STAT 30900 : MATHEMATICAL COMPUTATIONS I

Fall 2019

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Homework 4

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Solutions by

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1

Let $A \in \mathbb{R}^{m\times n}$ where $m \geq n$ and $\mathrm{rank}(A) = n$. Suppose GECP is performed on A to get

$$\Pi_1 A \Pi_2 = LU$$

where $L \in \mathbb{R}^{m\times n}$ is unit lower triangular, $U \in \mathbb{R}^{n\times n}$ is upper triangular, and $\Pi_1 \in \mathbb{R}^{m\times m}$, $\Pi_2 \in \mathbb{R}^{n\times n}$ are permutation matrices.

(a) Show that U is nonsingular and that L is of the form

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{R}^{n\times n}$ is nonsingular.

Proof. Since the permutation matrices Π_1 and Π_2 are full rank,

$$n = \mathrm{rank}(A) = \mathrm{rank}(\Pi_1 A \Pi_2) = \mathrm{rank}(LU) \leq \mathrm{rank}(U)$$

where the last inequality comes from HW1 1 (b). Also, $rank(U) \leq n$ since $U \in \mathbb{R}^{n\times n}$. So $\mathrm{rank}(U) = n$, i.e., U is nonsingular. Therefore,

$$n = \mathrm{rank}(LU) = \mathrm{rank}(L),$$

i.e., L has n linear independent rows. By suitable permutation, we have the first n rows of L are linear independent, so L is of the form $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ such that $L_1 \in \mathbb{R}^{n\times n}$ is nonsingular. □

(b) We will see how the LU factorization may be used to solve the least squares problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2.$$

(i) Show that the problem may be solved via

$$U\widetilde{\mathbf{x}} = \mathbf{y}, \quad L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}},$$

where $\widetilde{\mathbf{b}} = \Pi_1 \mathbf{b}$ and $\widetilde{\mathbf{x}} = \Pi_2^\top \mathbf{x}$.

Proof. Since the permutation matrices Π_1 and Π_2 are orthonomral and invertibel,

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2 &\overset{\widetilde{\mathbf{x}}=\Pi_2^\top \mathbf{x}}{=\!=\!=} \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^n} \|A\Pi_2\mathbf{x} - \mathbf{b}\|_2 \\
&\overset{\widetilde{\mathbf{b}}=\Pi_1^\top \mathbf{b}}{=\!=\!=} \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^n} \|\Pi_1 A\Pi_2\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}\|_2 \\
&= \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^n} \|LU\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}\|_2 \\
&\overset{\mathbf{y}=U\widetilde{\mathbf{x}}}{=\!=\!=} \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^n} \|L\mathbf{y} - \widetilde{\mathbf{b}}\|_2
\end{aligned}$$

And the normal equation for this optimization problem is given by

$$L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}}.$$

Solution (cont.)

So to solve the orginal problem, we can first solve $L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}}$ for $\mathbf{y}$ and then solve $U\widetilde{\mathbf{x}} = \mathbf{y}$ for $\widetilde{\mathbf{x}}$, and finally $\mathbf{x} = \Pi_2 \widetilde{\mathbf{x}}$. □

(ii) Describe how you would compute the solution $\mathbf{y}$ in

$$L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}}.$$

Since L has full rank, the QR decomposition of L is given by $L = Q\begin{bmatrix} R \\ 0 \end{bmatrix}$ where $Q \in \mathbb{R}^{m\times m}$ is unitary, $R \in \mathbb{R}^{n\times n}$ is upper triangular. Then $L^\top L = \begin{bmatrix} R^\top & 0 \end{bmatrix} Q^\top Q \begin{bmatrix} R \\ 0 \end{bmatrix} = R^\top R$. So we first slove $R^\top \mathbf{z} = L^\top \widetilde{\mathbf{b}}$ for $\mathbf{z}$ by forward substitution. Then we solve $R\mathbf{y} = \mathbf{z}$ for $\mathbf{y}$ by back substitution.

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Let $\varepsilon > 0$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}.$$

(a) Why is it a bad idea to solve the normal equation associated with A, i.e.

$$A^\top A\mathbf{x} = A^\top \mathbf{b}$$

when ε is small?

