STAT 150: Stochastic Processes

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Homework 7

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Solutions by

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PK Exercises 5.1.4

Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t.

(a) What is $Pr\{X(t) = k\}$ for $k = 0, 1 \cdots$?

(1) $t > 0$

$\because$

$$X(0) = 0$$

$$X(t) - X(0) \sim Poisson(t\lambda)$$

$\therefore \quad \forall\ k \in \mathbb{N},$

$$\begin{aligned} Pr\{X(t) = k\} &= Pr\{X(t) - X(0) = k\} \\ &= \frac{(t\lambda)^k}{k!} e^{-t\lambda} \end{aligned}$$

(2) $t = 0$

$$Pr\{X(0) = 0\} = 1$$

and

$$Pr\{X(0) = k\} = 0 \qquad (\forall\ k \in \mathbb{N}^+)$$

(b) Consider fixed times $0 < s < t$. Determine the conditional probability $Pr\{X(t) = n + k | X(s) = n\}$ and the expected value $\mathbb{E}[X(t)X(s)]$.

$\because \quad X(t) - X(s)$ and $X(s) - X(0)$ are independent

$\therefore$

$$\begin{aligned} Pr\{X(t) = n + k | X(s) = n\} &= Pr\{X(t) - X(s) = k | X(s) - X(0) = n\} \\ &= Pr\{X(t) - X(s) = k\} \\ &= \frac{(t-s)^k \lambda^k}{k!} e^{-(t-s)\lambda} \\ \mathbb{E}[X(t)X(s)] &= \mathbb{E}[X(t) - X(s)][X(s) - X(0)] + \mathbb{E}X(s)^2 \\ &= \mathbb{E}[X(t) - X(s)]\mathbb{E}[X(s) - X(0)] + Var[X(s) - X(0)] + \{\mathbb{E}[X(s) - X(0)]\}^2 \\ &= (t-s)\lambda \cdot s\lambda + s\lambda + s^2\lambda^2 \\ &= ts\lambda^2 + s\lambda \end{aligned}$$

PK Problems 5.1.2

Suppose that the minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t. Argue that $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

Let $\{X_1(t)\}, \{X_2(t)\}$ denote the Poisson precesses of minor and major defects respetively. Then
(1) for any time points $t_0 = 0 < t_1 < \cdots < t_n$,

$$X_1(t_1) - X_1(t_0)\ , X_1(t_2) - X_1(t_1), \cdots,\ X_1(t_n) - X_1(t_{n-1})$$

are independent and

$$X_2(t_1) - X_2(t_0)\ , X_2(t_2) - X_2(t_1), \cdots,\ X_2(t_n) - X_2(t_{n-1})$$

are independent;
(2) for $s \geqslant 0$ and $t > 0$, $X_1(s+t) - X_1(s) \sim Poisson(\alpha t)$ and $X_1(s+t) - X_1(s) \sim Poisson(\beta t)$;
(3) $X_1(0) = X_2(0) = 0$;
and $X(t) = X_1(t) + X_2(t)$.
(1*)
$\because\quad \{X_1(t)\}$ and $\{X_2(t)\}$ are independent
$\therefore\quad$ for any time points $t_0 = 0 < t_1 < \cdots < t_n$,

$$\begin{aligned}
X(t_1) - X(t_0) &= X_1(t_1) - X_1(t_0) + X_2(t_1) - X_2(t_0) \\
X(t_2) - X(t_1) &= X_1(t_2) - X_1(t_1) + X_2(t_2) - X_2(t_1) \\
&\vdots \\
X(t_n) - X(t_{n-1}) &= X_1(t_n) - X_1(t_{n-1}) + X_2(t_n) - X_2(t_{n-1})
\end{aligned}$$

are independent
(2*) for $s \geqslant 0$ and $t > 0$,
$\because\quad \forall\ k \in \mathbb{N}$,

$$\begin{aligned}
Pr\{X(s+t) - X(s) = k\} &= Pr\{X_1(s+t) - X_1(s) + X_2(s+t) - X_2(s) = k\} \\
&= \sum_{i=0}^{k} Pr\{X_1(s+t) - X_1(s) = i, X_2(s+t) - X_2(s) = k-i\} \\
&= \sum_{i=0}^{k} Pr\{X_1(s+t) - X_1(s) = i\} Pr\{X_2(s+t) - X_2(s) = k-i\} \\
&= \sum_{i=0}^{k} \frac{(\alpha t)^i e^{-\alpha t}}{i!} \frac{(\beta t)^{k-i} e^{-\beta t}}{(k-i)!} \\
&= \frac{t^k e^{-(\alpha+\beta)t}}{k!} \sum_{i=0}^{k} \binom{k}{i} \alpha^i \beta^{k-i} \\
&= \frac{(\alpha+\beta)^k t^k}{k!} e^{-(\alpha+\beta)t}
\end{aligned}$$

