Q1 (ab) Octave's "format rat" gives

$$A = QR$$

where

$$Q = \begin{bmatrix} -6/7 & 30/7\sqrt{73} \\ -3/7 & -34/7\sqrt{73} \\ -2/7 & -39/7\sqrt{73} \end{bmatrix} = \boxed{\begin{matrix} [\, e_1 \mid e_2 \,] \\ (a) \uparrow \end{matrix}}$$

$$R = \begin{bmatrix} -7/6 & -9/14 \\ 0 & -\sqrt{73}/84 \end{bmatrix}.$$

(Check $Q^*Q = I$ & $R^*R = A^*A$.)

So $e_1$ and $e_2$ are ON basis for $R(A)$.

(c) $P = QQ^* = A(A^*A)^{-1}A^*$

$$\boxed{P = \frac{1}{73}\begin{bmatrix} 72 & 6 & -6 \\ 6 & 37 & 36 \\ -6 & 36 & 37 \end{bmatrix}}$$

(checked $P^2 = P = P^*$)

Q2. Restate: Project $e^{-x}$ onto the span of $v_1(x) = 1$ and $v_2(x) = x$ in $L^2(0,1)$. Solution: Form Gram matrix

$$V^*V = \left[ \langle v_i, v_j \rangle \right] = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

the projection is

$$P = V(V^*V)^{-1}V^*$$

$$= \left[ 1 \,\middle|\, x \right] \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 1^* \\ \hline x^* \end{bmatrix}.$$

Apply to $f(x) = e^{-x}$ so

$$\langle 1, f \rangle = \int_0^1 e^{-x}\,dx = 1 - e^{-1}$$

$$\langle x, f \rangle = \int_0^1 x e^{-x}\,dx = 1 - 2e^{-1}$$

$$Pf(x) = \left[ 1 \,\middle|\, x \right] \begin{bmatrix} -2 + 8e^{-1} \\ 6 - 6e^{-1} \end{bmatrix} = \left[ 1 \,\middle|\, x \right] \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\boxed{Pf(x) = \underbrace{-2 + 8e^{-1}}_{a_0} + \underbrace{(6 - 6e^{-1})}_{a_1} x.}$$

Q3, (a) Form Gram matrix $V^*V$ where

$$V = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \quad \text{in } L^2(-1,1).$$

Because of even-odd symmetry,

$$V^*V = \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix}$$

Octave gives $V^*V = R^*R$ where

$$R = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2/9} \\ 0 & \sqrt{2/3} & 0 \\ 0 & 0 & \sqrt{8/45} \end{bmatrix}$$

(b) so $V = QR \Rightarrow Q = VR^{-1}$ has columns

$$\boxed{e_1 = \frac{1}{\sqrt{2}} \quad e_2 = \frac{\sqrt{3}}{\sqrt{2}}x \quad e_3 = \frac{1}{\sqrt{8}}(\sqrt{5} + \sqrt{45}x^2)}$$

Q4. (a) Let $f \in H^1$. Then by FTA,

$$|f(x)-f(y)| = \left|\int_x^y f'(s)\,ds\right|$$

$$\leq \int_x^y |f'(s)| \cdot 1 \cdot ds$$

$$\leq \sqrt{\int_x^y |f'(s)|^2 ds}\,\sqrt{\int_x^y 1^2 ds}$$

$$\leq \sqrt{|x-y|}\,\sqrt{\int_0^1 |f'(s)|^2 ds}$$

$$\leq \sqrt{|x-y|}\,\|f\|,$$

Hence $f(x)-f(y) \to 0$ as $x \to y$ and $f$ is (uniformly) continuous on $(0,1)$, hence extends uniquely to $[0,1]$ where it must be bounded.

