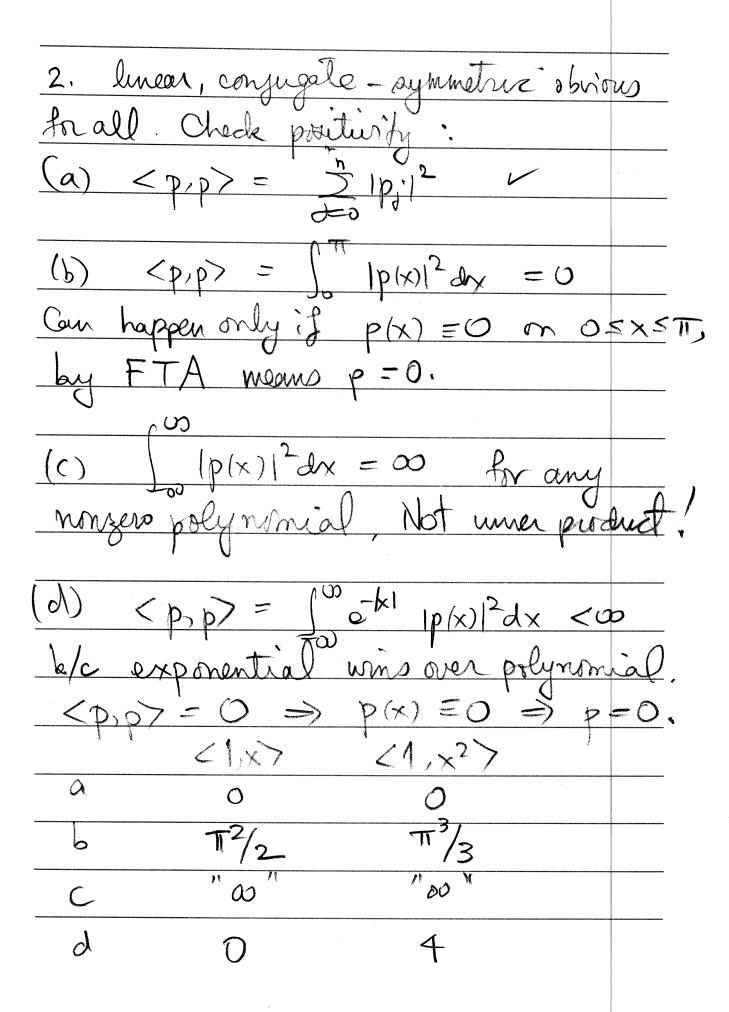
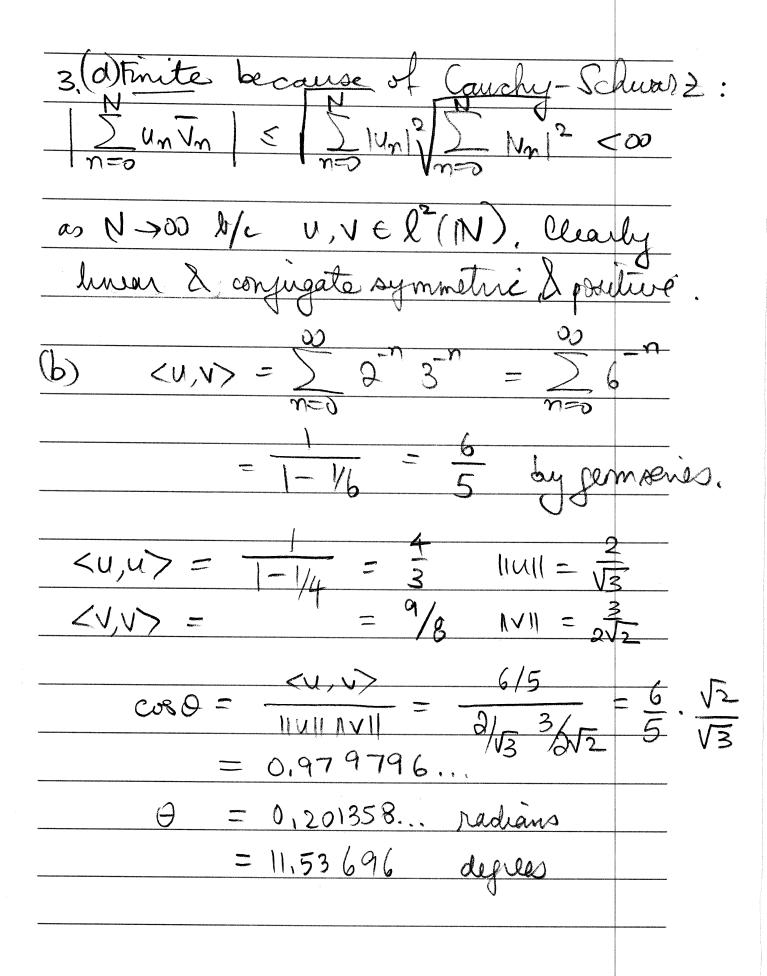
(1) limer m u / (2) conjugate-symmetrie in (4, V) <u, u> = U* 8 -1 u $= \begin{bmatrix} \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = [U, V2] -V, + 8U, = 814,12-4,42+814212 $> 7(|u_1|^2 + |u_2|^2) 70$ and = 0 iff u=0, because $|u_1u_2| \le |u_1||u_2| \le \frac{1}{2}|u_1|^2 + \frac{1}{2}|u_2|^2$





