STAT 30400 : Distribution Theory

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Homework 2

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Solutions by

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STAT 30400, Homework 2

1. (15 pts) Let X_1 and X_2 be independent standard normal random variables, and let $Y = X_1^2 + X_2^2$, $Z = \frac{X_1}{X_2}$ and $W = \frac{X_1}{\sqrt{X_1^2+X_1^2}}$.

   (a) Find the joint density of Y and Z, and find the marginal densities of Y and Z.

The inverse transform from (X_1, X_2) to (Y, Z) is given by

$$\begin{cases} X_1 = Z\sqrt{\frac{Y}{1+Z^2}} \\ X_2 = \sqrt{\frac{Y}{1+Z^2}} \end{cases}$$

when $X_2 \geq 0$, and

$$\begin{cases} X_1 = -Z\sqrt{\frac{Y}{1+Z^2}} \\ X_2 = -\sqrt{\frac{Y}{1+Z^2}} \end{cases}$$

when $X_2 < 0$.
The determinant of Jocabian of this inverse transform is

$$\begin{aligned} J &= \left| \begin{pmatrix} \pm\frac{Z}{2}\sqrt{\frac{1}{Y(1+Z^2)}} & \pm\frac{\sqrt{Y}}{(1+Z^2)^{\frac{3}{2}}} \\ \pm\frac{1}{2}\sqrt{\frac{1}{Y(1+Z^2)}} & \mp\frac{\sqrt{Y}Z}{(1+Z^2)^{\frac{3}{2}}} \end{pmatrix} \right| \\ &= -\frac{1}{2(1+Z^2)} \end{aligned}$$

The range of (Y, Z) when $X_2 \geq 0$ is the same as the one when $X_2 < 0$. Therefore, the joint density of Y and Z is given by

$$\begin{aligned} f_{(Y,Z)}(y,z) &= 2\phi\left(z\sqrt{\frac{y}{1+z^2}}\right)\phi\left(\sqrt{\frac{y}{1+z^2}}\right)\left|-\frac{1}{2(1+z^2)}\right|\mathbb{1}_{\{y\geq 0\}} \\ &= \frac{1}{2\pi(1+z^2)}e^{-\frac{1}{2}y}\mathbb{1}_{\{y\geq 0\}} \end{aligned}$$

So the marginal densities are

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{(Y,Z)}(y,z)\mathrm{d}z \\ &= \frac{1}{2}e^{-\frac{1}{2}y}\mathbb{1}_{\{y\geq 0\}} \\ f_Z(z) &= \int_{-\infty}^{\infty} f_{(Y,Z)}(y,z)\mathrm{d}y \\ &= \frac{1}{\pi(1+z^2)} \end{aligned}$$

(b) Use QQ plots on simulated data to demonstrate that the marginal densities you derived are correct. Show the plots and the work you have done to construct them. (Hint: you need to simulate draws for Y in two ways: using X_1 and X_2, and using the derived distribution of Y)

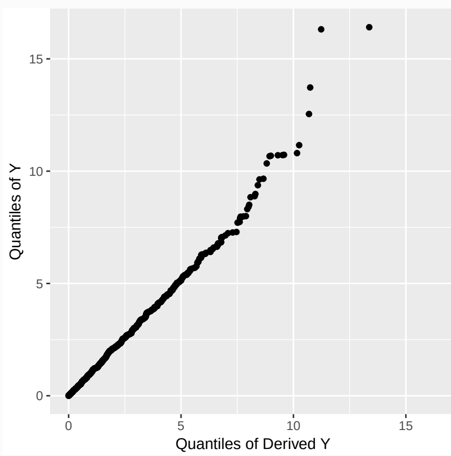
For the derived Y, we have

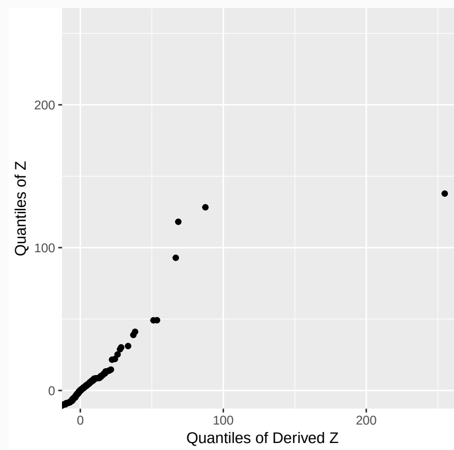
$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{1}{2} e^{-\frac{1}{2}x} \mathbb{1}_{\{x \geq 0\}} \mathrm{d}x \\ &= (1 - e^{-\frac{1}{2}y}) \mathbb{1}_{\{y \geq 0\}} \\ F_Y^{-1}(u) &= -2 \ln(1-u), \ u \in [0,1) \end{aligned}$$

