TTIC 31250
An Introduction to the Theory of Machine Learning

Avrim Blum

04/13/20

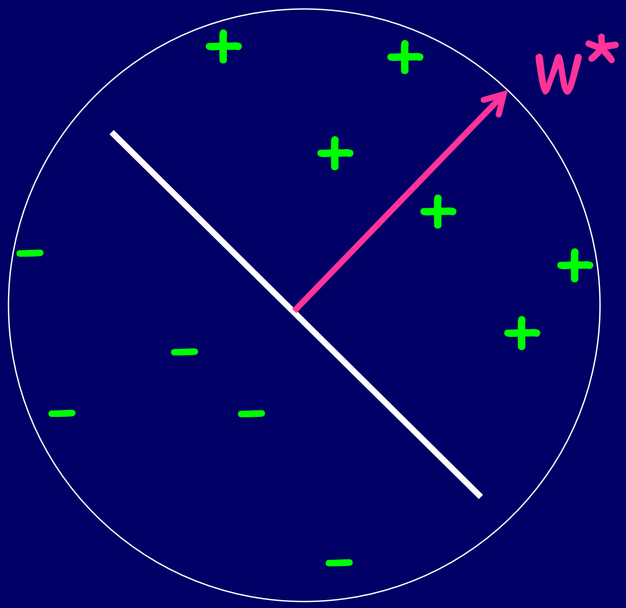
Lecture 3: The Perceptron Algorithm

Perceptron algorithm

Algorithm for learning a "large margin" linear separator in $\mathbb{R}^d$.

Online setting:

- Examples arrive one at a time.
- Given x, predict label y.
- Told correct answer.



Goal: bound number of mistakes under assumption there exists w^* such that $w^* \cdot x \geq 1$ on positives and $w^* \cdot x \leq -1$ on negatives.

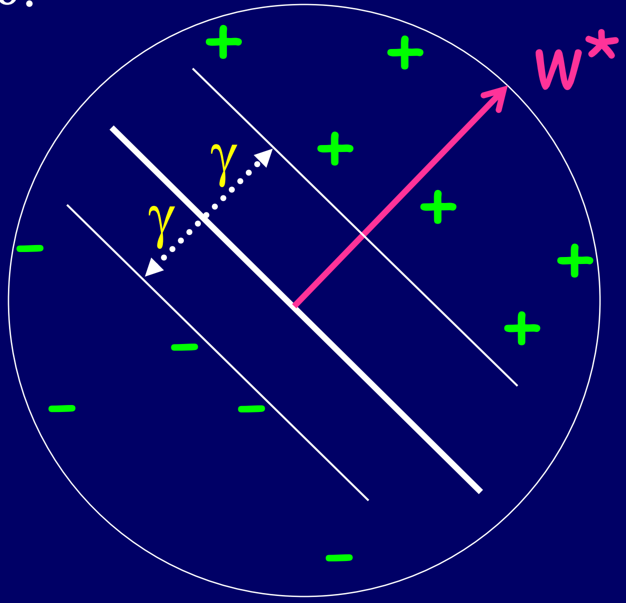
Perceptron alg: makes at most $\|w^*\|^2 \max(\|x\|^2)$ mistakes.

Perceptron algorithm

Perceptron alg makes $\leq \|w^*\|^2 \max(\|x\|^2)$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives.

How to think about this:

- $\frac{w^* \cdot x}{\|w^*\|}$ is distance of x to hyperplane $w^* \cdot x = 0$.
- Our assumption is equivalent to assuming exists a separator of margin $\gamma = \frac{1}{\|w^*\|}$.
- If points all lie in a ball of radius R, then mistake bound is at most R^2/γ^2.
- Notice this is scale-invariant.

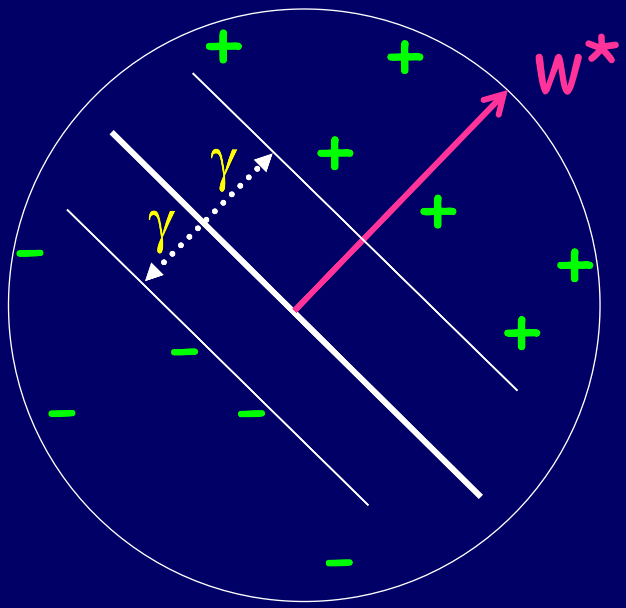


Perceptron algorithm

Perceptron alg makes $\leq \|w^*\|^2 \max(\ \|x\|^2\ )$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives.

