Question 1 (a) Compute the complex exponential Fourier coefficients $\hat{f}(k)$ of

$$f(x) = \mathrm{e}^{rx}$$

for the interval $|x| \leq \pi$.

(b) For the case $r = -1/2$ plot partial sums versus f for $N = 10$, 20, 30 on the larger interval $|x| \leq 2\pi$. Explain the regions of your plot where convergence appears to be fast versus slow.

Question 2 (a) Compute the complex exponential Fourier coefficients $\hat{f}(k)$ of

$$f(x) = x^2$$

for the interval $|x| \leq \pi$.

(b) Show that the Fourier series converges uniformly for $|x| \leq \pi$.

(c) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(d) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Question 3 (a) Solve the heat equation

$$u_t = u_{xx}$$

for $0 \leq x \leq 1$ with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = x(1-x)$.

(b) Express the solution as an integral operator

$$u(x,t) = \int_0^1 K_t(x,y) u(y,0)\ \mathrm{d}y$$

and find the kernel $K_t(x,y)$.

Question 4 Let $-\pi < a < b < \pi$ and $Q(x)$ be a polynomial of degree d. Evaluate the complex exponential Fourier coefficients of $f(x) = Q(x)$ for $a < x < b$ and $f(x) = 0$ otherwise.

Question 5 (a) Compute the complex exponential Fourier coefficient $\hat{\varphi}_j(k)$ over the interval $[-1, 1]$ of the four functions φ_j defined in Question 5 of Problem Set 02.

(b) Explain the relations between the four sequences $\hat{\varphi}_j(k)$ in terms of the scaling and shifting relations between the functions φ_j.

(c) Express the projection P from Question 5 of Problem Set 02 in the form

$$Pf(x) = \sum_{-\infty}^{\infty} \hat{P}(x, k)\hat{f}(k)$$

and find the coefficient functions $\hat{P}(x, k)$.

Question 6 (a) Let f and g be 2π-periodic piecewise smooth functions such that

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

and

$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k)\mathrm{e}^{\mathrm{i}kx}.$$

Define $h = f \star g$ by

$$h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)\mathrm{e}^{\mathrm{i}kx}.$$

Express $\hat{f}$ and $\hat{g}$ as integrals, combine them, and reverse the order of integration and summation to obtain an integral formula for h in terms of f and g.

(b) Let $g \in L^2(-\pi, \pi)$ have complex exponential Fourier coefficients $\hat{g}(k)$. Show that (cf. `https://arxiv.org/abs/0806.0150`)

$$\sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka}\hat{g}(k)$$

if and only if

$$g(0) = \frac{1}{2a} \int_{-a}^{a} g(y)\ \mathrm{d}y.$$

Note that $\frac{\sin(ka)}{ka} \to 1$ as $a \to 0$.

(c) Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = (\pi - 1)/2.$$