STAT 150: Stochastic Processes

Fall 2017

•

Homework 1

Due Sep 6 at 11:59pm

•

Solutions by

Jinhong Du

3033483677

1.3.11

Let X and Y be independent random variables sharing the geometric distribution whose mass function is

$$p(k) = (1-\pi)\pi^k \qquad k = 0, 1, \cdots$$

where $0 < \pi < 1$. Let $U = \min\{X, Y\},\ V = \max\{X, Y\}$ and $W = V - U$. Determine the joint probability mass function for U and W and show that U and W are independent.

Proof.

$\because \quad X, Y \overset{iid}{\sim} G(\pi)$

$\therefore \quad$ the mass distribution functions of U and V are

$$\begin{aligned}
p_U(k) &= P(\min\{X,Y\} = k) \\
&= P(\min\{X,Y\} = k, \max\{X,Y\} \geqslant k) \\
&= P(\min\{X,Y\} = k, \max\{X,Y\} > k) + P(\min\{X,Y\} = k, \max\{X,Y\} = k) \\
&= \binom{2}{1} p(k) \sum_{j=k+1}^{+\infty} p(j) + p(k)p(k) \\
&= 2(1-\pi)\pi^{2k+1} + (1-\pi)^2\pi^{2k} \\
&= \pi^{2k}(1-\pi^2) \\
p_V(k) &= P(\max\{X,Y\} = k) \\
&= P(\min\{X,Y\} \leqslant k, \max\{X,Y\} = k) \\
&= P(\min\{X,Y\} < k, \max\{X,Y\} = k) + P(\min\{X,Y\} = k, \max\{X,Y\} = k) \\
&= \binom{2}{1} \sum_{j=0}^{k-1} p(j)p(k) + p(k)p(k) \\
&= 2(1-\pi)(1-\pi^k)\pi^k + (1-\pi)^2\pi^{2k} \\
&= \pi^k(\pi-1)(\pi^k + \pi^{k+1} - 2)
\end{aligned}$$

The joint probability mass function for U and V is

$$\begin{aligned}
p_{U,V}(u,v) &= P(\min\{X,Y\} = u, \max\{X,Y\} = v) \\
&= \begin{cases} \binom{2}{1} p(u)p(v), & u < v,\ u, v \in N \\ p(u)p(v), & u = v,\ u, v \in N \\ 0, & otherwise \end{cases} \\
&= \begin{cases} 2(1-\pi)^2\pi^{u+v}, & u < v,\ u, v \in N \\ (1-\pi)^2\pi^{2u}, & u = v,\ u, v \in N \\ 0, & otherwise \end{cases}
\end{aligned}$$

let $V = W + U$, we get

$$p_{U,W}(u,w) = \begin{cases} 2(1-\pi)^2\pi^{2u+w}, & w > 0,\ u, w \in N \\ (1-\pi)^2\pi^{2u}, & w = 0,\ u \in N \\ 0, & otherwise \end{cases}$$

Solution (cont.)

The mass distribution function of W is

$$\begin{aligned} p_W(w) &= \sum_{u=0}^{+\infty} p_{U,W}(u,w) \\ &= \begin{cases} \dfrac{2(1-\pi)^2\pi^w}{1-\pi^2}, & w>0,\ w\in N \\ \dfrac{(1-\pi)^2}{1-\pi^2}, & w=0 \\ 0, & otherwise \end{cases} \end{aligned}$$

$\because \quad p_U(u)p_W(w) = p_{U,W}(u,w)$

$\therefore \quad U$, V are independent. □

2.1.2

A card is picked at random from N cards labeled $1, 2, \cdots, N$, and the number that appears is X. A second card is picked at random from cards numbered $1, 2, \cdots, X$ and its number is Y. Determine the conditional distribution of X given $Y = y$, for $y = 1, 2, \cdots$.

Proof.

$$\begin{aligned} P(X=x|Y=y) &= \frac{P(X=x,Y=y)}{P(Y=y)} \\ &= \frac{P(Y=y|X=x)P(X=x)}{\sum\limits_{j=y}^{N} P(Y=y|X=j)} \\ &= \begin{cases} \dfrac{P(Y=y|X=x)P(X=x)}{\sum\limits_{j=y}^{N} P(Y=y|X=j)P(X=j)} & ,y\leqslant x\leqslant N \\ 0 & ,x<y \end{cases} \\ &= \begin{cases} \dfrac{\frac{1}{x}\cdot\frac{1}{N}}{\sum\limits_{j=y}^{N}\frac{1}{j}\cdot\frac{1}{N}} & ,y\leqslant x\leqslant N \\ 0 & ,x<y \end{cases} \\ &= \begin{cases} \dfrac{1}{x\sum\limits_{j=y}^{N}\frac{1}{j}} & ,y\leqslant x\leqslant N \\ 0 & ,x<y \end{cases} \end{aligned}$$

□

2.3.3

Suppose that $\xi_1, \xi_2, \cdots$ are independent and identically distributed with $Pr\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2 \cdots$ and follow the geometric probability mass function

$$p_N(k) = \alpha(1-\alpha)^k \qquad \text{for } k = 0, 1, \cdots$$

where $0 < \alpha < 1$. Form the random sum $Z = \xi_1 + \cdots + \xi_N$.

