STAT 150: Stochastic Processes

Fall 2017

Homework 5

Solutions by

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PK Problems 4.4.2

Determine the stationary distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix} \end{array}$$

Let $\pi = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$ denotes the stationary distribution. Then

$$\begin{cases} \pi P = \pi \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases}$$

i.e.]

$$\begin{pmatrix} -1 & 0 & \frac{1}{4} & \frac{1}{3} \\ 0 & -1 & \frac{3}{4} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & -1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 0 & -1 \end{pmatrix} \pi^T = 0$$

and

$$\sum_{i=0}^{3} \pi_i = 1$$

i.e.

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{22}{45} \\ 0 & 1 & 0 & -\frac{17}{15} \\ 0 & 0 & 1 & -\frac{28}{45} \\ 0 & 0 & 0 & 0 \end{pmatrix} \pi^T = 0$$

i.e.

$$\begin{cases} \sum_{i=0}^{3} \pi_3 = 1 \\ \pi_0 = \frac{22}{45}\pi_3 \\ \pi_1 = \frac{17}{15}\pi_3 \\ \pi_2 = \frac{28}{45}\pi_3 \end{cases}$$

Therefore,

$$\begin{cases} \pi_0 = \frac{11}{73} \\ \pi_1 = \frac{51}{146} \\ \pi_2 = \frac{14}{73} \\ \pi_3 = \frac{45}{146} \end{cases}$$

PK Problems 4.4.5

Let $\mathbf{P}$ be the transition probability maxtrix of a finite-state Markov chain. Let $\mathbf{M}= \|m_{ij}\|$ be the matrix of mean return times.

(a) Use a first step argument to establish that

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} m_{kj}.$$

Let $T = \min\{n \geqslant 0; X_n \geqslant r\}$ and suppose that state j is the absorbing state, i.e. $P_{jj} = 1$.

$\because$ state space is finite

$\therefore$

$$\begin{aligned}
m_{ij} &= \mathbb{E}(T|X_0 = i) \\
&= \mathbb{E}\left(\sum_{n=0}^{T-1} g(X_n)|X_0 = i\right) \\
&= \sum_{t=1}^{\infty} t\mathbb{P}\left(T = t|X_0 = i\right) \\
&= \sum_{t=1}^{\infty} t\left[\sum_{k \neq j} P_{ik}\mathbb{P}\left(T = t|X_0 = k\right) + P_{jj}\mathbb{P}(T = t|X_0 = j)\right] \\
&= \sum_{t=1}^{\infty} t\left[\sum_{k \neq j} P_{ik}\mathbb{P}\left(T = t|X_0 = k\right) + \mathbb{P}(T = t|X_0 = j)\right] \\
&= \sum_{k \neq j} P_{ik} \sum_{t=1}^{\infty} t\mathbb{P}\left(T = t|X_0 = k\right) + \sum_{t=1}^{\infty} t\mathbb{P}(T = t|X_0 = j) \\
&= \sum_{k \neq j} P_{ik}\mathbb{E}\left(T|X_0 = k\right) + 1 \\
&= 1 + \sum_{k \neq j} P_{ik} m_{kj}
\end{aligned}$$

(b) Multiply both sides of the preceding by π_i and sum to obtain

$$\sum_i \pi_i m_{ij} = \sum_i \pi_i + \sum_{k \neq j} \sum_i \pi_i P_{ik} m_{kj}.$$

Simplify this to show (see equation 4.26)

$$\pi_j m_{jj} = 1, \qquad \text{or} \qquad \pi_j = \frac{1}{m_{jj}}.$$

$\because$

$$\sum_i \pi_i m_{ij} = \sum_i \pi_i + \sum_{k \neq j} \sum_i \pi_i P_{ik} m_{kj}$$

$$\sum_i \pi_i = 1$$

Solution (cont.)

$$\sum_i \pi_i P_{ik} = \pi_k$$

$\therefore$

$$\sum_i \pi_i m_{ij} = 1 + \sum_{k\neq j} \pi_k m_{kj}$$

i.e.

$$\pi_j m_{jj} = 1$$

or

$$\pi_j = \frac{1}{m_{jj}}$$

since $m_{jj} \geqslant 1$.

