CS 189: Introduction to Machine Learning

Fall 2017

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Homework 9

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Solutions by

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Question 1

(a)

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(b)

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. Jinhong Du

Question 2

(a)

For any sample x_i, suppose that x_i is in class 1, then

$$R(f(x_i)|x_i) = \sum_{j=1}^{c} L(f(x_i), j)\mathbb{P}(y_i = j|x_i)$$

If we make the first decision (a),

$$\begin{aligned} R(f(x_i)|x_i) &= \sum_{j=2}^{c} L(f(x_i), j)\mathbb{P}(y_i = j|x_i) \\ &= \lambda_s[1 - \mathbb{P}(y_i = 1|x_i)] \\ &\leqslant \lambda_s \frac{\lambda_r}{\lambda_s} \\ &= \lambda_r \end{aligned}$$

In such condition, $\lambda_r < \lambda_s$, so $R(f(x_i)|x_i) \leqslant \lambda_r < \lambda_s$. For other policies, the risk must bigger than $\lambda_s[1 - \mathbb{P}(y_i = j|x_i)] \geqslant \lambda_s[1 - \mathbb{P}(y_i = 1|x_i)]$, $\forall\, j \in \{1, 2, \cdots, c\}$. So the policy in such condition minimize the risk.
If we make the secend decision (b),

$$\begin{aligned} R(f(x_i)|x_i) &= \sum_{j=1}^{c} L(f(x_i), j)\mathbb{P}(y_i = j|x_i) \\ &= \lambda_s \end{aligned}$$

It is the same for all policies.
Therefore, the policy obtains the minimum risk.

(b)

If $\lambda_r = 0$, in both decisions,

$$R(f(x_i)|x_i) \equiv \lambda_s$$

Since $\lambda_r = 0$ means that the doubt sample won't give contribution to risk function. So the risk function only depends on the loss incurred for making a misclassication.
If $\lambda_r > \lambda_s$, then

$$\mathbb{P}(Y = i|x) < 1 - \frac{\lambda_r}{\lambda_s}$$

therefore

$$R(f(x_i)|x_i) \equiv \lambda_s$$

Since intuitively, the loss incurred for choosing doubt should be less than or equals the loss for making a misclassification. Otherwise, $\lambda_r > \lambda_s$ means that choosing wrong is better than choosing doubt.

Question 3

(a)

$$\mathbb{P}(X|L=i) = \frac{1}{\sqrt{2\pi}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\mu_i)^T\Sigma^{-1}(X-\mu_i)}$$

(1) MLE

The decision boundary is

$$\mathbb{P}(X|L=1) = \mathbb{P}(X|L=2)$$

i.e.

$$-\frac{1}{2}(X-\mu_1)^T\Sigma^{-1}(X-\mu_1) = -\frac{1}{2}(X-\mu_2)^T\Sigma^{-1}(X-\mu_2)$$

i.e.

$$f_{MLE}(X) = (\mu_1-\mu_2)^T\Sigma^{-1}X - \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}\mu_2) = 0$$

The decision rule is

$$\hat{L}_{MLE}(X) = \begin{cases} 1 & ,[\Sigma^{-1}(\mu_1-\mu_2)]^TX > \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}\mu_2) \\ 2 & ,[\Sigma^{-1}(\mu_1-\mu_2)]^TX \leqslant \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}\mu_2) \end{cases}$$

(2) MAP

$\because$

$$\begin{aligned}
\mathbb{P}(L=1|X) &= \frac{\mathbb{P}(X|L=1)\mathbb{P}(L=1)}{\mathbb{P}(X|L=1)\mathbb{P}(L=1)+\mathbb{P}(X|L=2)\mathbb{P}(L=2)} \\
&= \frac{\mathbb{P}(X|L=1)\pi_1}{\mathbb{P}(X|L=1)\pi_1+\mathbb{P}(X|L=2)\pi_2} \\
&= \frac{1}{1+\frac{\mathbb{P}(X|L=2)\pi_2}{\mathbb{P}(X|L=1)\pi_1}} \\
&= \frac{1}{1+e^{\ln\mathbb{P}(X|L=2)-\ln\mathbb{P}(X|L=1)+\ln\pi_2-\ln\pi_1}} \\
&= \frac{1}{1+e^{(\mu_2-\mu_1)^T\Sigma^{-1}X-\frac{1}{2}(\mu_2^T\Sigma^{-1}\mu_2-\mu_1^T\Sigma^{-1}\mu_1)+\ln\pi_2-\ln\pi_1}}
\end{aligned}$$

$\therefore$ the MAP decision boundary is

$$f_{MAP}(X) = (\mu_2-\mu_1)^T\Sigma^{-1}X - \frac{1}{2}(\mu_2^T\Sigma^{-1}\mu_2 - \mu_1^T\Sigma^{-1}\mu_1) + \ln\pi_2 - \ln\pi_1 = 0$$

The MAP decision rule is

$$\hat{L}_{MAP}(X) = \begin{cases} 1 & ,[\Sigma^{-1}(\mu_1-\mu_2)]^TX > \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}\mu_2) + \ln\pi_2 - \ln\pi_1 \\ 2 & ,[\Sigma^{-1}(\mu_1-\mu_2)]^TX \leqslant \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}\mu_2) + \ln\pi_2 - \ln\pi_1 \end{cases}$$

When $\pi_1 = \pi_2 = \frac{1}{2}$, then these two decision rules are the same.

(b)

$$\begin{aligned}
\mathbb{E}X &= \pi_1\mu_1 + \pi_2\mu_2 \\
&= \frac{\mu_1+\mu_2}{2} \\
\mathbb{E}[XX^T|\text{class } 1] &= Var[X|\text{class } 1] + (\mathbb{E}[X|\text{class } 1])(\mathbb{E}[X|\text{class } 1])^T \\
&= \Sigma + \mu_1\mu_1^T \\
\mathbb{E}[XX^T|\text{class } 2] &= Var[X|\text{class } 2] + (\mathbb{E}[X|\text{class } 2])(\mathbb{E}[X|\text{class } 2])^T \\
&= \Sigma + \mu_2\mu_2^T \\
\Sigma_{XX} &= \mathbb{E}[(X-\mathbb{E}X)(X-\mathbb{E}X)^T] \\
&= \mathbb{E}[XX^T] - (\mathbb{E}X)(\mathbb{E}X)^T \\
&= \pi_1\mathbb{E}[XX^T|\text{class } 1] + \pi_2\mathbb{E}[XX^T|\text{class } 2] - \frac{1}{4}(\mu_1+\mu_2)(\mu_1+\mu_2)^T \\
&= \Sigma + \frac{1}{2}\mu_1\mu_1^T + \frac{1}{2}\mu_2\mu_2^T - \frac{1}{4}(\mu_1+\mu_2)(\mu_1+\mu_2)^T \\
&= \Sigma - \frac{1}{4}(\mu_1-\mu_2)(\mu_1-\mu_2)^T \\
\mathbb{E}Y &= \pi_1\begin{pmatrix}1\\0\end{pmatrix} + \pi_2\begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix} \\
\Sigma_{XY} &= \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T] \\
&= \pi_1\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T|\text{class } 1] + \pi_2\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T|\text{class } 2] \\
&= \pi_1\begin{pmatrix}\frac{\mu_1-\mu_2}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix} + \pi_2\begin{pmatrix}\frac{\mu_2-\mu_1}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix}\mu_1-\mu_2 & \mu_2-\mu_1\end{pmatrix} \\
\Sigma_{YY} &= \mathbb{E}[(Y-\mathbb{E}Y)(Y-\mathbb{E}Y)^T] \\
&= \pi_1\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix} + \pi_2\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\-\frac{1}{4} & \frac{1}{4}\end{pmatrix}
\end{aligned}$$

(c)

$$
\begin{aligned}
\rho &= \max_{\substack{u\in\mathbb{R}^d \\ v\in\mathbb{R}^2}} \rho(u^T X, v^T Y) \\
&= \max_{\substack{u\in\mathbb{R}^d \\ v\in\mathbb{R}^2}} \frac{Cov(u^T X Y^T v)}{\sqrt{Var(u^T X) Var(v^T Y)}} \\
&= \max_{\substack{u\in\mathbb{R}^d \\ v\in\mathbb{R}^2}} \frac{u^T \Sigma_{XY} v}{\sqrt{u^T \Sigma_{XX} u}\sqrt{v^T \Sigma_{YY} v}} \\
&= \max_{\substack{u\in\mathbb{R}^d \\ v\in\mathbb{R}^2}} \frac{u^T(\mu_1-\mu_2)(v_1-v_2)}{2\sqrt{u^T\Sigma_{XX}u}|v_1-v_2|} \\
&= \max_{u\in\mathbb{R}^d} \frac{u^T(\mu_1-\mu_2)}{2\sqrt{u^T\Sigma_{XX}u}}
\end{aligned}
$$

$$
\begin{aligned}
&\xlongequal{u'=\Sigma_{XX}^{\frac{1}{2}}u} \max_{u'\in\mathbb{R}^d} \frac{u'^T\Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)}{2\sqrt{u'^T u'}} \\
&= \max_{\substack{u_2\in\mathbb{R}^d \\ \|u_2\|=1}} \frac{1}{2} u_2^T \Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)
\end{aligned}
$$

Let $v' = \Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)$, then by Cauchy Schwarz iequality,

$$
\begin{aligned}
(u_2^T v')^2 &\leqslant \|u_2\|_2^2 \|v'\|_2^2 \\
&= (\mu_1-\mu_2)^T \Sigma_{XX}^{-1}(\mu_1-\mu_2)
\end{aligned}
$$

the equation holds only when $u_2 = \pm\frac{v'}{\|v'\|} = \pm\frac{\Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)}{\|v'\|}$, where c is a constant. i.e. $\rho(u^T X, v^T Y)$ is maximized when

$$
\begin{aligned}
u &= \Sigma_{XX}^{-\frac{1}{2}} u' \\
&= c\Sigma_{XX}^{-1}(\mu_1-\mu_2) \\
&= c(\Sigma - \frac{1}{4}(\mu_1-\mu_2)(\mu_1-\mu_2)^T)^{-1}(\mu_1-\mu_2) \\
&= c(\Sigma^{-1} - \frac{\frac{1}{4}\Sigma^{-1}(\mu_1-\mu_2)(\mu_1-\mu_2)^T\Sigma^{-1}}{1+\frac{1}{4}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)})(\mu_1-\mu_2) \\
&= c\Sigma^{-1}(\mu_1-\mu_2) - c\frac{\frac{1}{4}\Sigma^{-1}(\mu_1-\mu_2)\left[(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)\right]}{1+\frac{1}{4}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)} \\
&= \frac{c}{1+\frac{1}{4}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)}\Sigma^{-1}(\mu_1-\mu_2)
\end{aligned}
$$

$\therefore$ u^* is proportional to $\Sigma^{-1}(\mu_1-\mu_2)$

u^* is proportional to the coefficient of X in $f(X)$, i.e. $f(X) = du^{*T}X$ where d is a constant

(d)

$$
\begin{aligned}
\mathbb{E}X &= \pi_1\mu_1 + \pi_2\mu_2 \\
\mathbb{E}[XX^T|\text{class 1}] &= Var[X|\text{class 1}] + (\mathbb{E}[X|\text{class 1}])(\mathbb{E}[X|\text{class 1}])^T \\
&= \Sigma + \mu_1\mu_1^T \\
\mathbb{E}[XX^T|\text{class 2}] &= Var[X|\text{class 2}] + (\mathbb{E}[X|\text{class 2}])(\mathbb{E}[X|\text{class 2}])^T \\
&= \Sigma + \mu_2\mu_2^T \\
\Sigma_{XX} &= \mathbb{E}[(X-\mathbb{E}X)(X-\mathbb{E}X)^T] \\
&= \mathbb{E}[XX^T] - (\mathbb{E}X)(\mathbb{E}X)^T \\
&= \pi_1\mathbb{E}[XX^T|\text{class 1}] + \pi_2\mathbb{E}[XX^T|\text{class 2}] - (\pi_1\mu_1+\pi_2\mu_2)(\pi_1\mu_1+\pi_2\mu_2)^T \\
&= \Sigma + \pi_1\mu_1\mu_1^T + (1-\pi_1)\mu_2\mu_2^T - \pi_1^2\mu_1\mu_1^T - 2\pi_1\pi_2\mu_1\mu_2^T - \pi_2^2\mu_2\mu_2^T \\
&= \Sigma + \pi_1\pi_2(\mu_1-\mu_2)(\mu_1-\mu_2)^T \\
\mathbb{E}Y &= \pi_1\begin{pmatrix}1\\0\end{pmatrix} + \pi_2\begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix}\pi_1\\\pi_2\end{pmatrix} \\
\Sigma_{XY} &= \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T] \\
&= \pi_1\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T|\text{class 1}] + \pi_2\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)^T|\text{class 2}] \\
&= \pi_1\pi_2(\mu_1-\mu_2)\begin{pmatrix}\pi_2 & -\pi_2\end{pmatrix} + \pi_1\pi_1(\mu_2-\mu_1)\begin{pmatrix}-\pi_1 & \pi_1\end{pmatrix} \\
&= \pi_1\pi_2\begin{pmatrix}\mu_1-\mu_2 & \mu_2-\mu_1\end{pmatrix} \\
\Sigma_{YY} &= \mathbb{E}[(Y-\mathbb{E}Y)(Y-\mathbb{E}Y)^T] \\
&= \pi_1\begin{pmatrix}\pi_2\\-\pi_2\end{pmatrix}\begin{pmatrix}\pi_2 & -\pi_2\end{pmatrix} + \pi_2\begin{pmatrix}-\pi_1\\\pi_1\end{pmatrix}\begin{pmatrix}-\pi_1 & \pi_1\end{pmatrix} \\
&= \begin{pmatrix}\pi_1\pi_2 & -\pi_1\pi_2\\-\pi_1\pi_2 & \pi_1\pi_2\end{pmatrix}
\end{aligned}
$$

Solution (cont.)

