TOPIC. Sums and limits. This section considers the question of when you can interchange the order of summation in a doubly indexed infinite series; the answer is given by Fubini's Theorem. We also consider the question of when you can bring a limit inside an infinite series; the answer is given in various forms by the Monotone Convergence Theorem, the Dominated Convergence Theorem, and the Sandwich Theorem. The aforementioned theorems apply not only to infinite series, but also to integrals and expectations; we will use them extensively in the rest of the course.

Question 1. Suppose

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

is an array of numbers with infinitely many rows and columns. We ask: is it true that

$$\sum_{m=1}^{\infty}\Bigl(\sum_{n=1}^{\infty} x_{m,n}\Bigr) = \sum_{n=1}^{\infty}\Bigl(\sum_{m=1}^{\infty} x_{m,n}\Bigr), \tag{1}$$

i.e., can we switch the order of summation? Notice that the expression in ()'s on the LHS is the sum, call it r_m, of the elements in the m^{th} row of the array; the LHS is the sum, call it r, of these row sums. Similarly, the expression in ()'s on the RHS is the sum c_n of the elements in the n^{th} column, and the RHS itself is the sum c of these column sums. r and c are called ***iterated sums***.

Example 1. (a) There are cases where $r = c$. For example, for the array

$$\begin{matrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix} \qquad \left(x_{m,n} = \begin{cases} 1, & \text{if } m \le n, \\ 0, & \text{otherwise} \end{cases}\right)$$

one has $r_m = \lim_{N\to\infty}(\sum_{n=1}^{N} x_{m,n}) = \infty$ for each m, so

$$r = \infty + \infty + \infty + \cdots = \infty.$$

Moreover $c_n = \lim_{M\to\infty}(\sum_{m=1}^{M} x_{m,n}) = n$ for each n, so

$$c = 1 + 2 + 3 + \cdots = \infty = r.$$

(b) Sadly, there are cases where $r \neq c$. For example, for the array

$$\begin{matrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix} \qquad \left(x_{m,n} = \begin{cases} 1, & \text{if } n = m, \\ -1, & \text{if } n = m+1, \\ 0, & \text{otherwise} \end{cases}\right)$$

one has $r_m = 0$ for each m, so

$$r = 0 + 0 + 0 + \cdots = 0,$$

whereas $c_1 = 1$ and $c_n = 0$ for all $n \geq 2$, so

$$c = 1 + 0 + 0 + 0 + \cdots = 1 \neq r. \qquad \bullet$$

So far we have tacitly assumed that r and c exist. The statement "r exists" means that in the recipe

$$r := \lim_{M\to\infty}\Bigl[\sum_{m=1}^{M}\Bigl(\lim_{N\to\infty}\Bigl[\sum_{n=1}^{N} x_{m,n}\Bigr]\Bigr)\Bigr],$$

all the indicated sums and limits exist (possibly as $+\infty$ or $-\infty$); a similar interpretation applies to the statement "c exists". There are, of course, cases where r and c don't exist; see, e.g., Exercise 3.

The following theorem says that you can switch the order of summation for a doubly infinite array provided all the array elements are nonnegative (as in Example 1 (a)).

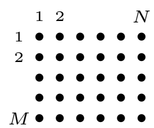
$r := \sum_{m=1}^{\infty} r_m$ with $r_m := \sum_{n=1}^{\infty} x_{m,n}$.
$c := \sum_{n=1}^{\infty} c_n$ with $c_n := \sum_{m=1}^{\infty} x_{m,n}$.

Theorem 1 (Fubini I – the nonnegative case). *Suppose* $0 \le x_{m,n} \le \infty$ *for* $m = 1, 2, \dots$ *and* $n = 1, 2, \dots$ *. Let the iterated sums* r *and* c *be defined as above. Then* r *and* c *both exist and equal*

$$s := \sup\Big\{ \sum_{(m,n)\in I} x_{m,n} : I \textit{ is a finite subset of } \mathbb{N}\times\mathbb{N} \Big\}. \tag{2}$$

Proof • *r and c exist*: $r_m := \lim_{N\to\infty}\big(\sum_{n=1}^{N} x_{m,n}\big)$ exists because the partial sums involved are nondecreasing in N; similarly $r := \lim_{M\to\infty}\big(\sum_{m=1}^{M} r_m\big)$ exists because the partial sums involved are nondecreasing in M. Parallel statements apply to the columns.

• $r \le s$: Let M and N be positive integers. Set $I = \{ (m,n) \in \mathbb{N}\times\mathbb{N} : 1 \le m \le M,\ 1 \le n \le N \}$. Then



$$\sum_{m=1}^{M}\Big(\sum_{n=1}^{N} x_{m,n}\Big) = \sum_{(m,n)\in I} x_{m,n} \le s.$$

As $N \to \infty$, the LHS above tends to $\sum_{m=1}^{M} r_m$ (see Exercise 1), so

$$\sum_{m=1}^{M} r_m \le s.$$

As $M \to \infty$, the LHS above tends to r, so $r \le s$.

• $r \ge s$: Let I be a finite subset of $\mathbb{N}\times\mathbb{N}$. There exist positive integers M and N such that

$$I \subset \{ (m,n) \in \mathbb{N}\times\mathbb{N} : 1 \le m \le M,\ 1 \le n \le N \}$$
$$\Longrightarrow \sum_{(m,n)\in I} x_{m,n} \le \sum_{m=1}^{M}\Big(\sum_{n=1}^{N} x_{m,n}\Big) \le \sum_{m=1}^{M} r_m \le r.$$

Since this is true for each I, we have $s \le r$.

