1 (a) Since $\cos(-t) = \cos(t)$, g is even. Since $\cos(t+2\pi) = \cos(t)$, g is 2π-periodic. Since f is continuous and cos is continuous, g is the composition of continuous functions and therefore continuous.

(b) By Fejer's theorem, the averaged Fourier sums

$$\sigma_n g(t) = \frac{1}{n+1}\sum_{j=0}^{n} \sum_{|k|\le j} \hat{g}(k) \frac{e^{ikt}}{\sqrt{2\pi}}$$

converge uniformly to g on $[-\pi, \pi]$. And

$$\sigma_n g(t) = \sum_{|k|\le n}\left(\frac{n+1-|k|}{\sqrt{2\pi}(n+1)}\hat{g}(k)\right)e^{ikt}$$

is a trigonometric polynomial with

$$q_{nk} = \frac{1}{\sqrt{2\pi}}\,\frac{n+1-|k|}{n+1}\,\hat{g}(\pm k).$$

(c) Since $\cos(a\pm b) = \cos a\cos b \mp \sin a \sin b$, addition gives

$$\cos(a+b)+\cos(a-b) = 2\cos a\cos b.$$

Putting $a = (n-1)t$ and $b = t$ gives

$$T_n(x) + T_{n-2}(x) = 2x\,T_{n-1}(x).$$

Since $T_0(x) = 1$ and $T_1(x) = x$ are polynomials in x, so are all the T_n's, and

$$\boxed{T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)}$$

Since the coefficients of this two-term recurrence are independent of n, it can be solved explicitly. The characteristic equation ~~has roots~~

$$r^2 - 2xr + 1 = 0$$

has roots

$$r = x \pm \sqrt{x^2-1} = r_\pm$$

so

$$T_n(x) = A\,r_+^n + B\,r_-^n$$

where A and B are determined so that

$$T_0 = 1, \quad T_1 = x.$$

Hence $A = B = 1/2$ and

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$$

$$= \sum_{k=0\,(n-k\ \text{even})}^{n} \binom{n}{k} x^k (x^2-1)^{\frac{n-k}{2}}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} (x^2-1)^k$$

is a polynomial of degree n in x.

(d) Given $\varepsilon > 0$, choose n so that

$$|q_n(t) - f(\cos t)| \le \varepsilon \qquad |t| \le \pi$$

where

$$q_n(t) = \sum_{|k| \le n} q_{nk} e^{ikt}$$

$$= \sum_0^n a_k \cos(kt)$$

$$= \sum_0^n a_k T_k(\cos t)$$

is an even trigonometric polynomial from part (b). Back in the $x = \cos t$ variable,

$$\left| \sum_0^n a_k T_k(x) - f(x) \right| \le \varepsilon \qquad |x| \le 1.$$

2. Let f and g be continuous functions on $[-1,1]$ with the same moments. Then $h(x) = f(x) - g(x)$ has all moments equal to zero. If

$$p(x) = \sum_{j=0}^{n} p_j x^j$$

is any degree n polynomial, then

$$\langle h, p\rangle = \int_{-1}^{1} h(x)\overline{p(x)}\,dx$$
$$= \sum_{j=0}^{n} \overline{p_j} \int_{-1}^{1} h(x) x^j \,dx$$
$$= 0.$$

By Weierstrass, let $|h(x) - p(x)| \le \varepsilon$ for $|x| \le 1$. Then

$$\cancel{0 = \langle h, p\rangle = \langle h-p, p\rangle + \langle p, p\rangle}$$

$$\langle h, h\rangle = |\langle h, h-p\rangle| \quad \text{since } \langle h, p\rangle = 0$$
$$\le \|h\|\,\|h-p\| \quad \text{by Cauchy-Schwarz}$$
$$\le 2\varepsilon\|h\|$$

so

$$\|h\|(\|h\| - 2\varepsilon) \le 0.$$

Hence $\|h\| \le 2\varepsilon$ and since ε was arbitrary $\|h\| = 0$. Since h is continuous, $h(x) \equiv 0$ $|x| \le 1$.

3. $$\int_0^\infty x^k e^{-x^{1/4}} \sin(x^{1/4})\,dx$$

$$= 4\int_0^\infty y^{4k+3} e^{-y} \sin(y)\,dy$$

$$= 4\,\mathrm{Im}\int_0^\infty y^{4k+3} e^{(-1+i)y}\,dy$$

Integration by parts gives

$$\int_0^\infty y^{4k+3} \frac{d}{dy}\frac{e^{(-1+i)y}}{(-1+i)}\,dy =$$

$$= \frac{(4k+3)}{(-1+i)}\int_0^\infty y^{4k+2} e^{(-1+i)y}\,dy$$

$$= \frac{(4k+3)!}{(-1+i)^{4k+3}}\int_0^\infty e^{(-1+i)y}\,dy$$

$$= (4k+3)!\,(-1+i)^{-(4k+4)}.$$

But

$$(-1+i) = \sqrt{2}\,e^{i\pi/4}$$

so $(-1+i)^4 = 4e^{-i\pi} = -4$

and the result is therefore <u>real</u>.

