Stochasitc Processes

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Content

Chapter 1 Discrete-time Markov Chains

1 Discrete-time Markov Chains

1.1 Definition

1.1.1 Markov Property

A $\textbf{discrete}-\textbf{time}$ markov chains is a stochastic process with discrete index set $\mathbb{N} = \{0, 1, 2, \cdots\}$, state space $\mathbb{S}$(either infinite like $\mathbb{N}$ or finite) and the Markov property

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

where $x, x_1, \cdots, x_n \in \mathbb{S}$ when both conditional probabilities are well defined, i.e. when $\mathbb{P}(X_1 = i_1, ..., X_n = i_n) > 0$.

Strong Markov Property. If $\{X_n : n \geqslant 0\}$ is a Markov chain and τ is the stopping time, then condition on $X_\tau = i$, $\{X_{\tau+n} : n \geqslant 0\}$ is still a Markov chain. In particular, $(X_0, X_1, \cdots, X_{\tau-1})$ is independent of $(X_{\tau+1}, X_{\tau+2}, \cdots)$.

1.1.2 Time-homogeneous

When $\textbf{time}-\textbf{homogeneous}$, $\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i) = P_{ij}$ is irrelevant with time n. We mainly discuss time-homogeneous chains.

1.1.3 Chapman-Kolmogorov Equations

The one-step transition probability of X_{n+1} being in state j given that X_n is in state i is

$$P_{ij}^{n,n+1} = \mathbb{P}\{X_{n+1} = j | X_n = i\}$$

When the Markov chain is time-homogeneous, then

$$P_{ij}^{n,n+1} = P_{ij}, \qquad \forall\ n \in \mathbb{N}$$

is called **stationary** transition probability and it satisfies

$$\sum_{j \in \mathbb{S}} P_{ij} = 1, \qquad \forall\ i \in \mathbb{S}$$

The matrix of one-step transition probabilities P_{ij} is

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ P_{i0} & P_{i1} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which satisfies

(1) $\forall\ i,j \in \mathbb{S},\ P_{ij} \geqslant 0$

(2) $\forall\ i \in \mathbb{S},\ \sum_{j\in S} P_{ij} = 1$

The n-step transition probability from state i at time m to state j at time $m+n$ is

$$P_{ij}^{(m,m+n)} = \mathbb{P}\{X_{m+n} = j | X_m = i\}$$

The matrix of n-step transition probabilities matrix from time m to time $m+n$ is denoted by $\mathbf{P}^{(m,m+n)}$. And $\mathbf{P}^{(n,n)} = \mathbf{I}$. For time-homogeneous chain, $\forall\ m \in \mathbb{N}$, $\mathbf{P}^{(n)} = \mathbf{P}^{(m,m+n)}$ and $\mathbf{P}^{(0)} = I$.

$\forall\ i,j,m,m,r \geqslant 0$,

$$P_{ij}^{(m,m+n+r)} = \sum_{k\in\mathbb{S}} P_{ik}^{(m,m+n)} P_{kj}^{(m+n,m+n+r)}$$

and

$$\mathbf{P}^{(m,m+n+r)} = \mathbf{P}^{(m,m+n)} \cdot \mathbf{P}^{(m+n,m+n+r)}$$

For time-homogeneous chain, above equations becomes

$$P_{ij}^{(m+n)} = \sum_{k\in\mathbb{S}} P_{ik}^{(m)} P_{kj}^{(n)}$$

and

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)} \cdot \mathbf{P}^{(n)}$$

Specially, $\forall\ n \in \mathbb{N}$,

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

Below, we only discuss time-homogeneous Markov chains.

1.1.4 Initial Distribution

Let vector

$$\mu^{(n)} = \left(\mu_i^{(n)}\right)_{i\in\mathbb{S}}$$

denote the distribution of X_n, where

$$\mu_i^{(n)} = \mathbb{P}\{X_n = i\}$$

Then we have

$$\mu^{(m+n)} = \mu^{(m)}\mathbf{P}^n$$

The statistical properties of a homogeneous Markov chain are encoded by $\mu^{(0)}$ and $\mathbf{P}$.

1.2 Generating Functions

1.2.1 Definition

The generating fucntions for a sequence $\{x_i : i \in \mathbb{N}\}$ is defined by

$$G_x(s) = \sum_{n=0}^{\infty} x_n s^n$$

for $s \in \mathbb{R}$ when the summation converges.

The probability generating functions of a $\mathbb{N}$-value random variable X is defined by

$$\begin{aligned} G_X(s) &= \mathbb{E}(s^X) \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{X = n\} s^n \end{aligned}$$

Since $\sum\limits_{n=0}^{\infty} \mathbb{P}\{X = n\} = 1$ if $|s| < 1$, $G_X(s)$ converges if $|s| < 1$.

The probability generating fucntions are useful when dealing with the discrete Markov chains.

$$\begin{aligned} G_X(0) &= \mathbb{P}\{X = 0\} \\ G_X(1) &= \sum_{n=0}^{\infty} \mathbb{P}\{X = n\} \\ &= 1 - \mathbb{P}\{X = \infty\} \end{aligned}$$

the second equation comes from Abel's Theorem. X is called to be defective if $\mathbb{P}\{X = \infty\} > 0$.

1.2.2 Properties

(1) Convergence

There exists a radius of convergence $R \in [0, \infty)$ s.t. the summation absolutely converges at $|s| < R$ and does not converges at $|s| > R$.

$\forall\ r \in \mathbb{R}$, $0 < r < R$, the summation uniformly converges in $|s| < r$.

(2) Differentiation

$G_x(s)$ can be differentiated or integrated term-by-term for all $|s| < R$ for unlimited times.

(3) Uniqueness

If $G_x(s) = G_y(s)$ $(\forall\ |s| < R')$, then $x_n = y_n$ $(\forall\ n \in \mathbb{N})$ since we have

$$x_n = \frac{1}{n!} G_x^{(n)}(0)$$

(4) Abel's Theorem

If $x_n \geqslant 0$ $(\forall\ n \in \mathbb{N})$ and $G_x(s)$ $(\forall\ |s| < 1)$ is finite, then

$$\lim_{s \uparrow 1} G_x(s) = \sum_{n=0}^{\infty} x_n \in \mathbb{R} \cup \{\infty\}$$

For $\mathbb{N}$-value random variable

(5) Convolution

Convolution of 2 real sequences $x = (x_n : n \in \mathbb{N})$ and $y = (y_n : n \in \mathbb{N})$ is defined to be $z = (z_n : n \in \mathbb{N})$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. If x and y have generating functions $G_x(s)$ and $G_y(s)$, then the generating function $G_z(s)$ of $z = x * y$ is

$$\begin{aligned} G_z(s) &= \sum_{n=0}^{\infty} z_n s^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x_i y_{n-i} \right) s^n \\ &= G_x(s) G_y(s) \end{aligned}$$

Convolution of 2 independent random variables X and Y is given by

$$G_{X+Y}(s) = G_X(s) G_Y(s)$$

If $\{X_n : n \geqslant 0\}$ is an independent and identically distributed sequence of $\mathbb{N}$-valued random variables. Suppose that the generating function of X_i is $G_X(s)$. Let N be a $\mathbb{N}$-value random variable, independently of $\{X_n : n \geqslant 0\}$ with the generating function $G_N(s)$. Then

$$S = \begin{cases} \sum_{n=1}^{N} X_n & ,N > 0 \\ 0 & ,N = 0 \end{cases}$$

has a generating functions $G_S(s) = G_N(G_X(s))$.

