Stochastic Processes

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Week 12

Solutions by

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4.1

A store that stocks a certain commodity uses the following (s, S) ordering policy; if its supply at the beginning of a time period is x, then it orders $\begin{cases} 0 & ,\text{if } x \geqslant s \\ S - x & ,\text{if } x < s. \end{cases}$ The order is immediately filled. The daily demands are independent and equal j with probability α_j. All demands that cannot be immediately met are lost. Let X_n denote the inventory level at the end of the nth time period. Argue that $\{X_n, n \geqslant 1\}$ is a Markov chain and compute its transition probabilities.

Let Y_n denote the inventory level after order is filled of the nth time period.
$\{X_n, n \geqslant 1\}$ is a Markov chain since X_{n+1} is only related to the previous inventory level and the demands at the $(n+1)$th time period, i.e., $\forall\ i_0, \cdots, i_n, j \in \mathbb{N},\ i_0, \cdots, i_n, j \leqslant S$

$$
\begin{aligned}
&\Pr\{X_{n+1} = j | X_n = i_n, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\} \\
=&\begin{cases} \Pr\{X_{n+1} = j | Y_n = i_n\} & , i_n \geqslant s \\ \Pr\{X_{n+1} = j | Y_n = S\} & , i_n < s \end{cases} \\
=&\begin{cases} \alpha_{j-i_n} & , i_n \geqslant s, j > 0 \text{ and } j - i_n \geqslant 0 \\ \alpha_{S-j} & , i_n < s, j > 0 \text{ and } S - j \geqslant 0 \\ \sum\limits_{i=i_n}^{\infty} \alpha_i & , i_n \geqslant s \text{ and } j = 0 \\ \sum\limits_{i=S}^{\infty} \alpha_i & , i_n < s \text{ and } j = 0 \\ 0 & , \text{otherwise} \end{cases} \\
=&\Pr\{X_{n+1} = j | X_n = i\}
\end{aligned}
$$

$\therefore$ the transition matrix is given by

$$
\begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S-1 \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-1 & S \end{array} \\
\begin{pmatrix}
\sum\limits_{i=S}^{\infty} \alpha_i & \alpha_{S-1} & \cdots & \alpha_{S-s} & \alpha_{S-s-1} & \cdots & \alpha_1 & \alpha_0 \\
\sum\limits_{i=S}^{\infty} \alpha_i & \alpha_{S-1} & \cdots & \alpha_{S-s} & \alpha_{S-s-1} & \cdots & \alpha_0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sum\limits_{i=s}^{\infty} \alpha_i & \alpha_{s-1} & \cdots & \alpha_0 & 0 & \cdots & 0 & 0 \\
\sum\limits_{i=s-1}^{\infty} \alpha_i & \alpha_s & \cdots & \alpha_1 & \alpha_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sum\limits_{i=S-1}^{\infty} \alpha_i & \alpha_{S-2} & \cdots & \alpha_{S-s-1} & \alpha_{S-s-2} & \cdots & \alpha_0 & 0 \\
\sum\limits_{i=S}^{\infty} \alpha_i & \alpha_{S-1} & \cdots & \alpha_{S-s} & \alpha_{S-s-1} & \cdots & \alpha_1 & \alpha_0
\end{pmatrix}
\end{array}
$$

4.2

For a Markov chain prove that

$$\mathbb{P}\{X_n = j | X_{n_1} = i_1, \cdots, X_{n_k} = i_k\} = \mathbb{P}\{X_n = j | X_{n_k} = i_k\}$$

whenever $n_1 < n_2 < \cdots < n_k < n$.

$$\begin{aligned}
&\mathbb{P}\{X_n = j | X_{n_1} = i_1, \cdots, X_{n_k} = i_k\} \\
=&\frac{\mathbb{P}\{X_n = j, X_{n_1} = i_1, \cdots, X_{n_k} = i_k\}}{\mathbb{P}\{X_{n_1} = i_1, \cdots, X_{n_k} = i_k\}} \\
=&\frac{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_n = j, X_{n-1} = j_{n-1}, \cdots, X_{n_k+1} = j_{n_k+1}, X_{n_k} = i_k, \cdots, X_0 = j_0\}}{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n_k \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_{n_k} = i_k, \cdots, X_0 = j_0\}} \\
=&\frac{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_n = j | X_{n-1} = j_{n-1}, \cdots, X_0 = j_0\} \mathbb{P}\{X_{n-1} = j_{n-1}, \cdots, X_0 = j_0\}}{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n_k \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_{n_k} = i_k | X_{n_k-1} = j_{n_k-1}, \cdots, X_0 = j_0\} \mathbb{P}\{X_{n_k-1} = j_{n_k-1}, \cdots, X_0 = j_0\}} \\
=&\frac{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_n = j | X_{n-1} = j_{n-1}\} \mathbb{P}\{X_{n-1} = j_{n-1}, \cdots, X_0 = j_0\}}{\sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n_k \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_{n_k} = i_k | X_{n_k-1} = j_{n_k-1}\} \mathbb{P}\{X_{n_k-1} = j_{n_k-1}, \cdots, X_0 = j_0\}} \\
=&\frac{\sum\limits_{j_{n-1} \in S} \mathbb{P}\{X_n = j | X_{n-1} = j_{n-1}\} \sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n-1 \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_{n-1} = j_{n-1}, \cdots, X_0 = j_0\}}{\sum\limits_{j_{n_k-1} \in S} \mathbb{P}\{X_{n_k} = i_k | X_{n_k-1} = j_{n_k-1}\} \sum\limits_{\substack{j_m \in S \\ 0 \leqslant m < n_k-1 \\ m \neq n_1, n_2, \cdots, n_k}} \mathbb{P}\{X_{n_k-1} = j_{n_k-1}, \cdots, X_0 = j_0\}} \\
=&\frac{\sum\limits_{j_{n-1} \in S} \mathbb{P}\{X_n = j | X_{n-1} = j_{n-1}\} \mathbb{P}\{X_{n-1} = j_{n-1}\}}{\sum\limits_{j_{n_k-1} \in S} \mathbb{P}\{X_{n_k} = i_k | X_{n_k-1} = j_{n_k-1}\} \mathbb{P}\{X_{n_k-1} = j_{n_k-1}\}} \\
=&\frac{\mathbb{P}\{X_n = j\}}{\mathbb{P}\{X_{n_k} = i_k\}} \\
=&\mathbb{P}\{X_n = j | X_{n_k} = i_k\}
\end{aligned}$$

4.4

Show that

$$P_{ij}^n = \sum_{k=0}^{n} f_{ij}^k P_{jj}^{n-k}$$

$$\begin{aligned}
P_{ij}^n &= \mathbb{P}\{X_n = j | X_0 = i\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{X_n = j, X_k = j, X_{k-1} \neq j, \cdots, X_1 \neq j | X_0 = i\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{X_n = j | X_k = j, X_{k-1} \neq j, \cdots, X_1 \neq j, X_0 = i\} \mathbb{P}\{X_k = j, X_{k-1} \neq j, \cdots, X_1 \neq j | X_0 = i\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{X_n = j | X_k = j\} \mathbb{P}\{X_k = j, X_{k-1} \neq j, \cdots, X_1 \neq j | X_0 = i\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{X_k = j, X_{k-1} \neq j, \cdots, X_1 \neq j | X_0 = i\} \mathbb{P}\{X_{n-k} = j | X_0 = j\} \\
&= \sum_{k=0}^{n} f_{ij}^k P_{jj}^{n-k}
\end{aligned}$$