TOPIC. Densities. This section discusses densities. We begin with a review of the concept of a density of a random variable and the relation between densities and dfs. We then turn to the so-called change-of-variable formula, which (under appropriate conditions) gives the density of a transformation $Y = h(X)$ of a variable X having a density; we first consider the case where h is one-to-one, and then the case where h is many-to-one. Finally, we consider the generalization of the foregoing results to random vectors.

The density of a random variable. A real-valued random variable is said to ***have a density*** if there exists a function $f\colon \mathbb{R} \to [0, \infty)$ such that

$$P[\, X \in B \,] = \int_{x \in B} f(x)\, dx \tag{1}$$

for subsets B of $\mathbb{R}$; f is called the ***density*** of X. In a measure theory course, f and B would be required to be Borel measurable and the integral in (1) would be taken to be a Lebesgue integral. In this course, we will only consider the case where f is piecewise continuous, B is a simple set like an interval or a countable disjoint union of intervals, and the integral in (1) is taken to be a Riemann integral.

Formula (1) has the following heuristic interpretation. Let $x \in \mathbb{R}$ be a continuity point of f and let dx be a positive infinitesimal. Then

$$P[\, x \le X \le x + dx \,] = \int_x^{x+dx} f(\xi)\, d\xi = f(x)\, dx \tag{2}$$

because f is constant over the interval $[\, x, x + dx \,]$. Consequently

$$f(x) = \frac{P[\, x \le X \le x + dx \,]}{dx} = \frac{\text{mass in } [x, x + dx]}{\text{length of } [x, x + dx]} \tag{3}$$

is what a physicist would call the "density" of probability mass in the vicinity of the point x.

(1): $P[\, X \in B \,] = \int_{x \in B} f(x)\, dx$.

Note that (1) implies that

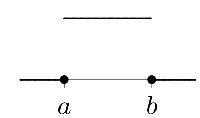
$$\int_{\mathbb{R}} f(x)\, dx = 1. \tag{4}$$

Sometimes a density f is specified only up to a multiplicative constant c, i.e., $f = cg$ for a known function g and some number c; since (4) says $c \int g(x)\, dx = 1$, one has

$$c = 1 \Big/ \int g(x)\, dx. \tag{5}$$

Example 1. (A) X is said to be ***uniformly distributed over the interval*** $\boldsymbol{(a, b)}$, written $\boldsymbol{X \sim}$ **Uniform**$\boldsymbol{(a, b)}$, if it has a density which is constant over (a, b) and equal to 0 outside that interval. According to (5) the constant value on (a, b) must be $c := 1/\int_a^b 1\, dx = 1/(b - a)$ so the density is

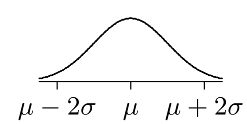
$$u(x; a, b) := \begin{cases} \dfrac{1}{b - a}, & \text{if } a < x < b \\ 0, & \text{if } x \le a \text{ or } x \ge b. \end{cases} \tag{6}$$



The "standard" case, where $a = 0$ and $b = 1$, played a prominent role in the preceding two sections.

(B) X is said to be ***normally distributed with parameters*** $\boldsymbol{\mu}$ ***and*** $\boldsymbol{\sigma^2 > 0}$, written $\boldsymbol{X \sim N(\mu, \sigma^2)}$, if X has density

$$n(x; \mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x - \mu)^2}{2\sigma^2}\Big) \tag{7}$$



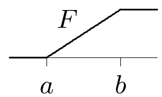
for $-\infty < x < \infty$. The "standard" case has $\mu = 0$ and $\sigma^2 = 1$. •

Densities and distribution functions. If X has a density, say f, then the df F of X is obtained by integration:

$$F(x) := P[\,X \le x\,] = P[\,X \in (-\infty, x]\,] = \int_{-\infty}^{x} f(\xi)\,d\xi \tag{8}$$

for $-\infty < x < \infty$. For example if $X \sim \mathrm{Uniform}(a, b)$, then

$$F(x) = \int_{-\infty}^{x} u(\xi; a, b)\,d\xi = \begin{cases} 0, & \text{if } x \le a, \\ \dfrac{x-a}{b-a}, & \text{if } a < x < b, \\ 1, & \text{if } b \le x. \end{cases}$$



Note that this F is not differentiable at a and at b, although it is continuous at those points; moreover F is continuously differentiable over the three intervals $(-\infty, a)$, (a, b) and (b, ∞). This example motivates the following theorem.

