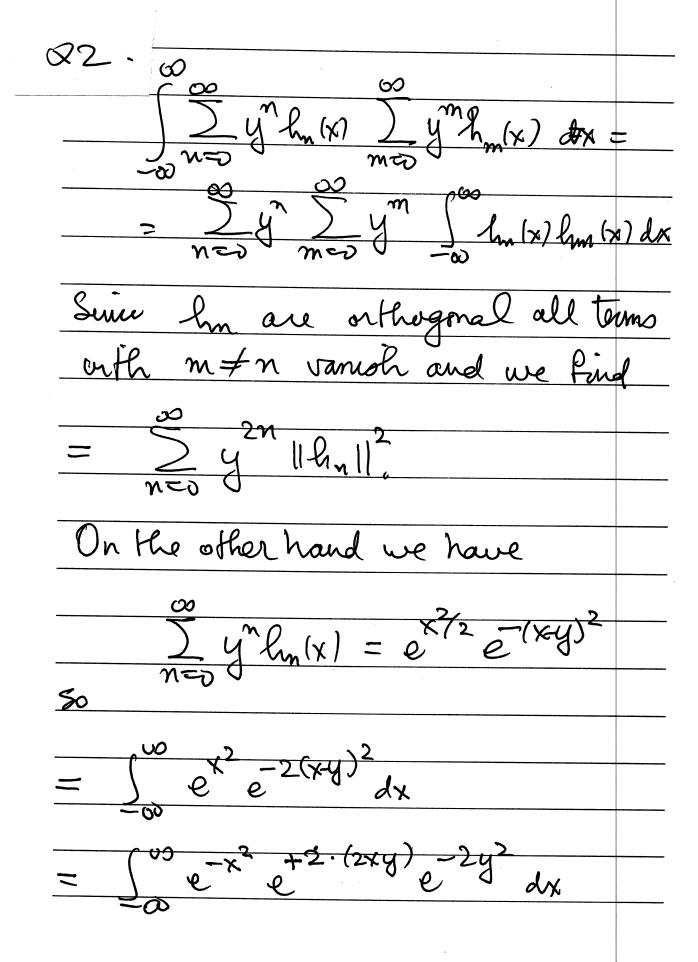
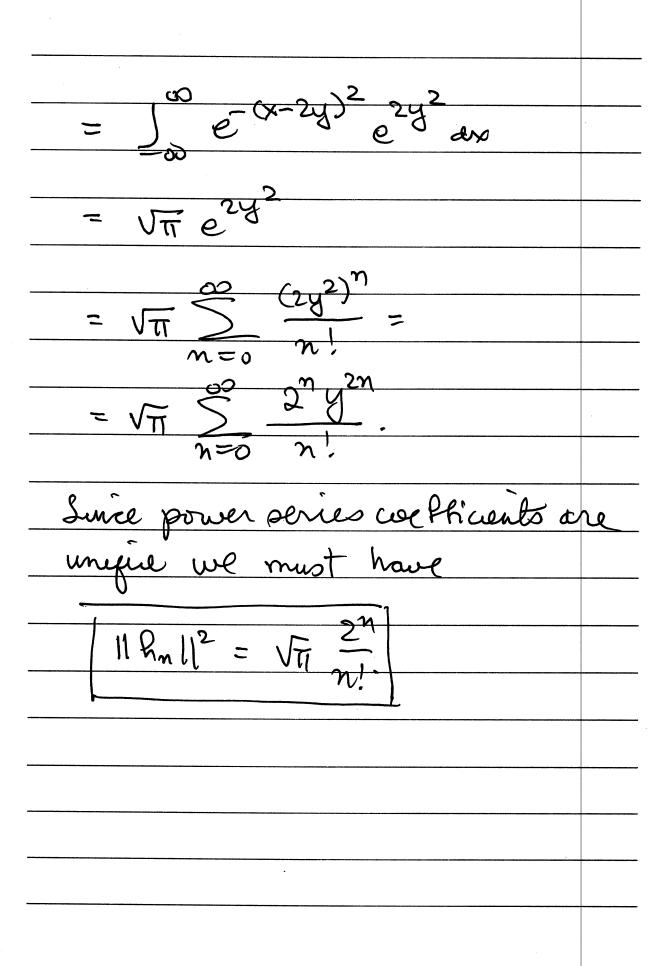
Problem Set UB Solutions 118,517 Q1.) fur) (q"(n)-x2q(x)) dx f(x)g'(x) - f'(x)g(x) + 5 (f(x) -x2 fex)) g(x) dx x2 is real-valued Surie fand gare "nice" smooth functions we can essume $f(x)g'(x) \rightarrow 0 \quad \Rightarrow \quad |x| \rightarrow \infty$ $f'(x)g(x) \rightarrow 0$ and we have shown < f, Kg7 = < Kf,g>