$$
\begin{aligned}
\rho &= \max_{\substack{u\in\mathbb{R}^d\\ v\in\mathbb{R}^2}} \rho(u^T X, v^T Y)\\
&= \max_{\substack{u\in\mathbb{R}^d\\ v\in\mathbb{R}^2}} \frac{Cov(u^T XY^T v)}{\sqrt{Var(u^T X)Var(v^T Y)}}\\
&= \max_{\substack{u\in\mathbb{R}^d\\ v\in\mathbb{R}^2}} \frac{u^T\Sigma_{XY}v}{\sqrt{u^T\Sigma_{XX}uv^T\Sigma_{YY}v}}\\
&= \max_{\substack{u\in\mathbb{R}^d\\ v\in\mathbb{R}^2}} \frac{\sqrt{\pi_1\pi_2}u^T(\mu_1-\mu_2)(v_1-v_2)}{\sqrt{u^T\Sigma^{-\frac{1}{2}}u}|v_1-v_2|}\\
&= \max_{u\in\mathbb{R}^d} \frac{\sqrt{\pi_1\pi_2}u^T(\mu_1-\mu_2)}{\sqrt{u^T\Sigma_{XX}u}}\\
&\overset{u'=\Sigma^{\frac{1}{2}}u}{=\!=\!=} \max_{u\in\mathbb{R}^d} \frac{\sqrt{\pi_1\pi_2}u'^T\Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)}{\|u'\|}\\
&= \max_{\substack{u_2\in\mathbb{R}^d\\ \|u_2\|=1}} \sqrt{\pi_1\pi_2}u_2^T\Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)
\end{aligned}
$$

Let $v' = \Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)$, then by Cauchy Schwarz iequality,

$$
\begin{aligned}
(u_2^T v')^2 &\leqslant \|u_2\|_2^2\|v'\|_2^2\\
&= (\mu_1-\mu_2)^T\Sigma_{XX}^{-1}(\mu_1-\mu_2)
\end{aligned}
$$

the equation holds only when $u_2 = \pm\frac{v'}{\|v'\|} = \pm\frac{\Sigma_{XX}^{-\frac{1}{2}}(\mu_1-\mu_2)}{\|v'\|}$, where c is a constant. i.e. $\rho(u^T X, v^T Y)$ is maximized when

$$
\begin{aligned}
u &= \Sigma_{XX}^{-\frac{1}{2}}u'\\
&= c\Sigma_{XX}^{-1}(\mu_1-\mu_2)\\
&= c(\Sigma-\pi_1\pi_2(\mu_1-\mu_2)(\mu_1-\mu_2)^T)^{-1}(\mu_1-\mu_2)\\
&= c(\Sigma^{-1}-\frac{\pi_1\pi_2\Sigma^{-1}(\mu_1-\mu_2)(\mu_1-\mu_2)^T\Sigma^{-1}}{1+\pi_1\pi_2(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)})(\mu_1-\mu_2)\\
&= c\Sigma^{-1}(\mu_1-\mu_2) - c\frac{\pi_1\pi_2\Sigma^{-1}(\mu_1-\mu_2)\left[(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)\right]}{1+\pi_1\pi_2(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)}\\
&= \frac{c}{1+\pi_1\pi_2(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)}\Sigma^{-1}(\mu_1-\mu_2)
\end{aligned}
$$

$\therefore$ u^* is proportional to $\Sigma^{-1}(\mu_1-\mu_2)$

u^* is proportional to the coefficient of X in $f(X)$, i.e. $f(X) = du^{*T}X$ where d is a constant

(e)

(1) Compute the sample covariance matrix

$$\hat{\mu}_l = \frac{1}{n} \sum_{x_i \in \text{class } l} x_i$$

$$\hat{\Sigma} = \sum_{l=1}^{2} \frac{\pi_l}{|\text{class } l|} \sum_{x_i \in \text{class } l} (x_i - \hat{\mu}_l)(x_i - \hat{\mu}_l)^T$$

Suppose that $\hat{\mu}_1 < \hat{\mu}_2$ and $\mu_1 + \mu_2 \geqslant 0$.

(2) Subtract the mean

$$X'_{train} = X_{train} - \frac{\hat{\mu}_2 - \hat{\mu}_1}{2}$$

(3) Estimate u^* by

$$\hat{\Sigma}^{-1}(\frac{\hat{\mu}_2 + \hat{\mu}_1}{2})$$

(4) Project X_{test} into the subspace of the range of X that contributes most to predicting Y

$$u^{*T} X_{test}$$

(5) Predict

If $u^{*T} X_{test} < 0$ then predict Y to be class 1; otherwise predict Y to be class 2.

Question 4

(a)

```python
class Generative_Model(object):
    def __init__(self,
                 num_gd_replicates = 20,
                 total_step_count = 1000,
                 step_size = lambda i: 1/(1+i),
                 err = 1e-5):
        self.num_gd_replicates = num_gd_replicates
        self.total_step_count = total_step_count
        self.step_size = step_size
        self.err = err


    def gradient1(self, obj_loc, d, sen_loc):
        g = np.zeros_like(sen_loc)
        for i in range(len(g)):
            g[i] = -np.dot(((np.linalg.norm(obj_loc-sen_loc[i],
              axis=1)-d[:,i])/np.linalg.norm(obj_loc-sen_loc[i],
              axis=1)).T,(obj_loc-sen_loc[i]))
        if np.any(np.isnan(g)):
            return 0
        else:
            return g

    def gradient_descend_step1(self,obj_loc, d, sen_loc, step_count):
        sen_loc = sen_loc - self.step_size(step_count) *
          self.gradient1(obj_loc, d, sen_loc)
        return sen_loc

    def gradient_descend1(self,obj_loc, d, sen_loc):
        positions = [np.array(sen_loc)]
        for k in range(self.total_step_count):
            new = self.gradient_descend_step1(obj_loc, d,
              positions[-1], k)
            if np.linalg.norm(positions[-1]-new)<self.err:
                break
            else:
                positions.append(new)
        return np.array(positions)

    def gradient2(self,obj_loc, d, sen_loc):
```

Solution (cont.)

```
        g = np.dot((((np.linalg.norm(obj_loc-sen_loc,axis=1)-d)
          /np.linalg.norm(obj_loc-sen_loc,axis=1))).T,
          (obj_loc-sen_loc))
        if np.any(np.isnan(g)):
            return 0
        else:
            return g/len(g)

    def gradient_descend_step2(self,obj_loc, d, sen_loc, step_count):
        """Computes the new point after the update at x."""
        obj_loc = obj_loc - self.step_size(step_count) *
          self.gradient2(obj_loc, d, sen_loc)
        return obj_loc

    def gradient_descend2(self,obj_loc, d, sen_loc):
        positions = [np.array(obj_loc)]
        for k in range(self.total_step_count):
            new = self.gradient_descend_step2(
              positions[-1], d, sen_loc, k)
            if np.max(np.linalg.norm(positions[-1]-new,axis=1))
              <self.err:
                break
            else:
                positions.append(new)
        return positions

    def evaluate(self,obj_loc,d):
        est_obj_loc = np.zeros_like(obj_loc)
        for j in range(len(est_obj_loc)):
            p = []
            for i in range(self.num_gd_replicates):
                initial_position = np.random.randn(1,2)*100
                optimal_solutions = self.gradient_descend2(
                  initial_position, d[j], self.est_sen_loc)
                p += [optimal_solutions[-1]]
            p = np.array(p)
            est_obj_loc[j] = np.unique(np.round(p,2),axis=0)[
              np.argmax(np.unique(np.round(p,2),axis=0,
                return_counts=True)[1])]
        error = np.mean(np.linalg.norm(est_obj_loc - obj_loc,axis=1))
        return {'MSE':error,'est_obj_loc':est_obj_loc}

    def train(self,obj_loc,sen_loc,d):
```

Solution (cont.)

```
        m,n = np.shape(sen_loc)
        p = []
        for i in range(self.num_gd_replicates):
            initial_position = np.random.randn(m,n)*100
            optimal_solutions = self.gradient_descend1(obj_loc,
                d, initial_position)
            p += [optimal_solutions[-1]]
        p = np.array(p)
        self.est_sen_loc = np.unique(np.round(p,2),axis=0)[
          np.argmax(np.unique(np.round(p,2),axis=0,
            return_counts=True)[1])]
        self.train_error = self.evaluate(obj_loc,d)['MSE']

    def test(self,obj_loc,d):
        self.test_error = self.evaluate(obj_loc,d)['MSE']
        return self.test_error

class Linear_Model(object):
    def __init__(self):
        pass
    def train(self,X,Y,lamb=0):
        m,n = np.shape(X)
        self.A = np.linalg.solve(X.T.dot(X)+lamb*
          np.identity(n),X.T.dot(Y))
        yhat = X.dot(self.A)
        self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))

    def test(self,X,Y):
        m,n = np.shape(X)
        yhat = X.dot(self.A)
        self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
        return {'A':self.A,'error':self.test_error}

import itertools
class Second_Model(object):
    def __init__(self):
        pass
    def train(self,X,Y,lamb=0):
        m,n = np.shape(X)
        X2 = np.ones((m,1))
        X2 = np.hstack((X2,X))
        for i in range(2,3):
            for j in itertools.combinations_with_replacement(
```

Solution (cont.)

```
                np.arange(n), i):
                xn = np.ones((1,m))
                for k in j:
                    xn = xn*X[:,k]
                X2 = np.hstack((X2,xn.reshape(m,1)))
        self.A = np.linalg.solve(X2.T.dot(X2)+lamb*
          np.identity(len(X2.T)),X2.T.dot(Y))
        yhat = X2.dot(self.A)
        self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
    def test(self,X,Y):
        m,n = np.shape(X)
        X2 = np.ones((m,1))
        X2 = np.hstack((X2,X))
        for i in range(2,3):
            for j in itertools.combinations_with_replacement(
                np.arange(n), i):
                xn = np.ones((1,m))
                for k in j:
                    xn = xn*X[:,k]
                X2 = np.hstack((X2,xn.reshape(m,1)))
        yhat = X2.dot(self.A)
        self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
        return {'A':self.A,'error':self.test_error}

class Third_Model(object):
    def __init__(self):
        pass
    def train(self,X,Y,lamb=0):
        m,n = np.shape(X)
        X2 = np.ones((m,1))
        X2 = np.hstack((X2,X))
        for i in range(2,4):
            for j in itertools.combinations_with_replacement(
                np.arange(n), i):
                xn = np.ones((1,m))
                for k in j:
                    xn = xn*X[:,k]
                X2 = np.hstack((X2,xn.reshape(m,1)))
        self.A = np.linalg.solve(X2.T.dot(X2)+lamb*
          np.identity(len(X2.T)),X2.T.dot(Y))
        yhat = X2.dot(self.A)
        self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
    def test(self,X,Y):
```

Solution (cont.)

```python
        m,n = np.shape(X)
        X2 = np.ones((m,1))
        X2 = np.hstack((X2,X))
        for i in range(2,4):
            for j in itertools.combinations_with_replacement(
              np.arange(n), i):
                xn = np.ones((1,m))
                for k in j:
                    xn = xn*X[:,k]
                X2 = np.hstack((X2,xn.reshape(m,1)))
        yhat = X2.dot(self.A)
        self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
        return {'A':self.A,'error':self.test_error}

# Gradient descent optimization
# The learning rate is specified by eta
class GDOptimizer(object):
    def __init__(self, eta):
        self.eta = eta

    def initialize(self, layers):
        pass

    def update(self, layers, g, a):
        m = a[0].shape[1]
        for layer, curGrad, curA in zip(layers, g, a):
            update = np.dot(curGrad,curA.T)
            updateB = np.sum(curGrad,1).reshape(layer.b.shape)
            layer.updateWeights(-self.eta/m *
              np.dot(curGrad,curA.T))
            layer.updateBias(-self.eta/m *
              np.sum(curGrad,1).reshape(layer.b.shape))

# Cost function used to compute prediction errors
class QuadraticCost(object):
    @staticmethod
    def fx(y,yp):
        return 0.5 * np.square(yp-y)

    # Derivative of the cost function with respect to yp
    @staticmethod
    def dx(y,yp):
        return y - yp
```

Solution (cont.)

```

    # Sigmoid function fully implemented as an example
    class SigmoidActivation(object):
        @staticmethod
        def fx(z):
            return 1 / (1 + np.exp(-z))

        @staticmethod
        def dx(z):
            return SigmoidActivation.fx(z) * (1 -
              SigmoidActivation.fx(z))

    # Hyperbolic tangent function
    class TanhActivation(object):

        # Compute tanh for each element in the input z
        @staticmethod
        def fx(z):
            return np.tanh(z)

        # Compute the derivative of the tanh function with respect to z
        @staticmethod
        def dx(z):
            return 1 - np.square(np.tanh(z))

    # Rectified linear unit
    class ReLUActivation(object):
        @staticmethod
        def fx(z):
            return np.maximum(0,z)

        @staticmethod
        def dx(z):
            return (z>0).astype('float')

    # Linear activation
    class LinearActivation(object):
        @staticmethod
        def fx(z):
            return z

        @staticmethod
        def dx(z):
```

Solution (cont.)

```
            return np.ones(z.shape)

class DenseLayer(object):

    def __init__(self, numNodes, activation):
        self.numNodes = numNodes
        self.activation = activation

    def getNumNodes(self):
        return self.numNodes

    def initialize(self, fanIn, scale=1.0):
        s = scale * np.sqrt(6.0 / (self.numNodes + fanIn))
        self.W = np.random.normal(0, s,
                                  (self.numNodes, fanIn))
        #self.b = np.zeros((self.numNodes,1))
        self.b = np.random.uniform(-1,1,(self.numNodes,1))

    # Apply the activation function of the layer on the input z
    def a(self, z):
        return self.activation.fx(z)

    def z(self, a):
        #print('a:\n'+str(a))
        #print('Wa:\n'+str(self.W.dot(a)))
        return self.W.dot(a) + self.b

    def dx(self, z):
        return self.activation.dx(z)

    # Update the weights of the layer by adding dW to the weights
    def updateWeights(self, dW):
        self.W = self.W + dW

    # Update the bias of the layer by adding db to the bias
    def updateBias(self, db):
        self.b = self.b + db

class Model(object):

    def __init__(self, inputSize):
        self.layers = []
        self.inputSize = inputSize
```

Solution (cont.)

```

    # Add a layer to the end of the network
    def addLayer(self, layer):
        self.layers.append(layer)

    # Get the output size of the layer at the given index
    def getLayerSize(self, index):
        if index >= len(self.layers):
            return self.layers[-1].getNumNodes()
        elif index < 0:
            return self.inputSize
        else:
            return self.layers[index].getNumNodes()

    def initialize(self, cost, initializeLayers=True):
        self.cost = cost
        if initializeLayers:
            for i in range(0,len(self.layers)):
                if i == len(self.layers) - 1:
                    self.layers[i].initialize(
                      self.getLayerSize(i-1))
                else:
                    self.layers[i].initialize(
                      self.getLayerSize(i-1))

    def evaluate(self, x):
        curA = x.T
        a = [curA]
        z = []
        for layer in self.layers:
            z.append(layer.z(curA))
            curA = layer.a(z[-1])
            a.append(curA)
        yp = a.pop()
        return yp, a, z

    def predict(self, a):
        a,_,_ = self.evaluate(a)
        return a.T

    def train(self, x, y, numEpochs, optimizer):

        # Initialize some stuff
```

Solution (cont.)

```
        n = x.shape[0]
        hist = []
        optimizer.initialize(self.layers)

        # Run for the specified number of epochs
        for epoch in range(0,numEpochs):

            yp, a, z = self.evaluate(x)

            # Compute the error
            C = self.cost.fx(yp,y.T)
            d = self.cost.dx(yp,y.T)
            grad = []

            # Backpropogate the error
            idx = len(self.layers)
            for layer, curZ in zip(reversed(self.layers),
              reversed(z)):
                idx = idx - 1
                grad.insert(0,np.multiply(d,layer.dx(curZ)))
                d = np.dot(layer.W.T,grad[0])

            # Update the errors
            optimizer.update(self.layers, grad, a)

            # Compute the error at the end of the epoch
            yh = self.predict(x)
            C = self.cost.fx(yh,y)
            C = np.mean(C)
            hist.append(C)
        return hist

    def trainBatch(self, x, y, batchSize, numEpochs, optimizer):

        x = x.copy()
        y = y.copy()
        hist = []
        n = x.shape[0]

        for epoch in np.arange(0,numEpochs):

            # Shuffle the data
            r = np.arange(0,x.shape[0])
```

Solution (cont.)

```
            x = x[r,:]
            y = y[r,:]
            e = []

            # Split the data in chunks and run SGD
            for i in range(0,n,batchSize):
                end = min(i+batchSize,n)
                batchX = x[i:end,:]
                batchY = y[i:end,:]
                e += self.train(batchX, batchY, 1, optimizer)
            hist.append(np.mean(e))

        return hist
```

(b)

(1) Generating model:

Strengths: Good to generalize to shifted data.

