TOPIC. Characteristic functions. This lecture begins our study of the characteristic function $\phi_X(t) := Ee^{itX} = E\cos(tX) + iE\sin(tX)$ ($t \in \mathbb{R}$) of a real random variable X. Characteristic functions are of interest for a variety of reasons, among them: ϕ_X uniquely determines the distribution of X; random variables X_n converge in distribution to X iff $\phi_{X_n}(t) \to \phi_X(t)$ for each $t \in \mathbb{R}$; and the characteristic function of a sum of independent random variables is the product of the characteristic functions of the summands. These properties make characteristic functions an ideal tool for proving limit theorems for sums of independent random variables.

The integral of a complex-valued function. Let

$$\mathbb{C} := \{u + iv : u, v \in \mathbb{R}\}$$

($i = \sqrt{-1}$) be the ***complex plane***. Let Ω be a sample space. A mapping

$$Z = U + iV$$

from Ω to $\mathbb{C}$ is said to be a ***complex-valued random variable*** on Ω if the mappings U and V are real-valued random variables. Z is said to be ***integrable*** with respect to a measure μ on Ω, written $Z \in L(\mu)$, if U and V are each integrable with respect to μ. The ***integral*** of $Z \in L(\mu)$ is

$$\int_\Omega Z\,d\mu := \int_\Omega U\,d\mu + i\int_\Omega V\,d\mu. \tag{1}$$

Here, e.g., $\int_\Omega U\,d\mu$ would be

$\int_{-\infty}^{\infty} u(x)f(x)\,dx$	when μ has density f with respect to Lebesgue measure on $\Omega = \mathbb{R}$,
or	
$\sum_{n=1}^{\infty} u(n)f(n)$	when μ places mass $f(n)$ at each point $n \in \Omega = \mathbb{N}$,
or	
$E(U(X))$	when μ is the distribution of a Ω-valued random variable X.

$Z = U + iV$ is integrable $\iff U$ and V are integrable

In general, since

$$\max(|U|, |V|) \le |Z| := \sqrt{U^2 + V^2} \le |U| + |V|,$$

it follows that

$$Z \text{ is integrable} \iff |Z| \text{ is integrable.} \tag{2}$$

Generating functions. Let X be a real-valued random variable defined on a sample space Ω and let P be a probability measure on Ω. Consider

$$Z := e^{zX}$$

for $z := u + iv$ in $\mathbb{C}$. We ask: "When is Z integrable with respect to P?" Since

$$e^{zX} = e^{uX}e^{ivX} = e^{uX}\big(\cos(vX) + i\sin(vX)\big),$$

we have $|Z| = |e^{uX}| \times |e^{ivX}| = e^{uX} \times 1 = e^{uX}$ and thus

$$E(|Z|) < \infty \iff M_X(u) := E(e^{uX}) < \infty. \tag{3}$$

Here M_X is the (real) moment generating function of X; it was studied in the previous section. M_X maps $\mathbb{R}$ into $[0, \infty]$. By Theorem 12.1,

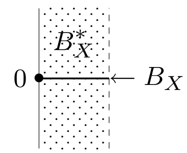
$$B_X := \{\, u \in \mathbb{R} : M_X(u) < \infty \,\} \tag{4}$$

is a (possibly degenerate) interval containing 0. According to (3),

$0 \in B_X := \{\, u \in \mathbb{R} : M_X(u) := E(e^{uX}) < \infty \,\}$

$Z = e^{zX}$ is integrable if and only if $z \in B_X^*$, where

$$B_X^* := \{\, z \in \mathbb{C} : \Re(z) \in B_X \,\}. \tag{5}$$



The function $G_X \colon B_X^* \to \mathbb{C}$ defined by

$$G_X(z) = E(e^{zX}) \tag{6}$$

is called the ***complex moment generating function*** of X, and the function $\phi_X \colon \mathbb{R} \to \mathbb{C}$ defined by

$$\phi_X(t) := G_X(it) = Ee^{itX} = E\cos(tX) + iE\sin(tX) \tag{7}$$

is called the ***characteristic function*** of X. We are primarily interested in characteristic functions; the real and complex generating functions will serve as convenient means to obtain the characteristic function in the case where B_X is nondegenerate.