Notice that $A^\top A = \begin{bmatrix} 3 & 3 \\ 3 & 3+2\epsilon^2 \end{bmatrix}$. When ϵ is small, ϵ^2 is much smaller. Then ϵ^2 may be canceled out due to floating point errors, which makes $A^\top A$ to be singular. So it is a bad idea to solve the normal equation associated with A.

(b) Show that the LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}.$$

Proof. Since $\operatorname{rank}(A) = 2$, assume that $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \\ l_{31} & l_{32} \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ such that the condensed LU decomposition of A is given by

$$A = LU = \begin{bmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} \end{bmatrix}$$

So

$$\begin{aligned} u_{11} = u_{12} &= 1 \\ l_{21} = l_{31} &= 1 \\ u_{22} &= \epsilon \\ l_{32} &= -1 \end{aligned}$$

i.e.,

$$A = LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$$

□

(c) Why is it a much better idea to solve the normal equation associated with L, i.e.

$$L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}}?$$

This shows that the method in Problem **1** is a more stable method than using the normal equation in (a) directly.

Since ϵ only exists in U, by computing $L^\top L\mathbf{y} = L^\top \widetilde{\mathbf{b}}$, we will not get a term ϵ^2 and $L^\top L$ is nonsingular. So it is a much better idea to solve the normal equation associated with L.

(d) Show that the Moore–Penrose pseudoinverse of A is

$$A^\dagger = \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}.$$

Proof. Since A has full column rank, the pseudoinverse of A is given by

$$\begin{aligned} A^\dagger &= (A^\top A)^{-1}A^\top \\ &= \begin{bmatrix} 3 & 3 \\ 3 & 3+2\epsilon^2 \end{bmatrix}^{-1} A^\top \\ &= \begin{bmatrix} \frac{1}{3}+\frac{1}{2\epsilon^2} & -\frac{1}{2\epsilon^2} \\ -\frac{1}{2\epsilon^2} & \frac{1}{2\epsilon^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon & 1-\epsilon \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix} \end{aligned}$$

□

(e) Describe a method to compute $A^\dagger$ given L and U. Verify that your method is correct by checking it against the expression in (d).

Proof. Suppose A has full collumn rank, and has LU decomposition $A = LU$. Then L has full column rank and U is nonsingular. In this problem,

$$\begin{aligned} A^\dagger &= (A^\top A)^{-1}A^\top \\ &= (U^\top L^\top LU)^{-1}U^\top L^\top \\ &= U^{-1}(L^\top L)^{-1}L^\top \end{aligned}$$

$$\begin{aligned} U^{-1}(L^\top L)^{-1}L^\top &= \begin{bmatrix} 1 & -\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{2}\epsilon^{-1} \\ 0 & \frac{1}{2}\epsilon^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix} \end{aligned}$$

□

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We will now discuss an alternative method to solve the least squares problem in Problem **1** that is more efficient when $m - n < n$.

(a) Show that the least squares problem in Problem **1** is equivalent to

$$\min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} I_n \\ S \end{bmatrix} \mathbf{z} - \widetilde{\mathbf{b}} \right\|_2$$

where $S = L_2 L_1^{-1}$ and $L_1\mathbf{y} = \mathbf{z}$. Here and below, I_n denotes the $n \times n$ identity matrix.

Proof. Let $A = \Pi_1 L U \Pi_2$ be the LU decomposition of A as in Problem 1, where $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \in \mathbb{R}^{m\times n}$ is unit lower triangular and $U \in \mathbb{R}^{n\times n}$ is upper triangular. Fron Problem 1, we have

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2 &= \min_{\mathbf{y}\in\mathbb{R}^n} \|L\mathbf{y} - \widetilde{\mathbf{b}}\|_2 \\
&\overset{\mathbf{z}=L_1\mathbf{y}}{=\!=\!=} \min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} L_1^{-1}\mathbf{z} - \widetilde{\mathbf{b}} \right\|_2 \\
&= \min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} I_n \\ L_2 L_1^{-1} \end{bmatrix} \mathbf{z} - \widetilde{\mathbf{b}} \right\|_2
\end{aligned}$$

where $\mathbf{y} = U\Pi_2^\top \mathbf{x}$ and $\widetilde{\mathbf{b}} = \Pi_1 \mathbf{b}$. □