Theorem 1. *Let X be a random variable with df F and let x_1, x_2, ... be a set of isolated points in $\mathbb{R}$. The following two conditions are equivalent:*

D1 *X has a density which is continuous except at the x_i's;*

D2 *F is continuous at the x_i's and continuously differentiable on the intervals between them;*

and imply that

D3 *the function*

$$f(x) := \begin{cases} F'(x), & \textit{if } x \textit{ is not one of the } x_i\textit{'s}, \\ \textit{arbitrary}, & \textit{otherwise} \end{cases} \tag{9}$$

serves as the density of X.

The main point is that X will have a density if its df is continuous and piecewise continuously differentiable, and in that case the density is the derivative F' of F at the points where F' exists. We used this result in the preceding section, in deriving formula (2.10).

D1 X has a density which is continuous except at the x_i's.

D2 F is continuous at the x_i's and continuously differentiable on the intervals between them;

D3 X has density $f(x) := F'(x)$ except for $x = x_1, x_2, \ldots$.

Proof of Theorem 1. We give a partial proof of the theorem.

• D1 $\Longrightarrow$ D2: Suppose that X has density f and that f is continuous at x. Then

$$\lim_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \downarrow 0} \frac{1}{h} \int_{x}^{x+h} f(\xi)\,d\xi = f(x) \tag{10}$$

by the rule for differentiating an integral with respect to its upper limit of integration; this shows that F is differentiable from the right at x, with right-hand derivative $f(x)$.

• D2 $\Longrightarrow$ D1 *and* D3: Suppose that F is continuously differentiable on $\mathbb{R}$. Then $F'(x) \ge 0$ for all x, since F is nondecreasing. Moreover by the fundamental theorem of calculus

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

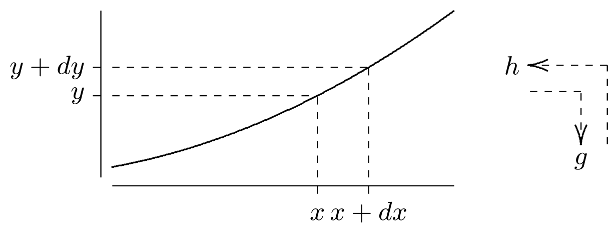
for all $a < b$. This says that

$$P[\,X \in B\,] = \int_{x \in B} F'(x)\,dx \tag{11}$$

for sets B of the form $(a, b]$; it follows by taking sums and limits that (11) holds for all subsets B of interest and hence that F' serves as a density for X. ∎

If X has a density, then $F(x)$ will necessarily be continuous in x. But beware — there are continuous distribution functions F for which $F'(x)$ exists for "Lebesgue almost all x" (that term being defined in measure theory) but $\int F'(x)\,dx < 1$. In such cases the rule "the derivative of the df serves as the density" fails, because the df is not smooth enough.

Transformation of variables. The arguments in this section are informal, but insightful. We'll prove the results rigorously later on. Let X be a random variable with density f_X. Suppose that h is a smooth increasing function defined on an interval in which X takes all its values. Put $Y = h(X)$. It is plausible that Y will have a density f_Y — we're starting with a smooth distribution of probability mass and transforming it smoothly, so we should end up with a smooth distribution of probability mass. To find f_Y, let a point y and the positive infinitesimal dy be given. Define x and dx as in the following picture:



In terms of the inverse g to h

$$\begin{aligned} x &= g(y) \\ x + dx &= g(y+dy) = g(y) + g'(y)\,dy = x + g'(y)\,dy. \end{aligned}$$