Example 1. (a) Suppose $X \sim N(0,1)$. By Example 12.1 (a),

$$M_X(u) = e^{u^2/2} \tag{8}$$

for all $u \in \mathbb{R}$ and $B_X = \mathbb{R}$. We see later on (see Example 4 (a)) that (8) implies that $G_X(z) = e^{z^2/2}$ for $z \in B_X^* = \mathbb{C}$, and thus that $\phi_X(t) = e^{-t^2/2}$ for $t \in \mathbb{R}$.

(b) Suppose next that $X \sim G(r,1)$. By Example 12.1 (b),

$$M_X(u) = \begin{cases} 1/(1-u)^r, & \text{if } -\infty < u < 1 \\ \infty, & \text{if } 1 \le u < \infty \end{cases} \tag{9}$$

and $B_X = (-\infty, 1)$. We will see later on (see Example 4 (b)) that (9) implies that $G_X(z) = 1/(1-z)^r$ for $z \in B_X^* = \{\, z \in \mathbb{C} : \Re(z) < 1 \,\}$, and thus that $\phi_X(t) = 1/(1-it)^r$ for $t \in \mathbb{R}$. •

Properties of the integral. Return now to the general complex-valued random variable $Z = U + iV$ on a sample space Ω, and recall that the integral of Z with respect to a measure μ on Ω is defined as

$$\int_\Omega Z\,d\mu = \int_\Omega U\,d\mu + i\int_\Omega V\,d\mu \tag{10}$$

provided Z is integrable. The following theorem presents the basic properties of $\int_\Omega Z\,d\mu$.

Theorem 1. *The complex integral in* (10) *has the following properties.*

(I1) $\int$ ***is linear****: if Z_1 and Z_2 are complex-valued integrable random variables and c_1 and c_2 are complex numbers, then the complex-valued random variable $c_1Z_1 + c_2Z_2$ is integrable and*

$$\int_\Omega (c_1Z_1 + c_2Z_2)\,d\mu = c_1\int_\Omega Z_1\,d\mu + c_2\int_\Omega Z_2\,d\mu. \tag{11}$$

(I2) $\int$ ***decreases absolute values****: if Z is an integrable complex-valued random variable, then*

$$\Big|\int_\Omega Z\,d\mu\Big| \le \int_\Omega |Z|\,d\mu. \tag{12}$$

(I3) ***the Dominated Convergence Theorem (DCT) holds for*** $\int$*: if Z and Z_1, Z_2, ... are complex-valued random variables such that* (i) *Z_n tends to Z pointwise (or just μ-almost-everywhere) on Ω as $n \to \infty$ and* (ii) *there exists an integrable real-valued random variable D on Ω such that $|Z_n| \le D$ for all n, then each Z_n is integrable, Z is integrable, and*

$$\int_\Omega Z_n\,d\mu \to \int_\Omega Z\,d\mu \ \text{ as } n \to \infty. \tag{13}$$

Proof Properties (I1) and (I3) follow easily from the corresponding properties of the integral for real-valued random variables. For (I2), write Z as $U + iV$. If U and V each take only finitely many values, then they can be put in the form $U = \sum_k u_k I_{A_k}$ and $V = \sum_k v_k I_{A_k}$ for finitely many disjoint sets A_k. In this case $Z = \sum_k (u_k + iv_k) I_{A_k}$ and $|Z| = \sum_k |u_k + iv_k| I_{A_k}$, whence

$$\begin{aligned} \Bigl|\int_\Omega Z\,d\mu\Bigr| &= \Bigl|\sum\nolimits_k (u_k + iv_k)\mu(A_k)\Bigr| \\ &\le \sum\nolimits_k |u_k + iv_k|\mu(A_k) = \int_\Omega |Z|\,d\mu \end{aligned}$$

by the triangle inequality in $\mathbb{C}$. For the general case, one writes $U = \lim_n U_n$ and $V = \lim_n V_n$ as pointwise limits of finitely-valued functions U_n and V_n, each of which is dominated by $|Z|$ (see Exercise 3). Then $Z_n := U_n + iV_n$ converges pointwise to Z and $|Z_n| \le 2|Z|$ for all n. Thus

$$\Bigl|\int_\Omega Z\,d\mu\Bigr| = \lim_n \Bigl|\int_\Omega Z_n\,d\mu\Bigr| \le \lim_n \int_\Omega |Z_n|\,d\mu = \int_\Omega |Z|\,d\mu$$

by two applications of the DCT (the first for complex-valued random variables, the second for real-valued ones). ∎

