TTIC 31250 An Introduction to the Theory of Machine Learning

Homework # 5 **Solutions**

Exercises:

1. **[Zero-sum Games]** Consider the following zero-sum game. Player A (Alice) hides either a nickel (5 cents) or a quarter (25 cents) behind her back. Then, player B (Bob) guesses which it is. If Bob guesses correctly, he wins the coin. If Bob guesses incorrectly, he has to pay Alice 15 cents. In other words, the amount that Alice wins can be summarized by the following payoff matrix:

Alice hides \ Bob guesses	N	Q
N	-5	$+15$
Q	$+15$	-25

This seems like a fair game since 15 cents is the average of 5 cents and 25 cents, but we will see that one of the players in fact has an advantage.

(a) What is the value to Alice of the strategy "with probability 1/2 hide a nickel and with probability 1/2 hide a quarter"? (The value of a strategy is its value assuming that the opponent knows it and plays a best-response to it).

Solution: In this case Bob will guess Quarter, so the value to Alice is -5 cents.

(b) What is Alice's minimax optimal strategy, and what is its value?

Solution: We want to set the probability p of hiding a nickel to maximize the minimum of $-5p + 15(1-p)$ and $15p - 25(1-p)$. Since one of these is decreasing with p and one is increasing with p, the minimum is maximized when the two terms are equal. So, we get $15 - 20p = -25 + 40p$ or $p = 2/3$. So the minimax optimal strategy for Alice is to hide a nickel with probability $2/3$ and hide a quarter with probability $1/3$. The value to Alice of this strategy is $5/3$.

(c) What is Bob's minimax optimal strategy, and what is its value to Bob?

Solution: We want to set the probability p of guessing a nickel to minimize the maximum of $-5p + 15(1-p)$ and $15p - 25(1-p)$. Since one of these is decreasing with p and one is increasing with p, the maximum is minimized when the two terms are equal, giving us $p = 2/3$ as before. So the minimax optimal strategy for Bob is to guess a nickel with probability $2/3$ and guess a quarter with probability $1/3$. The value to Bob of this strategy is $-5/3$ (an average loss of $5/3$ cents).

(d) Is it better to be Alice or Bob in this game?

Solution: Alice

Problems:

2. **[On approximate Nash equilibria]** Consider a two-player general-sum game. Let us for concreteness focus on games where each player has n actions, and use R to denote the payoff matrix for the row player and C to denote the payoff matrix for the column player. (So if the row-player plays action i and the column-player plays action j, then the row-player gets R_{ij} and the column-player gets C_{ij}. Recall that a Nash Equilibrium is a pair of distributions p and q (one for each player) such that neither player has any incentive to deviate from its distribution assuming that the other player doesn't deviate from its distribution either. Formally, a pair of distributions p (for the row player) and q (for the column player) is a Nash equilibrium if the following holds: assuming the column player plays at random from q, the expected payoff to the row player for each row i with $p_i > 0$ is equal to the maximum payoff out of all the rows ($e_i^T Rq = \max_{i'} e_{i'}^T Rq$); and, assuming the row player plays at random from p, the expected payoff to the column player for each column j with $q_j > 0$ is equal to the maximum payoff out of all the columns ($p^T Ce_j = \max_{j'} p^T Ce_{j'}$). (Here, e_i denotes the column-vector with a 1 in position i and 0 everywhere else).

   Now, assume we have a game in which all payoffs are in the range $[0, 1]$. Define a pair of distributions p, q to be an "ϵ-Nash" equilibrium if each player has *at most* ϵ incentive to deviate. That is, the expected payoff to the row player for each row i with $p_i > 0$ is within ϵ of the maximum payoff out of all the rows, and vice-versa for the column player.

   Using the fact that Nash equilibria must exist, show that there must exist an ϵ-Nash equilibrium in which each player has positive probability on at most $O(\frac{1}{\epsilon^2}\log n)$ actions (rows or columns).

   Hint #1: what is a good randomized way to get a sparse approximation to a probability distribution p that was handed to you?

   Hint #2: your solution will require using Hoeffding bounds and the union bound.