(a) Determine the mean and variance of Z.

$\because$

$$\begin{aligned}
E\xi_i &= 1 \cdot Pr\{\xi_i = 1\} - 1 \cdot Pr\{\xi_i = -1\} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0 \\
Var\xi_i &= 1^2 Pr\{\xi_i = 1\} + (0-1)^2 Pr\{\xi_i = -1\} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1 \\
EN &= \sum_{k=0}^{+\infty} k p_N(k) \\
&= \sum_{k=0}^{+\infty} k\alpha(1-\alpha)^k \\
&= \frac{1-\alpha}{\alpha} \\
VarN &= EN^2 - (EN)^2 \\
&= \sum_{k=0}^{+\infty} k^2 p_N(k) - \left(\frac{1-\alpha}{\alpha}\right)^2 \\
&= \sum_{k=0}^{+\infty} k^2 \alpha(1-\alpha)^k - \left(\frac{1-\alpha}{\alpha}\right)^2 \\
&= \frac{(2-\alpha)(1-\alpha)}{\alpha^2} - \left(\frac{\alpha}{1-\alpha}\right)^2 \\
&= \frac{1-\alpha}{\alpha^2}
\end{aligned}$$

Assume that $Z = 0$ when $N = 0$.

$\therefore$

$$\begin{aligned}
EZ &= \sum_{k=0}^{+\infty} E(Z|N=k) Pr\{N=k\} \\
&= \sum_{k=1}^{+\infty} E(\xi_1 + \cdots + \xi_k) Pr\{N=k\} \\
&= \sum_{k=1}^{+\infty} \sum_{j=1}^{k} E\xi_j Pr\{N=k\} \\
&= 0
\end{aligned}$$

Solution (cont.)

$$\begin{aligned}
VarZ &= E(Z-EZ)^2 \\
&= EZ^2 \\
&= \sum_{k=1}^{+\infty} E(Z^2|N=k)Pr\{N=k\} \\
&= \sum_{k=1}^{+\infty} Var(Z|N=k)Pr\{N=k\} \\
&= \sum_{k=1}^{+\infty}\sum_{j=1}^{k} Var(\xi_j)Pr\{N=k\} \\
&= \sum_{k=1}^{+\infty} k\alpha(1-\alpha)^k \\
&= \alpha(1-\alpha)\sum_{k=1}^{+\infty} k(1-\alpha)^{k-1} \\
&= \alpha(1-\alpha)\frac{\mathrm{d}}{\mathrm{d}(1-\alpha)}\left[\sum_{k=1}^{+\infty}(1-\alpha)^k\right] \\
&= \alpha(1-\alpha)\frac{1}{\alpha^2} \\
&= \frac{1-\alpha}{\alpha}
\end{aligned}$$

(b) Evaluate the higher moments $m_3 = E[Z^3]$ and $m_4 = E[Z^4]$.

Hint: Express Z^4 in terms of the ξ_i's where $\xi_i^2 = 1$ and $E[\xi_i\xi_j] = 0$.

$\because$

$$\begin{aligned}
E\xi_i^2 &= Var\xi_i + (E\xi_i)^2 \\
&= 1 \\
E(\xi_i\xi_j) &= E\xi_i E\xi_j \\
&= 0 \\
E\xi_i^3 &= 1^3 Pr\{\xi_i = 1\} + (-1)^3 Pr\{\xi_i = 0\} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0 \\
\xi_i^2 &= 1
\end{aligned}$$

Solution (cont.)

