MATH 118:
Fourier Analysis and Wavelets

Fall 2017

Problem Set 2

Solutions by

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Question 1

Use Gram-Schmidt orthogonalization to find an orthonormal basis for the span of $\{e^{-x}, e^{-2x}, e^{-3x}\}$ in $L^2(0,\infty)$ with inner product

$$< f, g >= \int_0^\infty f(x)\overline{g}(x)\mathrm{d}x.$$

$\because$

$$\begin{aligned}
\|e^{-x}\|^2 &=< e^{-x}, e^{-x} > \\
&= \int_0^\infty e^{-x}\overline{e^{-x}}\mathrm{d}x \\
&= \int_0^\infty e^{-2x}\mathrm{d}x \\
&= -\frac{1}{2}e^{-2x}\Big|_0^\infty \\
&= \frac{1}{2}
\end{aligned}$$

$\therefore$ the first vector of the orthonormal basis is

$$\begin{aligned}
u_1 &= \frac{e^{-x}}{\sqrt{\|e^{-x}\|}} \\
&= \sqrt{2}e^{-x} \\
v_2 &= e^{-2x}- < e^{-2x}, u_1 > u_1 \\
&= e^{-2x} - \sqrt{2}e^{-x}\int_0^\infty e^{-2t}\overline{\sqrt{2}e^{-t}}\mathrm{d}t \\
&= e^{-2x} - 2e^{-x}\int_0^\infty e^{-3t}\mathrm{d}t \\
&= e^{-2x} + \frac{2}{3}e^{-x}e^{-3t}\Big|_0^\infty \\
&= e^{-2x} - \frac{2}{3}e^{-x} \\
\|v_2\|^2 &= \int_0^\infty (e^{-2t} - \frac{2}{3}e^{-t})\overline{(e^{-2t} - \frac{2}{3}e^{-t})}\mathrm{d}t \\
&= \int_0^\infty \left(e^{-4t} - \frac{4}{3}e^{-3t} + \frac{4}{9}e^{-2t}\right)\mathrm{d}t \\
&= -\frac{1}{4}e^{-4t} + \frac{4}{9}e^{-3t} - \frac{2}{9}e^{-2t}\Big|_0^\infty \\
&= \frac{1}{36}
\end{aligned}$$

$\therefore$ the second vector of the orthonormal basis is

$$\begin{aligned}
u_2 &= \frac{v_2}{\|v_2\|} \\
&= \frac{e^{-2x} - \frac{2}{3}e^{-x}}{\frac{1}{6}} \\
&= 6e^{-2x} - 4e^{-x}
\end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
v_3 &= e^{-3x} - < e^{-3x}, u_1 > u_1 - < e^{-3x}, u_2 > u_2 \\
&= e^{-3x} - \sqrt{2}e^{-x} \int_0^{\infty} e^{-3t}\overline{\sqrt{2}e^{-t}}\mathrm{d}t \\
&\quad - (6e^{-2x} - 4e^{-x}) \int_0^{\infty} e^{-3t}\overline{(6e^{-2t} - 4e^{-t})}\mathrm{d}t \\
&= e^{-3x} + \frac{1}{2}e^{-x}e^{-4t}\Big|_0^{\infty} \\
&\quad - (6e^{-2x} - 4e^{-x}) \left(-\frac{6}{5}e^{-5t} + e^{-4t}\right)\Big|_0^{\infty} \\
&= e^{-3x} - \frac{1}{2}e^{-x} - \frac{1}{5}(6e^{-2x} - 4e^{-x}) \\
&= e^{-3x} - \frac{6}{5}e^{-2x} + \frac{3}{10}e^{-x} \\
\|v_3\|^2 &= < v_3, v_3 > \\
&= \int_0^{\infty} \left(e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}\right) \overline{\left(e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}\right)}\mathrm{d}t \\
&= \int_0^{\infty} (e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t})\overline{(e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t})}\mathrm{d}t \\
&= -\frac{1}{3}e^{-3t} + \frac{12}{25}e^{-5t} - \frac{51}{100}e^{-4t} + \frac{6}{25}e^{-3t} - \frac{9}{200}e^{-2t} \\
&= \frac{1}{600}
\end{aligned}
$$

