STAT 150: Stochastic Processes

Fall 2017

Homework 2

Solutions by

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3.1.4

The random variables $\xi_1, \xi_2, \cdots$ are independent and with the common probability mass function

$k =$	0	1	2	3
$P\{\xi_k\} =$	0.1	0.3	0.2	0.4

Set $X_0 = 0$, and let $X_n = \max\{\xi_1, \cdots, \xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

$\because \quad \forall\, i \in \mathbb{N}$

$$\begin{aligned} Pr\{\xi_i \leqslant 0\} &= 0.1 \\ Pr\{\xi_i \leqslant 1\} &= 0.4 \\ Pr\{\xi_i \leqslant 2\} &= 0.6 \\ Pr\{\xi_i \leqslant 3\} &= 1 \end{aligned}$$

$\therefore$

$$\begin{aligned} P_{00} &= Pr\{X_{n+1} = 0 | X_n = 0\} \\ &= Pr\{\xi_{n+1} \leqslant 0\} \\ &= 0.1 \\ P_{11} &= Pr\{X_{n+1} = 1 | X_n = 1\} \\ &= Pr\{\xi_{n+1} \leqslant 1\} \\ &= 0.4 \\ P_{22} &= Pr\{X_{n+1} = 2 | X_n = 2\} \\ &= Pr\{\xi_{n+1} \leqslant 2\} \\ &= 0.6 \\ P_{33} &= Pr\{X_{n+1} = 3 | X_n = 3\} \\ &= Pr\{\xi_{n+1} \leqslant 3\} \\ &= 1 \\ P_{01} &= Pr\{X_{n+1} = 1 | X_n = 0\} \\ &= Pr\{\xi_{n+1} = 1\} \\ &= 0.3 \end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
P_{02} &= Pr\{X_{n+1} = 2|X_n = 0\}\\
&= Pr\{\xi_{n+1} = 2\}\\
&= 0.2\\
P_{03} &= Pr\{X_{n+1} = 3|X_n = 0\}\\
&= Pr\{\xi_{n+1} = 3\}\\
&= 0.4\\
P_{12} &= Pr\{X_{n+1} = 2|X_n = 1\}\\
&= Pr\{\xi_{n+1} = 2\}\\
&= 0.2\\
P_{13} &= Pr\{X_{n+1} = 3|X_n = 1\}\\
&= Pr\{\xi_{n+1} = 3\}\\
&= 0.4\\
P_{23} &= Pr\{X_{n+1} = 3|X_n = 2\}\\
&= Pr\{\xi_{n+1} = 3\}\\
&= 0.4
\end{aligned}
$$

$\because \quad \xi_i \geqslant 0$

$\therefore \quad X_{n+1} \geqslant X_n$, i.e, $\forall\ i, j \in \{0, 1, 2, 3\}, i > j$

$$P_{ij} = 0$$

$\therefore \quad$ the transition probability matrix for Markov chain $\{X_n\}$ is

$$
P = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3.2.4

Suppose X_n is a two-state Markov chain whose trainsition probability matrix is

$$
\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{Vmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{Vmatrix} \end{array}
$$

Then, $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. Dtermine the trainsition probability matrix.

