Problem Set 03 Solutions

Q1. Let

$$A = \begin{bmatrix} \text{---} \; a_1^* \; \text{---} \\ \vdots \\ \text{---} \; a_n^* \; \text{---} \end{bmatrix} \qquad a_j \in \mathbb{C}^n$$

Suppose

a_j form ON basis of $\mathbb{C}^n$

so $a_j^* a_k = \delta_{jk}$. Then

$$AA^* = \begin{bmatrix} \text{---} \; a_1^* \; \text{---} \\ \vdots \\ \text{---} \; a_n^* \; \text{---} \end{bmatrix} \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \langle a_i, a_j \rangle \end{bmatrix}$$

$$= [\delta_{ij}] = I.$$

Suppose $AA^* = I$. Then A is a square invertible matrix since $\det A \neq 0$. Hence left-multiplying by A^{-1} gives

$$A^{-1}AA^* = A^* = A^{-1}.$$

$\Rightarrow$ Hence $I = A^{-1}A = A^*A$ as well.

Hence $$\|Ax\|^2 = \langle Ax, Ax\rangle$$
$$= \langle A^*Ax, x\rangle$$
$$= \langle x, x\rangle$$
$$= \|x\|^2$$
and $\|Ax\| = \|x\|$ for all $x \in \mathbb{C}^n$.

Finally suppose $\|Ax\| = \|x\|$ for all x. Then
$$\langle A^*Ax, x\rangle = \langle x, x\rangle \quad \text{for all } x.$$
Since $x \neq 0 \Rightarrow Ax \neq 0$, A is invertible.
Since A^*A is symmetric it is diagonalizable.
Let
$$A^*A = Q\Lambda Q^*$$
where $Q^*Q = QQ^* = I$. Then
$$\langle y, y\rangle = \langle A^*Ax, x\rangle = \langle \Lambda y, y\rangle \ \forall y$$
where $y = Q^*x$ so $\|y\| = \|x\|$. Setting $y = e_j$ gives $\lambda_j = 1$ so $\Lambda = I$.

Hence $A^*A = Q\Lambda Q^* = QQ^* = I$.
Since A is invertible $A^* = A^{-1}$
and $AA^* = I$, or equivalently
the rows of A form an ON
basis in $\mathbb{C}^n$.

Q2. Let $A^*z = 0$. Then

$$\langle A^* z, v\rangle = \langle z, Av\rangle = 0$$

so $z \in N(A^*) \Rightarrow z \perp R(A)$ and

$$N(A^*) \subset R(A)^\perp.$$

Conversely if $z \perp R(A)$ then

$$\langle z, Av\rangle = \langle A^* z, v\rangle = 0$$

for all $v \in V$. Letting $v = A^* z \in V$ gives $\|A^* z\|^2 = 0$ so $A^* z = 0$ and

$$R(A)^\perp \subset N(A^*).$$

Hence

$$\boxed{N(A^*) = R(A)^\perp.}$$

Q3. Suppose $Ax = b$ for some $x \in V$.
Then for any $w \in N(A^*)$ we have

$$0 = \langle A^* w, x \rangle = \langle w, Ax \rangle = \langle w, b \rangle$$

so (b) cannot hold.
Conversely if $b \notin R(A) = N(A^*)^\perp$
then there exists $w \in N(A^*)$ with
$\langle w, b \rangle \neq 0$.

Q4. Finding the coefficients $p_0 \dots p_n$ of P is equivalent to the linear system

$$p_0 + p_1 x_0 + \dots + p_n x_0^n = f_0$$
$$\vdots \qquad\qquad \vdots \qquad \text{i.e. } Xp = f$$
$$p_0 + p_1 x_n + \dots + p_n x_n^n = f_n$$

where $X_{ij} = x_i^j$. Suppose $\exists\, p \in \mathbb{C}^{n+1}$ with

$$Xp = 0.$$

Then $p(x)$ is a degree n polynomial with $n+1$ distinct zeroes $x_0, \dots, x_n$. By the FTA, $p = 0$. Hence $N(X) = \{0\}$. Since $\det X = \det X^* \neq 0$ we know $N(X^*) = \{0\}$. By the Fredholm Alternative, $R(X) = \mathbb{C}^{n+1}$ so there is a polynomial p interpolating any f. Since $N(X) = \{0\}$ p is unique.

