MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 3

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Solutions by

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Question 1

Suppose A is a complex $n \times n$ matrix. Show that the following are equivalent:

(a) The rows of A form an orthonormal basis in $\mathbb{C}^n$.

(b) $AA^* = I$.

(c) $\|Ax\| = \|x\|$ for all $x \in \mathbb{C}^n$.

Proof.

(a)→(b)

Suppose that $A = (\ a_1^T, a_2^T, \cdots, a_n^T\ )^T$ where $a_i = (\ a_{i1}, a_{i2}, \cdots, a_{in}\ )^T$ is the row vector of A. Therefore $\{a_1, a_2, \cdots, a_n\}$ is an orthonormal basis in $\mathbb{C}^n$.

$\because \quad < a_i, a_j >= a_i^T (a_j^T)^* = a_i^T \overline{a_j} = \delta_{ij}$

$\therefore$

$$
\begin{aligned}
AA^* &= \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} (\ \overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}\ ) \\
&= \begin{pmatrix} a_1^T\overline{a_1} & a_1^T\overline{a_2} & \cdots & a_1^T\overline{a_n} \\ a_2^T\overline{a_1} & a_2^T\overline{a_2} & \cdots & a_2^T\overline{a_n} \\ \vdots & \vdots & \vdots & \vdots \\ a_n^T\overline{a_1} & a_n^T\overline{a_2} & \cdots & a_n^T\overline{a_n} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\
&= I
\end{aligned}
$$

(b)→(c)

$\because$

$$
\begin{aligned}
A^*A = I \\
AA^{-1} = I
\end{aligned}
$$

$\therefore$

$$A^{-1} = A^*$$

$\therefore \quad \forall\ x \in \mathbb{C}^n$

$$
\begin{aligned}
\|Ax\|^2 &=< Ax, Ax > \\
&=< x, A^*Ax > \\
&=< x, x > \\
&= \|x\|^2
\end{aligned}
$$

$\because$

$$\|Ax\| \geqslant 0, \|x\| \geqslant 0$$

Solution (cont.)

$\therefore$

$$\|Ax\| = \|x\|, \qquad \forall\ x \in \mathbb{C}^n$$

(c)→(a)

$\because \quad \|Ax\| = \|x\| \quad \forall\ x \in \mathbb{C}^n$

$\therefore \quad Ax = 0$ iff $x = 0$, i.e. $\{a_1, a_2, \cdots, a_n\}$ form a basis in $\mathbb{C}^n$

$\because \quad \forall\ x, y \in \mathbb{C}^n$

$$\begin{aligned}
\|Ax\|^2 &= (Ax)^T\overline{Ax} \\
&= x^T A^T \overline{Ax} \\
\|x\|^2 &= x^T I \overline{x} \\
x^T\overline{y} &= \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) \\
&= \frac{1}{2}(\|A(x+y)\|^2 - \|Ax\|^2 - \|Ay\|^2) \\
&=< Ax, Ay > \\
&= x^T A^T \overline{Ay}
\end{aligned}$$

Set $x = e_i = (\delta_{1i}, \cdots, \delta_{ni})^T$, $y = e_j = (\delta_{1j}, \cdots, \delta_{nj})^T \in \mathbb{R}^n$, we got

$$\begin{aligned}
e_i^T e_j &= e_i^T A^T \overline{A} e_j \\
&= \delta_{ij}
\end{aligned}$$

i.e.

$$A^T\overline{A} = I$$

$$AA^* = I$$

$\therefore \quad \forall\ i, j \in \mathbb{N},\ 1 \leqslant i, j \leqslant n,\ i \neq j,$

$$\begin{aligned}
< a_i, a_j > &= 0 \\
< a_i, a_i > &= 1
\end{aligned}$$

$\therefore \quad \{a_1, a_2, \cdots, a_n\}$ is an orthonormal basis in $\mathbb{C}^n$ $\square$

Question 2

Suppose $A : V \to W$ is a linear map between two inner product spaces. Show that the nullspace of A^* is exactly the perpendicular complement of the range of A.

Proof.

Solution (cont.)

