STAT 150: Stochastic Processes

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Homework 10

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Solutions by

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PK Problems 7.1.4

Let γ_t be the excess life and δ_t the age in a renewal process having interoccurrence distribution function $F(x)$. Determine the conditional probability $\Pr\{\gamma_t > y|\delta_t = x\}$ and the conditional mean $\mathbb{E}[\gamma_t|\delta_t = x]$.

Let T_n denote the total time until the nth event happens.
If $x \in [0, t]$,

$$\begin{aligned}
\Pr\{\gamma_t > y|\delta_t = x\} &= \Pr\{T_{N(t)+1} - t > y|t - T_{N(t)} = x\} \\
&= \sum_{n=0}^{\infty} \Pr\{T_{N(t)+1} - t > y|t - T_{N(t)} = x, N(t) = n\} \Pr\{N(t) = n|t - T_{N(t)} = x\} \\
&= \sum_{n=0}^{\infty} \Pr\{X_{n+1} > x + y|T_n = t - x, N(t) = n\} \Pr\{N(t) = n|t - T_{N(t)} = x\} \\
&= \sum_{n=0}^{\infty} \Pr\{X_{n+1} > x + y|X_{n+1} > x\} \Pr\{N(t) = n|t - T_{N(t)} = x\} \\
&= \sum_{n=0}^{\infty} \Pr\{X_1 > x + y|X_1 > x\} \Pr\{N(t) = n|t - T_{N(t)} = x\} \\
&= \Pr\{X_1 > x + y|X_1 > x\} \\
&= \frac{\Pr\{X_1 > x + y\}}{\Pr\{X_1 > x\}} \\
&= \frac{1 - F(x + y)}{1 - F(x)}
\end{aligned}$$

If $x < 0$ or $x > t$,

$$\Pr\{\gamma_t > y|\delta_t = x\} = 0$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\gamma_t|\delta_t = x] &= \int_0^{\infty} y \Pr\{\gamma_t = y|\delta_t = x\} \mathrm{d}y \\
&= \int_0^{\infty} \Pr\{\gamma_t > y|\delta_t = x\} \mathrm{d}y \\
&= \begin{cases} 0 & , x < 0 \text{ or } x > t \\ \int_0^{\infty} \frac{1 - F(x + y)}{1 - F(x)} \mathrm{d}y & , x \in [0, t] \end{cases}
\end{aligned}$$

PK Problems 7.2.2

Let $X_1, X_2, \ldots$ be the interoccurrence times in a renewal process. Suppose $\Pr\{X_k = 1\} = p$ and $\Pr\{X_k = 2\} = q = 1 - p$. Verify that

$$M(n) = \mathbb{E}[N(n)] = \frac{n}{1 + q} - \frac{q^2}{(1 + q)^2}[1 - (-q)^n]$$

for $n = 2, 4, 6, \ldots$.

Let

$$p_k = \Pr\{X_1 = k\}$$

$\because$

$$\begin{aligned} p_1 &= p \\ p_2 &= 1 - p \end{aligned}$$

$\therefore$

$$\begin{aligned} M(n) &= F(n) + \sum_{k=1}^{n-1} p_k M(n-k) \\ M(1) &= F(1) \\ &= p \\ M(2) &= F(2) + p_1 M(1) \\ &= 1 + p^2 \\ &= 1 + (1-q)^2 \\ &= \frac{2}{1+q} - \frac{q^2}{(1+q)^2}(1-q^2) \end{aligned}$$

For $n \geqslant 3$,

$$M(n) = 1 + pM(n-1) + qM(n-2)$$

$\therefore$ for $n \geqslant 3$,

$$\begin{aligned} M(n) - M(n-1) &= 1 - q[M(n-1) - M(n-2)] \\ &= 1 + (-q)^1 + \cdots + (-q)^{n-3} + (-q)^{n-2}[M(2) - M(1)] \\ &= \frac{1-(-q)^{n-2}}{1+q} + (-q)^{n-2}(1+p^2-p) \end{aligned}$$

$\because$ for $n \geqslant 3$,

$$\begin{aligned} M(n) &= M(2) + \sum_{k=3}^{n}\left[\frac{1-(-q)^{k-2}}{1+q} + (-q)^{k-2}(1+p^2-p)\right] \\ &= 1 + p^2 + \frac{n-2}{1+q} + \left[(1+p^2-p) - \frac{1}{1+q}\right]\sum_{k=1}^{n-2}(-q)^k \\ &= \frac{n-2}{1+q} + 1 + (1-q)^2 + \frac{q^3}{1+q}\sum_{k=1}^{n-2}(-q)^k \\ &= \frac{n-2}{1+q} + 1 + (1-q)^2 - \frac{q^4}{(1+q)^2}[1-(-q)^{n-2}] \\ &= \frac{n}{1+q} + \frac{1}{(1+q)^2}\left[-2(1+q) + (1+q)^2 + (1-q)^2(1+q)^2\right. \\ &\quad \left. - q^4 + (-q)^n\right] \\ &= \frac{n}{1+q} - \frac{q^2}{(1+q)^2}[1-(-q)^{n-2}] \end{aligned}$$

PK Problems 7.3.3

Pulses arrive at a counter according to a Poisson process of rate λ. All physically realizable counters are imperfect, incapable of detecting all signals that enter their detection chambers. After a particle or signal arrives, a counter must recuperate, or renew itself, in preparation for the next arrival. Signals arriving during the readjustment period, called dead time or locked time, are lost. We must distinguish between the arriving particles and the recorded particles. The experimenter observes only the particles recorded; from this observation he desires to infer the properties of the arrival process.

