Stochastic Processes

Fall 2017

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Week 6

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Solutions by

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3.1

Is it true that

(a) $N(t) < n$ if and only if $S_n > t$?

$\because$

$$N(t) \geqslant n \quad \Longleftrightarrow \quad S_n \leqslant t$$

$\therefore$

$$N(t) < n \quad \Longleftrightarrow \quad S_n > t$$

(b) $N(t) \leqslant n$ if and only if $S_n \geqslant t$?

It is false. when $S_n < t$ and $S_{n+1} > t$, $N(t) = n \leqslant n$ but $S_n < t$.

(c) $N(t) > n$ if and only if $S_n < t$?

It is false. Since the complement of this statement, i.e. (b), is false.

3.2

In defining a renewal process we suppose that $F(\infty)$, the probability that an interarrival time is finite, equals 1. If $F(\infty) < 1$, then after each renewal there is a positive probability $1 - F(\infty)$ that there will be no further renewals. Argue that when $F(\infty) < 1$ the total number of renewals, call it $N(\infty)$, is such that $1 + N(\infty)$ has a geometric distribution with mean $\frac{1}{1-F(\infty)}$.

$\because$ the interarrival time of the renewal process is independent identically distributed

$$\mathbb{P}(T_n < \infty) = F(\infty)$$

$\therefore$

$$\begin{aligned}
\mathbb{P}(1 + N(\infty) = n) &= \mathbb{P}(T_1 < \infty, \cdots, T_{n-1} < \infty, T_n = \infty) \\
&= \mathbb{P}(T_n = \infty) \prod_{i=1}^{n-1} \mathbb{P}(T_i < \infty) \\
&= [1 - F(\infty)][F(\infty)]^{n-1} \\
\mathbb{E}[1 + N(\infty)] &= \sum_{n=1}^{\infty} n\mathbb{P}(1 + N(\infty) = n) \\
&= \sum_{n=1}^{\infty} n[1 - F(\infty)]F(\infty)^{n-1} \\
&= [1 - F(\infty)] \frac{\mathrm{d}}{\mathrm{d}F(\infty)} \sum_{n=1}^{\infty} [F(\infty)]^n \\
&= [1 - F(\infty)] \frac{\mathrm{d}}{\mathrm{d}F(\infty)} \frac{F(\infty)}{1 - F(\infty)} \\
&= \frac{1}{1 - F(\infty)}
\end{aligned}$$

$\therefore$ $1 + N(\infty)$ has a geometric distribution with mean $\dfrac{1}{1 - F(\infty)}$.