then for $U \sim Uniform(0,1)$, $F_Y^{-1}(U) \sim Y$.
For the derived Z, we have

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} \frac{1}{\pi(1+x^2)} \mathrm{d}x \\ &= \frac{1}{\pi} \arctan z + \frac{1}{2} \\ F_Z^{-1}(u) &= \tan\left[\left(u - \frac{1}{2}\right)\pi\right], \ u \in (0,1) \end{aligned}$$

then for $U \sim Uniform(0,1)$, $F_Z^{-1}(U) \sim Z$.





From the QQ plots, the derived marginal distributions are correct.
Code:

```r
set.seed(1)
n <- 1000
X_1 <- rnorm(n, 0, 1)
X_2 <- rnorm(n, 0, 1)
Y <- X_1^2 + X_2^2
Z <- X_1 / X_2

U <- runif(n, 0, 1)
Y_derived <- -2 * log(1-U)
Z_derived <- tan((U-1/2)*pi)
```

Solution (cont.)

```
library(ggplot2)
df <- as.data.frame(qqplot(Y_derived, Y, plot.it=FALSE));

ggplot(df) + geom_point(aes(x=x, y=y)) +
    coord_fixed(ratio = 1,
                xlim=c(0,max(Y,Y_derived)), ylim=c(0,max(Y,Y_derived))) +
    xlab('Quantiles of Derived Y') +
    ylab('Quantiles of Y')
```

(c) Find the distribution of $\arcsin(W)$.

Let $X = X_1$, then the inverse transform from (X_1, X_2) to (W, X) is given by

$$\begin{cases} X_1 = X \\ X_2 = \sqrt{\frac{X^2}{W^2} - X^2} \end{cases}$$

The determinant of Jocabian is

$$J = \left| \begin{pmatrix} 0 & 1 \\ -\frac{X^2}{W^3\sqrt{\frac{X^2}{W^2}-X^2}} & \frac{X}{W^2\sqrt{\frac{X^2}{W^2}-X^2}} \end{pmatrix} \right| = \frac{X^2}{W^3\sqrt{\frac{X^2}{W^2} - X^2}}$$

Therefore, the joint density of W and X is given by

$$f_{(W,X)}(w,x) = \phi(x)\phi\left(\sqrt{\frac{x^2}{w^2} - w^2}\right) \left| \frac{x^2}{w^3\sqrt{\frac{x^2}{w^2} - x^2}} \right| \mathbb{1}_{\{w\in(-1,0)\cup(0,1)\}}.$$

So

$$\begin{aligned} f_W(w) &= \int_{\mathbb{R}} \phi(x)\phi\left(\sqrt{\frac{x^2}{w^2} - w^2}\right) \frac{|x|}{w^2\sqrt{1-w^2}} \mathbb{1}_{\{w\in(-1,0)\cup(0,1)\}} \mathrm{d}x \\ &= 2\int_0^{+\infty} \frac{1}{2\sqrt{1-w^2}} \mathbb{1}_{\{w\in(-1,0)\cup(0,1)\}} \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{x}{w}\right)^2} \mathrm{d}\left(\frac{x}{w}\right)^2 \\ &= -\frac{1}{\pi\sqrt{1-w^2}} \mathbb{1}_{\{w\in(-1,0)\cup(0,1)\}} e^{-\frac{1}{x}} \Big|_0^{+\infty} \\ &= \frac{1}{\pi\sqrt{1-w^2}} \mathbb{1}_{\{w\in(-1,0)\cup(0,1)\}}. \end{aligned}$$