$\therefore$ the third vector of the orthonormal basis is

$$
\begin{aligned}
u_3 &= \frac{u_3}{\|u_3\|} \\
&= \frac{e^{-3x} - \frac{6}{5}e^{-2x} + \frac{3}{10}e^{-x}}{\sqrt{\frac{1}{600}}} \\
&= 10\sqrt{6}e^{-3x} - 12\sqrt{6}e^{-2x} + 3\sqrt{6}e^{-x}
\end{aligned}
$$

$\therefore$ the orthonormal basis is $\{u_1,\ u_2,\ u_3\}$

Question 2

(a) Find the orthogonal projection $Pf(x)$ of

$$f(x) = xe^{-\frac{x}{2}}$$

onto the subspace of Question 1.

$$
\begin{aligned}
\int_0^\infty te^{-t}\mathrm{d}t &= -\int_0^\infty t\mathrm{d}e^{-t} \\
&= -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}\mathrm{d}t \\
&= 1 \\
Pf(x) &=< f, u_1 > u_1 + < f, u_2 > u_2 + < f, u_3 > u_3 \\
&= \int_0^\infty xe^{-\frac{x}{2}}\overline{\sqrt{2}e^{-x}}\mathrm{d}x \cdot u_1 + \int_0^\infty xe^{-\frac{x}{2}}\overline{(6e^{-2x} - 4e^{-x})}\mathrm{d}x \cdot u_2 \\
&\quad + \int_0^\infty xe^{-\frac{x}{2}}\overline{(10\sqrt{6}e^{-3x} - 12\sqrt{6}e^{-2x} + 3\sqrt{6}e^{-x})}\mathrm{d}x \cdot u_3 \\
&= \sqrt{2}\int_0^\infty xe^{-\frac{3}{2}x}\mathrm{d}x \cdot u_1 + 2\int_0^\infty x(3e^{-\frac{5}{2}x} - 2e^{-\frac{3}{2}x})\mathrm{d}x \cdot u_2 \\
&\quad + \sqrt{6}\int_0^\infty x(10e^{-\frac{7}{2}x} - 12e^{-\frac{5}{2}x} + 3e^{-\frac{3}{2}x})\mathrm{d}x \cdot u_3 \\
&= \frac{4\sqrt{2}}{9}\int_0^\infty te^{-t}\mathrm{d}t \cdot u_1 + \left(\frac{24}{25} - \frac{16}{9}\right)\int_0^\infty te^{-t}\mathrm{d}t \cdot u_2 \\
&\quad + \left(\frac{40\sqrt{6}}{49} - \frac{48\sqrt{6}}{25} + \frac{12\sqrt{6}}{9}\right)\int_0^\infty te^{-t}\mathrm{d}t \cdot u_3 \\
&= \frac{4\sqrt{2}}{9}u_1 + \left(\frac{24}{25} - \frac{16}{9}\right)u_2 + \left(\frac{40\sqrt{6}}{49} - \frac{48\sqrt{6}}{25} + \frac{12\sqrt{6}}{9}\right)u_3
\end{aligned}
$$

(b) Express P in the form of an integral operator

$$
Pf(x) = \int_0^\infty K(x,y)f(y)\mathrm{d}y
$$

and find the kernel $K(x,y)$.

$\because$

$$
\begin{aligned}
Pf(x) &=< f, u_1 > u_1 + < f, u_2 > u_2 + < f, u_3 > u_3 \\
&= \int_0^\infty ye^{-\frac{y}{2}}\overline{\sqrt{2}e^{-y}}\mathrm{d}y \cdot u_1 + \int_0^\infty ye^{-\frac{y}{2}}\overline{(6e^{-2y} - 4e^{-y})}\mathrm{d}y \cdot u_2 \\
&\quad + \int_0^\infty ye^{-\frac{y}{2}}\overline{(10\sqrt{6}e^{-3y} - 12\sqrt{6}e^{-2y} + 3\sqrt{6}e^{-y})}\mathrm{d}y \cdot u_3 \\
&= \int_0^\infty u_1(x)\sqrt{2}e^{-y}f(y)\mathrm{d}y + \int_0^\infty u_2(x)(6e^{-2y} - 4e^{-y})f(y)\mathrm{d}y \\
&\quad + \int_0^\infty u_3(x)(10\sqrt{6}e^{-3y} - 12\sqrt{6}e^{-2y} + 3\sqrt{6}e^{-y})f(y)\mathrm{d}y \\
&= \int_0^\infty (u_1(x)u_1(y) + u_2(x)u_2(y) + u_3(x)u_3(y))f(y)\mathrm{d}y \\
&= \int_0^\infty K(x,y)f(y)\mathrm{d}y
\end{aligned}
$$

