MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 5

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Solutions by

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Question 1

(a) Compute the complex exponential Fourier coefficients $\hat{f}(k)$ of

$$f(x) = e^{rx}$$

for the interval $|x| \leqslant \pi$.

When $r = 0$,

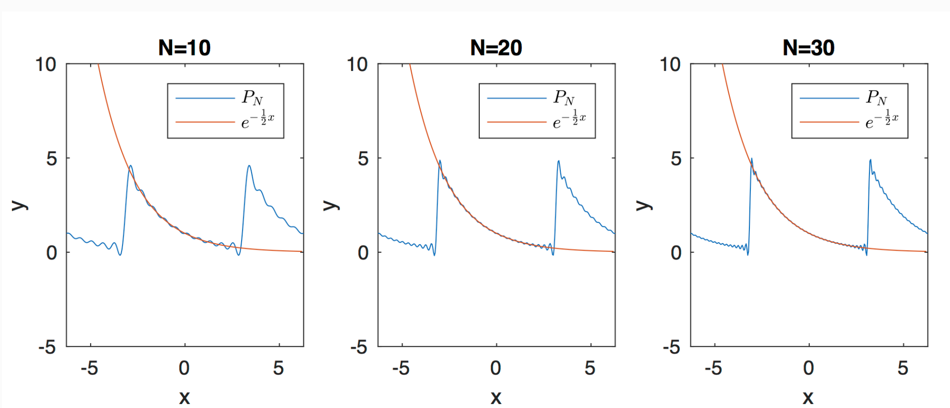
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ikt} \mathrm{d}t$$
$$= \begin{cases} \sqrt{2\pi} & , k = 0 \\ 0 & , k \neq 0 \end{cases}$$

When $r \neq 0$, $\forall\, k \in \mathbb{Z}$,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{(r-ik)t} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}(r-ik)} e^{(r-ik)t} \bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\sqrt{2\pi}(r-ik)} (e^{r\pi} - e^{-r\pi})(-1)^k \\ &= \frac{r+ik}{\sqrt{2\pi}(r^2+k^2)} (e^{r\pi} - e^{-r\pi})(-1)^k \end{aligned}$$

(b) For the case $r = -\frac{1}{2}$ plot partial sums versus f for $N = 10, 20, 30$ on the larger interval $|x| \leqslant 2\pi$. Explain the regions of your plot where convergence appears to be fast versus slow.

$$f_N(x) = \frac{1}{2\pi} \sum_{k=-N}^{N} \frac{r+ik}{r^2+k^2} (e^{r\pi} - e^{-r\pi})(-1)^k e^{ikx}$$



We can see that the middle part of the plot, i.e. the part that is close to 0, converges faster than two side of the interval $[-\pi, \pi]$. And it is the same in other periodic intervals $[-\pi + 2k\pi, \pi + 2k\pi]$.

Solution (cont.)

The accuracy of the approximation gets worse as x get closer to a point of discontinuity $\pm\pi$. And also Gibbs phenomenon appears at such points.

Question 2

(a) Compute the complex exponential Fourier coefficients $\hat{f}(x)$ of

$$f(x) = x^2$$

for the interval $|x| \leqslant \pi$.

When $k = 0$,

$$\begin{aligned}\hat{f}(0) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-i\cdot 0\cdot t} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} t^2 \mathrm{d}t \\ &= \frac{\sqrt{2}\pi^{\frac{5}{2}}}{3}\end{aligned}$$

When $k \neq 0$,

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} t^2 e^{-ikt} \mathrm{d}t \\ &= -\frac{1}{\sqrt{2\pi}\cdot (ik)^3} e^{-ikt} [(ikt)^2 + 2(ikt) + 2] \Bigg|_{t=-\pi}^{\pi} \\ &= \frac{2\sqrt{2\pi}(-1)^k}{k^2}\end{aligned}$$

(b) Show that the Fourier series converges uniformly for $|x| \leqslant \pi$.

$\because$ $f(x) = x^2 \in L^2(-\pi, \pi)$ and $f(x)$ is continuous at $x \in [-\pi, \pi]$, $f'(x) = x$

$\therefore$ from Chernof Theorem,

$$P_N f(x) \to f(x) \qquad a.e.$$

$\because$

$$P_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{N} \hat{f}(k) e^{-ikx}$$

Solution (cont.)

