dim(\operatorname{im}(A)^{\perp}) &= n - \dim(\operatorname{im}(A))\\ &= n - \operatorname{rank}(A)\\ &= n - \operatorname{rank}(A^*)\\ &= \operatorname{nullity}(A^*)\\ &= \dim(A^*),\end{aligned}$$

we have $\operatorname{im}(A)^{\perp} = \ker(A^*)$.
(2) Similarly, we have $\ker(A) = \operatorname{im}(A^*)^{\perp}$ by replace A with A^* in the above result. Therefore, $\operatorname{im}(A^*) = (\operatorname{im}(A^*)^{\perp})^{\perp} = \ker(A)^{\perp}$. □

(c) Deduce the Fredholm alternative:

$$\mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A) \quad \text{and} \quad \mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A).$$

In other words any $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$ can be written uniquely as

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_0 \in \ker(A),\ \mathbf{x}_1 \in \operatorname{im}(A^*),\ \mathbf{x}_0^*\mathbf{x}_1 = 0,\\ \mathbf{y} &= \mathbf{y}_0 + \mathbf{y}_1, \quad \mathbf{y}_0 \in \ker(A^*),\ \mathbf{y}_1 \in \operatorname{im}(A),\ \mathbf{y}_0^*\mathbf{y}_1 = 0.\end{aligned}$$

Proof. Since $\ker(A^*)$ and $\operatorname{im}(A^*)$ are the subspaces of $\mathbb{R}_m$ and $\mathbb{R}_n$, respectivly, from (a), we have

$$\begin{aligned}\mathbb{C}^m &= \ker(A^*) \oplus \ker(A^*)^{\perp}\\ &= \ker(A^*) \oplus \operatorname{im}(A)\\ \mathbb{C}^n &= \operatorname{im}(A^*) \oplus \operatorname{im}(A^*)^{\perp}\\ &= \operatorname{im}(A^*) \oplus \ker(A)\end{aligned}$$

□

(d) Show that

$$\mathbf{x}_0 = P_{\ker(A)}\mathbf{x}, \quad \mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}, \quad \mathbf{y}_0 = P_{\ker(A^*)}\mathbf{y}, \quad \mathbf{y}_1 = P_{\operatorname{im}(A)}\mathbf{y}.$$

Proof. $\forall\ \mathbf{x} \in \mathbb{C}^n$, $\exists\ \mathbf{x}_0 \in \ker(A),\ \mathbf{x}_1 \in \operatorname{im}(A^*)$ uniquely such that $\mathbf{x}_0^*\mathbf{x}_1 = 0$. Then

$$P_{\ker(A)}\mathbf{x} = P_{\ker(A)}\mathbf{x}_0 + P_{\ker(A)}\mathbf{x}_1 = P_{\ker(A)}\mathbf{x}_0,$$

since $\mathbf{x}_1 \in \operatorname{im}(A^*) = \ker(A)^{\perp}$. Also,

$$P_{\operatorname{im}(A^*)}\mathbf{x} = P_{\operatorname{im}(A^*)}\mathbf{x}_0 + P_{\operatorname{im}(A^*)}\mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}_1,$$

since $\mathbf{x}_0 \in \ker(A) = \operatorname{im}(A^*)^{\perp}$.
$\forall\ \mathbf{y} \in \mathbb{C}^m$, $\exists\ \mathbf{y}_0 \in \ker(A^*),\ \mathbf{y}_1 \in \operatorname{im}(A)$ uniquely such that $\mathbf{y}_0^*\mathbf{y}_1 = 0$. Then

$$P_{\ker(A^*)}\mathbf{y} = P_{\ker(A^*)}\mathbf{y}_0 + P_{\ker(A^*)}\mathbf{y}_1 = P_{\ker(A^*)}\mathbf{y}_0,$$

Solution (cont.)

since $\mathbf{y}_1 \in \text{im}(A) = \ker(A^*)^\perp$. Also,

$$P_{\text{im}(A)}\mathbf{y} = P_{\text{im}(A)}\mathbf{y}_0 + P_{\text{im}(A)}\mathbf{y}_1 = P_{\text{im}(A)}\mathbf{y}_1,$$

since $\mathbf{y}_0 \in \ker(A^*) = \text{im}(A)^\perp$. $\square$

(e) Consider the least squares problem for some $\mathbf{b} \in \mathbb{C}^m$,

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2. \tag{6.4}$$

Show that for any $\mathbf{x} \in \mathbb{C}^n$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 \geq \|\mathbf{b}_0\|_2$$

where $\mathbf{b}_0 = P_{\ker(A^*)}\mathbf{b}$. Deduce that $\mathbf{x} \in \mathbb{C}^n$ is a solution to (6.4) if and only if

$$A\mathbf{x} = \mathbf{b}_1 \quad \text{or, equivalently,} \quad \mathbf{b} - A\mathbf{x} = \mathbf{b}_0. \tag{6.5}$$

Why is $A\mathbf{x} = \mathbf{b}_1$ consistent?

