MATH 118:
Fourier Analysis and Wavelets

Fall 2017

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Problem Set 1

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Solutions by

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Question 1

Prove or disprove:

$$< u, v >= u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v$$

is an inner product on $\mathbb{C}^2$. Check the properties.

Proof.

(1) Positivity:

$\forall\ v = (v_1, v_2)^T \in \mathbb{C}^2$

$$\begin{aligned}
< v, v > &= v^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v \\
&= \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= 8v_1^2 - 2v_1v_2 + 8v_2^2 \\
&= 7v_1^2 + 7v_2^2 + (v_1 - v_2)^2 \\
&\geqslant 0
\end{aligned}$$

$< v, v >= 0$ iff $7v_1^2 = 7v_2^2 = (v_1 - v_2)^2 = 0$, i.e. $v_1 = v_2 = 0$

(2) Conjugate symmetry:

$\forall\ u = (u_1, u_2)^T,\ v = (v_1, v_2)^T \in \mathbb{C}^2$

$$\begin{aligned}
\overline{< u, v >} &= \overline{u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v} \\
&= \overline{8\overline{u_1}v_1 - \overline{u_1}v_2 - \overline{u_2}v_1 + 8\overline{u_2}v_2} \\
&= \overline{v^T \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} (u^*)^T} \\
&= \overline{v^T} \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} \overline{(u^*)^T} \\
&= v^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} u \\
&=< v, w >
\end{aligned}$$

(3) Homogeneity:

$\forall\ \alpha \in \mathbb{C},\ u = (u_1, u_2)^T,\ v = (v_1, v_2)^T \in \mathbb{C}^2,$

$$\begin{aligned}
< \alpha u, v > &= (\alpha u)^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v \\
&= \begin{pmatrix} \overline{\alpha u_1} \\ \overline{\alpha u_2} \end{pmatrix} \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} z \\
&= \overline{\alpha} u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} z \\
&= \overline{\alpha} < u, v >
\end{aligned}$$

Solution (cont.)

Not satisfied.

(4) Linearity:

$\forall\ u = (u_1, u_2)^T,\ v = (v_1, v_2)^T,\ w = (w_1, w_2)^T \in \mathbb{C}^2$

$$
\begin{aligned}
< u + v, w > &= (u+v)^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} w \\
&= (u^* + v^*) \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} w \\
&= u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} w + u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} w \\
&= < u, w > + < v, w >
\end{aligned}
$$

Therefore, it is not an inner product on $\mathbb{C}^2$. □

Question 2

Which of the following define an inner product on degree-n polynomials

$$p(x) = p_0 + p_1 x + \cdots + p_n x^n?$$

Justify your answers with proof or counterexample. Evaluate $< 1, x >$ and $< 1, x^2 >$ for each case.

(a)

$$< p, q >= \sum_{j=0}^{n} p_j \overline{q_j}$$

Proof.
(1) Positivity:
$\forall\ p(x) \in C[x]$,

$$\begin{aligned} < p, p > &= \sum_{j=0}^{n} p_j \overline{p_j} \\ &= \sum_{j=0}^{n} |p_j|^2 \\ &\geqslant 0 \end{aligned}$$

$< p, p >= 0$ iff $|p_1| = |p_2| = \cdots = |p_n| = 0$, i.e. $p(x) \equiv 0$.
(2) Conjugate symmetry:
$\forall\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned} \overline{< p, q >} &= \overline{\sum_{j=0}^{n} p_j \overline{q_j}} \\ &= \sum_{j=0}^{n} \overline{p_j} q_j \\ &=< q, p > \end{aligned}$$

(3) Homogeneity:
$\forall\ \alpha \in \mathbb{C},\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned} < \alpha p, q > &= \sum_{j=0}^{n} \alpha p_j \overline{q_j} \\ &= \alpha \sum_{j=0}^{n} p_j \overline{q_j} \\ &= \alpha < p, q > \end{aligned}$$

(4) Linearity:
$\forall\ p(x),\ q(x),\ r(x) \in C[x]$,

$$\begin{aligned} < p + q, r > &= \sum_{j=0}^{n} (p_j + q_j) \overline{r_j} \\ &= \sum_{j=0}^{n} p_j \overline{r_j} + \sum_{j=0}^{n} q_j \overline{r_j} \\ &=< p, r > + < q, r > \end{aligned}$$

Solution (cont.)