Question 1

Let $\{X_n\}$ be a Markov chain on $\mathbb{N}$ with transition probability given by

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \in \mathbb{Z}^+.$$

Show that

(a) The chain is irreducible and transient;

$\because \quad \forall\, i \in \mathbb{N}^+$,

$$\begin{cases} p_{i,i+1} + p_{i,i-1} = 1 \\ p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1} \end{cases}$$

$\therefore$

$$\begin{cases} p_{i,i+1} = \dfrac{(i+1)^2}{i^2+(i+1)^2} \\ p_{i,i-1} = \dfrac{i^2}{i^2+(i+1)^2} \end{cases}$$

$\because \quad \forall\, i \in \mathbb{N}^+$,

$$p_{0,1}, p_{i,i+1}, p_{i,i-1} > 0$$

$\therefore \quad \forall\, i, j \in \mathbb{N},\ i \neq j$,

$$\begin{aligned} \mathbb{P}(X_n = j \text{ for some } n \geqslant 1 | X_0 = i) &\geqslant \begin{cases} p_{i,i+1}p_{i+1,i+2}\cdots p_{j-1,j} & i < j \\ p_{i,i-1}p_{i-1,i-2}\cdots p_{j+1,j} & i > j \end{cases} \\ &> 0 \end{aligned}$$

i.e. the chain is irreducible

For the equations

$$y_i = \sum_{j\neq 0} P_{ij} y_j \qquad i \neq 0$$

Solution (cont.)

i.e.

$$\begin{cases} y_1 = p_{12} y_2 \\ y_2 = p_{21} y_1 + p_{23} y_3 \\ y_3 = p_{32} y_2 + p_{34} y_4 \\ \vdots \end{cases}$$

We have

$$y_2 = \frac{1}{p_{12}} y_1 = \frac{5}{4} y_1 = \sum_{i=1}^{2} \frac{1}{i^2} y_1$$
$$y_3 = \frac{1 - p_{12}p_{21}}{p_{12}p_{23}} y_1 = \frac{49}{36} y_1 = \sum_{i=1}^{3} \frac{1}{i^2} y_1$$
$$y_4 = \frac{1 - p_{12}p_{21} - p_{23}p_{32}}{p_{12}p_{23}p_{34}} y_1$$

$\because \quad \forall\ k \in \mathbb{N}^+, k > 2,$

$$\begin{aligned} p_{k,k+1}(y_{k+1} - y_k) &= p_{k,k-1}(y_k - y_{k-1}) \\ y_{k+1} - y_k &= \left(\frac{k}{k+1}\right)^2 (y_k - y_{k-1}) \\ &= \left(\frac{3}{k+1}\right)^2 (y_3 - y_2) \\ &= \left(\frac{3}{k+1}\right)^2 \frac{1 - p_{12}p_{21} - p_{23}}{p_{12}p_{23}} y_1 \\ &= \frac{1}{(k+1)^2} y_1 \end{aligned}$$

$\therefore \quad \forall\ k \in \mathbb{N}^+,$

$$\begin{aligned} y_{k+1} &= \sum_{i=3}^{k} \frac{1}{(i+1)^2} y_1 + y_3 \\ &= \sum_{i=1}^{k} \frac{1}{i^2} y_1 \end{aligned}$$

$\because$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

$\therefore$

$$\lim_{k \to \infty} y_k = \frac{\pi^2}{6} y_1$$

$\because \quad$ the equations

$$y_i = \sum_{j \neq 0} P_{ij} y_j \qquad i \neq 0$$

have a non-zero solution satisfying $0 < |y_i| \leqslant 1 (\forall\ i \in \mathbb{N}^+)$ if

$$0 < |y_1| \leqslant \frac{6}{\pi^2} < 1 \qquad \forall\ i \in \mathbb{N}^+$$

$\therefore \quad$ the chain is transient

(b) $\mathbb{P}(X_n = 0 \text{ for some } n \in \mathbb{Z}^+ | X_0 = k) \to 0$ as $k \to \infty$;

Let $f_{i,j} = \mathbb{P}(X_n = j \text{ for some } n | X_0 = i)$.

