Math 118, Fall 2017 Problem Set 01

Question 1 Prove or disprove:

$$< u, v >= u^* \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} v$$

is an inner product on C^2. Check the properties.

Question 2 Which of the following define an inner product on degree-n polynomials

$$p(x) = p_0 + p_1 x + \cdots + p_n x^n?$$

(a)

$$< p, q >= \sum_{j=0}^{n} p_j \bar{q}_j$$

(b)

$$< p, q >= \int_0^{\pi} p(x)\bar{q}(x) \, \mathrm{d}x$$

(c)

$$< p, q >= \int_{-\infty}^{\infty} p(x)\bar{q}(x) \, \mathrm{d}x$$

(d)

$$< p, q >= \int_{-\infty}^{\infty} p(x)\bar{q}(x)\mathrm{e}^{-|x|} \, \mathrm{d}x$$

Justify your answers with proof or counterexample. Evaluate $< 1, x >$ and $< 1, x^2 >$ for each case.

Question 3 (a) Prove or disprove:

$$< u, v >= \sum_{n=0}^{\infty} u_n \bar{v}_n$$

is an inner product on $l^2(N)$. Check the properties.

(b) For $u_n = 2^{-n}$ and $v_n = 3^{-n}$ compute $< u, v >$ and the angle between u and v.

Question 4 (a) Prove the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for any x and y vectors in a *real* inner product space with norm $\|\|$.

(b) Prove

$$< x, y >= \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right)$$

for nonzero vectors x and y.

(c) Given a subspace A of a real inner-product space V, and a vector $x \in V$ which is not in A, show that there is a constant $\gamma < 1$ such that

$$| < a, x > | \leq \gamma \|a\|\|x\|$$

for all $a \in A$.

Question 5 For $p = 1, 2, \ldots$ define the Sobolev space $H^p = H^p(-\pi, \pi)$ by

$$H^p = \{g \in L^2 = L^2(-\pi, \pi) | g \text{ is } 2\pi\text{-periodic and } g', g'', \ldots, g^{(p)} \in L^2\},$$

with

$$< f, g >_p = \int_{-\pi}^{\pi} f(x)\bar{g}(x) + f'(x)\bar{g}'(x) + \cdots + f^{(p)}(x)\bar{g}^{(p)}(x)\ \mathrm{d}x.$$

For $p = 0$ we set $H^0 = L^2$ with the usual L^2 inner product $<,>$.

(a) Show that $< f, g >_p$ defines an inner product on H^p.

(b) Compute the norm $\|f\|_p = \sqrt{< f, f >_p}$ in H^p of $f(x) = \mathrm{e}^{\mathrm{i}ax}$ and the angle in H^p between f and $g(x) = \mathrm{e}^{ibx}$ for $a, b \in Z$.

(c) For $f \in L^2$ define a generalized derivative $f^{(p)}$ by the requirement

$$< f^{(p)}, g >= (-1)^p < f, g^{(p)} >$$

for all $g \in H^p$. Let $f \in L^2$ be given by $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x < 0$. For $1 \leq q \leq p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g \in H^p$.

(d) Fix $-\pi < a < b < \pi$ and let $f \in L^2$ be given by $f(x) = 1$ for $a < x < b$ and $f(x) = 0$ otherwise. For $1 \leq q \leq p$, compute the generalized derivatives

$$< f^{(q)}, g >$$

for all $g \in H^p$.

(e) Compute the generalized derivatives of $f(x) = Q(x)$ for $a < x < b$ and $f(x) = 0$ otherwise, where Q is a degree-n polynomial.