MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 4

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Solutions by

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Question 1

(a) Compute an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} = QR$$

$\because \quad a_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}^T, a_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}^T$

$\therefore$

$$\begin{aligned} e_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{a_1}{\sqrt{1+\frac{1}{4}+\frac{1}{9}}} \\ &= \frac{6}{7}a_1 \\ &= \begin{pmatrix} \frac{6}{7} & \frac{3}{7} & \frac{2}{7} \end{pmatrix}^T \end{aligned}$$

$\because$

$$\begin{aligned} < a_2, e_1 > &= \frac{1}{2}\cdot\frac{6}{7} + \frac{1}{3}\cdot\frac{3}{7} + \frac{1}{4}\cdot\frac{2}{7} \\ &= \frac{9}{14} \end{aligned}$$

$\therefore$

$$\begin{aligned} e_2 &= \frac{a_2 - < a_2, e_1 > e_1}{\|a_2 - < a_2, e_1 > e_1\|} \\ &= \frac{a_2 - \frac{9}{14}e_1}{\|a_2 - \frac{9}{14}e_1\|} \\ &= \frac{\begin{pmatrix} -\frac{5}{98} & \frac{17}{294} & \frac{13}{196} \end{pmatrix}^T}{\left\|\begin{pmatrix} -\frac{5}{98} & \frac{17}{294} & \frac{13}{196} \end{pmatrix}^T\right\|} \\ &= \begin{pmatrix} -\frac{30}{7\sqrt{73}} & \frac{-34}{7\sqrt{73}} & \frac{-39}{7\sqrt{73}} \end{pmatrix}^T \end{aligned}$$

(b) find the orthonormal and upper-triangular matrices Q and R.

Let $Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^T = \begin{pmatrix} e_1 & e_2 \end{pmatrix}^T$, $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$

$\because$

$$\begin{cases} a_1 = \frac{7}{6}e_1 \\ a_2 = \frac{9}{14}e_1 + \frac{\sqrt{73}}{84}e_2 \end{cases}$$

Solution (cont.)

$\therefore$

$$r_{11} = \frac{7}{6}$$
$$r_{12} = \frac{9}{14}$$
$$r_{22} = \frac{\sqrt{73}}{84}$$

i.e.

$$Q = \begin{bmatrix} \frac{6}{7} & -\frac{30}{7\sqrt{73}} \\ \frac{3}{7} & \frac{-34}{7\sqrt{73}} \\ \frac{2}{7} & \frac{-39}{7\sqrt{73}} \end{bmatrix} \qquad R = \begin{bmatrix} \frac{7}{6} & \frac{9}{14} \\ 0 & \frac{\sqrt{73}}{84} \end{bmatrix}$$

(c) Compute the orthogonal projection P onto the range of A.

$$\begin{aligned} P &= e_1 e_1^* + e_2 e_2^* \\ &= \begin{bmatrix} \frac{36}{49} & \frac{18}{49} & \frac{12}{49} \\ \frac{18}{49} & \frac{9}{49} & \frac{12}{49} \\ \frac{12}{49} & \frac{18}{49} & \frac{4}{49} \end{bmatrix} + \begin{bmatrix} \frac{313}{1244} & \frac{-1020}{3577} & \frac{-1170}{3577} \\ \frac{-1020}{3577} & \frac{403}{1247} & \frac{787}{2123} \\ \frac{-1170}{3577} & \frac{787}{2123} & \frac{344}{809} \end{bmatrix} \\ &= \begin{bmatrix} \frac{72}{73} & \frac{6}{73} & \frac{-6}{73} \\ \frac{6}{73} & \frac{37}{73} & \frac{36}{73} \\ \frac{-6}{73} & \frac{36}{73} & \frac{37}{73} \end{bmatrix} \end{aligned}$$