$\because \quad \forall\ a_1, a_2, b_1, b_2 \in \{0, 1\},\ a_2 \neq b_1$

$$
\begin{aligned}
Pr\{Z_{n+1} = (b_1, b_2)|Z_n = (a_1, a_2)\} &= Pr\{X_{n+1} = b_2, X_n = b_1|X_n = a_2, X_{n-1} = a_1\} \\
&= 0 \\
Pr\{Z_{n+1} = (0, 0)|Z_n = (0, 0)\} &= Pr\{X_{n+1} = 0, X_n = 0|X_n = 0, X_{n-1} = 0\} \\
&= \frac{Pr\{X_{n+1} = 0, X_n = 0, X_{n-1} = 0\}}{Pr\{X_n = 0, X_{n-1} = 0\}} \\
&= Pr\{X_{n+1} = 0|X_n = 0, X_{n-1} = 0\} \\
&= Pr\{X_{n+1} = 0|X_n = 0\} \\
&= \alpha \\
Pr\{Z_{n+1} = (0, 1)|Z_n = (0, 0)\} &= Pr\{X_{n+1} = 0, X_n = 1|X_n = 0, X_{n-1} = 0\} \\
&= \frac{Pr\{X_{n+1} = 1, X_n = 0, X_{n-1} = 0\}}{Pr\{X_n = 0, X_{n-1} = 0\}} \\
&= Pr\{X_{n+1} = 1|X_n = 0, X_{n-1} = 0\} \\
&= Pr\{X_{n+1} = 1|X_n = 0\} \\
&= 1 - \alpha \\
Pr\{Z_{n+1} = (0, 0)|Z_n = (1, 0)\} &= Pr\{X_{n+1} = 0, X_n = 0|X_n = 0, X_{n-1} = 1\} \\
&= \frac{Pr\{X_{n+1} = 0, X_n = 0, X_{n-1} = 1\}}{Pr\{X_n = 0, X_{n-1} = 1\}} \\
&= Pr\{X_{n+1} = 0|X_n = 0, X_{n-1} = 1\} \\
&= Pr\{X_{n+1} = 0|X_n = 0\} \\
&= \alpha \\
Pr\{Z_{n+1} = (0, 1)|Z_n = (1, 0)\} &= Pr\{X_{n+1} = 0, X_n = 1|X_n = 0, X_{n-1} = 1\} \\
&= \frac{Pr\{X_{n+1} = 1, X_n = 0, X_{n-1} = 1\}}{Pr\{X_n = 0, X_{n-1} = 1\}} \\
&= Pr\{X_{n+1} = 1|X_n = 0, X_{n-1} = 1\} \\
&= Pr\{X_{n+1} = 1|X_n = 0\} \\
&= 1 - \alpha \\
Pr\{Z_{n+1} = (1, 0)|Z_n = (0, 1)\} &= Pr\{X_{n+1} = 0, X_n = 1|X_n = 1, X_{n-1} = 0\} \\
&= \frac{Pr\{X_{n+1} = 0, X_n = 1, X_{n-1} = 0\}}{Pr\{X_n = 1, X_{n-1} = 0\}} \\
&= Pr\{X_{n+1} = 0|X_n = 1, X_{n-1} = 0\} \\
&= Pr\{X_{n+1} = 0|X_n = 1\} \\
&= 1 - \beta
\end{aligned}
$$

Solution (cont.)

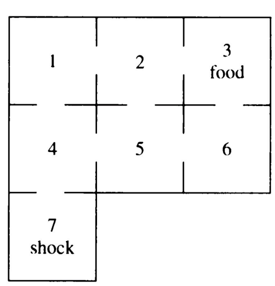
$$
\begin{aligned}
Pr\{Z_{n+1}=(1,1)|Z_n=(0,1)\} &= Pr\{X_{n+1}=1, X_n=1|X_n=1, X_{n-1}=0\} \\
&= \frac{Pr\{X_{n+1}=1, X_n=1, X_{n-1}=0\}}{Pr\{X_n=1, X_{n-1}=0\}} \\
&= Pr\{X_{n+1}=1|X_n=1, X_{n-1}=0\} \\
&= Pr\{X_{n+1}=1|X_n=1\} \\
&= \beta \\
Pr\{Z_{n+1}=(1,0)|Z_n=(1,1)\} &= Pr\{X_{n+1}=0, X_n=1|X_n=1, X_{n-1}=1\} \\
&= \frac{Pr\{X_{n+1}=0, X_n=1, X_{n-1}=1\}}{Pr\{X_n=1, X_{n-1}=1\}} \\
&= Pr\{X_{n+1}=0|X_n=1, X_{n-1}=1\} \\
&= Pr\{X_{n+1}=0|X_n=1\} \\
&= 1-\beta \\
Pr\{Z_{n+1}=(1,1)|Z_n=(1,1)\} &= Pr\{X_{n+1}=1, X_n=1|X_n=1, X_{n-1}=1\} \\
&= \frac{Pr\{X_{n+1}=1, X_n=1, X_{n-1}=1\}}{Pr\{X_n=1, X_{n-1}=1\}} \\
&= Pr\{X_{n+1}=1|X_n=1, X_{n-1}=1\} \\
&= Pr\{X_{n+1}=1|X_n=1\} \\
&= \beta
\end{aligned}
$$

$\therefore$ the transition probability matrix for Markov chain $\{Z_n\}$ is

$$
\mathbf{P} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array}
\begin{array}{c}
\begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\
\begin{pmatrix}
\alpha & 1-\alpha & 0 & 0 \\
0 & 0 & 1-\beta & \beta \\
\alpha & 1-\alpha & 0 & 0 \\
0 & 0 & 1-\beta & \beta
\end{pmatrix}
\end{array}
$$

3.4.5

A white rat is put into compartment 4 of maze shown here:



It moves through the compartments at random; i.e., if there are k ways to leave a compartment, it chooses each of these with probability $\frac{1}{k}$. What is the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7?