Proof. For orthogonal vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $\|\mathbf{x} \pm \mathbf{y}\|_2^2 = \mathbf{x}^*\mathbf{x} \pm 2\mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{y} = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2$. Then

$$\begin{aligned}
\|\mathbf{b} - A\mathbf{x}\|_2^2 &= \|P_{\ker(A^*)}\mathbf{b} + P_{\ker(A^*)^\perp}\mathbf{b} - A\mathbf{x}\|_2^2 \\
&= \|P_{\ker(A^*)}\mathbf{b} + P_{\text{im}(A)}\mathbf{b} - A\mathbf{x}\|_2^2 \\
&= \|P_{\ker(A^*)}\mathbf{b}\|_2^2 + \|P_{\text{im}(A)}\mathbf{b} - A\mathbf{x}\|_2^2 \\
&\geq \|P_{\ker(A^*)}\mathbf{b}\|_2^2,
\end{aligned}$$

i.e. $\|\mathbf{b} - A\mathbf{x}\|_2 \geq \|\mathbf{b}_0\|_2$.
Thus, $\mathbf{x} \in \mathbb{C}^n$ is a solution to (6.4) if and only if $\|P_{\text{im}(A)}\mathbf{b} - A\mathbf{x}\|_2 = 0$, i.e. $A\mathbf{x} = P_{\text{im}(A)}\mathbf{b} \triangleq \mathbf{b}_1$ or, equivalently, $\mathbf{b} - A\mathbf{x} = \mathbf{b}_0$.
$A\mathbf{x} = \mathbf{b}_1$ is consistent since $A\mathbf{x} \in \text{im}(A)$ and we can only choose $\mathbf{x}$ to approximate $\mathbf{b}$ in the subspace $\text{im}(A)$, which is orthogonal to $\ker(A^*)$. While in $\text{im}(A)$, we can always find a $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}_1 = P_{\text{im}(A)}\mathbf{b}$ since $P_{\text{im}(A)}\mathbf{b}$ is exactly in $\text{im}(A)$. $\square$

(f) Show that (6.5) is equivalent (i.e., if and only if) to the normal equation

$$A^*A\mathbf{x} = A^*\mathbf{b}. \tag{6.6}$$

Caveat: In numerical analysis, it is in general a terrible idea to solve a least squares problem via its normal equation. Nonetheless (6.6) can be useful in mathematical arguments. We will discuss in the lectures the very limited number of scenarios when it makes sense to solve (6.6) via Cholesky decomposition.

Proof. Notice that $\mathbf{b} = P_{\ker(A^*)}\mathbf{b} + P_{\ker(A^*)^\perp}\mathbf{b} = P_{\ker(A^*)}\mathbf{b} + P_{\text{im}(A)}\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1$.
$\Longrightarrow$

Solution (cont.)

$$
\begin{aligned}
A^*\mathbf{b} &= A^*(P_{\ker(A^*)}\mathbf{b} + P_{\mathrm{im}(A)}\mathbf{b}) \\
&= 0 + A^*P_{\mathrm{im}(A)}\mathbf{b}) \\
&= A^*\mathbf{b}_1 \\
&= A^*A\mathbf{x}
\end{aligned}
$$

$\Longleftarrow$

Since

$$A^*(\mathbf{b} - A\mathbf{x}) = 0,$$

we have $\mathbf{b} - A\mathbf{x} \in \ker(A^*)$. Moreover $Ax \in \mathrm{im}(A) = \ker(A^*)^\perp$ and $\mathbf{b}_1 \in \ker(A^*)^\perp$, then $\mathbf{b} - A\mathbf{x} = \mathbf{b}_0 + (\mathbf{b}_1 - A\mathbf{x}) = \mathbf{b}_0 \in \ker(A^*)$. So, equivalently, $A\mathbf{x} = \mathbf{b}_1$ from (e). □

(g) Show that the pseudoinverse solution

$$\min\left\{\|\mathbf{x}\|_2 : \mathbf{x} \in \underset{\mathbf{x}\in\mathbb{C}^n}{\operatorname{argmin}}\|\mathbf{b} - A\mathbf{x}\|_2\right\}$$

is given by

$$\mathbf{x}_1 = P_{\mathrm{im}(A^*)}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{C}^n$ satisfies (6.5).