Hence $\int_0^\infty x^k e^{-x^{1/4}} \sin(x^{1/4})\,dx = 0 \quad k=0,1,\ldots$

Q4. (a)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\pi}^{0}(-1/2)\,e^{-ikx}\,dx + \int_{0}^{\pi}(+1/2)\,e^{-ikx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1/2}{-ik}\left[-e^{-ikx}\Big|_{-\pi}^{0} + e^{-ikx}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1/2}{ik}\left[-1+(-1)^k + (-1)^k - 1\right]$$

$\hat{f}(0)=0$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{ik}\left[(-1)^k-1\right]$$ are the <u>exponential</u> Fourier coefficients, so

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{\substack{-\infty \\ k\neq 0}}^{\infty}\frac{1}{\sqrt{2\pi}}\,\underbrace{\frac{1}{ik}}_{\text{odd}}\,\underbrace{[(-1)^k-1]}_{\text{even}}\,\underbrace{e^{ikx}}_{\text{even + odd}}$$

$$= \frac{1}{2\pi}\sum_{k\neq 0}\frac{1}{ik}[(-1)^k-1]\; i\sin(kx)$$

$$= \frac{i}{2\pi}\sum_{k=1}^{\infty}\frac{2}{ik}\left[(-1)^k-1\right]\sin(kx)$$

so

$$\boxed{\hat{f}(k) = \frac{1}{\pi k}\left[(-1)^k-1\right]}$$ are the <u>sine</u> coeffs.

$$f(x) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin((2k+1)x)$$

if we select odd terms.

$$f(x) = \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$$

(b) $g_N(x) = \frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \ldots\right)$

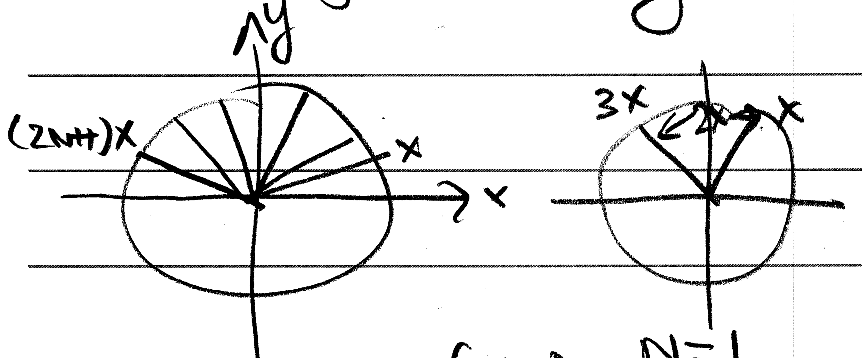
$$g_N'(x) = -\frac{2}{\pi}\left(\cos x + \cos 3x + \ldots + \cos(2N+1)x\right)$$

$= 0$

when points $e^{ix}, e^{3ix}, \ldots, e^{(2N+1)ix}$

are equispaced around the unit circle symmetrically about the y-axis i

spacing is $2x$



so $\boxed{x = \frac{\pi}{2N+2}}$ exactly!

$N=1$

since then $\pi - \frac{(2N+1)\pi}{2N+2} = \frac{\pi}{2N+2}$

So $x_N = \frac{\pi}{2(N+1)}$.

(c) $g_N(x_N) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{N} \frac{1}{2k+1} \sin(2k+1)\frac{\pi}{2(N+1)}$

$= \frac{1}{2} - \frac{2}{\pi}\frac{1}{2N+2}\sum_{k=0}^{N} \frac{2N+2}{2k+1} \sin\left(\frac{2k+1}{2N+2}\pi\right)$

$\rightarrow \frac{1}{2} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,dx$ as $N \to \infty$.

By Taylor series,

$\int_0^{\pi} \frac{\sin x}{x}\,dx = \int_0^{\pi} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}(-1)^k\,dx$

$= \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)(2k+1)!}$

$= \pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + \frac{\pi^9}{9\cdot 9!} - \cdots$

Need quite a few terms,

$= 1.851$

So $\frac{1}{2} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,dx =$