1.3 Random Variables

1.3.1 First visit time

Let the random variable T_j be the first visit time to state j

$$T_j = \begin{cases} \infty, & \{n : n \geqslant 1, X_n = j\} = \varnothing \\ \min\{n : n \geqslant 1, X_n = j\}, & \textit{elsewhere} \end{cases}$$

The probability of the first visit to state j after n steps starting at state i is given by

$$f_{ij}^{(n)} = \mathbb{P}(T_j = n | X_0 = i)$$

When $i = j$, it will become the probability of the first return after n steps to state i.

We have $\forall\ n \in \mathbb{N}^+$,

$$\begin{aligned} P_{ij}^{(n)} &= \sum_{m=1}^{n} f_{ij}^{(m)} P_{jj}^{(n-m)} \\ f_{ij}^{(n)} &= \sum_{k \neq j} P_{ik} f_{kj}^{(n-1)} \mathbb{1}_{\{n>1\}} + P_{ij} \mathbb{1}_{\{n=1\}} \end{aligned}$$

The probability of the transition from state j to state i exists is

$$\begin{aligned} f_{ij} &= \mathbb{P}\{T_j < \infty | X_0 = i\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{T_j = n | X_0 = i\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{X_n = j, X_k \neq j, 1 \leqslant k < n | X_0 = i\} \\ &= \sum_{n=0}^{\infty} f_{ij}^{(n)} \end{aligned}$$

where $f_{ij}^{(0)} = 0$. When $i = j$, it will become the probability of returning to state i.

Mean return time. The expected return time of state i is

$$\begin{aligned} \mu_i &= \begin{cases} \mathbb{E}(T_i | X_0 = i) & , i \text{ is recurrent} \\ \infty & , i \text{ is transient} \end{cases} \\ &= \begin{cases} \sum\limits_{n=1}^{\infty} n f_{ii}^{(n)} & , i \text{ is recurrent} \\ \infty & , i \text{ is transient} \end{cases} \end{aligned}$$

Define $N_{ik} = \sum\limits_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\} \cap \{T_k \geqslant n\}}$ as the number of visits to the state i before visits to state k. Given $X_0 = k$, The mean number of visits to the state i between two successive visits to state k is

$$\begin{aligned} \rho_{ik} &= \mathbb{E}(N_{ik} | X_0 = k) \\ &= \mathbb{E}\left(\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\} \cap \{T_k \geqslant n\}} \middle| X_0 = k \right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(X_n = i, T_k \geqslant n \big| X_0 = k)$$

Clearly,

$$T_k = \sum_{i \in \mathbb{S}} N_{ik}$$

Therefore,

$$\mu_i = \begin{cases} \sum_{k \in \mathbb{S}} \rho_{ik} & , i \text{ is recurrent} \\ \infty & , i \text{ is transient} \end{cases}$$

1.3.2 Visits

Let random variable V_i denotes the number of times that the process visits state i, then

$$V_i = |\{n \in \mathbb{N} : X_n = i\}|$$

Define

$$\eta_{ij} = \mathbb{P}\{V_j = \infty | X_0 = i\}$$

then

(1) $\eta_{ii} = 1 \qquad \Longleftrightarrow \qquad 1 - f_{ii} = \mathbb{P}\{V_j = 0 | X_0 = i\} > 0$

(2) $\eta_{ii} = \begin{cases} 1 & \text{, if } i \text{ is recurrent} \\ 0 & \text{, if } i \text{ is transient} \end{cases}$

(3) $\eta_{ij} = \begin{cases} f_{ij} & \text{, if } i \text{ is recurrent} \\ 0 & \text{, if } i \text{ is transient} \end{cases}$

(4) $\eta_{ij} = \eta_{ji} = 1$ if $i \to j$ and i is recurrent.

(5) $\eta_{ij} = 1$ if and only if $f_{ii} = f_{jj} = 1$.

Given $X_0 = i$, the number of returns will also be V_i.

Mean number of returns. When $f_{ii} < 1$,

$$\mathbb{P}(V_i = n | X_0 = i) = f_{ii}^n (1 - f_{ii})$$

and

$$\begin{aligned} \mathbb{E}(V_i | X_0 = i) &= \frac{f_{ii}}{1 - f_{ii}} \\ \mathbb{E}(V_i | X_0 = i) &= \mathbb{E}(\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}} | X_0 = i) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(X_n = i | X_0 = i) \\ &= \sum_{n=1}^{\infty} P_{ii}^{(n)} \end{aligned}$$

It is because that the number of time periods for state i to state i has a geometric distribution with parameter f_{ii}. And we also have the mean time spent at state i starting at j (including time 0)

$$\begin{aligned}\sum_{n=0}^{\infty} P_{ii}^{(n)} &= \frac{1}{1-f_{ii}} \\ &= \sum_{i=0}^{\infty} f_{ii}^{n}\end{aligned}$$

1.3.3 First passages

Let the random variable $T_A = \min\{n \geqslant 0 : X_n \in A\}$ denote the first passages time into A where A is a subset of the state space $\mathbb{S}$.

Define $\eta_i = \mathbb{P}\{T_A < \infty | X_0 = i\}$, then

$$\eta_i = \begin{cases} 1 & , \text{ if } x \in A \\ \sum\limits_{k \in \mathbb{S}} P_{ik}\eta_k & , \text{ if } x \notin A \end{cases}$$

Mean number of first passage.

Define the mean number of first passage as $\rho_{iA} = \mathbb{E}(T_A | X_0 = i)$, then

$$\mathbb{E}(T_A | X_0 = i) = \begin{cases} 0 & , \text{ if } x \in A \\ 1 + \sum\limits_{k \in \mathbb{S}} P_{ik}\rho_{kA} & , \text{ if } x \notin A \end{cases}$$

1.3.4 Last exits

Let $l_{ij}^{(n)} = \mathbb{P}\{X_n = j, X_k \neq i \text{ for } 1 \leqslant k \leqslant n | X_0 = i\}$ denote the probability that the chain passes from state i to j in n steps without revisiting i and its generating function is

$$L_{ij}(s) = \sum_{n=1}^{\infty} l_{ij}^{(n)} s^n$$

then for $i \neq j$,

$$P_{ij}(s) = P_{ii}(s) L_{ij}(s)$$

1.3.5 Stopping time

A stopping time random variable τ for a chain satisfies

(1) The event $\{\tau = k\}$ can be decided by $X_1, \cdots, X_k$;

(2) $\mathbb{P}\{\tau < \infty\} = 1$.

1.4 Classification of States

1.4.1 Reducibility

State j is **accessible** from state i, $i \longrightarrow j$, if $\exists n \in N_+ \ s.t. \ P_{ij}^{(n)} > 0 \iff f_{ij} > 0$.

State i and state j **communicate** with each other, $i \longleftrightarrow j$, if they are accessible to each other $\iff$ $f_{ij} f_{ji} > 0$. Any state communicates with itself.

The concept of communication divides the state space up into a number of separate classes. A **class** is the set of states whose communicated states are also in this set, and any two states in it communicate with each other (Or we say every subset is **closed**).

The Markov chain is said to be **irreducible** if there is only one class. The reducible chain will have absorbing state i such that $P_{ii} = 1$.

1.4.2 Recurrence & Transience

State i is **recurrent** if

$$\mathbb{P}\{X_n = i \text{ for some } n \geqslant 1 | X_0 = i\} = 1$$

also, if and only if any of the following conditions holds

(1) $f_{ii} = \sum_{n=0}^{+\infty} f_{ii}^{(n)} = 1$.

(2) $\sum_{n=0}^{+\infty} P_{ii}^{(n)} = \infty$.

