Homework 1 - COMM 671

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2022-09-20

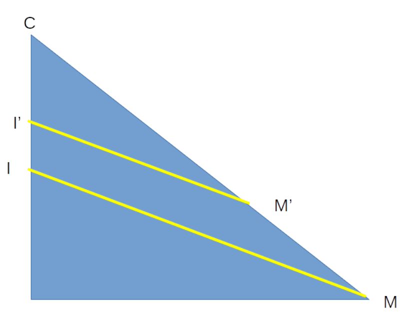
1 Show that Expected Utility Theory implies that indifference curves are parallel straight lines in the Machina triangle

I'll start by showing that EUT implies indifference curves are *straight* lines. Suppose we have a Machina Triangle like the one below and the agent is indifferent between points I and M. By the independence axiom, the agent must also be indifferent between $N = pM + (1-p)I$ and $J = pI + (1-p)M$ for any $p \in [0, 1]$. But then she is indifferent between any points along the straight line connecting I and M, so the indifference curve is a straight line.

To show they're parallel, consider

$$M' = pM + (1-p)C$$
$$I' = pI + (1-p)C$$

Since the agent is indifferent between M and I, the independence axiom implies they must also be indifferent between M' and I'. Since we used the same value p to construct the new compound lotteries, the indifference line $I'M'$ must be parallel to the line IM by similar triangles.



2 Roulette wheel with N outcomes

(a) $U(W) = \log(W)$, $\sum_n a_n = 1$.

$$U(W) = \sum_n p_i \log(a_i),$$
$$L = \sum_n p_i \log(a_i) - \lambda \left(\sum_n a_i - 1 \right)$$
$$\frac{\partial L}{\partial a_i} = \frac{p_i}{a_i} - \lambda = 0$$
$$\implies a_i = \frac{p_i}{\lambda}$$
$$\text{Set } \lambda = 1 \implies a_i = p_i$$

So the prizes should be set to have the same payout as the probabilities.
Certainty equivalent: $\sum_n p_i \log(p_i)$
(b) $U(W) = \frac{1}{1-\gamma} W^{1-\gamma}, 0 < \gamma$

$$
\begin{aligned}
U(W) &= \frac{1}{1-\gamma}\left(\sum_n p_i a_i\right)^{1-\gamma} \\
L &= \frac{1}{1-\gamma}\left(\sum_n p_i a_i\right)^{1-\gamma} - \lambda\left(\sum_n a_i - 1\right) \\
\frac{\partial L}{\partial a_i} &= p_i (a_i p_i)^{-\gamma} - \lambda = 0 \\
\implies a_i &= \frac{p_i^{1/\gamma - 1}}{\lambda^{1/\gamma}}
\end{aligned}
$$

So the prizes should be set to have payouts proportional to the probabilities to the power of $1/\gamma - 1$.
Certainty equivalent: $\frac{1}{1-\gamma}(\sum_n p_i^2)^{1-\gamma}$
(c) $U(W) = -e^{-bW}$

$$
\begin{aligned}
U(W) &= \sum_n -e^{-bp_i a_i} \\
L &= \sum_n \left(-e^{-bp_i a_i}\right) - \lambda\left(\sum_n a_i - 1\right) \\
\frac{\partial L}{\partial a_i} &= a_i p_i b e^{-bp_i a_i} - \lambda = 0 \\
\frac{\partial L}{\partial a_j} &= a_j p_j b e^{-bp_j a_j} - \lambda = 0 \\
\implies \frac{a_i}{a_j} &= \frac{p_i e^{-bp_i a_i}}{p_j e^{-bp_j a_j}}
\end{aligned}
$$

so the prizes should be proportional to the probabilities.
Certainty equivalent: $\sum_n -e^{-bp_i^2}$

3 Two lotteries

(a) $U(W) = -\frac{1}{W}$
Expected utility from lottery 1:

$$E[U(W)] = 0.5U(10) + 0.5U(50) = 0.5\left(-\frac{1}{10}\right) + 0.5\left(-\frac{1}{50}\right) = -0.06$$

