118, F17 Problem Set 08 Solutions

Q1.

$$\int_{-\infty}^{\infty} f(x)\, \overline{\left(g''(x) - x^2 g(x)\right)}\, dx$$

$$= f(x)\,\overline{g'(x)} - f'(x)\,\overline{g(x)}\,\Big|_{-\infty}^{\infty}$$

$$+ \int_{-\infty}^{\infty} \left(f''(x) - x^2 f(x)\right) \overline{g(x)}\, dx$$

since

x^2 is real-valued.

Since f and g are "nice" smooth functions we can assume

$f(x)\,g'(x) \to 0$ as $|x| \to \infty$

$f'(x)\,g(x) \to 0$

and we have shown

$$\langle f, Kg \rangle = \langle Kf, g \rangle.$$

Q2.

$$\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} y^n h_n(x) \sum_{m=0}^{\infty} y^m h_m(x)\, dx =$$

$$= \sum_{n=0}^{\infty} y^n \sum_{m=0}^{\infty} y^m \int_{-\infty}^{\infty} h_n(x) h_m(x)\, dx$$

Since h_n are orthogonal all terms with $m \neq n$ vanish and we find

$$= \sum_{n=0}^{\infty} y^{2n} \|h_n\|^2.$$

On the other hand we have

$$\sum_{n=0}^{\infty} y^n h_n(x) = e^{x^2/2} e^{-(x-y)^2}$$

So

$$= \int_{-\infty}^{\infty} e^{x^2} e^{-2(x-y)^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{+2\cdot(2xy)} e^{-2y^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-(x-2y)^2} e^{2y^2} \, dx$$

$$= \sqrt{\pi}\, e^{2y^2}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2y^2)^n}{n!} =$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n y^{2n}}{n!} .$$

Since power series coefficients are unique we must have

$$\boxed{\| h_n \|^2 = \sqrt{\pi}\, \frac{2^n}{n!}}$$

Q3(a) Since $\deg H_n(x) = n$ exactly

(b) c

$$H_{n+1}(x) = \underset{+1}{2x}\,\underset{\deg n}{H_n(x)} - 2n\,\underset{\deg n-1}{H_{n-1}}$$

(cannot cancel))

we can write

$$x^n = c_{nn} H_n(x) + q(x)$$

where $\deg(q) < n$. By induction, H_n form a basis.

(b) $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$.

$$\Rightarrow \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} H_1(x)^2 e^{-x^2}\,dx$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} h_1(x)^2\,dx = \frac{1}{4}\|h_1\|^2$$

$$= \frac{1}{4}\,\frac{2^1}{1!}\sqrt{\pi} = \boxed{\frac{1}{2}\sqrt{\pi}}$$

(c) By definition;

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$$

so

$$e^{x^2} H_n(x) = (-1)^n D^n e^{-x^2}.$$

Hence

$$\int_0^x e^{s^2} H_n(s)\,ds =$$

$$= (-1)^n \int_0^x D^n e^{-s^2}\,ds$$

$$= (-1)^n \left(D^{n-1} e^{-x^2} - C_n\right)$$

$$= C_n - e^{x^2} H_{n-1}(x),$$

for $n \geq 1$. (For $n = 0$, $\int_0^x e^{-s^2}\,ds$ $= \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$ is known to be not an elementary function.)

(d) Since H_n form a basis,

$$P(x) = \sum_{m=0}^{n} a_m H_m(x)$$

for some constants a_m.

$$\left(a_m = \frac{\int_{-\infty}^{\infty} P(s) H_m(s) e^{-s^2} ds}{\int_{-\infty}^{\infty} H_m(s)^2 e^{-s^2} ds} \right)$$

Then

$$\int_0^x P(s) e^{s^2} ds = \int_0^x a_0 H_0(s) e^{s^2} ds$$

$$+ \sum_{m=1}^{n} a_m \left(C_m - e^{x^2} H_{m-1}(x) \right)$$

is expressible as an elementary function iff

$$a_0 = \text{const.} \int_{-\infty}^{\infty} P(s) H_0(s) e^{-s^2} ds = 0.$$

Q4. (a) Since $K h_n = -(2n+1) h_n$ and h_n form a basis for $L^2(\mathbb{R})$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, h_n \rangle}{\|h_n\|^2} h_n(x).$$

Hence

$$-Kf(x) = \sum_{n=0}^{\infty} \frac{\langle f, h_n \rangle}{\|h_n\|^2} (2n+1) h_n(x)$$

and

$$-\langle Kf, f \rangle = \sum_{n=0}^{\infty} (2n+1) \frac{\langle f, h_n \rangle^2}{\|h_n\|^2}.$$

On the other hand,

$$-\langle Kf, f \rangle = -\int_{-\infty}^{\infty} (f'' - x^2 f) f \, dx$$

$$= + \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \, dx$$

by integration by parts.

We ignore the boundary terms at $\pm\infty$ because for $\int f'(x)^2$ and $\int x^2 f(x)^2$ to converge we must have more regularity than just $f \in L^2(\mathbb{R})$.

(b) Since $2n+1 \geq 1$ we have

$$\int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \, dx \geq$$

$$\geq \sum_{n=0}^{\infty} \frac{\langle f, h_n \rangle^2}{\|h_n\|^2} = \|f\|^2.$$

Q5. We know

$$\sum_{n=0}^{\infty} y^n h_n(x) = e^{x^2/2} e^{-(x-y)^2}$$

and $h_n(x) = \frac{1}{n!} H_n(x) e^{-x^2/2}$.