Weaknesses: Need lots of calculation.

(2) Linear model:

Strengths: Simple to model and compute.

Weaknesses: Underfitting for the original data.

(3) Second-Order Polynomial Regression Model

Strengths: Better than Linear Model. Have small testing error in shifted data.

Weaknesses: Bad to be generlized to shifted data.

(4) Third-Order Polynomial Regression Model

Strengths: More complicated to represent the data.

Weaknesses: Overfitting. Causing large errors when testing. Bad to be generlized to shifted data.

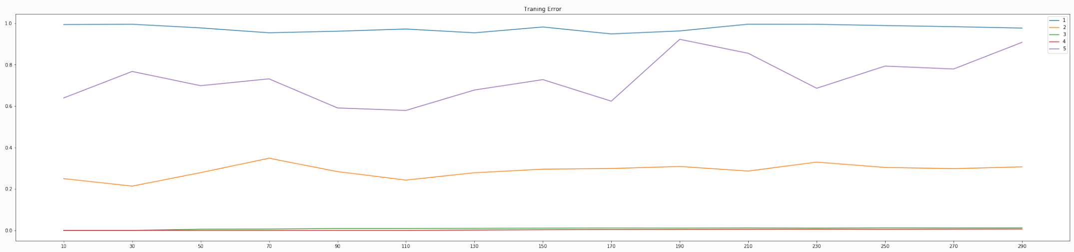
(5) Neural Network Model

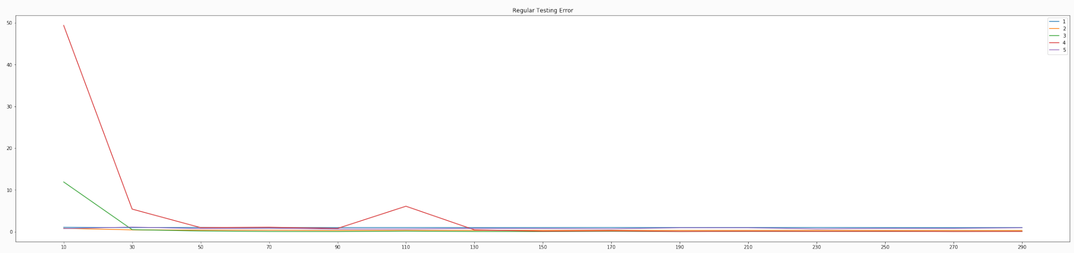
Strengths: Easy to chain. Good to generalize to shifted data.

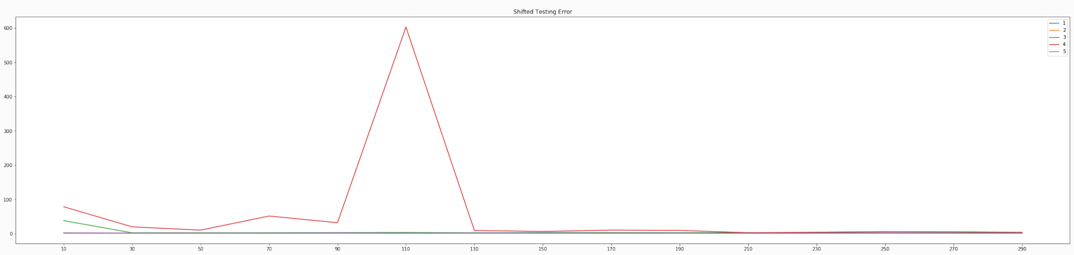
Weaknesses: Not stable.

Error Plot

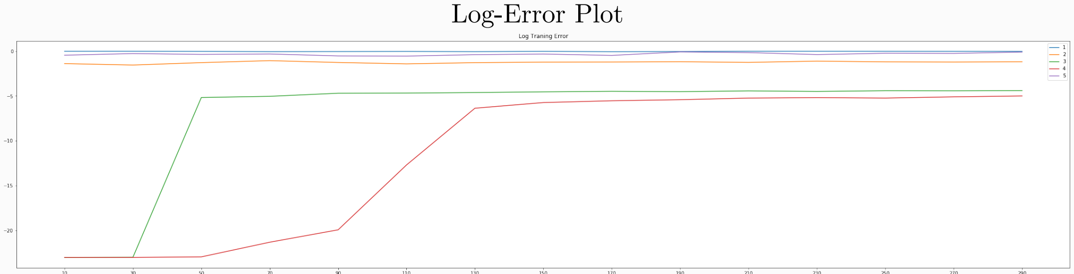
Solution (cont.)

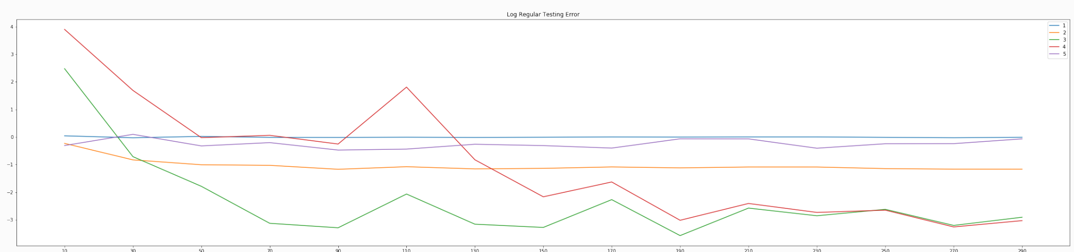


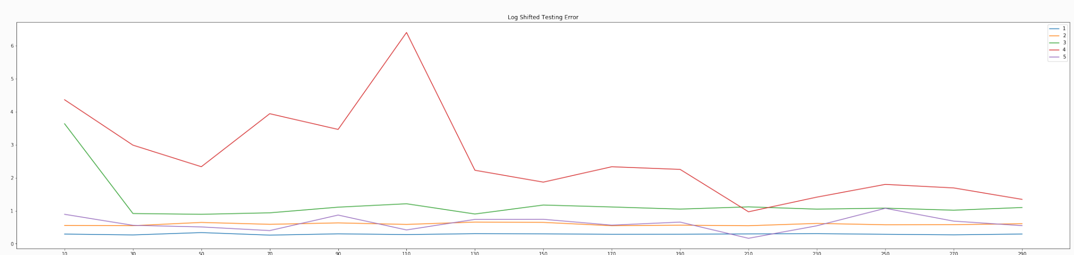




Log-Error Plot

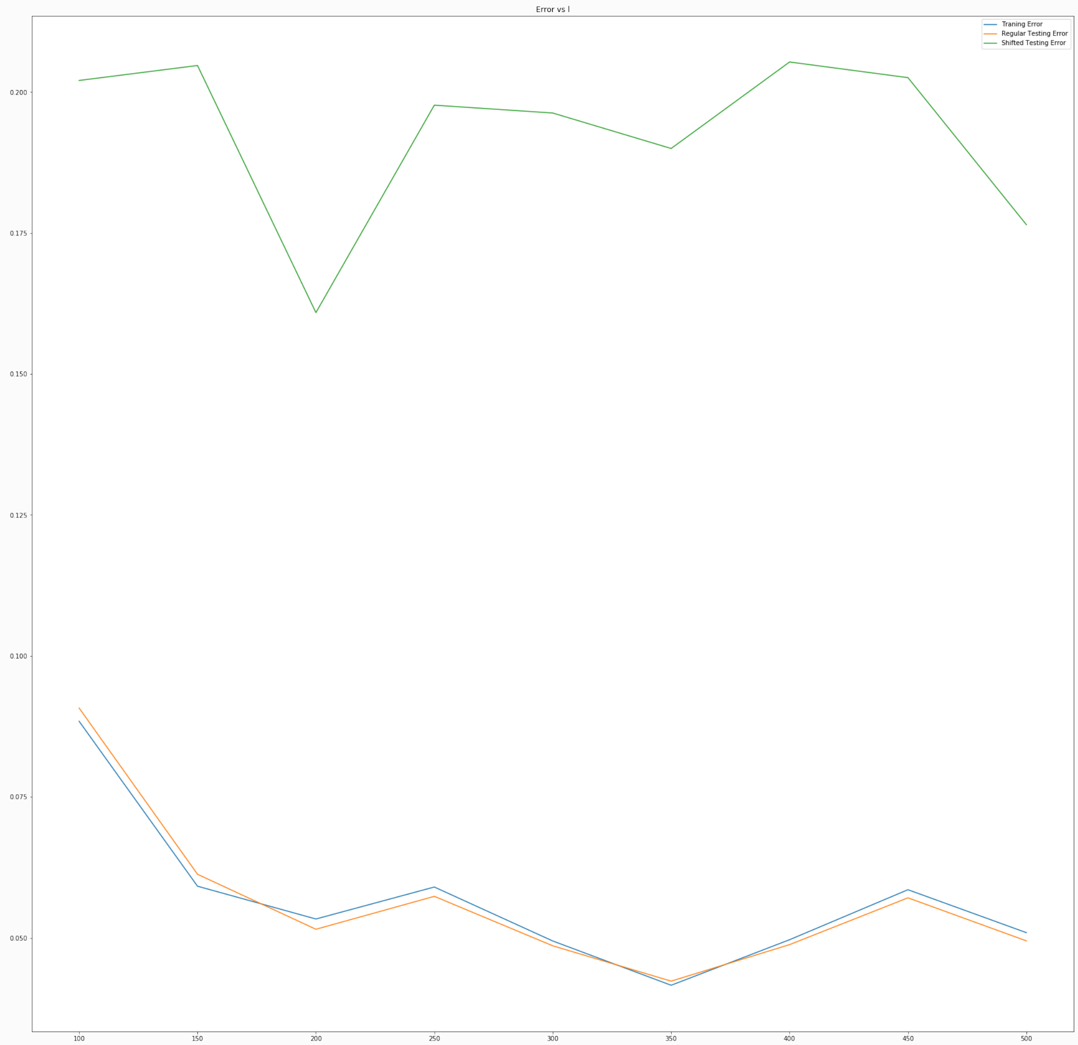






(c)

We can see that shifted testing error is much bigger than others. Shifted testing data is the true testing data since the model never sees such data. So we should choose $l = 150$ such that the shifted testing error is minimized.




```python
l = np.arange(100,550,50)
n = len(l)
resultc = np.zeros((3,n))

for i in range(n):
    modelc1 = Model(xtrain.shape[1])
    modelc1.addLayer(DenseLayer(l[i],ReLUActivation()))
    modelc1.addLayer(DenseLayer(l[i],ReLUActivation()))
    modelc1.addLayer(DenseLayer(2,LinearActivation()))
    modelc1.initialize(QuadraticCost())
    hist = modelc1.train(xtrain,ytrain,500,GDOptimizer(eta=0.0001))
    yHat = modelc1.predict(xtrain)
```

Solution (cont.)

