STAT 150: Stochastic Processes

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Homework 9

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Solutions by

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GS 6.9.8

Let X be the simple symmetric random walk on the integers in continuous time, so that

$$p_{i,i+1}(h) = p_{i,i-1}(h) = \frac{1}{2}\lambda h + o(h).$$

Show that the walk is persistent. Let T be the time spent visiting m during an excursion from 0, Find the distribution of T.

$\because$

$$p_{i,i+1}(h) = p_{i,i-1}(h) = \frac{1}{2}\lambda h + o(h).$$

$\therefore$

$$\begin{aligned}
g_{i,i+1} &= \lim_{h\to 0}\frac{p_{i,i+1}(h)}{h} = \frac{\lambda}{2}\\
g_{i,i-1} &= \lim_{h\to 0}\frac{p_{i,i-1}(h)}{h} = \frac{\lambda}{2}\\
g_{i,i} &= \lim_{h\to 0}\frac{p_{i,i}(h)}{h} = \frac{\lambda}{2}\\
&= \lim_{h\to 0}\frac{(1-p_{i,i+1}(h)-p_{i,i-1}(h))-1}{h}\\
&= -\lambda\\
g_i &= -g_{i,i}\\
&= \lambda > 0
\end{aligned}$$

Let $Y_n = X(T_n+)$ denote the jump chain of X where T_n is the time of the nth changes of X. Then Y_n is a random walk with transition probability

$$\begin{aligned}
p_{i,i+1} &= \frac{g_{i,i+1}}{g_i} = \frac{1}{2}\\
p_{i,i-1} &= \frac{g_{i,i-1}}{g_i} = \frac{1}{2}
\end{aligned}$$

$\because$

$$\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{P}(Y_n = 0|Y_0 = 0) &= \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}}\\
&\approx \sum_{n=1}^{\infty}\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4n\pi}}{\left(\frac{n}{e}\right)^{2n}2n\pi}\frac{1}{2^{2n}}\\
&= \sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi n}}\\
&= \infty
\end{aligned}$$

$\therefore\quad Y_n$ is recurrent and $g_i > 0$

$\therefore\quad X$ is recurrent

Since the chain is symmetric, we can suppose that $m > 0$. Given the initial $X(0) = 0$, let $q_i (i = 1, \cdots, m-1)$ denote the probability that the chain visits m for the first time before hitting state 0 begining at state i at some time $t > 0$. Let q_0 denote the probability that the chain visits m for the first time during an excursion

Solution (cont.)

given $P(0) = 0$. The initial step must be to the right otherwise m won't be visited in this excursion.

$$\begin{cases} q_0 = \frac{1}{2} q_1 \\ q_1 = \frac{1}{2} q_0 + \frac{1}{2} q_2 \\ q_2 = \frac{1}{2} q_1 + \frac{1}{2} q_3 \\ \qquad \vdots \\ q_{m-2} = \frac{1}{2} q_{m-3} + \frac{1}{2} q_{m-1} \\ q_{m-1} = \frac{1}{2} \end{cases}$$

We have

$$q_0 = \frac{1}{2m}$$

Let $r_i (i = 1, 2, \cdots, m)$ denotes the probability that starting at state i, the chain visits state 0 for the first time before returning to i.

$$\begin{cases} r_m = \frac{1}{2} r_{m-1} \\ r_{m-1} = \frac{1}{2} r_m + \frac{1}{2} r_{m-2} \\ \qquad \vdots \\ r_2 = \frac{1}{2} r_1 + \frac{1}{2} r_3 \\ r_1 = \frac{1}{2} \end{cases}$$

we have

$$r_m = \frac{1}{2m}$$

Therefore, the probability that starting at state m, the chain returns state m before hitting state i, is $1 - r_m = 1 - \frac{1}{2m}$.

Let N denote the number of visits to state m during an excursion, then $\forall\, n \in \mathbb{N}^+$,

$$\mathbb{P}(N \geqslant n) = \left(1 - \frac{1}{2m}\right)^{n-1} \frac{1}{2m}$$

$$\mathbb{P}(N = 0) = 1 - \frac{1}{2m}$$

$\therefore$

$$\begin{aligned} \mathbb{P}(N = n) &= \mathbb{P}(N \geqslant n) - \mathbb{P}(N \geqslant n+1) \\ &= \left(1 - \frac{1}{2m}\right)^{n-1} \left(\frac{1}{2m}\right)^2 \end{aligned}$$

$\because$

$$T = \sum_{n=0}^{N} T_n$$

where T_n is the time spent at the nth visit to state m during an excursion and $T_n \sim Exp(\lambda)$. And $\forall\, n \in \mathbb{N}^+$,

$$\sum_{i=1}^{n} T_n \sim \Gamma(n, \lambda)$$

Solution (cont.)