Q3(a) Since deg
$$H_n(x) = n$$
 exactly

(b) c

 $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}$
 $+1$ deg n deg $n-1$

(cannot cancel))

we can write

$$x^m = C_{nn} H_n(x) + q_1(x)$$

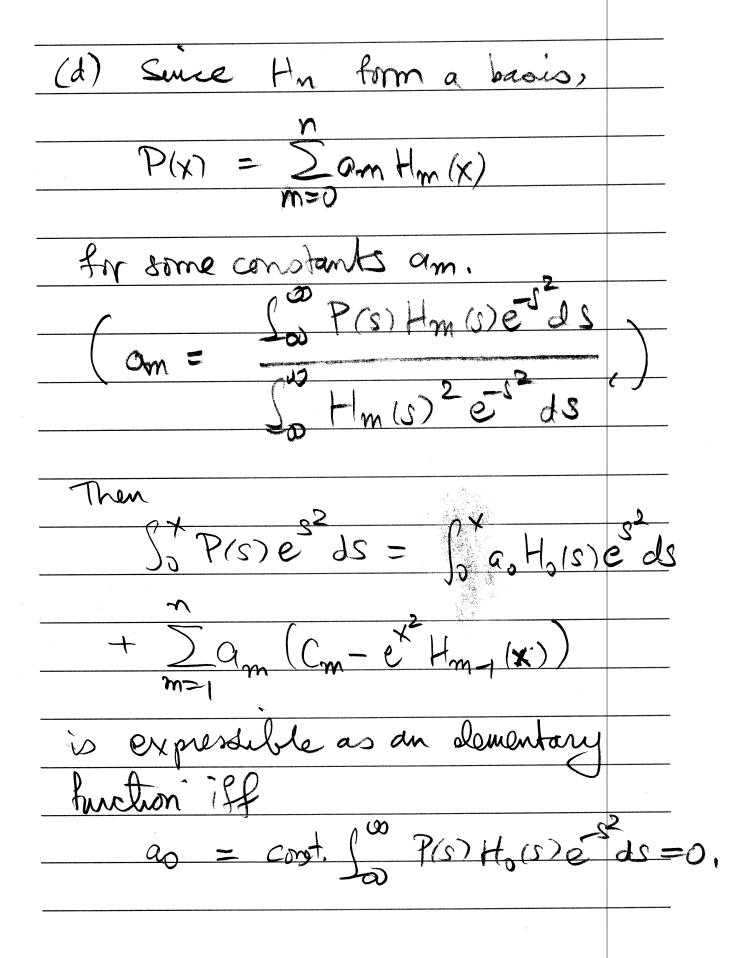
where deg $(q) < n$. By induction,

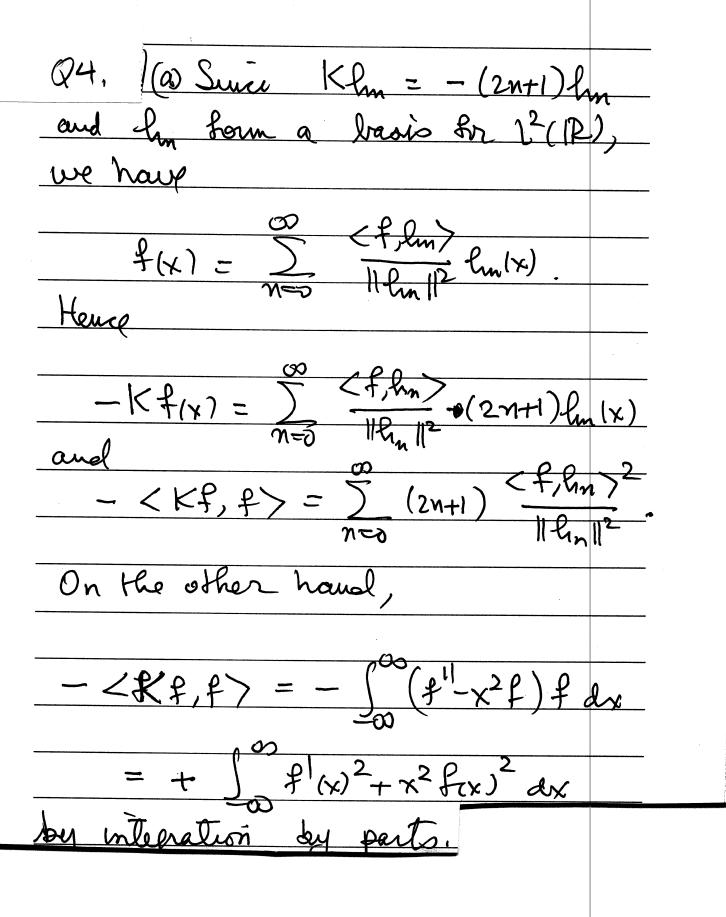
 H_n form a basis.

(b) $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$.

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(c) By definition; Hn(x) = (-1) ex Dre-x ex Hulx) = (-1) D'ex. (es Hy w) ds = = (-1) Dress = (-1) (Dn-1e-x-- Cm = Cn - ex Hn-1(x), for not an elementary function.)