Let $V = \arcsin W$, we have

$$\begin{aligned} f_V(v) &= f_W(\sin v) \cdot |\cos(v)| \cdot \mathbb{1}_{\{v\in(-\frac{\pi}{2},\frac{\pi}{2})\}} \\ &= \frac{|\cos v|}{\pi\sqrt{1-\sin^2 v}} \mathbb{1}_{\{v\in(-\frac{\pi}{2},\frac{\pi}{2})\}} \\ &= \frac{1}{\pi} \mathbb{1}_{\{v\in(-\frac{\pi}{2},\frac{\pi}{2})\}}, \end{aligned}$$

i.e., $\arcsin W \sim Uniform(-\frac{\pi}{2}, \frac{\pi}{2})$

2. (10 pts) This exercise is related to the notion of Bayesian credible intervals we discussed in class. Let F be a distribution function with continuous, unimodal density f such that $f(x) > 0$, $\forall x \in \mathbb{R}$. Let m be the unique mode, and let $0 < \alpha < 2\min(F(m), 1 - F(m))$. Show that if (a, b) is the shortest interval such that $F(b) - F(a) = \alpha$, then $f(a) = f(b)$.

First we need to prove that there exist finite interval (a, b) such that $F(b) - F(a) = \alpha$. Since $0 < \alpha < 2\min(F(m), 1 - F(m))$,

$$0 < F(m) - \frac{\alpha}{2} < F(m) + \frac{\alpha}{2} < 1.$$

Since $f(x) > 0$ ($\forall\, x \in \mathbb{R}$), $F(x)$ is strictly increasing and thus F^{-1} exists, and $F^{-1}(0) = -\infty$, $F^{-1}(1) = \infty$. So interval $(F^{-1}(F(m) - \frac{\alpha}{2}), F^{-1}(F(m) + \frac{\alpha}{2}))$ is a finite interval with $F(F^{-1}(F(m) + \frac{\alpha}{2}))) - F(F^{-1}(F(m) - \frac{\alpha}{2})) = \alpha$.

Suppose that (a, b) is an interval such that $F(b) - F(a) = \alpha$.

1. If $f(a) < f(b)$, since f is continuous, then for $\epsilon = \frac{f(b)-f(a)}{2}$, $\exists\, \delta > 0$, s.t. $\forall\, x \in (0, \delta)$, $|f(a) - f(a+x)| < \epsilon$ and $|f(b) - f(b + x)| < \epsilon$.
Then

$$f(a + x) < f(a) + \epsilon = f(b) - \epsilon < f(b + x)$$

for $x \in (0, \delta)$. So

$$\begin{aligned}
&[F(b + \delta) - F(a + \delta)] - [F(b) - F(a)]\\
=&[F(b + \delta) - F(b)] - [F(a + \delta) - F(a)]\\
=&\int_b^{b+\delta} f(x)\mathrm{d}x - \int_a^{a+\delta} f(x)\mathrm{d}x\\
=&\int_a^{a+\delta} [f(x + (b - a)) - f(x)]\mathrm{d}x\\
>&0,
\end{aligned}$$

which means that the area under $F(x)$ in the interval $(a + \delta, b + \delta)$ is larger than α, i.e., there exists a subinterval of it such that the area under $F(x)$ equals to α, i.e. (a, b) is not the shortest interval.

2. If $f(a) > f(b)$, since f is continuous, then for $\epsilon = \frac{f(a)-f(b)}{2}$, $\exists\, \delta > 0$, s.t. $\forall\, x \in (0, \delta)$, $|f(a) - f(a-x)| < \epsilon$ and $|f(b) - f(b - x)| < \epsilon$.
Then

$$f(a - x) > f(a) - \epsilon = f(b) + \epsilon > f(b - x)$$

for $x \in (0, \delta)$. So

$$\begin{aligned}
&[F(b - \delta) - F(a - \delta)] - [F(b) - F(a)]\\
=&[F(b - \delta) - F(b)] - [F(a - \delta) - F(a)]\\
=&\int_b^{b-\delta} f(x)\mathrm{d}x - \int_a^{a-\delta} f(x)\mathrm{d}x\\
=&-\int_{a-\delta}^{a} [f(x + (b - a)) - f(x)]\mathrm{d}x\\
>&0,
\end{aligned}$$

which means that the area under $F(x)$ in the interval $(a - \delta, b - \delta)$ is larger than α, i.e., there exists a subinterval of it such that the area under $F(x)$ equals to α, i.e. (a, b) is not the shortest interval.
Therefore, if (a, b) is the shortest interval such that $F(b) - F(a) = \alpha$, then $f(a) = f(b)$.

