STAT 30400 : Distribution Theory

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Homework 7

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Solutions by

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1. (15 pts) For the following variables, find the moment generating function, and use them to find the mean and the variance. Also find the complex generating functions, and the characteristic functions.

   (a) X is distributed Binomial(n, p);

$$\begin{aligned}
m(t) &= \mathbb{E}e^{tX} \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{tk} \\
&= (pe^t + 1 - p)^n, \qquad \forall\, t \in \mathbb{R} \\
m'(t) &= npe^t(pe^t + 1 - p)^{n-1} \\
m^{(2)}(t) &= n(n-1)p^2 e^t (pe^t + 1 - p)^{n-2} + npe^t(pe^t + 1 - p)^{n-1} \\
\mathbb{E}X &= m'(0) = np \\
\mathbb{E}X^2 &= m^{(2)}(0) = n(n-1)p^2 + np \\
Var(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \\
&= np(1-p) \\
G(z) &= \mathbb{E}e^{zX} \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{zk} \\
&= (pe^z + 1 - p)^n \\
\phi(t) &= \mathbb{E}e^{itX} \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{itk} \\
&= (pe^{it} + 1 - p)^n
\end{aligned}$$

   (b) X has a double exponential distribution characterized by the density,

$$f(x) = e^{-e^{-x}} e^{-x}, \qquad -\infty < x < \infty.$$

$$\begin{aligned}
m(t) &= \mathbb{E}e^{tX} \\
&= \int_{\mathbb{R}} e^{tx} e^{-e^{-x}} e^{-x} \mathrm{d}x \\
&= -\int_{\mathbb{R}} e^{tx} e^{-e^{-x}} \mathrm{d}e^{-x} \\
&\xlongequal{y=e^{-x}} -\int_{\infty}^{0} y^{-t} e^{-y} \mathrm{d}y \\
&= \int_{0}^{\infty} y^{-t} e^{-y} \mathrm{d}y \\
&= \Gamma(1-t), \qquad t < 1
\end{aligned}$$

Solution (cont.)

$$\begin{aligned}
m'(t) &= \int_0^\infty y^{-t}[-\ln(y)]e^{-y}\mathrm{d}y \\
m^{(2)}(t) &= \int_0^\infty y^{-t}[-\ln(y)]^2 e^{-y}\mathrm{d}y \\
\mathbb{E}X = m'(0) &= -\int_0^\infty e^{-y}\ln y\mathrm{d}y \\
\mathbb{E}X^2 = m^{(2)}(0) &= \int_0^\infty e^{-y}(\ln y)^2\mathrm{d}y \\
&= \frac{\pi^2}{6} + m'(0)^2 \\
Var(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\pi^2}{6} \\
G(z) &= \mathbb{E}e^{zX} \\
&= \int_{\mathbb{R}} e^{zx}e^{-e^{-x}}e^{-x}\mathrm{d}x \\
&= \Gamma(1-z) \\
\phi(t) &= \mathbb{E}e^{itX} \\
&= \int_{\mathbb{R}} e^{itx}e^{-e^{-x}}e^{-x}\mathrm{d}x \\
&= \Gamma(1-it)
\end{aligned}$$

(c) X is distributed uniformly on $[-a,a]$, where $a>0$.

$$\begin{aligned}
m(t) &= \mathbb{E}e^{tX} \\
&= \frac{1}{2a}\int_{-a}^{a} e^{tx}\mathrm{d}x \\
&= \begin{cases} \frac{e^{ta}-e^{-ta}}{2at} & ,t\neq 0 \\ 1 & ,t=0 \end{cases} \\
m'(t) &= \frac{a(e^{ta}+e^{-ta})t - (e^{ta}-e^{-ta})}{2at^2} \\
&= \frac{(1+at)e^{-at}-(1-at)e^{at}}{2at^2}, \qquad t\neq 0 \\
\mathbb{E}X = m'(0) &= \lim_{t\to 0}\frac{(1+at)e^{-at}-(1-at)e^{at}}{2at^2} \\
&= \lim_{t\to 0}\frac{-a^2te^{-at}+a^2te^{at}}{4at} \\
&= \lim_{t\to 0}\frac{-a^2e^{-at}+a^3te^{-at}+a^2e^{at}+a^3te^{at}}{4a} \\
&= 0
\end{aligned}$$

Solution (cont.)

