118.F17 Problem Set 07 Solutions

Q1 (a)

$$\left| \frac{1}{2\pi} \sum_{|k| \geq 6} e^{-tk^2} e^{ikx} \right| \leq \frac{1}{2\pi} \sum_{|k| \geq 6} e^{-tk^2}$$

$$\leq \frac{1}{\pi} \sum_{k=6}^{\infty} e^{-\delta 6 k} \qquad t \geq \delta$$

$$= \frac{1}{\pi} \frac{e^{-36\delta}}{1 - e^{-6\delta}} \leq 10^{-14} \text{ if}$$

$$e^{-36\delta} \leq 10^{-14}$$

$$-36\delta \leq \ln(10^{-14})$$

$$36\delta \geq -\ln(10^{-14})$$

$$36\delta \geq 14 \ln 10$$

$$\delta \geq \frac{14}{36} \ln 10 = \boxed{0.895}$$

(b) $\left| \sum_{|k|\geq 6} e^{-(x-2\pi k)^2/4t} \right|$

$$\leq 2 \sum_{k=6}^{\infty} e^{-(2k-1)^2\pi^2/4\Delta}$$

$$\leq 2 \sum_{k=6}^{\infty} e^{-(2k-1)11\pi^2/4\Delta}$$

$$= 2 e^{11\pi^2/4\Delta} \sum_{k=6}^{\infty} \left(e^{-11\pi^2/2\Delta}\right)^k$$

$$= 2 e^{11\pi^2/4\Delta} \cdot \frac{e^{-33\pi^2/2\Delta}}{1-e^{-11\pi^2/2\Delta}}$$

$$= \frac{2}{1-e^{-11\pi^2/2\Delta}} \cdot e^{-55\pi^2/4\Delta} \leq 10^{-14}$$

if

$$e^{-55\pi^2/4\Delta} \leq 10^{-14}$$

$$-55\pi^2/4\Delta \leq \ln(10^{-14})$$

$$55\pi^2/4\Delta \geq 14 \ln 10$$

$$\Delta \leq \frac{55\pi^2}{56 \ln 10} = \boxed{4.21}$$

Fortunately $\delta < \Delta$.

(c) Use (a) for $t \geq \varepsilon$

and (b) for $t \leq \varepsilon$

where $\delta < \varepsilon < \Delta$.

Get 14-digit accuracy for 11 terms (each costing one exponential per x value).

Q2 (a) The PSF with $x=0$, $T=1$ reads

$$\sum_{-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{-\infty}^{\infty} \hat{f}(2\pi k),$$

Given $f : [0,\infty) \to \mathbb{C}$, extend f to be <u>even</u> by

$$f_e(-x) = f(x) \quad \text{for } x > 0.$$

Then

$$\sum_{-\infty}^{\infty} f_e(n) = f(0) + 2\sum_{1}^{\infty} f(n)$$

and

$$\hat{f}_e(k) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(-x)\, e^{-ikx}\, dx + \int_0^{\infty} f(x)\, e^{-ikx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x)\left(e^{ikx} + e^{-ikx}\right) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos(kx)\, dx$$

Integrating by parts twice gives

$$\int_0^\infty f(x)\, \frac{d}{dx}\frac{\sin(kx)}{k}\, dx = -\int_0^\infty f'(x)\, \frac{\sin kx}{k}\, dx$$

$$= +\int_0^\infty f'(x)\, \frac{d}{dx}\frac{+\cos kx}{k^2}\, dx$$

$$= f'(x)\, \frac{\cos kx}{k^2}\Big|_0^\infty - \int_0^\infty f''(x)\, \frac{\cos kx}{k^2}\, dx$$

$$= -f'(0)\frac{1}{k^2} - \frac{1}{k^2}\int_0^\infty f''(x)\cos kx\, dx$$

and iterating the procedure gives

$$\int_0^\infty f(x)\cos kx\, dx = -f'(0)\frac{1}{k^2} - \frac{1}{k^2}\cdot$$

$$\left(-f'''(0)\frac{1}{k^2} - \frac{1}{k^2}\int_0^\infty f''''(x)\cos kx\right)$$

$$= -f'(0)\frac{1}{k^2} + f'''(0)\frac{1}{k^4} + \frac{1}{k^4}\int_0^\infty f''''\cos kx\, dx.$$