The transition probability matrix for Markov chain $\{X_n\}$ is

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let $u_i = u_i(3)$ denote the probability of absorption in the food compartment 3, given that the rat is dropped initially in compartment i.

$$\begin{aligned} u_1 &= \frac{1}{2}u_2 + \frac{1}{2}u_4 \\ u_2 &= \frac{1}{3} + \frac{1}{3}u_1 + \frac{1}{3}u_5 \\ u_4 &= \frac{1}{3}u_1 + \frac{1}{3}u_5 \\ u_5 &= \frac{1}{3}u_2 + \frac{1}{3}u_4 + \frac{1}{3}u_6 \\ u_6 &= \frac{1}{2} + \frac{1}{2}u_5 \end{aligned}$$

$\because$

$$\begin{aligned} u_5 &= \frac{1}{3}u_2 + \frac{1}{3}u_4 + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}u_5\right) \\ &= \frac{1}{6} + \frac{1}{3}u_2 + \frac{1}{3}u_4 + \frac{1}{6}u_5 \end{aligned}$$

$\therefore$

$$u_5 = \frac{1}{5} + \frac{2}{5}u_2 + \frac{2}{5}u_4$$

$\because$

$$\begin{aligned} u_2 &= \frac{1}{3} + \frac{1}{3}u_1 + \frac{1}{3}\left(\frac{1}{5} + \frac{2}{5}u_2 + \frac{2}{5}u_4\right) \\ &= \frac{2}{5} + \frac{1}{3}u_1 + \frac{2}{15}u_2 + \frac{2}{15}u_4 \end{aligned}$$

$\therefore$

$$u_2 = \frac{6}{13} + \frac{5}{13}u_1 + \frac{2}{13}u_4$$

$\therefore$

$$\begin{aligned} u_4 &= \frac{1}{3}\left(\frac{1}{2}u_2 + \frac{1}{2}u_4\right) + \frac{1}{3}\left(\frac{1}{5} + \frac{2}{5}u_2 + \frac{2}{5}u_4\right) \\ &= \frac{1}{15} + \frac{3}{10}u_2 + \frac{3}{10}u_4 \end{aligned}$$

Solution (cont.)

$\therefore$

$$u_4 = \frac{2}{21} + \frac{3}{7}u_2$$

By solving

$$\begin{cases} u_1 = \frac{1}{2}u_2 + \frac{1}{2}u_4 \\ u_2 = \frac{6}{13} + \frac{5}{13}u_1 + \frac{2}{13}u_4 \\ u_4 = \frac{2}{21} + \frac{3}{7}u_2 \end{cases}$$

we get

$$\begin{cases} u_1 = \frac{7}{12} \\ u_2 = \frac{3}{4} \\ u_4 = \frac{5}{12} \end{cases}$$

$\therefore$ the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7 is $\dfrac{5}{12}$.

1

Find the generating function of the following random variables.

(a) The geometric distributed X with mass function

$$\mathbb{P}(X = k) = (1-p)^k p, \qquad k = 0, 1, \cdots$$

where $0 < p < 1$.

(b) The negative binomial distributed Y with mass function

$$\mathbb{P}(Y = k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r, \qquad k = r, r+1, \cdots$$

where $0 < p < 1$ and $r \in \mathbb{Z}^+$.

Deduce the mean and variance in each cases.