Therefore, it defines an inner product on degree-n polynomials.

$$\begin{aligned}
<1, x> &= 1\cdot 0 + 0\cdot 1 \\
&= 0 \\
<1, x^2> &= 1\cdot 0 + 0\cdot 0 + 0\cdot 1 \\
&= 0
\end{aligned}$$

□

(b)

$$<p, q> = \int_0^\pi p(x)\overline{q(x)}\mathrm{d}x$$

Proof.
(1) Positivity:
$\forall\ p(x) \in C[x]$,

$$\begin{aligned}
<p, p> &= \int_0^\pi p(x)\overline{p(x)}\mathrm{d}x \\
&= \int_0^\pi |p(x)|^2\mathrm{d}x \\
&\geqslant 0
\end{aligned}$$

$<p, p> = 0$ iff $|p(x)| = 0$ $(a.e.)$, i.e. $p(x) \equiv 0$ ($p(x)$ is continuous at $\mathbb{R}$).
(2) Conjugate symmetry:
$\forall\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned}
\overline{<p, q>} &= \overline{\int_0^\pi p(x)\overline{q(x)}\mathrm{d}x} \\
&= \overline{\int_0^\pi \sum_{i,j=0}^{n} p_i\overline{q_j}x^{i+j}\mathrm{d}x} \\
&= \overline{\sum_{i,j=0}^{n} \int_0^\pi p_i\overline{q_j}x^{i+j}\mathrm{d}x} \\
&= \overline{\sum_{i,j=0}^{n} \frac{1}{i+j+1} p_i\overline{q_j}\pi^{i+j+1}} \\
&= \sum_{i,j=0}^{n} \frac{1}{i+j+1} \overline{p_i}q_j\pi^{i+j+1} \\
&= \sum_{i,j=0}^{n} \int_0^\pi \overline{p_i}q_j x^{i+j}\mathrm{d}x \\
&= \int_0^\pi \sum_{i,j=0}^{n} \overline{p_i}q_j x^{i+j}\mathrm{d}x \\
&= \int_0^\pi q(x)\overline{p(x)}\mathrm{d}x \\
&= <q, p>
\end{aligned}$$

Solution (cont.)

(3) Homogeneity:
$\forall\ \alpha \in \mathbb{C},\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned} < \alpha p, q > &= \int_0^\pi \alpha p(x)\overline{q(x)}\mathrm{d}x \\ &= \alpha \int_0^\pi p(x)\overline{q(x)}\mathrm{d}x \\ &= \alpha < p, q > \end{aligned}$$

(4) Linearity:
$\forall\ p(x),\ q(x),\ r(x) \in C[x]$,

$$\begin{aligned} < p+q, r > &= \int_0^\pi [p(x)+q(x)]\overline{r(x)}\mathrm{d}x \\ &= \int_0^\pi p(x)\overline{r(x)}\mathrm{d}x + \int_0^\pi q(x)\overline{r(x)}\mathrm{d}x \\ &=< p, r > + < q, r > \end{aligned}$$

Therefore, it defines an inner product on degree-n polynomials.

$$\begin{aligned} < 1, x > &= \int_0^\pi x\mathrm{d}x \\ &= \frac{1}{2}\pi^2 \\ < 1, x^2 > &= \int_0^\pi x^2\mathrm{d}x \\ &= \frac{1}{3}\pi^3 \end{aligned}$$

□

(c)

$$< p, q >= \int_{-\infty}^{\infty} p(x)\overline{q(x)}\mathrm{d}x$$

Proof.
(1) Positivity:
$\forall\ p(x) \in C[x]$,

$$\begin{aligned} < p, p > &= \int_{-\infty}^{\infty} p(x)\overline{p(x)}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} |p(x)|^2\mathrm{d}x \\ &\geqslant 0 \end{aligned}$$

$< p, p >= 0$ iff $|p(x)| = 0$ $(a.e.)$, i.e. $p(x) \equiv 0$ ($p(x)$ is continuous at $\mathbb{R}$).
(2) Conjugate symmetry:

Solution (cont.)