3. (10 pts) Let X be a standard Cauchy random variable.

(a) Find a representing function for X. What are the first and third quartiles of X? (i.e. 0.25 and 0.75 quantiles). Show your derivations.

The density and the cumulative distribution function of the standard Cauchy distribution is given by

$$\begin{aligned} f(x) &= \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbb{R} \\ F(x) &= \int_{-\infty}^{x} f(y)\mathrm{d}y \\ &= \frac{1}{\pi} \arctan y \Big|_{-\infty}^{x} \\ &= \frac{1}{\pi} \arctan x + \frac{1}{2}, \qquad x \in \mathbb{R} \end{aligned}$$

Let $y = F(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}$, then the representing function is given by

$$F^{-1}(y) = x = \tan\left[\left(y - \frac{1}{2}\right)\pi\right], \qquad y \in (0,1).$$

Thus,

$$\begin{aligned} F^{-1}(\frac{1}{4}) &= \tan\left(-\frac{\pi}{4}\right) = -1 \\ F^{-1}(\frac{3}{4}) &= \tan\left(\frac{\pi}{4}\right) = 1 \end{aligned}$$

(b) Show that $\mathbb{P}(X \geq x) \approx \frac{1}{\pi x}$ as $x \to \infty$.

Proof.

$$\begin{aligned} \mathbb{P}(X \geq x) &= \int_{x}^{+\infty} f(y)\mathrm{d}y \\ &= 1 - F(x) \\ &= \frac{1}{2} - \frac{1}{\pi}\arctan x \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan x\right) \end{aligned}$$

Since

$$\begin{aligned} \lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{x^2}{1+x^2} \\ &= 1, \end{aligned}$$

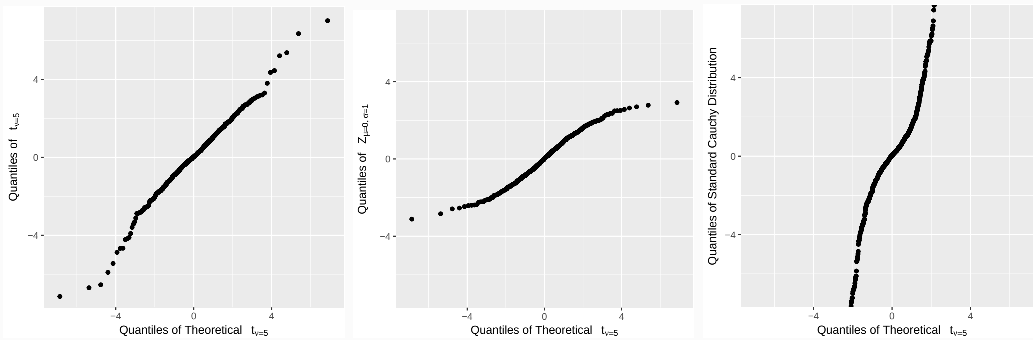
we have

$$\mathbb{P}(X \geq x) = \frac{1}{\pi x} + o(1)$$

where $o(1)$ denotes a quantity that tends to 0 as $x \to \infty$. So $\mathbb{P}(X \geq x) \approx \frac{1}{\pi x}$ as $x \to \infty$. □

(c) Use R or another statistical package to make a QQ plot of a supposedly Student t random variable with 5 degrees of freedom against the theoretical t distribution. We want to do something similar to the *qqnorm* function in R. Show this QQ plot for data coming from the t distribution with 5 degrees of freedom (in R you can use *rt* to simulate from the t distribution), also from the normal (in R you can use *rnorm* to simulate from the normal distribution), and from the Cauchy distribution(*rcauchy*). Comment on the three QQ plots.

The QQ plots are given by



From the above QQ plots, we can know that - (1) points in the first QQ plot is almost a line since the data are sampled from exactly t_5 distribution; (2) the t_5 distribution has heavy tails versus the standard normal distribution; (3) the standard Cauchy distribution has heavy tails versus the t_5 distribution; (4) the standard Cauchy distribution is symmetric as the third QQ plot is symmetric and the t distributions are symmetric.