(a)

$$\begin{aligned} f_X(s) &= Es^X \\ &= \sum_{k=0}^{+\infty} P(X=k)s^k \\ &= \sum_{k=0}^{+\infty} (1-p)^k p s^k \end{aligned}$$

Solution (cont.)

$$= \frac{p}{1-(1-p)s}$$

$\because$

$$f'_X(s) = \frac{(1-p)p}{[1-(1-p)s]^2}$$
$$f^{(2)}_X(s) = \frac{2(1-p)^2p}{[1-(1-p)s]^3}$$

$\therefore$

$$\begin{aligned}
EX &= f'_X(1) \\
&= \frac{1-p}{p} \\
VarE &= f^{(2)}_X(1) + f'_X(1) - [f'_X(1)]^2 \\
&= \frac{2(1-p)^2}{p^2} + \frac{1-p}{p} - \frac{(1-p)^2}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

(b)

$$\begin{aligned}
f_Y(s) &= Es^Y \\
&= \sum_{k=r}^{+\infty} P(Y=k)s^k \\
&= \sum_{k=r}^{+\infty} \binom{k-1}{r-1}(1-p)^{k-r}p^r s^k \\
&= p^r s^r \sum_{k=r}^{+\infty} \binom{k-1}{r-1}(1-p)^{k-r}s^{k-r} \\
&\xlongequal{n=k-r} p^r s^r \sum_{n=0}^{+\infty} \binom{n+r-1}{r-1}(1-p)^n s^n \\
&= \frac{p^r s^r}{[1-(1-p)s]^r}
\end{aligned}$$

$\because$

$$f'_X(s) = \frac{rp^r s^{r-1}}{[1-(1-p)s]^{r+1}}$$
$$f^{(2)}_X(s) = \frac{rp^r s^{r-2}[r-1+2(1-p)s]}{[1-(1-p)s]^{r+2}}$$

$\therefore$

$$\begin{aligned}
EX &= f'_X(1) \\
&= \frac{r}{p} \\
VarE &= f^{(2)}_X(1) + f'_X(1) - [f'_X(1)]^2 \\
&= \frac{r[r+1-2p]}{p^2} + \frac{r}{p} - \frac{r^2}{p^2} \\
&= \frac{r(1-p)}{p^2}
\end{aligned}$$

Let X and Y be independent random variables taking values in $\mathbb{N}$, such that

$$\mathbb{P}(X = k|X + Y = n) = \binom{n}{k} p^k (1-p)^{n-k}$$

for some $0 < p < 1$ and all $0 \leqslant k \leqslant n$. Show that X and Y have Poisson distribution.

$\because$

$$\begin{aligned} P(X = k|X + Y = n) &= \frac{P(X = k, X + Y = n)}{P(X + Y = n)} \\ &= \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} \end{aligned}$$

$\therefore$

$$\begin{aligned} \frac{P(X = k + 1|X + Y = n)}{P(X = k|X + Y = n)} &= \frac{P(X = k + 1)P(Y = n - k - 1)}{P(X = k)P(Y = n - k)} \\ &= \frac{\binom{n}{k+1} p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} \\ &= \frac{n-k}{k+1} \frac{p}{1-p} \end{aligned}$$

$\therefore$

$$\begin{aligned} \frac{P(X = k|X + Y = n - 1)}{P(X = k - 1|X + Y = n - 1)} &= \frac{P(X = k)P(Y = n - k - 1)}{P(X = k - 1)P(Y = n - k)} \\ &= \frac{\binom{n-1}{k} p^k (1-p)^{n-k-1}}{\binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}} \\ &= \frac{n-k}{k} \frac{p}{1-p} \end{aligned}$$

$\therefore$

$$\begin{aligned} \frac{\frac{P(X = k + 1)P(Y = n - k - 1)}{P(X = k)P(Y = n - k)}}{\frac{P(X = k)P(Y = n - k - 1)}{P(X = k - 1)P(Y = n - k)}} &= \frac{k}{k + 1} \\ (k + 1)\frac{P(X = k + 1)}{P(X = k)} &= k\frac{P(X = k)}{P(X = k - 1)} \\ &= \frac{P(X = 1)}{P(X = 0)} \end{aligned}$$

Let $\dfrac{P(X = 1)}{P(X = 0)} = a$, then $\forall\ k \in \mathbb{N}$

$$\begin{aligned} P(X = k + 1) &= \frac{a}{k + 1} P(X = k) \\ &= \cdots \\ &= \frac{a^{k+1}}{(k + 1)!} P(X = 0) \end{aligned}$$

Solution (cont.)