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\begin{aligned}
m^{(2)}(t) &= \lim_{t\to 0} \frac{m'(t) - m'(0)}{t} \\
&= \lim_{t\to 0} \frac{(1+at)e^{-at} - (1-at)e^{at}}{2at^3} \\
&= \lim_{t\to 0} \frac{-a^2te^{-at} + a^2te^{at}}{6at^2} \\
&= \lim_{t\to 0} \frac{-a^2e^{-at} + a^3te^{-at} + a^2e^{at} + a^3te^{at}}{12at} \\
&= \lim_{t\to 0} \frac{a^3e^{-at} + a^3te^{-at} - a^4te^{-at} + a^3e^{at} + a^3e^{at} - a^4te^{at}}{12a} \\
&= \frac{a^2}{3} \\
G(z) &= \mathbb{E}e^{zX} \\
&= \begin{cases} \frac{e^{za} - e^{-za}}{2az} & , z \neq 0 \\ 1 & , z = 0 \end{cases} \\
\phi(t) &= \mathbb{E}e^{zX} \\
&= \begin{cases} \frac{e^{ita} - e^{-ita}}{2ita} & , t \neq 0 \\ 1 & , t = 0 \end{cases}
\end{aligned}
$$

2. (10 pts) Let $\phi_1, \ldots, \phi_n$ denote characteristic functions for distributions on the real line. Let $a_1, \ldots, a_n$ denote nonnegative constants such that $a_1 + \ldots + a_n = 1$. Show that $\sum_{i=1}^n a_i\phi_i$ is also a characteristic function.

Proof. Suppose $\phi_1, \ldots, \phi_n$ are characteristic functions for $X_1, \ldots, X_n$, then $\phi_i = \mathbb{E}e^{itX_i}$. Define a random variable as

$$Y = \begin{cases} X_1 & , a_1 \\ \vdots & \\ X_n & , a_n \end{cases}$$

then

$$\begin{aligned} \phi_Y(t) &= \mathbb{E}e^{itY} \\ &= \sum_{i=1}^n \mathbb{E}(e^{itY}|Y = X_i)\mathbb{P}(Y = X_i) \\ &= a_i\mathbb{E}e^{itX_i} \\ &= \phi(t). \end{aligned}$$

So $\sum_{i=1}^n a_i\phi_i$ is a characteristic function of Y. □

3. (10 pts) Let X be a random variable with characteristic function ϕ given by

$$\phi(t) = \frac{1}{3}[\cos t + \cos(\pi t) + \cos(2\pi t)]$$

What is the distribution of X? (Hint: what distribution has $cos(t)$ as a characteristic function?)

For Bernoulli random variable $X_1 = \begin{cases} 1 & , \frac{1}{2} \\ -1 & , \frac{1}{2} \end{cases}$, we have

$$\begin{aligned} \phi_{X_1}(t) &= \mathbb{E}e^{itX_1} \\ &= \frac{1}{2}e^{it1} + \frac{1}{2}e^{-it1} \\ &= \cos(t). \end{aligned}$$

Similarly, for $X_2 = \begin{cases} \pi & , \frac{1}{2} \\ -\pi & , \frac{1}{2} \end{cases}$, $X_3 = \begin{cases} 2\pi & , \frac{1}{2} \\ -2\pi & , \frac{1}{2} \end{cases}$,

$$\begin{aligned} \phi_{X_2}(t) &= \cos(\pi t) \\ \phi_{X_3}(t) &= \cos(2\pi t). \end{aligned}$$

Let $X = \begin{cases} X_1 & , \frac{1}{3} \\ X_2 & , \frac{1}{3} \\ X_3 & , \frac{1}{3} \end{cases}$, then from problem 2 we know that

$$\begin{aligned} \phi_X(t) &= \frac{1}{3}(\phi_{X_1}(t) + \phi_{X_2}(t) + \phi_{X_2}(t)) \\ &= \frac{1}{3}[\cos t + \cos(\pi t) + \cos(2\pi t)]. \end{aligned}$$

Also, since X can be determined uniquely by the characteristic function, such X is unique.

4. (15 pts)

(a) Show that the characteristic function of the *Triangular* distribution, TR(a), is,

$$\phi(t) = \frac{2(1-\cos(at))}{a^2t^2}.$$

The density of the triangular distribution is,

$$f(x) = \frac{1}{a}\left(1 - \frac{|x|}{a}\right), \qquad -a < x < a.$$

[Hint: What is the distribution of $X+Y$ for X and Y iid Unif$(-\frac{a}{2}, \frac{a}{2})$.]

Proof. Let $X, Y \overset{iid}{\sim} Uniform(-\frac{a}{2}, \frac{a}{2})$. The characteristic function of X or Y is given by

$$\begin{aligned}\phi_X(t) = \phi_Y(t) = \mathbb{E}e^{itX} &= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{a} e^{itx}\mathrm{d}x \\ &= \frac{1}{ita}[e^{it\frac{a}{2}} - e^{-it\frac{a}{2}}] \\ &= \frac{\sin\left(\frac{a}{2}t\right)}{\frac{a}{2}t}.\end{aligned}$$