Example 2. Let $\phi(t) = E(e^{itX})$ be the characteristic function of a real-valued random variable X. By (12)

$$|\phi(t)| = |E(e^{itX})| \le E(|e^{itX}|) = E1 = 1; \tag{14}$$

thus $|\phi|$ is bounded by 1. Moreover

$$\phi \text{ is continuous.} \tag{15}$$

To see this, suppose $t_n \to t \in \mathbb{R}$. Then $Z_n := e^{it_nX} \to Z := e^{itX}$ pointwise on Ω, and $|Z_n| \le D := 1$ for all n. Since $E(D) = 1 < \infty$, the complex DCT implies that $\phi(t_n) = E(Z_n) \to E(Z) = \phi(t)$. •

Fubini's Theorem also holds for complex-valued random variables. The following result is a special case.

Theorem 2. *Let Z and $Z_1, Z_2, \ldots$ be complex-valued random variables on a sample space Ω equipped with a measure μ. Suppose that*

$$Z = \sum\nolimits_{n=0}^\infty Z_n \quad \text{and} \quad \int_\Omega \Bigl(\sum\nolimits_{n=0}^\infty |Z_n|\Bigr)\,d\mu < \infty. \tag{16}$$

Then each Z_n is integrable, Z is integrable, and

$$\int_\Omega Z\,d\mu = \sum\nolimits_{n=0}^\infty \Bigl(\int_\Omega Z_n\,d\mu\Bigr). \tag{17}$$

Proof Each Z_n is integrable because

$$\sum\nolimits_{n=0}^\infty \Bigl(\int_\Omega |Z_n|\,d\mu\Bigr) = \int_\Omega \Bigl(\sum\nolimits_{n=0}^\infty |Z_n|\Bigr)\,d\mu < \infty.$$

Put $S = Z$ and $S_n = \sum_{k=0}^n Z_k$ for $n = 0, 1, \ldots$. Then S_n converges pointwise to S as n tends to ∞, and $|S_n| \le D := \sum_{k=0}^\infty |Z_k|$ for all n. Since D is integrable, the complex DCT implies that each S_n is integrable, $S = Z$ is integrable, and $\int_\Omega S_n\,d\mu \to \int_\Omega S\,d\mu$. (17) follows since $\int_\Omega S_n\,d\mu = \sum_{k=0}^n \int_\Omega Z_k\,d\mu$. ∎

Analyticity of the complex generating function. Suppose X is a random variable such that the region B_X where M_X is finite contains a nonempty open interval. Let B_X° be the largest such interval; B_X° is all of B_X except possibly for the endpoints. Put

$$\mathcal{D}_X = \{\, u + iv \in \mathbb{C} : u \in B_X^\circ \,\}; \tag{18}$$

$\mathcal{D}_X$ is the largest open subset of $\mathbb{C}$ contained in B_X^*. We are going to show that the complex generating function G_X has a power series expansion about each point of $\mathcal{D}_X$. In the language of complex variables, this says that G_X is ***analytic*** on $\mathcal{D}_X$.

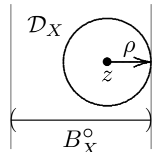
For complex numbers z and real numbers $r > 0$ let

$$B(z, r) := \{\, \zeta \in \mathbb{C} : |\zeta - z| < r \,\}$$

be the open ball about z of radius r. The following result generalizes Theorems 12.2, 12.3, and part of 12.5.

Theorem 3. *Let X be real random variable such that the interior $\mathcal{D}_X$ of the domain B_X^* of G_X is nonempty.* (i) *Let $z \in \mathcal{D}_X$ and set*

$$\rho = \sup\{\, r > 0 : B(z,r) \subset \mathcal{D}_X \,\}.$$



Then $X^n e^{zX}$ is integrable for $n = 0, 1, 2, \ldots$, and

$$G_X(\zeta) = \sum_{n=0}^{\infty} E(X^n e^{zX})\,\frac{(\zeta - z)^n}{n!} \tag{19}$$

for each $\zeta \in \mathcal{D}_X$ such that $|\zeta - z| < \rho$. (ii) *G_X is infinitely differentiable in $\mathcal{D}_X$ and*

$$G_X^{(n)}(z) = E(X^n e^{zX}) \quad \text{for all } z \in \mathcal{D}_X \text{ and } n \in \mathbb{N}. \tag{20}$$

