1(a) $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \int_0^{\pi} e^{-x} \cos(kx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} e^{(ik-1)x} + e^{(-ik-1)x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{(ik-1)\pi} - 1}{ik-1} + \frac{e^{(-ik-1)\pi} - 1}{-ik-1} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+k^2} \Big[(e^{(ik-1)\pi} - 1)(-ik-1) + (e^{(-ik-1)\pi} - 1)(ik-1) \Big]$$

$$\boxed{\hat{f}(k) = \sqrt{\frac{2}{\pi}} \; \frac{1 - (-1)^k e^{-\pi}}{1+k^2}}$$

(b) Since $\cos(k\pi) = (-1)^k$, we evaluate

$$1 = e^{-0} = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{1-(-1)^k e^{-\pi}}{1+k^2}\, e^{ik\cdot 0}$$

$$= \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{1-(-1)^k e^{-\pi}}{1+k^2} \cos(k\cdot 0)$$

and

$$e^{-\pi} = \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{1-(-1)^k e^{-\pi}}{1+k^2} \cos(k\pi)$$

and subtract.

Since $\quad 1-(-1)^k = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$

this gives

$$1 - e^{-\pi} = \frac{2}{\pi} \sum_{\substack{-\infty \\ k\ \text{odd}}}^{\infty} \frac{1-(-1)^k e^{-\pi}}{1+k^2}$$

$$= \frac{4}{\pi} \sum_{k\ \text{odd}>0} \frac{1+e^{-\pi}}{1+k^2}$$

and

$$\sum_{k\geq 0}^{\infty} \frac{1}{1+(2k+1)^2} = \frac{\pi}{4}\, \frac{1-e^{-\pi}}{1+e^{-\pi}}$$

$$= \frac{\pi}{4} \tanh\left(\frac{\pi}{2}\right).$$

(c) Theorem: Suppose f is periodic and piecewise continuous, and has left and right derivatives at x. Then

$$S_N f(x) \to \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

Verification: $f(x) = e^{-|x|}$ is periodic (by periodic extension) and piecewise continuous (since it is continuous). Its derivatives exist everywhere and

$$f'(x) = -\operatorname{sgn}(x)\, e^{-|x|}$$

except at $x = 0$ and $|x| = \pi$ where left and right derivatives exist.

2(a) Since $\langle f_1, f_2 \rangle = 0$ already we need only normalize:

$$\boxed{\begin{aligned} \varphi_1 &= \frac{f_1}{\|f_1\|_2} = \frac{1}{\sqrt{2}} \\ \varphi_2 &= \frac{f_2}{\|f_2\|_2} = \frac{1}{\sqrt{2}}\, \mathrm{sgn}(x). \end{aligned}}$$

(b) $Pf = \langle f, \varphi_1 \rangle \varphi_1 + \langle f, \varphi_2 \rangle \varphi_2$

$$= \frac{1}{2}\int_{-1}^{1} e^x\,dx + \frac{1}{2}\int_{-1}^{1} e^x\,\mathrm{sgn}(x)\,dx\ \mathrm{sgn}(x)$$

$$= \frac{1}{2}\left(e^1 - e^{-1}\right) + \frac{1}{2}\left[(e^1 - 1) - (1 - e^{-1})\right] \mathrm{sgn}(x)$$

$$\boxed{Pf(x) = \sinh(1) + (\cosh(1) - 1)\,\mathrm{sgn}(x)}$$