$\therefore \quad \forall\, t > 0,$

$$\begin{aligned}
\mathbb{P}(T = t) &= \mathbb{P}(T = t, N \geqslant 1) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\sum_{i=1}^{n} T_n = t)\mathbb{P}(N = n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} \left(1 - \frac{1}{2m}\right)^{n-1} \left(\frac{1}{2m}\right)^2 \\
&= \lambda e^{-\lambda t} \left(\frac{1}{2m}\right)^2 \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} t^{n-1} \left(1 - \frac{1}{2m}\right)^{n-1} \\
&= \lambda e^{-\lambda t} \left(\frac{1}{2m}\right)^2 e^{\lambda\left(1 - \frac{1}{2m}\right)t} \\
&= \frac{1}{2m} \cdot \frac{\lambda}{2m} e^{-\frac{\lambda}{2m} t}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(T = 0) &= \mathbb{P}(N = 0) + \sum_{n=1}^{\infty} \mathbb{P}(\sum_{i=1}^{n} T_n = 0)\mathbb{P}(N = n) \\
&= 1 - \frac{1}{2m}
\end{aligned}$$

GS 6.9.9

Let i be a transient state of a continuous-time Markov chain X with $X(0) = i$. Show that the total time spent in state i has an exponential distribution.

Let

$$f_{ii} = \mathbb{P}(\text{the chain ever returns to } i | X(0) = i)$$

$\because \quad$ state i is a transient state

$\therefore$

$$0 \leqslant f_{ii} < 1$$

Let N denote the number of sojourns in i. Since the chain begins in i, $N \geqslant 1$ and there will be $n - 1$ returns to state i when $N = n$. Then N has an geometric distribution with parameter f_{ii}, i.e. $\forall\, n \in \mathbb{N}^+$,

$$\mathbb{P}(N = n) = f_{ii}^{n-1}(1 - f_{ii})$$

$\because \quad T_i$, the time spent at each sojourn has an exponential distribution with parameter λ. The total time spent in state i is $T = \sum\limits_{i=1}^{N} T_i$. And $\forall\, n \in \mathbb{N}^+$,

$$\sum_{i=1}^{n} T_n \sim \Gamma(n, \lambda)$$

Solution (cont.)

$\therefore \quad \forall\, t \geqslant 0,$

$$
\begin{aligned}
\mathbb{P}(T=t) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} T_i = t\right) \mathbb{P}(N=n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\sum_{i=1}^{n} T_n = t) \mathbb{P}(N=n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} f_{ii}^{n-1}(1-f_{ii}) \\
&= \lambda e^{-\lambda t}(1-f_{ii}) \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} t^{n-1} f_{ii}^{n} \\
&= (1-f_{ii})\lambda e^{-(1-f_{ii})\lambda t}
\end{aligned}
$$

i.e. $T \sim Exp((1-f_{ii})\lambda)$

PK Exercises 7.1.2

Consider a renewal process in which the interoccurrence times have an exponential distribution with parameter λ:

$$f(x) = \lambda e^{-\lambda x}, \text{ and } F(x) = 1 - e^{-\lambda x} \text{ for } x > 0.$$

Calculate $F_2(t)$ by carrying out the appropriate convolution [see the equation just piror to (7.3)], and then determine $Pr\{N(t) = 1\}$ from equation (7.5).

Let X_i denote the ith interarrival time and W_i denote the waiting time until the ith event occurs. Then

$$W_1 = X_1$$

$$W_2 = X_1 + X_2$$

$\because$

$$f_{X_i}(x) = f(x) = \lambda e^{-\lambda x}$$

$$F_1(x) = F_{W_1}(x) = F_{X_i}(x) = F(x) = 1 - e^{-\lambda x}$$

for $x > 0$ and X_1, X_2 are independent

$\therefore$

$$
\begin{aligned}
F_2(t) &= \int_0^t F(t-x)\mathrm{d}F_1(x) \\
&= \int_0^t F(t-x)\mathrm{d}F(x) \\
&= \int_0^t (1-e^{-\lambda(t-x)})\lambda e^{-\lambda x}\mathrm{d}x \\
&= \int_0^t \lambda e^{-\lambda x}\mathrm{d}x - \int_0^t \lambda e^{-\lambda t}\mathrm{d}x \\
&= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}
\end{aligned}
$$

Solution (cont.)

for $t > 0$.

