Problem Set 9

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April 2, 2022

1 Overidentifying restrictions test for nonlinear GMM

(a) Find efficient weight matrix and asymptotic variance

The efficient weight matrix for nonlinear GMM is $A^{*\prime}A^* = Var\big(g(W_i,\theta_0)\big)^{-1} = \Sigma_0^{-1}$. The asymptotic variance of the GMM estimator is

$$(\Gamma_0' A' A \Gamma_0)^{-1}\Gamma_0' A' A \Sigma_0 A' A \Gamma_0 (\Gamma_0' A' A \Gamma_0)^{-1}$$

with the efficient weight matrix, this becomes

$$\begin{aligned} &= (\Gamma_0'\Sigma_0^{-1}\Gamma_0)^{-1}\Gamma_0'\Sigma_0^{-1}\Sigma_0\Sigma_0^{-1}\Gamma_0(\Gamma_0'\Sigma_0^{-1}\Gamma_0)^{-1} \\ &= (\Gamma_0'\Sigma_0^{-1}\Gamma_0)^{-1} \end{aligned}$$

(b) Show that when the model holds, $J_n \to_d \chi^2_{l-d}$

$J_n = nQ_n^*(\hat\theta_n^*)$, where $Q_n^*(\theta) = \big(n^{-1}\sum_{i=1}^n g(W_i,\theta)\big)' A_n^{*\prime} A_n^* \big(n^{-1}\sum_{i=1}^n g(W_i,\theta)\big)$. Let $S_0 \equiv E[g(W_i,\theta_0)g(W_i,\theta_0)']$. Using a mean value expansion,

$$\begin{aligned} n^{-1/2}\sum_{i=1}^n g(W_i,\hat\theta_n^*) &= n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) + n^{-1}\sum_{i=1}^n \frac{\partial g(W_i,\tilde\theta_n^*)}{\partial\theta'}\sqrt{n}(\hat\theta_n^* - \theta_0) \\ &= n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) + E\frac{\partial g(W_i,\theta_0)}{\partial\theta'}\sqrt{n}(\hat\theta_n^* - \theta_0) + o_P(1) \\ &= \big(I_l - \Gamma_0(\Gamma_0' S_0^{-1}\Gamma_0)^{-1}\Gamma_0' S_0^{-1}\big) n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) + o_P(1) \\ \implies A_n^* n^{-1/2}\sum_{i=1}^n g(W_i,\hat\theta_n^*) &= \big(S_0^{-1/2} - S_0^{-1/2}\Gamma_0(\Gamma_0' S_0^{-1}\Gamma_0)^{-1}\Gamma_0' S_0^{-1}\big) n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) + o_P(1) \\ &= \big(I_l - S_0^{-1/2}\Gamma_0(\Gamma_0' S_0^{-1}\Gamma_0)^{-1}\Gamma_0' S_0^{-1}\big) S_0^{-1/2} n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) + o_P(1) \end{aligned}$$

Then since $S_0^{-1/2}n^{-1/2}\sum_{i=1}^n g(W_i,\theta_0) \to_d N(0,I_l)$, if $H \equiv S_0^{-1/2}\Gamma_0$, $J_n = N(0,I_l)'\big(I_l - H(H'H)^{-1}H'\big)N(0,I_l) + o_P(1)$, with rank $l-d$.

(c) Find $\lim_{n\to\infty} P(J_n > \chi^2_{l-d,1-\alpha})$

Since $J_n \sim \chi^2_{l-d}$, $\lim_{n\to\infty} P(J_n > \chi^2_{l-d,1-\alpha}) = \alpha$

2 Efficient IVs for nonlinear GMM

(a) Find asymptotic variance of $\hat\theta_n^h$

The criterion function for this GMM is

$$Q_n(\theta) = \frac{1}{2}||A_n n^{-1}\sum_{i=1}^{n} g(W_i,\theta)h(Z_i)||^2$$

and

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = -(\Gamma_n' A' A \Gamma_n)^{-1}\Gamma_n' A' A\big(n^{-1/2}\sum_{i=1}^{n} g(W_i,\theta_0)h(Z_i)\big) + o_p(1)$$

Since by assumption,

$$n^{-1/2}\sum_{i=1}^{n} g(W_i,\theta_0)h(Z_i) \to_d N(0,\Omega_n)$$

it follows that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, (\Gamma_n' A' A \Gamma_n)^{-1}\Gamma_n' A' A \Omega_n A' A \Gamma_n (\Gamma_n' A' A \Gamma_n)^{-1})$$

then if we choose $A'A = \Omega_n^{-1}$,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, (\Gamma_n' \Omega_n^{-1}\Gamma_n)^{-1})$$

(b) Find asymptotic variance of $\hat{\theta}_n^*$

$\hat{\theta}_n^*$ is the GMM estimator corresponding to the unconditional moment restriction using $h_*(Z_i)$ as the instrument: $E(g(W_i,\theta_0)h_*(Z_i)) = 0$. Let Γ_n^* be the Γ_n moment associated with this instrument. Then

$$\begin{aligned}
\Gamma_n^* &= \frac{\partial}{\partial\theta'}E(g(W_i,\theta_0)h_*(Z_i)) \\
&= \frac{\partial}{\partial\theta'}E\big[g(W_i,\theta_0)\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E\big(\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i\big)\big] \\
&= E\big[\frac{\partial}{\partial\theta'}g(W_i,\theta_0)\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E\big(\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i\big)\big] \\
&= E\big[\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E(\frac{\partial g(W_i,\theta_0)}{\partial\theta'}|Z_i)E(\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i)\big]
\end{aligned}$$