$\forall\ w \in Null(A^*),\ \forall\ a \in Range(A),\ \exists\ v \in V$, s.t. $a = Av$

$$\begin{aligned} < w, a > &=< w, Av > \\ &=< A^*w, v > \\ &=< 0, v > \\ &= 0 \end{aligned}$$

And $\forall\ w_2 \in W \setminus Null(A^*),\ A^*w_2 \neq 0, \quad \exists v_0 \in V, v_2 \neq 0$ s.t. $Av_2 \in Range(A)$

$$\begin{aligned} < w_2, Av_2 > &=< A^*w_1, v_2 > \\ &\neq 0 \end{aligned}$$

Therefore $Null(A^*) = Range(A)^\perp$ □

Question 3

Prove the Fredholm Alternative: Suppose $A : V \to W$ is a linear map between two inner product spaces. Let $b \in W$. Then either

(a) $Ax = b$ for some $x \in V$ or

(b) There is $w \in W$ with $A^*w = 0$ and $< b, w > \neq 0$.

Proof.
From Question 2, we have $Null(A^*) = Range(A)^\perp$.
Then from Theorem 0.25, $W = Range(A) \bigcup Range(A)^\perp = Range(A) \bigcup Null(A^*)$.
(1) $\forall\ x \in Range(A),\ x \in V,\ Ax = b$, then $\forall\ w \in Null(A^*)$, we have

$$\begin{aligned} A^*w &= 0 \\ < A^*w, Ax > &=< A^*w, b > \\ 0 &=< A^*w, b > \\ 0 &=< w, b > \end{aligned}$$

therefore, x with condition (a) won't satisfy condition (b).
(2) $\forall\ x \notin Range(A) = Null(A^*)^\perp$, then $\exists\ w \in Null(A^*)$, s.t. $A^*w = 0$ and $< b, w > \neq 0$.

□

Question 4

Use the Fredholm Alternative and the Fundamental Theorem of Algebra to prove the existence and uniqueness of polynomial interpolation: given $n+1$ distinct real numbers $x_0, x_1, \cdots x_n$ and $n+1$ complex numbers $f_0, f_1, \cdots f_n$, there exists a unique degree-n polynomial $P(x) = p_0 + p_1 x + \cdots + p_n x^n$ such that $P(x_j) = f_j$ for $0 \leqslant j \leqslant n$.

Proof.

Existence

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_0 & x_0^2 & \cdots & x_0^n \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$XP = F$$

To get solutions of this equation of P.

$$X^* = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0^n & x_1^n & x_2^2 & \cdots & x_n^n \end{pmatrix}$$

Let

$$X^* Q = 0$$

We get no non-zero solution because $\begin{cases} q_0 + \cdots + q_n = 0 \\ x_0 q_0 + \cdots + x_n q_n = 0 \\ x_i \neq x_j & i, j \in \mathbb{N},\ 0 \leqslant i, j \leqslant n,\ i \neq j \end{cases}$

$\therefore \quad dim(Null(X^*)) = 0$

$\therefore \quad$ from Question 3, $\exists\ P \in \mathbb{C}^{n+1}$ s.t. $XP = F$

Uniqueness

Solution (cont.)

$\because$ subtract (i-1)-th column multiplies x_0 from i-th column,

$$\begin{aligned}
|X| &= \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1 - x_0 & x_1^2 - x_1 x_0 & \cdots & x_1^n - x_1^n x_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n - x_0 & x_n^2 - x_n x_1 & \cdots & x_n^n - x_n^{n-1} x_0 \end{vmatrix} \\
&= \prod_{i=1}^{n}(x_i - x_0) \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} \\
&= \cdots \\
&= \prod_{0 \leqslant i < j \leqslant n} (x_i - x_j) \\
&\neq 0
\end{aligned}$$

$\therefore X$ is invertible, i.e. the equation has unique solution $P = X^{-1}F$. $\square$

Question 5

Prove that a projection P on an inner product space is an orthogonal projection if and only if $P^* = P$.

Proof.

Suppose that $P : V \to V_0$

$\Longleftarrow$

$\because$ P is a projection on an inner space

$\therefore$

$$P^2 = P$$

$\because$

$$P^* = P$$

$\therefore$

$$P^* P = P^2 = P = P^*$$

$\therefore$

$$P^*(I - P) = 0$$

Solution (cont.)