Let $Z = X+Y$, then the inverse transform from X, Y to X, Z is given by $\begin{cases} X = X \\ Y = Z - X \end{cases}$, the determinant of Jacobian of the inverse transform is given by $\left|\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right| = 1$. Then

$$\begin{aligned} f_{(X,Z)}(x,z) &= f_X(x) f_Y(z-x)|1| \\ &= \frac{1}{a^2}\mathbb{1}_{-\frac{a}{2}<x<\frac{a}{2}}\mathbb{1}_{-\frac{a}{2}<z-x<\frac{a}{2}} \\ f_Z(z) &= \int_{\mathbb{R}} f_{(X,Z)}(x,z)\mathrm{d}x \\ &= \begin{cases} \frac{1}{a^2}\int_{z-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}x & , 0 < z < a \\ \frac{1}{a^2}\int_{-\frac{a}{2}}^{z+\frac{a}{2}} \mathrm{d}x & , -a < z \le 0 \end{cases} \\ &= \begin{cases} \frac{1}{a}\left(1 - \frac{z}{a}\right) & , 0 < z < a \\ \frac{1}{a}\left(1 + \frac{z}{a}\right) & , -a < z \le 0 \end{cases} \\ &= \frac{1}{a}\left(1 - \frac{|z|}{a}\right), \ z \in (-a, a), \end{aligned}$$

i.e., $Z \sim$ TR(a).

$$\begin{aligned} \phi_Z(t) &= \phi_{X+Y}(t) \\ &= \phi_X(t)\phi_Y(t) \\ &= \left(\frac{\sin\left(\frac{a}{2}t\right)}{\frac{a}{2}t}\right)^2 \\ &= \frac{4\sin^2\left(\frac{a}{2}t\right)}{a^2t^2} \\ &= \frac{2(1-\cos(at))}{a^2t^2} \end{aligned}$$

□

(b) Show that the characteristic function of the *Inverse Triangular* distribution, $\mathrm{IT}(a)$, is equal to,

$$\phi(t) = \left(1 - \frac{|t|}{a}\right)^+.$$

The density of the inverse triangular distribution is,

$$f(x) = \frac{1 - \cos(ax)}{\pi a x^2}, \qquad -\infty < x < \infty.$$

Proof. Since

$$\int_{\mathbb{R}} \frac{2[1 - \cos(ax)]}{x^2 t^2} \mathrm{d}t = 2\pi < \infty,$$

by the inverse formula, from (a) we have

$$\begin{aligned}
\frac{1}{a}\left(1 - \frac{|x|}{a}\right) \mathbb{1}_{(-a,a)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{2(1 - \cos(at))}{a^2 t^2} \mathrm{d}t \\
\left(1 - \frac{|x|}{a}\right)^+ &= \int_{-\infty}^{\infty} e^{-itx} \frac{1 - \cos(at)}{\pi a t^2} \mathrm{d}t \\
&\overset{t'=-t}{=\!=} \int_{\infty}^{-\infty} e^{it'x} \frac{1 - \cos(-at')}{\pi a (-t')^2} \mathrm{d}(-t') \\
&= \int_{-\infty}^{\infty} e^{it'x} \frac{1 - \cos(at')}{\pi a t'^2} \mathrm{d}t' \\
&= \mathbb{E} e^{-ixX} \\
&= \phi_X(x)
\end{aligned}$$

where X comes from the inverse triangular distribution. So $\phi(t) = \left(1 - \frac{|t|}{a}\right)^+$. ☐

(c) Show that the characteristic functions of $\mathrm{IT}(\frac{4}{3})$ and the mixture 0.5 IT(1)+0.5 IT(2) agree on a open interval containing zero even though these are not the same distributions. Why this doesn't contradict the uniqueness theorem for characteristic functions?

Proof. Let $X \sim \mathrm{IT}(\frac{4}{3})$ and $Y \sim 0.5\mathrm{IT}(1) + 0.5\mathrm{IT}(2)$.

$$\begin{aligned}
\phi_X(t) &= \left(1 - \frac{3|t|}{4}\right)^+ \\
\phi_Y(t) &= \frac{1}{2}(1 - |t|)^+ + \frac{1}{2}\left(1 - \frac{|t|}{2}\right)^+
\end{aligned}$$

For $|t| < 1$, we have

$$\phi_X(t) = \left(1 - \frac{3|t|}{4}\right) = \phi_Y(t).$$

This is not contradict the uniqueness theorem for characteristic functions, since $\phi_X(t)$ and $\phi_Y(t)$ are not equal everywhere in $\mathbb{R}$. ☐

5. Give a counterexample of

$$X_n \xrightarrow{D} X \quad \Longrightarrow \quad f_n(x) \to f(x)$$

if f_n $(n = 1, 2, \ldots)$ and f all exist.

Let $f(x) = \mathbb{1}_{(-1,1)}$, $f_n(x) = [1 - \cos(2\pi n x)]\mathbb{1}_{(0,1)}$, then for $x \in (0, 1)$,

$$\begin{aligned}
F_n(x) &= \int_{-\infty}^{x} f_n(t)\mathrm{d}t \\
&= \int_{0}^{x} [1 - \cos(2\pi n t)]\mathrm{d}t \\
&= x - \frac{1}{2\pi n}\sin(2\pi n x), \\
F(x) &= x,
\end{aligned}$$

$F_n(x) = F(x) = 0$ for $x \le 0$ and $F_n(x) = F(x) = 1$ for $x \ge 1$. So

$$\lim_{n\to\infty} F_n(x) = x = F(x)$$

i.e., $X_n \xrightarrow{D} X$. However, $\lim_{n\to\infty} f_n(x)$ does not exist.