$\therefore$

$$\begin{aligned} Pr\{N(t) = 1\} &= F_1(t) - F_2(t) \\ &= (1 - e^{-\lambda x}) - (1 - e^{-\lambda t} - \lambda t e^{-\lambda t}) \\ &= \lambda t e^{-\lambda t} \end{aligned}$$

for $t > 0$.

PK Exercises 7.1.4

Consider a renewal process for which the lifetimes $X_1, X_2, \cdots$ are discrete random variables having the Poisson distribution with mean λ. That is,

$$Pr\{X_k = n\} = \frac{e^{-\lambda}\lambda^n}{n!} \qquad \text{for } n = 0, 1, \cdots$$

(a) What is the distribution of the waiting time W_k?

$\because \quad X_1, X_2, \cdots \overset{i.i.d.}{\sim} Poisson(\lambda)$

$\therefore \quad W_k = \sum_{i=1}^{k} X_i \sim Poisson(k\lambda)$, i.e.

$$Pr\{W_k = n\} = \frac{e^{-k\lambda}(k\lambda)^n}{n!}$$

for $n = 0, 1, \cdots$.

(b) Determine $Pr\{N(t) = k\}$.

$$\begin{aligned} Pr\{N(t) = k\} &= F_k(t) - F_{k+1}(t) \\ &= \sum_{n=0}^{\lfloor t \rfloor} \frac{e^{-k\lambda}(k\lambda)^n}{n!} - \sum_{n=0}^{\lfloor t \rfloor} \frac{e^{-(k+1)\lambda}[(k+1)\lambda]^n}{n!} \end{aligned}$$

PK Problems 7.1.2

From equation (7.5), and for $k \geqslant 1$, verify that

$$\begin{aligned} Pr\{N(t)=k\} &= Pr\{W_k \leqslant t < W_{k+1}\} \\ &= \int_0^t [1-F(t-x)]\mathrm{d}F_k(x), \end{aligned}$$

and carry out the evaluation when the interoccurrence times are exponentially distributed with parameter λ. so that $\mathrm{d}F_k$ is the gamma density

$$\mathrm{d}F_k(z) = \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z}\mathrm{d}z \quad \text{for } z > 0.$$

$$\begin{aligned} Pr\{N(t)=k\} &= Pr\{N(t) \geqslant k\} - Pr\{N(t) \geqslant k+1\} \\ &= Pr\{W_k \leqslant t\} - Pr\{W_{k+1} \leqslant t\} \\ &= Pr\{W_k \leqslant t, W_k \leqslant W_{k+1}\} - Pr\{W_{k+1} \leqslant t, W_k \leqslant W_{k+1}\} \\ &= Pr\{W_k \leqslant t, W_k \leqslant W_{k+1}, W_{k+1} > t\} \\ &= Pr\{W_k \leqslant t < W_{k+1}\} \\ Pr\{N(t)=k\} &= Pr\{W_k \leqslant t\} - Pr\{W_{k+1} \leqslant t\} \\ &= F_k(t) - \int_0^t F_k(t-x)\mathrm{d}F(x) \\ &= F_k(t) - F_k(t-x)F(x)\Big|_0^t + \int_0^t F(x)\mathrm{d}F_k(t-x) \\ &= \int_0^t 1\mathrm{d}F_k(x) - 0 - \int_0^t F(t-x)\mathrm{d}F_k(x) \\ &= \int_0^t [1-F(t-x)]\mathrm{d}F_k(x) \end{aligned}$$

since

$$F(0) = F_k(0) = 0$$

$\because$

$$F(x) = 1 - e^{-\lambda x}$$

$$\mathrm{d}F_k(z) = \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z}\mathrm{d}z \quad \text{for } z > 0$$

$\therefore \quad \forall\ k \in \mathbb{N}^+$,

$$\begin{aligned} Pr\{N(t)=k\} &= \int_0^t [1-F(t-x)]\mathrm{d}F_k(x) \\ &= \int_0^t e^{-\lambda(t-x)} \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}\mathrm{d}x \\ &= \int_0^t e^{-\lambda t} \frac{\lambda^k x^{k-1}}{(k-1)!}\mathrm{d}x \\ &= \frac{e^{-\lambda t}\lambda^k}{k!} x^k\Big|_0^t \\ &= \frac{e^{-\lambda t}(\lambda t)^k}{k!} \end{aligned}$$

PK Problems 7.2.3

Determine $M(n)$ when the interoccurrence times have the geometric distribution

$$Pr\{X_1 = k\} = p_k = \beta(1-\beta)^{k-1} \quad \text{for } k = 1, 2, \cdots$$

where $0 < \beta < 1$.