Actually, Student's t-distribution becomes the standar Cauchy distribution when the degrees of freedom is equal to one and converges to the normal distribution as the degrees of freedom go to infinity.

Code:

```
set.seed(0)
n <- 1000
t_5 <- rt(n, 5)
Z <- rnorm(n)
Cauchy <- rcauchy(n)

library(ggplot2)
df <- as.data.frame(qqplot(qt(ppoints(n), df = 5), t_5, plot.it=FALSE))
ggplot(df) + geom_point(aes(x=x, y=y)) +
    coord_fixed(ratio = 1,
                xlim=c(-7,7), ylim=c(-7,7)) +
    xlab(expression('Quantiles of Theoretical ' ~~ {t}[nu == 5])) +
    ylab(expression('Quantiles of ' ~~ {t}[nu == 5]))
```

Solution (cont.)

```r
df <- as.data.frame(qqplot(qt(ppoints(n), df = 5), Z, plot.it=FALSE))
ggplot(df) + geom_point(aes(x=x, y=y)) +
    coord_fixed(ratio = 1,
                xlim=c(-7,7), ylim=c(-7,7)) +
    xlab(expression('Quantiles of Theoretical ' ~~ {t}[nu == 5])) +
    ylab(expression('Quantiles of ' ~~ {Z}[list(mu == 0, sigma == 1)]))

df <- as.data.frame(qqplot(qt(ppoints(n), df = 5), Cauchy, plot.it=FALSE))
ggplot(df) + geom_point(aes(x=x, y=y)) +
    coord_fixed(ratio = 1,
                xlim=c(-7,7), ylim=c(-7,7)) +
    xlab(expression('Quantiles of Theoretical ' ~~ {t}[nu == 5])) +
    ylab('Quantiles of Standard Cauchy Distribution')
```

4. (15 pts) Let $X_1, \ldots, X_n$ be independent random variables $\sim Exp(\lambda)$.

(a) Find the density function of $R = X_{(n)} - X_{(1)}$.

The density function, the cumulative distribution function and the representing function of X_i $(i = 1, \ldots, n)$ is given by

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x} \mathbb{1}_{\{x \geq 0\}} \\ F(x) &= \int_{-\infty}^{x} f(y) \mathrm{d}y \\ &= -e^{-\lambda y}\big|_0^x \cdot \mathbb{1}_{\{x \geq 0\}} \\ &= [1 - e^{-\lambda x}] \mathbb{1}_{\{x \geq 0\}} \\ F^{-1}(y) &= -\frac{1}{\lambda} \ln(1-y), \qquad y \in (0,1). \end{aligned}$$

According to Theorem 7.9, $(X_{(1)}, \ldots, X_{(n)})$ has an absolutely continuous distribution with density function

$$n! f(x_1) \cdots f(x_n), \qquad 0 < x_1 < \ldots < x_n < +\infty$$

So the joint distribution of $(X_{(1)}, X_{(n)})$ is given by

$$\begin{aligned} f_{(X_{(1)}, X_{(n)})}(x_1, x_n) &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} n! f(x_1) \cdots f(x_n) \mathbb{1}_{\{x_1 < x_2 < \ldots < x_{n-1} < x_n\}} \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \\ &= \frac{n!}{(n-2)!} [F(x_n) - F(x_1)]^{n-2} f(x_1) f(x_n) \mathbb{1}_{\{0 < x_1 < x_n\}} \\ &= n(n-1)\lambda^2 (e^{-\lambda x_1} - e^{-\lambda x_n})^{n-2} e^{-\lambda(x_1 + x_n)} \mathbb{1}_{\{0 \leq x_1 < x_n\}} \end{aligned}$$