$\therefore$

$$\begin{aligned}
m_3 &= EZ^3 \\
&= \sum_{k=0}^{+\infty} E(Z^3|N=k)Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty} E(\xi_1+\cdots+\xi_k)^3 Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty} E\left(\sum_{j=1}^{k}\xi_j^3 + \binom{3}{1}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\xi_i^2\xi_j + \binom{3}{1}\binom{2}{1}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\sum_{\substack{m=1\\ i\neq m\\ j\neq m}}^{k}\xi_i\xi_j\xi_m\right) \\
&\quad \cdot Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty}\left(\sum_{j=1}^{k}E\xi_j^3 + \sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}E\xi_i^2E\xi_j + \sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\sum_{\substack{m=1\\ i\neq m\\ j\neq m}}^{k}E\xi_iE\xi_jE\xi_m\right) \\
&\quad \cdot Pr\{N=k\} \\
&= 0 \\
m_4 &= EZ^4 \\
&= \sum_{k=0}^{+\infty} E(Z^4|N=k)Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty} E\left(\xi_1+\cdots+\xi_k\right)^4 Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty} E\left(\binom{4}{1}\binom{3}{1}\binom{2}{1}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\sum_{\substack{l=1\\ i\neq l\\ j\neq l}}^{k}\sum_{\substack{m=1\\ i\neq m\\ j\neq m\\ l\neq m}}^{k}\xi_i\xi_j\xi_l\xi_m + \binom{4}{1}\binom{3}{1}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\xi_i\xi_j\cdot 1 + \right. \\
&\quad \left. \frac{1}{2}\binom{4}{2}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}1\cdot 1 + \binom{4}{1}\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\xi_i\xi_j\cdot 1 + \sum_{i=1}^{k}1\cdot 1\right)\cdot Pr\{N=k\} \\
&= \sum_{k=0}^{+\infty}(3k^2-3k+k)\alpha(1-\alpha)^k \\
&= \sum_{k=0}^{+\infty}(3k^2-2k)\alpha(1-\alpha)^k \\
&= E(3N^2-2N) \\
&= 3EN^2-2EN
\end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
&= 3[(EN)^2 + VarN] - 2EN \\
&= 3\left[\left(\frac{1-\alpha}{\alpha}\right)^2 + \frac{1-\alpha}{\alpha^2}\right] - 2\frac{1-\alpha}{\alpha} \\
&= \frac{5\alpha^2 - 11\alpha + 6}{\alpha^2}
\end{aligned}
$$

2.4.3

Let X be a Possion distribution with parameter $\lambda > 0$. Suppose λ itself is random, following an exponential density with parameter θ.

(a) What is the marginal distribution of X.

$\because \quad \lambda \sim Exp(\theta)$

$\therefore$

$$
f_{X|\lambda}(k|y) = \begin{cases} \dfrac{y^k}{k!}e^{-y}, & k \in \mathbb{N} \\ 0, & otherwise \end{cases}
$$

$$
f_\lambda(y) = \begin{cases} \theta e^{-\theta y}, & t \geqslant 0 \\ 0, & t < 0 \end{cases}
$$

$$
\begin{aligned}
f_{X\lambda}(k,y) &= f_{X|\lambda}(k|y) f_\lambda(y) \\
&= \begin{cases} \dfrac{\theta y^k}{k!}e^{-(\theta+1)y}, & k \in \mathbb{N}, \theta \geqslant 0 \\ 0, & otherwise \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_X(k) &= \int_R f_{X\lambda}(k,y)\mathrm{d}y \\
&= \int_0^{+\infty} \frac{\theta y^k}{k!} e^{-(\theta+1)y}\mathrm{d}y \\
&= \frac{\theta}{k!(\theta+1)^{k+1}} \int_0^{+\infty} [(\theta+1)y]^k e^{-(\theta+1)y}\mathrm{d}[(\theta+1)y] \\
&= \frac{\theta}{k!(\theta+1)^{k+1}} \Gamma(k+1) \\
&= \frac{\theta}{(\theta+1)^{k+1}} \qquad k \in \mathbb{N} \\
f_X(k) &= 0 \qquad\qquad\quad k \notin \mathbb{N}
\end{aligned}
$$

(b) Determine the conditional density for λ given $X = k$.

$$
\begin{aligned}
f_{\lambda|X}(y|k) &= \frac{f_{X,\lambda}(k,y)}{f_X(k)} \\
&= \begin{cases} \dfrac{\frac{\theta y^k}{k!}e^{-(\theta+1)y}}{\frac{\theta}{(\theta+1)^{k+1}}}, & y \geqslant 0, k \in \mathbb{N} \\ 0, & otherwise \end{cases} \\
&= \begin{cases} \dfrac{(\theta+1)^{k+1}y^k}{k!}e^{-(\theta+1)y}, & y \geqslant 0, k \in \mathbb{N} \\ 0, & otherwise \end{cases}
\end{aligned}
$$

2.4.4

Suppose X and Y are independent random variables having the same Possion distribution with parameter λ, but where λ is also random, being exponentially distributed with parameter θ. What is the conditional distribution for X given that $X + Y = n$?

