STAT 30400 : Distribution Theory

Fall 2019

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Homework 6

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Solutions by

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STAT 30400, Homework 6

1. (10 pts) Let X denote a random variable distributed χ^2_k ($k \geq 1$). Find the moment generating function of X and the first two moments.

Since $X \sim \chi^2_k$, we have that

$$f_X(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x>0}$$

$$\begin{aligned} m(t) = \mathbb{E}e^{tX} &= \int_{\mathbb{R}} e^{tx} \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x>0} \mathrm{d}x \\ &= \int_0^\infty \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\left(\frac{1}{2}-t\right)x} \mathrm{d}x. \end{aligned}$$

If $t > \frac{1}{2}$, $x^{\frac{k}{2}-1}e^{-\left(\frac{1}{2}-t\right)x} \to \infty$ as $x \to 0$, so the integral diverges. If $t = \frac{1}{2}$, $\int_0^\infty x^{\frac{k}{2}-1}\mathrm{d}x = \frac{2}{k}x^{\frac{k}{2}}\Big|_0^\infty$ also diverges for $k \geq 1$. If $t < \frac{1}{2}$, then

$$\begin{aligned} m(t) &= \int_0^\infty \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\left(\frac{1}{2}-t\right)x} \mathrm{d}x \\ &\overset{y=(1-2t)x}{=\!=\!=\!=} \int_0^\infty \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} (1-2t)^{-\frac{k}{2}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} \mathrm{d}y \\ &= (1-2t)^{-\frac{k}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} m^{(1)}(t) &= -\frac{k}{2}(1-2t)^{-\frac{k}{2}-1} \cdot (-2) = k(1-2t)^{-\frac{k}{2}-1} \\ \mathbb{E}X &= m^{(1)}(0) = k \\ m^{(2)}(t) &= k\left(-\frac{k}{2}-1\right)(1-2t)^{-\frac{k}{2}-2} \cdot (-2) = k(k+2)(1-2t)^{-\frac{k}{2}-2} \\ \mathbb{E}X^2 &= m^{(2)}(0) = k(k+2). \end{aligned}$$

2. (10 pts) Let X denote a discrete random variable with probability function

$$p(x) = \theta(1-\theta)^x, \qquad x = 0, 1, 2, \ldots$$

where $0 < \theta < 1$. Find the moment generating function of X and the first three moments.

$$\begin{aligned}
m(t) = \mathbb{E}e^{tX} &= \sum_{x=0}^{\infty} e^{tx}\theta(1-\theta)^x \\
&= \theta \sum_{x=0}^{\infty} [(1-\theta)e^t]^x \\
&= \begin{cases} \frac{\theta}{1-(1-\theta)e^t} & \text{, if } (1-\theta)e^t < 1 \text{, i.e. } t < -\log(1-\theta) \\ \infty & \text{, otherwise} \end{cases} \\
m^{(1)}(t) &= \frac{\theta(1-\theta)e^t}{[1-(1-\theta)e^t]^2} = \frac{\theta}{[1-(1-\theta)e^t]^2} - m(t) \\
m^{(2)}(t) &= \frac{2\theta(1-\theta)e^t}{[1-(1-\theta)e^t]^3} - m^{(1)}(t) \\
m^{(3)}(t) &= \theta(1-\theta)\frac{e^t[1-(1-\theta)e^t]^3 + 2\theta(1-\theta)^2e^t3[1-(1-\theta)e^t]^2}{[1-(1-\theta)e^t]^6} - m^{(2)}(t) \\
&= \theta(1-\theta)e^t\frac{1-(1-\theta)e^t + 6\theta(1-\theta)^1}{[1-(1-\theta)e^t]^4} - m^{(2)}(t)
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}X = m^{(1)}(0) &= \frac{1-\theta}{\theta} \\
\mathbb{E}X^2 = m^{(2)}(0) &= \frac{(2-\theta)(1-\theta)}{\theta^2} \\
\mathbb{E}X^3 = m^{(3)}(0) &= \frac{(6-6\theta+\theta^2)(1-\theta)}{\theta^3}
\end{aligned}$$

3. (10 pts) Calculate the moment generating function for $|X|$, where X is a standard normal random variable, and use it to derive the mean and the variance of $|X|$. Optional: show that,

$$\mathbb{E}|X|^{2n+1} = 2^n n! \sqrt{\frac{2}{\pi}}.$$

Since

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \\ f_{|X|}(x) &= \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbb{1}_{x \geq 0}, \end{aligned}$$

we have the moment-generating function for $|X|$ is given by

$$\begin{aligned} m(t) &= \mathbb{E}e^{t|X|} \\ &= \int_{\mathbb{R}} e^{tx} \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbb{1}_{x\geq 0} \mathrm{d}x \\ &= \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} \mathrm{d}x \\ &\overset{y=x-t}{=\!=\!=} \int_{-t}^\infty \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} e^{\frac{t^2}{2}} \mathrm{d}y \\ &= 2[1 - F_X(-t)] e^{\frac{t^2}{2}}, \qquad t \in \mathbb{R} \end{aligned}$$