Suppose that each arriving pulse locks the counter for a fixed time τ. Determine the probability $p(t)$ that the counter is free at time t.

When $t \geqslant \tau$,

$$\begin{aligned} p(t) &= \Pr\{\text{no events happen in } (t-\tau, t]\} \\ &= \Pr\{N(t) - N(t-\tau) = 0\} \\ &= e^{-\lambda\tau} \end{aligned}$$

When $t < \tau$,

$$\begin{aligned} p(t) &= \Pr\{\text{no events happen in } (0, t]\} \\ &= \Pr\{N(t) - N(0) = 0\} \\ &= e^{-\lambda t} \end{aligned}$$

Therefore, for $t \geqslant 0$,

$$p(t) = e^{-\lambda \min\{t,\tau\}}$$

PK Problems 7.3.4

This problem is designed to aid in the understanding of lengthbiased sampling. Let X be a uniformly distributed random variable on $[0,1]$. Then, X divides $[0,1]$ into the subintervals $[0,X]$ and $(X,1]$. By symmetry, each subinterval has mean length $\frac{1}{2}$. Now pick one of these subintervals at random in the following way: Let Y be independent of X and uniformly distributed on $[0,1]$, and pick the subinterval $[0,X]$ or $(X,1]$ that Y falls in. Let L be the length of the subinterval so chosen. Formally,

$$L = \begin{cases} X & , \text{ if } Y \leqslant X \\ 1-X & , \text{ if } Y > X \end{cases}$$

Determine the mean of L.

$\because$

$$L = \begin{cases} X & , \text{ if } Y \leqslant X \\ 1-X & , \text{ if } Y > X \end{cases}$$

Solution (cont.)

$\therefore$ from the law of total probability, $\forall\, l \in [0,1]$,

$$\begin{aligned}\Pr\{L = l\} &= \Pr\{L = l | Y \leqslant X, X = l\} \Pr\{Y \leqslant X, X = l\} \\ &\quad + \Pr\{L = l | Y > X, X = 1-l\} \Pr\{Y > X, X = 1-l\} \\ &= \Pr\{Y \leqslant l\} + \Pr\{Y > 1-l\} \\ &= 2l\end{aligned}$$

$\therefore$

$$\begin{aligned}\mathbb{E}L &= \int_0^1 l \Pr\{L = l\} \mathrm{d}l \\ &= \frac{2}{3} l^3 \Big|_0^1 \\ &= \frac{2}{3}\end{aligned}$$

Problem 1

Suppose that the interarrival times $\{X_n : n \geqslant 1\}$ that specify a renewal process N are uniformly distributed on $[0,1]$. Find the mean and variance of $N(t)$ for $0 \leqslant t \leqslant 1$.

For $t \in [0,1]$,

$$\begin{aligned}F_1(t) &= t \\ F_2(t) &= \int_0^x F_1(t-x) \mathrm{d}F(x) \\ &= \int_0^x (t-x) \mathrm{d}x \\ &= \frac{1}{2} t^2 \\ F_3(t) &= \int_0^x F_2(t-x) \mathrm{d}F(x) \\ &= \int_0^x \frac{1}{2}(t-x)^2 \mathrm{d}x \\ &= \frac{1}{3!} t^3\end{aligned}$$

Suppose that for $k \in \mathbb{N}^+,\ k \leqslant n$,

$$F_k(t) = \frac{t^k}{k!}$$

Solution (cont.)

then for $k = n + 1$,

$$\begin{aligned} F_{n+1}(t) &= \int_0^x F_n(t-x)\mathrm{d}F(x) \\ &= \int_0^x \frac{1}{2}\frac{(t-x)^n}{n!}\mathrm{d}x \\ &= \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

Therefore, by induction, $\forall\ k \in \mathbb{N}^+$,

$$F_k(t) = \frac{t^k}{k!}$$

$\therefore \quad \forall\ t \in [0, 1]$,

$$\begin{aligned} \mathbb{E}[N(t)] &= \sum_{n=1}^{\infty} F_n(t) \\ &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\ &= e^t - 1 \\ \Pr\{N(t) = n\} &= \Pr\{N(t) \geqslant n\} - \Pr\{N(t) \geqslant n+1\} \\ &= F_n(t) - F_{n+1}(t) \\ &= \frac{t^n}{n!} - \frac{t^{n+1}}{(n+1)!} \\ \mathbb{E}[N(t)]^2 &= \sum_{n=1}^{\infty} n^2 \Pr\{N(t) = n\} \\ &= \sum_{n=1}^{\infty} n^2 \left[\frac{t^n}{n!} - \frac{t^{n+1}}{(n+1)!}\right] \\ &= \sum_{n=1}^{\infty} (2n-1)\frac{t^n}{n!} \\ &= 2t\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{t^n}{n!} \\ &= 2te^t - (e^t - 1) \\ &= (2t-1)e^t + 1 \\ Var[N(t)] &= \mathbb{E}[N(t)]^2 - \{\mathbb{E}[N(t)]\}^2 \\ &= (2t-1)e^t + 1 - (e^t - 1)^2 \\ &= -e^{2t} + (2t+1)e^t \end{aligned}$$