Stochastic Processes

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Week 9

Solutions by

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3.3

(**The inspection paradox**) Express in words what the random variable $X_{N(t)+1}$ represents (**Hint**: It is the length of which renewal interval?)? Show that

$$\mathbb{P}\{X_{N(t)+1} \geqslant x\} \geqslant \overline{F}(x).$$

Compute the above exactly when $F(x) = 1 - e^{-\lambda x}$.

Since $X_{N(t)}$ represents the last interarrival time corresponding to the last renewal prior to or at time t(or the previous renewal interval to the one that contains the point t), $X_{N(t)+1}$ represents the first interarrival time corresponding to the first renewal after time t(or the renewal interval that contains the point t).
Let $F(x)$ be the distribution function of X_1 and $F_{S_{N(t)}}(s)$ be the distribution function of $S_{N(t)}$.

$$\begin{aligned}
\mathbb{P}\{X_{N(t)+1} \geqslant x\} &= \int_0^\infty \mathbb{P}\{X_{N(t)+1} \geqslant x | S_{N(t)} = s\}\mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_{\mathbb{R}} \mathbb{P}\{X_{N(t)+1} \geqslant x | X_{N(t)+1} \geqslant t - s\}\mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_{\mathbb{R}} \frac{\mathbb{P}\{X_{N(t)+1} \geqslant x, X_{N(t)+1} \geqslant t - s\}}{\mathbb{P}\{X_{N(t)+1} \geqslant t - s\}}\mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_{\mathbb{R}} \frac{1 - F(\max\{x, t-s\})}{1 - F(t-s)}\mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_{\mathbb{R}} \min\left\{\frac{1 - F(x)}{1 - F(t-s)}, \frac{1 - F(t-s)}{1 - F(t-s)}\right\}\mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_{\mathbb{R}} \min\left\{\frac{1 - F(x)}{1 - F(t-s)}, 1\right\}\mathrm{d}F_{S_{N(t)}}(s) \\
&\geqslant \int_{\mathbb{R}} [1 - F(x)]\mathrm{d}F_{S_{N(t)}}(s) \\
&= 1 - F(x) \\
&= \overline{F}(x)
\end{aligned}$$

Given $F(x) = 1 - e^{-\lambda x}$, i.e. $X_i \sim Exp(\lambda)$, then $S_n \sim \Gamma(n, \lambda)$,

$$\begin{aligned}
F_n(x) &= \int_0^x \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}\mathrm{d}y \\
m(t) &= \sum_{n=1}^\infty F_n(t) \\
&= \int_0^t \sum_{n=1}^\infty \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}\mathrm{d}y \\
&= \lambda t
\end{aligned}$$

$\because$

$$\begin{aligned}
F_{S_{N(t)}}(x) &= \left[\overline{F}(t) + \int_0^x \overline{F}(t-y)\mathrm{d}m(y)\right]\mathbb{1}_{[0,t]}(x) \\
&= \left[e^{-\lambda t} + \int_0^x \lambda e^{-\lambda(t-y)}\mathrm{d}y\right]\mathbb{1}_{[0,t]}(x) \\
&= e^{-\lambda(t-x)}\mathbb{1}_{[0,t]}(x)
\end{aligned}$$

and $x = 0$ is a discotinous point of $F_{S_{N(t)}}$

$$\mathbb{P}\{S_{N(t)} = 0\} = e^{-\lambda t}$$

Solution (cont.)