(3) $\mathbb{P}\{T_i = \infty | X_0 = i\} = 0$.

If this holds, then

(1) If $f_{ji} > 0$ ($\forall \ j \in S$), then $\sum_{n=0}^{\infty} P_{ji}^{(n)} = \infty$.

(2) If $i \longrightarrow j$, then $f_{ji} = 1$.

State i is **transient** if

$$\mathbb{P}\{X_n = i \text{ for some } n \geqslant 1 | X_0 = i\} < 1$$

also, if and only if any of the following conditions holds

(1) $f_{ii} = \sum_{n=0}^{+\infty} f_{ii}^{(n)} < 1$.

(2) $\sum_{n=0}^{+\infty} P_{ii}^{(n)} < \infty$.

(3) $\mathbb{P}\{T_i = \infty | X_0 = i\} > 0$.

If this holds, then

(1) $\sum_{n=0}^{\infty} P_{ij}^{(n)} < \infty$ ($\forall \ i \in S$).

(2) $\lim_{n\to\infty} P_{ij}^{(n)} = 0$.

The above result can be abtained by defining generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^{(n)} s^n$$
$$F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n$$

and abtaining the equations

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s)P_{jj}(s)$$

Relationship:

(1) Communication

If state i is recurrent and communicates with state j, then state j is recurrent and $f_{ij} = 1$.

If state i is transient and communicates with state j, then state j is transient and $f_{ij} < 1$.

(2) Finite-state

In a finite-state Markov chain, not all states can be transient, i.e. , there is at least one state is recurrent.

(3) Irreducibility

If the chain is irreducible and the state i is recurrent, then every state is recurrent.

If the chain is irreducible and the state i is transient, then every state is transient.

(4) Finite-state & Irreducibility

All states of a finite irreducible Markov chain are recurrent.

1.4.3 Positive Recurrent & Null Recurrent

State i is **positive recurrent** if $\mu_i < \infty$ and state i is recurrent.
State i is **null recurrent** if $\mu_i = \infty$ and state i is recurrent.
If i is recurrent, then i is null recurrent if and only if $P_{ii}^{(n)} \to 0 \quad (n \to \infty)$. If this holds, then $P_{ji}^{(n)} \to 0$ $(\forall\ j \in \mathbb{S})$.
Relationship:

(1) Communication

If state i is positive recurrent and communicates with state j, then state j is positive recurrent.

If state i is null recurrent and communicates with state j, then state j is null recurrent.

(2) Finite-state

If $\mathbb{S}$ is finite, then at least one state is recurrent and all recurrent states are positive recurrent.

(3) Irreducibility

If the chain is irreducible and the state i is positive recurrent, then every state is positive recurrent.

If the chain is irreducible and the state i is null recurrent, then every state is null recurrent.

(4) Finite-state & Irreducibility

All states of a finite irreducible Markov chain are positive recurrent.

1.4.4 Decomposition Theorem

The state space $\mathbb{S}$ is be partitioned uniquely as

$$\mathbb{S} = T \cup C_1 \cup C_2 \cup \cdots$$

where T is the set of transient states and C_i are the irreducible closed set of recurrent states.

1.4.5 Periodicity

Let $d(i) = \gcd\{n > 0 : \Pr(X_n = i \mid X_0 = i) > 0\}$ denote the period of state i. $d(i) = \infty$ if $P_{ii}^n = 0$ ($\forall\ n \in \mathbb{N}$). A state i is said to be aperiodic if $d(i) = 1$ and periodic if $d(i) > 1$.

A Markov chian is said to be aperiodic if all states all aperiodic and periodic otherwise.

Relationship:

(1) Communication

If state i is communicates with state j, then $d(i) = d(j)$.

(2) Irreducible

Let $\{X_n, n \geqslant 1\}$ be an irreducible Markov chain, then all states of the chain will all be periodic or aperiodic.

1.4.6 Ergodicity

A state i is said to be ergodic if it is aperiodic and positive recurrent.

If all states in an irreducible Markov chain are ergodic, then the chain is said to be ergodic.

Relationship:

(1) Finite & Irreducible & Aperiodic

A finite state irreducible Markov chain is ergodic if it has an aperiodic state.

1.5 First Step Analysis

1.5.1 Definition

Let $\{X_n : n \geqslant 0\}$ be a finite-state Markov chain with $\mathbb{S} = \{0, 1, \cdots, N\}$ and transition matrix

$$\mathbf{P}_{N\times N} = \begin{pmatrix} \mathbf{Q}_{r\times r} & \mathbf{P}_{r\times(N-r)} \\ \mathbf{0}_{(N-r)\times r} & \mathbf{I}_{(N-r)\times(N-r)} \end{pmatrix}$$

i.e., states $0, 1, \cdots, r$ are transient and states $r+1, \cdots, N$ are absorbing for some $r \in \mathbb{N}, r < N$.

Let

$$T = \min\{n \geqslant 0 : X_n > r\}$$

be the absorbing time and

$$w_i = \mathbb{E}\left[\sum_{n=0}^{T-1} g(X_n) \middle| X_0 = i\right]$$

be the mean total amount with rate $g(i)$ for $i \in \{0, \cdots, r\}$ starting at state i before absorbed, then $\forall\, i \in \{0, \cdots, r\}$,

$$w_i = g(i) + \sum_{j=0}^{r} P_{ij} w_j$$

1.5.2 Probability of Absorption In a State

When

$$g(i) = P_{ik}$$

for $k \in \{r+1, \cdots, N\}$,

$$w_i = P_{ik} + \sum_{j=0}^{r} P_{ij} w_j$$

denote the probability of absorption in state k.

1.5.3 Mean Time Until Absorption

When

$$g(i) \equiv 1$$

for $k \in \{0, \cdots, r\}$,

$$\begin{aligned} w_i &= 1 + \sum_{j=0}^{r} P_{ij} w_j \\ &= \mathbb{E}(T | X_0 = i) \end{aligned}$$

denote the mean time until absorption.

1.5.4 Mean Number of Visits Prior to Absorption

When

$$g(i) = \begin{cases} 1 & \text{, if } i = k \\ 0 & \text{, if } i \neq k \end{cases}$$

for $k \in \{0, \cdots, r\}$,

$$w_i = \delta_{ik} + \sum_{j=0}^{r} P_{ij} w_j$$

denote the mean number of visits to state $k (0 \leqslant k \leqslant r)$ prior to absorption.

1.6 Limiting Theorems

1.6.1 Stationary Distribution

The stationary distribution of the chain

$$\pi = \left(\pi_i\right)_{i \in \mathbb{S}}$$

satisfies

(1) $\sum_{i \in \mathbb{S}} \pi_i = 1$ and $\pi_i \geqslant 0$ ($\forall\ i \in \mathbb{S}$);

(2) $\pi_i = \sum_{j \in \mathbb{S}} \pi_j P_{ij}$ or $\pi = \pi \mathbf{P}$.

For an irreducible recurrent chain, the vector $\rho_k = \left(\rho_{ik}\right)_{i \in \mathbb{S}}$, the mean number of visits to the state i between two successive visits to state k, satisfies

$$\rho_{ik} < \infty$$

and

$$\rho_k = \rho_k \mathbf{P}$$

Exsitence & Uniqueness. An irreducible chain has a stationary distribution π if and only if all the states are positive recurrent. If this holds, then π is the unique stationary distribution and is given by

$$\pi_i = \frac{1}{\mu_i}$$

where μ_i is the mean return time.