(b) Write

$$\widetilde{\mathbf{b}} = \begin{bmatrix} \widetilde{\mathbf{b}}_1 \\ \widetilde{\mathbf{b}}_2 \end{bmatrix}$$

where $\widetilde{\mathbf{b}}_1 \in \mathbb{R}^n$ and $\widetilde{\mathbf{b}}_2 \in \mathbb{R}^{m-n}$. Show that the solution $\mathbf{z}$ is given by

$$\mathbf{z} = \widetilde{\mathbf{b}}_1 + S^\top (I_{m-n} + SS^\top)^{-1}(\widetilde{\mathbf{b}}_2 - S\widetilde{\mathbf{b}}_1).$$

Proof.

$$\left\| \begin{bmatrix} I_n \\ L_2 L_1^{-1} \end{bmatrix} \mathbf{z} - \widetilde{\mathbf{b}} \right\|_2^2 = \left\| \begin{bmatrix} I_n\mathbf{z} \\ S\mathbf{z} \end{bmatrix} - \begin{bmatrix} \widetilde{\mathbf{b}}_1 \\ \widetilde{\mathbf{b}}_2 \end{bmatrix} \right\|_2^2 = \|\mathbf{z} - \widetilde{\mathbf{b}}_1\|_2^2 + \|S\mathbf{z} - \widetilde{\mathbf{b}}_2\|_2^2 \triangleq f(\mathbf{z})$$

Let

$$\nabla_{\mathbf{z}} f(\mathbf{z}) = 2\mathbf{z} - 2\widetilde{\mathbf{b}}_1 + 2S^\top S\mathbf{z} - 2S^\top \widetilde{\mathbf{b}}_2 = 0,$$

we have

$$\begin{aligned}
(I_n + S^\top S)\mathbf{z} &= \widetilde{\mathbf{b}}_1 + S^\top \widetilde{\mathbf{b}}_2 \\
\mathbf{z} &= (I_n + S^\top S)^{-1}(\widetilde{\mathbf{b}}_1 + S^\top \widetilde{\mathbf{b}}_2) \\
&= (I_n + S^\top S)^{-1}\widetilde{\mathbf{b}}_1 + (I_n + S^\top S)^{-1} S^\top \widetilde{\mathbf{b}}_2 \\
&= (I_n + S^\top S)^{-1}(I_n + S^\top S - S^\top S)\widetilde{\mathbf{b}}_1 + (I_n + S^\top S)^{-1} S^\top \widetilde{\mathbf{b}}_2 \\
&= \widetilde{\mathbf{b}}_1 - (I_n + S^\top S)^{-1} S^\top S\widetilde{\mathbf{b}}_1 + (I_n + S^\top S)^{-1} S^\top \widetilde{\mathbf{b}}_2 \\
&= \widetilde{\mathbf{b}}_1 + (I_n + S^\top S)^{-1} S^\top (\widetilde{\mathbf{b}}_2 - S\widetilde{\mathbf{b}}_1).
\end{aligned}$$

Solution (cont.)

Since

$$S^\top(I_{m-n} + SS^\top) = S^\top + S^\top SS^\top = (I_n + S^\top S)S^\top$$

we have

$$(I_n + S^\top S)^{-1}S^\top = S^\top(I_{m-n} + SS^\top)^{-1}.$$

So

$$\mathbf{z} = \widetilde{\mathbf{b}}_1 + S^\top(I_{m-n} + SS^\top)^{-1}(\widetilde{\mathbf{b}}_2 - S\widetilde{\mathbf{b}}_1).$$

□

(c) Explain why when $m - n < n$, the method in (a) is much more efficient than the method in Problem **1**. For example, what happens when $m = n + 1$?