$\forall\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned}
\overline{<p,q>} &= \overline{\int_{-\infty}^{\infty} p(x)\overline{q(x)}\mathrm{d}x} \\
&= \overline{\int_{-\infty}^{\infty} \sum_{i,j=0}^{n} p_i\overline{q_j}x^{i+j}\mathrm{d}x} \\
&= \overline{\sum_{i,j=0}^{n} p_i\overline{q_j}\ (L)\int_{-\infty}^{\infty} x^{i+j}\mathrm{d}x} \\
&= \sum_{i,j=0}^{n} q_i\overline{p_j}\ (L)\int_{-\infty}^{\infty} x^{i+j}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} q(x)\overline{p(x)}\mathrm{d}x \\
&=<q,p>
\end{aligned}$$

When the Lebesgue integration is ∞, the equation still holds.

(3) Homogeneity:

$\forall\ \alpha \in \mathbb{C},\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned}
<\alpha p,q> &= \int_{-\infty}^{\infty} \alpha p(x)\overline{q(x)}\mathrm{d}x \\
&= \alpha\int_{\infty}^{\infty} p(x)\overline{q(x)}\mathrm{d}x \\
&= \alpha <p,q>
\end{aligned}$$

(4) Linearity:

$\forall\ p(x),\ q(x),\ r(x) \in C[x]$,

$$\begin{aligned}
<p+q,r> &= \int_{-\infty}^{\infty} [p(x)+q(x)]\overline{r(x)}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} p(x)\overline{r(x)}\mathrm{d}x + \int_{0}^{\pi} q(x)\overline{r(x)}\mathrm{d}x \\
&=<p,r>+<q,r>
\end{aligned}$$

Therefore, it defines an inner product on degree-n polynomials.

$$\begin{aligned}
<1,x> &= \int_{-\infty}^{\infty} x\mathrm{d}x \\
&= \infty \\
<1,x^2> &= \int_{-\infty}^{\infty} x^2\mathrm{d}x \\
&= \infty
\end{aligned}$$

□

(d)

$$<p,q>= \int_{-\infty}^{\infty} p(x)\overline{q(x)}e^{-|x|}\mathrm{d}x$$

Proof.
(1) Positivity:
$\forall\ p(x) \in C[x]$,

$$\begin{aligned} < p, p > &= \int_{-\infty}^{\infty} p(x)\overline{p(x)}e^{-|x|}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} |p(x)|^2 e^{-|x|}\mathrm{d}x \\ &\geqslant 0 \end{aligned}$$

$< p, p >= 0$ iff $|p(x)| = 0$ $(a.e.)$, i.e. $p(x) \equiv 0$ ($p(x)$ is continuous at $\mathbb{R}$).
(2) Conjugate symmetry:
$\forall\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned} \overline{< p, q >} &= \overline{\int_{-\infty}^{\infty} p(x)\overline{q(x)}e^{-|x|}\mathrm{d}x} \\ &= \overline{\int_{-\infty}^{\infty} \sum_{i,j=0}^{n} p_i\overline{q_j}x^{i+j}e^{-|x|}\mathrm{d}x} \\ &= \overline{\sum_{i,j=0}^{n} p_i\overline{q_j}\ (L)\int_{-\infty}^{\infty} x^{i+j}e^{-|x|}\mathrm{d}x} \\ &= \overline{\sum_{i,j=0}^{n} q_i\overline{p_j}\ (L)\int_{-\infty}^{\infty} x^{i+j}e^{-|x|}\mathrm{d}x} \\ &= \int_{-\infty}^{\infty} q(x)\overline{p(x)}e^{-|x|}\mathrm{d}x \\ &=< q, p > \end{aligned}$$

It is because

$$\begin{aligned} \int_{-\infty}^{\infty} x^k e^{-|x|}\mathrm{d}x &= 2\int_{0}^{\infty} x^k e^{-x}\mathrm{d}x \\ &= 2(-kx^{k-1}e^{-x} - k(k-1)x^{k-2}e^{-x} - \cdots - k!e^{-x})\big|_0^{\infty} \\ &= 2k! < \infty \end{aligned}$$

(3) Homogeneity:
$\forall\ \alpha \in \mathbb{C},\ p(x),\ q(x) \in C[x]$,

$$\begin{aligned} < \alpha p, q > &= \int_{-\infty}^{\infty} \alpha p(x)\overline{q(x)}e^{-|x|}\mathrm{d}x \\ &= \alpha\int_{-\infty}^{\infty} p(x)\overline{q(x)}e^{-|x|}\mathrm{d}x \\ &= \alpha < p, q > \end{aligned}$$

