Problem Set 8

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March 22, 2022

1 PwCo $\implies$ UnCo

(a)

$\forall x \in [a,b], f_n(x) \to_p f(x), f(x)$ monotone increasing and continuous on $[a,b]$. Let $\delta > 0$ such that $B(x,\delta) = (x+\delta, x-\delta) \cap [a,b]$ and let $y \in B(x,\delta)$. $y \in B(x,\delta) \implies |y-x| < \delta$, therefore we have

$$\begin{aligned} x-\delta \le y \le x+\delta \to & f_n(x-\delta) \le f_n(y) \le f_n(x+\delta) \\ & f(x-\delta) \le f(y) \le f(x+\delta) \end{aligned}$$

because f_n and f are monotonically increasing. Then

$$\begin{aligned} |f_n(y) - f(y)| &\le \max\{|f_n(x+\delta) - f(x-\delta)|, |f_n(x-\delta) - f(x+\delta)|\} \\ &\le |f_n(x+\delta) - f(x-\delta)| + |f_n(x-\delta) - f(x+\delta)| \end{aligned}$$

by the triangle inequality. Therefore

$$\sup |f_n(y) - f(y)| \le |f_n(x+\delta) - f(x-\delta)| + |f_n(x-\delta) - f(x+\delta)|$$

(b)

In (a), we showed that any $\delta-$ball in $[a,b]$ would satisfy the above inequality. The difference here is that we're adding a finite sequence of x's and taking the maximum over all those x's. Since from (a) this is true for any x_j, it's true for the maximum.

(c)

$|f_n(x_j+\delta) - f(x_j-\delta)| \le |f_n(x_j+\delta) - f(x_j+\delta)| + f(x_j+\delta) - f(x_j-\delta)|$

$f(x)$ is continuous and satisfies asymptotic uniform equicontinuity. We can pick $\delta > 0$ small enough so that for n large:

$$\sup_{x \in [a,b]} \sup_{x-\delta < x' < x+\delta} |f_n(x') - f_n(x)| < \epsilon/4$$

Because $f_n(x) \to_p f(x)$ we have:

$$\begin{aligned} & |f_n(x_j) - f(x_j)| < \epsilon/4J, \forall j = 1,2,...,J \\ \implies & \max_{j=1,2,\ldots,J} |f_n(x_j) - f(x_j)| < \epsilon/4 \end{aligned}$$

We can choose δ small enough to have $\max_{j=1,2,\ldots,J} |f_n(x_j) - f(x_j)| < \epsilon/4$. Therefore

$$\begin{aligned} & |f_n(x_j+\delta) - f(x_j-\delta)| \le |f_n(x_j+\delta) - f(x_j+\delta)| + |f(x_j+\delta) - f(x_j-\delta)| \\ \to & |f(x_j+\delta) - f(x_j-\delta) < \epsilon/4 \end{aligned}$$

(d)

From (c), $|f_n(x_j) - f(x_j)| < \epsilon/4J$. Since $f_n(x_j) \to_p f(x_j), f_n(x_j+\delta) \to_p f(x_j+\delta)$. Hence for n large, $|f_n(x_j+\delta) - f(x_j+\delta)| < \epsilon/4J$. Furthermore, we know that $x_n \to_p x$ iff $\forall \epsilon, \delta > 0, \exists N_\epsilon \in \mathbb{N}$ s.t. $P(|x_n - x| > \epsilon) < \delta, \forall n \ge N_\epsilon$. Pick $\delta = \epsilon/4J$ and

$$P(|f_n(x_j+\delta) - f(x_j+\delta)| > \epsilon/4J) < \epsilon/2J, \forall n \ge N_\epsilon$$

(e)

$$
\begin{aligned}
\max|f_n(x) - f(x)| &\leq \max_{j=1,2,\ldots,J}\{|f_n(x_j+\delta) - f(x_j-\delta)|\} + \max_j\{|f_n(x_j-\delta) - f(x_j+\delta)|\} \\
&\leq \max_j\{|f_n(x_j+\delta) - f(x_j+\delta)|\} + \max_j\{|f(x_j+\delta) - f(x_j-\delta)|\} + \max_j\{|f_n(x_j-\delta) - f(x_j-\delta)|\} + \max_j
\end{aligned}
$$

from (c),

$$
\begin{aligned}
\sup_{x\in[a,b]}|f_n(x) - f(x)| &\leq \max_j\{|f_n(x_j+\delta) - f(x_j+\delta)|\} + \max_j\{|f_n(x_j-\delta) - f(x_j-\delta)|\} + \epsilon/2 \\
&\leq \sum_{j=1}^{J}|f_n(x_j+\delta) - f(x_j+\delta)| + \sum_{j=1}^{J}|f_n(x_j-\delta) - f(x_j-\delta)| + \epsilon/2
\end{aligned}
$$

Now we can compute probabilities:

$$
\begin{aligned}
P(\sup_{x\in[a,b]}|f_n(x) - f(x)| > \epsilon) &\leq P(\sum_j|f_n(x_j+\delta) - f(x_j+\delta)| + \sum_j|f_n(x_j-\delta) - f(x_j-\delta)| + \epsilon/2) > \epsilon \\
&\leq P(\sum_j|f_n(x_j+\delta) - f(x_j+\delta)| + \sum_j|f_n(x_j-\delta) - f(x_j-\delta)|) > \epsilon/2 \\
&\leq P(\sum_j|f_n(x_j+\delta) - f(x_j+\delta)| > \epsilon/4) + P(\sum_j|f_n(x_j-\delta) - f(x_j-\delta)| > \epsilon/4) \\
&\leq \sum_j P(|f_n(x_j+\delta) - f(x_j+\delta)| > \epsilon/4J) + \sum_j P(|f_n(x_j-\delta) - f(x_j-\delta)| > \epsilon/4J \\
&\leq \sum_j \epsilon/2J + \sum_j \epsilon/2J = \epsilon
\end{aligned}
$$

2 Uniform convergence of empirical CDFs

(a)

Since F_n and F are CDFs, they are monotonically increasing and by the weak law of large numbers, $F_n(x) \to_p F(x)$. So from question 1,

$$
\sup_{x\in[a,b]}|F_n(x) - F(x)| \to_p 0
$$

for any interval $[a,b]$.

(b)

Pick $a, b \in \mathbb{R}$ such that $F(a) < \epsilon/8$ and $1 - F(b) < \epsilon/8$. For $\delta > 0, \exists N_\epsilon \in \mathbb{N}$ such that $P(F_n(a) < \epsilon/6) < \delta/3, n \leq N_\epsilon$

$$
\begin{aligned}
P(\sup_{x\in\mathbb{R}}|F_n(x) - F(x)| > \epsilon) &\leq P(|F_n(a)| + |F(a)| + \sup_{x\in[a,b]}|F_n(x) - F(x)| + |1 - F_n(b)| + |1 - F(b)| > \epsilon) \\
&\leq P(|F_n(a)| + F(a) > \epsilon/3) + P(\sup_{x\in[a,b]}|F_n(x) - F(x)| > \epsilon/3) + P(1 - F_n(b) + 1 - F(b) > \epsilon/3) \\
&< \delta/3 + \delta/3 + \delta/3 \\
&= \delta
\end{aligned}
$$