(b) $$\langle f, g\rangle = \int_0^1 f(x)g(x) + f'(x)g'(x)\,dx$$

$$\int_{x_0}^1 f'(x)g'(x)\,dx = f(x)g'(x)\Big|_{x_0}^1 - \int_{x_0}^1 f g''$$

$$\int_0^{x_0} f' g' = f g' \Big|_0^{x_0} - \int_0^{x_0} f g''$$

So

$$\langle f, g\rangle = f(1)g'(1) - f(x_0)g'(x_0^+)$$

$$+ f(x_0)g'(x_0^-) - f(0)g'(0) + \int_0^1 f(g - g'')\,dx.$$

(c) Solve boundary value problem

$$-g'' + g = 0 \qquad x \neq x_0$$

$$g'(0) = g'(1) = 0$$

$$g'(x_0^-) - g'(x_0^+) = 1$$

$$g(x_0^-) - g(x_0^+) = 0.$$

Clearly

$$g(x) = \begin{cases} Ae^{x} + Be^{-x} & x < x_0 \\ Ce^{x} + De^{-x} & x > x_0 \end{cases}$$

so setting up a 4x4 linear system gives

$$\boxed{g(x) = \begin{cases} -\frac{1}{2}\,\frac{e^{2-x_0} - e^{x_0}}{1-e^2}\left(e^{x} + e^{-x}\right) & x < x_0 \\ -\frac{1}{2}\,\frac{e^{x_0} + e^{-x_0}}{1-e^2}\left(e^{x} + e^{2-x}\right) & x > x_0. \end{cases}}$$

Q5. (a)

$$P_n^2 f(x) = \sum_{j=0}^{n} L_j(x) P_n f(x_j)$$

$$= \sum_{j=0}^{n} L_j(x) \sum_{k=0}^{n} L_k(x_j) f(x_k)$$

$$= \sum_{j=0}^{n} L_j(x) \sum_{k=0}^{n} \delta_{kj} f(x_k)$$

$$= \sum_{j=0}^{n} L_j(x) f(x_j) = P_n f(x).$$

Hence $P_n^2 = P_n$ is a projection.

(b) For $0 \le j \le n$ let $g_j \in H^1$ be such that

$$\langle f, g_j \rangle = f(x_j) \quad \forall f \in H^1.$$

Then

$$P_n f(x) = \sum_{j=0}^{n} L_j(x) \langle f, g_j \rangle$$

so

$$\langle P_n f, g \rangle = \langle \sum_{j=0}^{n} L_j \langle f, g_j \rangle, g \rangle$$

$$= \sum_{j=0}^{n} \langle f, g_j \rangle \langle L_j, g \rangle$$

$$= \langle f, \sum_{j=0}^{n} g_j \langle L_j, g \rangle \rangle = \langle f, P_n^* g \rangle.$$

Hence

$$P_n^* g(x) = \sum_{j=0}^{n} g_j(x) \langle L_j, g \rangle$$

$$= \sum_{j=0}^{n} g_j(x) \int_0^1 L_j(y) g(y) + L_j'(y) g'(y)\, dy$$

$$= \int_0^1 \sum_{j=0}^{n} g_j(x) L_j(y)\, g(y) + \sum_{j=0}^{n} g_j(x) L_j'(y) g'(y)\, dy.$$

(c) Since the range of $P_n^*$ is different from the range of $P_n$ we have $P_n^* \neq P_n$. Since $P_n$ is not selfadjoint it is not orthogonal.

(d) The Gram matrix is

$$V^*V = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle & \langle 1,x^3\rangle \\ \text{etc.} & & & \\ & & & \\ & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 4/3 & 5/4 & 6/5 \\ 1/3 & 5/4 & 23/15 & 25/15 \\ 1/4 & 6/5 & 25/15 & 68/35 \end{bmatrix}$$

$$= R^*R$$

where

$$S = R^{-1} = \begin{bmatrix} 1 & -0.48038 & 0.28630 & -0.01701 \\ 0 & 0.96077 & -1.71780 & 2.24532 \\ 0 & 0 & 1.71780 & -6.63390 \\ 0 & 0 & 0 & 4.42260 \end{bmatrix}$$

to 5-digit accuracy. Hence

$$\boxed{\begin{aligned} e_1(x) &= 1 \\ e_2(x) &= S_{12} + S_{22}x \\ e_3(x) &= S_{13} + S_{23}x + S_{33}x^2 \\ e_4(x) &= S_{14} + S_{24}x + S_{34}x^2 + S_{44}x^3 \end{aligned}}$$

is an ON basis (in $H^1$) for cubic polynomials (the range of $P_3$).