Proof. From (e), $\mathbf{x} \in \mathbb{C}^n$ is a solution to (6.4) if and only if $A\mathbf{x} = \mathbf{b}_1$. Decompose $\mathbf{x}$ as $\mathbf{x} = P_{\mathrm{im}(A^*)}\mathbf{x} + P_{\mathrm{im}(A^*)^\perp}\mathbf{x} = P_{\mathrm{im}(A^*)}\mathbf{x} + P_{\ker(A)}\mathbf{x}$, then

$$
\begin{aligned}
A\mathbf{x} &= A(P_{\mathrm{im}(A^*)}\mathbf{x} + P_{\ker(A)}\mathbf{x}) \\
\mathbf{b}_1 &= AP_{\mathrm{im}(A^*)}\mathbf{x}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{x}\|_2^2 &= \|P_{\mathrm{im}(A^*)}\mathbf{x} + P_{\ker(A)}\mathbf{x}\|_2^2 \\
&= \|P_{\mathrm{im}(A^*)}\mathbf{x}\|_2^2 + \|P_{\ker(A)}\mathbf{x}\|_2^2 \\
&\geq \|P_{\mathrm{im}(A^*)}\mathbf{x}\|_2^2
\end{aligned}
$$

i.e., for solution $\mathbf{x}$, $P_{\mathrm{im}(A^*)}\mathbf{x}$ is also a solution of (6.5) which achieve smaller or equal 2-norm.
If there exists another solution $\mathbf{y}$ of (6.5), then

$$A(\mathbf{x} - \mathbf{y}) = \mathbf{b}_1 - \mathbf{b}_1 = 0,$$

i.e., $\mathbf{x} - \mathbf{y} \in \ker(A)$. Then $P_{\mathrm{im}(A^*)}\mathbf{y} = P_{\mathrm{im}(A^*)}\mathbf{x}$, which means that the solution to $\min\{\|\mathbf{x}\|_2 : \mathbf{x} \in \arg\min_{\mathbf{x}\in\mathbb{C}^n}\|\mathbf{b} - A\mathbf{x}\|_2\}$ is unique.
Therefore, the pseudoinverse solution is given by $\mathbf{x}_1 = P_{\mathrm{im}(A^*)}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{C}^n$ satisfying (6.5). □

(h) Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e., $A^*A = AA^*$. Show that

$$\ker(A^*) = \ker(A) \quad \text{and} \quad \operatorname{im}(A^*) = \operatorname{im}(A)$$

and deduce that for a normal matrix,

$$\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A).$$

Proof. $\forall\ \mathbf{x} \in \ker(A^*)$, $A^*\mathbf{x} = 0$, and therefore

$$A^*A\mathbf{x} = AA^*\mathbf{x} = 0.$$

So $A\mathbf{x} \in \ker(A^*) = \operatorname{im}(A)^\perp$. Also, $A\mathbf{x} \in \operatorname{im}(A)$, which means that $A\mathbf{x} = \mathbf{0}_n$, i.e., $\mathbf{x} \in \ker(A)$. Therefore, $\ker(A^*) \subseteq \ker(A)$.
$\forall\ \mathbf{x} \in \ker(A)$, $A\mathbf{x} = 0$, and therefore

$$AA^*\mathbf{x} = A^*A\mathbf{x} = 0.$$

So $A^*\mathbf{x} \in \ker(A) = \operatorname{im}(A^*)^\perp$. Also, $A^*\mathbf{x} \in \operatorname{im}(A^*)$, which means that $A^*\mathbf{x} = \mathbf{0}_n$, i.e., $\mathbf{x} \in \ker(A^*)$. Therefore, $\ker(A^*) \supseteq \ker(A)$.
Therefore, $\ker(A) = \ker(A^*)$ and

$$\operatorname{im}(A^*) = \ker(A)^\perp = \ker(A^*)^\perp = \operatorname{im}(A).$$

Then from (c) we have $\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A^*) = \ker(A) \oplus \operatorname{im}(A)$. ☐