Stochastic Processes

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Week 14

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Solutions by

Jinhong Du

15338039

4.14

Show that in a finite Markov chain there are no null recurrent states and not all states can be transient.

Suppose that the finite Markov chain is $X = \{X(n) : n \in \mathbb{N}\}$ with the state space is S.
From Decomposition Theorem, we have $S = T \cup R_1 \cup R_2 \cup \cdots \cup R_m$, where T is the set of transient states and R_j $(j = 1 \cdots, m)$ are the irreducible closed sets of recurrent states.
(1) Suppose that R_j is a null recurrent class. Consider the sub chain $Y = \{Y(n) : n \in \mathbb{N}\}$ on R_j, we have

$$1 = \lim_{n\to\infty} \sum_{j\in R_j} P_{ij}^{(n)} = \sum_{j\in R_j} \lim_{n\to\infty} P_{ij}^{(n)} = 0$$

since R_j is a finite set. Contradiction. Therefore, X has no null recurrent states.
(2) Suppose that X has only transient states. We also have

$$1 = \lim_{n\to\infty} \sum_{j\in S} P_{ij}^{(n)} = \sum_{j\in S} \lim_{n\to\infty} P_{ij}^{(n)} = 0$$

Contradiction. Therefore, not all states of X can be transient.
Therefore, in a finite Markov chain there are no null recurrent states and not all states can be transient.

4.17

Consider a positive recurrent irreducible periodic Markov chain and let π_j denote the long-run proportion of time in state j, $j > 0$. Prove that $\pi_j, j \geqslant 0$, satisfy $\pi_j = \sum_i \pi_i P_{ij}$, $\sum_j \pi_j = 1$.

Suppose that the chain $X = \{X_n : n \in \mathbb{N}^+\}$ is positive recurrent irreducible periodic, then the limiting distribution and unique stationary distribution $v = \left(v_j\right)_{j\in\mathbb{S}}$ exists.

$$v_j = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mu_j} > 0$$

where $\mu_i = \sum_{n=0}^{\infty} n f_{ii}^{(n)}$ is the mean return time to state i. Let $T_i^{(k)}$ denote the kth return time to state i.

$\because$ from Strong Law of Large Number, with probability one,

$$\begin{aligned}
\pi_j &= \lim_{m\to\infty} \frac{1}{m} \sum_{m}^{i=1} \mathbb{1}_{\{X_j=i\}} \\
&= \lim_{m\to\infty} \frac{m}{T_i^{(m)}} \\
&= \frac{1}{\mu_j}
\end{aligned}$$

$\therefore$ with probability one,

$$\begin{aligned}
\pi_j &= v_j \\
&= \sum_i v_i P_{ij} \\
&= \sum_i \pi_i P_{ij} \\
\sum_j \pi_j &= \sum_j v_j \\
&= 1
\end{aligned}$$