$\because$

$$\sum_{k=0}^{\infty} P(X=k)=1$$

$\therefore$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{a^k}{k!}P(X=0) &= e^a P(X=0)\\ &= 1\end{aligned}$$

$\therefore$

$$P(X=0)=e^{-a}$$

$\therefore \quad \forall\ k \in \mathbb{N},$

$$P(X=k)=\frac{a^k}{k!}e^{-a}$$

i.e. $X \sim P(a)$

$\because$

$$\frac{P(X=k)P(Y=n-k-1)}{P(X=k-1)P(Y=n-k)}=\frac{n-k}{k}\frac{p}{1-p}$$

$\therefore$

$$\begin{aligned}\frac{P(Y=n-k-1)}{P(Y=n-k)} &= \frac{\frac{a^{k-1}}{(k-1)!}e^{-a}}{\frac{a^k}{k!}e^{-a}}\frac{n-k}{k}\frac{p}{1-p}\\ &= \frac{n-k}{a}\frac{p}{1-p}\end{aligned}$$

i.e.

$$\begin{aligned}P(Y=k) &\xlongequal{n=2k} \frac{a}{k}\frac{1-p}{p}P(Y=k-1)\\ &\dots\\ &= \frac{a^k}{k!}\left(\frac{1-p}{p}\right)^k P(Y=0)\end{aligned}$$

$\therefore \quad$ by

$$\sum_{k=0}^{\infty} P(Y=k)=1$$

we have $P(Y=0)=e^{-a\frac{1-p}{p}}$, $P(Y=k)=\dfrac{\left(a\frac{1-p}{p}\right)^k}{k!}e^{-a\frac{1-p}{p}}$, i.e. $Y \sim P\left(a\frac{1-p}{p}\right)$

3

Let $\{X_n\}$ be a Markov Chain. Which of the following are Markov chains?

(a) $\{X_{n+r}\}$ for $r \in \mathbb{Z}^+$

$\because$ $\{X_n\}$ is a Markov Chain

$\therefore$ $\forall\ n \in \mathbb{N}$ and all states $i_0, \cdots, i_m, i, j$,

$$P\{X_{m+1} = j | X_0 = i_0, \cdots, X_{m-1} = i_{m-1}, X_m = i\} = P\{X_{m+1} = j | X_m = i\}$$

Let $n = m - r$, then $\forall$ states $i_0, \cdots, i_{n+r-1}, i', j'$

$$P\{X_{n+r+1} = j' | X_0 = i_0, \cdots, X_{n+r-1} = i_{n+r-1}, X_{n+r} = i'\} = P\{X_{n+r+1} = j' | X_{n+r} = i'\}$$

$\therefore$ $\{X_{n+r}\}$ is a Markov Chain

(b) $\{X_{rn}\}$ for $r \in \mathbb{Z}^+$

$\because$ $\{X_n\}$ is a Markov Chain

$\therefore$ $\forall\ n \in \mathbb{N}$ and all states $i_0, \cdots, i_n, i, j$,

$$P\{X_{n+1} = j | X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i\} = P\{X_{n+1} = j | X_n = i\}$$

$\therefore$ $\forall\ r \in \mathbb{Z}^+$

$$\begin{aligned} & P\{X_0 = i_0, X_r = i_r, \cdots, X_{r(n+1)} = i_{r(n+1)}\} \\ = & \sum_{\substack{i_m \\ 0<m<r(n+1) \\ m\neq r,\cdots,rn}} P\{X_0 = i_0, X_1 = i_1, \cdots, X_{r(n+1) = i_{r(n+1)}}\} \\ = & \sum_{\substack{i_m \\ 0<m<r(n+1) \\ m\neq r,\cdots,rn}} P\{X_{r(n+1)} = i_{r(n+1)} | X_{r(n+1)-1} = i_{r(n+1)-1}\} \cdots P\{X_1 = i_1 | X_0 = i_0\} P\{X_0 = i_0\} \qquad (1) \end{aligned}$$

The summation symbol indicates summation for all probability given $X_m = i_m$ when $0 < m < r(n+1)$ and $m \neq r, 2r, \cdots, rn$.

Solution (cont.)