$\therefore\quad X(s+t) - X(s) \sim Poisson((\alpha+\beta)t)$
(3*)

$$X(0) = X_1(0) + X_2(0) = 0$$

Therefore $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

PK Problems 5.1.4

Let X and Y be independent random variables, Poisson distributed with parameter α and β, respectively. Show that the generating function of their sum $N = X + Y$ is given by

$$g_N(s) = e^{-(\alpha+\beta)(1-s)}$$

(**Hint:** Verify and use the fact that the generating function of a sum of independent random variables is the product of their respective generating functions. See Chapter 3, Section 3.9.2.)

$\because$ $X \sim Poisson(\alpha),\ Y \sim Poisson(\beta)$, X and Y are independent
$\therefore$ $\forall\ n \in \mathbb{N}$,

$$\begin{aligned}
Pr\{N = n\} &= Pr\{X + Y = n\} \\
&= \sum_{k=0}^{n} Pr\{X = k, Y = n - k\} \\
&= \sum_{k=0}^{n} Pr\{X = k\} Pr\{Y = n - k\} \\
&= \sum_{k=0}^{n} \frac{\alpha^k e^{-\alpha}}{k!} \frac{\beta^{n-k} e^{-\beta}}{(n-k)!} \\
&= \frac{e^{-(\alpha+\beta)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \\
&= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!}
\end{aligned}$$

i.e. $N \sim Poisson(\alpha + \beta)$
$\therefore$

$$\begin{aligned}
g_N(s) &= \sum_{n=0}^{\infty} \mathbb{P}(N = n) s^n \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} s^n \\
&= e^{-(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)^n s^n}{n!} \\
&= e^{-(\alpha+\beta)} e^{(\alpha+\beta)s} \\
&= e^{-(\alpha+\beta)(1-s)}
\end{aligned}$$

PK Problems 5.1.6

Let $\{X(t); t \geqslant 0\}$ be a Poisson process of rate λ. For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t + s) = n$.

$$
\begin{aligned}
Pr\{X(t)=k|X(t+s)=n\} &= \frac{Pr\{X(t)=k, X(t+s)=n\}}{Pr\{X(t+s)=n\}} \\
&= \frac{Pr\{X(t)-X(0)=k, X(t+s)-X(t)=n-k\}}{Pr\{X(t+s)=n\}} \\
&= \frac{Pr\{X(t)-X(0)=k\}Pr\{X(t+s)-X(t)=n-k\}}{Pr\{X(t+s)=n\}} \\
&= \frac{\frac{(t\lambda)^k}{k!}e^{-\lambda t}\cdot\frac{(s\lambda)^{n-k}}{(n-k)!}e^{-\lambda s}}{\frac{(t+s)^n\lambda^n}{n!}e^{-\lambda(t+s)}} \\
&= \binom{n}{k}\frac{t^k s^{n-k}}{(t+s)^n}
\end{aligned}
$$

PK Problems 5.3.5

Let $X(t)$ be a Poisson process with parameter λ. Independently, let T be a random variable with the exponential density

$$f_T(t) = \theta e^{-\theta t} \qquad \text{for } t > 0.$$

Determine the probablity mass function for $X(T)$.

(**Hint:** Use the law of total probability and Chapter 1,(1.54). Alternatively, use the results of Chapter 1, Section 1.52.)