Q5. $\langle Px - x, Py \rangle =$

$$= \langle P^*(P-I)x, y \rangle = 0 \quad \forall x, y$$

$$\Leftrightarrow \quad P^*(P-I) = 0$$

$$\Leftrightarrow \quad P^*P = P^*,$$

Taking adjoints,

$$\Leftrightarrow \quad P^*P = P$$

so

$$\Leftrightarrow \quad P = P^*.$$

6(a) $\int_{-\infty}^{\infty} K_t(x-y) f(y)\,dy =$

$$= \frac{t}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+(x-y)^2} f(y)\,dy$$

Let $y = x+ts$ $dy = t\,ds$ so

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+s^2} f(x+ts)\,ds.$$

Since f is continuous,

$$\frac{1}{\pi}\frac{1}{1+s^2} f(x+ts) \to \frac{1}{\pi}\frac{1}{1+s^2} f(x)$$

for each s, and since f is bounded

$$\left|\frac{1}{\pi}\frac{1}{1+s^2} f(x+ts)\right| \le \frac{M}{1+s^2}$$

is dominated. Hence the limit as $t \downarrow 0$ is

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{1+s^2}\,ds = f(x).$$

(b) Let $f(x) = 1$ be bounded and continuous. By (a)

$$\int K_t(x-y) f(y)\,dy \to 1 \quad \forall x.$$

But

$$\int K_t(x-y) f(y)\,dy = \int K_t(x-y)\,dy$$

$$= \int_{-\infty}^{\infty} K_t(y)\,dy \qquad x-y=z$$

is independent of t:

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ds}{1+s^2} = 1.$$

Q7. $$\int_{-\infty}^{\infty} \frac{e^{-|x-y|/t}}{2t} f(y)\,dy \qquad \begin{aligned} y &= x+ts \\ dy &= t\,ds \end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-|s|}}{2t} f(x+ts)\, t\, ds$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|s|} f(x+ts)\, ds.$$

Since $e^{-|s|} f(x+ts) \to e^{-|s|} f(x) \quad \forall s$

and

$$|e^{-|s|} f(x+ts)| \le M e^{-|s|},$$

we have

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-|s|} f(x+ts)\,ds \to \frac{1}{2}\int_{-\infty}^{\infty} e^{-|s|}\,ds\, f(x)$$

$$= f(x)$$

as $t \to 0$.

Question 1 Suppose A is a complex $n \times n$ matrix. Show that the following are equivalent:

(a) The rows of A form an orthonormal basis in C^n.
(b) $AA^* = I$.
(c) $\|Ax\| = \|x\|$ for all $x \in C^n$.

Question 2 Suppose $A : V \to W$ is a linear map between two inner product spaces. Show that the nullspace of A^* is exactly the perpendicular complement of the range of A.

Question 3 Prove the Fredholm Alternative: Suppose $A : V \to W$ is a linear map between two inner product spaces. Let $b \in W$. Then either

(a) $Ax = b$ for some $x \in V$ or
(b) There is $w \in W$ with $A^*w = 0$ and $< b, w > \neq 0$.

Question 4 Use the Fredholm Alternative and the Fundamental Theorem of Algebra to prove the existence and uniqueness of polynomial interpolation: given $n+1$ distinct real numbers $x_0, x_1, \dots x_n$ and $n+1$ complex numbers $f_j, f_1, \dots f_n$, there exists a unique degree-n polynomial $P(x) = p_0 + p_1 x + \cdots + p_n x^n$ such that $P(x_j) = f_j$ for $0 \leq j \leq n$.

Question 5 Prove that a projection P on an inner product space is an orthogonal projection if and only if $P^* = P$.

Question 6 (a) Let

$$K_t(x) = \frac{t}{\pi(t^2 + x^2)}$$

for $t > 0$ and $x \in R$. Use the Dominated Convergence Theorem to show that

$$\int_{-\infty}^{\infty} K_t(x-y) f(y) \, \mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.

(b) Use (a) to evaluate

$$\int_{-\infty}^{\infty} K_t(x-y) \, \mathrm{d}y$$

Question 7 Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|x-y|/t}}{2t} f(y)\ \mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.