1.6.2 Limiting Distribution

If the limiting distribution exists, then it is given by

(1) $\left(\lim_{n \to \infty} P_{ij}^{(n)}\right)_{j \in \mathbb{S}}$

(2) $\sum_{i \in \mathbb{S}} \lim_{n \to \infty} P_{ij}^{(n)} = 1$

which is irrelevant to initial state i.

For an irreducible aperiodic chain, we have that $\forall\ i, j \in \mathbb{S}$,

$$P_{ij}^{(n)} \to \frac{1}{\mu_j} \qquad (n \to \infty)$$

For an irreducible periodic chain with period d, $\{Y_n = X_{nd} : n \geqslant 0\}$ is an aperiodic chain and it follows that $\forall\ i \in \mathbb{S}$,

$$P_{ii}^{(nd)} \to \frac{d}{\mu_j} \qquad (n \to \infty)$$

For any aperiodic state j of a Markov chain and $i \leftrightarrow j$,

$$P_{ij}^{(n)} \to \frac{1}{\mu_j} \qquad (n \to \infty)$$

Furthermore, $\forall\ i \in \mathbb{S}, i \neq j$,

$$P_{ij}^{(n)} \to \frac{f_{ij}}{\mu_j} \qquad (n \to \infty)$$

and

$$\tau_{ij}^{(n)} \to \frac{f_{ij}}{\mu_j} \qquad (n \to \infty)$$

where

$$\tau_{ij}^{(n)} = \frac{1}{n} \sum_{m=1}^{n} P_{ij}^{(m)}$$

denotes the mean proportion of elapsed time up to the nth step during which the chain was in state j, starting at state i.

For transient and null recurrent states, the above limiting will be 0 and hence limiting distribution cannot exsit for an irreducible chain with such states.

Exsitence & Uniqueness. An irreducible aperiodic chain has a stationary distribution π if and only if all the states are positive recurrent. If this holds, then π is the unique stationary distribution as well as limiting distribution and is given by

$$\pi_i = \lim_{n\to\infty} P_{ji}^{(n)} = \frac{1}{\mu_i}$$

$\forall\ j \in \mathbb{S}$, where μ_i is the mean return time.

1.6.3 Other Theorems

$\forall\ j \in \mathbb{S}$,

$$\frac{N_j(t)}{t} \xrightarrow{P} \frac{1}{\mu_j}$$

where $\{N_j(t)\}$ denotes the renewal processes at state j.

Let r be a bounded function on the state space, then

$$\frac{\sum_{m=1}^{n} r(X_m)}{n} \xrightarrow{P} \sum_{i\in\mathbb{S}} r(i)\pi_i$$

1.7 Reversibility

1.7.1 Definition

Let $X = \{X_n : n \geqslant 0\}$ be an irreducible Markov chain with transient matrix $\mathbf{P}$ and stationary distribution π. Suppose that $X_0 \overset{d}{=} \pi$, then the time-reversed chain of X is given by

$$Y_n = X_{N-n} \qquad 0 \leqslant n \leqslant N$$

for given $N \in \mathbb{N}$ and extended to $n > N$.

$Y = \{Y_n : n \geqslant 0\}$ is a Markov chian with

$$\mathbb{P}\{Y_{n+1} = j | Y_n = i\} = \frac{\pi_j}{\pi_i} P_{ji}$$

We say that X is **reversible** if X and Y habe the same transition probabilities, i.e., the **detail balance equations** holds, i.e.,$\forall\ i, j \in \mathbb{S}$,

$$\pi_i P_{ij} = P_{ji} \pi_j$$

The transient matrix of the reversed chain is also $\mathbf{P}$.

1.7.2 Property

Irreducibility & **Reversibility** & **Stationary Distribution**. Let $X = \{X_n :\geqslant 0\}$ be a Markov chain with transition matrix $\mathbf{P}$. Suppose that a vector $\pi = \left(\pi_i \geqslant 0 : i \in \mathbb{S}\right)$ satisfies

$$\sum_{i \in \mathbb{S}} \pi_i = 1$$

and $\forall\ i, j \in \mathbb{S}$,

$$\pi_i P_{ij} = P_{ji} \pi_j$$

Then π is a stationary distribution for X and X is reversible with respect to π.

2 Branching Processes

2.1 Definition

Branching processes are Markov chains of a special type.

Assumptions:

(1) The number of offsprings of different individuals of the branching process form a collection of independent random variables;

(2) The number of offsprings of every individual have the same probability mass fucntion and generating fucntion.

Let $Z_0 = 1$, $Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}$ denote the number of individuals in generation n where $X_{n,i}$ is the number of individuals in generation n who are descendants of the ith individuals in generation $n-1$. Then $X_{n,i}$ are independent identically distributed. Suppose that $X_{n,i} \stackrel{d}{=} X$, then

$$\begin{aligned}
\mathbb{P}\{X_{n,i} = k\} &= \mathbb{P}\{X = k\} \\
&= P_i \\
\mathbb{E}X_{n,i} &= \mathbb{E}X \\
&= \mu \\
VarX_{n,i} &= VarX \\
&= \sigma^2 \\
G_{X_1}(s) &= G_X(s)
\end{aligned}$$

Let $G_{Z_n}(s) = \mathbb{E}s^{Z_n}$. In particular,

$$G_{Z_1}(s) = G_{X_1}(s) = G_X(s)$$

2.2 Property

For $m, n \in \mathbb{N}^+$,

$$\begin{aligned}
G_{Z_{m+n}}(s) &= G_{Z_m}(G_{Z_n}(s)) \\
&= G_{Z_n}(G_{Z_m}(s)) \\
G_{Z_n}(s) &= G_{Z_1}(G_{Z_1}(\cdots G_{Z_1}(s))) \\
&= G_X(G_X(\cdots G_X(s)))
\end{aligned}$$

From the properties of the random sum,

$$\begin{aligned}
\mathbb{E}Z_{n+1} &= \mu \mathbb{E}Z_n \\
VarZ_{n+1} &= \sigma^2 \mathbb{E}Z_n + \mu^2 VarZ_n
\end{aligned}$$

we have

$$
\begin{aligned}
\mathbb{E}Z_n &= \mu^n \\
Var Z_n &= \begin{cases} n\sigma^2 & ,\mu = 1 \\ \dfrac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & ,\mu \neq 1 \end{cases}
\end{aligned}
$$

2.3 Extinction Probabilities

Since

$$\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$$

we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\{Z_n = 0\} &= \mathbb{P}\{\bigcup_{n=1}^{\infty}\{Z_n = 0\}\} \\
&= \mathbb{P}\{\text{ultimate extinction}\} \\
&= \eta
\end{aligned}
$$

where η is the smallest non-negative solution of $G_X(s) = s$.

