STAT 30100 : Mathematical Statistics-1

Winter 2020

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Homework 0

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Solutions by

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STAT 30100, Homework 0

This assignment covers the handouts `probmeasures.pdf` and `charfn.pdf`.

1. In each case determine whether or not ν is absolutely continuous with respect to μ . Say why or why not. If it is, give the density.

   (a) Ω is the positive integers, $\mathcal{F}$ is the collection of all subsets of Ω, μ is counting measure, ν is the measure associated with the random variable $X = \#$ fair coin tosses needed to get first head.

Since $\Omega = \{1, 2, \ldots\}$ and μ is the counting measure, we have $\forall\ A \in \mathcal{F}$, $\mu(A) = 0$ if and only if $A = \varnothing$. It is clear that $\mu(\varnothing) = 0$. To see the other side, if $A \neq \varnothing$, then $\exists\ k \in \Omega$ such that $k \in A$, i.e., at least one element in A, i.e. $\mu(A) \geq 1 > 0$, which is a contradiction.
Since ν is a measure and $\nu(\varnothing) = 0$, we have that ν is absolutely continuous with respect to μ.
$\forall\ k \in \Omega$,

$$\nu(\{k\}) = f(k)\mu(\{k\}),$$

i.e.,

$$\frac{1}{2^k} = f(k),$$

and therefore $f(k) = \frac{1}{2^k}$ for $k \in \Omega$. So

$$f(x) = \begin{cases} \frac{1}{2^x} & , \ x \in \Omega \\ 0 & , \ x \notin \Omega \end{cases}.$$

   (b) $\Omega = \mathbb{R}^1$, $\mathcal{F}$ is the collection of Borel subsets of $\mathbb{R}^1$, μ is the measure associated with a uniform(0,1) random variable, ν is the measure associated with a normal(0,1) random variable.

Let $A = [2, 3] \subset \mathbb{R}$, then $A \in \mathcal{F}$. Since

$$\mu(A) = 0$$
$$\nu(A) = \int_2^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \mathrm{d}x > 0,$$

ν is not absolutely continuous with respect to μ.

2. Suppose measures μ and ν on $(\Omega, \mathcal{F})$ are mutually singular with $\mu(\Omega) > 0$ and $\nu(\Omega) > 0$. Show that this implies that neither measure is absolutely continuous with respect to the other.

Proof. Since μ and ν are mutually singular, there exists $S_\mu, S_\nu \in \mathcal{F}$ such that $\mu(\Omega \backslash S_\mu) = 0$, $\nu(\Omega \backslash S_\nu) = 0$ and $S_\mu \cap S_\nu = \varnothing$.
Since $\mu(\Omega) = \mu(\Omega \setminus S_\mu) + \mu(S_\mu)$, we have $\mu(S_\mu) = \mu(\Omega) > 0$. Analogously, $\nu(S_\nu) = \nu(\Omega) > 0$.
Since $S_\nu \subseteq \Omega \setminus S_\mu$, we have

$$\nu(S_\nu) \leq \nu(\Omega \setminus S_\mu) \leq \nu(\Omega),$$

i.e., $\nu(\Omega \setminus S_\mu) = \nu(\Omega) > 0$, which implies that ν is not absolutely continuous with respect to μ.
Since $S_\mu \subseteq \Omega \setminus S_\nu$, we have

$$\mu(S_\mu) \leq \mu(\Omega \setminus S_\nu) \leq \mu(\Omega),$$

i.e., $\mu(\Omega \setminus S_\nu) = \mu(\Omega) > 0$, which implies that μ is not absolutely continuous with respect to ν. $\square$

3. Verify properties (1) and (2) in the handout on characteristic functions.

Proof. (1) $\forall\ t \in \mathbb{R}$,

$$\begin{aligned} \phi_X(0) &= \mathbb{E}e^{i \cdot 0 \cdot X} = \mathbb{E}1 = 1 \\ |\phi_X(t)| &= |\mathbb{E}e^{itX}| \leq \mathbb{E}|e^{itX}| = \mathbb{E}1 = 1 \end{aligned}$$

(2) $\forall\ a, b \in \mathbb{R}$,

$$\begin{aligned} \phi_{a+bX}(t) &= \mathbb{E}e^{it(a+bX)} \\ &= e^{ita}\mathbb{E}e^{i(tb)X} \\ &= e^{ita}\phi_X(tb) \end{aligned}$$

$\square$