Question 2

Find a_0 and a_1 minimizing

$$F(a_0, a_1) = \int_0^1 |a_0 + a_1 x - e^{-x}|^2 \mathrm{d}x.$$

$$\begin{aligned} F(a_0, a_1) &= \int_0^1 |a_0 + a_1 x - e^{-x}|^2 \mathrm{d}x \\ &= \int_0^1 (a_0^2 + 2a_0 a_1 x + a_1^2 x^2 - 2a_0 e^{-x} - 2a_1 x e^{-x} + e^{-2x}) \mathrm{d}x \\ &= \left(a_0^2 x + a_0 a_1 x^2 + \frac{a_1^2}{3} x^3 + 2a_0 e^{-x} + 2a_1 (x+1) e^{-x} - \frac{1}{2} e^{-2x} \right) \Big|_0^1 \\ &= a_0^2 + a_0 a_1 + \frac{a_1^2}{3} + 2a_0 (e^{-1} - 1) + 2a_1 (2e^{-1} - 1) - \frac{1}{2}(e^{-2} - 1) \end{aligned}$$

Solution (cont.)

Let

$$\begin{cases} \frac{\partial F}{\partial a_0} = 2a_0 + a_1 + 2(e^{-1} - 1) = 0 \\ \frac{\partial F}{\partial a_1} = a_0 + \dfrac{2a_1}{3} + 2(2e^{-1} - 1) = 0 \end{cases}$$

We get

$$\begin{cases} \hat{a_0} = 8e^{-1} - 2 \\ \hat{a_1} = -18e^{-1} + 6 \end{cases}$$

Because

$$F_{a_0a_1}^2 - F_{a_0a_0}F_{a_1a_1} < 0$$
$$F_{a_0a_0} = 2 > 0$$

Therefore $\hat{a_0}, \hat{a_1}$ minimize $F(a_0, a_1)$.

Question 3

(a) Find an orthonormal basis for the 3-dimensional subspace of $L^2(-1,1)$ spanned by 1,x and x^2.

$$\begin{aligned} e_1 &= \frac{1}{\|1\|} \\ &= \frac{1}{\sqrt{\int_{-1}^{1} \mathrm{d}x}} \\ &= \frac{\sqrt{2}}{2} \\ v_2 &= x - < x, e_1 > e_1 \\ &= x - \frac{1}{2}\int_{-1}^{1} x \mathrm{d}x \\ &= x \\ e_2 &= \frac{v_2}{\|v_2\|} \\ &= \frac{v_2}{\sqrt{\int_{-1}^{1} x^2 \mathrm{d}x}} \\ &= \frac{\sqrt{6}}{2} x \end{aligned}$$

Solution (cont.)

$$\begin{aligned} v_3 &= x^2 - < x^2, e_1 > e_1 - < x^2, e_2 > e_2 \\ &= x^2 - \frac{1}{2}\int_{-1}^{1} x^2 \mathrm{d}x - \int_{-1}^{1} \frac{\sqrt{6}}{2} x^3 \mathrm{d}x e_2 \\ &= x^2 - \frac{1}{3} \\ e_3 &= \frac{v_3}{\|v_3\|} \\ &= \frac{v_3}{\sqrt{\int_{-1}^{1}(x^2 - \frac{1}{3})^2 \mathrm{d}x}} \\ &= \frac{\sqrt{10}}{4}(3x^2 - 1) \end{aligned}$$

(b) Interpret as a QR factorization.

$\because$

$$\begin{cases} 1 = \sqrt{2}e_1 \\ x = \frac{\sqrt{6}}{3}e_2 \\ x^2 = \sqrt{2}e_1 + \frac{2\sqrt{10}}{15}e_3 \end{cases}$$

$\therefore$

$$\begin{bmatrix} 1 & x & x^2 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & \frac{2\sqrt{10}}{15} \end{bmatrix}$$

$\iff$

$$X = QR$$

where Q is a othonormal matrix and R is a upper-triangular matrix.

Question 4

Let

$$H^1 = H^1(0,1) = \{f \in L^2(0,1) | f' \in L^2(0,1)\}$$

with inner product

$$< f, g >= \int_0^1 f(x)g(x) + f'(x)g'(x)\mathrm{d}x.$$

(For simplicity assume all functions are real-valued.)

(a) Show that every $f \in H^1$ is continuous and bounded on $(0,1)$.

Proof.