• $r = s = c$: We have just shown $r = s$. A similar argument shows $c = s$. ∎

$r := \sum_{m=1}^{\infty} r_m$ with $r_m := \sum_{n=1}^{\infty} x_{m,n}$.
$c := \sum_{n=1}^{\infty} c_n$ with $c_n := \sum_{m=1}^{\infty} x_{m,n}$.
(2): $s := \sup\{ \sum_{(m,n)\in I} x_{m,n} : I$ is a finite subset of $\mathbb{N}\times\mathbb{N} \}$.

For nonnegative summands, the supremum in (2) is called the ***double sum*** of the $x_{m,n}$'s, denoted $\sum_{m,n} x_{m,n}$. For summands of arbitrary sign, the double sum is defined as

$$s = \sum_{m,n} x_{m,n} := \Big[\sum_{m,n} x^{+}_{m,n}\Big] - \Big[\sum_{m,n} x^{-}_{m,n}\Big] = s_+ - s_-, \tag{3}$$

provided at least one of the two double sums s_+ and s_- on the RHS is finite; otherwise s is said not to exist. In (3) $x^{+}_{m,n}$ and $x^{-}_{m,n}$ denote respectively the positive and negative parts of $x_{m,n}$. (Note though that s_+ and s_- may not be the positive and negative parts of s.)

Here is the main theorem. It implies that you can switch the order of summation for the $x_{m,n}$'s if the iterated sum — taken in either order — of the $|x_{m,n}|$'s is finite.

Theorem 2 (Fubini II – the quasi-integrable case). *Suppose* $-\infty \le x_{m,n} \le \infty$ *for* $m = 1, 2, \dots$ *and* $n = 1, 2, \dots$ *. The double sum* s *of the* $x_{m,n}$*'s exists if and only if at least one of the following iterated sums is finite:*

$$r_- := \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} x^{-}_{m,n}\Big), \quad r_+ := \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} x^{+}_{m,n}\Big),$$
$$c_- := \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} x^{-}_{m,n}\Big), \quad c_+ := \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} x^{+}_{m,n}\Big).$$

If the double sum exists, then so do the iterated sums r *and* c*, and* $r = s = c$*.*

According to Fubini I, one has $r_- = s_- = c_-$, and $r_+ = s_+ = c_+$. This proves the first assertion of the theorem and implies that in principle it doesn't matter whether you verify one of the row conditions, or one of the column conditions. However, it may be a little easier to work with the rows in one application, but with the columns in

$r_- := \sum_{m=1}^\infty \big(\sum_{n=1}^\infty x_{m,n}^-\big).$ $\qquad r_+ := \sum_{m=1}^\infty \big(\sum_{n=1}^\infty x_{m,n}^+\big)$

another. There are cases (see, e.g., Exercise 4) where r and c exist and are equal, but s does not exist; hence the conditions of the theorem are sufficient, but not necessary, for being able to interchange the order of summation.

Proof I will do the case where $r_- < \infty$, which implies

$$\textstyle\sum_{n=1}^\infty x_{m,n}^- < \infty \text{ for all } m, \text{ and } x_{m,n}^- < \infty \text{ for all } m \text{ and } n. \tag{4}$$

I need to show that r and c exist and equal s. Since $x_{m,n} = x_{m,n}^+ - x_{m,n}^-$, we have

$$r_{m,N} := \sum_{n=1}^N x_{m,n} = \sum_{n=1}^N x_{m,n}^+ - \sum_{n=1}^N x_{m,n}^-;$$

note that the difference on the RHS is well defined since the second term is finite by (4). As $N \to \infty$,

$$\sum_{n=1}^N x_{m,n}^+ \to (r_+)_m := \sum_{n=1}^\infty x_{m,n}^+$$
$$\sum_{n=1}^N x_{m,n}^- \to (r_-)_m := \sum_{n=1}^\infty x_{m,n}^- < \infty$$

so

$$r_m := \textstyle\lim_{N\to\infty} r_{m,N} \text{ exists and equals } (r_+)_m - (r_-)_m. \tag{5}$$

Adding (5) for $m = 1, \ldots, M$ gives

$$\sum_{m=1}^M r_m = \sum_{m=1}^M (r_+)_m - \sum_{m=1}^M (r_-)_m.$$

As $M \to \infty$,

$$\sum_{m=1}^M (r_+)_m \to r_+ = \sum_{m=1}^\infty (r_+)_m$$
$$\sum_{m=1}^M (r_-)_m \to r_- = \sum_{m=1}^\infty (r_-)_m < \infty$$

so

$$r = \textstyle\lim_{N\to\infty} \sum_{m=1}^M r_m \text{ exists and equals } r_+ - r_- = s_+ - s_- = s.$$

Since $c_- = r_- < \infty$, a similar argument shows c exists and equals s.∎

Generalizations. Using measure theory, one can show that Fubini I and II hold not just for sums, but also for integrals and expectations, and combinations of such. For example

$$\int \Big[\int f(x,y)\,dx\Big]\,dy = \int \Big[\int f(x,y)\,dy\Big]\,dx = \iint f(x,y)\,dx\,dy \tag{6}$$

provided f is nonnegative, or

$$\int \Big[\int |f(x,y)|\,dx\Big]\,dy < \infty, \text{ or } \int \Big[\int |f(x,y)|\,dy\Big]\,dx < \infty. \tag{7}$$

Similarly for a random variable X,

$$E\Big[\int f(t,X)\,dt\Big] = \int E\big[f(t,X)\big]\,dt \tag{8}$$

provided f is nonnegative, or

$$E\Big[\int |f(t,X)|\,dt\Big] < \infty, \text{ or } \int E\big[|f(t,X)|\big]\,dt < \infty. \tag{9}$$

There are some additional technical conditions that are needed for these results, namely, the function f must be jointly measurable in its two arguments and the integrations have to be taken with respect to σ-finite measures. We'll ignore these condition in this course.