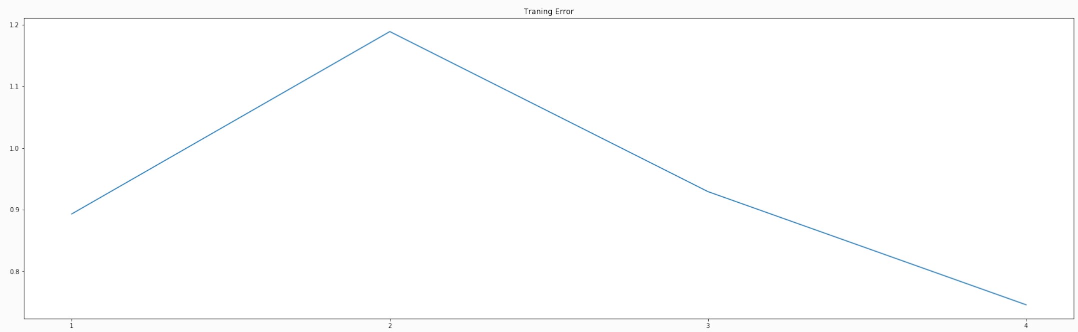
```
        resultc[0,i] = np.mean(np.linalg.norm(yHat - ytrain,axis=1))
        resultc[1,i] = np.mean(np.linalg.norm(modelc1.predict(xtest1)
         - ytest1,axis=1))
        resultc[2,i] = np.mean(np.linalg.norm(modelc1.predict(xtest2)
         - ytest2,axis=1))

plt.figure(figsize=(30,30))
for i in range(3):
        plt.plot(np.arange(9),resultc[i,:],label=s[i])
plt.title('Error vs l')
plt.legend()
plt.xticks(np.arange(9),l)
plt.show()
```

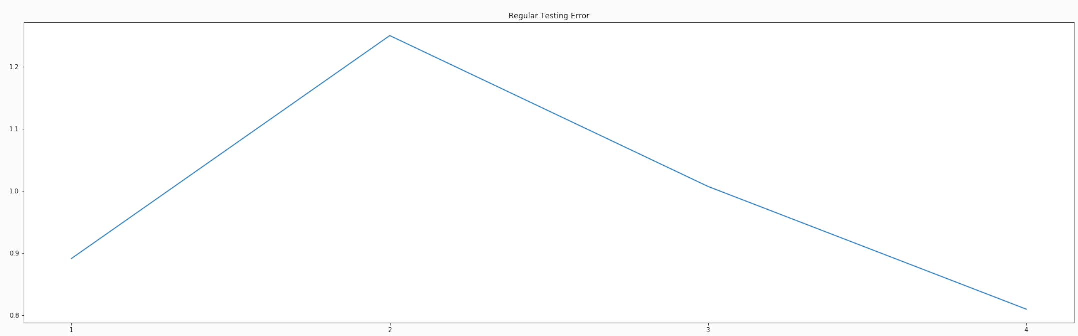
(d)

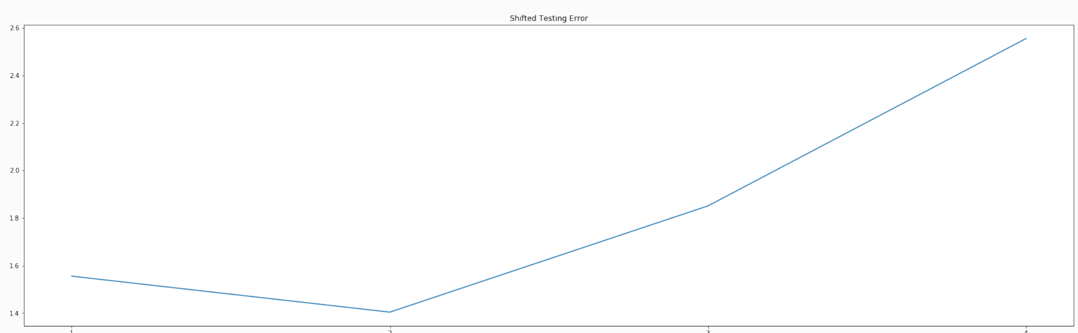
$$n = 7l + (k-1)l^2 + 2l = 9l + (k-1)l^2$$

To get $n = 10000$, we have $l = \begin{cases} \dfrac{-9+\sqrt{81+10000(k-1)}}{2(k-1)} & ,k \geqslant 1 \\ \frac{10000}{9} & ,k = 1 \end{cases}$

The best choice for k is 2. It is because that when $k \leqslant 2$ the traning error and regular testing error increase while the shifted testing error decreases as k increases. When $k > 2$ the traning error and regular testing error decrease while the shifted testing error increases as k increases. So when $k < 2$, the model may be underfitting and when $k > 2$, the model may be overfitting. Therefore the best choice should be $k = 2$.








```
k2 = [i for i in range(1,5)]
n = len(l2)
resultd = np.zeros((3,n))

for i in range(n):
    modeld1 = Model(xtrain.shape[1])
```

Solution (cont.)

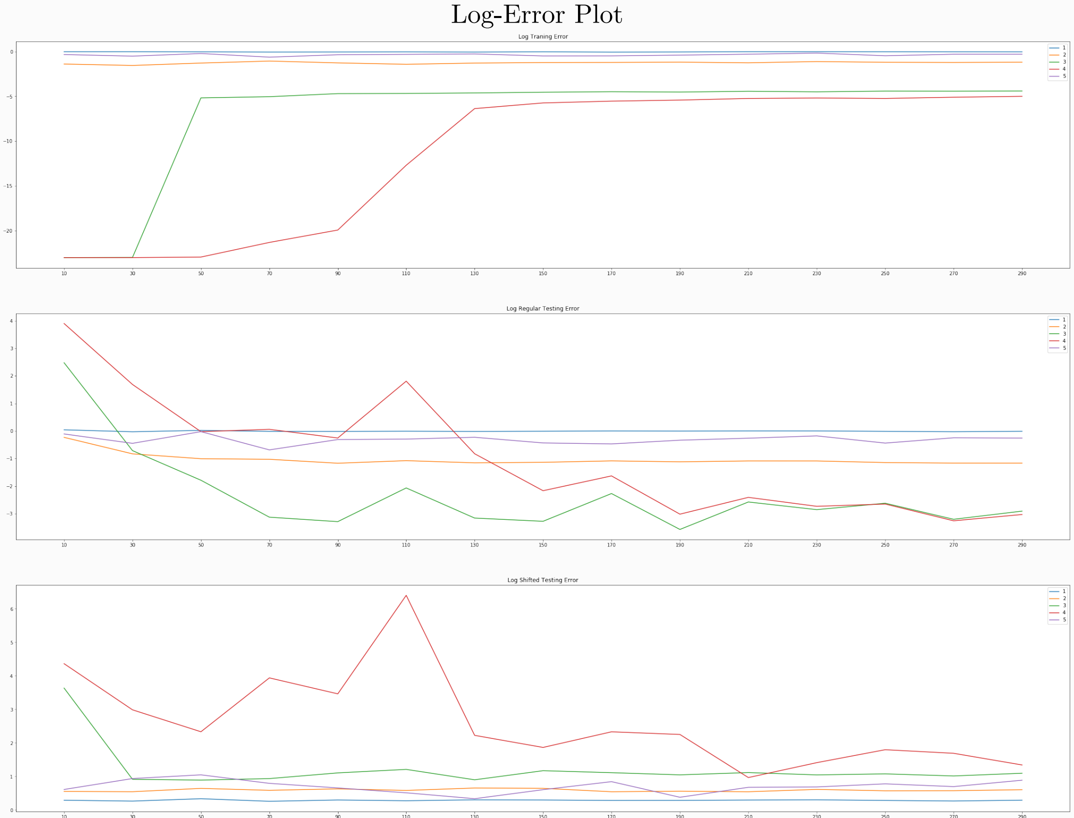
```
    for _ in range(k2[i]):
        modeld1.addLayer(DenseLayer(l2[i],ReLUActivation()))
    modeld1.addLayer(DenseLayer(2,LinearActivation()))
    modeld1.initialize(QuadraticCost())
    hist = modeld1.train(xtrain,ytrain,500,GDOptimizer(eta=0.0001))
    yHat = modeld1.predict(xtrain)
    resultd[0,i] = np.mean(np.linalg.norm(yHat - ytrain,axis=1))
    resultd[1,i] = np.mean(np.linalg.norm(modeld1.predict(xtest1)
     - ytest1,axis=1))
    resultd[2,i] = np.mean(np.linalg.norm(modeld1.predict(xtest2)
     - ytest2,axis=1))

plt.figure(figsize=(30,30))
s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
for i in range(3):
    plt.subplot(3,1,i+1)
    plt.plot(np.arange(4),resultd[i,:])
    plt.xticks(np.arange(4),k2)
    plt.title(s[i])
plt.show()
```

(e)

The nerual network is still not very stable but it has been improved.

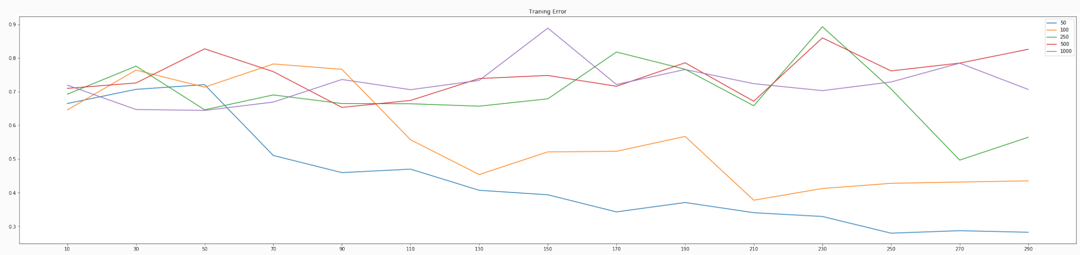
Error Plot

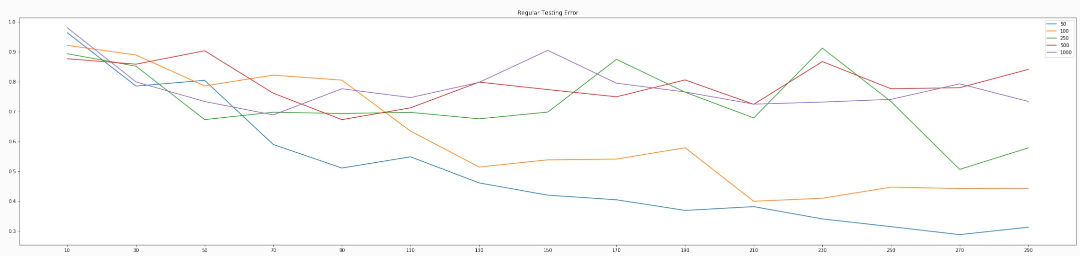
Log-Error Plot

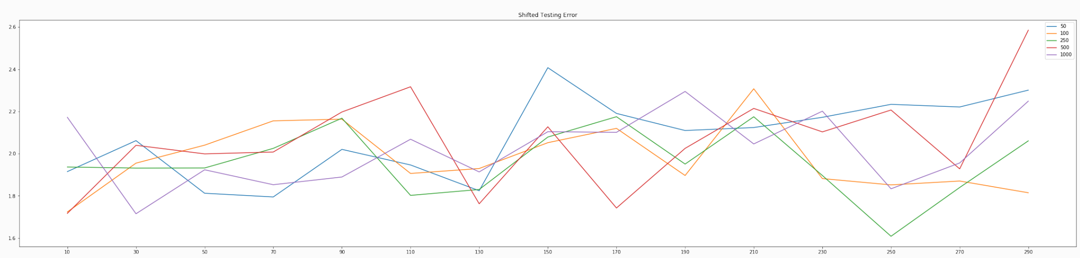


(f)

In defferent sizes of data set, SGD with defferent batch sizes have different impact on the model training. Sometimes it will be better to just use gradient descend (batch size = 1000). But it is not stable and sometimes need more iterations to train the network.








```
for i in range(15):
    for _ in range(5):
        model5 = Model(ztrain[i][0].shape[1])
        model5.addLayer(DenseLayer(100,ReLUActivation()))
        model5.addLayer(DenseLayer(100,ReLUActivation()))
        model5.addLayer(DenseLayer(2,LinearActivation()))
        model5.initialize(QuadraticCost())
        hist = model5.train(ztrain[i][0],ztrain[i][1],500,
          GDOptimizer(eta=0.0001))
        yHat = model5.predict(ztrain[i][0])
        resultf[0,4,i] += np.mean(np.linalg.norm(yHat - ztrain[i][1],
          axis=1))
        resultf[1,4,i] += np.mean(np.linalg.norm(
          model5.predict(xtest1) - ytest1,axis=1))
        resultf[2,4,i] += np.mean(np.linalg.norm(
          model5.predict(xtest2) - ytest2,axis=1))

batchsize = [50,100,250,500]
for j in range(len(batchsize)):
```

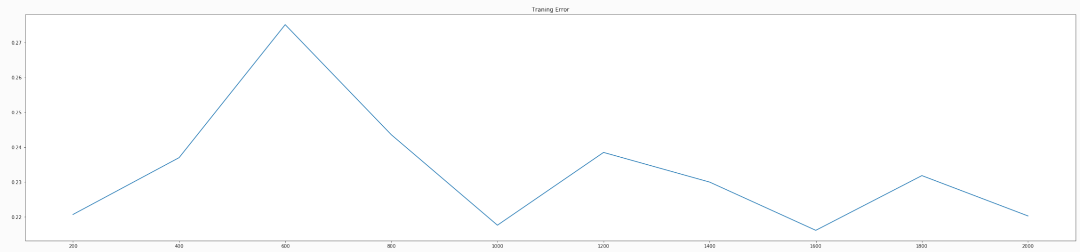
Solution (cont.)

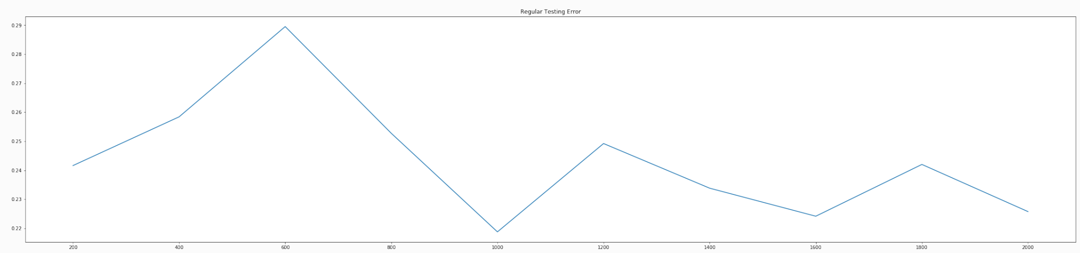
```
        for i in range(15):
            for _ in range(5):
                model5 = Model(ztrain[i][0].shape[1])
                model5.addLayer(DenseLayer(100,ReLUActivation()))
                model5.addLayer(DenseLayer(100,ReLUActivation()))
                model5.addLayer(DenseLayer(2,LinearActivation()))
                model5.initialize(QuadraticCost())
                hist = model5.trainBatch(ztrain[i][0],ztrain[i][1],
                  batchsize[j],500,GDOptimizer(eta=0.0001))
                yHat = model5.predict(ztrain[i][0])
                resultf[0,j,i] += np.mean(np.linalg.norm(yHat
                  - ztrain[i][1],axis=1))
                resultf[1,j,i] += np.mean(np.linalg.norm(
                  model5.predict(xtest1) - ytest1,axis=1))
                resultf[2,j,i] += np.mean(np.linalg.norm(
                  model5.predict(xtest2) - ytest2,axis=1))

resultf /= 5
```