Summing up,

$$f(0) + 2\sum_{1}^{\infty} f(n) = 2\sum_{n=0}^{\infty} f(n) - f(0)$$

$$= \sqrt{2\pi} \sum_{-\infty}^{\infty} \hat{f}(2\pi k)$$

$$= \sqrt{2\pi}\left(\frac{2}{\sqrt{2\pi}}\int_0^{\infty} f(x)\,dx\right.$$

$$\left. + \sum_{k\neq 0} \hat{f}(2\pi k)\right)$$

$$= 2\int_0^{\infty} f(x)\,dx + \sqrt{2\pi}\sum_{k\neq 0} \hat{f}_e(2\pi k),$$

$$= 2\int_0^{\infty} f(x)\,dx + \sum_{k\neq 0} 2\int_0^{\infty} f(x)\cos(2\pi kx)$$

Hence

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\,dx$$

$$+ 2\sum_{k=1}^{\infty} \int_0^{\infty} f(x)\cos(2\pi kx)\,dx.$$

Since
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$
and
$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},$$
we have

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\,dx$$

$$+ 2 \sum_{k=1}^{\infty} - f'(0) \frac{1}{(2\pi k)^2} + f'''(0) \frac{1}{(2\pi k)^4}$$

$$+ \frac{1}{(2\pi k)^4} \int_0^{\infty} f'''' \cos(2\pi k x)\,dx$$

$$\boxed{= \frac{1}{2} f(0) + \int_0^{\infty} f(x)\,dx - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) - \cdots}$$

(b) For $f(x) = e^{-tx}$ we have

$$\sum_{n=0}^{\infty} e^{-tn} = \frac{1}{1-e^{-t}}$$

$$= \frac{1}{2} + \frac{1}{t} - \sum_{1}^{\infty} \frac{B_{2k}}{(2k)!} (-t)^{2k-1},$$

so multiplying by t gives

$$\frac{t}{1-e^{-t}} = 1 + \frac{t}{2} + \sum_{1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k},$$

Thus we can read off B_{2k} from the Taylor expansion of $t/1-e^{-t}$:

$$\frac{t}{1-e^{-t}} = 1 + \frac{t}{2} + \frac{B_2}{2!} t^2 + \frac{B_4}{4!} t^4 + \ldots$$

Cross-multiply by $1-e^{-t} = t - \frac{t^2}{2!} + \frac{t^3}{3!} - \ldots$

to get

$$1 = \left(1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \dots\right) \cdot$$

$$\cdot \left(1 + \frac{t}{2} + \frac{B_2}{2!}t^2 + \frac{B_4}{4!}t^4 + \dots\right).$$

Equating coefficients,

$1 = 1$ ✓

$0 = \frac{1}{2} - \frac{1}{2!}$ ✓

$0 = \frac{1}{3!} - \frac{1}{2}\frac{1}{2!} + \frac{B_2}{2!}$ $\qquad B_2 = \frac{1}{6}$

$0 = -\frac{1}{4!} + \frac{1}{3!}\frac{1}{2} - \frac{1}{2!}B_2 + \frac{B_4}{4!}$ $\qquad B_4 = 1$

and in general

$$\boxed{\frac{B_{2k}}{(2k)!} = \frac{B_{2k-2}}{(2k-2)!} + \frac{B_{2k-4}}{(2k-4)!} - \dots \pm \frac{1}{2}\cdot\frac{1}{(2k-1)!} \mp \frac{1}{(2k)!}.}$$