(4) Linearity:
$\forall\ p(x),\ q(x),\ r(x) \in C[x]$,

$$\begin{aligned} < p + q, r > &= \int_{-\infty}^{\infty} [p(x) + q(x)]\overline{r(x)}e^{-|x|}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} p(x)\overline{r(x)}e^{-|x|}\mathrm{d}x + \int_{-\infty}^{\infty} q(x)\overline{r(x)}e^{-|x|}\mathrm{d}x \\ &=< p, r > + < q, r > \end{aligned}$$

Solution (cont.)

Therefore, it defines an inner product on degree-n polynomials.

$$\begin{aligned} < 1, x > &= \int_{-\infty}^{\infty} x e^{-|x|} \mathrm{d}x \\ &= 0 \\ < 1, x^2 > &= \int_{-\infty}^{\infty} x^2 e^{-|x|} \mathrm{d}x \\ &= 4 \end{aligned}$$

$\square$

Question 3

(a) Prove or disprove:

$$< u, v >= \sum_{n=0}^{\infty} u_n \overline{v_n}$$

is an inner product on $l^2(N)$. Check the properties.

Proof.
(1) Positivity:
$\forall\ u \in l^2(N)$,

$$\begin{aligned} < u, u > &= \sum_{n=0}^{\infty} u_n \overline{u_n} \\ &= \sum_{n=0}^{\infty} |u_n|^2 \\ &\geqslant 0 \end{aligned}$$

$< u, u >= 0$ iff $|u_1(x)| = |u_2(x)| = \cdots = 0$, i.e. $u(x) \equiv 0$.
(2) Conjugate symmetry:
$\forall\ u, v \in l^2(N)$,

$$\begin{aligned} \overline{< u, v >} &= \overline{\sum_{n=0}^{\infty} u_n \overline{v_n}} \\ &= \sum_{n=0}^{\infty} \overline{u_n} v_n \\ &=< v, u > \end{aligned}$$

(3) Homogeneity:
$\forall\ \alpha \in \mathbb{C},\ u,\ v \in l^2(N)$,

$$\begin{aligned} < \alpha u, v > &= \sum_{n=0}^{\infty} \alpha u_n \overline{v_n} \\ &= \alpha \sum_{n=0}^{\infty} u_n \overline{v_n} \\ &= \alpha < u, v > \end{aligned}$$

(4) Linearity:
$\forall\ u,\ v,\ w \in l^2(N)$,

Solution (cont.)

$\because$

$$\begin{aligned}
< u+v, w > &= \sum_{n=0}^{\infty}(u_n+v_n)\overline{w_n} \\
< u, w > + < v, w > &= \sum_{n=0}^{\infty} u_n\overline{w_n} + \sum_{n=0}^{\infty} v_n\overline{w_n} \\
&\leqslant \sum_{n=0}^{\infty} |u_n||\overline{w_n}| + \sum_{n=0}^{\infty} |v_n||\overline{w_n}| \\
&\leqslant \sum_{n=0}^{\infty} |u_n|^2 + \sum_{n=0}^{\infty} |w_n|^2 + \sum_{n=0}^{\infty} |v_n|^2 + \sum_{n=0}^{\infty} |w_n|^2 \\
&< \infty
\end{aligned}$$

$\therefore$

$$< u+v, w > = < u, w > + < v, w >$$

Therefore, it defines an inner product on degree-n polynomials. □

(b) For $u_n = 2^{-n}$ and $v_n = 3^{-n}$ compute $< u, v >$ and the angle between u and v.

Proof.

$$\begin{aligned}
< u, v > &= \sum_{n=0}^{\infty} u_n\overline{v_n} \\
&= \sum_{n=0}^{\infty} 2^{-n}3^{-n} \\
&= \sum_{n=0}^{\infty} 6^{-n} \\
&= \frac{6}{5} \\
\cos\angle(u, v) &= \frac{< u, v >}{\|u\|\|v\|} \\
&= \frac{6}{5\sqrt{\sum\limits_{n=0}^{\infty} 2^{-2n}}\sqrt{\sum\limits_{n=0}^{\infty} 3^{-2n}}} \\
&= \frac{6}{5\cdot\sqrt{\frac{4}{3}\cdot\frac{9}{8}}} \\
&= \frac{2\sqrt{6}}{5} \\
\angle(u, v) &= \arccos\frac{2\sqrt{6}}{5}
\end{aligned}$$

□

Question 4

(a) Prove the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for any x and y vectors in a real inner product space with norm $\|\cdot\|$.