Hence

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x) = e^{x^2} e^{-(x-y)^2}$$

$$= e^{2xy} e^{-y^2}.$$

Setting $x = s$ and $y = it$ gives

$$\boxed{e^{2its} = e^{-t^2} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H_n(s)}$$

a useful expansion for the Fourier kernel.

Q 6.

$$E_N = \left| e^{2its} - \sum_{n=0}^{N-1} f_n(t) H_n(s) \right|$$

$$\leq \sum_{n=N}^{\infty} |f_n(t)| \, |H_n(s)|$$

$$\leq (1.09) \sum_{n=N}^{\infty} \left| e^{-t^2} \frac{(it)^n}{n!} \right| 2^{n/2} \sqrt{n!} \, e^{s^2/2}$$

By Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^{n+1/2} \qquad \text{(actually } \gtrsim \text{)}$$

so

$$\frac{1}{\sqrt{n!}} \leq \left(\frac{e}{n}\right)^{n/2} \leq \left(\frac{e}{N}\right)^{n/2}$$

for $n \geq N$. Hence

$$E_N \leq (1.09) \sum_{n=N}^{\infty} \left| e^{-t^2} (it)^n \left(\frac{e}{N}\right)^{n/2} \right| 2^{n/2} e^{s^2/2}$$

$$\leq (1.09) e^{-t^2} e^{s^2/2} \sum_{n=N}^{\infty} \left(\sqrt{\frac{2e}{N}} |t| \right)^n$$

$$\le 2(1.09)\, e^{s^2/2}\, e^{-t^2} \left(\sqrt{\tfrac{2e}{N}}\,|t|\right)^N$$

if $\sqrt{\frac{2e}{N}}|t| \le \frac{1}{2}$. Where does the maximum of

$$g(t) = t^N e^{-t^2}$$

occur? Set

$$g'(t) = Nt^{N-1} e^{-t^2} - 2t^{N+1} e^{-t^2} = 0$$

to get

$$2t^2 = N \quad t = \sqrt{\tfrac{N}{2}} \gg 1,$$

or $t=0$ where $g(t)=0$. Thus for $|t|\le 1$ the max of $g(t)$ is $g(1)=e^{-1}$ and

$$|E_N| \le e^{1} \cdot 2(1.09) \left(\frac{2e}{N}\right)^{N/2}$$

$$\le 10 \left(\frac{2e}{N}\right)^{N/2} \qquad \text{for } |s|\le 2,$$

$$\le 10^{-10}$$

if $N \geq 30$.

Question 1: Show that $K = D^2 - x^2$ is a symmetric operator on $L^2(R)$: for nice smooth functions $f,\ g \in L^2(R)$ we have

$$\int_{-\infty}^{\infty} f(x)Kg(x)^* \mathrm{d}x =< f, Kg >=< Kf, g > .$$

Question 2: Show that

$$\|h_n\|^2 = \frac{\sqrt{\pi}}{n!}2^n.$$

(Hint: Square the expansion

$$\sum_{n=0}^{\infty} y^n h_n(x) = \mathrm{e}^{x^2/2}\mathrm{e}^{-(x-y)^2}$$

and integrate.)

Question 3: (a) Show that the Hermite polynomials of degree less than or equal to n form a basis for the vector space of all polynomials of degree less than or equal to n.

(b) Calculate the first three Hermite polynomials and use them to compute

$$\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^2} \mathrm{d}x.$$

(c) Show that

$$\int_0^x \mathrm{e}^{-s^2} H_n(s)\ \mathrm{d}s = C_n - \mathrm{e}^{-x^2} H_{n-1}(x)$$

for some constant C_n, whenever $n \geq 1$.

(d) Show that the indefinite integral

$$\int_0^x P(s)\mathrm{e}^{-s^2}\ \mathrm{d}s$$

can be evaluated explicitly whenever P is a polynomial with

$$\int_{-\infty}^{\infty} P(s)H_0(s)\mathrm{e}^{-s^2}\ \mathrm{d}s = 0.$$

Question 4: (a) Show that

$$-<Kf, f>=\int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \mathrm{d}x = \sum_{n=0}^{\infty}(2n+1)\frac{<f, h_n>^2}{\|h_n\|^2}$$

for real-valued $f \in L^2(R)$. (b) Prove the weak Heisenberg inequality

$$\int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2 \mathrm{d}x \geq \int_{-\infty}^{\infty} f(x)^2 \mathrm{d}x$$

for such f.

Question 5: Show that

$$\mathrm{e}^{2its} = \mathrm{e}^{-t^2} \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} H_n(s)$$

(Hint: Seek an expansion of the form

$$\mathrm{e}^{2its} = \sum_{n=0}^{\infty} f_n(t) H_n(s)$$

and use orthogonality of the H_n's.)

Question 6: Use Cramer's inequality

$$|H_n(s)| \leq 1.09\ 2^{n/2}\sqrt{n!}\mathrm{e}^{s^2/2}$$

and Stirling's approximation to show that the error in N terms of the approximation in Question 5 is bounded by

$$|\mathrm{e}^{2its} - \sum_{n=0}^{N-1} f_n(t) H_n(s)| \leq 10 \left(\frac{2\mathrm{e}}{N}\right)^{N/2}$$

for $N > 10$, $|t| \leq 1$, and $|s| \leq 2$. How many terms are required to get 10-digit accuracy?