Proof (i) We have $G_X(\zeta) = E(e^{\zeta X})$ with

$$e^{\zeta X} = e^{zX} e^{(\zeta - z)X} = \sum_{n=0}^{\infty} e^{zX}\,\frac{X^n(\zeta - z)^n}{n!}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\left| e^{zX}\,\frac{X^n(\zeta - z)^n}{n!}\right| &\le |e^{zX}| \sum_{n=0}^{\infty} \frac{|X(\zeta - z)|^n}{n!} \\ &= e^{uX} e^{r|X|} \le e^{(u-r)X} + e^{(u+r)X} := D,\end{aligned}$$

where $u = \Re(z)$ and $r = |\zeta - z|$. D is integrable because $u - r$ and $u + r$ each lie in B_X. (19) now follows from Theorem 2. (ii) (20) follows by differentiating both sides of (19). ∎

Example 3. Suppose $X \sim N(0,1)$. We have seen that $E(X^n) = 0$ if n is odd, whereas $E(X^{2k}) = (2k)!/(2^k k!)$ for $k = 0, 1, \ldots$. Since $\mathcal{D}_X = \mathbb{C}$, it follows from (19) with $z = 0$ that

$$G_X(\zeta) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^k k!}\,\frac{\zeta^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(\zeta^2/2)^k}{k!} = e^{\zeta^2/2} \tag{21}$$

for all $\zeta \in \mathbb{C}$. •

The substitution principle. The formula

$$G_X(z) = \sum_{n=0}^{\infty} E(X^n)\frac{z^n}{n!} \tag{22}$$

can be used to compute the complex generating function if $B_X = \mathbb{R}$, as in the preceding example, but not otherwise. We describe an approach that works whenever B_X has nonempty interior. The technique is based on the following uniqueness theorem from complex variables.

Theorem 4. *Let $\mathcal{D}$ be an open connected subset of $\mathbb{C}$ and let f and g be differentiable complex-valued functions defined on $\mathcal{D}$. Then $f(z) = g(z)$ for all $z \in \mathcal{D}$ provided $f(z) = g(z)$ for all z in some subset Δ of $\mathcal{D}$ having a limit point in $\mathcal{D}$.*

By definition, z is a limit point of Δ if there exists an infinite sequence of points $z_n \in \Delta$ such that $z_n \to z$ as $n \to \infty$, and, moreover, $z_n \neq z$ for all n.

Theorem 5. *Let X be a real random variable such that B_X has nonempty interior. Suppose that H is a differentiable complex-valued function on $\mathcal{D}_X$ such that $H(u) = M_X(u)$ for all u in some subset of B_X° having a limit point in B_X° (in particular, for all u in B_X°). Then $G_X(z) = H(z)$ for all $z \in \mathcal{D}_X$.*

Proof This follows from Theorem 4 by taking $\mathcal{D} = \mathcal{D}_X$, $f(z) = H(z)$, and $g(z) = G_X(z)$. ∎

This is called the ***substitution principle***, because it states, in effect, that you can get $G_X(z)$ by substituting z for u in $M_X(u)$. Of course, it is necessary to check that the resulting function of z is differentiable. The formulas

$$\frac{d}{dz} z^n = n z^{n-1}, \quad \frac{d}{dz} e^z = e^z, \quad \frac{d}{dz}\log(z) = \frac{1}{z},$$
$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$$

are helpful in this regard.

Example 4. (a) Suppose $X \sim N(0,1)$. X has real moment generating function $M_X(u) = e^{u^2/2}$ and $\mathcal{D}_X = \mathbb{C}$. The function $H\colon \mathcal{D}_X \to \mathbb{C}$ defined by $H(z) = e^{z^2/2}$ is differentiable (the derivative is $ze^{z^2/2}$) and coincides with M_X on $B_X^\circ = \mathbb{R}$. According to the substitution principle, $G_X(z) = e^{z^2/2}$ for all $z \in \mathbb{C}$.

(b) Suppose $X \sim G(r,1)$. The real moment generating function M_X of X is finite in $B_X = (-\infty, 1)$, and has there the value

$$M_X(u) = 1/(1-u)^r = e^{-r\log(1-u)}.$$

Let H mapping $\mathcal{D}_X = \{\, z \in \mathbb{C} : \Re(z) < 1 \,\}$ to $\mathbb{C}$ be defined by

$$H(z) = e^{-r\log(1-z)}.$$

Here $\zeta := 1 - z = \rho e^{i\theta}$ with $\rho > 0$ and $-\pi/2 < \theta < \pi/2$, and $\log(\zeta) = \log(\rho) + i\theta$ is the so-called principal value of the logarithm of ζ. H is differentiable on $\mathcal{D}_X$ (the derivative is $rH(z)/(1-z)$) and agrees with M_X on $B_X^\circ = B_X$. According to the substitution principle, $G_X(z) = e^{-r\log(1-z)} = 1/(1-z)^r$ for all $z \in \mathcal{D}_X$. •

The substitution principle can be used (verify this!) to obtain all the entries in the table on the next page except for lines 9 and 12. The exercises in this section ask you to derive the entries on lines 1–5, 8, 10, 13, and 14. The remaining formulas will be obtained in the next section.

