Math 118 Sample Final Problems

$\boxed{1.}$ Let φ be a smooth function with

$$|\varphi(x)| \le \frac{C}{1+|x|^{10}} \quad \text{as } |x| \to \infty,$$

and

$$\hat{\varphi}(k) = \int_{-\infty}^{\infty} \varphi(x)\, e^{-ikx}\, dx$$

be its Fourier transform.

(a) Suppose $\varphi(0) = 0$. Show that

$$\psi(x) = \varphi(x)/x$$

is a smooth function with

$$|\psi(x)| \le \frac{C}{1+|x|^{10}} \quad \text{as} \quad |x| \to \infty.$$

(b) Show that

$$\hat{\psi}'(k) = i\,\hat{\varphi}(k)$$

(c) Show that

$$\int_{-\infty}^{\infty} \hat{\varphi}(k)\, dk = 0 = \varphi(0).$$

$\boxed{2.}$ Compute the Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

for

$$f(x) = -2x e^{-x^2}.$$

$\boxed{3.}$ Compute the Fourier transform of

$$f(x) = \frac{-2x}{(1+x^2)^2}$$

and use it to evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^4}\, dx.$$

$\boxed{4.}$ Compute the Fourier series coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \quad k \in \mathbb{Z}$$

for

$$f(x) = \pi^2 - x^2.$$

5. Compute the Fourier series coefficients of

$$f(x) = \begin{cases} 1 & |x| \le \pi/2 \\ 0 & \pi/2 \le |x| \le \pi \end{cases}.$$

Show without any work that the Fourier series of f converges to f <u>in L^2</u> but <u>not</u> <u>uniformly</u>.

6. (a) Is $\{ f_k(x) = e^{-(x-k)^2} \}_{k \in \mathbb{Z}}$ an orthogonal basis for $L^2(\mathbb{R})$?

(b) If not, how would you fix it?

(c) Suggest another way to fix it and compare the advantages.

$\boxed{7.}$ Let S_n be the span in $L^2(\mathbb{R})$ of the n functions

$$f_k(x) = x^{k-1} e^{-x^2/2}.$$

(a) Find the orthogonal complement T_2 of S_2 in S_4.

(b) Construct a 2-element ON basis $\{g_k\}$ for T_2.

(c) Find a matrix M which takes

$$S_4 \longrightarrow S_2 \oplus T_2.$$

$$\begin{array}{c} \| \\ \left\{ \sum_{j=1}^{4} a_j f_j \right\} \end{array} \longrightarrow \left\{ \sum_{j=1}^{2} a_j f_j + \sum_{j=1}^{2} b_j g_j \right\}$$

$$M : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \longmapsto \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

$\boxed{8.}$ Prove that a set of orthonormal vectors is linearly independent.

$\boxed{9.}$ Let $f \in L^2 [-\pi, \pi]$.
Prove that

$$|\hat{f}(k)| \to 0 \quad \text{as } |k| \to \infty.$$

Don't work too hard.

$\boxed{10.}$ Compute the Fourier series coefficients of

$$f(x) = e^{zx} \quad \text{on } |x| \le \pi$$

where z is an arbitrary complex number. Discuss convergence.

11. Evaluate the Fourier transforms of

(a) $f(x) = e^{-x^2}$

(b) $f(x) = -2x e^{-x^2}$

(c) $f(x) = (-2+4x^2)e^{-x^2}$

(d) $f(x) = \left(-2+4\left((x-\pi)/17\right)^2\right) \cdot e^{-(x-\pi)^2/17^2}$.

12. Suppose $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$, so

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}. \quad (*)$$

Show that sampling f at <u>twice</u> the resolution gives

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{j\pi}{2\Omega}\right) \boxed{}$$

where the box converges <u>faster than</u> $(*)$.

(Hint: look at even and odd terms.)

$\boxed{13.}$ (a) Evaluate

$$S_n(x) = \sum_{j=1}^{n} \sin(jx)$$

(b) Show that the Cesaro sum

$$\frac{1}{N+1} \sum_{n=0}^{N} S_n(x) \to \frac{1}{2} \cot(x/2)$$

as $N \to \infty$.

$\boxed{14.}$ Let $f(x) = x^2$ for $|x| \le \pi$

and $F(x) = x^4$ for $|x| \le \pi$.