Example 2. Suppose $A_1, A_2, \ldots$ is an infinite sequence of events. Let N be the random variable which records how many of these events occur:

$$N(\omega) = \sum_{n=1}^\infty I_{A_n}(\omega)$$

for each $\omega \in \Omega$. Since $I_{A_n} \geq 0$ for each n, we have

$$E(N) = E\Big(\sum_{n=1}^\infty I_{A_n}\Big) \underset{\text{Fub I}}{=} \sum_{n=1}^\infty E(I_{A_n}) = \sum_{n=1}^\infty P[A_n]. \tag{10}$$

If the sum on the RHS here is finite, then we must have $N(\omega) < \infty$, i.e., $\omega \in A_n$ for at most finitely many n, for almost all sample points ω. This result is called the first Borel-Cantelli Lemma (see page 7-16).

Question 2. Suppose $x_1 = (x_1(k))_{k=1}^\infty$, $x_2 = (x_2(k))_{k=1}^\infty$, $\ldots$ is an infinite sequence of infinite sequences such that $x(k) := \lim_{n\to\infty} x_n(k)$ exists for each k. We ask: is it true that

$$\lim_{n\to\infty}\Big(\sum_{k=1}^\infty x_n(k)\Big) = \sum_{k=1}^\infty x(k), \tag{11}$$

i.e., can we bring the limit on n inside the sum? The answer is — not without some conditions. For example, suppose the $x_n(k)$'s are given by the array

$$\begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix} \qquad x_n(k) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{otherwise,} \end{cases}$$

where n is the row index and k is the column index. Then

$$\sum_{k=1}^\infty x_n(k) = 1 \text{ for each } n \Longrightarrow \lim_{n\to\infty}\Big(\sum_{k=1}^\infty x_n(k)\Big) = 1.$$

However

$$x(k) = \lim_{n\to\infty} x_n(k) = 0 \text{ for each } k \Longrightarrow \sum_{k=1}^\infty x(k) = 0.$$

We are going to present some conditions under which (11) does hold. But first, here is some terminology. An infinite sequence $x = (x(k))_{k=1}^\infty$ of extended real numbers (i.e., finite numbers or $\pm\infty$) is said to be ***integrable***, written $x \in \mathcal{L}$, if $\sum_{k=1}^\infty |x(k)| < \infty$, i.e., if the series $\sum_{k=1}^\infty x(k)$ is absolutely convergent; the ***integral*** of an integrable x is

$$\int x := \sum_{k=1}^\infty x(k) = \lim_{K\to\infty}\Big(\sum_{k=1}^K x(k)\Big). \tag{12}$$

x is said to be ***quasi-integrable from below***, written $x \in \mathcal{Q}_-$, if the sequence $x^- = ((x(k))^-)_{k=1}^\infty$ is integrable, or, equivalently, if there is an integrable sequence $\ell = (\ell(k))_{k=1}^\infty$ such that $\ell \le x$ in the sense that $\ell(k) \le x(k)$ for all k. Similarly, x is said to be ***quasi-integrable from above***, written $x \in \mathcal{Q}_+$, if the sequence $x^+ = ((x(k))^+)_{k=1}^\infty$ is integrable, or, equivalently, if there is an integrable sequence $u = (u(k))_{k=1}^\infty$ such that $x \le u$. Finally, x is said to be ***quasi-integrable***, written $x \in \mathcal{Q}$, if it is quasi-integrable from below or above (or both). The integral of a quasi-integrable x is taken to be

$$\begin{aligned}\int x &:= \int x^+ - \int x^- = \sum_{k=1}^\infty x^+(k) - \sum_{k=1}^\infty x^-(k) \\ &= \sum_{k=1}^\infty x(k) := \lim_{K\to\infty}\sum_{k=1}^K x(k).\end{aligned} \tag{13}$$

Note that the collection $\mathcal{Q}$ of quasi-integrable sequences is $\mathcal{Q}_- \cup \mathcal{Q}_+$, whereas the collection $\mathcal{L}$ of integrable sequences is $\mathcal{Q}_- \cap \mathcal{Q}_+$. The theorems below only apply to integrable or quasi-integrable sequences.