Proof. In Problem 1, we need to calculate matrix product $L^\top L$, which requires $O(n^2m^2)$ multiplication operations. But here we just need to calculate matrix product $SS^\top$ for $S \in \mathbb{R}^{(m-n)\times n}$, which requires $O(n^2(m-n)^2)$ multiplication operations. So it is more efficient.
When $m = n + 1$, S reduces to be a row vector $\mathbb{R}^{1\times n}$. $1 + SS^\top$ is a scalar and its inverve is easily to compute. We don't need so much multiplication as in Problem 1 for computing $L^\top L$. □

Let $\mathbf{c} \in \mathbb{R}^n$ and consider the linearly constrained least squares problem/minimum norm linear system

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{w}\|_2 \\ \text{subject to} \quad & A^\top \mathbf{w} = \mathbf{c}. \end{aligned}$$

(a) If we write $\widetilde{\mathbf{c}} = \Pi_2^\top \mathbf{c}$ and $\widetilde{\mathbf{w}} = \Pi_1 \mathbf{w}$, show that

$$\widetilde{\mathbf{w}} = L(L^\top L)^{-1} U^{-\top} \widetilde{\mathbf{c}}$$

where $U^{-\top} = (U^{-1})^\top = (U^\top)^{-1}$, a standard notation that we will also use below. (*Hint*: You'd need to use something that you've already determined in an earlier part).

Proof. Let $\Pi_1 A \Pi_2 = LU$ be the GECP performed on A, then $A = \Pi_1^\top L U \Pi_2^\top$. Since

$$A^\top \mathbf{w} = \Pi_2 U^\top L^\top \Pi \mathbf{w} = \mathbf{c},$$

we have

$$U^\top L^\top \widetilde{\mathbf{w}} = \widetilde{\mathbf{c}}.$$

From problem 1, if A has full column rank, then U is nonsingular and L has full column rank. So $L^\top L$ is nonsingular and

$$\begin{aligned} L^\top \widetilde{\mathbf{w}} &= U^{-\top} \widetilde{\mathbf{c}} \\ L^\top [(LL^\top)(LL^\top)^{-1}] \widetilde{\mathbf{w}} &= U^{-\top} \widetilde{\mathbf{c}} \\ L^\top (LL^\top)^{-1} \widetilde{\mathbf{w}} &= (L^\top L)^{-1} U^{-\top} \widetilde{\mathbf{c}} \\ \widetilde{\mathbf{w}} &= L(L^\top L)^{-1} U^{-\top} \widetilde{\mathbf{c}} \end{aligned}$$

□

(b) Write

$$\widetilde{\mathbf{w}} = \begin{bmatrix} \widetilde{\mathbf{w}}_1 \\ \widetilde{\mathbf{w}}_2 \end{bmatrix}$$

where $\widetilde{\mathbf{w}}_1 \in \mathbb{R}^n$ and $\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}$. Show that

$$\widetilde{\mathbf{w}}_1 = L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} - S^\top \widetilde{\mathbf{w}}_2.$$

Proof.

$$\begin{aligned} \begin{bmatrix} \widetilde{\mathbf{w}}_1 \\ \widetilde{\mathbf{w}}_2 \end{bmatrix} &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (L_1^\top L_1 + L_2^\top L_2)^{-1} U^{-\top} \widetilde{\mathbf{c}} \\ &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} L_1^{-1} (I_n + L_1^{-\top} L_2^\top L_2 L_1^{-1})^{-1} L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} \\ &= \begin{bmatrix} (I_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} \\ S(I_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} \end{bmatrix}, \end{aligned}$$

Solution (cont.)

so

$$\widetilde{w}_1 + S^\top \widetilde{w}_2 = (I_n + S^\top S)(I_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} = L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}},$$

i.e., $\widetilde{\mathbf{w}}_1 = L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}} - S^\top \widetilde{\mathbf{w}}_2$. □

(c) Write $\mathbf{d} = L_1^{-\top} U^{-\top} \widetilde{\mathbf{c}}$. Deduce that $\widetilde{\mathbf{w}}_2$ may be obtained either as a solution to

$$\min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ I_{m-n} \end{bmatrix} \widetilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2$$

or as

$$\widetilde{\mathbf{w}}_2 = (I_{m-n} + SS^\top)^{-1} S\mathbf{d}.$$

Note that when $m - n < n$, this method is advantageous for the same reason in Problem **3**.