Since h is monotone,

$$y \le Y \le y + dy \iff x \le X \le x + dx\,. \tag{12}$$

Thus

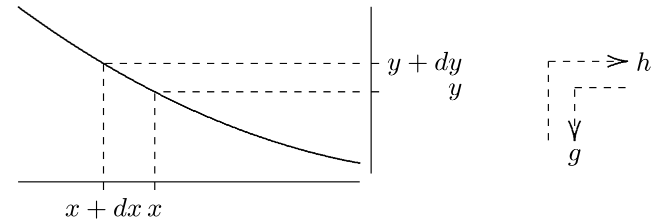
$$\begin{aligned} f_Y(y)\,dy &= P[\,y \le Y \le y+dy\,] && \text{(by (2) for } Y) \\ &= P[\,x \le X \le x + x\,] && \text{(by (12))} \\ &= f_X(x)\,dx\,; && \text{(by (2) for } X) \end{aligned}$$

dividing through by dy gives

$$f_Y(y) = f_X(x)\,\frac{dx}{dy} = f_X\big(g(y)\big)\,g'(y)\,. \tag{13}$$

$Y = h(X)$.   (13): h smooth, increasing $\Longrightarrow f_Y(y) = f_X(x)\,dx/dy$.

A similar analysis applies when the transformation h is decreasing, instead of increasing. The guiding picture in this case is



Note that now dx is negative. From

$$\begin{aligned} f_Y(y)\,dy &= P[\,y \le Y \le y+dy\,] \\ &= P[\,x+dx \le X \le x\,] = f_X(x)\,|dx| \end{aligned}$$

we get

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X\big(g(y)\big)\,\big|g'(y)\big|\,. \tag{14}$$

(14) is called the ***change-of-variables formula for densities***; it holds when h is smooth and monotone — either everywhere increasing, as in (13), or everywhere decreasing.

Example 2. Suppose $X \sim N(\mu, \sigma^2)$ and $Y = h(X)$ for $h(x) = a + bx$ with $b \neq 0$. This h is smooth and monotone, so formula (14) applies. The inverse transformation is $x = g(y) = (y-a)/b$, with derivative $dx/dy = g'(y) = 1/b$. According to (14), Y should have density

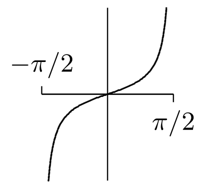
$$\begin{aligned} f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\frac{1}{|b|} \\ &= \frac{1}{\sqrt{2\pi b^2\sigma^2}}\exp\Big(-\frac{\big(y-(a+b\mu)\big)^2}{2b^2\sigma^2}\Big); \end{aligned} \tag{15}$$

thus $Y \sim N(\nu, \tau^2)$ for $\nu = a + b\mu$ and $\tau^2 = b^2\sigma^2$. •

X has a density. $\quad Y = h(X)$, $\quad h$ smooth and monotone.
(14): $f_Y(y) = f_X(x)\,|dx/dy|$ for $x = h^{-1}(y)$.

Example 3. Suppose $X \sim \text{Uniform}(-\pi/2, \pi/2)$, with density

$$f_X(x) = \begin{cases} 1/\pi, & \text{if } |x| < \pi/2 \\ 0, & \text{otherwise.} \end{cases}$$



$y = \tan(x)$

Consider $Y = h(X)$ for

$$h(x) = \tan(x).$$

This h is smooth and increasing over the range $(-\pi/2, \pi/2)$ of X, so (14) applies. The inverse transformation is

$$x = g(y) = \arctan(y)$$

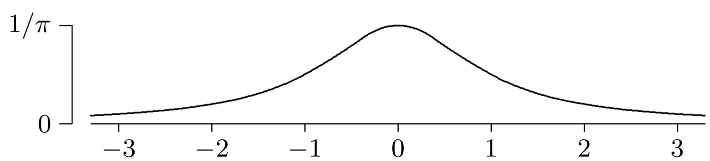
with derivative

$$\frac{dx}{dy} = g'(y) = \frac{1}{1+y^2}.$$

According to (14), Y should have density

$$f_Y(y) = f_X\big(g(y)\big)\,\big|g'(y)\big| = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\,\frac{1}{1+y^2} \tag{16}$$