Proof.

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &=< x+y, x+y > + < x-y, x-y > \\ &=< x,x > + < y,x > + < x,y > + < y,y > \\ &\quad + < x,x > - < y,x > - < x,y > + < y,y > \\ &= 2 < x,x > +2 < y,y > \\ &= 2\|x\|^2 + 2\|y\|^2\end{aligned}$$

$\square$

(b) Prove

$$< x,y >= \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right)$$

for nonzero vectors x and y.

Proof.

$\forall\ x,\ y \neq 0,\ \|x\|,\ \|y\| \neq 0$

$$\begin{aligned}\|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right) &= \|x\|\|y\| \left(1 - \frac{1}{2}\left\langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\right\rangle\right) \\ &= \|x\|\|y\| \left(1 - \frac{1}{2}\left\langle \frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle + \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle - \frac{1}{2}\left\langle \frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle\right) \\ &= \|x\|\|y\| \left(1 - \frac{1}{2} + \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle - \frac{1}{2}\right) \\ &= \|x\|\|y\| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \\ &=< x,y >\end{aligned}$$

$\square$

(c) Given a subspace A of a real inner-product space V, and a vector $x \in V$ which is not in A, show that there is a constant $\gamma < 1$ such that

$$| < a,x > | \leqslant \gamma \|a\|\|x\|$$

for all $a \in A$.

Proof.

$\because \quad A$ is a subspace of V, $x \in V \setminus A$

Solution (cont.)

$\therefore \quad 0 \in A,\, x \neq 0$

(1) $a = 0$: Inequality holds.

(2) $\forall\ a \in A,\ a \neq 0$:

By Question 4 (b), we have

$$| < a, x > | = \|a\|\|x\| \left| 1 - \frac{1}{2} \left\| \frac{a}{\|a\|} - \frac{x}{\|x\|} \right\|^2 \right|$$

$\because \quad \dfrac{a}{\|a\|} \in A,\ \dfrac{x}{\|x\|} \in V \setminus A,\ \left\| \dfrac{a}{\|a\|} \right\| = \left\| \dfrac{x}{\|x\|} \right\| = 1$

$\therefore$ the orthogonal projection of $\dfrac{x}{\|x\|}$ onto A is the unique vector $v_0 \in A$, $\dfrac{x}{\|x\|} - v_0 \perp A$ and

$$\left\| \frac{x}{\|x\|} - v_0 \right\| = \min_{a \in A} \left\| \frac{x}{\|x\|} - a \right\| \neq 0$$

$\therefore$

$$\left\| \frac{a}{\|a\|} - \frac{x}{\|x\|} \right\|^2 \geqslant \left\| \frac{x}{\|x\|} - v_0 \right\|^2$$

$\therefore$

$$\left\| \frac{x}{\|x\|} - v_0 \right\|^2 < \left\| \frac{x}{\|x\|} \right\|^2 = 1$$

We set $\gamma = \left| 1 - \dfrac{1}{2} \left\| \dfrac{x}{\|x\|} - v_0 \right\|^2 \right| < 1$, then

$$| < a, x > | \leqslant \gamma \|a\|\|x\|$$

$\square$

Question 5

For $p = 1, 2, \cdots$ define the Sobolev space $H^p = H^p(-\pi, \pi)$ by

$$H^p = \{g \in L^2 = L^2(-\pi, \pi) | g \text{ is } 2\pi\text{-periodic and } g', g'', \cdots, g^{(p)} \in L^2\},$$

with

$$< f, g >_p = \int_{-\pi}^{\pi} f(x)\overline{g}(x) + f'(x)\overline{g}'(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)\mathrm{d}x.$$

For $p = 0$ we set $H^0 = L^2$ with the usual L^2 inner product $< \cdot, \cdot >$.

(a) Show that $< f, g >_p$ defines an inner product on H^p.