$\because$ $X(t)$ is a Poisson process with parameter λ

$\therefore$ $X(T) \in \mathbb{N}$

$\therefore$ $\forall\ k \in \mathbb{N}$,

$$
\begin{aligned}
Pr\{X(T)=k\} &= \int_0^\infty Pr\{X(t)=k\}f_T(t)\mathrm{d}t \\
&= \int_0^\infty \frac{(t\lambda)^k e^{-t\lambda}}{k!}\theta e^{-\theta t}\mathrm{d}t \\
&= \frac{\theta\lambda^k}{k!}\int_0^\infty t^k e^{-(\lambda+\theta)t}\mathrm{d}t \\
&\overset{s=(\lambda+\theta)t}{=\!=\!=} \frac{\theta\lambda^k}{(\lambda+\theta)^{k+1}k!}\int_0^\infty s^k e^{-s}\mathrm{d}s \\
&= \frac{\theta\lambda^k}{(\lambda+\theta)^{k+1}k!}\Gamma(k+1) \\
&= \frac{\theta\lambda^k}{(\lambda+\theta)^{k+1}}
\end{aligned}
$$

and $\forall\ k \notin \mathbb{N}$,

$$Pr\{X(T)=k\} = 0$$

GS 6.8.2

Thinning. Insects land in the soup in the manner of a Poisson process with intensity λ, and each such insect is green with probability p, independently of the colours of all other insects. Show that the arrivals of green insects from a Poisson process with intensity λp.

Let $X(t)$ denotes the number of insects land in the soup in time t and $Y(t)$ denotes the number of green insects land in the soup in times t.

$\because$ $X(t)$ is a Poisson process with parameter λ

$\therefore$

(1) for any time points $t_0 = 0 < t_1 < \cdots < t_n$, $X(t_1) - X(t_0)$, $X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent

(2) $\forall\ s \geqslant 0, t > 0, k \in \mathbb{N}$,

$$\mathbb{P}(X(s+t) - X(s) = k) = \frac{(t\lambda)^k e^{-t\lambda}}{k!}$$

(3) $X(0) = 0$

$\therefore$

(1*) for any time points $t_0 = 0 < t_1 < \cdots < t_n$, $Y(t_1) - Y(t_0)$, $Y(t_2) - Y(t_1), \cdots, Y(t_n) - Y(t_{n-1})$ are independent since $Y(t_i) - Y(t_{i-1})$ only dependents on $X(t_i) - X(t_{i-1})$

(2*) $\forall\ s \geqslant 0, t > 0, k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}(Y(s+t) - Y(s) = k) &= \mathbb{P}(Y(s+t) - Y(s) = k, X(s+t) - X(s) \geqslant k) \\
&= \sum_{n=k}^{\infty} \mathbb{P}(Y(s+t) - Y(s) = k | X(s+t) - X(s) = n) \mathbb{P}(X(s+t) - X(s) = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(t\lambda)^n e^{-t\lambda}}{n!} \\
&= \frac{(t\lambda p)^k e^{-t\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} (t\lambda)^{n-k}}{(n-k)!} \\
&= \frac{(tp\lambda)^k e^{-t\lambda}}{k!} e^{(1-p)t\lambda} \\
&= \frac{(tp\lambda)^k e^{-tp\lambda}}{k!}
\end{aligned}$$

(3*)

$$0 \leqslant Y(0) \leqslant X(0) = 0$$

i.e.

$$Y(0) = 0$$

Therefore, $Y(t)$ is a Poisson process with parameter $p\lambda$.

GS 6.8.4

Let B be a simple birth process (6.8.11b) with $B(0) = I$; the birth rates are $\lambda_n = n\lambda$. Write down the forward system of equations for the process and deduce that

$$\mathbb{P}(B(t)=k)=\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I}, \qquad k\geqslant I$$

Show also that $\mathbb{E}(B(t))=Ie^{\lambda t}$ and $Var(B(t))=Ie^{2\lambda t}(1-e^{-\lambda t})$.