We have

$$
\begin{aligned}
G_X(1) &= \sum_{n=0}^{\infty} \mathbb{P}\{X = n\} \\
&= 1 \\
G_X(0) &= \mathbb{P}\{X = 0\} \\
&= P_0 \\
G_X'(1) &= \sum_{n=0}^{\infty} n\mathbb{P}\{X = n\} \\
&= \mu \\
G_X''(s) &= \mathbb{E}[X(X-1)s^{X-2}] \geqslant 0 \qquad 0 \leqslant s \leqslant 1
\end{aligned}
$$

Therefore,

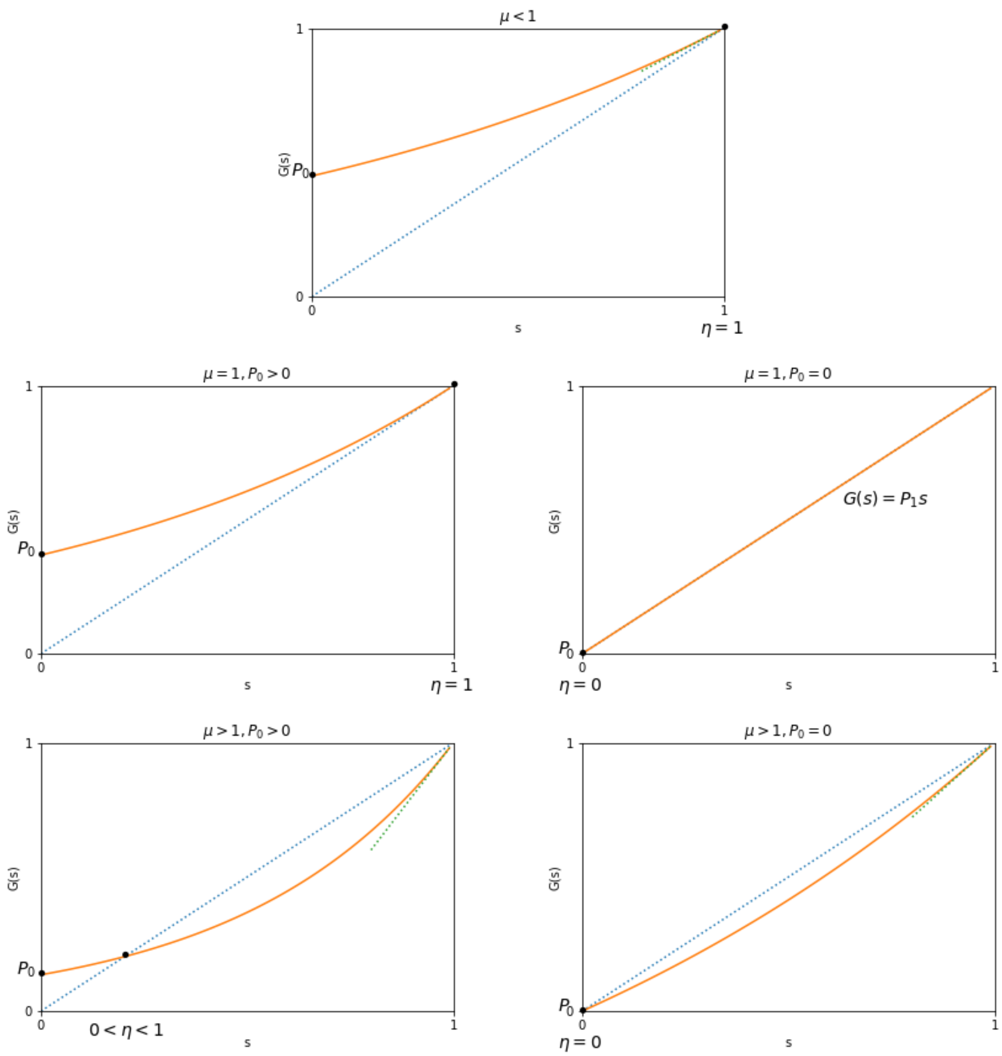
(1) If $\mu < 1$ (implies $P_0 > 0$), then $\eta = 1$.

(2) If $\mu = 1$ and $P_0 > 0$ (equivalently, $\mu = 1$ and $P_1 < 1$), then $\eta = 1$.

(3) If $\mu = 1$ and $P_0 = 0$ (equivalently, $\mu = 1$ and $P_1 = 1$), then $\eta = 0$.

(4) If $\mu > 1$ and $P_0 > 0$, then $0 < \eta < 1$.

(5) If $\mu > 1$ and $P_0 = 0$, then $\eta = 0$.



Intuitively, $P_0 = 0$ indicates the number of individuals won't go down during generations, i.e., $\eta = 0$ which implies (2) and (5).

If given $P_0 > 0$, then $\mu \leqslant 1 \iff$ extinction occurs almost surely.

Also, by first step analysis, we have $\forall\ n \in \mathbb{N}$,

$$u_{n+1} = \sum_{k=0}^{\infty} P_k u_n^k$$

where $\forall\ n \in \mathbb{N}$,

$$u_n = \mathbb{P}\{Z_n = 0\}$$

$$u_1 = P_0$$

$$u_0 = 0$$

Chapter 2 Continuous-time Markov Chains

3 Continuous-time Markov Processes

3.1 Definition

3.1.1 Markov Property

The process $X = \{X(t) : t \geqslant 0\}$ satisfies the Markov property if

$$\mathbb{P}\{X(t_n) = i_n | X(t_0) = i_0, \cdots, X(t_{n-1}) = i_{n-1}\} = \mathbb{P}\{X(t_n) = i_n | X(t_{n-1}) = i_{n-1}\}$$

$\forall\ i_0, \cdots, i_n \in \mathbb{S}, 0 \leqslant t_0 < \cdots < t_n$.

3.1.2 Time-homogeneous

The process is time-homogeneous if

$$\mathbb{P}\{X(s+t) = j | X(s) = i\} = \mathbb{P}\{X(t) = j | X(0) = i\}$$

3.1.3 Chapman-Kolmogorov Equations

For a homogeneous process, let

$$P_{ij}(t) = \mathbb{P}\{X(s+t) = j | X(s) = i\}$$

denote the transition probability. The transition matrix is given by

$$\mathbf{P}_t = \Big(P_{ij}(t)\Big)_{(i,j)\in\mathbb{S}\times\mathbb{S}}$$

we have $\{\mathbf{P}_t : t \geqslant 0\}$ is a stochatic semigroup,

(1) $\mathbf{P}_0 = I$

(2) $\forall\ i \in \mathbb{S},\ t \geqslant 0, \sum_{j\in\mathbb{S}} P_{ij}(t) = 1$ and $P_{ij}(t) \geqslant 0$

(3) $\forall\ i,k \in \mathbb{S},\ s,t \geqslant 0,$

$$P_{ik}(s+t) = \sum_{j\in\mathbb{S}} P_{ij}(s) P_{jk}(t)$$

or

$$\mathbf{P}_{s+t} = \mathbf{P}_s \mathbf{P}_t$$

Usually, we want the semigroup to be **standard**, i.e., $\mathbf{P}_t \to \mathbf{I}$ as $t \downarrow 0$, so that $\mathbf{P}_t$ is continuous and differentiable in $t \geqslant 0$. Thus the below limits exist.

$$q_{ii} = \lim_{t\to 0+} \frac{P_{ii}(t) - 1}{t} \leqslant \infty$$
$$q_{ij} = \lim_{t\to 0+} \frac{P_{ij}(t)}{t} < \infty \qquad (i \neq j)$$

3.1.4 Infinitesimal Matrix

Let $\mathbf{G} = \left(g_{ij}\right)_{(i,j)\in\mathbb{S}\times\mathbb{S}}$ denote the infinitesimal matrix (or the generator)where

$$g_{ij} = \begin{cases} q_{ii} & ,i \in \mathbb{S} \\ q_{ij} & ,i,j \in \mathbb{S}, i \neq j \end{cases}$$

we have

$$\lim_{t\to 0+} \frac{\mathbf{P}_t - \mathbf{I}}{t} = \mathbf{G}$$

and $\forall\ i \in \mathbb{S}$,

$$0 \leqslant \sum_{\substack{j\in\mathbb{S} \\ j\neq i}} g_{ij} \leqslant -g_{ii} \leqslant \infty$$

Especially, when $\mathbb{S}$ is finite,

$$\sum_{j\in\mathbb{S}} g_{ij} = 0$$

or

$$\mathbf{G}\mathbf{1}^T = \mathbf{0}^T$$

3.2 Properties

3.2.1 Forward Equations

$\forall\ i,j \in \mathbb{S}$,

$$\mathbf{P}_t' = \mathbf{P}_t\mathbf{G}$$

or

$$p_{ij}'(t) = \sum_{k\in\mathbf{S}} P_{ik}(t) g_{kj}$$

3.2.2 Backward Equations

$\forall\ i,j \in \mathbb{S}$,

$$\mathbf{P}_t' = \mathbf{G}\mathbf{P}_t$$

or

$$P_{ij}'(t) = \sum_{k\in\mathbf{S}} g_{ik} P_{kj}(t)$$

3.2.3 Generator & Transition Matrix

Generator can specify the transition matrix,

$$\begin{aligned} \mathbf{P}_t &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{G}^n \\ &= e^{t\mathbf{G}} \end{aligned}$$

3.3 Random Variables

3.3.1 Changing Time

Let T_n be the time of the nth change in value of X and set $T_0 = 0$.