Example 3. (a) The sequence $x = (1/(k(k+1)))_{k=1}^\infty$ is integrable, with integral

$$\int x = \sum_{k=1}^\infty x(k) = \lim_{K\to\infty}\sum_{k=1}^K\Big(\frac{1}{k} - \frac{1}{k+1}\Big) = 1.$$

(b) The sequence $x = (1)_{k=1}^\infty$ is not integrable, but it is quasi-integrable from below with integral

$$\int x = \sum_{k=1}^\infty 1 = \lim_{K\to\infty}\sum_{k=1}^K 1 = \infty.$$

(c) The infinite series $\sum_{k=1}^\infty (-1)^k/k$ is convergent in the usual sense, i.e., $c := \lim_{K\to\infty}\big(\sum_{k=1}^K (-1)^k/k\big)$ exists and is finite (by calculus, $c = -\log(2)$). However the sequence $x = ((-1)^k/k)_{k=1}^\infty$ is not quasi-integrable, since $\sum_{k=1}^\infty x^-(k) = \sum_{k\text{ odd}} 1/k = \infty = \sum_{k\text{ even}} 1/k = \sum_{k=1}^\infty x^+(k)$. The theorems that follow don't apply to this sequence. •

$x = (x(k))_{k=1}^\infty \in \mathcal{L} \Longleftrightarrow \sum_{k=1}^\infty |x(k)| < \infty$

$x = (x(k))_{k=1}^\infty \in \mathcal{Q}_- \Longleftrightarrow \sum_{k=1}^\infty x^-(k) < \infty$

$x = (x(k))_{k=1}^\infty \in \mathcal{Q}_- \iff \sum_{k=1}^\infty x^-(k) < \infty$.

For $x \in \mathcal{Q}_-$, $\int x = \sum_{k=1}^\infty x(k) = \lim_{K\to\infty} \sum_{k=1}^K x(k)$.

For extended real numbers η_1, η_2, η_3, …, and η, the notation $\eta_n \uparrow \eta$ means $\eta_1 \le \eta_2 \le \eta_3 < \cdots$ and $\eta = \lim_n \eta_n$. $\eta_n \downarrow \eta$ is defined similarly. For infinite sequences x_1, x_2, …, and x,

$x_n \uparrow x$ means $x_n(k) \uparrow x(k)$ for $k = 1, 2, \ldots$, while

$x_n \downarrow x$ means $x_n(k) \downarrow x(k)$ for $k = 1, 2, \ldots$.

Theorem 3 (The Monotone Convergence Theorem (MCT)). *Suppose x_1, x_2, … and x are infinite sequences of extended real numbers.*

MCT_-: *If $x_n \uparrow x$ and $x_1 \in \mathcal{Q}_-$, then $x_n \in \mathcal{Q}_-$ for all n, $x \in \mathcal{Q}_-$, and*

$$\int x_n \uparrow \int x. \tag{14}$$

MCT_+: *If $x_n \downarrow x$ and $x_1 \in \mathcal{Q}_+$, then $x_n \in \mathcal{Q}_+$ for all n, $x \in \mathcal{Q}_+$, and*

$$\int x_n \downarrow \int x. \tag{15}$$

Example 4. (a) Suppose

$$\begin{aligned} x_1 &= (1,\ 0,\ 0,\ 0,\ \ldots) \\ x_2 &= (1,\ 1,\ 0,\ 0,\ \ldots) \\ x_3 &= (1,\ 1,\ 1,\ 0,\ \ldots) \end{aligned} \qquad \left(x_n(k) = \begin{cases} 1, & \text{if } k \le n \\ 0, & \text{otherwise} \end{cases} \right)$$

etc. Then $x_n \uparrow x = (1,\ 1,\ 1,\ \ldots)$ and $x_1 \in \mathcal{L} \subset \mathcal{Q}_-$. MCT_- asserts that $\int x_n \uparrow \int x$. This is correct, since $\int x_n = n \uparrow \infty = \int x$.

(b) Suppose

$$\begin{aligned} x_1 &= (1,\ 1,\ 1,\ 1,\ \ldots) \\ x_2 &= (0,\ 1,\ 1,\ 1,\ \ldots) \\ x_3 &= (0,\ 0,\ 1,\ 1,\ \ldots) \end{aligned} \qquad \left(x_n(k) = \begin{cases} 1, & \text{if } k \ge n \\ 0, & \text{otherwise} \end{cases} \right)$$

etc. Then $x_n \downarrow x = (0,\ 0,\ 0,\ 0,\ \ldots)$. The x_n's and x are all quasi-integrable. However $\int x_n = \infty$ doesn't tend down to $\int x = 0$. This doesn't contradict MCT_+, because no x_n is in $\mathcal{Q}_+$. •

MCT_-: $x_n \uparrow x$ and $x_1 \in \mathcal{Q}_- \Longrightarrow x_n \in \mathcal{Q}_-$, $x \in \mathcal{Q}_-$, and $\int x_n \uparrow \int x$.

Proof I'll prove MCT_-; MCT_+ then follows by changing signs. To begin with, consider the case where x_1 is nonnegative (i.e., $x_1(k) \ge 0$ for all k). Then the x_n's and x are nonnegative, and hence trivially quasi-integrable. For indices $m < n$ we have

$$x_m(k) \le x_n(k) \le x(k)$$

for each k; adding over $k = 1, 2, \ldots$ gives $\int x_m \le \int x_n \le \int x$. Thus

$$L := \lim_n \int x_n \text{ exists and } L \le \int x.$$

To get the opposite inequality, let K be a positive integer. Then

$$\begin{aligned} &\sum_{k=1}^K x(k) = \sum_{k=1}^K \lim_n x_n(k) = \lim_n \sum_{k=1}^K x_n(k) \\ &\quad \le \lim_n \sum_{k=1}^\infty x_n(k) = \lim_n \int x_n = L; \end{aligned}$$

Letting $K \to \infty$ gives $\int x = \sum_{k=1}^\infty x(k) \le L$.