$\because$ for $k = 1, 2, \cdots$,

$$Pr\{X_1 = k\} = p_k = \beta(1-\beta)^{k-1}$$

$$M(0) = 0$$

$\therefore$ $\forall\, n \in \mathbb{N}^+$,

$$\begin{aligned} M(n) &= \sum_{k=1}^{n} p_k[1 + M(n-k)] + \sum_{k=n+1}^{\infty} p_k \cdot 0 \\ &= \sum_{k=1}^{n} p_k + \sum_{k=1}^{n-1} p_k M(n-k) \\ &= 1 - (1-\beta)^n + \sum_{k=1}^{n-1} \beta(1-\beta)^{k-1} M(n-k) \end{aligned}$$

$\therefore$

$$\begin{aligned} M(n) &= 1 - (1-\beta)^n + \sum_{k=2}^{n-1} \beta(1-\beta)^{k-1} M(n-k) + \beta M(n-1) \\ &= 1 - (1-\beta)^n + (1-\beta) \sum_{k=1}^{n-2} \beta(1-\beta)^{k-1} M(n-1-k) + \beta M(n-1) \\ &= 1 - (1-\beta)^n + (1-\beta)[M(n-1) - 1 + (1-\beta)^{n-1}] + \beta M(n-1) \\ &= \beta + M(n-1) \\ &= (n-1)\beta + M(1) \end{aligned}$$

$\because$

$$M(1) = p_1 = \beta$$

$\therefore$ $\forall\, n \in \mathbb{N}$,

$$M(n) = n\beta$$

Question 1

Suppose that $(X_n(t), t \geqslant 0)$, $n \geqslant 1$, are independent continuous time Markov chains, all with state space $S = \{0, 1\}$ and transition rates λ from 0 to 1 and μ from 1 to 0.

(a) Let $S_2(t) = X_1(t) + X_2(t)$. Show that $(S_2(t), t \geqslant 0)$ is a Markov chain and find its transition rate matrix.

Markov Chain

The infinitesimal matrix of $X_n(t)$ is

$$G = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$\because \quad X_n \in S$

$\therefore \quad$ the state space of $S_2(t)$ is $S' = \{0, 1, 2\}$

$\because \quad X_1(t), X_2(t)$ are independent Markov chains

$\therefore \quad \forall\ 0 \leqslant t_0 < t_1 < \cdots < t_n,\ i_0, \cdots, i_n \in S,$

$$\mathbb{P}(X_1(t_n) = i_n | X_1(t_0) = i_0, \cdots, X_1(t_{n-1}) = i_{n-1}) = \mathbb{P}(X_1(t_n) = i_n | X_1(t_{n-1}) = i_{n-1})$$
$$\mathbb{P}(X_2(t_n) = i_n | X_2(t_0) = i_0, \cdots, X_2(t_{n-1}) = i_{n-1}) = \mathbb{P}(X_2(t_n) = i_n | X_2(t_{n-1}) = i_{n-1})$$

$\because \quad S_2(t) = X_1(t) + X_2(t)$, i.e. $\forall\ j \in S'$, it can be written as $j = j_1 + j_2$ where $X_i(t) = j_i \in S$