3.3.2 Holding Time

Let $H_n = T_n - T_{n-1}$ $(n \in \mathbb{N}^+)$ be the nth hodling time. Then $\forall\ g_{ii} \neq 0$, $H_{n+1}|X_{T_n} = i \sim Exp(-g_{ii})$

3.4 Jump Chains

3.4.1 Definition

Then jump chain is given by $J_n = X(T_n+)$, then value of X immediately after its jump. $J = \{J_n : n \geqslant 0\}$ has transition probabilities

$$P_{ij}^J = \begin{cases} -\dfrac{g_{ij}}{g_{ii}} & ,i \neq j, g_{ii} \neq 0 \\ 0 & ,i = j, g_{ii} \neq 0 \\ \delta_{ij} & ,g_{ii} = 0 \end{cases}$$

3.4.2 Properties

If $g_{ii} = 0$,

$$\begin{aligned} \mathbb{E}(H_{n+1}|X_{T_n} = i) &= \infty \\ \mathbb{P}\{H_{n+1} = \infty|X_{T_n} = i\} &= \lim_{t\to\infty} \mathbb{P}\{H_{n+1} > t|X_{T_n} = i\} \\ &= 1 \end{aligned}$$

which means that starting at state i, the process remains in state i for ever with probability one.

If $g_{ii} = \infty$,

$$\begin{aligned} \mathbb{P}\{H_{n+1} = 0|X_{T_n} = i\} &= \lim_{t\to 0} \mathbb{P}\{H_{n+1} < t|X_{T_n} = i\} \\ &= 1 \end{aligned}$$

which means that starting at state i, the process jumps to other state immediately with probability one.

If $0 < -g_{ii} < \infty$,

$$H_{n+1}|X_{T_n} = i \sim Exp(-g_{ii})$$

which means that starting at state i, the process remains in state i for time $H_{n+1}|X_{T_n} = i$ and then jump to other state j for probability $P_{ij}^J = -\dfrac{g_{ij}}{g_{ii}}$.

3.5 Classification of States

3.5.1 Reducibility

$\forall\ i, j \in \mathbb{S}$, either $\forall\ t > 0$, $P_{ij}(t) = 0$ or $\forall\ t > 0$, $P_{ij}(t) > 0$.

The chain is called **irreducible** if $\forall\ i, j \in \mathbb{S}$, $\exists\ t \geqslant 0$, s.t. $P_{ij}(t) > 0$.

3.5.2 Recurrence & Transience

The definition for continuous-time Markov chains are similar to discrete-time Markov chains.

State i is **recurrent** for X if $\mathbb{P}\{\text{the set } \{t : X(t) = i\} \text{ is unbounded} | X(0) = i\} = 1$.

State i is **transient** for X if $\mathbb{P}\{\text{the set } \{t : X(t) = i\} \text{ is unbounded} | X(0) = i\} = 0$.

From the relationship between X and its jump chain, we have

(1) If $g_{ii} = 0$, the state i is recurrent.

(2) If $g_{ii} < 0$, then state i is recurrent $\iff$ state i is recurrent for jump chain J.

State i is recurrent for X $\iff$ $\int_0^\infty P_{ii}(t)\mathrm{d}t = \infty$. State i is transient for X $\iff$ $\int_0^\infty P_{ii}(t)\mathrm{d}t < \infty$.

Relationship:

(1) Finite-state & Irreducibility

If $\mathbb{S}$ is finite and the process is irreducible, then the process is positive recurrent.

3.6 Limiting Theorems

3.6.1 Stationary Distribution

The vector $\pi = \left(\pi_i\right)_{i \in \mathbb{S}}$ is a stationary distribution of the process if

(1) $\forall\ i \in \mathbb{S}$, $\sum\limits_{j \in \mathbb{S}} \pi_j = 1$ and $\pi_i \geqslant 0$.

(2) $\forall\ t \geqslant 0$, $\pi = \pi \mathbf{P}_t$

we have a useful way to find out the stationary distribution,

$$\pi = \pi \mathbf{P},\ \forall\ t \geqslant 0 \qquad \iff \qquad \pi \mathbf{G} = \mathbf{0}$$

3.6.2 Limiting Distribution

Let X be irreducible with a standard semigroup $\{\mathbf{P}_t\}$ of transition probabilities.

(1) If there exists a stationary distribution π then it is unique and it is the limiting distribution, i.e., $\forall\ i, j \in \mathbb{S}$,

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j$$

(2) If there is no stationary distribution then limiting distribution doesn't exist

$$\lim_{t \to \infty} P_{ij}(t) = 0$$

3.7 Explosion

The time of explosion is given by

$$T_\infty = \lim_{n\to\infty} T_n$$

The process X does not explode if any of the following holds:

(1) $\mathbb{S}$ is finite;

(2) $\sup_i g_{ii} < \infty$;

(3) $X(0) = i$ where i is a recurrent state for the jump chain J.

4 Birth Proceses

4.1 Definition

A special case of continuous Markov processes is the birth process, with rate sequences $\{\lambda_n : n \geqslant 0\}$, for which

(1) Independent Increments

$\forall\ 0 = t_0 < t_1 < \cdots < t_n$, the process increments

$$B(t_1) - B(t_0),\ B(t_2) - B(t_1),\ \cdots,\ B(t_n) - B(t_{n-1})$$

are independent, i.e.,

$$\begin{aligned}\mathbb{P}\{B(s+t) = n+k | B(s) = n\} &= \mathbb{P}\{B(s+t) - B(s) = k | B(s) - B(0) = 0\} \\ &= \mathbb{P}\{B(s+t) - B(s) = k\}\end{aligned}$$

(2) Nonstationary increments

$\forall\ s > 0, t \geqslant 0$, as $t \to 0+$,

$$\mathbb{P}\{B(s+t) - B(s) = k\} = \begin{cases} 1 - \lambda_{B(s)} t + o(t) & ,k = 0 \\ \lambda_{B(s)} t + o(t) & ,k = 1 \\ o(t) & ,k > 1 \end{cases}$$

(3) Initial Condition

$$B(0) = 0$$

The infinitesimal matrix is given by

$$\mathbf{G} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & \cdots \\ 0 & 0 & 0 & -\lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

4.2 Properties

4.2.1 Forward System of Equations

$$P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

where $P'_{ij}(0) = \delta_{ij},\ \lambda_{-1} = 0,\ i, j \in \mathbb{N},\ i \leqslant j,\ t \geqslant 0.$

4.2.2 Backward System of Equations

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

where $P'_{ij}(0) = \delta_{ij},\ \lambda_{-1} = 0,\ i, j \in \mathbb{N},\ i \leqslant j,\ t \geqslant 0.$