$\therefore$ the generalized RiemannStieltjes integral becomes

$$
\begin{aligned}
\mathbb{P}\{X_{N(t)+1} \geqslant x\} &= \int_{\mathbb{R}} \min\left\{\frac{1-F(x)}{1-F(t-s)}, 1\right\} \mathrm{d}F_{S_{N(t)}}(s) \\
&= \int_0^t \min\left\{\frac{e^{-\lambda x}}{e^{-\lambda(t-s)}}, 1\right\} f_{S_{N(t)}}(s)\mathrm{d}s + \min\left\{\frac{1-F(x)}{1-F(t)}, 1\right\} \mathbb{P}\{S_{N(t)} = 0\} \\
&= \int_0^t \min\{e^{-\lambda(x-t+s)}, 1\} f_{S_{N(t)}}(s)\mathrm{d}s + \min\left\{\frac{1-F(x)}{1-F(t)}, 1\right\} e^{-\lambda t} \\
&= \begin{cases} \int_0^{t-x} f_{S_{N(t)}}(s)\mathrm{d}s + \int_{t-x}^t e^{-\lambda(x-t+s)}\mathrm{d}F_{S_{N(t)}}(s) + e^{-\lambda t} & ,t > x \\ \int_0^t e^{-\lambda(x-t+s)} f_{S_{N(t)}}(s)\mathrm{d}s + e^{-\lambda x} & ,t \leqslant x \end{cases} \\
&= \begin{cases} F_{S_{N(t)}}(t-x) - F_{S_{N(t)}}(0) + \int_{t-x}^t e^{-\lambda(x-t+s)} f_{S_{N(t)}}(s)\mathrm{d}s + e^{-\lambda t} & ,t > x \\ \int_0^t e^{-\lambda(x-t+s)} f_{S_{N(t)}}(s)\mathrm{d}s + e^{-\lambda x} & ,t \leqslant x \end{cases} \\
&= \begin{cases} e^{-\lambda x} + \int_{t-x}^t \lambda e^{-\lambda x}\mathrm{d}s & ,t > x \\ \int_0^t \lambda e^{-\lambda x}\mathrm{d}s + e^{-\lambda x} & ,t \leqslant x \end{cases} \\
&= \begin{cases} (\lambda x + 1)e^{-\lambda x} & ,t > x \\ (\lambda t + 1)e^{-\lambda x} & ,t \leqslant x \end{cases} \\
&\geqslant e^{-\lambda x} \\
&= \overline{F}(x)
\end{aligned}
$$

Note: $X_{N(t)+1}$ will dependent on $t - S_{N(t)}$. Consider the buses arrives at one perticular bus stop one by one, and their interarrival times $X_n (n \in \mathbb{N}^+)$ are independent distributed with the same distribution, for example,

$$f_{X_1}(x) = \frac{1}{2}\mathbb{1}_{\{0.01, 1.99\}}(x)$$

i.e.

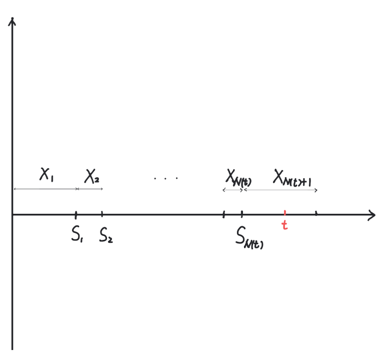
$$f_{X_1}(0.01) = f_{X_1}(1.99) = \frac{1}{2}$$

and $f_{X_1} = 0$ otherwise.

Then intuitively,

$$\mathbb{P}\{X_{N(t)+1} \geqslant 1\} \geqslant \overline{F}_{X_1}(1) = \frac{1}{2}$$

meaning that when a person arrives at the bus stop at time t and the previous bus he missed left at time $S_{N(t)}$, the probability of the $[N(t)+1]$th interarrival time is bigger than 1 is $\mathbb{P}\{X_{N(t)+1} \geqslant 1\}$, will be larger or equals to $\overline{F}(1)$ since there will be more likely for the man come to the longer interarrival interval $(S_{N(t)}, S_{N(t)+1}]$, i.e. $X_{N(t)+1} = 1.99$



3.4

Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x).$$

$\because$

$$X_1 > t \quad \Longrightarrow \quad N(t) = 0$$

$\therefore$

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)] \\ &= \mathbb{E}\{\mathbb{E}[N(t)]|X_1\} \\ &= \int_0^t \mathbb{E}[N(t)|X_1 = x]\mathrm{d}F(x) \end{aligned}$$

$\because$ given $X_1 = x$,

$$\begin{aligned} N(t) &= N(x) + N(t) - N(x) \\ &= 1 + N(t-x) \end{aligned}$$

$\therefore$

$$\begin{aligned} m(t) &= \int_0^t \mathbb{E}[1 + N(t-x)]\mathrm{d}F(x) \\ &= \int_0^t [1 + m(t-x)]\mathrm{d}F(x) \\ &= F(t) + \int_0^t m(t-x)\mathrm{d}F(x) \end{aligned}$$