$\therefore$

$$\begin{aligned} |P_N f(x) - f(x)| &= \frac{1}{\sqrt{2\pi}} \left| \sum_{|k|>N} \hat{f}(k) e^{-ikx} \right| \\ &\leqslant \frac{1}{\sqrt{2\pi}} \sum_{|k|>N} |\hat{f}(k)| \\ &\leqslant \sum_{|k|>N} \left| \frac{2\cos(k\pi)}{k^2} \right| \\ &\leqslant \sum_{|k|>N} \frac{2}{k^2} \\ &= 4 \sum_{k>N} \frac{1}{k^2} \end{aligned}$$

$\because$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

$\therefore$

$$\sum_{k>N} \frac{1}{k^2} \to 0 \qquad (N \to \infty)$$

$\therefore$

$$|P_N f(x) - f(x)| \to 0 \qquad (N \to \infty)$$

$\therefore$ $P_N f(x) \to f(x)$ uniformly for $|x| < \pi$

(c) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$\because$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

$\therefore$

$$f(\pi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) = \pi^2$$

$\therefore$

$$2 \sum_{k=1}^{\infty} \frac{2}{k^2} = \frac{2\pi^2}{3}$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(d) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

$\because$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

$\therefore$ from Parseval Equation,

$$\int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 2\sum_{k=1}^{\infty} \frac{8\pi}{k^4}$$

$\therefore$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Question 3

(a) Solve the heat equation

$$u_t = u_{xx}$$

for $0 \leqslant x \leqslant 1$ with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = x(1-x)$.

Use the orthonormal sine basis in $L^2(0,1) - \{e_k\}$ where $e_k = \sqrt{2}\sin(k\pi x) \qquad (k \in \mathbb{N}^+)$ since
When $i \neq j$,

$$\begin{aligned} < e_i, e_j > &= 2\int_0^1 \sin(i\pi x)\sin(j\pi x)\mathrm{d}x \\ &= \int_0^1 \{-\cos[(i+j)\pi x] + \cos[(i-j)\pi x]\}\mathrm{d}x \\ &= 0 \end{aligned}$$

When $i = j$,

$$\begin{aligned} < e_i, e_j > &= 2\int_0^1 \sin^2(i\pi x)\mathrm{d}x \\ &= \int_0^1 [1 - \cos(2i\pi x)]\mathrm{d}x \\ &= 1 \end{aligned}$$

Suppose that

$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}(k,t) e_k$$

$$\hat{u}(k,t) =< u(x,t), e_k >$$

Solution (cont.)

$\because$

$$\begin{aligned}\frac{\partial}{\partial t} < u, e_k > &=< u_t, e_k > \\ &=< u_{xx}, e_k > \\ &= \sqrt{2}\int_0^1 u_{yy}(y,t)\sin(k\pi y)\mathrm{d}y \\ &= \sqrt{2}u_y(y,t)\sin(k\pi y)\Big|_{y=0}^1 - \sqrt{2}k\pi u(y,t)\cos(k\pi y)\Big|_{y=0}^1 - k^2\pi^2 < u(y,t), e_k > \\ &= -k^2\pi^2 < u, e_k >\end{aligned}$$

$\therefore$

$$< u, e_k >= c_k e^{-k^2\pi^2 t}$$

$\therefore$

$$u(x,t) = \sqrt{2}\sum_{k=1}^{\infty} c_k e^{-k^2\pi^2 t}\sin(k\pi x)$$

$\because$

$$u(x,0) = \sqrt{2}\sum_{k=11}^{\infty} c_k \sin(k\pi x) = x(1-x)$$

$\therefore$

$$\begin{aligned}c_k &= \sqrt{2}\int_0^1 x(1-x)\sin(k\pi x)\mathrm{d}x \\ &= \frac{2\sqrt{2}[1-(-1)^k]}{k^3\pi^3}\end{aligned}$$

$\therefore$

$$u(x,t) = \sum_{k=1}^{\infty}\frac{4[1-(-1)^k]}{k^3\pi^3}e^{-k^2\pi^2 t}\sin(k\pi x)$$

(b) Express the solution as an integral operator

$$u(x,t) = \int_0^1 K_t(x,y)u(y,0)\mathrm{d}y$$

and find the kernel $K_t(x,y)$.