$\therefore$

$$
K(x,y) = u_1(x)u_1(y) + u_2(x)u_2(y) + u_3(x)u_3(y)
$$

Question 3

Let D be the unit disk in C,

$$L^2(D) = \{f : D \to C| \iint_D |f(x,y)|^2 \mathrm{d}x\mathrm{d}y < \infty\},$$

and

$$< f, g >= \iint_D f(x,y)\overline{g}(x,y)\mathrm{d}x\mathrm{d}y.$$

(a) Show that

$$\varphi_n(x,y) = (x+iy)^n$$

for $n \in \mathbb{N}$ is an orthogonal set in $L^2(D)$.

$\because$

$D' = \{(r,\theta)|0 \leqslant r \leqslant 1, 0 \leqslant \theta < 2\pi\}$

$\forall m, n \in \mathbb{N},\ m \neq n$

$$\begin{aligned}
< \varphi_m(x,y), \varphi_n(x,y) > &= \iint_D \varphi_m(x,y)\overline{\varphi_n}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint_D (x+iy)^m\overline{(x+iy)^n}\mathrm{d}x\mathrm{d}y \\
&= \iint_D (x+iy)^m(x-iy)^n\mathrm{d}x\mathrm{d}y \\
&= \begin{cases} \iint_{D'} |r|^{2n+1}(\cos\theta - i\sin\theta)^{n-m}\mathrm{d}r\mathrm{d}\theta, & n > m \\ \iint_{D'} |r|^{2m+1}(\cos\theta - i\sin\theta)^{m-n}\mathrm{d}r\mathrm{d}\theta, & n < m \end{cases} \\
&= \begin{cases} \int_0^1 r^{2n+1}\mathrm{d}r \int_0^{2\pi}\{\cos[(n-m)\theta] - i\sin[(n-m)\theta]\}\mathrm{d}\theta, & n > m \\ \int_0^1 r^{2m+1}\mathrm{d}r \int_0^{2\pi}\{\cos[(m-n)\theta] - i\sin[(m-n)\theta]\}\mathrm{d}\theta, & n < m \end{cases} \\
&= \begin{cases} \dfrac{1}{(2n+2)(n-m)}\{-\sin[(n-m)\theta] - i\cos[(n-m)\theta]\}\Big|_0^{2\pi}, & n > m \\ \dfrac{1}{(2m+2)(m-n)}\{-\sin[(m-n)\theta] - i\cos[(m-n)\theta]\}\Big|_0^{2\pi}, & n < m \end{cases} \\
&= 0
\end{aligned}$$

$\forall\ m \in \mathbb{N}$

$$\begin{aligned}
< \varphi_m(x,y), \varphi_m(x,y) > &= \iint_D \varphi_m(x,y)\overline{\varphi_m}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint_D (x+iy)^m\overline{(x+iy)^m}\mathrm{d}x\mathrm{d}y \\
&= \iint_D (x+iy)^m(x-iy)^m\mathrm{d}x\mathrm{d}y \\
&= \iint_{D'} r^{2m+1}\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^1 r^{2m+1}\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{\pi}{m+1} \neq 0
\end{aligned}$$

$\therefore \quad \varphi_n(x,y) (n \in \mathbb{N})$ is an othogonal set in $L^2(D)$.

(b) Normalize them.

$\forall\ n \in \mathbb{N}$

$$\begin{aligned}\psi_n(x,y) &= \frac{\varphi_n(x)}{\|\varphi_n(x)\|}\\ &= \frac{\varphi_n(x)}{\sqrt{<\varphi_n(x),\varphi_n(x)>}}\\ &= \sqrt{\frac{n+1}{\pi}}(x+iy)^n\end{aligned}$$

Then $\psi_n(x,y)(n \in \mathbb{N})$ is an orthonormal set in $L^2(D)$.

(c) Project

$$f(x,y) = \sqrt{x+iy}$$

onto the span of $\{\varphi_0, \cdots, \varphi_N\}$.