Table 1 — Characteristic Functions of Some Distributions

Distributions 1–5 (respectively, 6–14) have the indicated density with respect to counting measure (respectively, Lebesgue measure) on the indicated support. On lines 3 and 5, $q = 1 - p$.

	Distribution	*Density*	*Support*	*C.f.*
1	Degenerate	1	$\{\alpha\}$	$e^{i\alpha t}$
2	Symmetric Bernoulli	$1/2$	$\{-1,1\}$	$\cos(t)$
3	Binomial	$\binom{n}{x} p^x q^{n-x}$	$\{0,1,\ldots,n\}$	$(pe^{it}+q)^n$
4	Poisson	$e^{-\lambda}\dfrac{\lambda^x}{x!}$	$\{0,1,\ldots\}$	$\exp(\lambda(e^{it}-1))$
5	Negative binomial	$\binom{-r}{x}(-q)^x p^r$	$\{0,1,\ldots\}$	$\left(\dfrac{p}{1-qe^{it}}\right)^r$
6	Normal	$\dfrac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}$	$(-\infty,\infty)$	$\exp(it\mu - \sigma^2 t^2/2)$
7	Gamma	$\dfrac{\alpha^r x^{r-1} e^{-\alpha x}}{\Gamma(r)}$	$(0,\infty)$	$\dfrac{1}{(1-it/\alpha)^r}$
8	Two-sided exponential	$\alpha e^{-\alpha\lvert x\rvert}/2$	$(-\infty,\infty)$	$\dfrac{1}{1+t^2/\alpha^2}$
9	Symmetric Cauchy	$\dfrac{\alpha}{\pi(\alpha^2+x^2)}$	$(-\infty,\infty)$	$e^{-\alpha\lvert t\rvert}$
10	Symmetric uniform	$\dfrac{1}{2\alpha}$	$[-\alpha,\alpha]$	$\dfrac{\sin(\alpha t)}{\alpha t}$
11	Triangular	$\dfrac{1}{\alpha}\left(1-\dfrac{\lvert x\rvert}{\alpha}\right)$	$[-\alpha,\alpha]$	$\dfrac{2(1-\cos(\alpha t))}{\alpha^2 t^2}$
12	Inverse triangular	$\dfrac{1-\cos(\alpha x)}{\pi\alpha x^2}$	$(-\infty,\infty)$	$\left(1-\dfrac{\lvert t\rvert}{\alpha}\right)^+$
13	Double exponential	$e^{-e^{-x}}e^{-x}$	$(-\infty,\infty)$	$\Gamma(1-it)$
14	Logistic	$\dfrac{e^{-x}}{(1+e^{-x})^2}$	$(-\infty,\infty)$	$\Gamma(1+it)\Gamma(1-it)$

Exercise 1. Evaluate the integral $\int_{-\infty}^{\infty} \cos(tx)\, e^{-x^2/2}\, dx$ "without calculation"; here $t \in \mathbb{R}$. ◇

Exercise 2. Write out complete proofs of properties I1 and I3 of Theorem 1. ◇

Exercise 3. Let U be a real random variable. Show that there exists an infinite sequence $U_1, U_2, \ldots$ of real random variables such that: (i) $\lim_n U_n(\omega) = U(\omega)$ for all sample points ω and (ii) $|U_n(\omega)| \leq |U(\omega)|$ for $n \in \mathbb{N}$ and all sample points ω. [Hint: If $0 \leq U(\omega) \leq n$, let $U_n(\omega)$ be the result of rounding $U(\omega)$ down to the nearest multiple of $1/2^n$. ◇

