**Question 1** (a) Show that the Hermite polynomial $H_n(x)$ satisfies

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-k^2} (x - \mathrm{i}k)^n \, \mathrm{d}k.$$

(b) Show that

$$|(x + \mathrm{i}k)^n| \leq 2^n (|x|^n + |k|^n).$$

(c) Use Stirling's approximation $n! \approx (n/e)^n$ to show

$$\frac{|h_n(x)|}{\|h_n\|} \leq 2^{n+1} \mathrm{e}^{-x^2/2} + 2 \left( \frac{8e}{n} \right)^{n/2} \mathrm{e}^{-x^2/2} |x|^n.$$

(d) Show that

$$\frac{|h_n(x)|}{\|h_n\|} \leq 2 \left( \frac{16\mathrm{e}\pi}{\mathrm{e}^{2\pi}} \right)^{n/2}$$

for $|x| \geq \sqrt{2\pi n}$.

(e) Explain why scaled Hermite functions $h_0(cx), h_1(cx), \ldots, h_n(cx)$ might form a suitable basis for approximating functions $f \in L^2$ which are approximately band- and time-limited in the sense that

$$\int_{|x| \geq T} |f(x)|^2 \, \mathrm{d}x \leq \epsilon^2 \|f\|^2$$

and

$$\int_{|k| \geq K} |\hat{f}(k)|^2 \, \mathrm{d}k \leq \epsilon^2 \|\hat{f}\|^2.$$

How should $n$ and $c$ relate to $K$ and $T$?

**Question 2** (a) Show that

$$FDf(k) = \widehat{f'}(k) = \mathrm{i}k\widehat{f}(k) = \mathrm{i}kFf(k)$$

and

$$F(xf)(k) = \widehat{xf}(k) = \mathrm{i}\widehat{f}'(k) = \mathrm{i}DFf(k).$$

(b) Show that the differential operator

$$D_{ab}f(x) = \big( (a^2 - x^2) f'(x) \big)' - b^2 x^2 f(x)$$

satisfies

$$FD_{ab} = D_{ba}F.$$

(c) Show that $D_{ab}$ commutes with the orthogonal projection onto time-limited functions

$$P_a f(t) = f(t)$$

for $|t| \leq a$ and

$$P_a f(t) = 0$$

for $|t| > a$.

(d) Use (b) and (c) to show that $D_{ab}$ commutes with the integral operator

$$S_{ab} f(t) = P_a Q_b P_a f(t) = \frac{1}{\pi} \int_{-a}^{a} \frac{\sin b(t-s)}{t-s} f(s) \, \mathrm{d}s$$

where $Q_b = F^* P_b F$ is the orthogonal projection onto bandlimited functions.

(e) Explain why the eigenfunctions of $D_{ab}$ might be useful in representing approximately time- and band-limited functions.

**Question 3** (a) Use Fourier transform to find a bounded solution $u$ of

$$u_{xx} + u_{tt} = 0$$

in the upper half plane $x \in R$, $t > 0$, with boundary conditions

$$u(x, 0) = g(x)$$

where $g \in L^2(R)$ is bounded and continuous.

(b) Show that $u$ attains its boundary values in the sense that

$$u(x, t) \to g(x)$$

as $t \to 0$.

(c) Assume that $g' \in L^2(R)$ is also bounded and continuous. Argue directly from the Laplace equation that if the Dirichlet-Neumann operator $\Lambda$ is defined by

$$u_t(x, t) \to \Lambda g(x)$$

as $t \to 0$, then $\Lambda$ must satisfy

$$\Lambda^2 g(x) = -g''(x).$$

(d) Find the kernel of the Hilbert transform operator $H$ such that

$$\Lambda g = H(g').$$

**Question 4** Solve the integral equation

$$D^{-1/2}h(t) = \int_0^t \frac{1}{\sqrt{\pi(t-s)}} h(s)\mathrm{d}s = g(t)$$

where $g$ is a nice function with $g(0) = 0$. (Hint: Square $D^{-1/2}$.)

**Question 5** (a) Solve the initial-boundary value problem for the heat equation

$$u_t = u_{xx}$$

for $x > 0$, $t > 0$, with homogeneous initial conditions

$$u(x, 0) = 0$$

and boundary conditions

$$u(0, t) = g(t)$$

where $g$ is a nice function with $g(0) = 0$. (Hint: Try $u(x, t) = \int_0^t K_{t-s}(x)h(s)\mathrm{d}s$ where $K(x, t) = (4\pi t)^{-1/2}\mathrm{e}^{-x^2/4t}$, and solve an integral equation for $h$.)

(b) Assume that $g' \in L^2(R)$ is also bounded and continuous. Argue directly from the heat equation that if

$$u_x(x, t) \to \Lambda g(t)$$

as $x \to 0$, then the Dirichlet-Neumann operator $\Lambda$ must satisfy

$$\Lambda^2 g(t) = g'(t).$$

(c) Find the Dirichlet-Neumann operator $\Lambda$.