4.2.3 Probabilities

$$\mathbb{P}\{B(t) = n | B(0) = 0\} = \left(\prod_{i=0}^{n-1} \lambda_i\right) \sum_{i=0}^{n} B_{in} e^{-\lambda_i t}$$

where $\lambda_0, \cdots, \lambda_n$ are distinct and $\forall\ 0 \leqslant i \leqslant n$,

$$B_{in} = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{\lambda_j - \lambda_i}$$

For simple birth processes, i.e. $\lambda_n = n\lambda$ with initial condition $B(0) = I$, $B(t) \sim NB(I, 1 - e^{-\lambda t})$

$$\mathbb{P}\{B(t) = n\} = \binom{k-1}{I-1} (e^{-\lambda t})^I (1 - e^{-I\lambda t})^{k-I}$$

4.2.4 Explosion

$$\begin{aligned} \mathbb{P}\{B(t) < \infty\} = 1,\ \forall\ t \geqslant 0 \qquad &\iff \qquad \mathbb{P}\{N(t) < \infty,\ \forall\ t \geqslant 0\} = 1 \\ &\iff \qquad \sum_{i=0}^{\infty} \frac{1}{\lambda_i} = \infty \end{aligned}$$

5 Poisson Processes

5.1 Definition

5.1.1 Memorylessness

If $X \sim Poisson(\lambda)$ for $\lambda > 0$, then

$$\begin{aligned}
\mathbb{P}\{X = n\} &= \frac{\lambda^n}{n!}e^{-\lambda} \qquad n \in \mathbb{N} \\
\mathbb{E}X &= \lambda \\
VarX &= \lambda \\
G_X(s) &= e^{\lambda(s-1)}
\end{aligned}$$

If $Y \sim Exp(\lambda)$ for $\lambda > 0$, then

$$\begin{aligned}
f(y) &= \lambda e^{-\lambda y}\mathbb{1}_{[0,+\infty)}(y) \\
F(y) &= 1 - e^{-\lambda y}\mathbb{1}_{[0,+\infty)}(y) \\
\overline{F}(y) &= 1 - F(y) \\
&= e^{-\lambda y}\mathbb{1}_{[0,+\infty)}(y) \\
\mathbb{E}Y &= \frac{1}{\lambda} \\
VarY &= \frac{1}{\lambda^2}
\end{aligned}$$

Exponential distribution is the only continuous distribution with memoryless property, that is $\forall\ t,s > 0$,

$$\mathbb{P}\{Y > t+s | Y > t\} = \mathbb{P}\{Y > s\}$$

Connection between exponential and Poisson random variables: Consider an i.i.d. sequence $(Y_1, Y_2, \cdots)$ of exponential distributed random variables with parameters $\lambda > 0$. Set $X_n = \sum_{i=1}^{n} Y_i$ and let $\rho \subseteq (0,\infty)$ denote the random set given by $\{X_n : n \in \mathbb{N}^+\}$ which is called a spatial Poisson process of rate λ. Then for $t \geqslant 0$,

$$X = |\rho \cap (0,t)| \sim Poisson(\lambda)$$

5.1.2 Poisson Processes

Possion process is a special case of birth processes. A Poisson process of rate $\lambda > 0$ is an $\mathbb{N}$-valued stochastic process $\{N(t) : t \geqslant 0\}$ for which

(1) Independent Increments

$\forall\ 0 = t_0 < t_1 < \cdots < t_n$, the process increments

$$N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ \cdots,\ N(n_1) - N(t_{n-1})$$

are independent, i.e.,

$$\begin{aligned}\mathbb{P}\{N(s+t)=n+k|N(s)=n\} &= \mathbb{P}\{N(s+t)-N(s)=k|N(s)-N(0)=0\}\\ &= \mathbb{P}\{N(s+t)-N(s)=k\}\end{aligned}$$

(2) Stationary increments

$\forall\ s>0, t\geqslant 0$, $N(s+t)-N(s)\sim Poisson(\lambda t)$, i.e.,

$$\mathbb{P}\{N(s+t)-N(s)=k\}=\frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad k\in\mathbb{N}$$

Or equivalently, as $t\to 0+$,

$$\mathbb{P}\{N(s+t)-N(s)=k\}=\begin{cases}1-\lambda t+o(t) & ,k=0\\ \lambda t+o(t) & ,k=1\\ o(t) & ,k>1\end{cases}$$

(3) Initial Condition

$$N(0)=0$$

Poisson processes are also special cases of renewal processes.

The infinitesimal matrix is given by

$$\mathbf{G}=\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ \vdots\end{array}\begin{array}{c}\begin{array}{ccccc}0 & 1 & 2 & 3 & \cdots\end{array}\\ \begin{pmatrix}-\lambda & \lambda & 0 & 0 & \cdots\\ 0 & -\lambda & \lambda & 0 & \cdots\\ 0 & 0 & -\lambda & \lambda & \cdots\\ 0 & 0 & 0 & -\lambda & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\end{array}$$

5.2 Random Variables

5.2.1 Arrival Times

The nth arrival time (or waiting time) is given by

$$\begin{aligned}&T_0=0\\ &T_n=\inf\{t:N(t)=n\} \qquad n\in\mathbb{N}^+\end{aligned}$$

We have $T_n\sim\Gamma(n,\lambda)$ and

$$N(t)=\max\{n:T_n\leqslant t\}$$

$$\{N(t)\geqslant j\} \qquad \Longleftrightarrow \qquad \{T_j\leqslant t\}$$

5.2.2 Interarrival Times

The nth interarrival time (or sojourn time) is given by

$$X_n = T_n - T_{n-1} \qquad n \in \mathbb{N}^+$$

We have $X_1, X_2, \cdots \overset{i.i.d.}{\sim} Exp(\lambda)$ and

$$T_n = \sum_{i=1}^{n} X_n$$

5.3 Properties

5.3.1 Statistical Properties

The density function for $N(t)$ is

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad \forall\ k \in \mathbb{N}$$

The mean for $N(t)$ is

$$\mathbb{E}N(t) = \lambda t$$

5.3.2 Explosion

The Poisson Processes explodes with probability one, i.e.,

$$\mathbb{P}\{N(t) = \infty \text{ for some } t > 0\} = 1$$

6 Nonhomogeneous Poisson Processes

6.1 Definition

A nonhomogemeous Poisson process of rate function $\lambda(t)$ is an $\mathbb{N}$-valued stochastic process $\{N(t) : t \geqslant 0\}$ for which

(1) Independent Increments

$\forall\ 0 = t_0 < t_1 < \cdots < t_n$, the process increments

$$N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ \cdots,\ N(n_1) - N(t_{n-1})$$

are independent, i.e.,

$$\begin{aligned}\mathbb{P}\{N(s+t) = n+k | N(s) = n\} &= \mathbb{P}\{N(s+t) - N(s) = k | N(s) - N(0) = n\} \\ &= \mathbb{P}\{N(s+t) - N(s) = k\}\end{aligned}$$

(2) Nonstationary Increments

$\forall\ s > 0, t \geqslant 0$, $N(s+t) - N(s) \sim \mathit{Poisson}(\Lambda(s+t) - \Lambda(s))$, i.e.,

$$\mathbb{P}\{N(s+t) - N(s) = k\} = \frac{[\Lambda(s+t) - \Lambda(s)]^k e^{-[\Lambda(s+t)-\Lambda(s)]}}{k!} \qquad k \in \mathbb{N}$$

Or equivalently, as $t \to 0+$,

$$\mathbb{P}\{N(s+t) - N(s) = k\} = \begin{cases} 1 - \lambda(s)t + o(t) & ,k = 0 \\ \lambda(s)t + o(t) & ,k = 1 \\ o(t) & ,k > 1 \end{cases}$$

where

$$\Lambda(s) = \int_0^s \lambda(u)\mathrm{d}u$$

(3) Initial Condition

$$N(0) = 0$$

6.2 Connection between homogeneous and nonhomogemeous Poisson Processes

If $N(t)$ is a nonhomogemeous Poisson process and define the time scaled process $X(s) = X(\Lambda(t)) = N(t)$, then $X(s)$ is a homogeneous Poisson process with rate 1. And therefore, we can concentrate on the properties of homogeneous Poisson processes.