Now consider the general case, where x_1^- is assumed to be integrable. Since $x_1^-(k) \ge x_n^-(k) \ge x^-(k)$ for all k, this implies $\infty > \int x_1^- \ge \int x_n^- \ge \int x^-$, and hence that $x_n \in \mathcal{Q}_-$ and $x \in \mathcal{Q}_-$. Moreover $x_1^-(k)$ is finite for each k. Define infinite sequences y_1, y_2, …, and y by setting

$$y_n(k) = x_n(k) + x_1^-(k) \quad \text{and} \quad y(k) = x(k) + x_1^-(k)$$

for each k. Since the y_n's are nonnegative and tend up to y, we have

$$\int y_n \uparrow \int y$$

by the nonnegative case treated above. But

$$\begin{aligned} &\int y_n = \int (x_n + x_1^-) = \int x_n + \int x_1^-, \text{ and} \\ &\int y = \int (x + x_1^-) = \int x + \int x_1^-. \end{aligned}$$

Since $\int x_1^-$ is finite, it follows that $\int x_n \uparrow \int x$. ∎

Given an infinite sequence $x_1, x_2, \ldots$ of infinite sequences, one defines $\inf_n x_n$ and $\liminf_n x_n$ element by element, i.e.,

$$(\inf_n x_n)(k) := \inf_n x_n(k) \quad \text{and} \quad (\liminf_n x_n)(k) = \liminf_n x_n(k)$$

for $k = 1, 2, \ldots$. Similarly for sup's and limsup's.

Theorem 4 (Fatou's Lemma (FL)). *Let $x_1, x_2, \ldots$ be an infinite sequence of infinite sequences of extended real numbers.*

FL_-: *If there exists an integrable sequence ℓ such that $\ell \le x_n$ for all n, then $x_n \in \mathcal{Q}_-$ for all n, $\liminf_n x_n \in \mathcal{Q}_-$, and*

$$\int \liminf_n x_n \le \liminf_n \int x_n. \tag{16}$$

FL_+: *If there exists an integrable sequence u such that $x_n \le u$ for all n, then $x_n \in \mathcal{Q}_+$ for all n, $\limsup_n x_n \in \mathcal{Q}_+$, and*

$$\limsup_n \int x_n \le \int \limsup_n x_n. \tag{17}$$

Note that the hypothesis of the lower half (FL_-) of Fatou's Lemma is trivially satisfied if $x_n \ge 0$ for all n.

Proof of FL_-. Put $y_n = \inf_{p \ge n} x_p$ and $y = \liminf_n x_n$. Then

$$y_n \uparrow y \text{ (by definition)} \quad \text{and} \quad y_1 \in \mathcal{Q}_- \text{ (since } \ell \le y_1\text{)}.$$

Hence by MCT_-, $y_n \in \mathcal{Q}_-$ for all n, $y \in \mathcal{Q}_-$, and

$$\begin{aligned}\int(\liminf x_n) &= \int y \underset{\text{by (14)}}{=} \lim_n \int y_n = \lim_n \int(\inf_{p\ge n} x_p)\\ &\le \lim_n(\inf_{p \ge n} \int x_p) = \liminf_n \int x_n.\end{aligned}$$ ∎

Example 5. Reconsider the sequences $x_1, x_2, \ldots$ in Example 4b. Notice that $0 \le x_n$ for all n, and $\liminf_n x_n = x = \limsup_n x_n$, where $x = (0, 0, 0, \ldots)$. FL_- applies and correctly asserts that $0 = \int x \le \liminf_n \int x_n = \liminf_n \infty = \infty$. The conclusion (17) to FL_+ would be $\infty \le 0$, which is obviously false. This doesn't contradict FL_+, since there is no integrable sequence u such that $x_n \le u$ for all n. •

Theorem 5 (The Sandwich Theorem). *Let $\ell_1, \ell_2, \ldots, \ell, x_1, x_2, \ldots, x, u_1, u_2, \ldots$, and u be infinite sequences of real numbers such that*

S1 *$\ell_n \to \ell$, $x_n \to x$, and $u_n \to u$ as $n \to \infty$ (element by element),*

S2 *$\ell_n \le x_n \le u_n$ for each n, and*

S3 *the ℓ_n's, u_n's, ℓ, and u are integrable and*

$$\lim_{n\to\infty} \int \ell_n = \int \ell \quad \text{and} \quad \lim_{n\to\infty} \int u_n = \int u.$$

Then the x_n's and x are integrable and

S4 $\lim_{n\to\infty} \int x_n = \int x$.

Simply put, the x_n's can be integrated to the limit provided they are "sandwiched" between lower bounds ℓ_n's and upper bounds u_n's which can be integrated to the limit.