$\forall\ n \in \mathbb{N}, n \leqslant N$

$$\begin{aligned}<f(x,y),\varphi_n(x,y)> &= \iint_D f(x,y)\overline{\varphi_n}(x,y)\mathrm{d}x\mathrm{d}y\\ &= \iint_D \sqrt{x+iy}\cdot\overline{(x+iy)^n}\mathrm{d}x\mathrm{d}y\\ &= \iint_D \sqrt{x+iy}\cdot(x-iy)^n\mathrm{d}x\mathrm{d}y\\ &= \int_0^1\int_0^{2\pi}\sqrt{r}\sqrt{\cos\theta+i\sin\theta}\cdot r^n(\cos\theta-i\sin\theta)^n r\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^1 r^{n+\frac{3}{2}}\mathrm{d}r\int_0^{2\pi}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)[\cos(n\theta)-i\sin(n\theta)]\,\mathrm{d}\theta\\ &= \frac{2}{2n+5}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\theta\right)-i\sin\left(\frac{2n-1}{2}\theta\right)\right]\mathrm{d}\theta\\ &= \frac{2}{2n+5}\cdot\frac{4i}{2n-1}\\ &= \frac{8i}{(2n-1)(2n+5)}\end{aligned}$$

$\therefore$

$$\begin{aligned}Pf(x,y) &= \sum_{n=0}^{N}<f(x,y),\varphi_n(x,y)>\varphi_n(x,y)\\ &= \sum_{n=0}^{N}\frac{8i(x+iy)^n}{(2n-1)(2n+5)}\end{aligned}$$

Question 4

Find a sequence $f_n \in L^2(0,1)$ such that $f_n \to 0$ in $L^2(0,1)$ but not uniformly on $[0,1]$.

$$f_n(x) = \sqrt{n} I_{(0,\frac{1}{n^2}]}(x) \qquad n \in \mathbb{N}^+$$

$$\begin{aligned}\|f_n(x) - 0\|_2 &= \int_0^1 [f_n(x)]^2 \mathrm{d}x \\ &= \int_0^{\frac{1}{n^2}} n \mathrm{d}x \\ &= \frac{1}{n} \to 0 \qquad (n \to +\infty)\end{aligned}$$

$\forall x \in [0,1]$

$$f_n(x) = \sqrt{n} I_{(0,\frac{1}{n^2}]}(x) \to 0 \qquad (n \to \infty)$$

$\because$ given $\epsilon = \frac{1}{2}$, $x_n = \frac{1}{n^2}$, we have

$$f_n(x_n) = 1 \to 1 > \frac{1}{2}$$

$\therefore$ $f_n \not\rightrightarrows 0$

Question 5

Let $\varphi_j(x) = 0$ for all j whenever $|x| \geqslant 1$ and set

$$\varphi_0(x) = 1$$

$$\varphi_1(x) = sign(x)$$

$$\varphi_2(x) = \varphi_1(2x-1)$$

$$\varphi_3(x) = \varphi_1(2x+1)$$

(a) Sketch φ_j for $0 \leqslant j \leqslant 3$.

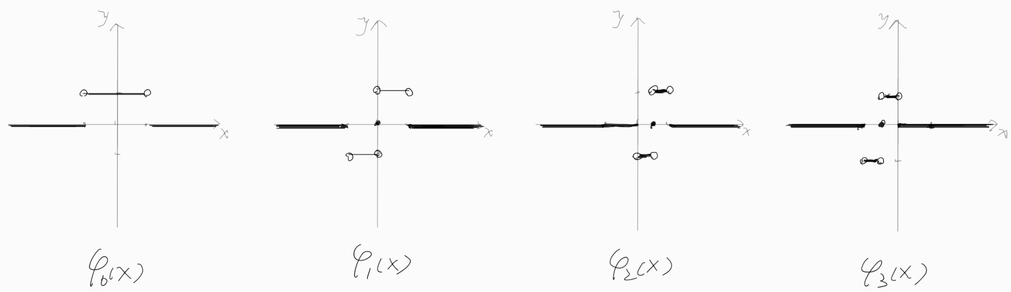
$$\varphi_0(x) = I_{(-1,1)}$$

$$\varphi_1(x) = \begin{cases} 1, & x \in (0,1) \\ -1, & x \in (-1,0) \\ 0, & otherwise \end{cases}$$

Solution (cont.)

$$\varphi_2(x) = \begin{cases} 1, & x \in \left(\dfrac{1}{2}, 1\right) \\ -1, & x \in \left(0, \dfrac{1}{2}\right) \\ 0, & otherwise \end{cases}$$

$$\varphi_3(x) = \begin{cases} 1, & x \in \left(-\dfrac{1}{2}, 0\right) \\ -1, & x \in \left(-1, -\dfrac{1}{2}\right) \\ 0, & otherwise \end{cases}$$



(b) Show that these functions are orthogonal in $L^2(-1, 1)$.