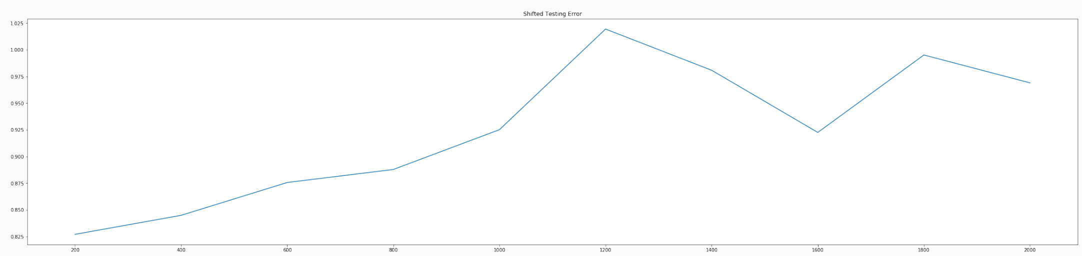
(g)

Use $k = 2$, $l = 46$, $learning\ rate = 0.0001$, $epoch = 1000$, $n_{train} = 1000$

We can see that $n_{train} = 1000$ is best for regular data set; $n_{train} = 200$ is best for shifted data set and therefore it's good to generlize to shifted data.








```
for i in range(15):
    for _ in range(10):
        model5 = Model(ztrain[i][0].shape[1])
        model5.addLayer(DenseLayer(150,ReLUActivation()))
        model5.addLayer(DenseLayer(150,ReLUActivation()))
        model5.addLayer(DenseLayer(2,LinearActivation()))
        model5.initialize(QuadraticCost())
        hist = model5.train(ztrain[i][0],ztrain[i][1],500,
          GDOptimizer(eta=0.0001))
        yHat = model5.predict(ztrain[i][0])
        resultg[0,i] += np.mean(np.linalg.norm(yHat - ztrain[i][1],
          axis=1))
        resultg[1,i] += np.mean(np.linalg.norm(model5.predict(xtest1)
          - ytest1,axis=1))
        resultg[2,i] += np.mean(np.linalg.norm(model5.predict(xtest2)
          - ytest2,axis=1))
resultg /= 10
```

Question 5

The training error on MNIST : 0.9138

```python
from __future__ import absolute_import
from __future__ import division
from __future__ import print_function

import argparse
import sys

from tensorflow.examples.tutorials.mnist import input_data

import tensorflow as tf

FLAGS = None


def main(_):
    # Import data
    mnist = input_data.read_data_sets(FLAGS.data_dir, one_hot=True)

    # Create the model
    x = tf.placeholder(tf.float32, [None, 784])
    W = tf.Variable(tf.zeros([784, 10]))
    b = tf.Variable(tf.zeros([10]))
    y = tf.matmul(x, W) + b

    # Define loss and optimizer
    y_ = tf.placeholder(tf.float32, [None, 10])

    # The raw formulation of cross-entropy,
    #
    #   tf.reduce_mean(-tf.reduce_sum(y_ * tf.log(tf.nn.softmax(y)),
    #                                 reduction_indices=[1]))
    #
    # can be numerically unstable.
    #
    # So here we use tf.nn.softmax_cross_entropy_with_logits on the raw
    # outputs of 'y', and then average across the batch.
    cross_entropy = tf.reduce_mean(
      tf.nn.softmax_cross_entropy_with_logits(labels=y_, logits=y))
    train_step =
     tf.train.GradientDescentOptimizer(0.5).minimize(cross_entropy)
```

Solution (cont.)

```python

    sess = tf.InteractiveSession()
    tf.global_variables_initializer().run()
    # Train
    for i in range(1000):
        batch_xs, batch_ys = mnist.train.next_batch(100)
        sess.run(train_step, feed_dict={x: batch_xs, y_: batch_ys})

        # Test trained model
        correct_prediction = tf.equal(tf.argmax(y, 1), tf.argmax(y_, 1))
        accuracy = tf.reduce_mean(tf.cast(correct_prediction, tf.float32))
        print(str(i)+': ',sess.run(accuracy,
            feed_dict={x: mnist.test.images,
            y_: mnist.test.labels}))

if __name__ == '__main__':
    parser = argparse.ArgumentParser()
    parser.add_argument('--data_dir', type=str,
                        default='/tmp/tensorflow/mnist/input_data',
                        help='Directory for storing input data')
    FLAGS, unparsed = parser.parse_known_args()
    tf.app.run(main=main, argv=[sys.argv[0]] + unparsed)
```

Question 6

Question What's the difference among various deep learning frameworks?

Solution

Framwork	Institution	Programming Language	Stars	Forks	Contributors
Thensorflow	Google	Python/C++/Go/$\cdots$	41628	19339	568
Caffe	BVLC	C++/Python	14956	9282	221
Keras	fchollet	Python	10727	3575	322
CNTK	Microsoft	C++	9063	2144	100
MXNet	DMLC	Python/C++/R/$\cdots$	7393	2745	241
Torch7	Facebook	Lua	6111	1784	113
Theano	U.Montreal	Python	5352	1868	271
Deeplearning4J	DeepLearning4J	Java/Scala	5053	1927	101
Leaf	AutumnAI	Rust	4562	216	14

Tensorflow

(1) Low-level core (C++/CUDA)

(2) Simple Python API to define the computational graph

(3) High-level API (TF-Learn, TF-Slim, soon Keras)

Theano

(1) Pioneered the use of a computational graph.

(2) General machine learning tool -¿ Use of Lasagne and Keras.

(3) Very popular in the research community, but not elsewhere. Falling behind. The development has been stopped.

Keras

(1) Easy-to-use Python library

(2) It wraps Theano and TensorFlow (it benefits from the advantages of both)

(3) Guiding principles: modularity, minimalism, extensibility, and Python-nativeness

(4) Less flexible

(5) Less projects available online than caffe

(6) Multi-GPU not 100% working

Caffe

(1) Applications in machine learning, vision, speech and multimedia.

(2) Good Python and MATLAB interfaces.

(3) No auto-differentiation.

(4) Need of examples to template own code.

HW9

October 28, 2017

1 Question 4

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import scipy.spatial
        import numpy as np
        import matplotlib
        import matplotlib.pyplot as plt
```

```
In [89]: ########################################################################
         #########  Data Generating Functions ###################################
         ########################################################################

         def generate_sensors(num_sensors = 7, spatial_dim = 2):
             """
             Generate sensor locations.
             Input:
             num_sensors: The number of sensors.
             spatial_dim: The spatial dimension.
             Output:
             sensor_loc: num_sensors * spatial_dim numpy array.
             """
             sensor_loc = 100*np.random.randn(num_sensors,spatial_dim)
             return sensor_loc

         def generate_dataset(sensor_loc, num_sensors = 7, spatial_dim = 2,
                          num_data = 1, original_dist = True, noise = 1):
             """
             Generate the locations of n points.

             Input:
             sensor_loc: num_sensors * spatial_dim numpy array. Location of sensor.
             num_sensors: The number of sensors.
             spatial_dim: The spatial dimension.
             num_data: The number of points.
             original_dist: Whether the data are generated from the original
             distribution.
```

```
Output:
obj_loc: num_data * spatial_dim numpy array. The location of the num_data objects.
distance: num_data * num_sensors numpy array. The distance between object and
the num_sensors sensors.
"""
assert num_sensors, spatial_dim == sensor_loc.shape

obj_loc = 100*np.random.randn(num_data, spatial_dim)
if not original_dist:
    obj_loc += 1000

distance = scipy.spatial.distance.cdist(obj_loc,
                                       sensor_loc,
                                       metric='euclidean')
distance += np.random.randn(num_data, num_sensors) * noise
return distance, obj_loc
```

1.1 Generative Model

```
In [172]: class Generative_Model(object):
              def __init__(self,
                           num_gd_replicates = 20,
                           total_step_count = 1000,
                           step_size = lambda i: 1/(1+i),
                           err = 1e-5):
                  self.num_gd_replicates = num_gd_replicates
                  self.total_step_count = total_step_count
                  self.step_size = step_size
                  self.err = err


              def gradient1(self, obj_loc, d, sen_loc):
                  g = np.zeros_like(sen_loc)
                  for i in range(len(g)):
                      g[i] = -np.dot(((np.linalg.norm(obj_loc-sen_loc[i],axis=1)-d[:,i])/np.lina
                  if np.any(np.isnan(g)):
                      return 0
                  else:
                      return g

              def gradient_descend_step1(self,obj_loc, d, sen_loc, step_count):
                  """Computes the new point after the update at x."""
                  sen_loc = sen_loc - self.step_size(step_count) * self.gradient1(obj_loc, d, se
                  return sen_loc


              def gradient_descend1(self,obj_loc, d, sen_loc):
```

```
        """Computes several updates towards the minimum of ||Ax-b|| from p.

        Params:
            obj_loc: object locations
            d: measurements of the distance from each sensor to objects
            sen_loc: initialization point of sensors
            total_step_count: number of iterations to calculate
            step_size: function for determining the step size at step i
        """
        positions = [np.array(sen_loc)]
        for k in range(self.total_step_count):
            new = self.gradient_descend_step1(obj_loc, d, positions[-1], k)
            if np.linalg.norm(positions[-1]-new)<self.err:
                break
            else:
                positions.append(new)
        return np.array(positions)

    def gradient2(self,obj_loc, d, sen_loc):
        g = np.dot((((np.linalg.norm(obj_loc-sen_loc,axis=1)-d)/np.linalg.norm(obj_loc
        if np.any(np.isnan(g)):
            return 0
        else:
            return g/len(g)

    def gradient_descend_step2(self,obj_loc, d, sen_loc, step_count):
        """Computes the new point after the update at x."""
        obj_loc = obj_loc - self.step_size(step_count) * self.gradient2(obj_loc, d, se
        return obj_loc

    def gradient_descend2(self,obj_loc, d, sen_loc):
        positions = [np.array(obj_loc)]
        for k in range(self.total_step_count):
            new = self.gradient_descend_step2(positions[-1], d, sen_loc, k)
            if np.max(np.linalg.norm(positions[-1]-new,axis=1))<self.err:
                break
            else:
                positions.append(new)
        return positions

    def evaluate(self,obj_loc,d):
        est_obj_loc = np.zeros_like(obj_loc)
        for j in range(len(est_obj_loc)):
            p = []
            for i in range(self.num_gd_replicates):
                initial_position = np.random.randn(1,2)*100
                optimal_solutions = self.gradient_descend2(initial_position, d[j], sel
                p += [optimal_solutions[-1]]
```

```
            p = np.array(p)
            est_obj_loc[j] = np.unique(np.round(p,2),axis=0)[np.argmax(np.unique(np.ro
        error = np.mean(np.linalg.norm(est_obj_loc - obj_loc,axis=1))
        return {'MSE':error,'est_obj_loc':est_obj_loc}

    def train(self,obj_loc,sen_loc,d):
        m,n = np.shape(sen_loc)
        p = []
        for i in range(self.num_gd_replicates):
            initial_position = np.random.randn(m,n)*100
            optimal_solutions = self.gradient_descend1(obj_loc, d, initial_position)
            p += [optimal_solutions[-1]]
        p = np.array(p)
        self.est_sen_loc = np.unique(np.round(p,2),axis=0)[np.argmax(np.unique(np.roun
        self.train_error = self.evaluate(obj_loc,d)['MSE']

    def test(self,obj_loc,d):
        self.test_error = self.evaluate(obj_loc,d)['MSE']
        return self.test_error
```

1.2 Linear Model

```
In [91]: class Linear_Model(object):
             def __init__(self):
                 pass
             def train(self,X,Y,lamb=0):
                 m,n = np.shape(X)
                 self.A = np.linalg.solve(X.T.dot(X)+lamb*np.identity(n),X.T.dot(Y))
                 yhat = X.dot(self.A)
                 self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))

             def test(self,X,Y):
                 m,n = np.shape(X)
                 yhat = X.dot(self.A)
                 self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
                 return {'A':self.A,'error':self.test_error}
```

1.3 Second-Order Polynomial Regression Model

```
In [92]: import itertools
         class Second_Model(object):
             def __init__(self):
                 pass
             def train(self,X,Y,lamb=0):
                 m,n = np.shape(X)
                 X2 = np.ones((m,1))
```

```
            X2 = np.hstack((X2,X))
            for i in range(2,3):
                for j in itertools.combinations_with_replacement(np.arange(n), i):
                    xn = np.ones((1,m))
                    for k in j:
                        xn = xn*X[:,k]
                    X2 = np.hstack((X2,xn.reshape(m,1)))
            self.A = np.linalg.solve(X2.T.dot(X2)+lamb*np.identity(len(X2.T)),X2.T.dot(Y))
            yhat = X2.dot(self.A)
            self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
        def test(self,X,Y):
            m,n = np.shape(X)
            X2 = np.ones((m,1))
            X2 = np.hstack((X2,X))
            for i in range(2,3):
                for j in itertools.combinations_with_replacement(np.arange(n), i):
                    xn = np.ones((1,m))
                    for k in j:
                        xn = xn*X[:,k]
                    X2 = np.hstack((X2,xn.reshape(m,1)))
            yhat = X2.dot(self.A)
            self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
            return {'A':self.A,'error':self.test_error}
```

1.4 Third-Order Polynomial Regression Model

```
In [93]: class Third_Model(object):
             def __init__(self):
                 pass
             def train(self,X,Y,lamb=0):
                 m,n = np.shape(X)
                 X2 = np.ones((m,1))
                 X2 = np.hstack((X2,X))
                 for i in range(2,4):
                     for j in itertools.combinations_with_replacement(np.arange(n), i):
                         xn = np.ones((1,m))
                         for k in j:
                             xn = xn*X[:,k]
                         X2 = np.hstack((X2,xn.reshape(m,1)))
                 self.A = np.linalg.solve(X2.T.dot(X2)+lamb*np.identity(len(X2.T)),X2.T.dot(Y))
                 yhat = X2.dot(self.A)
                 self.train_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
             def test(self,X,Y):
                 m,n = np.shape(X)
                 X2 = np.ones((m,1))
                 X2 = np.hstack((X2,X))
                 for i in range(2,4):
                     for j in itertools.combinations_with_replacement(np.arange(n), i):
```

```
                xn = np.ones((1,m))
                for k in j:
                    xn = xn*X[:,k]
                X2 = np.hstack((X2,xn.reshape(m,1)))
        yhat = X2.dot(self.A)
        self.test_error = np.mean(np.linalg.norm(yhat-Y,axis=1))
        return {'A':self.A,'error':self.test_error}
```

1.5 Neural Network Model