$\because$

$$\begin{aligned}u(x,t) &= 2\sum_{k=1}^{\infty}\int_0^1 u(y,0)\sin(k\pi y)\mathrm{d}y\cdot e^{-k^2\pi^2 t}\sin(k\pi x) \\ &= \int_0^1 2\sum_{k=1}^{\infty}\sin(k\pi y)e^{-k^2\pi^2 t}\sin(k\pi x)u(y,0)\mathrm{d}y \\ &= \int_0^1 K_t(x,y)u(y,0)\mathrm{d}y\end{aligned}$$

$\therefore$

$$K_t(x,y) = 2\sum_{k=1}^{\infty}\sin(k\pi y)\sin(k\pi x)e^{-k^2\pi^2 t}$$

Question 4

Let $-\pi < a < b < \pi$ and $Q(x)$ be a polynomial of degree d. Evaluate the complex exponential Fourier coefficients of $f(x) = Q(x)$ for $a < x < b$ and $f(x) = 0$ otherwise.

Suppose that $Q(x) = \sum_{n=0}^{d} a_n x^n \qquad x \in (a,b)$,

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} Q(t) e^{-ikt} \mathrm{d}t \\
&= -\frac{1}{\sqrt{2\pi} ik} Q(t) e^{-ikt} \Big|_{t=a}^{b} + \frac{1}{\sqrt{2\pi} ik} \int_{a}^{b} Q'(t) e^{-ikt} \mathrm{d}t \\
&= \cdots \\
&= -\frac{1}{\sqrt{2\pi} ik} Q(t) e^{-ikt} \Big|_{t=a}^{b} - \frac{1}{\sqrt{2\pi}(ik)^2} Q'(t) e^{-ikt} \Big|_{t=a}^{b} - \cdots - \frac{1}{\sqrt{2\pi}(ik)^{d+1}} Q^{(d)}(t) e^{-ikt} \Big|_{t=a}^{b} \\
&= \frac{e^{-ika} - e^{-ikb}}{\sqrt{2\pi}} \sum_{i=0}^{d} \frac{Q^{(i)}(b) - Q^{(i)}(a)}{(ik)^{i+1}}
\end{aligned}$$

Question 5

(a) Compute the complex exponential Fourier coefficient $\hat{\varphi}_j(k)$ over the interval $[1,1]$ of the four functions φ_j defined in Question 5 of Problem Set 02.

$$\begin{aligned}
\hat{\varphi}_0(k) &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_0(t) e^{-ik\pi t} \mathrm{d}t \\
&= \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-ik\pi t} \mathrm{d}t \\
&= \begin{cases} 0 & , k \neq 0 \\ \sqrt{2} & , k = 0 \end{cases} \\
\hat{\varphi}_1(k) &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_1(t) e^{-ik\pi t} \mathrm{d}t \\
&= \frac{1}{\sqrt{2}} \int_{0}^{1} e^{-ik\pi t} \mathrm{d}t - \frac{1}{\sqrt{2}} \int_{-1}^{0} e^{-ik\pi t} \mathrm{d}t \\
&= \begin{cases} \dfrac{\sqrt{2} i [(-1)^k - 1]}{\pi k} & , k \neq 0 \\ 0 & , k = 0 \end{cases}
\end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
\hat{\varphi}_2(k) &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\varphi_2(t)e^{-ik\pi t}\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\int_{\frac{1}{2}}^{1}e^{-ik\pi t}\mathrm{d}t - \frac{1}{\sqrt{2}}\int_{0}^{\frac{1}{2}}e^{-ik\pi t}\mathrm{d}t \\
&= \begin{cases} \dfrac{i(e^{-\frac{k}{2}\pi i}-1)^2}{\sqrt{2}\pi k} & ,k\neq 0 \\ 0 & ,k=0 \end{cases} \\
\hat{\varphi}_3(k) &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\varphi_3(t)e^{-ik\pi t}\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\int_{-\frac{1}{2}}^{0}e^{-ik\pi t}\mathrm{d}t - \frac{1}{\sqrt{2}}\int_{-1}^{-\frac{1}{2}}e^{-ik\pi t}\mathrm{d}t \\
&= \begin{cases} \dfrac{i(e^{\frac{k}{2}\pi i}-1)^2}{\sqrt{2}\pi k} & ,k\neq 0 \\ 0 & ,k=0 \end{cases}
\end{aligned}
$$