From Problem 2.4.3 we have

$$
\begin{aligned}
f_X(x) &= \frac{\theta}{(\theta+1)^{x+1}} I_{\mathbb{N}}(x) \\
f_Y(y) &= \frac{\theta}{(\theta+1)^{y+1}} I_{\mathbb{N}}(y)
\end{aligned}
$$

$\because$ X and Y are independent

$\therefore$

$$
f_{XY}(x,y) = \frac{\theta^2}{(\theta+1)^{x+y+2}} I_{\mathbb{N}}(x) I_{\mathbb{N}}(y)
$$

Let $Z = X + Y,\ J = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$

$$
\begin{aligned}
f_{XZ}(x,n) &= f_{XY}(x, n-x)\frac{1}{|J|} \\
&= \frac{\theta^2}{(\theta+1)^{n+2}} I_{\mathbb{N}}(x) I_{\mathbb{N}}(n-x)
\end{aligned}
$$

Solution (cont.)

$$
\begin{aligned}
f_Z(n) &= \int_R f_{XZ}(x,n)\mathrm{d}x \\
&= \begin{cases} \sum\limits_{x=0}^{n} \dfrac{\theta^2}{(\theta+1)^{n+2}}, & n \in \mathbb{N} \\ 0, & otherwise \end{cases} \\
&= \begin{cases} \dfrac{(n+1)\theta^2}{(\theta+1)^{n+2}}, & n \in \mathbb{N} \\ 0, & otherwise \end{cases} \\
f_{X|Z}(x|n) &= \frac{f_{XZ}(x,n)}{f_Z(n)} \\
&= \begin{cases} \dfrac{\dfrac{\theta^2}{(\theta+1)^{n+2}}}{\dfrac{(n+1)\theta^2}{(\theta+1)^{n+2}}}, & x,n \in \mathbb{N}, x \leqslant n \\ 0, & otherwise \end{cases} \\
&= \begin{cases} \dfrac{1}{n+1}, & x,n \in \mathbb{N}, x \leqslant n \\ 0, & otherwise \end{cases}
\end{aligned}
$$

2.4.6

Let $X_0, X_1, X_2 \cdots$ be independent identically distributed nonnegative random variables having a continuous distribution. Let N be the first index k for which $X_k > X_0$. That is, $N = 1$ if $X_1 > X_0$, $N = 2$ if $X_1 \leqslant X_0$ and $X_2 > X_0$, etc. Determine the probability mass function for N and the mean $E[N]$.(Interpretation: $X_0, X_1 \cdots$ are successive offers or bids on a car that you are trying to sell. Then, N is the index of the first bid that is better than the initial bid.)

Suppose that the probability distribution function of X_i is $f(x)$, the cumulative distribution function of X_i is $F(x)$. Because X_i is nonnegative random variables with continuous distribution, $f(x) \equiv 0$ when $x < 0$ and $F(0) = 0$.
Given n, let $A_n = \{X_1 \leqslant X_0, X_2 \leqslant X_0, \cdots, X_{n-1} \leqslant X_0, X_n > X_0\}$,

Solution (cont.)

$$
\begin{aligned}
f_N(n) &= P(A_n) \\
&= \int_{A_n} f_{X_0X_1\cdots X_n}(x_0, x_1, \cdots, x_n)\mathrm{d}x_0\cdots\mathrm{d}x_n \\
&= \int_{A_n} f_{X_0}(x_0)f_{X_1}(x_1)\cdots f_{X_n}(x_n)\mathrm{d}x_0\cdots\mathrm{d}x_n \\
&= \int_0^{+\infty}\int_0^{x_n}\left[\left(\int_0^{x_0} f(x_1)\mathrm{d}x_1\right)\cdots\left(\int_0^{x_0} f(x_{n-1})\mathrm{d}x_{n-1}\right)\right] \\
&\quad \cdot f(x_0)f(x_n)\mathrm{d}x_0\mathrm{d}x_n \\
&= \int_0^{+\infty}\int_0^{x_n}[F(x_0)]^{n-1}\mathrm{d}F(x_0)\mathrm{d}F(x_n) \\
&= \int_0^{+\infty}\frac{1}{n}[F(x_0)]^n\Big|_0^{x_n}\mathrm{d}F(x_n) \\
&= \int_0^{+\infty}\frac{1}{n}[F(x_n)]^n\mathrm{d}F(x_n) \\
&= \frac{1}{n(n+1)}[F(x_n)]^{n+1}\Big|_0^{+\infty} \\
&= \frac{1}{n(n+1)} \\
EN &= \sum_{n=1}^{+\infty}\frac{1}{n+1} \\
&= +\infty
\end{aligned}
$$