Algorithm:

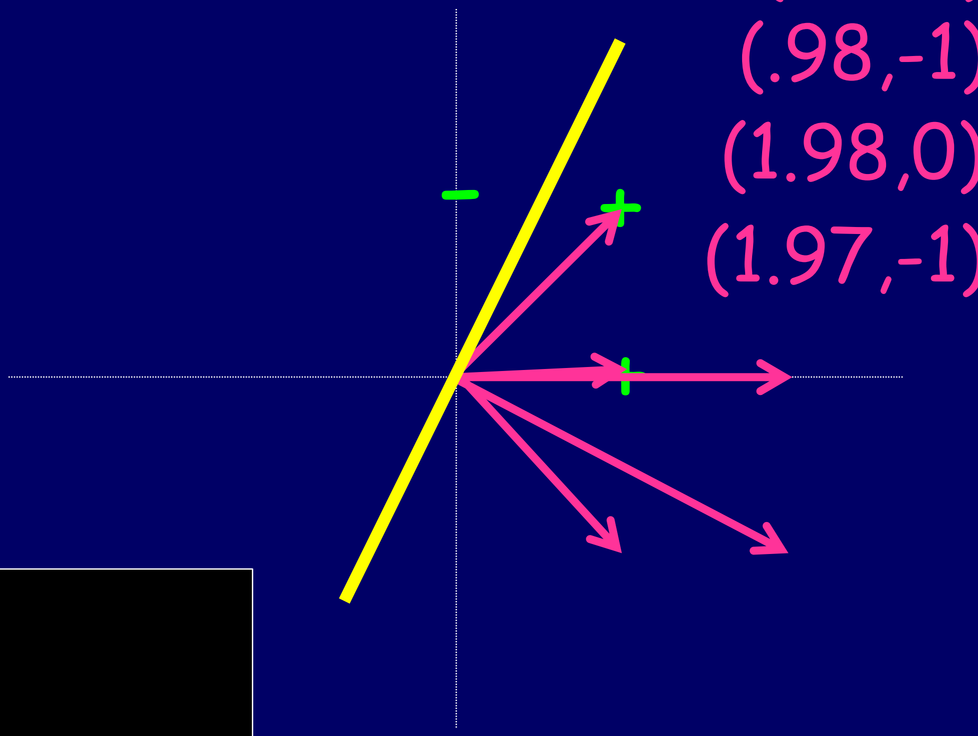
- Initialize $w = \vec{0}$. Predict positive if $w \cdot x > 0$, else predict negative.
- Mistake on positive: $w \leftarrow w + x$.
- Mistake on negative: $w \leftarrow w - x$.



Perceptron algorithm

Example:

(0.01,1) -
(1,1) +
(1,0) +
(0.01,1) -
(1,1) +
(1,0) +



Algorithm:

Initialize $w = \vec{0}$. Use $w \cdot x > 0$.

- Mistake on pos: $w \leftarrow w+x$.
- Mistake on neg: $w \leftarrow w-x$.

Analysis

Perceptron alg makes at most $\|w^*\|^2 R^2$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives, and all $\|x\| \leq R$.

Proof: consider w · w* and ||w||

- Each mistake increases w · w* by at least 1.

  $(w + x) \cdot w^* = w \cdot w^* + x \cdot w^* \geq w \cdot w^* + 1.$

  So after M mistakes, $w \cdot w^* \geq M$.

- Each mistake increases w·w by at most R^2.

  $(w + x) \cdot (w + x) = w \cdot w + 2(w \cdot x) + x \cdot x \leq w \cdot w + R^2.$

  So, after M mistakes, $\|w\|^2 \leq MR^2$, so $\|w\| \leq \sqrt{M}R$.

Since $\frac{w \cdot w^*}{\|w^*\|} \leq \|w\|$, get $\frac{M}{\|w^*\|} \leq \sqrt{M}R$ so $\sqrt{M} \leq \|w^*\|R$.

