Stochastic Processes

Fall 2017

•

Week 7

•

Solutions by

Jinhong Du

15338039

3.5

Prove that the renewal function $m(t), 0 \leqslant t < \infty$ uniquely determines the interarrival distribution F.

$\because$

$$X_1 > t \quad \Longrightarrow \quad N(t) = 0$$

$\therefore$

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)] \\ &= \mathbb{E}\{\mathbb{E}[N(t)]|X_1\} \\ &= \int_0^t \mathbb{E}[N(t)|X_1 = x]\mathrm{d}F(x) \\ &= \int_0^t \mathbb{E}[1 + N(t-x)]\mathrm{d}F(x) \\ &= \int_0^t [1 + m(t-x)]\mathrm{d}F(x) \\ &= F(t) + \int_0^t m(t-x)\mathrm{d}F(x) \end{aligned} \tag{1}$$

The Laplace transform of F is

$$\tilde{F}(s) = \int_0^\infty e^{-sx}\mathrm{d}F(x)$$

the Laplace transform of the convolution $(F * G)(t) = \int_0^\infty F(t-s)\mathrm{d}G(s)$ is

$$\begin{aligned} \widetilde{F * G}(s) &= \int_0^\infty e^{-st}\mathrm{d}\left(\int_0^\infty F(t-x)\mathrm{d}G(x)\right) \\ &= \int_0^\infty e^{-st}\int_0^\infty \mathrm{d}F(t-x)\mathrm{d}G(x) \\ &= \int_0^\infty \int_x^\infty e^{-st}\mathrm{d}F(t-x)\mathrm{d}G(x) \\ &\xlongequal{t-x=y} \int_0^\infty \int_0^\infty e^{-s(x+y)}\mathrm{d}F(y)\mathrm{d}G(x) \\ &= \int_0^\infty e^{-sy}\mathrm{d}F(y)\int_0^\infty e^{-sx}\mathrm{d}G(x) \\ &= \tilde{F}(s)\tilde{G}(s) \end{aligned}$$

$\therefore$ the Laplace transform of $m(t)$ is

$$\tilde{m}(s) = \tilde{F}(s) + \tilde{m}(s)\tilde{F}(s)$$

$\because$

$$\tilde{F}(s) = \frac{\tilde{m}(s)}{1 + \tilde{m}(s)} \tag{2}$$

$\therefore$ By the uniqueness of Laplace transforms, $m(t)$ uniquely determines F

3.6

Let $\{N(t), t \geqslant 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t) = n$, the event times $S_1, \cdots, S_n$ are distributed as the order statistics of a set of independent uniform $(0, t)$ random variables. Show that $\{N(t), t \geqslant 0\}$ is a Poisson process.

(**Hint:** Consider $\mathbb{E}[N(s)|N(t)]$ and then use the result of Problem 3.5.)

Let $U_{(1)}, \cdots, U_{(n)}$ denote the order statistics of n independent identical distributed random varables $U_1, \cdots, U_n$ with uniform distribution in $[0, t]$.

$\because$

$$\mathbb{E}[N(s)|N(t) = 0] = 0$$

for $n \in \mathbb{N}^+$,

$$\begin{aligned}
\mathbb{E}[N(s)|N(t) = n] &= \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{I}_{\{S_i \leqslant s\}} | N(t)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{I}_{\{U_{(i)} \leqslant s\}}\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\mathbf{I}_{\{U_{(i)} \leqslant s\}} \\
&= \sum_{i=1}^{n} \mathbb{P}\{U_{(i)} \leqslant s\} \\
&= \sum_{i=1}^{n} \frac{s}{t} \\
&= \frac{ns}{t}
\end{aligned}$$

$\therefore \quad \forall\ t, s \geqslant 0,$

$$\begin{aligned}
m(s) &= \mathbb{E}\{\mathbb{E}[N(s)|N(t)]\} \\
&= \frac{s}{t}\mathbb{E}[N(t)] \\
&= \frac{s}{t}m(t)
\end{aligned}$$

$\therefore$

$$m(s) = \lambda s$$

where λ is a constant

Suppose that $\{X(t), t \geqslant 0\}$ is a Poisson process with parameter λ.

$\because$

$$\begin{aligned}
m_X(t) &= \mathbb{E}[N(t)] \\
&= \sum_{n=0}^{\infty} n\mathbb{P}\{N(t) = n\} \\
&= \sum_{n=0}^{\infty} n\frac{(\lambda t)^n}{n!}e^{-\lambda t} \\
&= \lambda t
\end{aligned}$$

$\therefore \quad$ from 3.5, we have $\{N(t), t \geqslant 0\}$ is a Poisson process

3.7

If F is the uniform $(0,1)$ distribution function show that

$$m(t) = e^t - 1, \qquad 0 \leqslant t \leqslant 1$$

Now argue that the expected number of uniform $(0,1)$ random variables that need to be added until their sum exceeds 1 has mean e.

$\because \quad \forall\ t \in [0,1]$,

$$\begin{aligned} F_2(t) &= \int_0^t F(t-x)\mathrm{d}F(x) \\ &= \int_0^t (t-x)\mathrm{d}x \\ &= \frac{t^2}{2} \\ F_3(t) &= \int_0^t F_2(t-x)\mathrm{d}F(x) \\ &= \int_0^t \frac{(t-x)^2}{2}\mathrm{d}x \\ &= \frac{t^3}{3!} \end{aligned}$$

Suppose that for $n = k \in \mathbb{N}^+$,

$$F_k(t) = \frac{t^k}{k!}$$

then for $n = k+1$

$$\begin{aligned} F_{k+1}(t) &= \int_0^t F_k(t-x)\mathrm{d}F(x) \\ &= \int_0^t \frac{(t-x)^k}{k!}\mathrm{d}x \\ &= \frac{t^{k+1}}{(k+1)!} \end{aligned}$$

by induction we have $\forall\ t \in [0,1],\ n \in \mathbb{N}^+$,

$$F_n(t) = \frac{t^n}{n!}$$

$\therefore \quad \forall\ t \in [0,1]$,

$$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} F_n(t) \\ &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\ &= e^t - 1 \end{aligned}$$

Let N denote the number of uniform $(0,1)$ random variables that need to be added until their sum exceeds 1.

Solution (cont.)

$$
\begin{aligned}
\mathbb{E}N &= \sum_{n=2}^{\infty} n\mathbb{P}(N = n) \\
&= \sum_{n=2}^{\infty} n\mathbb{P}\{S_{n-1} \leqslant 1, S_n > 1\} \\
&= \sum_{n=2}^{\infty} n\left[\mathbb{P}\{S_{n-1} \leqslant 1\} - \mathbb{P}\{S_n \leqslant 1\}\right] \\
&= \sum_{n=2}^{\infty} n\left[F_{n-1}(1) - F_n(1)\right] \\
&= F_1(1) + \sum_{n=1}^{\infty} F_n(1) \\
&= F_1(1) + m(1) \\
&= 1 + (e - 1) \\
&= e
\end{aligned}
$$