Is it true that

$$\widehat{F''}(k) = 12 \hat{f}(k) ?$$

Why or why not?

$\boxed{15.}$ Evaluate the Fourier coefficients of

$$f(x) = e^{-i\alpha x} \qquad |x|<\pi,$$

extended to be 2π-periodic. Assume $\alpha \notin \mathbb{Z}$. Use your result to evaluate

$$\sum_{-\infty}^{\infty} \frac{e^{ikx}}{k+\alpha}$$

and discuss convergence. Conclude that

$$\sum_{-\infty}^{\infty} \frac{\sin(n+\alpha)x}{n+\alpha} = \pi \qquad 0<|x|<\pi$$

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and discuss why this formula is surprising.

$\boxed{16.}$ Show that any function

$f: [-\pi,\pi] \to \mathbb{R}$ such that f' is continuous and $\int f = 0$ satisfies

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx \le \int_{-\pi}^{\pi} |f'(x)|^2 \, dx.$$

$\boxed{17.}$ Define Laguerre polynomials by

$$L_n(x) = e^x D^n(x^n e^{-x}) \qquad 0 \le x < \infty$$

where $D = \frac{d}{dx}$. Show that the Laguerre <u>functions</u>

$$\ell_n(x) = e^{-x/2} L_n(x)$$

are <u>orthogonal</u> in $L^2(0,\infty)$.
(Hint: integrate by parts).
Evaluate $L_0(x)$, $L_1(x)$ and $L_2(x)$.

$\boxed{18.}$ Suppose f and g are smooth periodic functions on $[-\pi,\pi]$.
Let

$$g_m(x) = g(mx) \qquad \text{for } m \in \mathbb{Z}.$$

Assume $\int_{-\pi}^{\pi} g(x)\,dx = 0$. Show that

$$\int_{-\pi}^{\pi} f(x-y)\, g_m(y)\, dy \longrightarrow 0$$

uniformly as $m \to \infty$.

$\boxed{19.}$ (a) Use the Poisson sum formula to show that

$$\frac{2}{1-e^{-t}} = 1 + \sum_{n\in\mathbb{Z}} \frac{2t}{t^2+4\pi^2 n^2}, \qquad t>0$$

(b) Define the Bernoulli numbers b_n by

$$\frac{1}{1-e^{-t}} = \frac{-1}{t} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{b_n}{n!}(-t)^n \qquad t>0$$

Compute b_1 and b_3.

(c) The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Use (a) and (b) to show that

$$\zeta(2m) = \frac{(-1)^{m-1}\, 2^{2m-1}}{(2m-1)!}\, b_{2m-1}\, \pi^{2m}$$

and evaluate

$$\sum_1^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_1^{\infty} \frac{1}{n^4}.$$

$\boxed{20.}$ Use Plancherel's theorem to compute

$$\int_{-\infty}^{\infty} \frac{\sin(x)^2}{x^2}\,dx.$$

$\boxed{21.}$ Are the following filters

(A) linear (B) time-invariant (C) causal?

1. $L_1 f(t) = f(t) - \int_t^{t^2} f(s)\,ds$

2. $L_2 f(t) = \int_{-\infty}^{\infty} f(x)\, e^{-(x-t)^2}\,dx$

3. $L_3 f(x) = \int_0^{\infty} f(x-t)\, e^{-t}\,dt$

$\boxed{22.}$ Suppose $P: H \to H$ is a bounded operator on an inner product space satisfying

$$P = P^* = P^2.$$

Show that $P(I-P) = 0$. Give a nontrivial example. Give an example on $H = L^2(\mathbb{R})$ where P is an integral operator.

$\boxed{23.}$ Show that

$$\hat{\varphi}(\xi) = 2\sqrt{\frac{2}{\pi}}\ \frac{\sin^2(\xi/2)}{\xi^2\sqrt{1-\frac{2}{3}\sin^2(\xi/2)}}$$

has orthonormal shifts:

$$\langle \varphi(\cdot - k), \varphi(\cdot - \ell)\rangle = \delta_{k\ell}.$$

$\boxed{24.}$ Show that $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$ satisfies

$$\varphi(x) = \varphi(2x) + \sum_k \frac{2(-1)^k}{(2k+1)\pi}\,\varphi(2x-2k-1)$$

$\boxed{25.}$