Proof. Since $\widetilde{\mathbf{w}} = \Pi_1 \mathbf{w}$, the original problem can be formulated as $\min \|\widetilde{\mathbf{w}}\|_2$, s.t. $\widetilde{\mathbf{w}}_1 = \mathbf{d} - S^\top \widetilde{\mathbf{w}}_2$, i.e.

$$\begin{aligned}
\min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} \widetilde{\mathbf{w}}_1 \\ \widetilde{\mathbf{w}}_2 \end{bmatrix} \right\|_2 &= \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} \mathbf{d} - S^\top \widetilde{\mathbf{w}}_2 \\ \widetilde{\mathbf{w}}_2 \end{bmatrix} \right\|_2 \\
&= \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| S^\top \widetilde{\mathbf{w}}_2 - \mathbf{d} \right\|_2 + \|\widetilde{\mathbf{w}}_2\|_2 \\
&= \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \mathbf{d} - S^\top \widetilde{\mathbf{w}}_2 \right\|_2 + \|\widetilde{\mathbf{w}}_2\|_2 \\
&= \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \widetilde{\mathbf{w}}_2 - \mathbf{d} \\ \widetilde{\mathbf{w}}_2 \end{bmatrix} \right\|_2 \\
&= \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ I_{m-n} \end{bmatrix} \widetilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2
\end{aligned}$$

From Problem 3, the solution to

$$\min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ I_{m-n} \end{bmatrix} \widetilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2 = \min_{\widetilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} I_{m-n} \\ S^\top \end{bmatrix} \widetilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \end{bmatrix} \right\|_2$$

is $\widetilde{\mathbf{w}}_2 = S(I_n + S^\top S)^{-1}\mathbf{d} = (I_{m-n} + SS^\top)^{-1} S\mathbf{d}$ where the last equality comes from the solution of Problem 3 (b). □

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So far we have assumed that A has full column rank. Suppose now that $\mathrm{rank}(A) = r < \min\{m, n\}$.

(a) Show that the LU factorization obtained using GECP is of the form

$$\Pi_1 A \Pi_2 = LU = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where $L_1, U_1 \in \mathbb{R}^{r \times r}$ are triangular and nonsingular.

Proof. The GECP yields

$$\begin{aligned} \Pi_1 A \Pi_2 &= LU \\ &= \begin{bmatrix} L_1 & 0 \\ L_2 & I_{m-r} \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}. \end{aligned}$$

Since $L_{11} \in \mathbb{R}^{r \times r}$ is unit lower triangular matrix, L_{11} is nonsingular. Since $\mathrm{rank}(U) \geq \mathrm{rank}(LU) = \mathrm{rank}(A) = r$ and U has r nonzeros rows, i.e., $\mathrm{rank}(U) \leq r$, so $\mathrm{rank}(U) = r$. So first r rows of U are linear independent, and so does U_1. So $\mathrm{rank}(U_1) = r$, i.e., U_1 is nonsingular. Since U is upper triangular, U_1 is also upper triangular. □

(b) Show that the above equation may be rewritten in the form

$$\Pi_1 A \Pi_2 = \begin{bmatrix} I_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} I_r & S_2^\top \end{bmatrix}$$

for some matrices S_1 and S_2.

Proof. Since L_1, U_1 are nonsingular, we have

$$\begin{aligned} \Pi_1 A \Pi_2 = LU &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix} \\ &= \begin{bmatrix} L_1 \\ L_2 L_1^{-1} L_1 \end{bmatrix} \begin{bmatrix} U_1 & U_1 U_1^{-1} U_2 \end{bmatrix} \\ &= \begin{bmatrix} I_r \\ L_2 L_1^{-1} \end{bmatrix} L_1 U_1 \begin{bmatrix} I_r & U_1^{-1} U_2 \end{bmatrix}. \end{aligned}$$

So $S_1 = L_2 L_1^{-1}$, $S_2 = (U_1^{-1} U_2)^\top$.

□

(c) Hence show that the Moore–Penrose inverse of A is given by

$$A^\dagger = \Pi_2 \begin{bmatrix} I_r & S_2^\top \end{bmatrix}^\dagger U_1^{-1} L_1^{-1} \begin{bmatrix} I_r \\ S_1 \end{bmatrix}^\dagger \Pi_1.$$

Proof.

$$\begin{aligned} A^\dagger &= \left(\Pi_1^\top \begin{bmatrix} I_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} I_r & S_2^\top \end{bmatrix} \Pi_2^\top \right)^\dagger \\ &= \Pi_2 \begin{bmatrix} I_r & S_2^\top \end{bmatrix}^\dagger U_1^{-1} L_1^{-1} \begin{bmatrix} I_r \\ S_1 \end{bmatrix}^\dagger \Pi_1 \end{aligned}$$

□

(d) Using the general formula (derived in the lectures) for the Moore–Penrose inverse of a rank-retaining factorization, what do you get for $A^\dagger$?