$\because \quad f' \in L^2(0,1)$

$\therefore \quad \int_0^1 |f'(t)|^2 \mathrm{d}t < \infty$

$\because \quad \forall\ f \in H^1,\ f' \in L^2(0,1), \forall\ x_0 \in (0,1)$

$$\begin{aligned}
|f(x) - f(x_0)| &= \left| \int_x^{x_0} f'(t)\mathrm{d}t \right| \\
&\leqslant \int_x^{x_0} |f'(t)|\mathrm{d}t \\
&\leqslant \sqrt{\left| \int_x^{x_0} |f'(t)|^2 \mathrm{d}t \right|} \sqrt{\left| \int_x^{x_0} 1 \mathrm{d}t \right|} \\
&\leqslant \sqrt{\int_0^1 |f'(t)|^2 \mathrm{d}t} \cdot \sqrt{|x - x_0|} \\
&= \|f'\|_2 \sqrt{|x - x_0|} \to 0 \qquad (x \to x_0)
\end{aligned}$$

$\therefore \quad f$ is continuous on $(0,1)$

$\because \quad \forall\ x \in (0,1),$

$\therefore$

$$\begin{aligned}
|f(x)| &= \left| \int_{\frac{1}{2}}^x f'(t)\mathrm{d}t + f\left(\frac{1}{2}\right) \right| \\
&\leqslant \int_0^1 |f'(t)|\, \mathrm{d}t + \left| f\left(\frac{1}{2}\right) \right| \\
&\leqslant \int_0^1 |f'(t)|\, \mathrm{d}t + \left| f\left(\frac{1}{2}\right) \right| \\
&\leqslant \sqrt{\int_0^1 |f'(t)|^2\, \mathrm{d}t \cdot \int_0^1 \mathrm{d}t} + \left| f\left(\frac{1}{2}\right) \right| \\
&= \|f'\|_2 + \left| f\left(\frac{1}{2}\right) \right| < \infty
\end{aligned}$$

$\therefore \quad f$ is bounded on $(0,1)$ $\square$

(b) Let $g \in H^1$ and suppose also that g' and g'' are continuous except at some point $x_0 \in (0,1)$. Show that

$$< f, g >= f(1)g'(1) + f(x_0)(g'(x_0^-) - g'(x_0^+)) - f(0)g'(0) + \int_0^1 f(x)(g(x) - g''(x))\mathrm{d}x$$

for every $f \in H^1$.

$\because \quad f \in H^1,\ g \in H^1$

$\therefore \quad f, g$ are continuous on $(0,1)$

$\because \quad g', g''$ are continuous except at $x_0 \in (0,1)$

$\therefore \quad f(x)g'(x), f(x)g''(x)$ are continuous except at x_0

Solution (cont.)

$\therefore$

$$
\begin{aligned}
< f, g > &= \int_0^1 f(x)g(x) + f'(x)g'(x)\mathrm{d}x \\
&= \int_0^1 f(x)g(x)\mathrm{d}x + \int_0^1 f'(x)g'(x)\mathrm{d}x \\
&= \int_0^1 f(x)g(x)\mathrm{d}x + \int_0^{x_0} g'(x)\mathrm{d}f(x) + \int_{x_0}^1 g'(x)\mathrm{d}f(x) \\
&= \int_0^1 f(x)g(x)\mathrm{d}x + f(x)g'(x)\Big|_0^{x_0^-} - \int_0^{x_0} f(x)g''(x)\mathrm{d}x + f(x)g'(x)\Big|_{x_0^+}^1 - \int_{x_0}^1 f(x)g''(x)\mathrm{d}x \\
&= f(x)g'(x)\Big|_0^{x_0^-} + f(x)g'(x)\Big|_{x_0^+}^1 + \int_0^1 f(x)g(x)\mathrm{d}x - \int_0^1 f(x)g''(x)\mathrm{d}x \\
&= f(1)g'(1) + f(x_0)(g'(x_0^-) - g'(x_0^+)) - f(0)g'(0) + \int_0^1 f(x)(g(x) - g''(x))\mathrm{d}x
\end{aligned}
$$

(c) Find $g \in H^1$ such that

$$< f, g >= f(x_0)$$

for every $f \in H^1$.