Chapter 3 Renwal Processes

7 Renewal Processes

7.1 Definition

A renewal process is a generalization of the Poisson process. Let $\{X_n : n \in \mathbb{N}^+\}$ be a sequence of independent identically distributed $\mathbb{N}$-valued random variables with shared distribution function $F(x)$ and set $T_0 = 0$ and $T_n = \sum_{i=1}^{n} X_i$ $(n \in \mathbb{N}^+)$. Define the renewal process as

$$
\begin{aligned}
N : [0, \infty) &\longrightarrow \mathbb{N} \\
N(t) &= \max\{n \in \mathbb{N} : T_n \leqslant t\}
\end{aligned}
$$

7.2 Random Variables

7.2.1 Interarrival Times

$\forall\ n \in \mathbb{N}^+$, X_n is the nth interarrival time. The distribution function of X_n is

$$F_{X_n}(x) = F(x)$$

Let

$$
\begin{aligned}
\mathbb{E}X_n &= \mu \\
VarX_n &= \sigma^2
\end{aligned}
$$

7.2.2 Arrival Times

$\forall\ n \in \mathbb{N}^+$, T_n is the nth arrival time. We also have

$$
\begin{aligned}
\{N(t) \geqslant j\} \quad &\Longleftrightarrow \quad \{N(t) > j-1\} \\
&\Longleftrightarrow \quad \{T_j \leqslant t\} \\
\{N(t) \leqslant j-1\} \quad &\Longleftrightarrow \quad \{N(t) < j\} \\
&\Longleftrightarrow \quad \{T_j > t\} \\
\{T_{N(t)} = s\} \quad &\Longleftrightarrow \quad \{X_{N(t)+1} > t-s\} \\
T_{N(t)} \leqslant t < T_{N(t)+1} & \qquad \forall\ t \geqslant 0
\end{aligned}
$$

By convolution, we have

$$
\begin{aligned}
F_{T_1}(x) &= F(x) \\
F_{T_{k+1}}(x) &= \int_0^x F_{T_k}(x-y)\mathrm{d}F(y)
\end{aligned}
$$

7.2.3 Excess Lifetime

The excess lifetime at t is

$$\gamma_t = T_{N(t)+1} - t$$

$$\begin{aligned}
\mathbb{P}\{\gamma_t \leqslant y\} &= F(t+y) - \int_0^t [1 - F(t+y-x)]\mathrm{d}m(x) \\
&= F(t+y) - \sum_{k=1}^{\infty} \int_0^t [1 - F(t+y-x)]\mathrm{d}F_k(x) \qquad\qquad y \geqslant 0
\end{aligned}$$

7.2.4 Current Lifetime

The current lifetime (or age) at t is

$$\delta_t = t - T_{N(t)}$$

$$\begin{aligned}
\mathbb{P}\{\delta_t \leqslant y\} &= F(t) - \int_0^{t-y} [1 - F(t-x)]\mathrm{d}m(x) \\
&= F(t) - \sum_{k=1}^{\infty} \int_0^{t-y} [1 - F(t-x)]\mathrm{d}F_k(x) \qquad\qquad 0 \leqslant y \leqslant t
\end{aligned}$$

since

$$\mathbb{P}\{\delta_t \geqslant y\} = \mathbb{P}\{\gamma_{t-y} > y\}$$

7.2.5 Total Lifetime

The total lifetime at t is

$$\begin{aligned}
\beta_t &= \gamma_t + \delta_t \\
&= X_{N(t)+1}
\end{aligned}$$

7.2.6 Size Biased

$$\begin{aligned}
\mathbb{P}\{T_{N(t)} \leqslant s\} &= \sum_{n=0}^{\infty} \mathbb{P}\{T_n \leqslant s, T_{n+1} > t\} \\
&= \overline{F}(t) + \sum_{n=1}^{\infty} \int_0^{\infty} \mathbb{P}\{T_n \leqslant s, T_{n+1} > t | T_n = y\}\mathrm{d}F_{T_n}(y) \\
&= \overline{F}(t) + \sum_{n=1}^{\infty} \int_0^{s} \overline{F}(t-y)\mathrm{d}F_{T_n}(y) \\
&= \overline{F}(t) + \int_0^{s} \overline{F}(t-y)\mathrm{d}m(y) \\
\mathbb{P}\{T_{N(t)} = 0\} &= \overline{F}(t) \\
\mathrm{d}F_{S(N(t))}(y) &= \overline{F}(t-y)\mathrm{d}m(y) \qquad\qquad 0 < y < \infty
\end{aligned}$$

7.3 Properties

7.3.1 Wald's Equation

If $\{X_n : n \geqslant 0\}$ is an independent and identically distributed sequence of random variables with finite mean. Let M be stopping time with respect to X_i with $\mathbb{E}M < \infty$. Then

$$\mathbb{E}\left(\sum_{n=1}^{N} X_n\right) = \mathbb{E}N \cdot \mathbb{E}X_1$$

which is called Wald's Equation.

7.3.2 Explosion

$$\mathbb{P}\{N(t) < \infty\} = 1, \ \forall\, t \geqslant 0 \qquad \Longleftrightarrow \qquad \mu > 0$$

Below, we only consider the case when $\mathbb{P}\{X_1 > 0\} = 1$ so that the process won't explode.

7.3.3 Statistical Properties

The density function for $N(t)$ is

$$\mathbb{P}\{N(t) = k\} = F_{T_k}(t) - F_{T_{k+1}}(t)$$

The **renewal function** is given by

$$\begin{aligned} m(t) &= \mathbb{E}N(t) \\ &= \sum_{k=1}^{\infty} F_{T_k}(t) \qquad \forall\, t \geqslant 0 \end{aligned}$$

which is the unique solution of the **renewal equation**

$$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x) \qquad \forall\, t \geqslant 0$$

When $X_n \in \mathbb{N}$, the equation becomes

$$m(n) = F(n) + \sum_{k=0}^{n} \mathbb{P}\{X_1 = k\} m(n-k) \qquad \forall\, n \in \mathbb{N}$$

From Wald's Equation, we have

$$\mathbb{E}T_{N(t)+1} = \mu[m(t)+1]$$

7.4 Limiting Theorems

7.4.1 Asymptotic Distribution for $N(t)$

By the Strong Law of Large Number,

$$\frac{T_{N(t)}}{N(t)} \xrightarrow{P} \mu \qquad as \quad t \longrightarrow \infty$$

$$\frac{N(t)}{t} \xrightarrow{P} \frac{1}{\mu} \qquad as \quad t \longrightarrow \infty$$

$$\frac{\frac{N(t)}{t} - \frac{1}{\mu}}{\sqrt{\frac{\sigma^2}{t\mu^3}}} \xrightarrow{D} N(0,1) \qquad as \quad t \longrightarrow \infty$$

7.4.2 Elementary Renewal Theorem

$$\frac{m(t)}{t} \longrightarrow \frac{1}{\mu} \quad as \qquad t \longrightarrow \infty$$