$\because$

$$\begin{aligned} < \varphi_0(x), \varphi_1(x) > &= \int_{-1}^{1} \varphi_0(x)\overline{\varphi_1(x)}\mathrm{d}x \\ &= -\int_{-1}^{0} \mathrm{d}x + \int_{0}^{1} \mathrm{d}x \\ &= -1 + 1 \\ &= 0 \\ < \varphi_0(x), \varphi_2(x) > &= \int_{-1}^{1} \varphi_0(x)\overline{\varphi_2(x)}\mathrm{d}x \\ &= -\int_{0}^{\frac{1}{2}} \mathrm{d}x + \int_{\frac{1}{2}}^{1} \mathrm{d}x \\ &= -\frac{1}{2} + \frac{1}{2} \\ &= 0 \\ < \varphi_0(x), \varphi_3(x) > &= \int_{-1}^{1} \varphi_0(x)\overline{\varphi_3(x)}\mathrm{d}x \\ &= -\int_{-1}^{-\frac{1}{2}} \mathrm{d}x + \int_{-\frac{1}{2}}^{0} \mathrm{d}x \\ &= -\frac{1}{2} + \frac{1}{2} \\ &= 0 \end{aligned}$$

Solution (cont.)

$$
\begin{aligned}
<\varphi_1(x),\varphi_2(x)> &= \int_{-1}^{1}\varphi_1(x)\overline{\varphi_2(x)}\mathrm{d}x \\
&= -\int_0^{\frac{1}{2}}\mathrm{d}x + \int_{\frac{1}{2}}^{1}\mathrm{d}x \\
&= -\frac{1}{2}+\frac{1}{2} \\
&= 0 \\
<\varphi_1(x),\varphi_3(x)> &= \int_{-1}^{1}\varphi_1(x)\overline{\varphi_3(x)}\mathrm{d}x \\
&= \int_{-1}^{-\frac{1}{2}}\mathrm{d}x - \int_{-\frac{1}{2}}^{0}\mathrm{d}x \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
<\varphi_2(x),\varphi_3(x)> &= \int_{-1}^{1}\varphi_2(x)\overline{\varphi_3(x)}\mathrm{d}x \\
&= \int_{-1}^{-\frac{1}{2}}\mathrm{d}x - \int_{-\frac{1}{2}}^{0}\mathrm{d}x \\
&= 0 \\
<\varphi_0(x),\varphi_0(x)> &= \int_{-1}^{1}\varphi_0(x)\overline{\varphi_0(x)}\mathrm{d}x \\
&= \int_{-1}^{1}\mathrm{d}x \\
&= 2 \\
<\varphi_1(x),\varphi_1(x)> &= \int_{-1}^{1}\varphi_1(x)\overline{\varphi_1(x)}\mathrm{d}x \\
&= \int_{-1}^{1}\mathrm{d}x \\
&= 2 \\
<\varphi_2(x),\varphi_2(x)> &= \int_{-1}^{1}\varphi_2(x)\overline{\varphi_2(x)}\mathrm{d}x \\
&= \int_{0}^{1}\mathrm{d}x \\
&= 1 \\
<\varphi_3(x),\varphi_3(x)> &= \int_{-1}^{1}\varphi_3(x)\overline{\varphi_3(x)}\mathrm{d}x \\
&= \int_{-1}^{0}\mathrm{d}x \\
&= 1
\end{aligned}
$$

$\therefore$ these functions are orthogonal in $L^2(-1,1)$.

(c) Normalize them.