```
In [94]: # Gradient descent optimization
         # The learning rate is specified by eta
         class GDOptimizer(object):
             def __init__(self, eta):
                 self.eta = eta

             def initialize(self, layers):
                 pass

             # This function performs one gradient descent step
             # layers is a list of dense layers in the network
             # g is a list of gradients going into each layer before the nonlinear activation
             # a is a list of of the activations of each node in the previous layer going
             def update(self, layers, g, a):
                 m = a[0].shape[1]
                 for layer, curGrad, curA in zip(layers, g, a):
                     update = np.dot(curGrad,curA.T)
                     updateB = np.sum(curGrad,1).reshape(layer.b.shape)
                     layer.updateWeights(-self.eta/m * np.dot(curGrad,curA.T))
                     layer.updateBias(-self.eta/m * np.sum(curGrad,1).reshape(layer.b.shape))

         # Cost function used to compute prediction errors
         class QuadraticCost(object):

             # Compute the squared error between the prediction yp and the observation y
             # This method should compute the cost per element such that the output is the
             # same shape as y and yp
             @staticmethod
             def fx(y,yp):
                 return 0.5 * np.square(yp-y)

             # Derivative of the cost function with respect to yp
             @staticmethod
             def dx(y,yp):
                 return y - yp

         # Sigmoid function fully implemented as an example
         class SigmoidActivation(object):
```

```
    @staticmethod
    def fx(z):
        return 1 / (1 + np.exp(-z))

    @staticmethod
    def dx(z):
        return SigmoidActivation.fx(z) * (1 - SigmoidActivation.fx(z))

# Hyperbolic tangent function
class TanhActivation(object):

    # Compute tanh for each element in the input z
    @staticmethod
    def fx(z):
        return np.tanh(z)

    # Compute the derivative of the tanh function with respect to z
    @staticmethod
    def dx(z):
        return 1 - np.square(np.tanh(z))

# Rectified linear unit
class ReLUActivation(object):
    @staticmethod
    def fx(z):
        return np.maximum(0,z)

    @staticmethod
    def dx(z):
        return (z>0).astype('float')

# Linear activation
class LinearActivation(object):
    @staticmethod
    def fx(z):
        return z

    @staticmethod
    def dx(z):
        return np.ones(z.shape)

# This class represents a single hidden or output layer in the neural network
class DenseLayer(object):

    # numNodes: number of hidden units in the layer
    # activation: the activation function to use in this layer
    def __init__(self, numNodes, activation):
        self.numNodes = numNodes
```

```
        self.activation = activation

    def getNumNodes(self):
        return self.numNodes

    # Initialize the weight matrix of this layer based on the size of the matrix W
    def initialize(self, fanIn, scale=1.0):
        s = scale * np.sqrt(6.0 / (self.numNodes + fanIn))
        self.W = np.random.normal(0, s,
                                   (self.numNodes,fanIn))
        #self.b = np.zeros((self.numNodes,1))
        self.b = np.random.uniform(-1,1,(self.numNodes,1))

    # Apply the activation function of the layer on the input z
    def a(self, z):
        return self.activation.fx(z)

    # Compute the linear part of the layer
    # The input a is an n x k matrix where n is the number of samples
    # and k is the dimension of the previous layer (or the input to the network)
    def z(self, a):
        #print('a:\n'+str(a))
        #print('Wa:\n'+str(self.W.dot(a)))
        return self.W.dot(a) + self.b # Note, this is implemented where we assume a is

    # Compute the derivative of the layer's activation function with respect to z
    # where z is the output of the above function.
    # This derivative does not contain the derivative of the matrix multiplication
    # in the layer.  That part is computed below in the model class.
    def dx(self, z):
        return self.activation.dx(z)

    # Update the weights of the layer by adding dW to the weights
    def updateWeights(self, dW):
        self.W = self.W + dW

    # Update the bias of the layer by adding db to the bias
    def updateBias(self, db):
        self.b = self.b + db

# This class handles stacking layers together to form the completed neural network
class Model(object):

    # inputSize: the dimension of the inputs that go into the network
    def __init__(self, inputSize):
        self.layers = []
        self.inputSize = inputSize
```

```python
    # Add a layer to the end of the network
    def addLayer(self, layer):
        self.layers.append(layer)

    # Get the output size of the layer at the given index
    def getLayerSize(self, index):
        if index >= len(self.layers):
            return self.layers[-1].getNumNodes()
        elif index < 0:
            return self.inputSize
        else:
            return self.layers[index].getNumNodes()

    # Initialize the weights of all of the layers in the network and set the cost
    # function to use for optimization
    def initialize(self, cost, initializeLayers=True):
        self.cost = cost
        if initializeLayers:
            for i in range(0,len(self.layers)):
                if i == len(self.layers) - 1:
                    self.layers[i].initialize(self.getLayerSize(i-1))
                else:
                    self.layers[i].initialize(self.getLayerSize(i-1))

    # Compute the output of the network given some input a
    # The matrix a has shape n x k where n is the number of samples and
    # k is the dimension
    # This function returns
    # yp - the output of the network
    # a - a list of inputs for each layer of the newtork where
    #     a[i] is the input to layer i
    # z - a list of values for each layer after evaluating layer.z(a) but
    #     before evaluating the nonlinear function for the layer
    def evaluate(self, x):
        curA = x.T
        a = [curA]
        z = []
        for layer in self.layers:
            z.append(layer.z(curA))
            curA = layer.a(z[-1])
            a.append(curA)
        yp = a.pop()
        return yp, a, z

    # Compute the output of the network given some input a
    # The matrix a has shape n x k where n is the number of samples and
    # k is the dimension
    def predict(self, a):
```

```
        a,_,_ = self.evaluate(a)
        return a.T

    # Train the network given the inputs x and the corresponding observations y
    # The network should be trained for numEpochs iterations using the supplied
    # optimizer
    def train(self, x, y, numEpochs, optimizer):

        # Initialize some stuff
        n = x.shape[0]
        hist = []
        optimizer.initialize(self.layers)

        # Run for the specified number of epochs
        for epoch in range(0,numEpochs):

            # Feed forward
            # Save the output of each layer in the list a
            # After the network has been evaluated, a should contain the
            # input x and the output of each layer except for the last layer
            yp, a, z = self.evaluate(x)

            # Compute the error
            C = self.cost.fx(yp,y.T)
            d = self.cost.dx(yp,y.T)
            grad = []

            # Backpropogate the error
            idx = len(self.layers)
            for layer, curZ in zip(reversed(self.layers),reversed(z)):
                idx = idx - 1
                # Here, we compute dMSE/dz_i because in the update
                # function for the optimizer, we do not give it
                # the z values we compute from evaluating the network
                grad.insert(0,np.multiply(d,layer.dx(curZ)))
                d = np.dot(layer.W.T,grad[0])

            # Update the errors
            optimizer.update(self.layers, grad, a)

            # Compute the error at the end of the epoch
            yh = self.predict(x)
            C = self.cost.fx(yh,y)
            C = np.mean(C)
            hist.append(C)
        return hist

    def trainBatch(self, x, y, batchSize, numEpochs, optimizer):
```

```
        # Copy the data so that we don't affect the original one when shuffling
        x = x.copy()
        y = y.copy()
        hist = []
        n = x.shape[0]

        for epoch in np.arange(0,numEpochs):

            # Shuffle the data
            r = np.arange(0,x.shape[0])
            x = x[r,:]
            y = y[r,:]
            e = []

            # Split the data in chunks and run SGD
            for i in range(0,n,batchSize):
                end = min(i+batchSize,n)
                batchX = x[i:end,:]
                batchY = y[i:end,:]
                e += self.train(batchX, batchY, 1, optimizer)
            hist.append(np.mean(e))

        return hist
```

1.6 (b)

1.7 Generate Data

```
In [330]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = {i:generate_dataset(sensor_loc=sensor_loc,num_data=10+20*i) for i in range(15)
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=1000)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=1000,original_dist=False)

          zsensor_loc = (sensor_loc - np.average(sensor_loc))/np.std(sensor_loc)

          xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
          ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
          ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
          ztrain = {i:[(train[0][0] - np.average(train[0][0])) / np.std(train[0][0]+1e-6),
                       (train[0][1] - np.average(train[0][1])) / np.std(train[0][1]+1e-6)] for i
          result = np.zeros((3,5,15))

In [331]: zsensor_loc = (sensor_loc - np.average(sensor_loc))/np.std(sensor_loc)

In [345]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
```

```
        ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
        ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
        ztrain = {i:[(train[i][0] - np.average(train[i][0])) / np.std(train[i][0]+1e-6),
                     (train[i][1] - np.average(train[i][1])) / np.std(train[i][1]+1e-6)] for i
        result = np.zeros((3,5,15))
```

```
In [74]: np.shape(train[0][0])
```

```
Out[74]: (10, 7)
```

```
In [351]: #model1 = [Generative_Model() for _ in range(15)]
          for i in range(15):
              model1[i].train(ztrain[i][1],zsensor_loc,ztrain[i][0])
              result[0,0,i] = model1[i].train_error
              print(i)
              model1[i].test(ytest1, xtest1)
              result[1,0,i] = model1[i].test_error
              print(i)
              model1[i].test(ytest2, xtest2)
              result[2,0,i] = model1[i].test_error
              print(i)
```

```
0
0
0
1
1
1
2
2
2
3
3
3
4
4
4
5
5
5
6
6
6
7
7
7
8
8
8
```

```
9
9
9
10
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12
13
13
13
14
14
14
```

In [352]:
```python
import pickle
with open('result','wb') as f:
    pickle.dump({'result':result},f,True)
```

In [333]:
```python
with open('result','rb') as f:
    result = pickle.load(f)['result']
```

In [367]:
```python
result[:,3,:]
```

Out[367]:
```
array([[  1.70209589e-13,   2.53009930e-13,   6.40847141e-12,
          4.46966993e-10,   2.08082862e-09,   3.04794575e-06,
          1.71872998e-03,   3.23326160e-03,   3.95691864e-03,
          4.45329000e-03,   5.34026340e-03,   5.61710652e-03,
          5.34789962e-03,   6.11200923e-03,   6.75728029e-03],
       [  4.94112804e+01,   5.40902723e+00,   9.76510054e-01,
          1.06257919e+00,   7.75477816e-01,   6.09656234e+00,
          4.37831537e-01,   1.14482557e-01,   1.95893033e-01,
          4.89012267e-02,   8.99328073e-02,   6.50131274e-02,
          7.05950582e-02,   3.84164237e-02,   4.82885821e-02],
       [  7.85630463e+01,   1.98350641e+01,   1.03112067e+01,
          5.14169197e+01,   3.19479968e+01,   6.02993199e+02,
          9.26404141e+00,   6.46171482e+00,   1.03004258e+01,
          9.51386164e+00,   2.63143591e+00,   4.10362085e+00,
          6.03794058e+00,   5.42340147e+00,   3.83103954e+00]])
```

In [355]:
```python
model2 = [Linear_Model() for _ in range(15)]
for i in range(15):
    model2[i].train(ztrain[i][0],ztrain[i][1])
    result[0,1,i] = model2[i].train_error
```

```
        model2[i].test(xtest1,ytest1)
        result[1,1,i] = model2[i].test_error
        model2[i].test(xtest2,ytest2)
        result[2,1,i] = model2[i].test_error
```

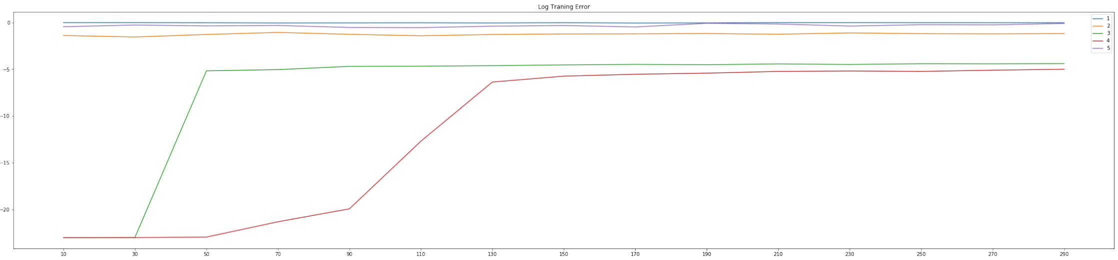
In [356]:
```
model3 =[Second_Model() for _ in range(15)]
for i in range(15):
    model3[i].train(ztrain[i][0],ztrain[i][1])
    result[0,2,i] = model3[i].train_error
    model3[i].test(xtest1,ytest1)
    result[1,2,i] = model3[i].test_error
    model3[i].test(xtest2,ytest2)
    result[2,2,i] = model3[i].test_error
```

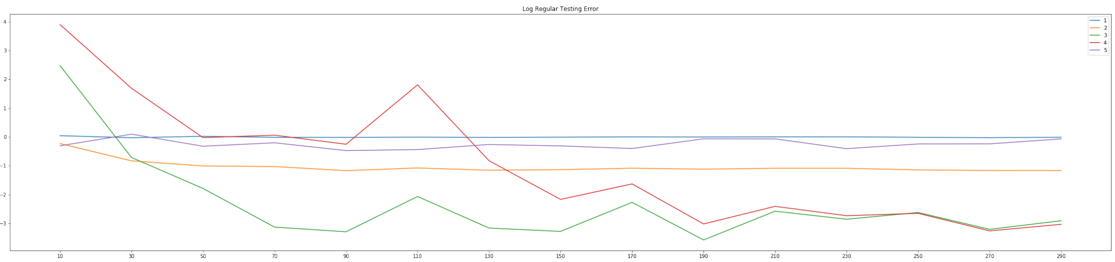
In [357]:
```
model4 =[Third_Model() for _ in range(15)]
for i in range(15):
    model4[i].train(ztrain[i][0],ztrain[i][1])
    result[0,3,i] = model4[i].train_error
    model4[i].test(xtest1,ytest1)
    result[1,3,i] = model4[i].test_error
    model4[i].test(xtest2,ytest2)
    result[2,3,i] = model4[i].test_error
```

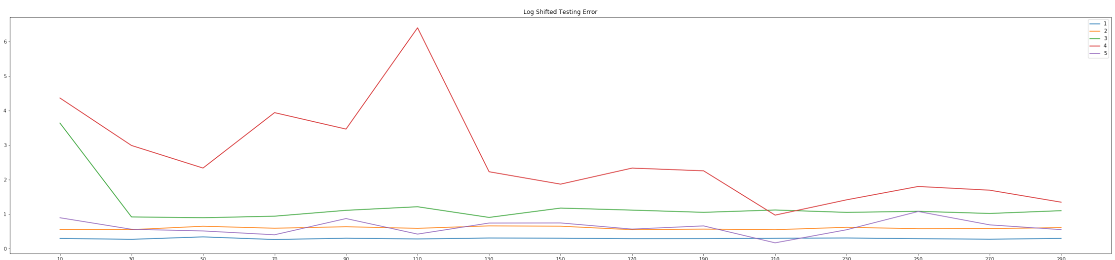
In [358]:
```
for i in range(15):
    model5 = Model(ztrain[i][0].shape[1])
    model5.addLayer(DenseLayer(100,ReLUActivation()))
    model5.addLayer(DenseLayer(100,ReLUActivation()))
    model5.addLayer(DenseLayer(2,LinearActivation()))
    model5.initialize(QuadraticCost())
    hist = model5.train(ztrain[i][0],ztrain[i][1],500,GDOptimizer(eta=0.0001))
    yHat = model5.predict(ztrain[i][0])
    result[0,4,i] = np.mean(np.linalg.norm(yHat - ztrain[i][1],axis=1))
    result[1,4,i] = np.mean(np.linalg.norm(model5.predict(xtest1) - ytest1,axis=1))
    result[2,4,i] = np.mean(np.linalg.norm(model5.predict(xtest2) - ytest2,axis=1))
```

In [359]:
```
plt.figure(figsize=(40,30))
s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
for i in range(3):
    plt.subplot(3,1,i+1)
    for j in range(5):
        plt.plot(np.arange(15),result[i,j,:],label=str(j+1))
    plt.title(s[i])
    plt.legend()
    plt.xticks(np.arange(15),np.arange(10,310,20))
plt.show()
```


