| | |
|---|---|
| Name | |
| 1a | |
| 1b | |
| 1c | |
| 2a | |
| 2b | |
| 3a | |
| 3b | |
| 3c | |
| Total | |

**(1a)** Compute the complex Fourier coefficients of

$$f(x) = \mathrm{sign}(x)$$

on the interval $-\pi < x < \pi$.

**(1b)** Let $u(x,t)$ be the solution of the Schrodinger equation

$$u_t = iu_{xx},$$

where $i = \sqrt{-1}$, which is $2\pi$-periodic in $x$ and satisfies the initial condition $u(x,0) = f(x)$ from 1a. Find the complex Fourier coefficients $\hat{u}(k,t)$ in terms of $\hat{f}(k)$.

**(1c)** Show that

$$u(x, 2\pi) = u(x, 0).$$

**(2a)** Suppose the discrete Fourier transform matrix $F$ is defined by

$$F_{jk} = \frac{1}{\sqrt{N}} e^{-2\pi i jk/N}$$

for $0 \leq j, k \leq N-1$. Show that $F^4 = I$ is the identity matrix and deduce all possible values for the eigenvalues of $F$.

**(2b)** Let

$$f_j = e^{-i\alpha j}$$

where $\alpha/2\pi$ is an irrational number. Evaluate $\hat{f} = Ff$.

**(3a)** Suppose $\varphi_n(x)$ is the orthonormal Hermite function

$$\varphi_n(x) = \frac{\pi^{-1/4}(-1)^n}{2^{n/2}\sqrt{n!}} e^{x^2/2} D^n e^{-x^2} = \frac{\pi^{-1/4}(-1)^n}{2^{n/2}\sqrt{n!}} \psi_n(x)$$

Show that

$$(D^2 - x^2)\psi_n(x) = -(2n+1)\psi_n(x).$$

**(3b)** Suppose

$$f(x) = \sum_{n=0}^{\infty} f_n \varphi_n(x)$$

where

$$\sum_{n=0}^{\infty} |(2n+1) f_n|^2 < \infty.$$

Evaluate the complex Fourier transform of $f$.

**(3c)** Suppose

$$-iu_t(x,t) = (D^2 - x^2)u(x,t)$$

for $t > 0$ and $x \in R$, and $u(x,0) = f(x)$ from 3b. Find a value of $t$ such that

$$u(x,t) = g(t)\hat{f}(x)$$

is proportional to the complex Fourier transform $\hat{f}$ of $f$.