Proof.
(1) Positivity:
$\forall\ f \in H^p$,

$$\begin{aligned} < f, f >_p &= \int_{-\pi}^{\pi} |f(x)|^2 + |f'(x)|^2 + \cdots + |f^{(p)}(x)|^2 \mathrm{d}x \\ &\geqslant 0 \end{aligned}$$

$< f, f >_p = 0$ iff $|f(x)|^2 = |f'(x)|^2 = \cdots = |f^{(p)}(x)|^2 = 0(a.e.)$, i.e. $f(x) \equiv 0$.
(2) Conjugate symmetry: $\forall\ f,\ g \in H^p$,

$$\begin{aligned} \overline{< f, g >} &= \overline{\int_{-\pi}^{\pi} f(x)\overline{g}(x) + f'(x)\overline{g}'(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)\mathrm{d}x} \\ &= \overline{\int_{-\pi}^{\pi} Re[f(x)\overline{g}(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)]\mathrm{d}x + \int_{-\pi}^{\pi} Im[f(x)\overline{g}(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)]\mathrm{d}x} \\ &= \int_{-\pi}^{\pi} Re[f(x)\overline{g}(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)]\mathrm{d}x - \int_{-\pi}^{\pi} Im[f(x)\overline{g}(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)]\mathrm{d}x \\ &= \int_{-\pi}^{\pi} \overline{f(x)\overline{g}(x) + f'(x)\overline{g}'(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)}\mathrm{d}x \\ &= \int_{-\pi}^{\pi} g(x)\overline{f}(x) + g'(x)\overline{f}'(x) + \cdots + g^{(p)}(x)\overline{f}^{(p)}(x)\mathrm{d}x \\ &= < g, f > \end{aligned}$$

(3) Homogenity:
$\forall\ c \in C,\ f,\ g \in H^p$,

$$\begin{aligned} < cf, g > &= \int_{-\pi}^{\pi} cf(x)\overline{g}(x) + cf'(x)\overline{g}'(x) + \cdots + cf^{(p)}(x)\overline{g}^{(p)}(x)\mathrm{d}x \\ &= \int_{-\pi}^{\pi} f(x)\overline{g}(x) + f'(x)\overline{g}'(x) + \cdots + f^{(p)}(x)\overline{g}^{(p)}(x)\mathrm{d}x \\ &= c < f, g > \end{aligned}$$

(4) Linearity:

Solution (cont.)

$\forall\ f,\ g,\ h \in H^p$,

$$\begin{aligned}
< f+g, h > &= \int_{-\pi}^{\pi} [f(x)+g(x)]\overline{h}(x) + [f'(x)+g'(x)]\overline{h}'(x) + \cdots + [f^{(p)}(x)+g^{(p)}(x)]\overline{h}^{(p)}(x)\mathrm{d}x \\
&= \int_{-\pi}^{\pi} f(x)\overline{h}(x) + f'(x)\overline{h}'(x) + \cdots + f^{(p)}(x)\overline{h}^{(p)}(x)\mathrm{d}x \\
&\quad + \int_{-\pi}^{\pi} g(x)\overline{h}(x) + f'(x)\overline{h}'(x) + \cdots + g^{(p)}(x)\overline{h}^{(p)}(x)\mathrm{d}x \\
&= < f, h > + < g, h >
\end{aligned}$$

Therefore, $< f, g >_p$ defines an inner product on H^p. □

(b) Compute the norm $\|f\|_p = \sqrt{< f, f >_p}$ in H^p of $f(x) = e^{iax}$ and the angle in H^p between f and $g(x) = e^{ibx}$ for $a,\ b \in Z$.

Proof.