$\therefore \quad \forall\ j_0, \cdots, j_n \in S'$,

$$\begin{aligned}
&\mathbb{P}(S_2(t_n) = j_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}) \\
= &\sum_{i_0, \cdots, i_n \in \mathbb{Z}} \mathbb{P}(S_2(t_n) = j_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}, X_1(t_0) = t_0, \cdots, X_1(t_n) = t_n) \cdot \\
&\mathbb{P}(X_1(t_0) = i_0, \cdots, X_1(t_n) = t_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}) \\
= &\sum_{i_0, \cdots, i_n \in \mathbb{Z}} \mathbb{P}(X_1(t_n) = i_n | X_1(t_0) = i_0, \cdots, X_1(t_{n-1}) = i_{n-1}) \cdot \\
&\mathbb{P}(X_2(t_n) = j_n - i_n | X_2(t_0) = j_0 - i_0, \cdots, X_2(t_{n-1}) = j_{n-1} - i_{n-1}) \\
&\mathbb{P}(X_1(t_0) = i_0, \cdots, X_1(t_n) = i_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}) \\
= &\sum_{i_0, \cdots, i_n \in \mathbb{Z}} \mathbb{P}(X_1(t_n) = i_n | X_1(t_{n-1}) = i_{n-1}) \mathbb{P}(X_2(t_n) = j_n - i_n | X_2(t_{n-1}) = j_{n-1} - i_{n-1}) \cdot \\
&\mathbb{P}(X_1(t_0) = i_0, \cdots, X_1(t_n) = i_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}) \\
= &\sum_{i_{n-1}, i_n \in \mathbb{Z}} \mathbb{P}(X_1(t_n) = i_n | X_1(t_{n-1}) = i_{n-1}) \mathbb{P}(X_2(t_n) = j_n - i_n | X_2(t_{n-1}) = j_{n-1} - i_{n-1}) \cdot \\
&\left[\sum_{i_0, \cdots, i_{n-2} \in \mathbb{Z}} \mathbb{P}(X_1(t_0) = i_0, \cdots, X_1(t_n) = i_n | S_2(t_0) = j_0, \cdots, S_2(t_{n-1}) = j_{n-1}) \right] \\
= &\sum_{i_{n-1}, i_n \in \mathbb{Z}} \mathbb{P}(X_1(t_n) = i_n | X_1(t_{n-1}) = i_{n-1}) \mathbb{P}(X_2(t_n) = j_n - i_n | X_2(t_{n-1}) = j_{n-1} - i_{n-1}) \cdot \\
&\mathbb{P}(X_1(t_{n-1}) = t_{n-1}, X_1(t_n) = i_n | S_2(t_{n-1}) = j_{n-1}) \\
= &\sum_{i_{n-1}, i_n \in \mathbb{Z}} \mathbb{P}(S_2(t_n) = j_n | S_2(t_{n-1}) = j_{n-1}, X_1(t_n) = i_n, X_1(t_{n-1}) = i_{n-1}) \cdot \\
&\mathbb{P}(X_1(t_{n-1}) = t_{n-1}, X_1(t_n) = i_n | S_2(t_{n-1}) = j_{n-1}) \\
&= \mathbb{P}(S_2(t_n) = j_n | S_2(t_{n-1}) = j_{n-1})
\end{aligned}$$

Above, we use Strong Markov Property of chain $X_1(t)$ and we assume that $\forall\ t \geqslant 0,\ j \notin S$,

$$\mathbb{P}(X_1(t) = j) = \mathbb{P}(X_2(t) = j) = 0$$

i.e. $(S_2(t), t \geqslant 0)$ is a Markov chain

Transition Rate Matrix

Solution (cont.)

$\forall\ i \in S'$, as $t \to 0$,

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 0) = 1 - \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 1) = 1 - \mu h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 0) = \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 1) = \mu h + o(h)$$

$$\begin{aligned}
&\mathbb{P}(S_2(t+h) = 2 | S_2(t) = 0) \\
=&\mathbb{P}(X_1(t+h) = 1 | X_1(t) = 0)\mathbb{P}(X_2(t+h) = 1 | X_2(t) = 0) \\
=&[\lambda h + o(h)]^2 \\
=&o(h) \\
&\mathbb{P}(S_2(t+h) = 0 | S_2(t) = 2) \\
=&\mathbb{P}(X_1(t+h) = 0 | X_1(t) = 1)\mathbb{P}(X_2(t+h) = 0 | X_2(t) = 1) \\
=&[\mu h + o(h)]^2 \\
=&o(h)
\end{aligned}$$

Only when for at most one $l_0 \in \{0,1\}$, X_{l_0} changes its state after $(t, t+h]$ and the other $X_l (l \neq l_0)$ remain the same states, i.e. $|j - i| \leqslant 1$, the probability $\mathbb{P}(S_2(t+h) = j | S_2(t) = i)$ won't be $o(h)$ as $h \to 0$. Therefore,

For $k \in \{0,1\}$,

$$\begin{aligned}
\mathbb{P}(S_2(t+h) = k+1 | S_2(t) = k) &= \binom{2-k}{1}[\lambda h + o(h)]\prod_{i=1}^{k}[1 - \mu h + o(h)]\prod_{m=1}^{1-k}[1 - \lambda h + o(h)] \\
&= (2-k)\lambda h + o(h)
\end{aligned}$$