(e) As usual

$$P_3 f(x) = \sum_{j=1}^{4} e_j(x) \langle f, e_j \rangle$$

$$= \sum_{j=1}^{4} e_j(x) \int_0^1 f e_j + f' e_j' \, dy$$

$$= \int_0^1 \left( \sum_{j=1}^{4} e_j(x) e_j(y) \right) f(y)$$

$$+ \left( \sum_{j=1}^{4} e_j(x) e_j'(y) \right) f'(y) \, dy$$

$$= \int_0^1 K(x,y) f(y) + K'(x,y) f'(y) \, dy.$$

(f) Follows from orthogonality.

**Question 1** (a) Compute an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = QR$$

(b) find the orthonormal and upper-triangular matrices $Q$ and $R$.
(c) Compute the orthogonal projection $P$ onto the range of $A$.

**Question 2** Find $a_0$ and $a_1$ minimizing

$$F(a_0, a_1) = \int_0^1 |a_0 + a_1 x - \mathrm{e}^{-x}|^2 \, \mathrm{d}x.$$

**Question 3** (a) Find an orthonormal basis for the 3-dimensional subspace of $L^2(-1,1)$ spanned by $1, x$ and $x^2$.
(b) Interpret as a $QR$ factorization.

**Question 4** Let

$$H^1 = H^1(0,1) = \{f \in L^2(0,1) | f' \in L^2(0,1)\}$$

with inner product

$$< f, g >= \int_0^1 f(x)g(x) + f'(x)g'(x) \, \mathrm{d}x.$$

(For simplicity assume all functions are real-valued.)
(a) Show that every $f \in H^1$ is continuous and bounded on (0,1).
(b) Let $g \in H^1$ and suppose also that $g'$ and $g''$ are continuous except at some point $x_0 \in (0,1)$. Show that

$$< f, g >= f(1)g'(1) + f(x_0)\left(g'(x_0^-) - g'(x_0^+)\right) - f(0)g'(0) + \int_0^1 f(x)\left(g(x) - g''(x)\right) \mathrm{d}x$$

for every $f \in H^1$.
(c) Find $g \in H^1$ such that

$$< f, g >= f(x_0)$$

for every $f \in H^1$.

**Question 5** Given $n+1$ distinct points $-1 < x_0 < x_1 < \ldots < x_n < 1$, let $P_n$ be the linear operator which takes $f \in H^1$ into the unique degree-$n$ polynomial

$$p_n(x) = P_n f(x) = \sum_{j=0}^{n} L_j(x) f(x_j)$$

which interpolates the $n+1$ values $f(x_j)$. Here $L_j(x)$ are the degree-$n$ polynomials satisfying

$$L_i(x_j) = \delta_{ij}.$$

(a) Show that $P_n$ is a projection.
(b) Find the adjoint operator $P_n^* g$ for $g \in H^1$.
(c) Show that $P_n$ is not an orthogonal projection.
(d) Find a basis $\{e_0, e_1, e_2, e_3\}$ for the range of $P_3$ which is orthogonal in the $H^1$ inner product.
(e) Find the orthogonal projection $Q_3$ onto the range of $P_3$. Express $Q_3$ as an integrodifferential operator

$$Q_3 f(x) = \int_0^1 K(x,y) f(y) + K'(x,y) f'(y) \, \mathrm{d}y$$

and compute the kernels $K$ and $K'$ in terms of $\{e_0, e_1, e_2, e_3\}$.
(f) Show that $q = Q_3 f$ minimizes the $H^1$ norm $\|q - f\|$ over $q$ in the range of $P_3$.