Expected utility from lottery 2:

$$E[U(W)] = U(X) = -\frac{1}{X} = -0.06 \implies X = \frac{50}{3}$$

(b) $U(W) = \log(W)$
Expected utility from lottery 1:

$$E[U(W)] = 0.5U(10) + 0.5U(50) = 0.5\log 10 + 0.5\log 50 \approx 3.107$$

Expected utility from lottery 2:

$$E[U(W)] = U(X) = \log X = 1.349 \implies X = e^{3.107} \approx 22.35$$

(c) $U(W) = \frac{1}{1-\gamma} W^{1-\gamma}, \gamma = 0.25; 0.75$
Expected utility from lottery 1:

$$E[U(W)] = 0.5U(10) + 0.5U(50) = 0.5\frac{1}{1-\gamma}10^{1-\gamma} + 0.5\frac{1}{1-\gamma}50^{1-\gamma} \approx 16.28, \gamma = 0.25; 8.87, \gamma = 0.75$$

Expected utility from lottery 2:

$$E[U(W)] = U(X) = \frac{1}{1-\gamma}X^{1-\gamma} = EU(\text{lottery } 1) \implies X = 28.12, \gamma = 0.25; 24.23, \gamma = 0.75$$

Increasing γ means that the certainty equivalent decreases which implies that higher γ is equivalent to higher risk aversion.

4 Proof of Theorem

Theorem: Suppose that utility functions u_1 and u_2 are twice continuously differentiable with continuous second derivatives and strictly increasing. The following are equivalent: 1) $A_1(y) \geq A_2(y)$ for every y. 2) $\pi_1(y, \tilde{z}) \geq \pi_2(y, \tilde{z})$ for every y and every random variable $\tilde{z}$. 3) u_1 is a concave transformation of u_2; that is, $u_1 = f \circ u_2$ for f concave and strictly increasing. 4) u_1 is at least as risk averse as u_2.

$1 \implies 3$

Suppose $A_1(y) \geq A_2(y)$ for every y. Suppose $u_1 = f \circ u_2 = u_1(u_2^{-1}(t))$. Then $\frac{d}{dt}u_1(u_2^{-1}(t)) = \frac{u_1'(u_2^{-1}(t))}{u_2'(u_2^{-1}(t))}$ which is decreasing (concave) iff $\log u_1'(x)/u_2'(x)$ is, which follows from $A_1(y) \geq A_2(y)$.

$3 \implies 2$

Suppose $u_1 = f \circ u_2$ for f concave and strictly increasing. $\pi_1(y, \tilde{z}) = y + E(\tilde{z}) - u_i^{-1}(E[u_i(y + \tilde{z})$. Then

$$\begin{aligned}
\pi_1(y, \tilde{z}) - \pi_2(y, \tilde{z}) &= u_2^{-1}(E[u_2(y + \tilde{z})]) - u_1^{-1}(E[u_1(y + \tilde{z})]) \\
&= u_2^{-1}(E[u_2(y + \tilde{z})]) - u_1^{-1}(E[u_1(u_2^{-1}(u_2(y + \tilde{z})))])
\end{aligned}$$

Since $u_1(u_2^{-1}(\cdot))$ is concave, $E[u_1(u_2^{-1}(u_2(y + \tilde{z})))] \leq u_1(u_2^{-1}(E[u_2(y + \tilde{z})]))$ so $\pi_1(y, \tilde{z}) \geq \pi_2(y, \tilde{z})$.

$2 \implies 1$ (actually not $1 \implies$ not 2)

If 1 does not hold, then 1 holds with A_1 and A_2 interchanged. Then 3 and 2 also hold with A_1 and A_2 interchanged, and therefore do not hold in the correct ordering. This completes the proof.

5 Mean variance does not satisfy independence

Suppose I have two lotteries L_1 and L_2. L_1 has high mean but also high variance, while L_2 has high mean and smaller variance, so L_2 is preferred to L_1. Then consider L_3 which has a high variance and is negatively correlated with L_1 but positively correlated with L_2. Then the combination lottery of L_1 and L_3 will be preferred to the combination of L_2 and L_3, which violates the independence axiom.