Stochastic Processes

Fall 2017

Week 4

Solutions by

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2.17

Let $X_1, X_2, \cdots, X_n$ be independent continuous random variables with common density function f. Let $X_{(i)}$ denote the ith smallest of $X_1, X_2, \cdots, X_n$.

(e) Let S_i denote the time of the ith event of the Poisson process $\{N(t), t \geqslant 0\}$. Find $\mathbb{E}[S_i | N(t) = n]$ for $i \leqslant n$ and $i > n$.

(1) $i \leqslant n$

$\because$ given that $N(t) = n$, $S_1, \cdots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$

$\therefore$,

$$f_{S_i|N(t)=n}(x) = \frac{n!}{(i-1)!(n-i)!} \cdot \left(\frac{x}{t}\right)^{i-1} \left(1 - \frac{x}{t}\right)^{n-i} \frac{1}{t} \mathbf{1}_{[0,t]}(x)$$

$\therefore$

$$\begin{aligned}
\mathbb{E}[S_i | N(t) = n] &= \int_0^t \frac{n!}{(i-1)!(n-i)!} \cdot \left(\frac{x}{t}\right)^{i-1} \left(1 - \frac{x}{t}\right)^{n-i} \frac{1}{t} x \mathrm{d}x \\
&= \frac{n!}{(i-1)!(n-i)!} t \cdot Beta(i+1, n-i+1) \\
&= \frac{n!}{(i-1)!(n-i)!} t \frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)} \\
&= \frac{i}{n+1} t
\end{aligned}$$

(2) $i > n$

$$\begin{aligned}
\mathbb{P}\{S_i \leqslant s | N(t) = n\} &= \mathbb{P}\{N(s) - N(t) \geqslant i - n | N(t) - N(0) = n\} \\
&= \mathbb{P}\{N(s) - N(t) \geqslant i - n\} \\
&= \mathbb{P}\{N(s-t) - N(0) \geqslant i - n\} && \text{(stationary increments)} \\
&= \mathbb{P}\{N(s-t) \geqslant i - n\} \\
&= \mathbb{P}\{S_{i-n} \leqslant s - t\} \\
&= \int_0^{s-t} \frac{(\lambda x)^{i-n-1}}{(i-n-1)!} \lambda e^{-\lambda x} \mathrm{d}x \\
f_{S_i|N(t)=n}(s) &= \frac{\lambda^{i-n-1}(s-t)^{i-n-1}}{(i-n-1)!} \lambda e^{-\lambda(s-t)} \\
\mathbb{E}[S_i | N(t) = n] &= \int_t^{\infty} \frac{\lambda^{i-n-1}(s-t)^{i-n-1}}{(i-n-1)!} \lambda e^{-\lambda(s-t)} s \mathrm{d}s \\
&\xlongequal{x=\lambda(s-t)} \frac{1}{\lambda} \int_0^{\infty} \frac{x^{i-n}}{(i-n-1)!} e^{-x} \mathrm{d}x + t \int_0^{\infty} \frac{x^{i-n-1}}{(i-n-1)!} e^{-x} \mathrm{d}x \\
&= \frac{\Gamma(i-n+1)}{\lambda(i-n-1)!} + t \frac{\Gamma(i-n)}{(i-n-1)!} \\
&= \frac{i-n}{\lambda} + t
\end{aligned}$$

2.30

Let $T_1, T_2, \cdots$ denote the interarrival times of events of a non-homogeneous Poisson process having intensity function $\lambda(t)$.

(a) Are the T_i independent?

Let $m(t) = \int_0^t \lambda(s)\mathrm{d}s$. We know that $\forall\ t, s > 0, N(t+s) - N(s) \sim Poisson(m(t+s) - m(s))$. That is,

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{[m(t+s) - m(s)]^n}{n!} e^{-[m(t+s)-m(s)]}$$

No, T_i's are not independent. For example, the conditional probability of $T_2 > t$ given $T_1 = s$ is

$$\begin{aligned}\mathbb{P}(T_2 > t | T_1 = s) &= \mathbb{P}(0 \text{ events in } (s, s+t] | T_1 = s) \\ &= \mathbb{P}(0 \text{ events in } (s, s+t]) \\ &= e^{-[m(s+t)-m(s)]}\end{aligned}$$

which depends on s, i.e. T_2 depends on T_1.

(b) Are the T_i identically distributed?

No. Since the rates are non-homogeneous, the T_i will not be identically distributed.

(c) Find the distribution of T_1.

Since $T_1 > t$ means no event occurs before time t, i.e., $N(t) = 0$, we can derive the distribution function of T_1, $F_{T_1}(t)$ as follows

$$\begin{aligned}F_{T_1}(t) &= \mathbb{P}(T_1 \leqslant t) \\ &= 1 - \mathbb{P}(T_1 > t) \\ &= 1 - \mathbb{P}(N(t) = 0) \\ &= 1 - e^{-m(t)}\end{aligned}$$

Therefore, the density function of T_1 is

$$\begin{aligned}f_{T_1}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} F_{T_1}(t) \\ &= \lambda(t) e^{-m(t)}\end{aligned}$$

(d) Find the distribution of T_2.

$$
\begin{aligned}
F_{T_2}(t) &= \mathbb{P}(T_2 \leqslant t) \\
&= 1 - \mathbb{P}(T_2 > t) \\
&= 1 - \int_0^\infty \mathbb{P}(T_2 > t | T_1 = s) f_{T_1}(s) \mathrm{d}s \\
&= 1 - \int_0^\infty e^{-[m(t+s)-m(s)]} \lambda(s) e^{-m(s)} \mathrm{d}s \\
&= 1 - \int_0^\infty \lambda(s) e^{-m(s+t)} \mathrm{d}s
\end{aligned}
$$

Therefore, the density function of T_2 is

$$
\begin{aligned}
f_{T_2}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} F_{T_2}(t) \\
&= \int_0^\infty \lambda(s) \frac{\mathrm{d}}{\mathrm{d}t} e^{-m(s+t)} \mathrm{d}s \\
&= \int_0^\infty \lambda(s) \lambda(s+t) e^{-m(s+t)} \mathrm{d}s
\end{aligned}
$$

2.39

Compute $Cov(X(s), X(t))$ for a compound Poisson process.

Suppose that $X_1, X_2, \cdots$ independent identical distributed with distribution F and each of them has mean μ and variance σ^2.
(1) $s = t$,

$$
\begin{aligned}
Cov(X(s), X(t)) &= Var[X(s)] \\
&= s(\mu^2 + \sigma^2)
\end{aligned}
$$

(2) $s \neq t$, suppose that $s < t$

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\begin{aligned}
Cov(X(s), X(t)) &= \frac{1}{2} \{Var[X(s)] + Var[X(t)] - Var[X(t) - X(s)]\} \\
&= \frac{1}{2} \left\{\lambda s(\mu^2 + \sigma^2) + \lambda t(\mu^2 + \sigma^2) - Var[X(t-s)]\right\} \\
&= \frac{1}{2} \left\{\lambda s(\mu^2 + \sigma^2) + \lambda t(\mu^2 + \sigma^2) - \lambda(t-s)(\mu^2 + \sigma^2)\right\} \\
&= \lambda s(\mu^2 + \sigma^2)
\end{aligned}
$$