$\because \quad \forall\ f \in H^1$,

$$< f, g >= f(1)g'(1) + f(x_0)(g'(x_0^-) - g'(x_0^+)) - f(0)g'(0) + \int_0^1 f(x)(g(x) - g''(x))\mathrm{d}x = f(x_0)$$

$\therefore$

$$
\begin{cases}
g'(1) = 0 \\
g'(x_0^-) - g'(x_0^+) = 1 \\
g'(0) = 0 \\
g(x) - g''(x) = 0 \qquad a.e. x \in (0, 1)
\end{cases}
$$

$\because \quad g \in H^1$, from Question 4 (a) we know that g is continuous

$\therefore$

$$g(x_0^-) = g(x_0^+)$$

To solve the ordinary differential equations set

$$
\begin{cases}
g(x) - g''(x) = 0 & (1) \\
g(x_0^-) = g(x_0^+) & (2) \\
g'(x_0^-) - g'(x_0^+) = 1 & (3) \\
g'(1) = 0 & (4) \\
g'(0) = 0 & (5)
\end{cases}
$$

Because the general solutions to (1) have form $a_1 e^x + a_2 e^{-x}$, let

$$
g(x) = \begin{cases}
ae^x + be^{-x} & , 0 < x < x_0 \\
ce^x + de^{-x} & , x_0 < x < 1
\end{cases}
$$

Solution (cont.)

Therefore

$$g'(x) = \begin{cases} ae^x - be^{-x} & , 0 < x < x_0 \\ ce^x - de^{-x} & , x_0 < x < 1 \end{cases}$$

From (4)(5) we have

$$\begin{cases} a = b \\ d = ce^2 \end{cases}$$

Then from (3)(4) we have

$$\begin{cases} a(e^{x_0} - e^{-x_0} - c(e^{x_0} - e^{2-x_0}) = 1 \\ a(e^{x_0} + e^{-x_0}) = c(e^{x_0} + e^{2-x_0}) = 0 \end{cases}$$

then we get

$$g(x) = \begin{cases} \dfrac{e^{x_0} + e^{2-x_0}}{2(e^2-1)} e^x + \dfrac{e^{x_0} + e^{2-x_0}}{2(e^2-1)} e^{-x} & , 0 < x < x_0 \\ \dfrac{e^{x_0} + e^{-x_0}}{2(e^2-1)} e^x + \dfrac{e^{x_0} + e^{-x_0}}{2(1-e^{-2})} e^{-x} & , x_0 < x < 1 \end{cases}$$

Question 5

Given $n+1$ distinct points $-1 < x_0 < x_1 < \cdots < x_n < 1$, let P_n be the linear operator which takes $f \in H^1$ into the unique degree-n polynomial

$$p_n(x) = P_n f(x) = \sum_{j=0}^{n} L_j(x) f(x_j)$$

which interpolates the $n+1$ values $f(x_j)$. Here $L_j(x)$ are the degree-n polynomials satisfying

$$L_i(x_j) = \delta_{ij}.$$

(a) Show that P_n is a projection.

$\because$

$$\begin{aligned} P_n^2 f(x) &= P_n \left(\sum_{j=0}^{n} L_j(x) f(x_j) \right) \\ &= \sum_{i=1}^{n} L_i(x) \left(\sum_{j=0}^{n} L_j(x_i) f(x_j) \right) \\ &= \sum_{i=1}^{n} L_i(x) L_i(x_i) f(x_i) \\ &= \sum_{i=1}^{n} L_i(x) f(x_i) \\ &= P_n f(x) \end{aligned}$$

Solution (cont.)

$\therefore \quad P_n$ is a projection

(b) Find the adjoint operator $P_n^* g$ for $g \in H^1$.