for $t \in \mathbb{R}$. Since

$$\begin{aligned} m^{(1)}(t) &= 2f_X(-t)e^{\frac{t^2}{2}} + 2[1-F_X(-t)]te^{\frac{t^2}{2}} = \sqrt{\frac{2}{\pi}} + 2[1-F_X(-t)]te^{\frac{t^2}{2}} \\ m^{(2)}(t) &= 2[1-F_X(-t)]e^{\frac{t^2}{2}} + tm^{(1)}(t) \\ m^{(3)}(t) &= m^{(1)}(t) + [m^{(1)}(t) + tm^{(2)}(t)] = tm^{(2)}(t) + 2m^{(1)}(t) \\ m^{(4)}(t) &= tm^{(3)}(t) + 3m^{(2)}(t) \\ &\vdots \\ m^{(n+1)}(t) &= tm^{(n)}(t) + nm^{(n-1)}(t) \end{aligned}$$

we have

$$\begin{aligned} \mathbb{E}|X| &= m^{(1)}(0) = \sqrt{\frac{2}{\pi}} \\ \mathbb{E}|X|^2 &= m^{(2)}(0) = 1 \end{aligned}$$

So $Var|X| = \mathbb{E}|X|^2 - (\mathbb{E}|X|)^2 = 1 - \frac{2}{\pi}$.
Assuming that for $k < n$, $\mathbb{E}|X|^{2k+1} = 2^k k! \sqrt{\frac{2}{\pi}}$, then for $k = n$,

$$\begin{aligned} m^{(2n+1)}(0) &= [tm^{(2n)}(t) + (2n)m^{(2n-1)}(t)]|_{t=0} \\ &= 2n \cdot m^{(2n-1)}(0) \\ &= 2^n n! \sqrt{\frac{2}{\pi}}. \end{aligned}$$

By induction, $\mathbb{E}|X|^{2n+1} = 2^n n! \sqrt{\frac{2}{\pi}}$.

4. (20 pts) Let Y be a random variable with distribution function F and moment-generating function M that is finite on $|t| < R$, with $R > 0$ chosen to be as large as possible. Let

$$\beta = \inf\{M(t) : 0 \leq t < R\}.$$

Suppose there exists a unique real number $\tau \in (0, R)$ such that $M(\tau) = \beta$.

(a) Show that $\mathbb{P}(Y \geq 0) \leq \beta$.

Proof. Since for $t \in (0, R)$,

$$\begin{aligned}\mathbb{P}(Y \geq 0) &= \mathbb{P}(e^{tY} \leq e^{t\cdot 0}) \\ &\leq \mathbb{E}e^{tY} \\ &= M(t),\end{aligned}$$

we have

$$\mathbb{P}(Y \geq 0) \leq \inf_{t\in(0,R)} M(t) = \beta$$

□

(b) Show that $M'(\tau) = 0$.

Proof. Since $M(t)$ is infinite differentiable on $|t| < R$,

$$\begin{aligned}M(t) &= \mathbb{E}e^{tY} = \mathbb{E}\sum_{n=0}^{\infty}\frac{(tY)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n\mathbb{E}Y^n}{n!} \\ M'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}e^{tY} = \mathbb{E}\frac{\mathrm{d}}{\mathrm{d}t}e^{tY} = \mathbb{E}(Ye^{tY})\end{aligned}$$

and the minimum of $M(t)$ is obtained at $\tau \in (0, R)$, we have that $M'(\tau) = 0$ □

(c) Let, for any real x, $G(x) = \frac{1}{\beta}\int_{-\infty}^{x}\exp(\tau y)\mathrm{d}F(y)$. Show that G is a distribution function. If X is a random variable with distribution function G, find its moment-generating function.

Proof. Since $e^{\tau y}$ is non-negative for all $y \in \mathbb{R}$ $\beta > 0$, we have that for $x_1, x_2 \in \mathbb{R}$, $x_1 \leq x_2$, $G(x_1) \leq G(x_1) + \frac{1}{\beta}\int_{x_1}^{x_2} e^{\tau y}\mathrm{d}F(y) = G(x_2)$, i.e. G is non-decreasing.
Since G is an indefinite integral, it is continuous. Also,

$$\begin{aligned}\lim_{x\to+\infty} G(x) &= \frac{1}{\beta}\int_{\mathbb{R}} e^{\tau y}\mathrm{d}F(y) \\ &= \frac{1}{\beta}M(\tau) \\ &= 1 \\ \lim_{x\to-\infty} G(x) &= 0.\end{aligned}$$

Therefore, G is a distribution function.

Solution (cont.)

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\begin{aligned}
M_X(t) = \mathbb{E}e^{tX} &= \int_{\mathbb{R}} e^{tx} \mathrm{d}G(x) \\
&= \frac{1}{\beta} \int_{\mathbb{R}} e^{tx} e^{\tau x} \mathrm{d}F(x) \\
&= \frac{1}{\beta} M(t+\tau)
\end{aligned}
$$

□

(d) For X in (c), find $\mathbb{E}(X)$.

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\begin{aligned}
\mathbb{E}X &= M_X'(t)|_{t=0} \\
&= \frac{1}{\beta} M'(t+\tau)|_{t=0} \\
&= \frac{1}{\beta} M'(\tau) \\
&= 0
\end{aligned}
$$