(b) Explain the relations between the four sequences $\hat{\varphi}_j(k)$ in terms of the scaling and shifting relations between the functions φ_j.

(1)

$$\varphi_2(x) = \varphi_1\left[2\left(x-\frac{1}{2}\right)\right]$$

$$
\begin{aligned}
\hat{\varphi}_2(k) &= \frac{1}{\sqrt{2}}\int_{\frac{1}{2}}^{1}e^{-ik\pi t}\mathrm{d}t - \frac{1}{\sqrt{2}}\int_{0}^{\frac{1}{2}}e^{-ik\pi t}\mathrm{d}t \\
&= \frac{e^{-ik\pi}}{2}\frac{1}{\sqrt{2}}\int_{0}^{1}e^{-ik\pi t}\mathrm{d}t - \frac{e^{-ik\pi}}{2}\frac{1}{\sqrt{2}}\int_{-1}^{0}e^{-ik\pi t}\mathrm{d}t \\
&= \frac{e^{-ik\pi}}{2}\hat{\varphi}_1\left(\frac{k}{2}\right)
\end{aligned}
$$

(2)

$$\varphi_3(x) = \varphi_1\left[2\left(x+\frac{1}{2}\right)\right]$$

$$
\begin{aligned}
\hat{\varphi}_3(k) &= \frac{1}{\sqrt{2}}\int_{\frac{1}{2}}^{1}e^{-ik\pi t}\mathrm{d}t - \frac{1}{\sqrt{2}}\int_{0}^{\frac{1}{2}}e^{-ik\pi t}\mathrm{d}t \\
&= \frac{e^{ik\pi}}{2}\frac{1}{\sqrt{2}}\int_{0}^{1}e^{-ik\pi t}\mathrm{d}t - \frac{e^{ik\pi}}{2}\frac{1}{\sqrt{2}}\int_{-1}^{0}e^{-ik\pi t}\mathrm{d}t \\
&= \frac{e^{ik\pi}}{2}\hat{\varphi}_1\left(\frac{k}{2}\right)
\end{aligned}
$$

(3)

$$\varphi_3(x) = \varphi_2\left(x-1\right)$$

Solution (cont.)

$$\begin{aligned}\hat{\varphi_3}(k) &= \frac{1}{\sqrt{2}}\int_{-\frac{1}{2}}^{0} e^{-ik\pi t}\mathrm{d}t - \frac{1}{\sqrt{2}}\int_{-1}^{-\frac{1}{2}} e^{-ik\pi t}\mathrm{d}t \\ &= e^{-ik\pi}\frac{1}{\sqrt{2}}\int_{0}^{\frac{1}{2}} e^{-ik\pi t}\mathrm{d}t - e^{-ik\pi}\frac{1}{\sqrt{2}}\int_{\frac{1}{2}}^{1} e^{-ik\pi t}\mathrm{d}t \\ &= e^{-ik\pi}\hat{\varphi}_2\,(k)\end{aligned}$$

Let 2-periodic function $g(x) = f(ax+b)$, $\forall\ a, b \in \mathbb{R}, a \neq 0$,

$$\begin{aligned}\widehat{g}(k) &= \frac{1}{\sqrt{2}}\int_{-1}^{1} f(at+b)e^{-ik\pi t}\mathrm{d}t \\ &= \frac{e^{ik\pi\frac{b}{a}}}{\sqrt{2}}\int_{-1}^{1} f(at+b)e^{-ik\pi\left(t+\frac{b}{a}\right)}\mathrm{d}t \\ &= \frac{e^{ik\pi\frac{b}{a}}}{a}\hat{f}\left(\frac{k}{a}\right)\end{aligned}$$