Proof x_n is integrable because $\ell_n \le x_n \le u_n$ and ℓ_n and u_n are integrable; similarly, x is integrable because $\ell \le x \le u$ and ℓ and u are integrable. Since

$$0 \le x_n - \ell_n \to x - \ell \quad \text{and} \quad 0 \le u_n - x_n \to u - x$$

Fatou's Lemma implies that

$$\begin{aligned}\int x - \int \ell &= \int(x - \ell) = \int \liminf_n (x_n - \ell_n)\\ &\underset{(\text{by } \mathsf{FL}_-)}{\le} \liminf_n \int (x_n - \ell_n) \underset{(\text{by S3})}{=} \liminf_n \int x_n - \int \ell,\end{aligned} \tag{18}$$

$$\begin{aligned}\int u - \int x &= \int(u - x) = \int \liminf_n (u_n - x_n)\\ &\le \liminf_n \int (u_n - x_n) = \int u - \limsup_n \int x_n.\end{aligned} \tag{19}$$

Hence

$$\limsup_n \int x_n \underset{(\text{by (19)})}{\le} \int x \underset{(\text{by (18)})}{\le} \liminf_n \int x_n.$$ ∎

Sandwich Theorem. If (S1) $\ell_n \to l$, $x_n \to x$, and $u_n \to u$,
(S2) $\ell_n \le x_n \le u_n$ for each n, and
(S3) $\int \ell_n \to \int l$ finite, and $\int u_n \to \int u$ finite,
then (S4) $\int x_n \to \int x$ finite.

Example 6. Let $P_1, P_2, \ldots$ and P be probability measures on the set $\mathbb{N}$ of positive integers, and let $f_1, f_2, \ldots$ and f be the corresponding probability mass functions. Thus

$$P_n[B] = \sum\nolimits_{k\in B} f_n(k) \quad \text{and} \quad P[B] = \sum\nolimits_{k\in B} f(k)$$

for each subset B of $\mathbb{N}$. Suppose

$$f(k) = \lim_{n\to\infty} f_n(k) \text{ for each } k \in \mathbb{N}.$$

For each B,

$$\begin{aligned}\big|P_n[B] - P[B]\big| &= \Big|\sum\nolimits_{k\in B} f_n(k) - \sum\nolimits_{k\in B} f(k)\Big| \\ &= \Big|\sum\nolimits_{k\in B}\big(f_n(k) - f(k)\big)\Big| \le \sum\nolimits_{k\in B}\big|f_n(k) - f(k)\big| \\ &\le \sum\nolimits_{k=1}^{\infty}\big|f_n(k) - f(k)\big| = \int \big|f_n - f\big| := v_n.\end{aligned}$$

The Sandwich Theorem implies that the bound v_n here tends to 0 as $n \to \infty$. Indeed since

$$\begin{aligned}&\ell_n := 0 \le x_n := |f_n - f| \le u_n := f_n + f,\\ &\ell_n \to \ell := 0, \quad x_n \to x := 0, \quad u_n \to u := 2f,\\ &\textstyle\int \ell_n = 0 \to 0 = \int \ell \quad \text{and} \quad \int u_n = \int f_n + \int f = 2 \to 2 = \int u,\end{aligned}$$

we have

$$v_n = \int x_n \to \int x = 0.$$

We've shown that if $P_n[B] \to P[B]$ for each one-point set B, then $P_n[B] \to P[B]$ uniformly for all subsets B of $\mathbb{N}$. •

Sandwich Theorem. If (S1) $\ell_n \to l$, $x_n \to x$, and $u_n \to u$,
(S2) $\ell_n \le x_n \le u_n$ for each n, and
(S3) $\int \ell_n \to \int l$ finite, and $\int u_n \to \int u$ finite,
then (S4) $\int x_n \to \int x$ finite.

The following theorem is used over and over.

Theorem 6 (The Dominated Convergence Theorem (DCT)). *Let $x_1, x_2, \ldots$ and x and d be infinite sequences of real numbers such that*

D1 $\lim_{n\to\infty} x_n(k) = x(k)$ *for each* k,
D2 $|x_n(k)| \le d(k)$ *for all* n *and* k, *and*
D3 d *is integrable.*

Then the x_n's and x are integrable and

D4 $\int x = \lim_{n\to\infty} \int x_n$.

Proof Apply the Sandwich Theorem with $\ell_n = -d = \ell$ and $u_n = d = u$. ∎

The sequence d above is called a ***dominator***. It is of course essential that the dominator be integrable and that it dominate every x_n. The conditions for the DCT are stronger than those of the Sandwich Theorem. For example, consider the sequences

$$\begin{aligned}x_1 &= (1, -1, \;\; 0, \;\; 0, \;\; 0, \ldots),\\ x_2 &= (0, \;\; 1, -1, \;\; 0, \;\; 0, \ldots),\\ x_3 &= (0, \;\; 0, \;\; 1, -1, \;\; 0, \ldots),\end{aligned} \qquad \left(x_n(k) = \begin{cases} 1, & \text{if } k = n,\\ -1, & \text{if } k = n+1,\\ 0, & \text{otherwise}\end{cases}\right)$$

etc. Here $x_n \to x = (0, 0, 0, \ldots)$ and $\int x_n = 0 \to 0 = \int x$. This conclusion can be deduced (somewhat artificially) from the Sandwich Theorem by taking $\ell_n = x_n = u_n$ and $\ell = x = u$. However, it can not be deduced from the DCT because there is no integrable dominator in this situation; indeed, any sequence d satisfying D2 has $d(k) \ge 1$ for all k, and so can't be integrable.