4. (a)
$$\|x \pm y\|^2 = \langle x \pm y, x \pm y \rangle$$

$$= \langle x, x \rangle \pm \langle y, x \rangle \pm \langle x, y \rangle + \langle y, y \rangle$$
80
$$\|x + y\|^2 + \|x - y\|^2 = 2 \|x\|^2 + 2 \|y\|^2.$$
(b) $\|x - y\|^2 = \langle x, y \rangle$

$$+ \langle y - y|^2 = \langle x, y \rangle$$

$$+ \langle y - y|^2 = \langle x, y \rangle$$

$$+ \langle y - y|^2 = \langle x, y \rangle$$

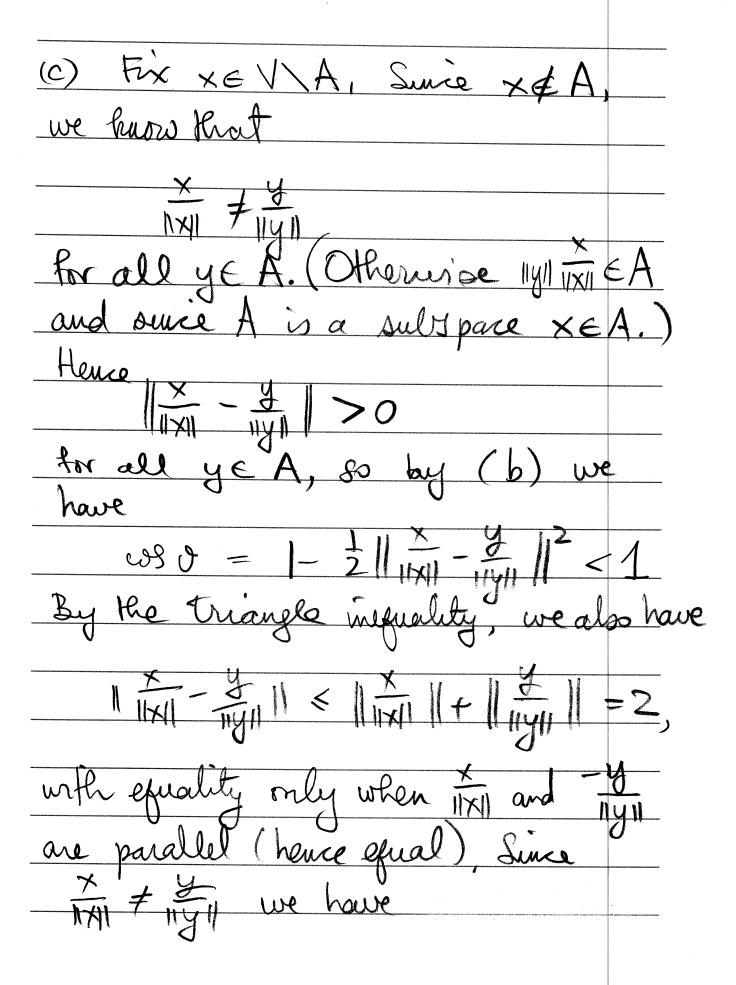
$$= 2 - 2 \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$$= 2 - 2 \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