$\because \quad \forall\ s\geqslant 0, t>0,$

$$p_{ij}=\mathbb{P}(B(s+t)=j|B(s)=i)=\mathbb{P}(B(t)=j|B(0)=i)$$

$\therefore \quad \forall\ j\geqslant i\geqslant I$, the forward system of equations are

$$\begin{aligned} p'_{ij}(t)&=\lambda_{j-1}p_{i,j-1}(t)-\lambda_j p_{ij}(t)\\ &=(j-1)\lambda p_{i,j-1}(t)-j\lambda p_{ij}(t)\end{aligned}$$

with $\lambda_{-1}=0$ and boundary condition $p_{ij}(0)=\delta_{ij}$

$$\begin{cases} p'_{I,I}(t)=-I\lambda p_{I,I}(t) & & (1)\\ p'_{In}(t)=(n-1)\lambda p_{0,n-1}(t)-n\lambda p_{0n}(t) & (n>I) & (2)\end{cases}$$

From (1) and boundary condition $p_{I,I}(0)=1$ we have

$$p_{I,I}(t)=e^{I\lambda t}$$

and

$$\begin{aligned}\mathbb{P}(B(t)=I)&=\mathbb{P}(B(t)=I|B(0)=I)\mathbb{P}(B(0)=I)\\ &=p_{I,I}(t)\\ &=\binom{I-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{I-I}\end{aligned}$$

Suppose that for $n=k(k\geqslant I)$, the following equation holds,

$$\mathbb{P}(B(t)=k)=\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I} \qquad (*)$$

Then

$$\begin{aligned} p_{I,k}&=\mathbb{P}(B(t)=k|B(0)=I)\\ &=\frac{\mathbb{P}(B(t)=k)}{\mathbb{P}(B(0)=I)}\\ &=\mathbb{P}(B(t)=k)\\ &=\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I}\end{aligned}$$

For $n=k+1$, we have

$$\begin{cases} p_{I,k+1}(0)=0\\ p'_{I,k+1}(t)=k\lambda p_{0,k}(t)-(k+1)\lambda p_{0,k+1}(t)\\ p_{0,k}=\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I}\end{cases}$$

Solution (cont.)

The general solution is

$$\begin{aligned}
p_{0,k+1} &= e^{-(k+1)\lambda t}\left[\int k\lambda\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I}e^{(k+1)\lambda t}\mathrm{d}t + c\right] \\
&\xlongequal{s=e^t} \binom{k-1}{I-1}ks^{-(k+1)}\left[\int (s-1)^{k-I}\mathrm{d}s + c\right] \\
&= \binom{k-1}{I-1}ks^{-(k+1)}\left[\frac{(s-1)^{k-I+1}}{k-I+1} + c\right] \\
&= \binom{k}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k+1-I} + c\binom{k-1}{I-1}ks^{-k+1}
\end{aligned}$$

From the initial condition $p_{I,k+1}(0) = 0$, we have

$$p_{0,k+1} = \binom{k}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k+1-I}$$

i.e. when $n = k+1$, $(*)$ still holds.
Therefore, by mathematical induction, $\forall\ k \geqslant I$,

$$\mathbb{P}(B(t)=k) = \binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I}$$

$\therefore$

$$\begin{aligned}
\mathbb{E}(B(t)) &= \sum_{k=I}^{\infty} k\mathbb{P}(B(t)=k) \\
&= \sum_{k=I}^{\infty} k\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I} \\
&\xlongequal{n=k-I} Ie^{-I\lambda t}\sum_{n=0}^{\infty}\binom{n+I}{n}(1-e^{-\lambda t})^{n} \\
&= \frac{Ie^{-I\lambda t}}{[1-(1-e^{-\lambda t})]^{I+1}} \\
&= Ie^{\lambda t} \\
\mathbb{E}[B(t)^2] &= \sum_{k=I}^{\infty} k^2\mathbb{P}(B(t)=k) \\
&= \sum_{k=I}^{\infty} k(k+1)\mathbb{P}(B(t)=k) - \mathbb{E}(B(t)) \\
&= \sum_{k=I}^{\infty} k(k+1)\binom{k-1}{I-1}e^{-I\lambda t}(1-e^{-\lambda t})^{k-I} - Ie^{\lambda t} \\
&= I(I+1)e^{-I\lambda t}\sum_{n=0}^{\infty}\binom{n+I+1}{n}(1-e^{-\lambda t})^{n} - Ie^{\lambda t} \\
&= I(I+1)e^{2\lambda t} - Ie^{\lambda t} \\
Var(B(t)) &= \mathbb{E}[B(t)^2] - [\mathbb{E}(B(t))]^2 \\
&= I(I+1)e^{2\lambda t} - Ie^{\lambda t} - I^2e^{2\lambda t} \\
&= Ie^{2\lambda t} - Ie^{\lambda t}
\end{aligned}$$