Proof.

$$\begin{aligned} (\Pi_1 A \Pi_2)^\dagger &= U^\top (UU^\top)^{-1} (L^\top L)^{-1} L^\top \\ A^\dagger &= \Pi_2 U^\top (UU^\top)^{-1} (L^\top L)^{-1} L^\top \Pi_1 \\ &= \Pi_2 \begin{bmatrix} U_1^\top \\ U_2^\top \end{bmatrix} (U_1 U_1^\top + U_2 U_2^\top)^{-1} (L_1^\top L_1 + L_2^\top L_2)^{-1} \begin{bmatrix} L_1^\top \\ L_2^\top \end{bmatrix} \Pi_1 \\ &= \Pi_2 \begin{bmatrix} U_1^\top \\ U_2^\top \end{bmatrix} U_1^{-\top} (I_r + U_1^{-1} U_2 U_2^\top U_1^{-\top})^{-1} U_1^{-1} L_1^{-1} (I_r + L_1^{-\top} L_2^\top L_2 L^{-1})^{-1} L_1^{-\top} \begin{bmatrix} L_1^\top \\ L_2^\top \end{bmatrix} \Pi_1 \\ &= \Pi_2 \begin{bmatrix} I_r \\ S_2 \end{bmatrix} (I_r + U_1^{-1} U_2 U_2^\top U_1^{-\top})^{-1} U_1^{-1} L_1^{-1} (I_r + L^{-\top} L_2^\top L_2 L^{-1})^{-1} \begin{bmatrix} I_r \\ S_1^\top \end{bmatrix} \Pi_1 \end{aligned}$$

which indicates that

$$\begin{aligned} \begin{bmatrix} I_r & S_2^\top \end{bmatrix}^\dagger &= \begin{bmatrix} I_r \\ S_2 \end{bmatrix} (I_r + U_1^{-1} U_2 U_2^\top U_1^{-\top})^{-1} \\ \begin{bmatrix} I_r \\ S_1 \end{bmatrix}^\dagger &= (I_r + L^{-\top} L_2^\top L_2 L^{-1})^{-1} \begin{bmatrix} I_r \\ S_1^\top \end{bmatrix} \end{aligned}$$

□

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Consider the block matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{n\times n}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{q\times q}$ and $n = p + q$. The Schur complements of A and D are

$$S = D - CA^{\dagger}B \quad \text{and} \quad T = A - BD^{\dagger}C$$

respectively.

(a) Verify that if A and S are nonsingular, then

$$X^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

and if D and T are nonsingular, then

$$X^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}.$$

Proof. If A and S are nonsingular, since

$$\begin{aligned}
&\begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\
&= \begin{bmatrix} I_p + A^{-1}BS^{-1}C - A^{-1}BS^{-1}C & A^{-1}B + A^{-1}BS^{-1}CA^{-1}B - A^{-1}BS^{-1}D \\ -S^{-1}C + S^{-1}C & -S^{-1}CA^{-1}B + S^{-1}D \end{bmatrix} \\
&= \begin{bmatrix} I_p & A^{-1}B - A^{-1}BS^{-1}(D - CA^{-1}B) \\ 0 & S^{-1}(D - CA^{-1}B) \end{bmatrix} \\
&= \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix},
\end{aligned}$$

we have that $X^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$.