$$
\begin{aligned}
\psi_0(x) &= \frac{\varphi_0(x)}{\|\varphi_0(x)\|} \\
&= \frac{\varphi_0(x)}{\sqrt{<\varphi_0(x), \varphi_0(x)>}} \\
&= \frac{\varphi_0(x)}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2} I_{(-1,1)} \\
\psi_1(x) &= \frac{\varphi_1(x)}{\|\varphi_1(x)\|} \\
&= \frac{\varphi_1(x)}{\sqrt{<\varphi_1(x), \varphi_1(x)>}} \\
&= \frac{\varphi_1(x)}{\sqrt{2}} \\
&= \begin{cases} \frac{\sqrt{2}}{2}, & x \in (0,1) \\ \frac{\sqrt{2}}{2}, & x \in (-1,0) \\ 0, & otherwise \end{cases} \\
\psi_2(x) &= \frac{\varphi_2(x)}{\|\varphi_2(x)\|} \\
&= \frac{\varphi_2(x)}{\sqrt{<\varphi_2(x), \varphi_2(x)>}} \\
&= \varphi_2(x) \\
&= \begin{cases} 1, & x \in \left(\frac{1}{2}, 1\right) \\ -1, & x \in \left(0, \frac{1}{2}\right) \\ 0, & otherwise \end{cases} \\
\psi_3(x) &= \frac{\varphi_3(x)}{\|\varphi_3(x)\|} \\
&= \frac{\varphi_3(x)}{\sqrt{<\varphi_3(x), \varphi_3(x)>}} \\
&= \varphi_3(x) \\
&= \begin{cases} 1, & x \in \left(-\frac{1}{2}, 0\right) \\ -1, & x \in \left(-1, -\frac{1}{2}\right) \\ 0, & otherwise \end{cases}
\end{aligned}
$$

(d) Compute the orthogonal projection Pf of $f(x) = x$ onto the span of $\{\varphi_j | 0 \leqslant j \leqslant 3\}$.

$$
\begin{aligned}
Pf &= \sum_{n=0}^{3} < f(x), \varphi_n(x) > \varphi_n(x) \\
&= \int_{-1}^{1} f(x)\overline{\varphi_0(x)}\mathrm{d}x \cdot \varphi_0(x) + \int_{-1}^{1} f(x)\overline{\varphi_1(x)}\mathrm{d}x \cdot \varphi_1(x) \\
&\quad + \int_{-1}^{1} f(x)\overline{\varphi_2(x)}\mathrm{d}x \cdot \varphi_2(x) + \int_{-1}^{1} f(x)\overline{\varphi_3(x)}\mathrm{d}x \cdot \varphi_3(x) \\
&= \int_{-1}^{1} x\mathrm{d}x \cdot \varphi_0(x) + \left[\int_{-1}^{0} (-x)\mathrm{d}x + \int_{0}^{1} x\mathrm{d}x\right] \cdot \varphi_1(x) \\
&\quad + \left[\int_{0}^{\frac{1}{2}} (-x)\mathrm{d}x + \int_{\frac{1}{2}}^{1} x\mathrm{d}x\right] \cdot \varphi_2(x) + \left[\int_{-1}^{-\frac{1}{2}} (-x)\mathrm{d}x + \int_{-\frac{1}{2}}^{0} x\mathrm{d}x\right] \cdot \varphi_3(x) \\
&= 0 + \varphi_1(x) - \frac{3}{4}\varphi_2(x) + \frac{3}{4}\varphi_3(x) \\
&= \begin{cases} -\frac{7}{4}, & x \in \left(-1, \frac{1}{2}\right) \\ -\frac{1}{4}, & x \in \left(-\frac{1}{2}, 0\right) \\ \frac{7}{4}, & x \in \left(0, \frac{1}{2}\right) \\ \frac{1}{4}, & x \in \left(\frac{1}{2}, 1\right) \\ 1, & x = \frac{1}{2} \\ -1, & x = -\frac{1}{2} \\ 0, & otherwise \end{cases}
\end{aligned}
$$