$\therefore \quad \forall\, x, y \in V$

$$
\begin{aligned}
< x - Px, Py > &=< P^*(x - Px), y > \\
&=< P^*(I - P)x, y > \\
&= 0
\end{aligned}
$$

$\therefore \quad P$ is an orthogonal projection

$\Longrightarrow$

$\because \quad P$ is an orthogonal projection

$\therefore \quad \forall\, x, y \in V$

$$
\begin{aligned}
< x - Px, Py > &= 0 \\
< P^*(x - Px), y > &= 0 \\
< P^*(I - P)x, y > &= 0
\end{aligned}
$$

$\therefore$

$$P^*(I - P) = 0$$

$\therefore$

$$
\begin{aligned}
P^* &= P^*P \\
&= (P^*P)^* \\
&= (P^*)^* \\
&= P
\end{aligned}
$$

$\square$

Question 6

(a) Let

$$K_t(x) = \frac{t}{\pi(t^2 + x^2)}$$

for $t > 0$ and $x \in R$. Use the Dominated Convergence Theorem to show that

$$\int_{-\infty}^{\infty} K_t(x - y) f(y) \mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.

Proof.

$\because \quad f(x)$ is bounded continuous function

$\therefore \quad \exists\, M > 0$ s.t. $|f(x)| \leqslant M \qquad x \in \mathbb{R}$

Solution (cont.)

$\because$

$$\begin{aligned}\int_{-\infty}^{\infty} K_t(x-y)f(y)\mathrm{d}y &= \int_{-\infty}^{\infty} \frac{t}{\pi[t^2+(x-y)^2]} f(y)\mathrm{d}y \\ &\xlongequal{z=\frac{x-y}{t}} \int_{-\infty}^{\infty} \frac{1}{\pi(1+z^2)} f(tz+x)\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi} f(tz+x)\mathrm{d}(\arctan z) \\ &\xlongequal{u=\pi\arctan z} \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(t\tan\frac{u}{\pi}+x\right)\mathrm{d}u\end{aligned}$$

(1)

$$f_t(u) = f\left(t\tan\frac{u}{\pi}+x\right) \longrightarrow f(x) \qquad (t\longrightarrow 0)$$

(2)

$$|f_t(u)| \leqslant |f(x)| \leqslant M$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} M\mathrm{d}u = M < \infty$$

$\therefore$ from **Dominated Convergence Theorem**, we have

$$\int_{-\infty}^{\infty} K_t(x-y)f(y)\mathrm{d}y \longrightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{d}u = f(x)$$

$\square$

(b) Use (a) to evaluate

$$\int_{-\infty}^{\infty} K_t(x-y)\mathrm{d}y.$$

$\because$

$$K_t(x-y) = \frac{t}{\pi(t^2+(x-y)^2)} > 0$$

$\therefore$ $\forall\ t>0$

$$\int_{-\infty}^{\infty} K_t(x-y)\mathrm{d}y > 0$$

Let $f(x)\equiv 1$, we have

$$\int_{-\infty}^{\infty} K_t(x-y)\mathrm{d}y \to 1 \qquad (t\to 0)$$

Question 7

Show that

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{|x-y|}{t}}}{2t} f(y)\mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.

Proof.

$\because$ $f(x)$ is bounded continuous function

$\therefore$ $\exists\ M > 0$ s.t. $|f(x)| \leqslant M \qquad x \in \mathbb{R}$

$\because$

$$\begin{aligned}\int_{-\infty}^{\infty} K_t(x-y)f(y)\mathrm{d}y &= \int_{-\infty}^{\infty} \frac{e^{-\frac{|x-y|}{t}}}{2t} f(y)\mathrm{d}y \\ &\xlongequal{z=\frac{x-y}{t}} \int_0^{\infty} e^{-z} f(x-tz)\mathrm{d}z \\ &\xlongequal{u=e^{-x}} \int_0^1 f(x+t\ln u)\mathrm{d}u\end{aligned}$$

(1)

$$f_t(u) = f\left(x + t\ln u\right) \longrightarrow f(x) \qquad (t \longrightarrow 0)$$

(2)

$$|f_t(u)| \leqslant |f(x)| \leqslant M$$

$$\int_0^1 M\mathrm{d}u = M < \infty$$

$\therefore$ from **Dominated Convergence Theorem**, we have

$$\int_{-\infty}^{\infty} K_t(x-y)f(y)\mathrm{d}y \longrightarrow \int_0^1 f(x)\mathrm{d}u = f(x)$$

$\square$