Lower bound

Perceptron alg makes at most $\|w^*\|^2 R^2$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives, and all $\|x\| \leq R$.

In general it's not possible to get $< R^2/\gamma^2$ mistakes with a deterministic algorithm.

Proof: consider R^2/γ^2 coordinate vectors scaled to length R.

$$w^* = (\pm x_1 \pm x_2 \pm \cdots \pm x_{R^2/\gamma^2})/R$$

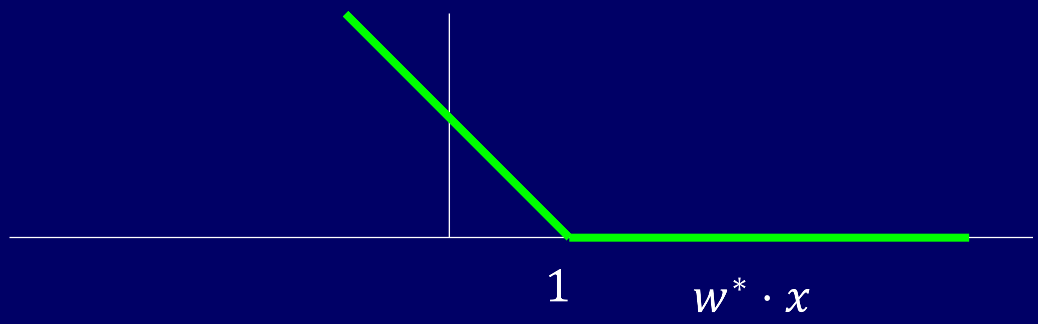
$|w^* \cdot x| = 1$ for all the input vectors, so can force all mistakes.

$\|w^*\| = \frac{\sqrt{R^2/\gamma^2}}{R} = \frac{1}{\gamma}$, so all margins are γ as desired.

What if no perfect separator?

In this case, a mistake could cause $|w \cdot w^*|$ to drop.

The hinge-loss of w* on positive x is $\max(0, 1 - w^* \cdot x)$: the amount by which the inequality $w^* \cdot x \geq 1$ is not satisfied.



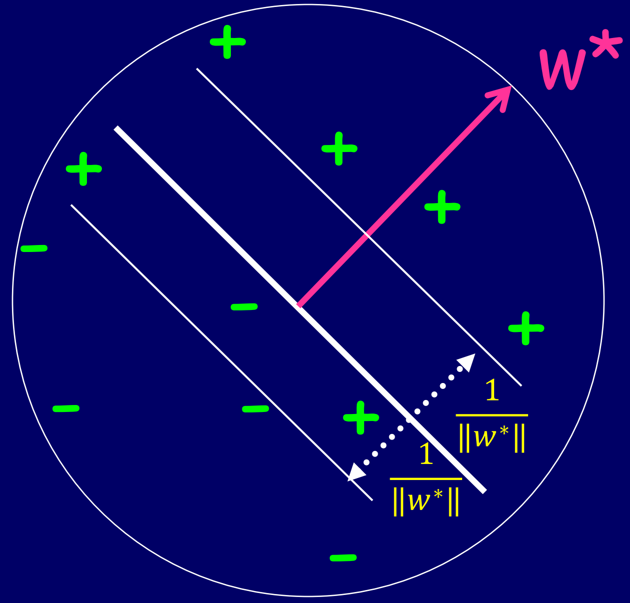
The hinge-loss of w* on negative x is $\max(0, 1 + w^* \cdot x)$: the amount by which the inequality $w^* \cdot x \leq -1$ is not satisfied.

What if no perfect separator?

In this case, a mistake could cause |w · w*| to drop.

Theorem: on any sequence of examples S, the Perceptron algo makes at most $\min_{w^*}\left[\|w^*\|^2 R^2 + 2L_{hinge}(w^*, S)\right]$ mistakes.

$L_{hinge}(w^*, S) =$ total hinge loss of w^* on set S.



Equivalently: how far you would have to move all the points to have them on the correct side by γ, in units of γ.

What if no perfect separator?

In this case, a mistake could cause |w · w*| to drop.

Theorem: on any sequence of examples S, the Perceptron algo makes at most $\min_{w^*}\left[\|w^*\|^2 R^2 + 2L_{hinge}(w^*, S)\right]$ mistakes.