(e) Express P in the form of an integral operator

$$
Pf(x) = \int_{-1}^{1} K(x, y)f(y)\mathrm{d}y
$$

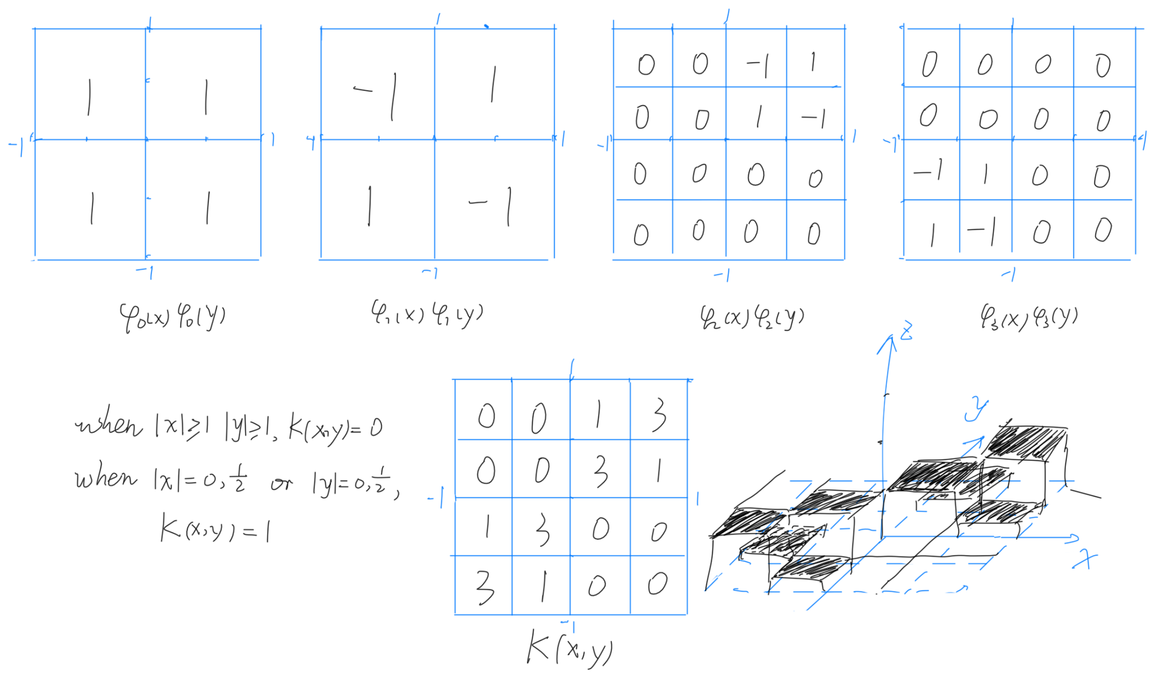
$$
\begin{aligned}
Pf &= \sum_{n=0}^{3} < f(x), \varphi_n(x) > \varphi_n(x) \\
&= \int_{-1}^{1} f(y)\overline{\varphi_0(y)}\mathrm{d}y \cdot \varphi_0(x) + \int_{-1}^{1} f(y)\overline{\varphi_1(y)}\mathrm{d}y \cdot \varphi_1(x) \\
&\quad + \int_{-1}^{1} f(y)\overline{\varphi_2(y)}\mathrm{d}y \cdot \varphi_2(x) + \int_{-1}^{1} f(y)\overline{\varphi_3(y)}\mathrm{d}y \cdot \varphi_3(x) \\
&= \int_{-1}^{1} \varphi_0(x)\varphi_0(y)f(y)\mathrm{d}y + \int_{-1}^{1} \varphi_1(x)\varphi_1(y)f(y)\mathrm{d}y \\
&\quad + \int_{-1}^{1} \varphi_2(x)\varphi_2(y)f(y)\mathrm{d}y + \int_{-1}^{1} \varphi_3(x)\varphi_3(y)f(y)\mathrm{d}y \\
&= \int_{-1}^{1} [1 + \varphi_1(x)\varphi_1(y) + \varphi_2(x)\varphi_2(y) + \varphi_3(x)\varphi_3(y)]f(y)\mathrm{d}y
\end{aligned}
$$

Solution (cont.)

$$= \int_{-1}^{1} K(x,y)f(y)\mathrm{d}y$$

(f) Sketch the kernel $K(x, y)$.

$$K(x,y) = 1 + \varphi_1(x)\varphi_1(y) + \varphi_2(x)\varphi_2(y) + \varphi_3(x)\varphi_3(y)$$



Question 6

Suppose $f \in L^2(0, 1)$ is differentiable and f is orthogonal to $g(x) = e^x + 1 - e$.

(a) Show that f' is orthogonal to $G(x) = e^x - 1 - (e - 1)x$.

$\because$ f is orthogonal to $g(x)$

$\therefore$ $< f, g >= 0$

$\therefore$

$$
\begin{aligned}
< f'(x), G(x) > &= \int_0^1 f'(x)\overline{G(x)}\mathrm{d}x \\
&= \int_0^1 G(x)\mathrm{d}f(x) \\
&= f(x)G(x)\Big|_0^1 - \int_0^1 f(x)\mathrm{d}G(x) \\
&= -\int_0^1 f(x)g(x)\mathrm{d}x \\
&= 0
\end{aligned}
$$

(b) Explain why.

It is because ∇ is linear operator and $< \nabla f, G >= -< f, \nabla G >= 0$.