$\therefore$

$$
\begin{aligned}
&P\{X_{r(n+1)} = i_{r(n+1)}|X_0 = i_0, X_r = i_r, \cdots, X_{rn} = i_{rn}\}\\
=&\frac{P\{X_0 = i_0, X_r = i_r, \cdots, X_{r(n+1)} = i_{r(n+1)}\}}{P\{X_0 = i_0, X_r = i_r, \cdots, X_{rn} = i_{rn}\}}\\
=&\frac{\sum\limits_{\substack{i_k\\ rn<k<r(n+1)}} \sum\limits_{\substack{i_m\\ 0<m<rn\\ m\neq r,\cdots,rn}} P\{X_{r(n+1)} = i_{r(n+1)}|X_{r(n+1)-1} = i_{r(n+1)-1}\}\cdots P\{X_1 = i_1|X_0 = i_0\}P\{X_0 = i_0\}}{\sum\limits_{\substack{i_m\\ 0<m<rn\\ m\neq r,\cdots,rn}} P\{X_{rn} = i_{rn}|X_{rn-1} = i_{rn-1}\}\cdots P\{X_1 = i_1|X_0 = i_0\}P\{X_0 = i_0\}}\\
=&\sum_{\substack{i_k\\ rn<k<r(n+1)}} P\{X_{r(n+1)} = i_{r(n+1)}|X_{r(n+1)-1} = i_{r(n+1)-1}\}\cdots P\{X_{rn+1} = i_{rn+1}|X_{rn} = i_{rn}\}\\
\xlongequal{From(1)}&P\{X_{r(n+1)=i_{r(n+1)}}|X_{rn} = i_{rn}\}
\end{aligned}
$$

$\therefore$ $\{X_{rn}\}$ ($\forall\ r \in \mathbb{Z}^+$) is a Markov Chain

(c) $\{(X_n, X_{n+r})\}$ for $r \in \mathbb{Z}^+$

$\because$ $\{X_n\}$ is a Markov Chain

$\therefore$ $\forall\ n \in \mathbb{N}$ and all states $i_0, \cdots, i_n, i, j$,

$$P\{X_{n+1} = j|X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i\} = P\{X_{n+1} = j|X_n = i\}$$

From (a), we have $\forall$ states $i_{m+1}, \cdots, i_{n+r-1}, i', j'$,

$$P\{X_{n+r+1} = j'|X_0 = i_0, \cdots, X_{n+r-1} = i_{n+r-1}, X_{n+r} = i'\} = P\{X_{n+r+1} = j'|X_{n+r} = i'\}$$

When $r = 1$,

$$
\begin{aligned}
&P\{(X_{n+1}, X_{n+r+1}) = (j, j')|(X_0, X_{r+1}) = (i_0, i_{r+1}), \cdots,\\
&\quad (X_{n-1}, X_{n+r-1}) = (i_{n-1}, i_{n+r-1}), (X_n, X_{n+r}) = (i, i')\}\\
=&P\{(X_{n+1}, X_{n+r+1}) = (j, j')|(X_n, X_{n+r}) = (i, i')\}
\end{aligned}
$$

$\therefore$ $\{(X_n, X_{n+r})\}$ is a Markov Chain

When $r > 1$, (X_{n+1}, X_{n+r+1}) is not independent of (X_{n-r+1}, X_{n+1}), the above equation won't hold. Therefore, $\{(X_n, X_{n+r})\}$ is not a Markov Chain.

4

There's a deck of n cards. Each card has a different pattern. Every minute, Tom will pick one of them at random, take a look at it, then put it back and shuffle the deck. So the chance that he sees any particular card in any given minute is $\frac{1}{n}$. What is the expectation of time past until Tom sees all the n patterns?

Let T_i $(i = 1, 2, \cdots, n)$ denotes the times past from when the $(i-1)^{th}$ pattern is first seen until i^{th} pattern is first seen. Let T denotes the time past until Tom sees all n patterns.

$$\begin{aligned}
ET_i &= \sum_{t=1}^{\infty} t \cdot Pr\{t \text{ minutes to see the } i^{th} \text{ pattern} \mid \text{have seen the } (i-1)^{th} \text{ pattern}\} \\
&= \sum_{t=1}^{\infty} t \left(\frac{i-1}{n}\right)^{t-1} \frac{n-i+1}{n} \\
&= \frac{n-i+1}{n} \frac{n^2}{(n-i+1)^2} \\
&= \frac{n}{n-i+1}
\end{aligned}$$

$\therefore$

$$\begin{aligned}
ET &= E(T_1 + T_2 + \cdots + T_n) \\
&= \sum_{i=1}^{n} \frac{n}{n-i+1}
\end{aligned}$$