Q3. (a) $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} e^{-ikx} \, dx$

Let $x = \sqrt{4t}\, y$ so $dx = \sqrt{4t}\, y$ and

$$\hat{f}(k) = \frac{1}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{-y^2} e^{-ik\sqrt{4t}\, y} \, dy$$

$$= \hat{g}\left(\sqrt{4t}\, k\right)$$

when

$$\hat{g}(k) = \frac{1}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{-y^2} e^{-iky} \, dy,$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-ik)^m}{m!} \int_{-\infty}^{\infty} e^{-y^2} y^m \, dy$$

$$= \frac{1}{\sqrt{2}\,\pi} \sum_{m=0}^{\infty} \frac{(-ik)^{2m}}{(2m)!} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} y^{2m} \, dy}_{\Gamma(m+1/2) = \frac{(2m-1)!}{2^{2m-1}(m-1)!}\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-ik)^{2m}}{(2m)\, 2^{2m-1}\, (m-1)!}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{(-ik/2)^2}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-k^2/4},$$

so $$\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\, e^{-tk^2}.}$$

(b) Let

$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{-y^2} e^{-iky}\, dy$$

so

$$\hat{g}'(k) = \int_{-\infty}^{\infty} e^{-y^2} (-iy) e^{-iky}\, dy$$

$$= \frac{i}{2} \int_{-\infty}^{\infty} \frac{d}{dy}\left(e^{-y^2}\right) e^{-iky}\, dy$$

$$= -\frac{i}{2} \int_{-\infty}^{\infty} e^{-y^2} (-ik) e^{-iky}\, dy$$

$$= -\frac{k}{2}\, \hat{g}(k)$$

Since

$$\hat{g}(0) = \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi},$$

$$\hat{g}(k) = e^{-k^2/4} \sqrt{\pi}$$

and

$$\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\, e^{-tk^2}}$$

(c) $G_t = +\frac{x^2}{4t^2}G + \left(-\frac{1}{2}\right)\frac{1}{t}G$

since $G = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$.

Similarly

$$G_x = -\frac{2x}{4t}G = -\frac{x}{2t}G$$

so

$$G_{xx} = -\frac{1}{2t}G - \frac{x}{2t}G_x$$

$$= -\frac{1}{2t}G - \frac{x}{2t}\left(-\frac{x}{2t}\right)G$$

$$= G_t.$$

Hence G satisfies the heat equation.

(d)
$$\int_{-\infty}^{\infty} G(x-y,t)\, f(y)\, dy$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\, dy$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} f(x+\sqrt{4t}\, s)\, ds.$$

Since f is continuous,

$$e^{-s^2} f(x+\sqrt{4t}\, s) \to e^{-s^2} f(x)$$

as $t \downarrow 0^+$. Since f is bounded, the integrand is dominated by a multiple of e^{-s^2} which is integrable. By the Dominated Convergence Theorem,

$$\int_{-\infty}^{\infty} G(x-y,t)\, f(y)\, dy \to f(x)$$

as $t \downarrow 0^+$.

(e) Multiply by e^{-ikx} and integrate:

$$\hat{u}_t = -k^2\hat{u} + \hat{\rho}(k,t)$$

Apply integrating factor e^{k^2t} so

$$\left(e^{k^2t}\hat{u}\right)_t = e^{k^2t}\hat{\rho}(k,t)$$

or

$$\hat{u}(k,t) = \int_0^t e^{-k^2(t-s)}\hat{\rho}(k,s)\,ds.$$

Fourier transform back to get

$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4(t-s)}}{\sqrt{4\pi(t-s)}}\,\rho(y,s)\,dy\,ds}$$

Q4.(a) $G(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$

$\hat{G}(k,t) = \dfrac{1}{\sqrt{2\pi}} e^{-tk^2}$

Solve $G = \hat{G}$ for t:

$t = \dfrac{1}{4t}$ $\quad t^2 = \dfrac{1}{4}$ $\quad \boxed{t = 1/2}$

$\Rightarrow \quad \hat{G}(k,t) = \dfrac{1}{\sqrt{2\pi}} e^{-k^2/2}$

$G(x,t) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$ ✓

$\lambda = 1$.