for $-\infty < y < \infty$. This is the ***standard Cauchy density*** with graph

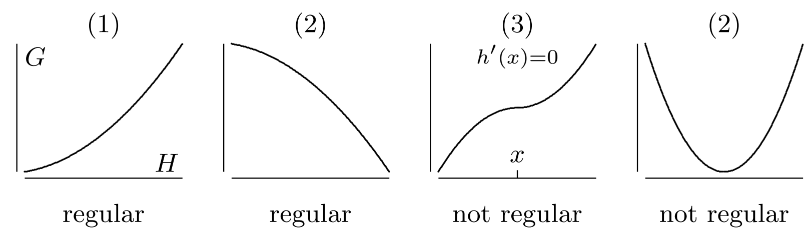


Note that the distribution has very heavy tails. •

Justification of the change-of-variables formula. We're going to use some results from calculus to justify the change-of-variables formula (14) for densities, and hence validate formulas (15) and (16). Let H and G be subsets of $\mathbb{R}$. A function h is said to be a ***regular transformation from H to G*** if

- R1 h is a one-to-one mapping of H onto G, and H and G are open intervals;
- R2 h is continuously differentiable throughout H; and
- R3 $h'(x) \neq 0$ for all $x \in H$.

Graphs of 4 mappings h from H to G



A regular h is necessarily continuous and strictly monotone. Its inverse $g = h^{-1}$ is itself regular as a mapping from G to H, and the derivatives g' and h' are related by the identity

$$h'(x)g'(y) = 1 \text{ if } y = h(x) \text{ or, equivalently, } x = g(y). \tag{17}$$

Moreover for any function $\phi\colon H \to [0,\infty)$ and any $B \subset G$, one has

$$\int_{x \in h^{-1}(B)} \phi(x)\,dx = \int_{y \in B} \phi\big(g(y)\big)\big|g'(y)\big|\,dy\,. \tag{18}$$

This is called the ***change-of-variables formula for integrals***. Note how the right-hand side can be obtained in a purely formal manner from the left-hand side by making the substitutions $x = g(y)$ and $dx = \big|g'(y)\big|\,dy$.

(18): In the regular case, $\int_{x\in h^{-1}(B)} \phi(x)\,dx = \int_{y\in B} \phi\big(g(y)\big)\big|g'(y)\big|\,dy$.

Theorem 2 (The change-of-variables formula for densities; the regular case). *Let h be a regular transformation from H to G and let $g = h^{-1}$ be the inverse to h. Let X be a random variable that has a density f_X and takes all its values in H. Put $Y = h(X)$. Then Y has density f_Y given by*

$$f_Y(y) = \begin{cases} f_X\big(g(y)\big)\,\big|g'(y)\big|, & \text{if } y \in G, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Proof Let f_Y be defined by (19). $f_Y(y)$ is clearly nonnegative for each $y \in \mathbb{R}$. We need to show that

$$P[\,Y \in B\,] = \int_B f_Y(y)\,dy \tag{20}$$

for subsets B of $\mathbb{R}$. If $B \subset G$ then

$$\begin{aligned} P[\,Y \in B\,] &= P[\,h(X) \in B\,] \\ &= P[\,X \in h^{-1}(B)\,] = \int_{h^{-1}(B)} f_X(x)\,dx \\ &= \int_B f_X\big(g(y)\big)\big|g'(y)\big|\,dy = \int_B f_Y(y)\,dy, \end{aligned} \tag{21}$$

the next to last step holding by (18). Moreover, if $B \subset G^c$ we have

$$P[\,Y \in B\,] = P[\emptyset\,] = 0 = \int_B 0\,dy = \int_B f_Y(y)\,dy. \tag{22}$$

Together (21) and (22) imply (20), since for any $B \subset \mathbb{R}$

$$\begin{aligned} P[\,Y \in B\,] &= P[\,Y \in B \cap G\,] + P[\,Y \in B \cap G^c\,] \\ &= \int_{B\cap G} f_Y(y)\,dy + \int_{B\cap G^c} f_Y(y)\,dy = \int_B f_Y(y)\,dy. \end{aligned}$$ ∎

Many-to-one transformations. So far in our discussion of the change-of-variable formula for densities the transformation h carrying X to Y has been one-to-one. We now consider the situation where that is not the case. We begin with an informal discussion of a particular case.