Proof sketch:

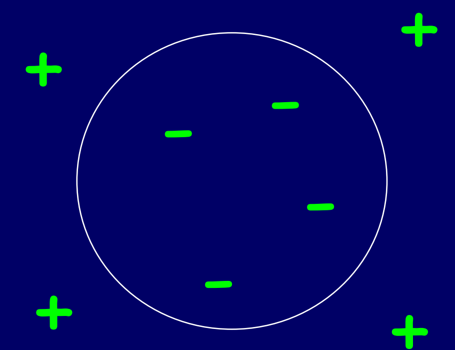
- After M mistakes, $w \cdot w^* \geq M - L_{hinge}(w^*, S)$.
- Still have: after M mistakes, $\|w\|^2 \leq MR^2$.
- Again use fact that $(w \cdot w^*)^2 \leq \|w\|^2 \|w^*\|^2$.
- Solve: $\left(M - L_{hinge}\right)^2 \leq MR^2 \|w^*\|^2$. Do some algebra.

$$M^2 - 2ML_{hinge} + L_{hinge}^2 \leq MR^2 \|w^*\|^2$$

$$M \leq R^2 \|w^*\|^2 + 2L_{hinge} - L_{hinge}^2 / M.$$

Kernel functions

What if the decision boundary between positive and negatives (e.g., spam and non-spam email) looks more like a circle than a linear separator?

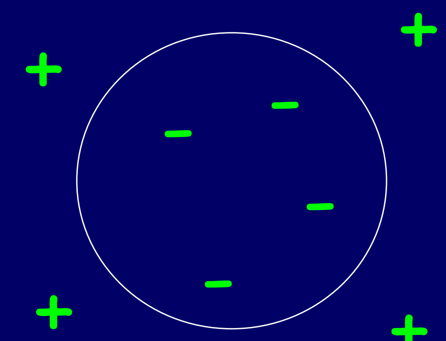


Idea: Kernel functions / "kernel trick":

- A pairwise function $K(x, x')$ is a kernel if there exists a function ϕ from input space to a new space (of possibly much higher dimension) such that $K(x, x') = \phi(x) \cdot \phi(x')$.
- Example: $K(x, x') = (1 + x \cdot x')^2$.
- Verify this is a kernel for special case that examples in $\mathbb{R}^2$:
- $K(x, x') = (1 + x_1 x_1' + x_2 x_2')^2 = 1 + 2x_1 x_1' + 2x_2 x_2' + x_1^2 x_1'^2 + 2x_1 x_2 x_1' x_2' + x_2^2 x_2'^2 = \phi(x) \cdot \phi(x')$ for $\phi(x) = (1, \sqrt{2} x_1, \sqrt{2}\, x_2, x_1^2, \sqrt{2}\, x_1 x_2, x_2^2)$.

Kernel functions

What if the decision boundary between positive and negatives (e.g., spam and non-spam email) looks more like a circle than a linear separator?



Idea: Kernel functions / “kernel trick”:

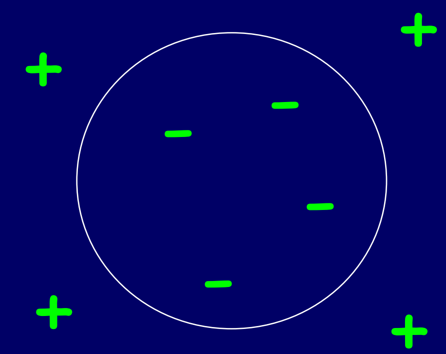
- If can modify Perceptron so that only interacts with data via taking dot-products, and then replace $x \cdot x'$ with $K(x, x')$, then algorithm will act as if data was in higher-dimensional ϕ-space.

- Called “kernelizing” the algorithm.

- E.g., for $\phi(x) = \left(1, \sqrt{2}x_1, \sqrt{2}\, x_2, x_1^2, \sqrt{2}\, x_1 x_2, x_2^2\right)$, the weight vector $w^* = (-100, 0, 0, 1, 0, 1)$ gives a circle of radius 10 as decision boundary $w^* \cdot \phi(x) = 0$.

Kernel functions

What if the decision boundary between positive and negatives (e.g., spam and non-spam email) looks more like a circle than a linear separator?



Idea: Kernel functions / “kernel trick”:

- If can modify Perceptron so that only interacts with data via taking dot-products, and then replace $x \cdot x'$ with $K(x, x')$, then algorithm will act as if data was in higher-dimensional ϕ-space.

- How to kernelize Perceptron?

- Easy: weight vector always a sum of previous examples (or their negations), e.g., $w = x^{(1)} + x^{(3)} - x^{(6)}$. So, to predict on new x, just compute $w \cdot x = x^{(1)} \cdot x + x^{(3)} \cdot x - x^{(6)} \cdot x$. Now replace dot-product with kernel.