Generalizations. Using measure theory, one can show that the Monotone Convergence Theorem, Fatou's Lemma, the Sandwich Theorem, and the Dominated Convergence Theorem hold not just for sums of sequences, but also for integrals of (measurable) functions and expectations of random variables. For example, suppose X_1, X_2, ... and X are random variables, all defined on some common probability space endowed with a probability measure P. The expectation version of MCT_- says that if

$$X_n(\omega) \uparrow X(\omega) \text{ for } (P\text{-almost}) \text{ all sample points } \omega \in \Omega$$

and

$$X_1 \in \mathcal{Q}_- \ \bigl(\text{meaning } E(X_1^-) < \infty\bigr),$$

then X_n and X are in $\mathcal{Q}_-$ (and so have expectations) and

$$E(X_n) \uparrow E(X); \tag{20}$$

according to the hypotheses $E(X_n)$ and $E(X)$ can't be $-\infty$; they can however be $+\infty$. The expectation version of Fatou's Lemma says in part that if the X_n's are nonnegative, then

$$E(\liminf_n X_n) \le \liminf_n E(X_n). \tag{21}$$

The expectation version of the DCT says that if

$$X_n(\omega) \to X(\omega) \text{ for } (P\text{-almost}) \text{ all sample points } \omega \in \Omega$$

and

there is an integrable random variable D such that
$|X_n(\omega)| \le D(\omega)$ for all n and for (P-almost) all $\omega \in \Omega$,

then X_n and X have finite expectations and

$$E(X_n) \to E(X). \tag{22}$$

The following definition and exercise cover some issues that the text assumes you are familiar with. Let x_1, x_2, ... and x be extended real-numbers. One says that ***x_n converges to x*** as $n \to \infty$, and writes $\boldsymbol{x = \lim_n x_n}$ or $\boldsymbol{x_n \to x}$, if for each real number $w < x$ one has $w \le x_n$ for all sufficiently large n, and, similarly, for each real number $y > x$ one has $x_n \le y$ for all sufficiently large n. If $x = \infty$ only the "w-condition" is required; if $x = -\infty$ only the "y-condition" is required. One has $x_n \to x \iff \liminf_n x_n = x = \limsup_n x_n$. For example, $\lim_n \bigl(1 + (-1)^n/n\bigr) = 1$, $\lim_n \sqrt{n} = \infty$, and $\lim_n \log(1/n) = -\infty$.

Exercise 1. Suppose x_1, x_2, ... and y_1, y_2, ... are extended real numbers such that $x_n + y_n$ is defined (i.e., not of the form $(\pm\infty) + (\mp\infty)$) for each n. (i) Suppose $y_n \to y$ and $(\liminf_n x_n) + y$ is defined. Show that $\liminf_n (x_n + y_n) = (\liminf_n x_n) + y$. (ii) Suppose $x_n \to x$ and $y_n \to y$ and $x + y$ is defined. Show that $x_n + y_n \to x + y$. ⋄

Exercise 2. Show that Fubini I is a special case of Fubini II. ⋄

Exercise 3. For $m = 1, 2, \ldots$ and $n = 1, 2, \ldots$, put

$$x_{m,n} = \begin{cases} (-1)^{n-1}, & \text{if } m \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the double sum s and the iterated sums r and c do not exist. ⋄

Exercise 4. For $(m, n) \in \mathbb{N} \times \mathbb{N}$, set

$$x_{m,n} = \begin{cases} 1, & \text{if } m = n, \\ -1, & \text{if } (m,n) = (2k-1, 2k) \text{ or } (2k, 2k-1) \\ & \qquad \text{for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the iterated sums r and c exist and are equal, but the double sum $\sum_{m,n} x_{m,n}$ does not exist. ⋄

Fubini's theorem was used implicitly in the argument leading up to Theorem 7.3. The following exercise makes the use explicit.

Exercise 5. Let X be a random variable with distribution function F and left-continuous representing function R. (a) Show that for each $u \in (0,1)$,

$$R^+(u) = \int_{x=0}^{\infty} I_{\{R(u)>x\}}\,dx$$

and use Fubini I to show that

$$\int_0^1 R^+(u)\,du = \int_0^{\infty} \big(1 - F(x)\big)\,dx.$$

(b) Similarly, use Fubini I to show that

$$\int_0^1 R^-(u)\,du = \int_{-\infty}^{0} F(x)\,dx.$$

[Hint: use the switching formula (1.5).] $\diamond$

Exercise 6. Use Fubini's theorem and induction on k to show that the volume of the unit ball

$$B := \{\,(x_1,\dots,x_k) : x_1^2 + \cdots + x_k^2 \le 1\,\}$$

in $\mathbb{R}^k$ equals

$$V_k := \frac{\pi^{k/2}}{\Gamma(k/2+1)}. \tag{23}$$ $\diamond$

$\int_{\mathbb{R}} \mathbb{E}\,\mathbb{1}_{\{x<X\le x+c\}}\,dx = \mathbb{E}\left[\int_{\mathbb{R}} \mathbb{1}_{\{x<X\le x+c\}}\,dx\right] = \mathbb{E}\,c = c$

Exercise 7. Let F be a distribution function and let c be a positive number. Show that $\int_{-\infty}^{\infty} \big(F(x+c) - F(x)\big)\,dx = c$. $\diamond$