If D and T are nonsingular, since

$$\begin{aligned}
&\begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\
&= \begin{bmatrix} T^{-1}A - T^{-1}BD^{-1}C & T^{-1}B - T^{-1}B \\ -D^{-1}CT^{-1}A + D^{-1}C + D^{-1}CT^{-1}BD^{-1}C & -D^{-1}CT^{-1}B + I_q + D^{-1}CT^{-1}B \end{bmatrix} \\
&= \begin{bmatrix} T^{-1}(A - BD^{-1}C) & 0 \\ D^{-1}C + D^{-1}CT^{-1}(BD^{-1}C - A) & I_q \end{bmatrix} \\
&= \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix},
\end{aligned}$$

we have that $X^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}$. □

(b) Show that

$$\det X = \begin{cases} \det(A)\det(D - CA^{-1}B) & \text{if } A \text{ nonsingular}, \\ \det(D)\det(A - BD^{-1}C) & \text{if } D \text{ nonsingular}. \end{cases}$$

Deduce that

$$\det(A + BC) = \det(A)\det(I + CA^{-1}B)$$

and use it to find the determinants of the following matrices

$$\begin{bmatrix} \frac{1+\lambda_1}{\lambda_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\lambda_2}{\lambda_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\lambda_n}{\lambda_n} \end{bmatrix}, \quad \begin{bmatrix} 1+\lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & 1+\lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & 1+\lambda_n \end{bmatrix}, \quad \begin{bmatrix} \lambda & \mu & \mu & \cdots & \mu \\ \mu & \lambda & \mu & \cdots & \mu \\ \mu & \mu & \lambda & \cdots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \mu & \mu & \cdots & \lambda \end{bmatrix}.$$

Proof. If A is nonsingular,

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \to \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & A^{-1}B \\ 0 & I_q \end{bmatrix}$$
$$\begin{bmatrix} I_p & A^{-1}B \\ 0 & I_q \end{bmatrix} \to \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

So

$$\begin{aligned} \det(X) &= \det\left(\begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}\right) \det\left(\begin{bmatrix} I_p & A^{-1}B \\ 0 & I_q \end{bmatrix}\right) \\ &= \det(A)\det(D - CA^{-1}B). \end{aligned}$$

If D is nonsingular,

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \to \begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix}$$
$$\begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix} \to \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

So

$$\begin{aligned} \det(X) &= \det\left(\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\right) \det\left(\begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix}\right) \\ &= \det(D)\det(A - BD^{-1}C). \end{aligned}$$

Let

$$X = \begin{bmatrix} A & B \\ C & -I_q \end{bmatrix} \to \begin{bmatrix} A + BC & 0 \\ C & -I_q \end{bmatrix} \to \begin{bmatrix} A + BC & 0 \\ 0 & -I_q \end{bmatrix}.$$

If A is nonsingular, we have

$$\det(X) = \det(A)\det(I + CA^{-1}B).$$

Also,

$$\det(X) = \det\left(\begin{bmatrix} A + BC & 0 \\ 0 & -I_q \end{bmatrix}\right) = \det(A + BC)\det(I_q),$$

Solution (cont.)

so $\det(A+BC) = \det(A)\det(I + CA^{-1}B)$.

Let

$$A = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top \qquad C = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$$

then

$$\begin{aligned} \det\left(\begin{bmatrix} \frac{1+\lambda_1}{\lambda_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\lambda_2}{\lambda_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\lambda_n}{\lambda_n} \end{bmatrix}\right) &= \det(A+BC) \\ &= \det(A)\det(1 + CA^{-1}B) \\ &= \left(\prod_{i=1}^n \frac{1}{\lambda_i}\right)\left(1 + \sum_{i=1}^n \lambda_i\right) \end{aligned}$$

Let

$$A = I_n \qquad B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top \qquad C = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}$$

then

$$\begin{aligned} \det\left(\begin{bmatrix} 1+\lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & 1+\lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & 1+\lambda_n \end{bmatrix}\right) &= \det(A+BC) \\ &= \det(A)\det(1 + CA^{-1}B) \\ &= \left(1 + \sum_{i=1}^n \lambda_i\right) \end{aligned}$$

If $\lambda = \mu$, $\det\left(\begin{bmatrix} \lambda & \mu & \mu & \cdots & \mu \\ \mu & \lambda & \mu & \cdots & \mu \\ \mu & \mu & \lambda & \cdots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \mu & \mu & \cdots & \lambda \end{bmatrix}\right) = 0$. Otherwise, let

$$A = (\lambda - \mu)I_n \qquad B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top \qquad C = \begin{bmatrix} \mu & \mu & \cdots & \mu \end{bmatrix}$$

Solution (cont.)