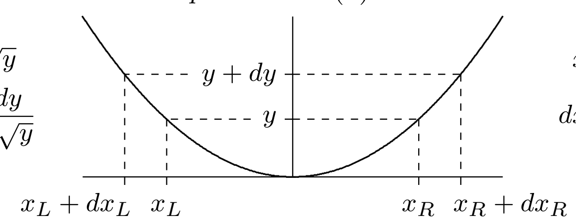
Example 4. Suppose $X \sim N(0,1)$, with density $f_X(x) = e^{-x^2/2}/\sqrt{2\pi}$ for $-\infty < x < \infty$. Put $Y = h(X)$ for $h(x) = x^2$; thus

$$Y = X^2. \tag{23}$$

The distribution of Y is called the ***Chisquare distribution with one degree of freedom***, denoted $\boldsymbol{\chi_1^2}$. Since we're making a smooth transformation of a smooth distribution, we expect Y to have a density f_Y. To find $f_Y(y)$, suppose first that $y > 0$. Let dy be a positive infinitesimal, and let x_L, dx_L, x_R, dx_R be defined as in the following picture:

Graph of $x \rightsquigarrow h(x) := x^2$

$$x_L = -\sqrt{y} \qquad dx_L = -\frac{dy}{2\sqrt{y}}$$



$$x_R = \sqrt{y} \qquad dx_R = \frac{dy}{2\sqrt{y}}$$

Now write

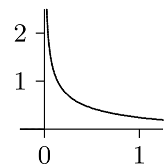
$$\begin{aligned} f_Y(y)\,dy &= P[\,y \le Y \le y + dy\,] \\ &= P[\,x_L + dx_L \le X \le x_L \ \text{ or } \ x_R \le X \le x_R + dx_R\,] \\ &= P[\,x_L + dx_L \le X \le x_L\,] + P[\,x_R \le X \le x_R + dx_R\,] \\ &= f_X(x_L)\,|dx_L| + f_X(x_R)\,dx_R \end{aligned}$$

and divide through by dy to get

$$f_Y(y) = f_X(x_L)\left|\frac{dx_L}{dy}\right| + f_X(x_R)\,\frac{dx_R}{dy} = \frac{1}{\sqrt{2\pi}}\,e^{-y/2}\,\frac{2}{2\sqrt{y}}\,.$$

However, if $y < 0$ then $f_Y(y) = P[\, y \le Y \le y + dy\,]/dy$ will be 0 since $Y \ge 0$. It thus appears that Y has density

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}}\, e^{-y/2}, & \text{if } y > 0, \\ 0, & \text{if } y < 0. \end{cases} \tag{24}$$



According to Theorem 3 below, these heuristics are correct: Y does indeed have a density and it is given by equation (24). •

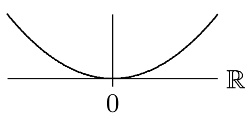
Let X be a real-valued random variable and let h be a mapping from $\mathbb{R}$ to $\mathbb{R}$. h is said to be ***X-essentially piecewise regular*** if

PR1 $\mathbb{R}$ can be written as the union of finitely or countably many disjoint subsets $H_0, H_1, \ldots$ such that

PR2 $P[\, X \in H_0\,] = 0$ and

PR3 for each $i \ge 1$, the restriction $h_i := h|H_i$ of h to H_i is a regular transformation from H_i to $G_i := h_i(H_i)$.

For example, for any random variable X having a density, the transformation $h\colon x \rightsquigarrow x^2$



is X-essentially regular. Indeed, the requirements above are satisfied for

$$H_0 = \{0\}, \;\; H_1 = (-\infty, 0), \;\text{ and }\; H_2 = (0, \infty)$$

because

the sets H_0, H_1, and H_2 are disjoint and $\mathbb{R} = H_0 \cup H_1 \cup H_2$,
$P[\, X \in H_0] = P[\, X = 0\,] = 0$,
$g_1\colon x \rightsquigarrow x^2$ is regular between H_1 and $G_1 := (0, \infty)$, and
$g_2\colon x \rightsquigarrow x^2$ is regular between H_2 and $G_2 := (0, \infty)$.