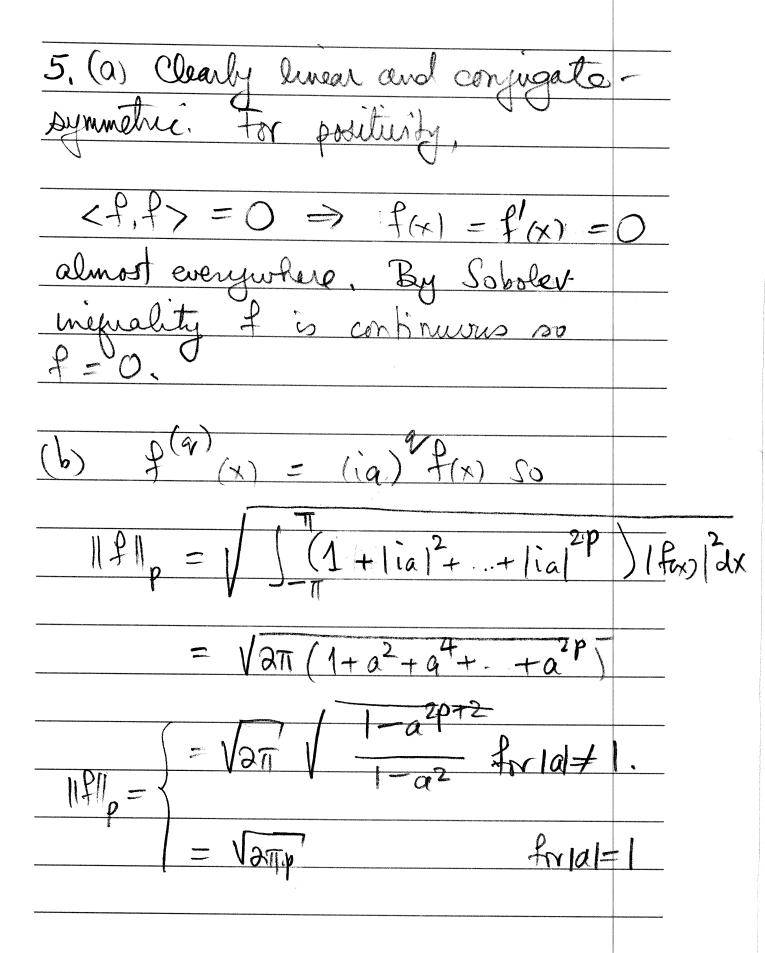
$$|x - y|^2$$

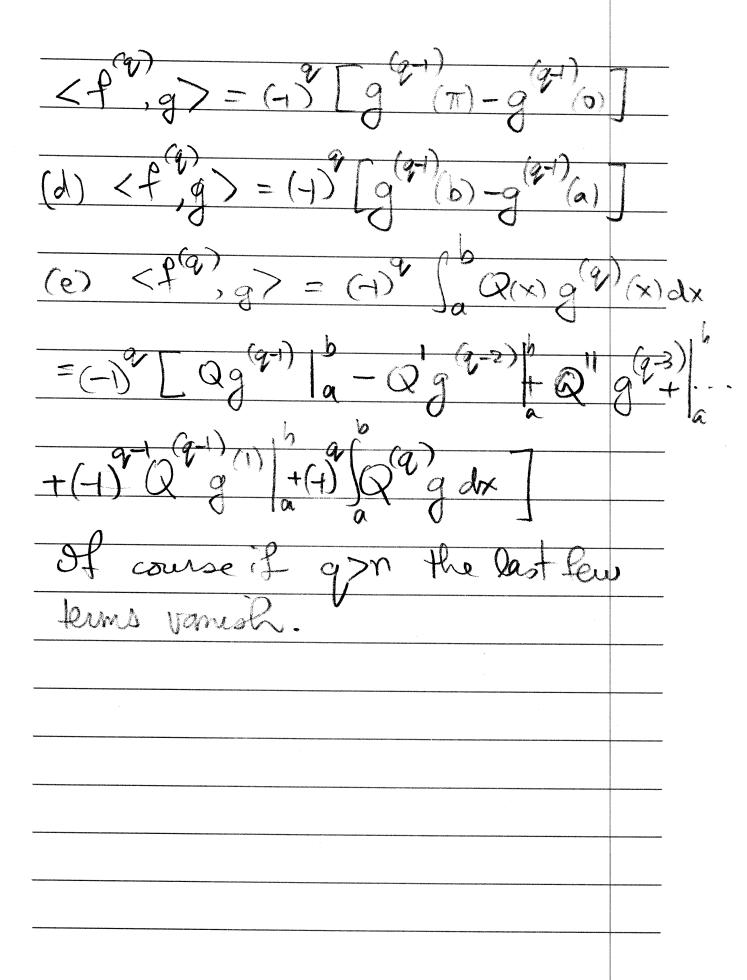
$$= |x|| |y|| (1 - \frac{1}{2} ||x| - |y||)$$
furing another formula for
$$|x - y|^2$$

$$|x - y|^2$$
in forms of the distance between $|x|$ and $|y|$.



