Problem Set 1

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1 Let A and B be symmetric positive definite $l \times l$ matrices

(a) Since A and B are symmetric positive definite $A^{1/2}$ and $B^{1/2}$ exist and are invertible and symmetric. ($\implies$) $(I - A)$ p.d. Consider an arbitrary non-zero l-vector x. Then

$$\begin{aligned} x'(A^{-1} - I)x &= x'A^{-1/2}(I - A)A^{-1/2}x \\ &> 0 \end{aligned}$$

because $A^{-1/2}x \neq 0$ and $(I - A) > 0$. Then $(A^{-1} - I)$ is positive definite.

($\impliedby$) $(A^{-1} - I)$ p.d. Then for non-zero l-vector x,

$$\begin{aligned} x'(I - A)x &= x'A^{1/2}(A^{-1} - I)A^{1/2}x \\ &> 0 \end{aligned}$$

because $A^{1/2}x \neq 0$ and $(A^{-1} - I)$ positive definite.

(b) ($\implies$) $(B - A)$ p.d. Then

$$(I - B^{-1/2}AB^{-1/2}) = B^{-1/2}(B - A)B^{-1/2}$$

From (a), if we say $C = B^{-1/2}AB^{-1/2}$, then if $I - C$ p.d. $C^{-1} - I$ also p.d. So then $(B^{1/2}A^{-1}B^{1/2} - I)$ positive definite and therefore

$$(A^{-1} - B^{-1}) = B^{-1/2}(B^{-1/2}A^{-1}B^{-1/2} - I)B^{-1/2}$$

is positive definite.

($\impliedby$) $(A^{-1} - B^{-1})$ p.d. Then

$$(I - A^{1/2}B^{-1}A^{1/2}) = A^{1/2}(A^{-1} - B^{-1})A^{1/2}$$

which is positive definite. From (a), $(B - A) = A^{1/2}(A^{-1/2}BA^{-1/2} - I)A^{1/2}$ is positive definite.

2 Prove (8) and (9) in Lecture 1

(8): $P(\beta_j \in CI_{n,j,1-\alpha}) \to 1-\alpha$

Note that $\sqrt{n}(\hat{\beta}_n(A_n) - \beta) \to_d N(0, V(A))$ and $\hat{V}_n(A) \to_p V(A)$. So then

$$\sqrt{n}(\hat{\beta}_{jn}(A_n) - \beta_j) \to_d N(0, V(A)_{jj})$$
$$[\hat{V}_n(A_n)]_{jj} \to_p [V(A)]_{jj}$$

Then by CLT and general Slutsky,

$$Z_{n,j} = \frac{\sqrt{n}}{\sqrt{[\hat{V}_n(A_n)]_{jj}}}(\hat{\beta}_{jn}(A_n) - \beta_j) \to_d N(0,1)$$

$\implies P(Z_{n,j} < z) = \Phi(z)$ as $n \to \infty$

$\implies P(|Z_{n,j}| < z_{1-\frac{\alpha}{2}}) = \Phi(z_{1-\frac{\alpha}{2}}) - \Phi(z_{\frac{\alpha}{2}}) = \Phi(z_{1-\alpha}) = 1-\alpha$

(9): $W_n \to_d \chi^2_k$

$$W_n = (\frac{\sqrt{n}}{\sqrt{\hat{V}_n(A_n)}}(\hat{\beta}_n(A_n) - \beta))'(\frac{\sqrt{n}}{\sqrt{\hat{V}_n(A_n)}}(\hat{\beta}_n(A_n) - \beta))$$

$\sqrt{n}(\hat{\beta}_n(A_n) - \beta) \to_d N(0, V(A))$ and $\hat{V}_n(A_n) \to_p V(A)$.

Then by general Slutsky,

$$\frac{\sqrt{n}}{\sqrt{\hat{V}_n(A_n)}}(\hat{\beta}_n(A_n) - \beta) \to_d N(0, I_k)$$
$$\implies W_n \to_d \chi^2_k$$

3 Show that $||R_{2,n}|| \to_p 0$

Done in lecture

4 Testing overidentifying restrictions

(a) Let $\Omega^{-1} = CC'$

$$\begin{aligned}
J_n(\hat{\beta}_n) &= n\bar{g}_n(\hat{\beta}_n)'\hat{\Omega}_n^{-1}\bar{g}_n(\hat{\beta}_n) \\
&= n\bar{g}_n(\hat{\beta}_n)'(CC^{-1})\hat{\Omega}_n^{-1}(C'^{-1}C')\bar{g}_n(\hat{\beta}_n) \\
&= n(C'\bar{g}_n(\hat{\beta}_n))'(C'\hat{\Omega}_nC)^{-1}(C'\bar{g}_n(\hat{\beta}_n))
\end{aligned}$$