Let $R = X_{(n)} - X_{(1)}$ and $Y = X_{(1)}$, then the inverse transform from $(X_{(1)}, X_{(n)})$ to (R, Y) is given by

$$\begin{cases} X_{(1)} = Y \\ X_{(n)} = R + Y \end{cases}$$

The determinant of the Jocabian of this inverse transform is given by

$$J = \left| \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right| = -1$$

Then the joint density of (R, Y) is given by

$$\begin{aligned} f_{(R,Y)}(r, y) &= f_{(X_{(1)}, X_{(n)})}(y, r + y) |J| \\ &= n(n-1)\lambda^2 (e^{-\lambda y} - e^{-\lambda(r+y)})^{n-2} e^{-\lambda(r+2y)} \mathbb{1}_{\{r \geq 0, y \geq 0\}} \\ &= n(n-1)\lambda^2 (1 - e^{-\lambda r})^{n-2} e^{-\lambda r} e^{-\lambda n y} \mathbb{1}_{\{r \geq 0, y \geq 0\}} \end{aligned}$$

Integrateing y, we have

$$\begin{aligned} f_R(r) &= \int_{\mathbb{R}} f_{(R,Y)}(r, y) \mathrm{d}y \\ &= (n-1)\lambda (1 - e^{-\lambda r})^{n-2} e^{-\lambda r} \mathbb{1}_{\{r \geq 0\}} \int_0^\infty \lambda n e^{-\lambda n y} \mathrm{d}y \\ &= (n-1)\lambda (1 - e^{-\lambda r})^{n-2} e^{-\lambda r} \mathbb{1}_{\{r \geq 0\}} (-e^{-\lambda n y})\big|_0^\infty \\ &= (n-1)\lambda (1 - e^{-\lambda r})^{n-2} e^{-\lambda r} \mathbb{1}_{\{r \geq 0\}} \end{aligned}$$

(b) Prove that $\min(X_1, X_2)$ and $X_1 - X_2$ are independent random variables.

Proof. Let $n = 2$, $Y = X_{(1)} = \min(X_1, X_2)$ and $Z = X_1 - X_2$.
If $X_1 \le X_2$, then $Y = X_1$, the determinant of Jocabian of transform from (X_1, X_2) to (Y, Z) is

$$J_1 = \left| \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right| = -1,$$

and $\mathcal{Y}_0 = \{y \ge 0, y - z \ge 0, y \le y - z\} = \{z \le 0 \le y\}$.
If $X_1 > X_2$, then $Y = X_2$, the determinant of Jocabian of transform from (X_1, X_2) to (Y, Z) is

$$J_2 = \left| \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right| = -1,$$

and $\mathcal{Y}_1 = \{y + z \ge 0, y \ge 0, y + z > y\} = \{z > 0, y \ge 0\}$.
Then

$$\begin{aligned} f_{(Y,Z)}(y, z) &= f_{X_1}(y) f_{X_2}(y - z) \frac{1}{|J_1|} \mathbb{1}_{\{z \le 0 \le y\}} + f_{X_1}(z + y) f_{X_2}(y) \frac{1}{|J_2|} \mathbb{1}_{\{y \ge 0, z > 0\}} \\ &= \lambda^2 e^{-\lambda(2y - z)} \mathbb{1}_{\{z \le 0\}} \mathbb{1}_{\{y \ge 0\}} + \lambda^2 e^{-\lambda(2y + z)} \mathbb{1}_{\{z > 0\}} \mathbb{1}_{\{y \ge 0\}} \\ &= (2\lambda e^{-2\lambda y} \mathbb{1}_{\{y \ge 0\}}) \cdot \left[\frac{\lambda}{2} (e^{\lambda z} \mathbb{1}_{\{z \le 0\}} + e^{-\lambda z} \mathbb{1}_{\{z > 0\}}) \right] \end{aligned}$$

Since $\int_{\mathbb{R}} 2\lambda e^{-2\lambda y} \mathbb{1}_{\{y \ge 0\}} \mathrm{d}y = 1$, we have that $f_Y(y) = 2\lambda e^{-2\lambda y} \mathbb{1}_{\{y \ge 0\}}$, $f_Z(z) = \frac{\lambda}{2}(e^{\lambda z} \mathbb{1}_{\{z \le 0\}} + e^{-\lambda z} \mathbb{1}_{\{z > 0\}})$ and $f_{(Y,Z)}(y, z) = f_Y(y) f_Z(z)$. Therefore, $\min(X_1, X_2)$ and $X_1 - X_2$ are independent random variables.

$\square$