Note that G_1 and G_2 overlap (in fact, coincide); that is allowed.

PR1 $\mathbb{R}$ is the union of disjoint sets $H_0, H_1, \ldots$.

PR2 $P[\, X \in H_0\,] = 0$.

PR3 for each $i \ge 1$, the restriction $h_i := h|H_i$ of h to H_i is a regular transformation from H_i to $G_i := h_i(H_i)$.

Theorem 3 (The change-of-variable formula for densities; the piecewise regular case). *Let X be a random variable having density f_X. Let $Y = h(X)$ for a piecewise regular transformation $h\colon \mathbb{R} \to \mathbb{R}$. Let H_0 and H_i, h_i, and G_i for $i = 1, \ldots$ be as in* PR1, PR2, *and* PR3. *Then Y has a density f_Y given by*

$$f_Y(y) = \sum\nolimits_{i \ge 1} f_Y^{(i)}(y) \tag{25}$$

where

$$f_Y^{(i)}(y) = \begin{cases} f_X\big(g_i(y)\big)\big|g_i'(y)\big|, & \text{if } y \in G_i, \\ 0, & \textit{otherwise;} \end{cases} \tag{26}$$

here $g_i := h_i^{-1}\colon G_i \to H_i$ is the inverse to h_i.

Simply put, to get the density of Y in the piecewise regular case, you apply the formula for the regular case to each chunk H_i of $\mathbb{R}$ over which h is regular, and then sum over chunks.

Proof of Theorem 3. f_Y is clearly nonnegative. We need to show that

$$P[\, Y \in B\,] = \int_B f_Y(y)\, dy \tag{27}$$

for subsets B of $\mathbb{R}$. For this, note that

$$\begin{aligned} P[\, Y \in B\,] &= P\Big[\bigcup\nolimits_{i \ge 0} \{X \in H_i,\, Y \in B\}\Big] && (\text{since } \mathbb{R} = \textstyle\bigcup_i H_i) \\ &= \sum\nolimits_{i \ge 0} P[\, X \in H_i,\, Y \in B\,] && (H_i\text{'s are disjoint}) \\ &= \sum\nolimits_{i \ge 1} P[\, X \in H_i,\, Y \in B\,] && (P[\, X \in H_0\,] = 0) \end{aligned}$$

PR3 for each $i \geq 1$, the restriction $h_i := h|H_i$ of h to H_i is a regular transformation from H_i to $G_i := h_i(H_i)$.

$$f_Y(y) \underset{(25)}{=} \sum_{i\geq 1} f_Y^{(i)}(y). \qquad f_Y^{(i)}(y) \underset{(26)}{=} \begin{cases} f_X\big(g_i(y)\big)\big|g_i'(y)\big|, & \text{if } y \in G_i, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y = h(X). \qquad P[\,Y \in B\,] = \sum_{i\geq 1} P[\,X \in H_i,\, Y \in B\,].$$

Now put

$$B_i = B \cap G_i \tag{28}$$

and note that

$$\begin{aligned} &X \in H_i \text{ and } Y := h(X) \in B \\ &\quad\Longleftrightarrow X \in H_i \text{ and } h_i(X) \in B \quad (\text{since } h \text{ is } h_i \text{ on } H_i) \\ &\quad\Longleftrightarrow X \in H_i \text{ and } h_i(X) \in B_i \quad (\text{since } h_i \text{ takes values in } G_i) \\ &\quad\Longleftrightarrow X \in h_i^{-1}(B_i). \end{aligned}$$

Thus

$$\begin{aligned} P[\,X \in H_i,\, Y \in B\,] &= P[\,X \in h_i^{-1}(B_i)\,] = \int_{h_i^{-1}(B_i)} f_X(x)\,dx \\ &= \int_{B_i} f_X\big(g_i(y)\big)\,\big|g_i'(y)\big|\,dy \qquad (\text{by PR3 and (18)}) \\ &= \int_B f_Y^{(i)}(y)\,dy \qquad (\text{by (28)}). \end{aligned}$$