(c) Express the projection P from Question 5 of Problem Set 02 in the form

$$Pf(x) = \sum_{-\infty}^{\infty}\hat{P}(x,k)\hat{f}(k)$$

and find the coefficient functions $\hat{P}(x,k)$.

$$\begin{aligned}Pf(x) &= \int_{-1}^{1}\left[\sum_{i=0}^{3}\varphi_i(x)\varphi_i(y)\right]f(y)\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}}\sum_{i=0}^{3}\int_{-1}^{1}\left[\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{iky}\right]\varphi_i(y)\mathrm{d}y\cdot\varphi_i(x) \\ &= \sum_{k=-\infty}^{\infty}\int_{-1}^{1}\sum_{i=0}^{3}e^{iky}\varphi_i(y)\varphi_i(x)\mathrm{d}y\end{aligned}$$

Therefore,

$$\hat{P}(x,k) = \int_{-1}^{1}\sum_{i=0}^{3}e^{iky}\varphi_i(y)\varphi_i(x)\mathrm{d}y = \sum_{i=0}^{3}\varphi_i(x)\varphi_i(-k)$$

Question 6

(a) Let f and g be 2π-periodic piecewise smooth functions such that

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{\infty}\hat{f}(k)e^{ikx}.$$

and

$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k) e^{ikx}.$$

Define $h = f * g$ by

$$h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k) e^{ikx}.$$

Express $\hat{f}$ and $\hat{g}$ as integrals, combine them, and reverse the order of integration and summation to obtain an integral formula for h in terms of f and g.

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t \\
\hat{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(t) e^{-ikt} \mathrm{d}t \\
\hat{f}(k)\hat{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(s) e^{-iks} \mathrm{d}s \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t) g(s) e^{-ik(t+s)} \mathrm{d}t \mathrm{d}s \\
&\xlongequal{\sigma = t+s} \frac{1}{2\pi} \iint\limits_{\substack{-\pi \leqslant t \leqslant \pi \\ t-\pi \leqslant \sigma \leqslant t+\pi}} f(t) g(\sigma - t) e^{-ik\sigma} \mathrm{d}t \mathrm{d}\sigma \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t) g(\sigma - t) e^{-ik\sigma} \mathrm{d}t \mathrm{d}\sigma
\end{aligned}$$

Let

$$p(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) g(x-t) \mathrm{d}t$$

then

$$\begin{aligned}
h(x) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(k) e^{ikx} \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t) g(\sigma - t) e^{ik(x-\sigma)} \mathrm{d}t \mathrm{d}\sigma \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} p(\sigma) e^{-ik\sigma} \mathrm{d}\sigma \cdot e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{p}(k) \cdot e^{ikx} \\
&= p(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) g(x-t) \mathrm{d}t
\end{aligned}$$

(b) Let $g \in L^2(\pi, \pi)$ have complex exponential Fourier coefficients $\hat{g}(k)$ Show that (cf. https://arxiv.org/abs/0806.0150)

$$\sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k)$$

if and only if

$$g(0) = \frac{1}{2a} \int_{-a}^{a} g(y) \mathrm{d}y.$$

Note that $\dfrac{\sin(ka)}{ka} \to 1$ as $a \to 0$.

Let

$$f(x) = \begin{cases} 1 & , |x| \leqslant a \\ 0 & , a < |x| \leqslant \pi \end{cases}$$

be a 2π periodic function.

When $k \neq 0$,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \mathrm{d}x \\ &= -\frac{1}{\sqrt{2\pi} ik} e^{-ikx}\Big|_{x=-a}^{a} \\ &= \frac{\sqrt{2}\sin(ak)}{\sqrt{\pi}k} \end{aligned}$$

When $k = 0$,

$$\begin{aligned} \hat{f}(0) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{d}x \\ &= \frac{\sqrt{2}a}{\sqrt{\pi}} \end{aligned}$$

Let $h = f * g$, then

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(k)e^{ikx} \\ h(0) &= \frac{a}{\pi}\hat{g}(0) + \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \frac{\sin(ak)}{\pi k}\hat{g}(k) \\ &= \frac{a}{\pi}\left(\hat{g}(0) + \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \frac{\sin(ak)}{ak}\hat{g}(k)\right) \end{aligned} \tag{1}$$

From (a) we have

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)g(x-t)\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(x-t)\mathrm{d}t \\ h(0) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(-t)\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(t)\mathrm{d}t \end{aligned} \tag{2}$$

$\Longrightarrow$

$\because$

$$\sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka}\hat{g}(k)$$

Solution (cont.)