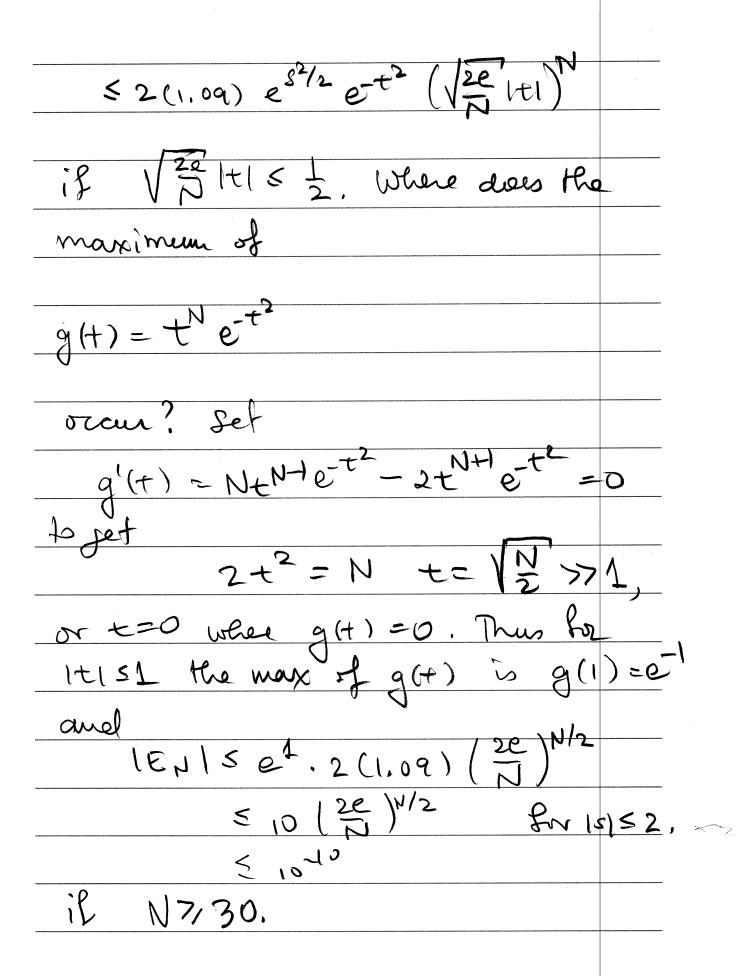




1.1	
We ignore the boundary terms at ±00	
We ignore the boundary terms at ±00 because for Sf(x)2 and Sx2 &	x) ²
to converge we must have mo	10
to converge we must have mo regularity tean just &G12(IR)	
(b) Since 2n+131 we have	
$\int_{00}^{\infty} f'(x)^2 + x^2 f(x)^2 dx >$	12 1
$\frac{7}{n=0} \frac{\langle f, lm \rangle^2}{ f ^2} = f ^2$	
n=0 lm 2	

Q5. We hnow	
$\frac{200}{200} = 0^{1/2} e^{-(xy)^2}$	
and $l_{m(x)} = \frac{1}{n!} H_{m(x)} e^{-x^2/2}$. Hence	
$\frac{\sum_{n=0}^{\infty} y^n}{n!} H_n(x) = e^{\chi^2} e^{-(\chi y)^2}$	
$= e^{2xy} e^{-y^2}$	
Susse Setting x = S and y = it	Sives
$\frac{e^{2its}}{e^{2its}} = e^{-t^2} \frac{\cos(it)^n}{n!} H_n(s)$	
a useful expansion for the Fourier barnel.	
Fourier kernel.	- Andrews Control of the Control of

$$\frac{Q6.}{E_{N}} = 1e^{2its} - \sum_{n=0}^{N-1} \int_{0}^{\infty} \int_$$



Question 1: Show that $K = D^2 - x^2$ is a symmetric operator on $L^2(R)$: for nice smooth functions $f, g \in L^2(R)$ we have

$$\int_{-\infty}^{\infty} f(x)Kg(x)^* dx = \langle f, Kg \rangle = \langle Kf, g \rangle.$$

Question 2: Show that

$$||h_n||^2 = \frac{\sqrt{\pi}}{n!} 2^n.$$

(Hint: Square the expansion

$$\sum_{n=0}^{\infty} y^n h_n(x) = e^{x^2/2} e^{-(x-y)^2}$$

and integrate.)

Question 3: (a) Show that the Hermite polynomials of degree less than or equal to n form a basis for the vector space of all polynomials of degree less than or equal to n.

