Question 1 Suppose you can only afford to evaluate 11 terms of either side of the Poisson Sum Formula

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} \mathrm{e}^{-(x-2\pi k)^2/4t} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \mathrm{e}^{-tk^2} \mathrm{e}^{\mathrm{i}kx}.$$

(a) Find δ such that the error in the right-hand side (truncated after 11 terms) is smaller than 10^{-14} for $t \geq \delta$ and $|x| \leq \pi$.

(b) Find $\Delta > \delta$ such that $\sqrt{4\pi t}$ times the error in the left hand side (truncated after 11 terms) is smaller than 10^{-14} for $0 < t \leq \Delta$ and $|x| \leq \pi$.

(c) Invent an efficient strategy for evaluating $K(x,t)$ accurately for any $t > 0$ and $|x| \leq \pi$.

Question 2 (a) Use the Poisson Sum Formula to prove the Euler-Maclaurin summation formula

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\ \mathrm{d}x - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) - \cdots$$

for a smooth function f. (Hint: extend f to be even.)

(b) Find formulas for the rest of the coefficients B_{2k} in

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\ \mathrm{d}x - \sum_{1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0)$$

by applying the formula to a suitable test function like $f(x) = \mathrm{e}^{-tx}$.

Question 3 Fix $t > 0$ and let

$$G(x,t) = \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi t}}.$$

(a) Compute $\hat{G}(k,t)$.

(b) Compute $\hat{G}(k,t)$ by a different method.

(c) Show that

$$G_t = G_{xx}$$

for $t > 0$.

(d) Let $f \in L^2(R)$ be continuous and bounded. Show that

$$\int_{-\infty}^{\infty} G(x-y,t) f(y)\ \mathrm{d}y \to f(x)$$

for every $x \in R$ as $t \to 0$.

(e) Solve the inhomogeneous initial-value problem

$$u_t = u_{xx} + \rho(x, t)$$

for $x \in R$, $t > 0$, subject to the intial condition

$$u(x, 0) = 0.$$

Question 4 (a) Find $t > 0$ such that the Gaussian $G(x, t)$ from Question 3 is an eigenfunction of the Fourier transform.

(b) Let F be the $N \times N$ *discrete* Fourier transform matrix with elements

$$F_{jk} = \frac{1}{\sqrt{N}} \mathrm{e}^{2\pi \mathrm{i} jk/N}$$

for $0 \leq j, k \leq N - 1$. Apply the Poisson Sum Formula to $G(x, t)$ and choose parameters x and T to find a formula for an eigenvector $g \in C^N$ and eigenvalue $\lambda \in C$ of F. (Hint: write the index of summation $k = p + qN$ and the sum over k as a double sum over $p = 0$ to $N - 1$ and $q \in Z$.)