This gives

$$\begin{aligned} P[\,Y \in B\,] &= \sum_{i\geq 1} \int_B f_Y^{(i)}(y)\,dy \\ &= \int_B \sum_{i\geq 1} f_Y^{(i)}(y)\,dy = \int_B f_Y(y)\,dy. \end{aligned}$$

The interchange of $\sum$ and $\int$ here is allowed because the integrands are nonnegative; see (?.?). ∎

$$f_Y(y) \underset{(25)}{=} \sum_{i\geq 1} f_Y^{(i)}(y). \qquad f_Y^{(i)}(y) = \begin{cases} f_X\big(g_i(y)\big)\big|g_i'(y)\big|, & \text{if } y \in G_i, \\ 0, & \text{otherwise.} \end{cases}$$

Another way to write (25) is

$$f_Y(y) = \sum_{i\geq 1:\, y\in G_i} f_X\big(g_i(y)\big)\big|g_i'(y)\big|. \tag{29}$$

The right-hand side here is the expression we evaluated to get formula (24) for the density of the χ_1^2 distribution in Example 4, so we have now validated that formula.

Exercise 1. Let X_1, X_2, ... be independent random variables, each taking the values 0 and 1 with equal probabilities. Put $X = 2\sum_{n=1} X_n/3^n$ and let F be the df of X. (a) Show that X takes all its values in $[0,1]$. (b) Show that F has the value $1/2$ on $(1/3, 2/3)$. (c) Show that F has the value $1/4$ on $(1/9, 2/9)$ and the value $3/4$ on $(7/9, 8/9)$. (d) Show that F is continuous. (e) Show that there are countably many disjoint open subintervals G_1, G_2, ... of $[0,1]$ such that $\sum_i$ length of $G_i = 1$ and $F'(x) = 0$ for all $x \in \bigcup_i G_i$. (f) Deduce $\int F'(x)\,dx = 0$. (g) Why doesn't this contradict $\mathsf{D2} \Longrightarrow \mathsf{D3}$ in Theorem 1? $\diamond$

Exercise 2. Suppose X has a continuous density f_X and $Y = h(X)$ for a mapping $h\colon \mathbb{R} \to \mathbb{R}$ that has a continuous strictly-positive derivative h'. Prove the change-of-variables formula (13) for this situation by finding the df F_Y of Y and then using Theorem 1 to produce the density. $\diamond$

Exercise 3. Suppose $Y = \log\bigl(U/(1-U)\bigr)$ for $U \sim \text{Uniform}(0,1)$. Use the change-of-variables formula for densities to obtain the density of Y from the density of U, and thereby give a new derivation of (2.10). $\diamond$

Exercise 4. The ***Pareto distribution*** with location parameter k and shape parameter $\alpha > 0$ has density

$$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}} \ \text{ for } x \geq k. \tag{30}$$

Suppose X is a random variable with density (30). Use the change-of-variables formula to find the density of k/X. $\diamond$

Exercise 5. Suppose Y is a standard Cauchy random variable. (a) What are the first and third quartiles of Y? (b) Show that $P[\,Y \geq y\,] \sim 1/(\pi y)$ as $y \to \infty$. $\diamond$

Exercise 6. The density (24) for the χ_1^2 distribution explodes to ∞ as $y \downarrow 0$. What is the intuitive explanation for that? $\diamond$

Exercise 7. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the sine function: $h(x) = \sin(x)$ for $x \in \mathbb{R}$. Show that g is X-essentially piecewise regular for any random variable X having a density. Find the density of $h(X)$ when $X \sim N(0,1)$. $\diamond$

Exercise 8. Draw simultaneous graphs of the densities of the Tukey distribution $\mathfrak{T}(\lambda)$ for the following sets of values of the shape parameter λ: (a) $\lambda = 1/4$ to 1 by $1/4$; (b) $\lambda = 5/4$ to 2 by $1/4$; (c) $\lambda = 9/4$ to 4 by $1/4$; (d) $\lambda = 0$ to -2 by $-1/4$. For most of these λ's there is no closed form expression for the density; however, that should cause no difficulties in making the plots. $\diamond$