```
In [360]: plt.figure(figsize=(40,30))
          for i in range(3):
              plt.subplot(3,1,i+1)
              for j in range(5):
                  plt.plot(np.arange(15),np.log(result[i,j,:]+1e-10),label=str(j+1))
              plt.legend()
              plt.title('Log '+s[i])
              plt.xticks(np.arange(15),np.arange(10,310,20))
          plt.show()
```







1.8 (c)

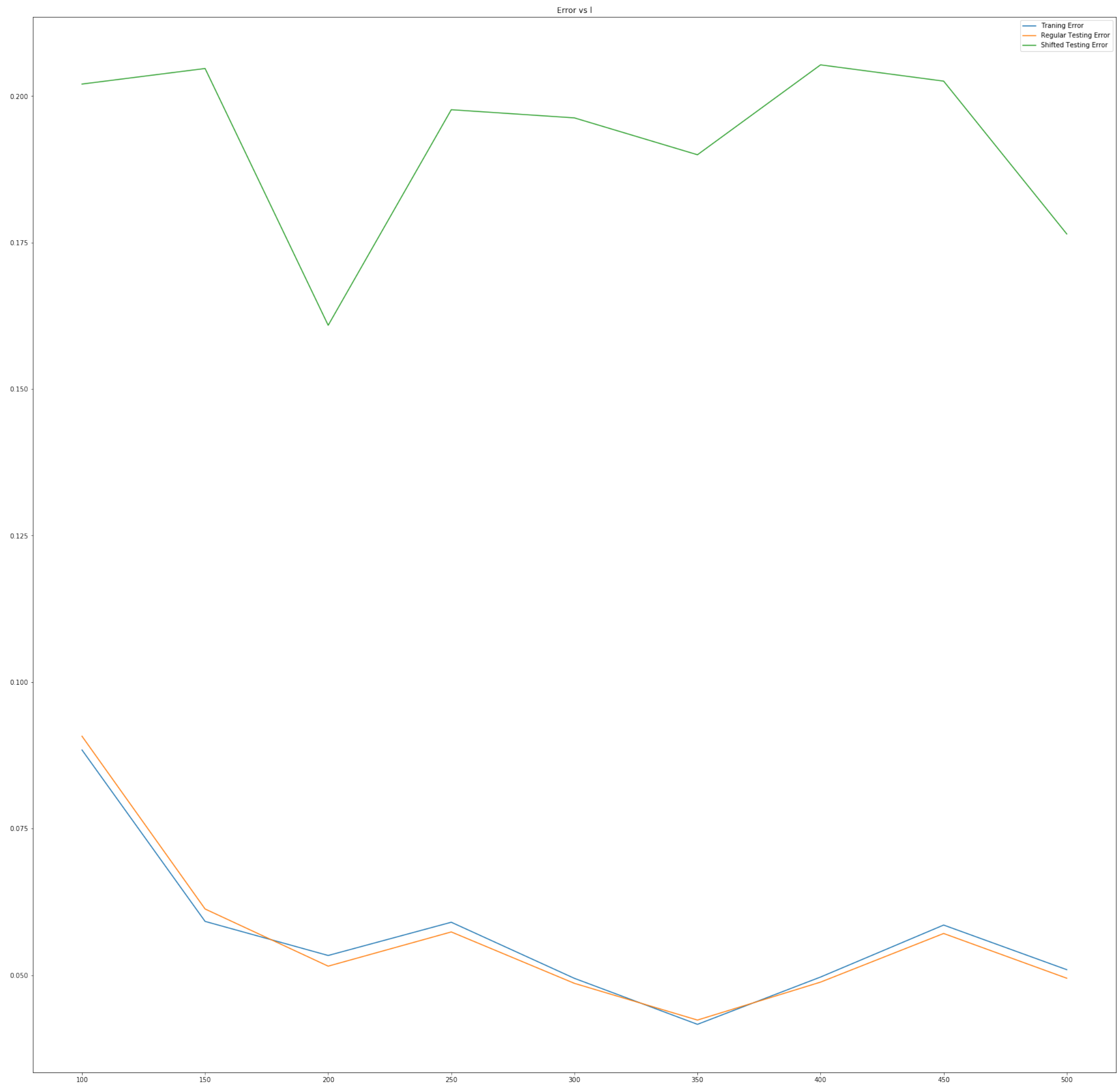
```
In [379]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = generate_dataset(sensor_loc=sensor_loc,num_data=200)
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=1000)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=1000,original_dist=False)
```

```
In [380]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
          ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
          ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
          xtrain = (train[0] - np.average(train[0])) / np.std(train[0]+1e-6);
          ytrain = (train[1] - np.average(train[1])) / np.std(train[1]+1e-6);
```

```
In [382]: l = np.arange(100,550,50)
          n = len(l)
          resultc = np.zeros((3,n))

          for i in range(n):
              for _ in range(10):
```

```
        modelc1 = Model(xtrain.shape[1])
        modelc1.addLayer(DenseLayer(l[i],ReLUActivation()))
        modelc1.addLayer(DenseLayer(l[i],ReLUActivation()))
        modelc1.addLayer(DenseLayer(2,LinearActivation()))
        modelc1.initialize(QuadraticCost())
        hist = modelc1.train(xtrain,ytrain,500,GDOptimizer(eta=0.0001))
        yHat = modelc1.predict(xtrain)
        resultc[0,i] = np.mean(np.linalg.norm(yHat - ytrain,axis=1))
        resultc[1,i] = np.mean(np.linalg.norm(modelc1.predict(xtest1) - ytest1,axis=1)
        resultc[2,i] = np.mean(np.linalg.norm(modelc1.predict(xtest2) - ytest2,axis=1)
resultc /= 10
```

In [383]:
```
plt.figure(figsize=(30,30))
for i in range(3):
    plt.plot(np.arange(9),resultc[i,:],label=s[i])
plt.title('Error vs l')
plt.legend()
plt.xticks(np.arange(9),l)
plt.show()
```




```
In [384]: import pickle
          with open('resultc','wb') as f:
              pickle.dump({'resultc':resultc},f,True)
```

1.9 (d)

```
In [232]: l2 = [round(10000/9)]
          for k in range(2,5):
              l2.append(round((-9+np.sqrt(81+10000*(k-1)))/2/(k-1)))
          l2 = np.array(l2,dtype=int)
          strk = ''
          for i in range(len(l2)):
              strk += str(i+1)+'\t'
          print('k: '+strk)
```

```
strl = ''
for i in range(len(l2)):
    strl += str(l2[i])+'\t'
print('l: '+strl)
```

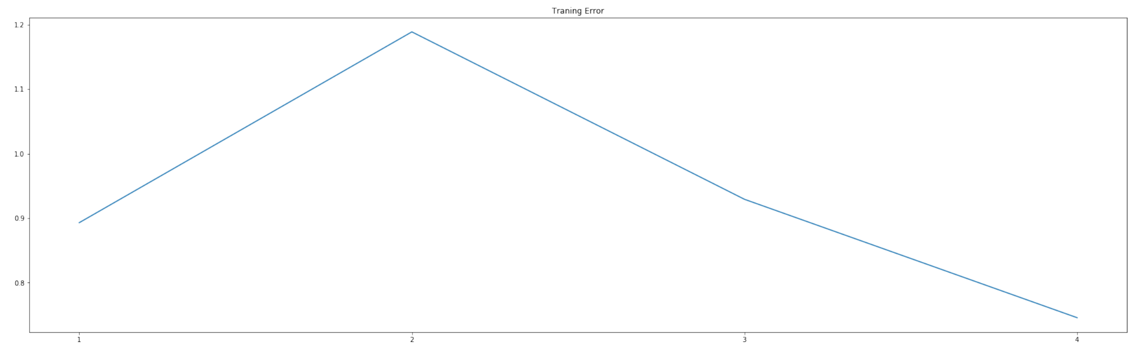
```
k: 1        2        3        4
l: 1111        46        33        27
```

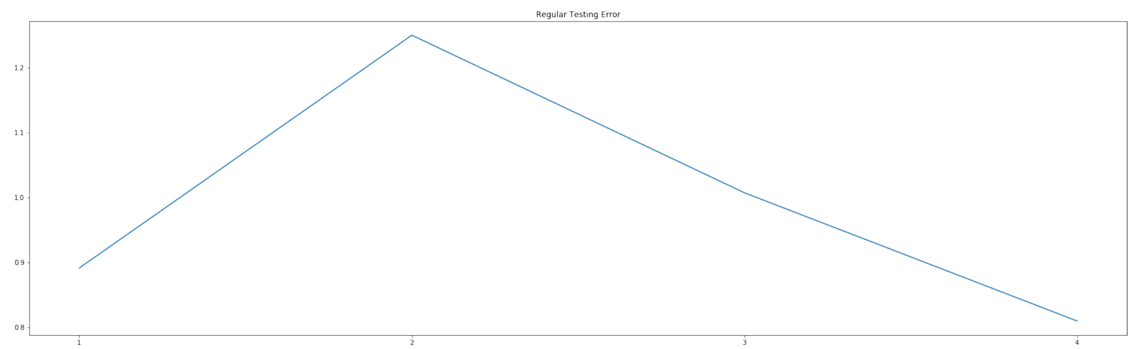
```
In [213]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = generate_dataset(sensor_loc=sensor_loc,num_data=200)
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=200)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=200,original_dist=False)
```

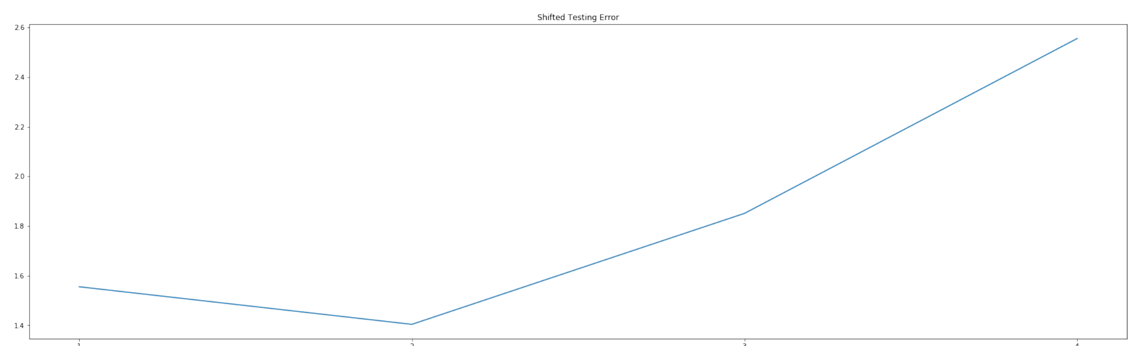
```
In [214]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
          ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
          ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
          xtrain = (train[0] - np.average(train[0])) / np.std(train[0]+1e-6);
          ytrain = (train[1] - np.average(train[1])) / np.std(train[1]+1e-6);
```

```
In [234]: k2 = [i for i in range(1,5)]
          n = len(l2)
          resultd = np.zeros((3,n))

          for i in range(n):
              modeld1 = Model(xtrain.shape[1])
              for _ in range(k2[i]):
                  modeld1.addLayer(DenseLayer(l2[i],ReLUActivation()))
              modeld1.addLayer(DenseLayer(2,LinearActivation()))
              modeld1.initialize(QuadraticCost())
              hist = modeld1.train(xtrain,ytrain,500,GDOptimizer(eta=0.0001))
              yHat = modeld1.predict(xtrain)
              resultd[0,i] = np.mean(np.linalg.norm(yHat - ytrain,axis=1))
              resultd[1,i] = np.mean(np.linalg.norm(modeld1.predict(xtest1) - ytest1,axis=1))
              resultd[2,i] = np.mean(np.linalg.norm(modeld1.predict(xtest2) - ytest2,axis=1))
```

```
In [240]: plt.figure(figsize=(30,30))
          s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
          for i in range(3):
              plt.subplot(3,1,i+1)
              plt.plot(np.arange(4),resultd[i,:])
              plt.xticks(np.arange(4),k2)
              plt.title(s[i])
          plt.show()
```








```
In [244]: import pickle
          with open('resultd','wb') as f:
              pickle.dump({'resultd':resultd},f,True)
```

1.10 (e)

```
In [241]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = {i:generate_dataset(sensor_loc=sensor_loc,num_data=10+20*i) for i in range(15)
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=1000)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=1000,original_dist=False)
```

