(1a) Let

$$f(x) = \frac{1}{\sqrt{|x|}}$$

for $0 \neq x \in R$. Prove or disprove that $f \in L^2(R)$.

(1b) Let

$$f(x) = \frac{1}{\sqrt{|x|}}$$

for $0 \neq x \in R$. Define and compute the Fourier transform $\hat{f}(k)$ (up to a multiplicative constant). Prove or disprove that $\hat{f} \in L^2(R)$.

(1c) Evaluate

$$C = \int_0^\infty \cos(x^2)\,dx.$$

(Hint: consider $\int \exp((i-t)x^2)$.)

(2a) Let ϕ be a nice function satisfying

$$\phi(x) = \sum_{k=0}^{m} p_k \phi(2x - k)$$

for all $x \in R$ and some finite set of constants $p_0, p_1, \ldots, p_m$. Find a function p such that the Fourier transform $\hat{\phi}(\lambda)$ satisfies

$$\hat{\phi}(\lambda) = p(\lambda)\hat{\phi}(\lambda/2)$$

for all $\lambda \in R$.

(2b) Assume $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. Compute the first moment

$$\mu_1 = \int_{-\infty}^{\infty} x\phi(x)\,dx$$

in terms of the constants p_k.

(3a) For $x \in R$ and $n \in Z$ define the Bessel function $J_n(x)$ by

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin\theta} e^{-in\theta}\, d\theta.$$

For each fixed $x \in R$ compute the Fourier series of $f(\theta) = e^{ix\sin\theta}$ in terms of $J_n(x)$ and prove that it converges uniformly for $|\theta| \leq \pi$.

(3b) Prove the addition formula

$$J_n(x+y) = \sum_{m \in Z} J_m(x) J_{n-m}(y).$$

Use uniform convergence if necessary.

(4a) Sum a geometric series to evaluate

$$e_n(x) = \sum_{j=-n}^{n} e^{ijx}.$$

(4b) Let f be a function in $L^2(-\pi, \pi)$. Show that

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin((n+1/2)(x-y))}{\sin((x-y)/2)} f(y)\, dy.$$

(4c) Show that

$$\frac{\sin x}{x} = \prod_{j=1}^{\infty} \cos\left(\frac{x}{2^j}\right)$$