$\therefore$ from (1),

$$
\begin{aligned}
h(0) &= \frac{a}{\pi}\left(\hat{g}(0) + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{\sin(ak)}{ak}\hat{g}(k)\right)\\
&= \frac{a}{\pi}\sum_{k=-\infty}^{\infty}\hat{g}(k)\\
&= \frac{\sqrt{2}a}{\sqrt{\pi}}g(0)
\end{aligned}
$$

it equals to (2),

$$\frac{\sqrt{2}a}{\sqrt{\pi}}g(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} g(t)\mathrm{d}t$$

i.e.

$$g(0) = \frac{1}{2a}\int_{-a}^{a} g(y)\mathrm{d}y$$

$\Longleftarrow$

$\because$

$$g(0) = \frac{1}{2a}\int_{-a}^{a} g(y)\mathrm{d}y$$

$\therefore$

$$\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\hat{g}(k) = \frac{1}{2a}\int_{-a}^{a} g(y)\mathrm{d}y$$

$\therefore$ from (2) and (1),

$$
\begin{aligned}
\frac{a}{\pi}\sum_{k=-\infty}^{\infty}\hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} g(y)\mathrm{d}y\\
&= h(0)\\
&= \frac{a}{\pi}\left(\hat{g}(0) + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{\sin(ak)}{ak}\hat{g}(k)\right)\\
&= \frac{a}{\pi}\sum_{k=-\infty}^{\infty}\frac{\sin(ak)}{ak}\hat{g}(k)
\end{aligned}
$$

i.e.

$$\sum_{k=-\infty}^{\infty}\hat{g}(k) = \sum_{k=-\infty}^{\infty}\frac{\sin(ak)}{ak}\hat{g}(k)$$

(c) Show that

$$\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2} = \sum_{n=1}^{\infty}\frac{\sin n}{n} = \frac{\pi-1}{2}.$$

For all $a \in (0,\pi)$, suppose that

$$f(x) = \begin{cases} 1 & , |x| \leqslant a\\ 0 & , a < |x| \leqslant \pi \end{cases}$$

Solution (cont.)

$\because$ when $k \neq 0$,

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \mathrm{d}x \\ &= -\frac{1}{\sqrt{2\pi} ik} e^{-ikx} \Big|_{x=-a}^{a} \\ &= \frac{\sqrt{2}\sin(ak)}{\sqrt{\pi}k}\end{aligned}$$

when $k = 0$,

$$\begin{aligned}\hat{f}(0) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{d}x \\ &= \frac{\sqrt{2}a}{\sqrt{\pi}}\end{aligned}$$

$\therefore$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(x) e^{ikx}$$

$\because$

$$f(0) = 1 = \sum_{k=-\infty}^{\infty} \hat{f}(x)$$

$\therefore$

$$a + 2\sum_{k=1}^{\infty} \frac{\sin(ak)}{k} = \pi$$

$\therefore$

$$\sum_{k=1}^{\infty} \frac{\sin(ak)}{k} = \frac{\pi - a}{2}$$

$\therefore$

$$\sum_{k=-\infty}^{\infty} \frac{\sin k}{k} = \pi$$

Given $a = 1$, we get

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}$$

From Passvel equation,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 &= \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin^2 n}{n^2} \\ &= \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x \\ &= 2\end{aligned}$$

$\therefore$

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 n}{n^2} = \pi$$

$\because$

$$|\hat{f}(0)|^2 = \frac{2}{\pi}$$

Solution (cont.)

$$\sum_{n=-\infty}^{-1} \frac{\sin^2 n}{n^2} = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$$

$\therefore$

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} = \frac{\pi - 1}{2}$$

$\therefore$

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} = \sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}$$