(1a) Find an orthonormal basis e_1, e_2 for the range of the matrix

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 7 & 3 \\ 1 & 8 & 3 \end{bmatrix} = [a_1|a_2|a_3]$$

$$e_1 = a_1/\|a_1\| = \boxed{\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}$$

$$e_2 = \left(a_2 - \langle a_2, e_1\rangle e_1\right)/\| \text{ " } \|$$

$$\langle a_2, e_1 \rangle = \frac{1}{3}(-4+14+8) = 6$$

$$\langle a_2, e_1\rangle e_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$a_2 - \langle a_2, e_1\rangle e_1 = \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} - \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$e_2 = \frac{1}{9}\begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

<u>Check</u>: $e_1 \perp e_2$

<u>Note</u>: $a_3 = \frac{1}{3}(a_1 + a_2)$ so $\mathrm{rank}(A) = 2$.

(1b) Find the 3×3 matrix P which projects orthonormally onto the range of A.

$$P = e_1 e_1^* + e_2 e_2^*$$

$$9P = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$\boxed{P = \frac{1}{9} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}}$$

<u>Check</u>: $P^* = P = P^2$

(1c) Find the closest point y in the range of A to

$$b = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}.$$

$$y = Pb = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 7 \\ 11 \end{bmatrix}$$

<u>Check</u>:

$$\langle b-y, e_1 \rangle = [1\ 2\ -2] \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} /3 = 0$$

$$\langle b-y, e_2 \rangle = [1\ 2\ -2] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} /3 = 0$$

✓

(2a) Let $u(x,t)$ be the solution of the wave equation

$$u_t = u_x$$

which is 2π-periodic in x and satisfies the initial condition $u(x,0) = g(x)$ where $g \in L^2(-\pi,\pi)$. Find the complex Fourier coefficients $\hat{u}(k,t)$ in terms of $\hat{g}$.

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u_t(x,t)\, e^{-ikx}\, dx = \hat{u}_t(k,t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u_x(x,t)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[u(x,t)\, e^{-ikx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u(x,t)(-ik)\, e^{-ikx} \right]$$

$$= ik\, \hat{u}(k,t)$$

Hence $\hat{u}(k,t) = e^{ikt}\, \hat{u}(k,0)$

and $\boxed{\hat{u}(k,t) = e^{ikt}\, \hat{g}(k).}$

(2b) Show that

$$\int_{-\pi}^{\pi} |u(x,t)|^2 dx = \int_{-\pi}^{\pi} |g(x)|^2 dx$$

for all $t \geq 0$.

By Parseval's equality,

$$\int_{-\pi}^{\pi} |u(x,t)|^2 dx = \sum_{-\infty}^{\infty} |\hat{u}(k,t)|^2$$

$$= \sum_{-\infty}^{\infty} |e^{ikt} \hat{g}(k)|^2$$

$$= \sum_{-\infty}^{\infty} |e^{ikt}|^2 |\hat{g}(k)|^2$$

$$= \int_{-\pi}^{\pi} |g(x)|^2 dx.$$

(2c) Sum the Fourier series to express $u(x,t)$ directly in terms of g.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{ikt} \hat{g}(k) e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k) e^{ik(x+t)}$$

$$\boxed{u(x,t) = g(x+t).}$$

Check: $u_t(x,t) = g'(x+t)$

$u_x(x,t) = g'(x+t).$

(2d) Show that u is 2π-periodic in t:

$$u(x, t + 2\pi) = u(x, t)$$

for all $t \geq 0$.

$$u(x, t+2\pi) = g(x+(t+2\pi))$$

$$= g((x+t)+2\pi)$$

$$= g(x+t)$$

$$= u(x,t).$$

(3a) Compute the complex Fourier coefficients on the interval $-\pi < x < \pi$ of the function $f(x) = x(\pi^2 - x^2)$. (Hint: $f(x)\mathrm{e}^{-ikx} = (\mathrm{i}D)(\pi^2 + D^2)\mathrm{e}^{-ikx}$ where $D = d/dk$ is independent of x.)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} iD(\pi^2+D^2)e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} iD(\pi^2+D^2) \left.\frac{e^{-ikx}}{-ik}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} iD(\pi^2+D^2) \frac{e^{-ik\pi}-e^{ik\pi}}{-ik}$$

$$= \frac{i}{\sqrt{2\pi}} D(\pi^2+D^2) \frac{-2i\sin(k\pi)}{-ik}$$

$$= \frac{2i}{\sqrt{2\pi}} D(\pi^2+D^2) \frac{\sin(k\pi)}{k}.$$

$$D\left(\frac{\sin k\pi}{k}\right) = D(k^{-1}\sin k\pi)$$

$$= -k^{-2}\sin k\pi$$

$$+ k^{-1}\pi\cos k\pi$$

$$D^2\left(\frac{\sin k\pi}{k}\right) = \left(+2k^{-3} - k^{-1}\pi^2\right)\sin k\pi$$

$$+ \left(-\pi k^{-2} - \pi k^{-2}\right)\cos k\pi$$

$$= \left(2k^{-3} \cancel{+} - k^{-1}\pi^2\right)\sin k\pi$$

$$- 2\pi k^{-2}\cos k\pi$$

$$\left(D^2 + \pi^2\right)\frac{\sin k\pi}{k} = 2k^{-3}\sin k\pi$$

$$- 2\pi k^{-2}\cos k\pi$$

$$D\left(D^2 + \pi^2\right)\frac{\sin k\pi}{k} = \left(-6k^{-4} + 2\pi k^{-2}\right)\sin k\pi$$

$$+\left(2\pi k^{-3} + 4\pi k^{-3}\right)\cos k\pi$$

At integer k,

$$\sin k\pi = 0$$

$$\cos k\pi = (-1)^k$$

so this

$$= \frac{6\pi}{k^3}(-1)^k.$$

Hence

$$\boxed{\hat{f}(k) = \frac{2i}{\sqrt{2\pi}}\,\frac{6\pi}{k^3}(-1)^k}$$

and $\hat{f}(0) = 0$ since $f(x)$ is odd.

(3b) Show that

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.$$

From (3a)

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{2i}{\sqrt{2\pi}} 6\pi \sum_{k\neq 0} \frac{(-1)^k}{k^3} e^{ikx}$$

$$= -6 \cdot 2 \sum_{1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx)$$

At $x = \pi/2$,

$$\sin(k\pi/2) = \begin{cases} 0 & k \text{ even} \\ (-1)^{(k-1)/2} & k \text{ odd} \end{cases}$$

So

$$= -12 \sum_{\substack{1 \\ k \text{ odd}}}^{\infty} \frac{(-1)^k}{k^3} (-1)^{(k-1)/2}$$

$$= 12 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

$$= f(\pi/2) = \frac{\pi}{2}\left(\pi^2 - \frac{1}{4}\pi^2\right) = \frac{3}{8}\pi^3.$$

Hence $\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.}$

(3c) State a theorem justifying (3b) and verify its hypotheses on $f(x) = x(\pi^2 - x^2)$.

Since f is a periodic function ($f(\pi) = f(-\pi)$) with f and f' both in L^2, its Fourier series converges uniformly on $|x| \leq \pi$.

Hence we can evaluate it at $x = \pi/2$.