$\because \quad \forall\ i \in \mathbb{N}^+$, by using first step analysis,

$$\begin{aligned} f_{i,0} &= P_{i0} + \sum_{j \neq 0} P_{ij} f_{j0} \\ &= \begin{cases} p_{i,i-1} f_{i-1,0} + p_{i,i+1} f_{i+1,0} & i > 1 \\ p_{1,0} + p_{1,2} f_{2,0} & i = 1 \end{cases} \end{aligned} \tag{1}$$

We first prove that $f_{i,0} = 1 (\forall\ i \in \mathbb{N}^+)$ is not the solution to these equations. Otherwise

$$\begin{aligned} f_{0,0} &= P_{00} + \sum_{j \neq 0} P_{0j} f_{j0} \\ &= p_{0,0} + p_{0,1} f_{1,0} \\ &= f_{1,0} \\ &= 1 \end{aligned} \tag{2}$$

which contradicts the transience of state 0. And from above analysis, we also have $f_{1,0} = f_{0,0} \neq 1$ and thus $f_{2,0} \neq 1$ since $f_{1,0} = p_{1,0} + p_{1,2} f_{2,0}$.

$\because \quad \forall\ k \in \mathbb{N}^+, k > 1$,

$$\begin{aligned} p_{k,k+1}(f_{k+1,0} - f_{k,0}) &= p_{k,k-1}(f_{k,0} - f_{k-1,0}) \\ f_{k+1,0} - f_{k,0} &= \left(\frac{k}{k+1}\right)^2 (f_{k,0} - f_{k-1,0}) \\ &= \left(\frac{2}{k+1}\right)^2 (f_{2,0} - f_{1,0}) \end{aligned}$$

$$f_{2,0} - f_{1,0} = -p_{1,0}(1 - f_{2,0}) < 0$$

$\therefore \quad \forall\ k \in \mathbb{N}^+, k > 1$,

$$f_{k+1,0} < f_{k,0}$$

$\because$

$$f_{k,0} \geqslant 0$$

$\therefore \quad \lim_{k \to \infty} f_{k,0}$ exists

Starting at state k, to return to 0, the chain must go through $(k, k-1), (k-1, k-2), \cdots, (1, 0)$. For the remaining moves, they should be pairs like $(i, i+1)$ and $(i+1, i)$. Define $f_{k0}(n) = \{X_n = 0 | X_0 = k\}$.

$\because \quad$ For every pair,

$$\begin{aligned} p_{i,i+1} p_{i+1,i} &= \frac{(i+1)^2}{i^2 + (i+1)^2} \cdot \frac{(i+1)^2}{(i+1)^2 + (i+2)^2} \\ &\leqslant \frac{1}{2} \end{aligned}$$

Solution (cont.)

$\therefore$

$$\begin{aligned}
f_{k+1,0} &\leqslant \sum_{n=0}^{\infty} f_{k0}(k+2n) \\
&\leqslant \sum_{n=0}^{\infty} \binom{k+2n}{k}\binom{2n}{n}\prod_{i=1}^{k} p_{i,i-1} \\
&\approx \sum_{n=0}^{\infty} \frac{\left(\frac{k+2n}{e}\right)^{k+2n}\sqrt{2\pi(k+2n)}\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi 2n}}{\left(\frac{k}{e}\right)^{k}\sqrt{2\pi k}\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi 2n}\left(\frac{n}{e}\right)^{2n}2\pi n}\frac{1}{2^n}\prod_{i=1}^{k} p_{i,i-1} \\
&\leqslant \sum_{n=0}^{\infty} \frac{(k+2n)^{k+2n}}{k^k}\sqrt{\frac{k+2n}{k}}\frac{2^{\frac{n}{k}}}{2\pi n} \\
&= 2\prod_{i=1}^{k} p_{i,i-1} \\
&\leqslant \frac{1}{2^{k-1}} \to 0 \qquad (k\to\infty)
\end{aligned}$$

$\therefore$

$$\lim_{k\to\infty} f_{k,0} = 0$$

(c) $\mathbb{P}(X_n > 0, \forall\, n \in \mathbb{Z}^+ | X_0 = 0) = \frac{6}{\pi^2}$.