then

$$\det\left(\begin{bmatrix}\lambda & \mu & \mu & \cdots & \mu\\ \mu & \lambda & \mu & \cdots & \mu\\ \mu & \mu & \lambda & \cdots & \mu\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mu & \mu & \mu & \cdots & \lambda\end{bmatrix}\right) = \det(A+BC)$$

$$= \det(A)\det(1+CA^{-1}B)$$

$$= (\lambda-\mu)^n\left(1+\frac{n\mu}{\lambda-\mu}\right)$$

□

(c) Show that if A has all principal matrices nonsingular so that we may perform Gaussian elimination without pivoting to A, then applying the first p steps of that to X yields

$$X = \begin{bmatrix}L_{11} & 0\\ L_{21} & I_q\end{bmatrix}\begin{bmatrix}I_p & 0\\ 0 & S\end{bmatrix}\begin{bmatrix}U_{11} & U_{12}\\ 0 & I_q\end{bmatrix}$$

where $A = L_{11}U_{11}$ is the LU factorization of A. What are L_{21} and U_{12} in terms of L_{11}, U_{11} and the blocks of X?

Proof. The first p steps of LU factorization of X yields

$$\begin{aligned}X &= \begin{bmatrix}L_{11} & 0\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}U_{11} & U_{12}\\ 0 & U_{22}\end{bmatrix}\\ &= \begin{bmatrix}L_{11} & 0\\ L_{21} & I_q\end{bmatrix}\begin{bmatrix}I_p & 0\\ 0 & L_{22}\end{bmatrix}\begin{bmatrix}I_p & 0\\ 0 & U_{22}\end{bmatrix}\begin{bmatrix}U_{11} & U_{12}\\ 0 & I_q\end{bmatrix}\\ &= \begin{bmatrix}L_{11} & 0\\ L_{21} & I_q\end{bmatrix}\begin{bmatrix}I_p & 0\\ 0 & L_{22}U_{22}\end{bmatrix}\begin{bmatrix}U_{11} & U_{12}\\ 0 & I_q\end{bmatrix}\end{aligned}$$

i.e., $S = L_{22}U_{22}$.
Since A has all princ matrices nonsingular, A is nonsingular. So L_{11} and U_{11} is nonsingular. Since

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}L_{11}U_{11} & L_{11}U_{12}\\ L_{21}U_{11} & L_{21}U_{12}+L_{22}U_{22}\end{bmatrix},$$

we have

$$\begin{aligned}U_{12} &= L_{11}^{-1}B\\ L_{21} &= CU_{11}^{-1}.\end{aligned}$$

□

(d) Suppose X is symmetric (so $C = B^\top$) and A is positive definite. Show that applying the first p steps of Cholesky factorization to X yields

$$X = \begin{bmatrix} R_{11}^\top \\ R_{12}^\top \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$$

where $A = R_{11}^\top R_{11}$ is the Cholesky factorization. What is R_{12} in terms of R_{11} and the blocks of X?

Proof. Since X is symmetric, the first p steps of Cholesky factorization of X yields

$$\begin{aligned} X &= \begin{bmatrix} R_{11}^\top & 0 \\ R_{12}^\top & R_{22}^\top \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \\ &= \begin{bmatrix} R_{11}^\top R_{11} & R_{11}^\top R_{12} \\ R_{12}^\top R_{11} & R_{12}^\top R_{12} + R_{22}^\top R_{22} \end{bmatrix} \\ &= \begin{bmatrix} R_{11}^\top R_{11} & R_{11}^\top R_{12} \\ R_{12}^\top R_{11} & R_{12}^\top R_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R_{22}^\top R_{22} \end{bmatrix} \\ &= \begin{bmatrix} R_{11}^\top \\ R_{12}^\top \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R_{22}^\top R_{22} \end{bmatrix} \end{aligned}$$

i.e., $S = R_{22}^\top R_{22}$.
Since A is positive definite, R_{11} is nonsingular. Also, since

$$\begin{bmatrix} A & B \\ B^\top & D \end{bmatrix} = \begin{bmatrix} R_{11}^\top R_{11} & R_{11}^\top R_{12} \\ R_{12}^\top R_{11} & R_{12}^\top R_{12} + R_{22}^\top R_{22} \end{bmatrix},$$

we have

$$R_{12} = R_{11}^{-\top} B.$$

□