Exercise 8. Let X and Y be two random variables such that for each $x \in \mathbb{R}$, the conditional distribution of Y given $X = x$ has a density, say, f_x; thus $P[\,Y \in B \mid X = x\,] = \int_B f_x(y)\,dy$ for each (Borel) subset B of $\mathbb{R}$. Let $f(y)$ be the result of averaging $f_x(y)$ with respect to the distribution μ of X, i.e.,

$$f(y) := \int_{-\infty}^{\infty} f_x(y)\mu(dx) = E\big(f_X(y)\big). \tag{24}$$

Show that Y has density f. $\diamond$

$F(u) \overset{?}{=} \int_{-\infty}^{u} \mathbb{E}\, f_X(y)\,dy = \mathbb{E}\left[\int_{-\infty}^{u} f_X(y)\,dy\right]$
$= \mathbb{E}\,\mathbb{P}(Y\le y \mid X=x)$
$= \mathbb{E}\,\mathbb{E}\,\mathbb{1}_{Y\le y} \mid X$
$= \mathbb{E}\,\mathbb{1}_{Y\le y}$
$= \mathbb{P}(Y\le y)$

Exercise 9. Deduce the MCT from FL. $\diamond$

The text gave a very simple proof of the DCT, but that came at the end of long series of arguments. The next exercise asks you to prove the DCT various ways, using progressively less and less machinery.

Exercise 10. (i) Deduce the DCT directly from Fatou's Lemma, using the fact that $\int x_n \to \int x \iff \liminf_n \int x_n = \int x = \limsup_n \int x_n$. (ii) Deduce the DCT from the MCT, using $x_n \to x \implies \sup_{n\ge m} |x_n - x| \downarrow 0$ as $m \to \infty$. (iii) Deduce the DCT "from scratch", using $\int |x_n - x| \le \sum_{k=1}^{K} |x_n(k) - x(k)| + 2\sup_{\ell > K} d(\ell)$ for each K. $\diamond$

Exercise 11. (a) Let U be a standard uniform random variable and let W a standard exponential random variable. Show that for $t \ge 0$

$$E(e^{-tU}) = \frac{1-e^{-t}}{t},$$
$$E(e^{-tW}) = \frac{1}{1+t},$$
$$E(W^2 e^{-tW}) = \frac{2}{(1+t)^3}\,;$$

use the convention that $(1 - e^{-0})/0 = 1$. (b) Let X and Y be two (possibly dependent) nonnegative real-valued random variables. Set $L(s,t) = E(e^{-sX-tY})$ for $s \ge 0$ and $t \ge 0$; L is called the joint Laplace transform of X and Y. Show that

$$L_s(0+,t) := \lim_{s\downarrow 0} \frac{L(s,t) - L(0,t)}{s} = -E(Xe^{-tY}), \tag{25}$$

$$\int_0^{\infty} \big(-L_s(0+,t)\big)\,dt = E\Big(\frac{X}{Y}\Big); \tag{26}$$

use the convention that $x/y = 0$ if $x = 0 = y$. (c) Let V and W be two independent standard exponential random variables. Show that $E\big((V^2+W^2)/(V+W)\big) = 4/3$. [Hint: use part (b), but don't completely evaluate $L(s,t)$.] $\diamond$

Exercise 12. Let X be an integrable random variable with an arbitrary distribution function F. For real numbers x, set

$$c(x) := E\big(|x - X|\big).$$

(a) Show that

$$\lim_{h \downarrow 0} \frac{c(x+h) - c(x)}{h} = E(I_{\{X \le x\}} - I_{\{x < X\}}) = 2F(x) - 1;$$

the limit on the left is taken as h tends down to 0 from above. [Hint: There are at least two ways to solve this problem; one uses the fact that for real numbers a and b, one has $\big||b| - |a|\big| \le |b - a|$.] (b) What is the necessary and sufficient condition on F for the function c to be differentiable at x? Explain briefly. ◇

The following exercise deals with a topic in Markov chains. The only facts you need to know about MC's are set out below. Let $X_0, X_1, \dots$ be an irreducible aperiodic Markov chain with countable state space I (which you may take to be $\mathbb{N}$) and transition probability matrix $\mathbb{P} = (\mathbb{P}_{ij})_{i \in I, j \in I}$; thus

$$\mathbb{P}_{ij} = P[\,X_n = j \mid X_{n-1} = i\,]$$

for each n. It follows from the Markov property that

$$P[\,X_n = j \mid X_0 = i\,] = (\mathbb{P}^n)_{ij}$$

where $\mathbb{P}^n$ denotes the n^{th} power of $\mathbb{P}$. Using renewal theory, one can show that

$$\pi_j := \lim_{n \to \infty} P[\,X_n = j \mid X_0 = i\,] \tag{27}$$

exists for each $j \in I$ and does not depend on the initial state i.

Exercise 13. Let the X_n's, I, $\mathbb{P}$, and π be as above. (a) Use Fatou's Lemma to show that

$$\sum\nolimits_{j \in I} \pi_j \le 1. \tag{28}$$

(b) Use Fatou's Lemma and Fubini's theorem to show that the π_k's satisfy the equations

$$\pi_k = \sum\nolimits_{j \in I} \pi_j \mathbb{P}_{jk}, \quad k \in I. \tag{29}$$

[Hint: Start by letting $n \to \infty$ in the equation $\mathbb{P}^{(n)}_{ik} = \sum_j \mathbb{P}^{(n-1)}_{ij} \mathbb{P}_{jk}$.] (c) Let $\nu = (\nu_j)_{j \in I}$ be a vector of nonnegative numbers such that $c := \sum_{j \in J} \nu_j < \infty$ and $\nu_k = \sum_{j \in J} \nu_j \mathbb{P}_{jk}$ for each $k \in J$. Use Fubini and the DCT to show that $\nu = c\pi$. [Hint: Start by showing $\nu = \nu \mathbb{P}^n$ for each n.] ◇