$\because \quad \forall\, k \in \mathbb{N}^+, k > 1,$

$$\begin{aligned}
p_{k,k+1}(f_{k+1,0} - f_{k,0}) &= p_{k,k-1}(f_{k,0} - f_{k-1,0}) \\
f_{k+1,0} - f_{k,0} &= \left(\frac{k}{k+1}\right)^2 (f_{k,0} - f_{k-1,0}) \\
&= \left(\frac{2}{k+1}\right)^2 (f_{2,0} - f_{1,0})
\end{aligned}$$

$$f_{2,0} - f_{1,0} = -p_{1,0}(1 - f_{2,0}) < 0$$

$\therefore$

$$\begin{aligned}
f_{k+1,0} &= (f_{2,0} - f_{1,0})\sum_{m=1}^{k}\left(\frac{2}{m+1}\right)^2 + f_{1,0} \\
&\to (f_{2,0} - f_{1,0})\sum_{m=2}^{\infty}\frac{4}{m^2} + f_{1,0} \\
&= -4p_{1,0}(1 - f_{2,0})\left(\frac{\pi^2}{6} - 1\right) + p_{1,0} + p_{1,2}f_{2,0} \\
&= 0
\end{aligned}$$

$\therefore$

$$f_{2,0} = \frac{\frac{\pi^2}{6} - \frac{5}{4}}{\frac{\pi^2}{6}}$$

Solution (cont.)

$\therefore$ from (b) (2),

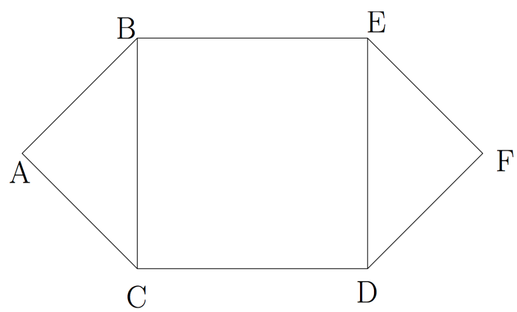
$$\begin{aligned} f_{0,0} &= f_{1,0} \\ &= p_{10} + p_{12} f_{2,0} \\ &= 1 - \frac{6}{\pi^2} \end{aligned}$$

i.e.

$$\mathbb{P}(X_n > 0, \forall\, n \in \mathbb{Z}^+ | X_0 = 0) = \frac{6}{\pi^2}$$

Question 2

Consider a particle moving between the vertices of the graph below, taking steps along the edges. Let X_n be the position of the particle at time n. At time $n+1$ the particle moves to one of the vertices adjoining X_n , with each of the adjoining vertices being equally likely, independently of previous moves. Explain briefly why $(X_n : n > 0)$ is a Markov chain on the vertices. Is this chain irreducible? Find an invariant distribution for this chain.



(1) X_n is a Markov chain on the vertices since it is determined by the transition matrix

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Solution (cont.)

(2) $\because$

$$\mathbf{P}^2 = \begin{pmatrix} + & + & + & + & + & 0 \\ + & + & + & + & 0 & + \\ + & + & + & 0 & + & + \\ + & + & 0 & + & + & + \\ + & 0 & + & + & + & + \\ 0 & + & + & + & + & + \end{pmatrix}$$

$$\mathbf{P}^3 = \begin{pmatrix} + & + & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \end{pmatrix}$$

$\therefore$ the chain is irreducible

(3) Let $\pi = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_5 \end{pmatrix}$ denotes the stationary distribution. Then

$$\sum_{i=0}^{5} \pi_i = 1$$

and

$$\pi P = \pi$$

i.e.

$$(P^T - I)^T \pi^T = 0$$

i.e.

$$A\pi^T = \begin{pmatrix} -1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & -1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -1 \end{pmatrix} \pi^T = 0$$

we have

$$Null(A) = span\left\{ \begin{pmatrix} 2 & 3 & 3 & 3 & 3 & 2 \end{pmatrix}^T \right\}$$

Therefore,

$$\begin{cases} \sum_{i=0}^{5} \pi_i = 1 \\ \pi_0 = \pi_5 \\ \pi_0 = \frac{2}{3}\pi_i \quad\quad i = 1, 2, 3, 4 \end{cases}$$

i.e.

$$\pi = \begin{pmatrix} \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} \end{pmatrix}$$

Question 3

A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13)$$

Suppose that $S = \{0, 1, 2, \cdots, 12\}$ and Y_n denotes the remainder of X_n devides 13 , then $\{Y_n\}$ is a Markov chain with transition matrix given by

$$\mathbf{P} = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}$$

It is easy to see that $\{Y_n\}$ is irreducible since $\forall\ i, j \in S$, $\exists$ a path from i to j.