For $k \in \{1,2\}$

$$\begin{aligned}
\mathbb{P}(S_2(t+h) = k-1 | S_2(t) = k) &= \binom{k}{1}[\mu h + o(h)]\prod_{i=1}^{k-1}[1 - \mu h + o(h)]\prod_{m=1}^{2-k}[1 - \lambda h + o(h)] \\
&= k\mu h + o(h)
\end{aligned}$$

For $k \in \{0,1,2\}$,

$$\begin{aligned}
\mathbb{P}(S_2(t+h) = k | S_2(t) = k) &= 1 - \sum_{\substack{j=0 \\ j \neq k}}^{2}\mathbb{P}(S_2(t+h) = j | S_2(t) = k) \\
&= 1 - (2-k)\lambda h - k\mu h + o(h)
\end{aligned}$$

$\therefore$ the transition rates matrix of $S_2(t)$ is

$$G_2 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \end{array}$$

(b) What is the limiting distribution of $S_2(t)$ as $t \to \infty$?

From (a), we have $\forall\ i, j \in S'$, $P_{ij}(t) > 0$ when $\lambda, \mu > 0$. So the $S_2(t)$ is irreducible.
Suppose the stationary distribution is $\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$
Let

$$\pi G_2 = 0$$

we have

$$\begin{cases} -2\lambda\pi_1 + 2\mu\pi_2 = 0 \\ 2\lambda\pi_1 - 2(\lambda+\mu)\pi_2 + 2\mu\pi_3 = 0 \\ 2\lambda\pi_2 - 2\mu\pi_3 = 0 \\ \sum\limits_{i=1}^{3} \pi_i = 1 \end{cases}$$

$\therefore$

$$\pi = \begin{pmatrix} \frac{\mu^2}{(\mu+\lambda)^2} & \frac{2\mu\lambda}{(\mu+\lambda)^2} & \frac{\lambda^2}{(\mu+\lambda)^2} \end{pmatrix}$$

Therefore, the limiting distribution exists and equals to π.

(c) What is the limiting distribution of $S_n(t) = \sum\limits_{k=1}^{n} X_k(t)$ as $t \to \infty$?

Solution One

Since S_{n-1} and X_n are independent, $S_n = S_{n-1} + X_n$ is Markov chain by induction. The state space of $S_n(t)$ is $S_n = \{0, 1, \cdots, n\}$.

$\because$ $X_1(t), X_2(t), \cdots, X_n(t)$ are independent

$\therefore$ $\forall\ t \geqslant 0, h > 0,\ k, l \in \mathbb{N}^+$,

$$\begin{aligned} &\mathbb{P}(S_n(t+h) = l | S_n(t) = k) \\ =&\mathbb{P}\left(\sum_{i=1}^{n} X_i(t+h) = l \middle| \sum_{i=1}^{n} X_i(t) = k\right) \\ =&\sum_{\substack{l_1+\cdots+l_n=l \\ 0\leqslant l_1,\cdots,l_n\leqslant 1}} \sum_{\substack{k_1+\cdots+k_n=k \\ 0\leqslant k_1,\cdots,k_n\leqslant 1}} \prod_{i=1}^{n} \mathbb{P}\left(X_i(t+h) - X_i(t) = l_i | X_i(t) - X_i(0) = k_i\right) \end{aligned}$$

$\because$ $\forall\ i \in S$,

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 0) = 1 - \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 1) = 1 - \mu h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 1 | X_i(t) = 0) = \lambda h + o(h)$$

$$\mathbb{P}(X_i(t+h) - X_i(t) = 0 | X_i(t) = 1) = \mu h + o(h)$$

$\therefore$ $\forall\ m \in S,\ k+m \in S$, and $m \geqslant 2$,

$$\begin{aligned} &\mathbb{P}(S_n(t+h) = k+m | S_n(t) = k) \\ =&\mathbb{P}(S_n(t+h) = k+m, \text{at least exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h) = X_{i_2}(t+h) = 1, \\ &\quad X_{i_1}(t) = X_{i_2}(t) = 0 | S_n(t) = k) \end{aligned}$$

Solution (cont.)

$$\begin{aligned}
&\leqslant \mathbb{P}(\text{exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h) = X_{i_2}(t+h) = 1, X_{i_1}(t) = X_{i_2}(t) = 0 | S_n(t) = k) \\
&= \binom{n}{2} [\lambda h + o(h)]^2 \\
&= o(h)
\end{aligned}$$

