Your Name:Eric Asamoah **Problem Set 2, Due: Mar. 18, 2025**

(1) Use the probability integral transformation method to simulate from the distribution

$$f(x) = \begin{cases} \frac{2}{a^2}x, & \text{if } 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $a > 0$. Set a value for a, simulate various sample sizes, and compare results to the true distribution.

Problem 1: Probability Integral Transformation

Given the probability density function:

$$f(x) = \begin{cases} \frac{2}{a^2}x, & 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

To generate random samples using the probability integral transformation:
1. Start with a uniform random variable $U \sim U(0,1)$. 2. Compute the cumulative distribution function (CDF):

$$F(x) = \int_0^x \frac{2}{a^2} t dt = \frac{x^2}{a^2}, \quad 0 \leq x \leq a. \tag{3}$$

3. Solve for X in terms of U:

$$X = a\sqrt{U}. \tag{4}$$

4. Generate samples X using $X = a\sqrt{U}$ for various values of a and sample sizes. 5. Compare histograms of simulated data with the true density function.

Problem 2: Finite Mixture Approach

The given density function is:

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x}. \tag{5}$$

This represents a mixture of two exponential distributions:
1. With probability $\frac{1}{3}$, draw X from $\text{Exp}(2)$. 2. With probability $\frac{2}{3}$, draw X from $\text{Exp}(3)$.
Algorithm: Finite Mixture Sampling [1] $i = 1$ to N Generate $U \sim U(0,1)$. $U \leq \frac{1}{3}$ Draw $X_i \sim \text{Exp}(2)$. Draw $X_i \sim \text{Exp}(3)$.

(2) Generate samples from the distribution

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x} \tag{6}$$

using the finite mixture approach.

Problem 3: Accept-Reject Algorithm for Beta(3,3)

To sample from $\text{Beta}(3,3)$ using accept-reject:
1. Proposal distribution: Use $g(x) = 1$ for $x \sim U(0,1)$ (i.e., a uniform distribution). 2. Target density: $f(x) \propto x^2(1-x)^2$. 3. Compute the upper bound M such that $Mg(x) \geq f(x)$. 4. Generate candidate $Y \sim U(0,1)$ and accept with probability:

$$\frac{f(Y)}{Mg(Y)}. \tag{7}$$

5. Repeat until 500 samples are drawn. 6. Compute sample mean and variance, comparing with theoretical values:

$$E[X] = \frac{3}{6} = 0.5, \quad \text{Var}(X) = \frac{3 \cdot 3}{(6)^2(7)} = \frac{9}{252} \approx 0.0357. \tag{8}$$