Question 1: Show that $K = D^2 - x^2$ is a symmetric operator on $L^2(R)$: for nice smooth functions $f,\ g \in L^2(R)$ we have

$$\int_{-\infty}^{\infty} f(x)Kg(x)^* \mathrm{d}x =< f, Kg >=< Kf, g > .$$

Question 2: Show that

$$\|h_n\|^2 = \frac{\sqrt{\pi}}{n!}2^n.$$

(Hint: Square the expansion

$$\sum_{n=0}^{\infty} y^n h_n(x) = \mathrm{e}^{x^2/2}\mathrm{e}^{-(x-y)^2}$$

and integrate.)

Question 3: (a) Show that the Hermite polynomials of degree less than or equal to n form a basis for the vector space of all polynomials of degree less than or equal to n.

(b) Calculate the first three Hermite polynomials and use them to compute

$$\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^2} \mathrm{d}x.$$

(c) Show that

$$\int_0^x \mathrm{e}^{-s^2} H_n(s)\ \mathrm{d}s = C_n - \mathrm{e}^{-x^2} H_{n-1}(x)$$

for some constant C_n, whenever $n \geq 1$.

(d) Show that the indefinite integral

$$\int_0^x P(s)\mathrm{e}^{-s^2}\ \mathrm{d}s$$

can be evaluated explicitly whenever P is a polynomial with

$$\int_{-\infty}^{\infty} P(s)H_0(s)\mathrm{e}^{-s^2}\ \mathrm{d}s = 0.$$

Question 4: (a) Show that

$$-< Kf, f >= \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \mathrm{d}x = \sum_{n=0}^{\infty} (2n+1) \frac{< f, h_n >^2}{\|h_n\|^2}$$

for real-valued $f \in L^2(R)$. (b) Prove the weak Heisenberg inequality

$$\int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \mathrm{d}x \geq \int_{-\infty}^{\infty} f(x)^2 \mathrm{d}x$$

for such f.

Question 5: Show that

$$\mathrm{e}^{2\mathrm{i}ts} = \mathrm{e}^{-t^2} \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} H_n(s)$$

(Hint: Seek an expansion of the form

$$\mathrm{e}^{2\mathrm{i}ts} = \sum_{n=0}^{\infty} f_n(t) H_n(s)$$

and use orthogonality of the H_n's.)

Question 6: Use Cramer's inequality

$$|H_n(s)| \leq 1.09 \; 2^{n/2} \sqrt{n!} \mathrm{e}^{s^2/2}$$

and Stirling's approximation to show that the error in N terms of the approximation in Question 5 is bounded by

$$|\mathrm{e}^{2\mathrm{i}ts} - \sum_{n=0}^{N-1} f_n(t) H_n(s)| \leq 10 \left(\frac{2\mathrm{e}}{N} \right)^{N/2}$$

for $N > 10$, $|t| \leq 1$, and $|s| \leq 2$. How many terms are required to get 10-digit accuracy?