1/11/11/2		œ
1/1/x11 - Tiy111 < 2 for all yeA. Putting it toge	ther	.)
		-
$-1 < \cos\theta < 1$		
so there is a constant $8 < 1$ and that		
such that		
100501 < 8 < 1		
for all yEA, Since	The second secon	
	-	
<x14> = cos8 11x11 11y11,</x14>		
we have a strengthened Cauchy	Sch	want
inequality	***************************************	
1 <x,y>1 < 8 x y .</x,y>		
	Management of the state of the	
	-	





Question 1 Prove or disprove:

$$< u, v> = u^* \left[egin{array}{cc} 8 & -1 \\ -1 & 8 \end{array}
ight] v$$

is an inner product on \mathbb{C}^2 . Check the properties.

Question 2 Which of the following define an inner product on degree-n polynomials

$$p(x) = p_0 + p_1 x + \dots + p_n x^n$$
?

(a)
$$\langle p, q \rangle = \sum_{j=0}^{n} p_j \bar{q}_j$$

(b)
$$\langle p, q \rangle = \int_0^{\pi} p(x)\bar{q}(x) dx$$

(c)
$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(x)\bar{q}(x) dx$$

(d)
$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(x)\bar{q}(x)e^{-|x|} dx$$

Justify your answers with proof or counterexample. Evaluate <1,x> and $<1,x^2>$ for each case.

Question 3 (a) Prove or disprove:

$$\langle u, v \rangle = \sum_{n=0}^{\infty} u_n \bar{v}_n$$

is an inner product on $l^2(N)$. Check the properties.

(b) For $u_n = 2^{-n}$ and $v_n = 3^{-n}$ compute $\langle u, v \rangle$ and the angle between u and v.

Question 4 (a) Prove the parallelogram identity

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

for any x and y vectors in a *real* inner product space with norm $\| \|$.

(b) Prove

$$|\langle x,y \rangle| = ||x|| ||y|| \left(1 - \frac{1}{2} \left\| \frac{x}{||x||} - \frac{y}{||y||} \right\|^2 \right)$$

for nonzero vectors x and y.

(c) Given a subspace A of a real inner-product space V, and a vector $x \in V$ which is not in A, show that there is a constant $\gamma < 1$ such that

$$|\langle a, x \rangle| \le \gamma ||a|| ||x||$$

for all $a \in A$.

Question 5 For p = 1, 2, ... define the Sobolev space $H^p = H^p(-\pi, \pi)$ by

$$H^p = \{g \in L^2 = L^2(-\pi, \pi) | g \text{ is } 2\pi\text{-periodic and } g', g'', \dots, g^{(p)} \in L^2\},$$

with

$$\langle f, g \rangle_p = \int_{-\pi}^{\pi} f(x)\bar{g}(x) + f'(x)\bar{g}'(x) + \dots + f^{(p)}(x)\bar{g}^{(p)}(x) dx.$$

For p = 0 we set $H^0 = L^2$ with the usual L^2 inner product <,>.

- (a) Show that $\langle f, g \rangle_p$ defines an inner product on H^p .
- (b) Compute the norm $||f||_p = \sqrt{\langle f, f \rangle_p}$ in H^p of $f(x) = e^{iax}$ and the angle in H^p between f and $g(x) = e^{ibx}$ for $a, b \in \mathbb{Z}$.
 - (c) For $f \in L^2$ define a generalized derivative $f^{(p)}$ by the requirement

$$< f^{(p)}, g> = (-1)^p < f, g^{(p)} >$$

for all $g \in H^p$. Let $f \in L^2$ be given by f(x) = 1 for x > 0 and f(x) = 0 for x < 0. For $1 \le q \le p$, compute the generalized derivatives

$$\langle f^{(q)}, g \rangle$$

for all $g \in H^p$.

(d) Fix $-\pi < a < b < \pi$ and let $f \in L^2$ be given by f(x) = 1 for a < x < b and f(x) = 0 otherwise. For $1 \le q \le p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g \in H^p$.

(e) Compute the generalized derivatives of f(x) = Q(x) for a < x < b and f(x) = 0 otherwise, where Q is a degree-n polynomial.