Stochastic Processes

Fall 2017

Week 2

Solutions by

Jinhong Du

15338039

2.3

For a Poisson process show, for $s < t$, that

$$\mathbb{P}\{N(s) = k | N(t) = n\} = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, \qquad k = 0, 1, \cdots, n$$

$\forall\ t > s > 0,\ n, k \in \mathbb{N}, k \leqslant n,$

$$\begin{aligned}
\mathbb{P}\{N(s) = k | N(t) = n\} &= \frac{\mathbb{P}\{N(s) = k, N(t) = n\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\mathbb{P}\{N(t) = n | N(s) = k\}\mathbb{P}\{N(s) = k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\mathbb{P}\{N(t) - N(s) = n - k | N(s) - N(0) = k\}\mathbb{P}\{N(s) = k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\mathbb{P}\{N(t) - N(s) = n - k\}\mathbb{P}\{N(s) = k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\mathbb{P}\{N(t - s) - N(0) = n - k\}\mathbb{P}\{N(s) = k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\dfrac{(t-s)^{n-k}\lambda^{n-k}}{(n-k)!} e^{-\lambda(t-s)} \cdot \dfrac{(s\lambda)^k}{k!} e^{-\lambda s}}{\dfrac{(t\lambda)^n}{n!} e^{-\lambda t}} \\
&= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}
\end{aligned}$$

2.4

Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ. Calculate $\mathbb{E}[N(t) \cdot N(t + s)]$.

$\because$ $N(t + s) - N(t)$ and $N(t) - N(0)$ are independent

$\therefore$

$$\begin{aligned}
\mathbb{E}[N(t)N(t+s)] &= \mathbb{E}[N(t+s) - N(t)][N(t) - N(0)] + \mathbb{E}[N(t)^2] \\
&= \mathbb{E}[N(t+s) - N(t)] \cdot \mathbb{E}[N(t) - N(0)] + Var[N(t) - N(0)] + \{\mathbb{E}[N(t) - N(0)]\}^2 \\
&= s\lambda \cdot t\lambda + t\lambda + t^2\lambda^2 \\
&= (s + t)t\lambda^2 + t\lambda
\end{aligned}$$