Stochastic Processes

Fall 2017

Week 1

Solutions by

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2.1

Show that Definition 2.1.1 of Poisson process implies Definition 2.1.2.

$\because$ $\{N(t), t \geqslant 0\}$ is a Poisson process by Definition 2.1.1

$\therefore$

(1) $N(0) = 0$;

(2) The process has independent increments

(3) $\forall\ t > 0, s \geqslant 0, n \in \mathbb{N}$, $\mathbb{P}\{N(t+s) - N(s) = n\} = \dfrac{(\lambda t)^n}{n!} e^{-\lambda t}$

$\therefore$

(1*) $N(0) = 0$;

(2*) From (2) the process has independent increments

From (3) we have $\forall\ t_2 > t_1 \geqslant 0$, the length of any interval $(t_1 + s, t_2 + s]$ is $\Delta t = t_2 - t_1$,

$$\mathbb{P}\{N(t_2 + s) - N(t_1 + s) = n\} = \frac{(\lambda \Delta t)^n}{n!} e^{-\lambda \Delta t}$$

i.e. the process has stationary increments

(3*) $\forall\ h > 0$,

$$\begin{aligned}
\mathbb{P}\{N(h) = 1\} &= \mathbb{P}\{N(h) - N(0) = 1\} \\
&= \lambda h e^{-h\lambda} \\
&= \lambda h \left(\sum_{k=0}^{\infty} \frac{(-\lambda h)^k}{k!} \right) \\
&= \lambda h \left(1 - \lambda h + \lambda^2 h^2 - \cdots \right) \\
&= \lambda h + o(h)
\end{aligned}$$

(4*) $\forall\ h > 0$,

$$\begin{aligned}
\mathbb{P}\{N(h) \geqslant 2\} &= 1 - \mathbb{P}\{N(h) = 1\} - \mathbb{P}\{N(h) = 0\} \\
&= 1 - \mathbb{P}\{N(h) - N(0) = 1\} - \mathbb{P}\{N(h) - N(0) = 0\} \\
&= 1 - [\lambda h + o(h)] - \frac{(\lambda h)^0}{0!} e^{-\lambda h} \\
&= o(h)
\end{aligned}$$

which satisfy Definition 2.1.2, i.e., Definition 2.1.1 of Poisson process implies Definition 2.1.2.

2.2

For another approach to proving that Definition 2.1.2 implies Definition 2.1.1.

(a) Prove, using Definition 2.1.2, that

$$P_0(t+s) = P_0(t)P_0(s)$$

$\because$ $\forall\ h > 0$,

$$\begin{aligned} P_0(t+s+h) &= \mathbb{P}\{N(t+s+h) = 0\} \\ &= \mathbb{P}\{N(t+s+h) - N(t+h) = 0, N(t+h) - N(t) = 0, N(t) - N(0) = 0\} \\ &= \mathbb{P}\{N(t+s+h) - N(t+h) = 0\}\mathbb{P}\{N(t+h) - N(t) = 0\} \\ &= \mathbb{P}\{N(t) - N(0) = 0\} \\ &= P_0(s)[1 - \lambda h + o(h)]P_0(t) \end{aligned}$$

$\therefore$ let $h \to 0+$, we have

$$P_0(t+s) = P_0(t)P_0(s)$$

(b) Use (a) to infer that the interarrival times X_1, $X_2 \cdots$ are independent exponential random variables with rate λ.

$\because$ $\forall\ m, n \in \mathbb{N}^+,\ m > n,\ t > 0$,

$$\{X_m > t\} = \left\{ N\left(2t + \sum_{\substack{i=1 \\ i \neq n}}^{m-1} X_i\right) - N\left(t + \sum_{\substack{i=1 \\ i \neq n}}^{m-1} X_i\right) = 0 \right\}$$

and

$$\{X_n > t\} = \left\{ N\left(t + \sum_{i=1}^{n-1} X_i\right) - N\left(\sum_{i=1}^{n-1} X_i\right) = 0 \right\}$$

$\therefore$ by stationary and independent increments, X_m and X_n are independent identically distributed

$\therefore$ $X_1, X_2, \cdots$ are independent identically distributed

$\because$ $\forall\ n \in \mathbb{N}^+,\ t > 0$

$$\begin{aligned} \mathbb{P}\{X_n > t\} &= \mathbb{P}\left\{ N\left(t + \sum_{k=1}^{n-1} X_k\right) - N\left(\sum_{k=1}^{n-1} X_k\right) = 0 \right\} \\ &= \mathbb{P}\{N(t) - N(0) = 0\} \\ &= P_0(t) \end{aligned}$$

$\therefore$ $\forall\ h > 0$,

$$\begin{aligned} \mathbb{P}\{X_n > t+h\} - \mathbb{P}\{X_n > t\} &= P_0(t+h) - P_0(t) \\ &= P_0(t)P_0(h) - P_0(t) \\ &= P_0(t)[-\lambda h + o(h)] \end{aligned}$$

$\therefore$

$$\begin{aligned} P_0'(t) &= \lim_{h \to 0+} \frac{P_0(t+h) - P_0(t)}{h} \\ &= P_0(t)\lambda \end{aligned}$$

Solution (cont.)

$\because$

$$\begin{cases} P_0'(t) = P_0(t)\lambda \\ P_0(0) = 1 \end{cases}$$

$\therefore$

$$P_0(t) = e^{-\lambda t}$$

i.e.

$$\mathbb{P}\{X_1 > t\} = e^{-\lambda t}$$

$\therefore \quad \forall\, n \in \mathbb{N}^+$, X_n has exponential distribution with rate λ

Therefore, the interarrival times X_1, $X_2 \cdots$ are independent exponential random variables with rate λ.

(c) Use (b) to show that $N(t)$ is Poisson distributed with mean λt.

Suppose that for $n \in \mathbb{N}$,

$$P_n(t) = \mathbb{P}\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{$*$}$$

Equation $(*)$ holds when $n = 1$. Then for $n + 1$,

$$\begin{aligned}
\mathbb{P}\{N(t) = n + 1\} &= \int_0^t \mathbb{P}\{N(t) = n + 1 | N(s) = n\}\mathbb{P}\{N(s) = n\}\mathrm{d}s \\
&= \int_0^t \mathbb{P}\{X_{n+1} = t - s\}\mathbb{P}\{N(s) = n\}\mathrm{d}s \\
&= \int_0^t \lambda e^{-\lambda(t-s)} \frac{(\lambda s)^n}{n!} e^{-\lambda s}\mathrm{d}s \\
&= \frac{\lambda^{n+1} e^{-\lambda t}}{n!} \int_0^t s^n \mathrm{d}s \\
&= \frac{(\lambda t)^{n+1}}{(n + 1)!} e^{-\lambda t}
\end{aligned}$$

Equation $(*)$ holds for $n + 1$. By induction, $N(t)$ is Poisson distribution with mean λt.