   Solution: Consider some Nash equilibrium (P, Q). Let S be a (multi-)set of k rows selected iid from P, and let T be a multi-set of k rows selected iid from Q. Let U_S denote the uniform distribution over S and let U_T denote the uniform distribution over T. The claim is that $k = O(\frac{1}{\epsilon^2}\log n)$ is sufficient so that with high probability, the pair (U_S, U_T) is an ϵ-Nash equilibrium (so such a pair must exist). In particular, by Hoeffding bounds, this value of k is sufficient so that with high probability, for every column c, its average payoff over the rows in S is within $\pm\epsilon/2$ of its expected payoff with respect to the distribution P. Similarly, with high probability, for every row r, its average payoff over the columns in T is within $\pm\epsilon/2$ of its expected payoff with respect to the distribution Q. So long as both conditions hold, this implies that the pair (U_S, U_T) has the property that each player has at most ϵ incentive to deviate. For instance, the row player has at most ϵ incentive to deviate because each row in S has payoff $\geq \max_{i'} e_{i'}^T Rq - \epsilon/2$ and each row not in S has payoff $\leq \max_{i'} e_{i'}^T Rq + \epsilon/2$. Similarly for the column player.

   Note: this fact yields an $n^{O(\frac{1}{\epsilon^2}\log n)}$-time algorithm for finding an ϵ-Nash equilibrium. No PTAS (algorithm running in time polynomial in n for any fixed $\epsilon > 0$) is known, however.

3. **Compression bounds.** For some learning algorithms, the hypothesis produced by running the algorithm on a training set of size n can be uniquely described by giving k of the training examples. E.g., if you are learning an interval on the line using the simple algorithm "take the smallest interval that encloses all the positive examples," then the hypothesis can be reconstructed from just being told the outermost positive examples, so $k = 2$. For a conservative Mistake-Bound learning algorithm, you can reconstruct the hypothesis produced by the algorithm by just looking at the examples on which a mistake was made, so $k \leq M$, where M is the algorithm's mistake-bound. (In this case, you would also care about the *order* in which those examples arrived.)

   Your job in this problem is to prove a PAC generalization guarantee based on k (essentially, proving that if k is small, then this is a legitimate notion of a "simple" hypothesis; these are called *compression bounds*). Specifically, assume we fix a reconstruction procedure, so that for a given sequence of examples S' we have a well-defined hypothesis $h_{S'}$. You will show that

$$\Pr_{S \sim D^n}\left(\exists S' \subseteq S, |S'| = k, \text{ such that } h_{S'} \text{ has 0 error on } S - S' \text{ but true error } > \epsilon\right) \leq \delta,$$

   so long as

$$n \geq \frac{1}{\epsilon}\left(k \ln n + \epsilon k + \ln \frac{1}{\delta}\right).$$

   (a) First, prove the following easier statement. Let's use $x_1, ..., x_n$ to denote the examples in S. Now suppose you are given a sequence of indices $i_1, ...i_k$. Define $A_{i_1,...,i_k}$ to be the event that $h_{(x_{i_1},...,x_{i_k})}$ has zero error on all examples $x_j \in S$ such that $j \notin \{i_1, ..., i_k\}$ and yet the true error of $h_{(x_{i_1},...,x_{i_k})}$ is more than ϵ. Prove that if $S \sim D^n$, the probability of event $A_{i_1,...,i_k}$ is at most $(1 - \epsilon)^{n-k}$.

   Solution: Since the examples in S are drawn iid, we can assume that $x_{i_1}, ..., x_{i_k}$ are drawn first, and prove the stronger statement that no matter what those examples are, the probability of the event $A_{i_1,...,i_k}$ is at most $(1 - \epsilon)^{n-k}$ over just the draw of the remaining examples. In particular, after those examples are drawn, $h_{(x_{i_1},...,x_{i_k})}$ is now fixed. If the true error of $h_{(x_{i_1},...,x_{i_k})}$ is less than ϵ, then the probability of $A_{i_1,...,i_k}$ over the remaining draw is 0. Else, if the true error is greater than or equal to ϵ, the probability of $A_{i_1,...,i_k}$ over the remaining draw is at most $(1 - \epsilon)^{n-k}$.

   (b) Now use this to prove the guarantee in the displayed equation above.

   Solution: The event we are concerned with is that for at least one sequence of indices $i_1, ...i_k$, the event $A_{i_1,...,i_k}$ occurs. By the union bound and part (a), this has probability at most $\binom{n}{k}(1 - \epsilon)^{n-k}$. Now we just set this to δ, take logs, and solve.