Exercise 4. Prove the following version of Fubini's theorem. Let $f = u + iv$ be a complex-valued mapping from $\mathbb{R}^2$ to $\mathbb{C}$; here $u(x,y) = \Re\big(f(x,y)\big)$ and $v(x,y) = \Im\big(f(x,y)\big)$ for real numbers x and y. Show that if $\iint |f(x,y)|\, dx\, dy < \infty$, then

$$\begin{aligned} \iint f(x,y)\, dx\, dy &== \int \Big[\int f(x,y)\, dy\Big]\, dx \\ &= \int \Big[\int f(x,y)\, dx\Big]\, dy; \end{aligned} \tag{23}$$

here the first integral is a double integral, and the next two integrals are iterated ones. ◇

Exercise 5. Suppose that Z is a discrete complex-valued random variable, taking the values $z_1, z_2, \ldots$ with probabilities $p_1, p_2, \ldots$. Show that Z is integrable iff $\sum_k |z_k| p_k < \infty$, in which case $EZ = \sum_k p_k z_k := \lim_{n\to\infty} \sum_{k=1}^n p_k z_k$. Use this formula to verify the characteristic functions on lines 1–5 of Table 1. ◇

Exercise 6 (*The fundamental theorem of calculus*). Suppose that Z is a continuously differentiable complex-valued function defined on a subinterval $[a,b]$ of $\mathbb{R}$. Show that

$$Z(b) - Z(a) = \int_a^b Z'(t)\, dt \tag{24}$$

[Hint: apply the FTC for real-valued functions to the real and imaginary parts of Z.] Use this formula to verify the characteristic function on line 10 in Table 1. ◇

Exercise 7. Deduce the results on lines 1–5 and 10 in Table 1 from the substitution principle. ◇

Exercise 8. Use the substitution principle to deduce the result on line 8 in Table 1. ◇

Exercise 9 (*On the complex gamma function*). Suppose $z \in \mathbb{C}$ with $\Re(z) > 0$. Show that the function $f(z,x) := x^{z-1}e^{-x}$ is integrable in x with respect to dx on $(0,\infty)$, and thus that

$$\Gamma(z) := \int_0^\infty x^{z-1}\, e^{-x}\, dx \tag{25}$$

is defined. Show further that

$$\Gamma^{(k)}(z) := \frac{d^k}{dz^k}\Gamma(z) = \int_0^\infty x^{z-1}(\log(x))^k e^{-x}\, dx \tag{26}$$

for all $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k = 1, 2, \ldots$. (The case $k = 1$ and z real was used in Exericse 8.8.) [Hint: For (26) you can use the DCT and induction, or mimic the argument used in Theorem 3 to establish the analyticity of complex moment generating functions. However, it is easiest to make use of that result, by showing that for $\Re(z) > 0$ one has $\Gamma(z) = G_Y(z-1)$, where $Y = \log(X)$ for a standard exponential random variable X and G_Y is the complex generating function of Y.]◇

Exercise 10. Deduce the last two lines in Table 1. [Hint: this can be done using the substitution principle and the differentiability of the complex Gamma function; there are other other ways to do it as well.] $\diamond$

Theorem 5 can be used to obtain the characteristic function of a random variable X from its MGF when 0 is in the interior of B_X. The case that $0 \notin B_X$ is covered by the following exercise.

Exercise 11. Let X be a random variable such that B_X has non-empty interior B_X°, but $0 \notin B_X^\circ$. The imaginary axis $I := \{\, iv : v \in \mathbb{R} \,\}$ of $\mathbb{C}$ is thus a (vertical) edge of region B_X^* in which the complex generating function G_X is defined. (a) Show that G_X is continuous at each point $z \in I$ and differentiable at each point z in the interior $\mathcal{D}_X$ of B_X^*. (b) Suppose that $H\colon B_X^* \to \mathbb{C}$ is continuous at each point of I and differentiable at each point of $\mathcal{D}_X$, and that $G_X(z) = H(z)$ for each point z in a subset Δ of $\mathcal{D}_X$ having a limit point in $\mathcal{D}_X$. Show that $G_X(z) = H(z)$ for all $z \in B_X^*$, and deduce that $\phi_X(t) = H(it)$ for all $t \in \mathbb{R}$. $\diamond$

Exercise 12. Let Y be a random variable with a chisquare distribution with 1 degree of freedom. Put $X = 1/Y$. Using a martingale argument, one can show that

$$E(e^{-sX}) = \exp\bigl(-\sqrt{2s}\,\bigr) \tag{27}$$

for $s \geq 0$. Use the result of the preceding exercise to show that X has characteristic function

$$\phi_X(t) = \exp\bigl(-2\sqrt{t}\,(1-i)\bigr). \tag{28}$$ $\diamond$