(b) The PSF reads

$$\sum_k e^{-(x+kT)^2/2} = \frac{\sqrt{2\pi}}{T} \sum_k e^{-(2\pi k/T)^2/2} e^{2\pi i k x/T}.$$

Following the hint, write

$$k = p + qN \qquad 0 \le p < N, \; q \in \mathbb{Z}$$

so that

$$\sum_k = \sum_{p=0}^{N-1} \sum_{q\in\mathbb{Z}}$$

and

$$e^{2\pi i k x/T} = e^{2\pi i (p+qN)x/T}.$$

If $x/T = j/N$ where $0 \le j < N$ then

$$e^{2\pi i (p+qN) j/N} = e^{2\pi i p j/N}$$

is (a) independent of q and

(b) $\sqrt{N}$ times the discrete Fourier transform matrix F. Hence

$$x + kT = T\left(\frac{x}{T} + k\right)$$
$$= T(j/N + k)$$

so

$$\sum_k e^{-(x+kT)^2/2} = \sum_k e^{-T^2(j/N+k)^2/2}$$

$$= \frac{\sqrt{2\pi}}{T} \sum_{p=0}^{N-1} e^{2\pi i j p/N} \sum_q e^{-(2\pi(p+qN))^2/2T^2}$$

If $T^2/2 = (2\pi)^2/2T^2 N^2$ so $T = \sqrt{2\pi N}$
then $T^2/2 = \pi N$ and

$$\sum_k e^{-\pi(j+kN)^2/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} e^{2\pi i j p/N} \sum_q e^{-\pi(p+qN)^2/N}$$

so

$$\boxed{g_j = \sum_k e^{-\pi(j+kN)^2/N}} \quad 0 \le j < N$$

is the desired eigenvector and $\boxed{\lambda = 1}$.

Question 1 Suppose you can only afford to evaluate 11 terms of either side of the Poisson Sum Formula

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} \mathrm{e}^{-(x-2\pi k)^2/4t} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \mathrm{e}^{-tk^2} \mathrm{e}^{ikx}.$$

(a) Find δ such that the error in the right-hand side (truncated after 11 terms) is smaller than 10^{-14} for $t \geq \delta$ and $|x| \leq \pi$.

(b) Find $\Delta > \delta$ such that $\sqrt{4\pi t}$ times the error in the left hand side (truncated after 11 terms) is smaller than 10^{-14} for $0 < t \leq \Delta$ and $|x| \leq \pi$.

(c) Invent an efficient strategy for evaluating $K(x,t)$ accurately for any $t > 0$ and $|x| \leq \pi$.

Question 2 (a) Use the Poisson Sum Formula to prove the Euler-Maclaurin summation formula

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\ \mathrm{d}x - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) - \cdots$$

for a smooth function f. (Hint: extend f to be even.)

(b) Find formulas for the rest of the coefficients B_{2k} in

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\ \mathrm{d}x - \sum_{1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0)$$

by applying the formula to a suitable test function like $f(x) = \mathrm{e}^{-tx}$.

Question 3 Fix $t > 0$ and let

$$G(x,t) = \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi t}}.$$

(a) Compute $\hat{G}(k,t)$.

(b) Compute $\hat{G}(k,t)$ by a different method.

(c) Show that

$$G_t = G_{xx}$$

for $t > 0$.

(d) Let $f \in L^2(R)$ be continuous and bounded. Show that

$$\int_{-\infty}^{\infty} G(x-y,t) f(y)\ \mathrm{d}y \to f(x)$$

for every $x \in R$ as $t \to 0$.

(e) Solve the inhomogeneous initial-value problem

$$u_t = u_{xx} + \rho(x,t)$$

for $x \in R$, $t > 0$, subject to the intial condition

$$u(x,0) = 0.$$

Question 4 (a) Find $t > 0$ such that the Gaussian $G(x,t)$ from Question 3 is an eigenfunction of the Fourier transform.

(b) Let F be the $N \times N$ *discrete* Fourier transform matrix with elements

$$F_{jk} = \frac{1}{\sqrt{N}} \mathrm{e}^{2\pi \mathrm{i} jk/N}$$

for $0 \leq j,k \leq N-1$. Apply the Poisson Sum Formula to $G(x,t)$ and choose parameters x and T to find a formula for an eigenvector $g \in C^N$ and eigenvalue $\lambda \in C$ of F. (Hint: write the index of summation $k = p + qN$ and the sum over k as a double sum over $p = 0$ to $N-1$ and $q \in Z$.)