Modern Multivariate Statistical Techniques

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Content

1 Ex 4.1

Consider the class of functions of the form $p(x) = \sum_{l=0}^{L-1} y_l I_{T_l}(x)$, where $h\sum_{l=0}^{L-1} y_l = 1$. Given an i.i.d. sample, $X_1, X_2, \cdots, X_n$ from $p(x)$, maximize the log-likelihood function, $L = \sum_{i=1}^{n} \log_e \left[\sum_{l=0}^{L-1} y_l I_{T_l}(x_i)\right]$, subject to the condition that $h\sum_{l=0}^{L-1} y_l = 1$. Show that the histogram

$$\hat{p}(x) = \frac{1}{nh}\sum_{l=0}^{L-1} N_l I_{T_l}(x)$$

where I_{T_l} denotes the indicator function of the lth bin and $N_l = \sum_{i=1}^{n} I_{T_l}(x_i)$ be the $L-1$ number of sample values that fall into T_l, $l = 0, 1, 2, \cdots, L-1$ and $n = \sum_{l=0}^{L-1} N_l$, is the unique ML estimator of p. [Hint: Use Lagrangian multipliers.]

Proof. Let

$$L_\lambda = \sum_{i=1}^{n} \log_e \left[\sum_{l=0}^{L-1} y_l I_{T_l}(x_i)\right] + \lambda\left(1 - h\sum_{l=0}^{L-1} y_l\right)$$

When $p(x) = y_k$, we have $x \in T_k$. Set

$$\begin{aligned}
\frac{\partial L_\lambda}{\partial y_k} &= \sum_{i=1}^{n} \frac{I_{T_k}(x_i)}{\sum_{l=0}^{L-1} y_l I_{T_l}(x_i)} - \lambda h \\
&= \sum_{\{i:x_i \in T_k\}} \frac{1}{y_k} - \lambda h = 0 \\
&= \frac{N_k}{y_k} - \lambda h = 0 \\
\frac{\partial L_\lambda}{\partial \lambda} &= 1 - h\sum_{l=0}^{L-1} y_l = 0
\end{aligned}$$

We got

$$\begin{aligned}
\lambda &= n \\
y_k &= \frac{N_k}{nh}
\end{aligned}$$

i.e.

$$\begin{aligned}
\hat{p}(x) &= \sum_{l=0}^{L-1} \frac{N_k}{nh} I_{T_l}(x) \\
&= \frac{1}{nh}\sum_{l=0}^{L-1} N_k I_{T_l}(x)
\end{aligned}$$

□