(b) Calculate the first three Hermite polynomials and use them to compute

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx.$$

(c) Show that

$$\int_0^x e^{-s^2} H_n(s) ds = C_n - e^{-x^2} H_{n-1}(x)$$

for some constant C_n , whenever $n \geq 1$.

(d) Show that the indefinite integral

$$\int_0^x P(s) \mathrm{e}^{-s^2} \, \mathrm{d}s$$

can be evaluated explicitly whenever P is a polynomial with

$$\int_{-\infty}^{\infty} P(s)H_0(s)e^{-s^2} ds = 0.$$

Question 4: (a) Show that

$$- \langle Kf, f \rangle = \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 dx = \sum_{n=0}^{\infty} (2n+1) \frac{\langle f, h_n \rangle^2}{\|h_n\|^2}$$

for real-valued $f \in L^2(R)$. (b) Prove the weak Heisenberg inequality

$$\int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 dx \ge \int_{-\infty}^{\infty} f(x)^2 dx$$

for such f.

Question 5: Show that

$$e^{2its} = e^{-t^2} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H_n(s)$$

(Hint: Seek an expansion of the form

$$e^{2its} = \sum_{n=0}^{\infty} f_n(t) H_n(s)$$

and use orthogonality of the H_n 's.)

Question 6: Use Cramer's inequality

$$|H_n(s)| \le 1.09 \ 2^{n/2} \sqrt{n!} e^{s^2/2}$$

and Stirling's approximation to show that the error in N terms of the approximation in Question 5 is bounded by

$$|e^{2its} - \sum_{n=0}^{N-1} f_n(t)H_n(s)| \le 10 \left(\frac{2e}{N}\right)^{N/2}$$

for $N>10,\, |t|\leq 1,\, {\rm and}\,\, |s|\leq 2.$ How many terms are required to get 10-digit accuracy?