$\because$ $\forall\ i \in S$, $\exists$ paths $i \to (i+1)\%13 \to (i+7)\%13 \to (i+12)\%13 \to i$ and $i \to (i+6)\%13 \to (i+12)\%13 \to i$

$\therefore$

$$d(i) \leqslant gcd(4, 3) = 1$$

$\therefore$

$$d(i) = 1$$

i.e. $\{Y_n\}$ is aperiodic

Let $\pi = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_{12} \end{pmatrix}$ denotes the stationary distribution. Then

$$\sum_{i=0}^{12} \pi_i = 1$$

and

$$\pi P = \pi$$

i.e.

$$(P^T - I)\pi^T = 0$$

We get

$$Null(\mathbf{P}^T - I) = span\left\{ \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T \right\}$$

$\therefore$

$$\pi = \begin{pmatrix} \frac{1}{13} & \frac{1}{13} & \cdots & \frac{1}{13} \end{pmatrix}$$

Solution (cont.)

$\therefore \quad \{Y_n\}$ is positive recurrent

$\therefore \quad$ the limiting distribution of $\{Y_n\}$ exists and equals to π

$\therefore$

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13) = \lim_{n\to\infty} \mathbb{P}(Y_n = 0) = \frac{1}{13}$$

Questin 4

A professor has N umbrellas, which he keeps either at home or in his office. He walks to and from his office each day, and takes an umbrella with him if and only if it is raining. Throughout each journey, it either rains, with probability p, or remains fine, with probability $1-p$, independently of the past weather. What is the long run proportion of journeys on which he gets wet?

Suppose $S = \{0, 1, \cdots, N\}$, X_n denotes the number of umbrellas at home, then $\{X_n\}$ is a Markov chian with transition probalility matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N-1 & N \end{array} \\ \begin{pmatrix} 1-p & p & 0 & \cdots & 0 & 0 \\ p(1-p) & (1-p)^2+p^2 & p(1-p) & \cdots & 0 & 0 \\ 0 & p(1-p) & (1-p)^2+p^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(1-p) & 1-p(1-p) \end{pmatrix} \end{array}$$

It is easy to prove that the chain is irreducible just like Question 1 and it is also aperiodic. Let $\pi = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_N \end{pmatrix}$ denotes the stationary distribution. Then

$$\sum_{i=0}^{N} \pi_i = 1$$

and

$$\pi P = \pi$$

i.e.

$$(P^T - I)\pi^T = 0$$

i.e.

$$\begin{pmatrix} -p & p(1-p) & 0 & \cdots & 0 & 0 \\ p & 2p(p-1) & p(1-p) & \cdots & 0 & 0 \\ 0 & p(1-p) & 2p(p-1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(1-p) & p(p-1) \end{pmatrix} \pi^T = 0$$

Solution (cont.)

$\therefore$

$$\begin{cases} \pi_1 = \cdots = \pi_N = \frac{1}{1-p}\pi_0 \\ \sum\limits_{i=0}^{N} \pi_i = 1 \end{cases}$$

i.e.

$$\pi = \begin{pmatrix} \dfrac{1-p}{N+1-p} & \dfrac{1}{N+1-p} & \cdots & \dfrac{1}{N+1-p} \end{pmatrix}$$

$\therefore$ the chain is positive recurrent and π is just the limiting distribution of the chain

$\because$ the long run proportion of journeys the professor gets wet when walking from home is $\dfrac{1-p}{N+1-p} \cdot p = \dfrac{(1-p)^2}{N+1-p}$, it's the same for $\{Y_n\}$ which Y_n is the number of umbrellas at office at time step n

$\therefore$ the long run proportion of journeys the professor gets wet is $\dfrac{(1-p)p}{N+1-p}$