(b)

$$
\begin{aligned}
C'\bar{g}_n(\hat{\beta}_n) &= C'\frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i'\hat{\beta}_n)Z_i \\
&= C'\frac{1}{n}\sum_{i=1}^{n}X_i'(\beta - \hat{\beta}_n)Z_i + C'\frac{1}{n}\sum_{i=1}^{n}u_iZ_i \\
C'\bar{g}_n(\beta) &= C'\frac{1}{n}\sum_{i=1}^{n}Z_i(Y_i - X_i'\beta) \\
&= C'\frac{1}{n}\sum_{i=1}^{n}u_iZ_i \\
\implies C'\bar{g}_n(\hat{\beta}_n) &= C'\frac{1}{n}\sum_{i=1}^{n}Z_iX_i'(\beta - \hat{\beta}_n) + C'\bar{g}_n(\beta) \\
&= C'\frac{1}{n}Z'X(\beta - \hat{\beta}_n) + C'\frac{1}{n}Z'u
\end{aligned}
$$

Since $\hat{\beta}_n = (X'Z\hat{\Omega}_n^{-1}Z'X)^{-1}X'Z\hat{\Omega}_n^{-1}Z'Y = \beta + (X'Z\hat{\Omega}_n^{-1}Z'X)^{-1}X'Z\hat{\Omega}_n^{-1}Z'u$, this implies

$$
\begin{aligned}
C'\bar{g}_n(\hat{\beta}_n) &= -\frac{1}{n}C'Z'X(X'Z\hat{\Omega}_n^{-1}Z'X)^{-1}X'Z\hat{\Omega}_n^{-1}Z'u + \frac{1}{n}C'Z'u \\
&= [I - C'Z'X(X'Z\hat{\Omega}_n^{-1}Z'X)^{-1}X'Z\hat{\Omega}_n^{-1}C'^{-1}]\frac{1}{n}C'Zu \\
&= D_nC'\bar{g}_n(\beta)
\end{aligned}
$$

(c)

$$
\begin{aligned}
D_n &= I - C'(\frac{Z'X}{n})(\frac{X'Z}{n}\hat{\Omega}^{-1}\frac{Z'X}{n})^{-1}(\frac{X'Z}{n})\hat{\Omega}_n^{-1}C'^{-1} \\
&= I - C'(\frac{Z'X}{n})(\frac{X'Z}{n}C'C\frac{Z'X}{n})^{-1}(\frac{X'Z}{n})\hat{\Omega}_n^{-1}C'^{-1}
\end{aligned}
$$

By LLN, $\frac{Z'X}{n} \to_p EZ_iX_i'$. Since $\hat{\Omega}_n \to_p \Omega, \hat{\Omega}_n^{-1} \to_p CC'$

$$
\begin{aligned}
\implies D_n &\to_p I - C'EZ_iX_i'(EX_iZ_i'C'CEZ_iX_i')^{-1}EX_iZ_i'C \\
&= I - R(R'R)^{-1}R'
\end{aligned}
$$

(d)
$\sqrt{n}C'\bar{g}_n(\beta) = C'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_iu_i$. By CLT, $\to_d N(0, Var(Z_iu_i)) = N(0,\Omega)$.
Note $C' = \Omega^{-1/2}, Z_i : l \times l$. Therefore $C'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_iu_i \to_d N(0, I_l)$
(e)

$$
\begin{aligned}
J_n(\hat{\beta}_n) &= n(C'\bar{g}_n(\hat{\beta}_n))'(C'\hat{\Omega}_n C)^{-1}(C'\bar{g}_n(\hat{\beta}_n)) \\
&= n(D_n C'\bar{g}_n(\beta))'(C'\hat{\Omega}_n C)^{-1}(D_n C'\bar{g}_n(\beta)) \\
&= (D_n\sqrt{n}C'\bar{g}_n(\beta))'(C'\hat{\Omega}_n C)^{-1}(D_n\sqrt{n}C'\bar{g}_n(\beta))
\end{aligned}
$$

From previous result, $D_n \to_p I - R(R'R)^{-1}R', \sqrt{n}C'\bar{g}_n(\beta) \to_d N(0, I_l), \hat{\Omega}_n \to_p \hat{\Omega}$

So we have $\sqrt{n}(\hat{\beta}_n) \to_d N'(I_l - R(R'R)^{-1}R')'(I_l - R(R'R)^{-1}R')N$

(f)

$R = C'EZ_iX_i'$ is an $l \times k$ matrix, so rank(R)=k, rank$(R(R'R)^{-1}R') = k$, so rank$(I_l - R(R'R)^{-1}R') = l - k$

(g)

$\sqrt{n}(\hat{\beta}_n) \to_d N'(I_l - R(R'R)^{-1}R')N \sim \chi^2$ rank$(I_l - R(R'R)^{-1}R') = \chi^2_{l-k}$