Abdulah Abrahim Problem Set 1 Due: Feb. 18, 2025

(1) The Poisson distribution has probability mass function

$$p(y_i|\theta) = \frac{\theta^{y_i} e^{-\theta}}{y_i!}, \qquad \theta > 0, \qquad y_i = 0, 1, \ldots \tag{1}$$

and let $y_1, \ldots, y_n$ be random sample from this distribution.

1. Show that the gamma distribution $\mathcal{G}(\alpha, \beta)$ is a conjugate prior distribution for the Poisson distribution.

2. Show that $\bar{y}$ is the MLE for θ.

3. Write the mean of the posterior distribution as a weighted average of the mean of the prior distribution and the MLE.

4. What happens to the weight on the prior mean as n becomes large?

Solution to Problem 1

1. The Gamma distribution is conjugate to the Poisson likelihood. The prior $\theta \sim \mathcal{G}(\alpha, \beta)$ has function:

   $$p(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}$$

   Given data $y_1, \ldots, y_n$, the posterior is:

   $$p(\theta|y) \propto \theta^{\sum y_i + \alpha - 1} e^{-(n+\beta)\theta}$$

   This is $\mathcal{G}(\alpha + \sum y_i, \beta + n)$, confirming it is a conjugate prior.

2. $\bar{y}$ is the MLE for θ, since log-likelihood is:

   $$\ell(\theta) = \sum y_i \log\theta - n\theta - \sum \log(y_i!)$$

   Taking derivative θ:

   $$\frac{d\ell}{d\theta} = \frac{\sum y_i}{\theta} - n = 0 \implies \hat{\theta} = \bar{y}$$

3. Posterior mean $E[\theta|y] = \frac{\alpha + \sum y_i}{\beta + n}$. Let prior mean $\mu_0 = \alpha/\beta$. Then:

   $$E[\theta|y] = \left(\frac{\beta}{\beta+n}\right)\mu_0 + \left(\frac{n}{\beta+n}\right)\bar{y}$$

4. As $n \to \infty$, the weight on μ_0 $(\beta/(\beta+n))$ approaches 0. The posterior mean converges to the MLE.

(2) Consider the following two sets of data obtained after tossing a die 100 and 1000 times, respectively:

n	1	2	3	4	5	6
100	19	12	17	18	20	14
1000	190	120	170	180	200	140

Suppose you are interested in θ_1, the probability of obtaining a one spot. Assume your prior for all the probabilities is a Dirichlet distribution, where each $\alpha_i = 2$. Compute the posterior distribution for θ_1 for each of the sample sizes in the table. Plot the resulting distribution and compare the results. Comment on the effect of having a larger sample.

Solution to Problem 2

The Dirichlet posterior for θ with prior $\alpha_i = 2$ and counts x_i is:

$$\theta|y \sim \text{Dirichlet}(\alpha_1 + x_1, \ldots, \alpha_6 + x_6)$$

For θ_1:

- **n=100**: Posterior $\alpha_1' = 2 + 19 = 21$, total $\alpha_0' = 12 + \sum(2 + x_i) = 12 + 100 + 6 * 2 = 112$
- **n=1000**: $\alpha_1' = 2 + 190 = 192$, $\alpha_0' = 12 + 1000 + 12 = 1012$

The marginal distribution of θ_1 is Beta(α_1', $\alpha_0' - \alpha_1'$). For n=100: Beta(21, 91); for n=1000: Beta(192, 820).

As sample size increases:

- Posterior variance decreases, posterior concentrates around MLE ($\hat{\theta}_1 = 0.19$ for n=100; 0.19 for n=1000), and Prior influence diminishes (weights 12 vs 100 and 12 vs 1000)