```
In [252]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
```

```
ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
ztrain = {i:[(train[i][0] - np.average(train[i][0])) / np.std(train[i][0]+1e-6),
             (train[i][1] - np.average(train[i][1])) / np.std(train[i][1]+1e-6)] for i
```

In [361]:
```
resulte = result
```

In [362]:
```
for i in range(15):
    model5 = Model(ztrain[i][0].shape[1])
    model5.addLayer(DenseLayer(150,ReLUActivation()))
    model5.addLayer(DenseLayer(150,ReLUActivation()))
    model5.addLayer(DenseLayer(2,LinearActivation()))
    model5.initialize(QuadraticCost())
    hist = model5.train(ztrain[i][0],ztrain[i][1],500,GDOptimizer(eta=0.0001))
    yHat = model5.predict(ztrain[i][0])
    resulte[0,4,i] = np.mean(np.linalg.norm(yHat - ztrain[i][1],axis=1))
    resulte[1,4,i] = np.mean(np.linalg.norm(model5.predict(xtest1) - ytest1,axis=1))
    resulte[2,4,i] = np.mean(np.linalg.norm(model5.predict(xtest2) - ytest2,axis=1))
```

In [363]:
```
plt.figure(figsize=(40,30))
s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
for i in range(3):
    plt.subplot(3,1,i+1)
    for j in range(5):
        plt.plot(np.arange(15),resulte[i,j,:],label=str(j+1))
    plt.title(s[i])
    plt.legend()
    plt.xticks(np.arange(15),np.arange(10,310,20))
plt.show()
```

In [364]:
```
plt.figure(figsize=(40,30))
for i in range(3):
    plt.subplot(3,1,i+1)
    for j in range(5):
        plt.plot(np.arange(15),np.log(resulte[i,j,:]+1e-10),label=str(j+1))
    plt.legend()
    plt.title('Log '+s[i])
    plt.xticks(np.arange(15),np.arange(10,310,20))
plt.show()
```


```
In [365]: import pickle
          with open('resulte','wb') as f:
              pickle.dump({'resulte':resulte},f,True)
```

1.11 (f)

```
In [317]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = {i:generate_dataset(sensor_loc=sensor_loc,num_data=10+20*i) for i in range(15)
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=1000)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=1000,original_dist=False)

In [318]: zsensor_loc = (sensor_loc - np.average(sensor_loc))/np.std(sensor_loc)

In [319]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
          ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
          ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
          ztrain = {i:[(train[i][0] - np.average(train[i][0])) / np.std(train[i][0]+1e-6),
                       (train[i][1] - np.average(train[i][1])) / np.std(train[i][1]+1e-6)] for i

In [368]: resultf = np.zeros((3,5,15))
```

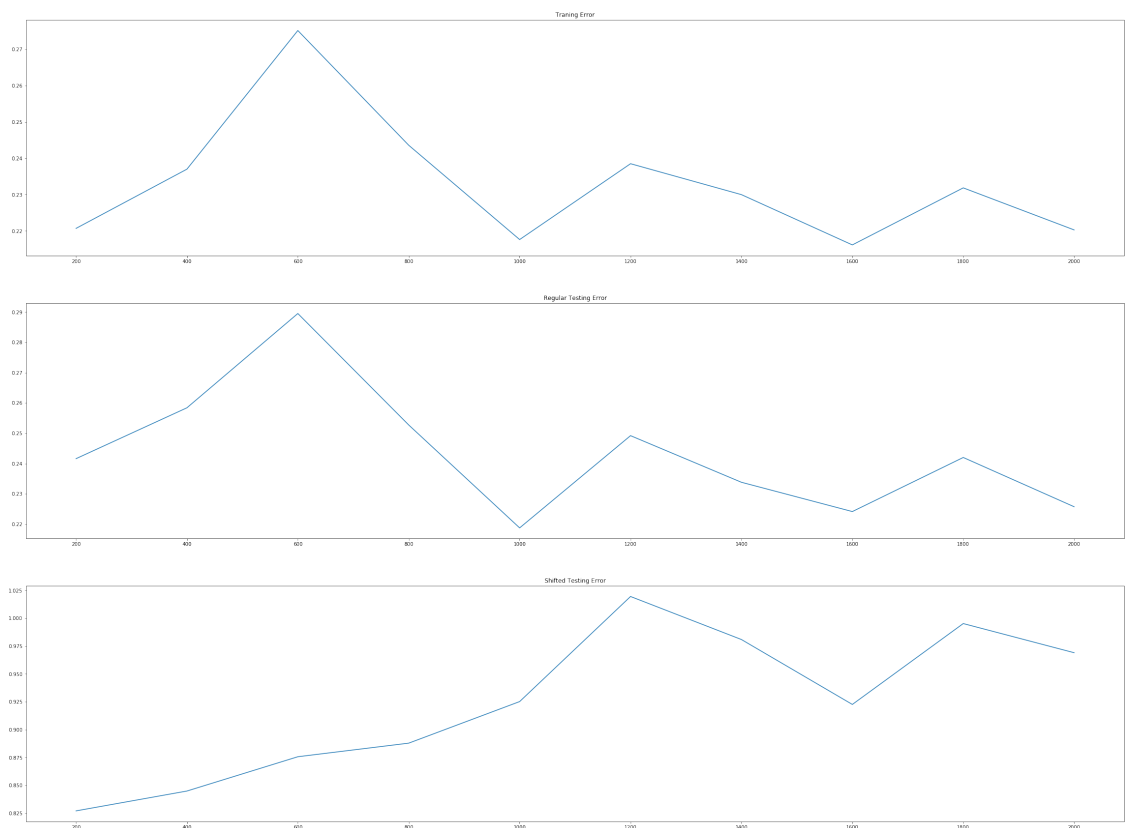
```
In [369]: for i in range(15):
              for _ in range(5):
                  model5 = Model(ztrain[i][0].shape[1])
                  model5.addLayer(DenseLayer(100,ReLUActivation()))
                  model5.addLayer(DenseLayer(100,ReLUActivation()))
                  model5.addLayer(DenseLayer(2,LinearActivation()))
                  model5.initialize(QuadraticCost())
                  hist = model5.train(ztrain[i][0],ztrain[i][1],500,GDOptimizer(eta=0.0001))
                  yHat = model5.predict(ztrain[i][0])
                  resultf[0,4,i] += np.mean(np.linalg.norm(yHat - ztrain[i][1],axis=1))
                  resultf[1,4,i] += np.mean(np.linalg.norm(model5.predict(xtest1) - ytest1,axis=
                  resultf[2,4,i] += np.mean(np.linalg.norm(model5.predict(xtest2) - ytest2,axis=

In [370]: batchsize = [50,100,250,500]
          for j in range(len(batchsize)):
              for i in range(15):
                  for _ in range(5):
                      model5 = Model(ztrain[i][0].shape[1])
                      model5.addLayer(DenseLayer(100,ReLUActivation()))
                      model5.addLayer(DenseLayer(100,ReLUActivation()))
                      model5.addLayer(DenseLayer(2,LinearActivation()))
                      model5.initialize(QuadraticCost())
                      hist = model5.trainBatch(ztrain[i][0],ztrain[i][1],batchsize[j],500,GDOpti
                      yHat = model5.predict(ztrain[i][0])
                      resultf[0,j,i] += np.mean(np.linalg.norm(yHat - ztrain[i][1],axis=1))
                      resultf[1,j,i] += np.mean(np.linalg.norm(model5.predict(xtest1) - ytest1,a
                      resultf[2,j,i] += np.mean(np.linalg.norm(model5.predict(xtest2) - ytest2,a

In [371]: resultf /= 5

In [372]: plt.figure(figsize=(40,30))
          s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
          b = batchsize + [1000]
          for i in range(3):
              plt.subplot(3,1,i+1)
              for j in range(5):
                  plt.plot(np.arange(15),resultf[i,j,:],label=str(b[j]))
              plt.title(s[i])
              plt.legend()
              plt.xticks(np.arange(15),np.arange(10,310,20))
          plt.show()
```

1.12 (g)

```
In [388]: np.random.seed(0)
          sensor_loc = generate_sensors()
          train = {i:generate_dataset(sensor_loc=sensor_loc,num_data=200*(i+1)) for i in range(1
          test1 = generate_dataset(sensor_loc=sensor_loc,num_data=1000)
          test2 = generate_dataset(sensor_loc=sensor_loc,num_data=1000,original_dist=False)

In [389]: xtest1 = (test1[0] - np.average(test1[0])) / np.std(test1[0]+1e-6);
          xtest2 = (test2[0] - np.average(test2[0])) / np.std(test2[0]+1e-6);
          ytest1 = (test1[1] - np.average(test1[1])) / np.std(test1[1]+1e-6);
          ytest2 = (test2[1] - np.average(test2[1])) / np.std(test2[1]+1e-6);
          ztrain = {i:[(train[i][0] - np.average(train[i][0])) / np.std(train[i][0]+1e-6),
                      (train[i][1] - np.average(train[i][1])) / np.std(train[i][1]+1e-6)] for i
          resultg = np.zeros((3,10))

In [390]: zsensor_loc = (sensor_loc - np.average(sensor_loc))/np.std(sensor_loc)

In [391]: for i in range(10):
              for _ in range(5):
                  model5 = Model(ztrain[i][0].shape[1])
                  model5.addLayer(DenseLayer(150,ReLUActivation()))
```

```
                model5.addLayer(DenseLayer(150,ReLUActivation()))
                model5.addLayer(DenseLayer(2,LinearActivation()))
                model5.initialize(QuadraticCost())
                hist = model5.train(ztrain[i][0],ztrain[i][1],1000,GDOptimizer(eta=0.0001))
                yHat = model5.predict(ztrain[i][0])
                resultg[0,i] += np.mean(np.linalg.norm(yHat - ztrain[i][1],axis=1))
                resultg[1,i] += np.mean(np.linalg.norm(model5.predict(xtest1) - ytest1,axis=1)
                resultg[2,i] += np.mean(np.linalg.norm(model5.predict(xtest2) - ytest2,axis=1)
        resultg /= 10
```

In [395]:
```
plt.figure(figsize=(40,30))
s = ['Traning Error', 'Regular Testing Error', 'Shifted Testing Error']
for i in range(3):
    plt.subplot(3,1,i+1)
    plt.plot(np.arange(10),resultg[i,:])
    plt.title(s[i])
    plt.xticks(np.arange(10),np.arange(200,2200,200))
plt.show()
```



In [ ]:
```
import pickle
with open('resultg','wb') as f:
    pickle.dump({'resultg':resultg},f,True)
```

2 Question 5

In [ ]:
```
from __future__ import absolute_import
from __future__ import division
from __future__ import print_function

import argparse
import sys

from tensorflow.examples.tutorials.mnist import input_data

import tensorflow as tf

FLAGS = None


def main(_):
    # Import data
    mnist = input_data.read_data_sets(FLAGS.data_dir, one_hot=True)

    # Create the model
    x = tf.placeholder(tf.float32, [None, 784])
    W = tf.Variable(tf.zeros([784, 10]))
    b = tf.Variable(tf.zeros([10]))
    y = tf.matmul(x, W) + b

    # Define loss and optimizer
    y_ = tf.placeholder(tf.float32, [None, 10])

    # The raw formulation of cross-entropy,
    #
    #   tf.reduce_mean(-tf.reduce_sum(y_ * tf.log(tf.nn.softmax(y)),
    #                                 reduction_indices=[1]))
    #
    # can be numerically unstable.
    #
    # So here we use tf.nn.softmax_cross_entropy_with_logits on the raw
    # outputs of 'y', and then average across the batch.
    cross_entropy = tf.reduce_mean(
      tf.nn.softmax_cross_entropy_with_logits(labels=y_, logits=y))
    train_step = tf.train.GradientDescentOptimizer(0.5).minimize(cross_entropy)

    sess = tf.InteractiveSession()
    tf.global_variables_initializer().run()
    # Train
    for i in range(1000):
        batch_xs, batch_ys = mnist.train.next_batch(100)
        sess.run(train_step, feed_dict={x: batch_xs, y_: batch_ys})
```

```python
            # Test trained model
            correct_prediction = tf.equal(tf.argmax(y, 1), tf.argmax(y_, 1))
            accuracy = tf.reduce_mean(tf.cast(correct_prediction, tf.float32))
            print(str(i)+': ',sess.run(accuracy, feed_dict={x: mnist.test.images,
                                                  y_: mnist.test.labels}))

    if __name__ == '__main__':
        parser = argparse.ArgumentParser()
        parser.add_argument('--data_dir', type=str, default='/tmp/tensorflow/mnist/input_dat
                          help='Directory for storing input data')
        FLAGS, unparsed = parser.parse_known_args()
        tf.app.run(main=main, argv=[sys.argv[0]] + unparsed)
```

```
Extracting /tmp/tensorflow/mnist/input_data/train-images-idx3-ubyte.gz
Extracting /tmp/tensorflow/mnist/input_data/train-labels-idx1-ubyte.gz
Extracting /tmp/tensorflow/mnist/input_data/t10k-images-idx3-ubyte.gz
Extracting /tmp/tensorflow/mnist/input_data/t10k-labels-idx1-ubyte.gz
0:  0.4075
1:  0.4029
2:  0.4927
3:  0.5023
4:  0.6827
5:  0.4972
6:  0.7257
7:  0.7076
8:  0.6808
9:  0.7821
10:  0.7311
11:  0.7984
12:  0.7478
13:  0.7959
14:  0.816
15:  0.8162
16:  0.8256
17:  0.8379
18:  0.8455
19:  0.7874
20:  0.8062
21:  0.8478
22:  0.8206
23:  0.8508
24:  0.8498
25:  0.8481
26:  0.8395
27:  0.83
28:  0.8551
29:  0.8696
```

```
30:  0.8515
31:  0.8429
32:  0.8581
33:  0.8546
34:  0.8602
35:  0.8588
36:  0.8337
37:  0.8494
38:  0.7981
39:  0.8557
40:  0.869
41:  0.8695
42:  0.8684
43:  0.8542
44:  0.8672
45:  0.8553
46:  0.8643
47:  0.8606
48:  0.8672
49:  0.8609
50:  0.8725
51:  0.8744
52:  0.8247
53:  0.8613
54:  0.8773
55:  0.878
56:  0.875
57:  0.8757
58:  0.8839
59:  0.8809
60:  0.8772
61:  0.876
62:  0.869
63:  0.8794
64:  0.8745
65:  0.8747
66:  0.8716
67:  0.8671
68:  0.8655
69:  0.8604
70:  0.8629
71:  0.8495
72:  0.8724
73:  0.8746
74:  0.8797
75:  0.8577
76:  0.8707
77:  0.884
```

```
78:  0.8814
79:  0.8832
80:  0.8829
81:  0.8853
82:  0.8755
83:  0.8857
84:  0.8859
85:  0.8867
86:  0.8852
87:  0.8861
88:  0.8859
89:  0.8828
90:  0.882
91:  0.8854
92:  0.8876
93:  0.8814
94:  0.891
95:  0.887
96:  0.8672
97:  0.8766
98:  0.889
99:  0.8943
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