Similarly, let Ω_n^* be the Ω_n moment associated with the instrument:

$$\begin{aligned}
\Omega_n^* &= E\big[g^2(W_i,\theta_0)h_*(Z_i)h_*(Z_i)'\big] \\
&= E\big[E[g^2(W_i,\theta_0)|Z_i]h_*(Z_i)h_*(Z_i)'\big] \\
&= E\big[E[g^2(W_i,\theta_0)|Z_i]\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E(\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i)\big(\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E[\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i]\big)'\big] \\
&= E\big[\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E[\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i]E[\frac{\partial g(W_i,\theta_0)}{\partial\theta'}|Z_i]'\big] \\
&= E\big[\frac{1}{E(g^2(W_i,\theta_0)|Z_i)}E[\frac{\partial g(W_i,\theta_0)}{\partial\theta}|Z_i]E[\frac{\partial g(W_i,\theta_0)}{\partial\theta'}|Z_i]\big] \\
&= \Gamma_n^*
\end{aligned}$$

then the asymptotic variance of $\hat{\theta}_n^*$ becomes

$$\begin{aligned}
(\Gamma_n^{*\prime}\Omega_n^{*-1}\Gamma_n^*)^{-1} &= (\Gamma_n^{*\prime}\Gamma_n^{*-1}\Gamma_n^*)^{-1} \\
&= (\Gamma_n^{*\prime})^{-1}
\end{aligned}$$

(c) Show $h_*(Z_i)$ is the efficient instrument

$$V^h - V^* = (\Gamma_n'\Omega_n^{-1}\Gamma_n)^{-1} - (\Gamma_n^{*\prime})^{-1}$$

Now let there exist ϵ and e such that $E(\epsilon e') = E(ee')$ which implies $E(\epsilon\epsilon') - E(ee') \geq 0$. Now I want to find ϵ and e such that $E(ee') = V^*$and $E(\epsilon\epsilon') = V^h$.

Let $e \equiv V^* g(W_i, \theta_0) h_*(Z_i)$. Then

$$\begin{aligned} E[ee'] &= E[V^* g(W_i, \theta_0) h_*(Z_i) h_*(Z_i)' g(W_i, \theta_0)' V^{*\prime}] \\ &= V^* E[g^2(W_i, \theta_0) h_*(Z_i) h_*(Z_i)'] V^* \\ &= V^* V^{*-1} V^* \\ &= V^* \end{aligned}$$

Now let $\epsilon \equiv V^h \Gamma_n' \Omega_n^{-1} g(W_i, \theta_0) h(Z_i)$. Then

$$\begin{aligned} E[\epsilon\epsilon'] &= E[V^h \Gamma_n' \Omega_n^{-1} g(W_i, \theta_0) h(Z_i) h(Z_i)' g(W_i, \theta_0) \Omega_n^{-1} \Gamma_n V^h] \\ &= V^h \Gamma_n' \Omega_n^{-1} E[g^2(W_i, \theta_0) h(Z_i) h(Z_i)'] \Omega_n^{-1} \Gamma_n V^h \\ &= V^h \Gamma_n' \Omega_n^{-1} \Gamma_n V^h \\ &= V^h V^{h-1} V^h \\ &= V^h \end{aligned}$$

$$\begin{aligned} E[\epsilon e'] &= E[V^h \Gamma_n' \Omega_n^{-1} g(W_i, \theta_0) h(Z_i) h_*(Z_i)' g(W_i, \theta_0) V^*] \\ &= V^h \Gamma_n' \Omega_n^{-1} E[g^2(W_i, \theta_0) h(Z_i) h_*(Z_i)'] V^* \\ &= V^h \Gamma_n' \Omega_n^{-1} E\big[g^2(W_i, \theta_0) h(Z_i) \frac{1}{E[g^2(W_i, \theta_0)|Z_i} E[\frac{\partial g(W_i, \theta_0)}{\partial \theta} |Z_i]'\big] V^* \\ &= V^h \Gamma_n' \Omega_n^{-1} E\big[E[g^2(W_i, \theta_0)|Z_i] \frac{h(Z_i)}{E[g^2(W_i, \theta_0)|Z_i]} E[\frac{\partial g(W_i, \theta_0)}{\partial \theta} |Z_i]'\big] V^* \\ &= V^h \Gamma_n' \Omega_n^{-1} E\big[E[h(Z_i) \frac{\partial g(W_i, \theta_0)}{\partial \theta} |Z_i]'\big] V^* \\ &= V^h V^{h-1} V^* \\ &= V^* \\ &= E[ee'] \end{aligned}$$

Then $V^h - V^* \geq 0$

(d) Steps for constructing a feasible version of $\hat{\theta}_n^*$ when $h_*(Z_i)$ is unknown

First, create a consistent but inefficient estimator $\hat{\theta}_n$ using $h(Z_i)$ as the instrument. Since $h(Z_i)$ is known, it can be used. We can then estimate $E[g^2(W_i, \hat{\theta}_n)|Z_i]$ and $E[\frac{\partial g^2(W_i, \hat{\theta}_n)}{\partial \theta}|Z_i]$, which are all we need to construct $\hat{h}_*(Z_i)$.