From Question 4(c), $\exists\ g_j(x) = \begin{cases} \dfrac{e^{x_j}+e^{2-x_j}}{2(e^2-1)}e^x + \dfrac{e^{x_j}+e^{2-x_j}}{2(e^2-1)}e^{-x} & ,0<x<x_j \\ \dfrac{e^{x_j}+e^{-x_j}}{2(e^2-1)}e^x + \dfrac{e^{x_j}+e^{-x_j}}{2(1-e^{-2})}e^{-x} & ,x_j<x<1 \end{cases} \in H^1$, s.t. $<f, g_j>= f(x_j)$

$\because$

$$\begin{aligned} P_n f(x) &= \sum_{j=1}^{n} L_j(x) f(x_j) \\ &= \sum_{j=1}^{n} L_j(x) < f(x), g_j(x) > \\ &= \sum_{j=1}^{n} L_j g_j^* f(x) \\ &= \left(\sum_{j=1}^{n} L_j g_j^* \right) f(x) \end{aligned}$$

$\therefore$

$$P_n = \sum_{j=1}^{n} L_j g_j^*$$

$\therefore$

$$\begin{aligned} P_n^* &= \left(\sum_{j=1}^{n} L_j g_j^* \right)^* \\ &= \sum_{j=1}^{n} \left(L_j g_j^* \right)^* \\ &= \sum_{j=1}^{n} g_j L_j^* \end{aligned}$$

(c) Show that P_n is not an orthogonal projection.

$\because$

$$g_j(x) = \begin{cases} \dfrac{e^{x_j}+e^{2-x_j}}{2(e^2-1)}e^x + \dfrac{e^{x_j}+e^{2-x_j}}{2(e^2-1)}e^{-x} & ,0<x<x_j \\ \dfrac{e^{x_j}+e^{-x_j}}{2(e^2-1)}e^x + \dfrac{e^{x_j}+e^{-x_j}}{2(1-e^{-2})}e^{-x} & ,x_j<x<1 \end{cases}$$

$\therefore \quad P_n^* f(x) = \sum_{j=1}^{n} g_j(x) < f(y), L_j(y) >$ piecewise exponential while $P_n f(x) = \sum_{j=1}^{n} L_j(x) f(x_j)$ is a polynomial

Solution (cont.)

i.e.

$$P_n^* \neq P_n$$

$\therefore$ From Problem Set 3, P_n is not an orthogonal projection

(d) Find a basis $\{e_0, e_1, e_2, e_3\}$ for the range of P_3 which is orthogonal in the H^1 inner product.

We point out that $\forall\ f \in H^1$,

$$P_3 f(x) = \sum_{j=0}^{3} \prod_{\substack{i=0 \\ i \neq j}}^{3} \frac{x - x_i}{x_j - x_i} f(x_j)$$

because we know that given $n+1$ distinct points we can only find a n-degree polynimal go through all of them:

$$\begin{cases} a_0 + a_1 x_0 \cdots + a_n x_0^n = f(x_0) \\ a_0 + a_1 x_0 \cdots + a_n x_0^n = f(x_1) \\ \qquad\qquad \vdots \\ a_0 + a_1 x_{n+1} \cdots + a_n x_{n+1}^n = f(x_{n+1}) \end{cases}$$

when $f(x_0), \cdots, f(x_{n+1})$ not be all 0.

$\because$

$$x_i \neq x_j \qquad (i \neq j)$$

$\therefore$ from what we have prove in PS4

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+1} & \cdots & x_{n+1}^n \end{vmatrix} \neq 0$$

So we have only one solution for $a_0, a_1, \cdots, a_n$, i.e. only one n-degree polynomial go through there $n+1$ points

$\therefore$ the range of P is cubic polynomial

$\therefore$ the range space of P is the span of $\{1, x, x^2, x^3\}$

Solution (cont.)

Gram-Schimidt

$$\begin{aligned}
< 1, 1 > &= \int_0^1 (1+0)\mathrm{d}x \\
&= 1 \\
e_0 &= \frac{1}{\sqrt{< 1, 1 >}} \\
&= 1 \\
u_1 &= x - < x, e_0 > e_0 \\
&= x - \int_0^1 x\mathrm{d}x \\
&= x - \frac{1}{2} \\
< u_1, u_1 > &= \int_0^1 \left[\left(x - \frac{1}{2} \right)^2 + 1 \right] \mathrm{d}x \\
&= \frac{13}{12}
\end{aligned}$$