$$\begin{aligned}
\|f\|_p &= \sqrt{< f, f >_p} \\
&= \sqrt{\int_{-\pi}^{\pi} |e^{iax}|^2 + (ia)(-ia)|e^{iax}|^2 + \cdots + (ia)^p(-ia)^p|e^{iax}|^2\mathrm{d}x} \\
&= \sqrt{2\pi[1 + a^2 + \cdots + a^{2p}]} \\
&= \sqrt{2\pi\frac{1-a^{2p}}{1-a^2}} \\
\|g\|_p &= \sqrt{< g, g >_p} \\
&= \sqrt{\int_{-\pi}^{\pi} |e^{ibx}|^2 + (ib)(-ib)|e^{ibx}|^2 + \cdots + (ib)^p(-ib)^p|e^{ibx}|^2\mathrm{d}x} \\
&= \sqrt{2\pi[1 + b^2 + \cdots + b^{2p}]} \\
&= \sqrt{2\pi \cdot \frac{1-b^{2p}}{1-b^2}} \\
< f, g >_p &= \int_{-\pi}^{\pi} e^{iax}e^{-ibx} + (ia)(-ib)e^{iax}e^{-ibx} + \cdots + (ia)^p(-ib)^p e^{iax}e^{-ibx}\mathrm{d}x \\
&= (1 + ab + \cdots + a^p b^p)\int_{-\pi}^{\pi} e^{i(a-b)x}\mathrm{d}x \\
&= \begin{cases} 0, & a \neq b,\ a, b \in \mathbb{Z} \\ 2\pi \cdot \dfrac{1-a^p b^p}{1-ab}, & a = b \end{cases} \\
\cos\angle(f, g) &= \frac{< f, g >_p}{\|f\|_p\|g\|_p} \\
&= \begin{cases} 0, & a \neq b \\ 1, & a = b \end{cases}
\end{aligned}$$

Solution (cont.)

$$\angle(f,g)=\begin{cases}\frac{\pi}{2}+2k\pi, & a\neq b\\ 0, & a=b\end{cases}$$

□

(c) For $f\in L^2$ define a generalized derivative $f^{(p)}$ by the requirement

$$< f^{(p)}, g >= (-1)^p < f, g^{(p)} >$$

for all $g\in H^p$. Let $f\in L^2$ be given by $f(x)=1$ for $x>0$ and $f(x)=0$ for $x<0$. For $1\leqslant q\leqslant p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g\in H^p$.

Proof.

$$\begin{aligned}< f^{(q)}, g > &= (-1)^q < f, g^{(q)} >\\ &= (-1)^q\int_{-\pi}^{\pi} f(x)\overline{g^{(q)}}(x)\mathrm{d}x\\ &= (-1)^q\int_{0}^{\pi}\overline{g^{(q)}}(x)\mathrm{d}x\\ &= (-1)^q[\overline{g^{(q-1)}}(\pi)-\overline{g^{(q-1)}}(0)]\end{aligned}$$

□

(d) Fix $-\pi<a<b<\pi$ and let $f\in L^2$ be given by $f(x)=1$ for $a<x<b$ and $f(x)=0$ otherwise. For $1\leqslant q\leqslant p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g\in H^p$.

Proof.

$$\begin{aligned}< f^{(q)}, g > &= (-1)^q < f, g^{(q)} >\\ &= (-1)^q\int_{-\pi}^{\pi} f(x)\overline{g^{(q)}}(x)\mathrm{d}x\\ &= (-1)^q\int_{a}^{b}\overline{g^{(q)}}(x)\mathrm{d}x\\ &= (-1)^q\overline{\int_{a}^{b} g^{(q)}(x)\mathrm{d}x}\\ &= (-1)^q\overline{[g^{(q-1)}(b)-g^{(q-1)}(a)]}\\ &= (-1)^q[\overline{g^{(q-1)}}(b)-\overline{g^{(q-1)}}(a)]\end{aligned}$$

□

(e) Compute the generalized derivatives of $f(x)=Q(x)$ for $a<x<b$ and $f(x)=0$ otherwise, where Q is a degree-n polynomial.

Proof. $\forall\ q \in N,\ q \leqslant n$

$$\begin{aligned} < f^{(q)}, g > &= (-1)^q < f, g^{(q)} > \\ &= (-1)^q \int_{-\pi}^{\pi} f(x)\overline{g^{(q)}}(x)\mathrm{d}x \\ &= (-1)^q \int_{a}^{b} Q(x)\overline{g^{(q)}}(x)\mathrm{d}x \\ &= (-1)^q \int_{a}^{b} Q(x)\mathrm{d}\overline{g^{(q-1)}}(x) \\ &= \sum_{j=0}^{n} (-1)^{q+j}[Q^{(j)}(b) \cdot \overline{g^{(q-1-j)}}(b) - Q^{(j)}(a) \cdot \overline{g^{(q-1-j)}}(a)] \end{aligned}$$

$\square$