$\forall\ m \in S,\ k - m \in S$, and $m \geqslant 2$,

$$\begin{aligned}
&\mathbb{P}(S_n(t+h) = k - m | S_n(t) = k) \\
=&\mathbb{P}(S_n(t+h) = k - m, \text{at least exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h) = X_{i_2}(t+h) = 0, \\
&\quad X_{i_1}(t) = X_{i_2}(t) = 1 | S_n(t) = k) \\
\leqslant&\mathbb{P}(\text{exists } i_1, i_2 \text{ s.t. } X_{i_1}(t+h) = X_{i_2}(t+h) = 0, X_{i_1}(t) = X_{i_2}(t) = 1 | S_n(t) = k) \\
=&\binom{n}{2} [\mu h + o(h)]^2 \\
=&o(h)
\end{aligned}$$

I.e., only when for at most one $l_0 \in S$, X_{l_0} changes its state after $(t, t+h]$ and other $X_l (l \neq l_0)$ remain the same states, i.e. $|j - i| \leqslant 1$, the probability $\mathbb{P}(S_n(t+h) = j | S_n(t) = i)$ won't be $o(h)$ as $h \to 0$.

$$\begin{aligned}
\mathbb{P}(S_n(t+h) = k+1 | S_n(t) = k) &= \binom{n-k}{1} [\lambda h + o(h)] \prod_{i=1}^{k} [1 - \mu h + o(h)] \prod_{m=1}^{n-k-1} [1 - \lambda h + o(h)] \\
&= (n-k)\lambda h + o(h) \\
\mathbb{P}(S_n(t+h) = k-1 | S_n(t) = k) &= \binom{k}{1} [\mu h + o(h)] \prod_{i=1}^{k-1} [1 - \mu h + o(h)] \prod_{m=1}^{n-k} [1 - \lambda h + o(h)] \\
&= k\mu h + o(h) \\
\mathbb{P}(S_n(t+h) = k | S_n(t) = k) &= 1 - \sum_{\substack{j=0 \\ j \neq k}}^{n} \mathbb{P}(S_n(t+h) = j | S_n(t) = k) \\
&= 1 - (n-k)\lambda h - k\mu h + o(h)
\end{aligned}$$

$\therefore$ the infinitesimal matrix for $S_n(t)$ is

$$G_n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\ \begin{pmatrix}
-n\lambda & n\lambda & 0 & 0 & \cdots & 0 & 0 \\
\mu & -(n-1)\lambda - \mu & (n-1)\lambda & 0 & \cdots & 0 & 0 \\
0 & 2\mu & -(n-2)\lambda - 2\mu & (n-2)\lambda & \cdots & 0 & 0 \\
0 & 0 & 3\mu & -(n-3)\lambda - 3\mu & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\lambda - (n-1)\mu & \lambda \\
0 & 0 & 0 & 0 & \cdots & n\mu & -n\mu
\end{pmatrix} \end{array}$$

Suppose the stationary distribution of S_n is $\pi_{S_n} = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \end{pmatrix}$. Let

$$\pi_{S_n} G_n = 0$$

Solution (cont.)

we have

$$\begin{cases} n\lambda\pi_0 = \mu\pi_1 \\ (n-1)\lambda\pi_1 = 2\mu\pi_2 \\ \vdots \\ \lambda\pi_{n-1} = n\mu\pi_n \\ \sum\limits_{i=0}^{n}\pi_i = 1 \end{cases}$$

then

$$\pi_{S_n} = \begin{pmatrix} \frac{\mu^n}{(\mu+\lambda)^n} & \frac{n\mu^{n-1}\lambda}{(\mu+\lambda)^n} & \cdots & \frac{\binom{n}{i}\mu^{n-i}\lambda^i}{(\mu+\lambda)^n} & \cdots & \frac{\lambda^n}{(\mu+\lambda)^n} \end{pmatrix}$$

$\because$ $S_n(t)$ is irreducible and the stationary distribution exists

$\therefore$ the limiting distribution of $S_n(t)$ exists and equals to π_{S_n}

Solution Two

Suppose that for $2 \leqslant k \leqslant n$, the limiting distribution of S_n is

$$\pi_{S_n} = \begin{pmatrix} \frac{\mu^n}{(\mu+\lambda)^n} & \frac{n\mu^{n-1}\lambda}{(\mu+\lambda)^n} & \cdots & \frac{\binom{n}{i}\mu^{n-i}\lambda^i}{(\mu+\lambda)^n} & \cdots & \frac{\lambda^n}{(\mu+\lambda)^n} \end{pmatrix}$$