$$\begin{aligned}
e_1 &= \frac{u_1}{\sqrt{< u_1, u_1 >}} \\
&= \frac{2\sqrt{39}}{13}x - \frac{\sqrt{39}}{13} \\
u_2 &= x^2 - < x^2, e_0 > e_0 - < x^2, e_1 > e_1 \\
&= x^2 - \int_0^1 (x^2 + 2x)\mathrm{d}x - \int_0^1 \left(\frac{2\sqrt{39}}{13}x^3 - \frac{\sqrt{39}}{13}x^2 + \frac{4\sqrt{39}}{13}x \right) \mathrm{d}x \cdot e_1 \\
&= x^2 - x + \frac{1}{6} \\
< u_2, u_2 > &= \int_0^1 \left[\left(x^2 - x + \frac{1}{6} \right)^2 + (2x-1)^2 \right] \mathrm{d}x \\
&= \frac{61}{180} \\
e_2 &= \frac{u_2}{\sqrt{< u_2, u_2 >}} \\
&= \frac{6\sqrt{305}}{61}x^2 - \frac{6\sqrt{305}}{61}x + \frac{\sqrt{305}}{61} \\
u_3 &= x^3 - < x^3, e_0 > e_0 - < x^3, e_1 > e_1 - < x^3, e_2 > e_2 \\
&= x^3 - \frac{3}{2}x^2 + \frac{33}{65}x - \frac{1}{260} \\
< u_3, u_3 > &= \frac{1861}{36400} \\
e_3 &= \frac{u_3}{\sqrt{< u_3, u_3 >}} \\
&= \frac{20\sqrt{169351}}{1861}x^3 - \frac{30\sqrt{169351}}{1861}x^2 + \frac{132\sqrt{169351}}{24193}x - \frac{1\sqrt{169351}}{24193}
\end{aligned}$$

(e) Find the orthogonal projection Q_3 onto the range of P_3 . Express Q_3 as an integrodierential operator

$$Q_3 f(x) = \int_0^1 K(x,y)f(y) + K'(x,y)f'(y)\mathrm{d}y$$

and compute the kernels K and K' in $\{e_0, e_1, e_2, e_3\}$.

$$\begin{aligned}
Q_3 &= e_0 e_0^* + e_1 e_1^* + e_2 e_2^* + e_3 e_3^* \\
Q_3 f(x) &= \sum_{j=0}^{3} e_j e_j^* f(x) \\
&= \sum_{j=0}^{3} e_j(x) < f(y), e_j(y) > \\
&= < f(x), \sum_{j=0}^{3} e_j(x) e_j(y) >
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left(\sum_{j=0}^{3} e_j(x) e_j(y) \right) f(y) + \frac{\partial}{\partial y} \left(\sum_{j=0}^{3} e_j(x) e_j(y) \right) f'(y)\mathrm{d}y \\
&= \int_0^1 K(x,y)f(y) + K'(x,y)f'(y)\mathrm{d}y \\
K(x,y) &= \sum_{j=0}^{3} e_j(x) e_j(y) \\
K'(x,y) &= \frac{2\sqrt{39}}{13} e_1(x) + \left(\frac{12\sqrt{305}}{61} y - \frac{6\sqrt{305}}{61} \right) e_2(x) \\
&\quad + \left(\frac{60\sqrt{169351}}{1861} y^2 - \frac{60\sqrt{169351}}{1861} y + \frac{132\sqrt{169351}}{24193} \right) e_3(x)
\end{aligned}$$

(f) Show that $q = Q_3 f$ minimizes the H^1 norm $\|q - f\|$ over q in the range of P_3.

$\because \quad q - Q_3 f \in Range(P_3)$, $Q_3 f - f \in Range(P_3)^{\perp}$,

$\therefore$

$$\begin{aligned}
\|q - f\|_{H^1} &= \|q - Q_3 f + Q_3 f - f\|_{H^1} \\
&= \|q - Q_3 f\|_{H^1} + \|Q_3 f - f\|_{H^1}
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\operatorname*{arg\,min}_{q \in Range(P_3)} \|q - f\|_{H^1} &= \operatorname*{arg\,min}_{q \in Range(P_3)} \|q - Q_3 f\|_{H^1} \\
&= Q_3 f
\end{aligned}$$

i.e. $q = Q_3 f$ minimizes the H^1 norm $\|q - f\|$ over q in the range of P_3.