$\because$ X_{n+1} and S_n are independent

$\therefore$ $\forall\ i \in S_n$,

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{P}(S_{n+1}(t) = 0 | S_{n+1}(0) = i) \\
=&\lim_{t\to\infty}\sum_{j=0}^{1}\mathbb{P}(S_n(t) = 0, X_{n+1} = 0 | S_n(0) = i-j, X_{n+1} = j)\mathbb{P}(X_{n+1} = j) \\
=&\sum_{j=0}^{1}\lim_{t\to\infty}\mathbb{P}(S_n(t) = 0 | S_n(0) = i-j)\mathbb{P}(X_{n+1} = 0 | X_{n+1} = j)\mathbb{P}(X_{n+1} = j) \\
=&\frac{\mu^n}{(\mu+\lambda)^n}\cdot\frac{\mu}{\mu+\lambda} \\
=&\frac{\mu^{n+1}}{(\mu+\lambda)^{n+1}} \\
&\lim_{t\to\infty}\mathbb{P}(S_{n+1}(t) = n+1 | S_{n+1}(0) = i) \\
=&\lim_{t\to\infty}\sum_{j=0}^{1}\mathbb{P}(S_n(t) = n, X_{n+1} = 1 | S_n(0) = i-j, X_{n+1} = j) \\
=&\sum_{j=0}^{1}\lim_{t\to\infty}\mathbb{P}(S_n(t) = n | S_n(0) = i-j)\mathbb{P}(X_{n+1} = 1 | X_{n+1} = j)\mathbb{P}(X_{n+1} = j) \\
=&\frac{\lambda^n}{(\mu+\lambda)^n}\cdot\frac{\lambda}{\mu+\lambda} \\
=&\frac{\lambda^{n+1}}{(\mu+\lambda)^{n+1}}
\end{aligned}$$

Solution (cont.)

$\forall\ k \in \mathbb{N},\ 0 < k \leqslant n,$

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}(S_{n+1}(t) = k | S_{n+1}(0) = i) \\
=&\lim_{t\to\infty} \sum_{l=0}^{1}\sum_{j=0}^{1} \mathbb{P}(S_n(t) = k - l, X_{n+1} = l | S_n(0) = i - j, X_{n+1} = j) \\
=&\sum_{l=0}^{1}\sum_{j=0}^{1} \lim_{t\to\infty} \mathbb{P}(S_n(t) = k - l | S_n(0) = i - j)\mathbb{P}(X_{n+1} = l | X_{n+1} = j)\mathbb{P}(X_{n+1} = j) \\
\\
=&\sum_{j=0}^{1} \mathbb{P}(X_{n+1} = j) \left[\frac{\binom{n}{k-1}\mu^{n-k+1}\lambda^{k-1}}{(\mu+\lambda)^n} \cdot \frac{\lambda}{\mu+\lambda} + \frac{\binom{n}{k}\mu^{n-k}\lambda^k}{(\mu+\lambda)^n} \cdot \frac{\mu}{\mu+\lambda} \right] \\
=&\frac{\left[\binom{n}{k-1} + \binom{n}{k}\right]\mu^{n+1-k}\lambda^k}{(\mu+\lambda)^{n+1}} \\
=&\frac{\binom{n+1}{k}\mu^{n+1-k}\lambda^k}{(\mu+\lambda)^{n+1}}
\end{aligned}$$

i.e. the limiting distribution of S_{n+1} is

$$\pi_{S_{n+1}} = \begin{pmatrix} \frac{\mu^{n+1}}{(\mu+\lambda)^{n+1}} & \frac{(n+1)\mu^n\lambda}{(\mu+\lambda)^{n+1}} & \cdots & \frac{\binom{n+1}{i}\mu^{n+1-i}\lambda^i}{(\mu+\lambda)^{n+1}} & \cdots & \frac{\lambda^{n+1}}{(\mu+\lambda)^{n+1}} \end{pmatrix}$$

by induction, $\forall\ n \in \mathbb{N}^+,\ n \geqslant 2,$

$$\pi_{S_n} = \begin{pmatrix} \frac{\mu^n}{(\mu+\lambda)^n} & \frac{n\mu^{n-1}\lambda}{(\mu+\lambda)^n} & \cdots & \frac{\binom{n}{i}\mu^{n-i}\lambda^i}{(\mu+\lambda)^n} & \cdots & \frac{\lambda^n}{(\mu+\lambda)^n} \end{pmatrix}$$