Problem Set 2

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Problem 1: Use the probability integral transformation method to simulate from the distribution

$$f(x) = \begin{cases} \frac{2}{a^2}x, & \textbf{if } 0 \leq x \leq a \\ 0, & \textbf{otherwise} \end{cases} \tag{1}$$

where $a > 0$**. Set a value for** a**, simulate various sample sizes, and compare results to the true distribution.**

1. **Find the cumulative distribution function (CDF):** The CDF is obtained by integrating the PDF:

$$F(x) = \int_0^x \frac{2}{a^2} t\, dt = \frac{x^2}{a^2}, \quad \text{for } 0 \leq x \leq a.$$

2. **Set the CDF equal to a uniform random variable U:** Let $U \sim \text{Uniform}(0, 1)$. Then:

$$F(X) = U \implies \frac{X^2}{a^2} = U.$$

3. **Solve for X:**

$$X = a\sqrt{U}.$$

4. **Simulate samples for various sample sizes:** - Choose $a = 2$. - Generate U from $\text{Uniform}(0, 1)$. - Compute $X = 2\sqrt{U}$.

5. - For sample sizes $n = 100, 1000, 10000$, F $F(x) = \frac{x^2}{4}$. - As n increases, the empirical CDF should converge to the true CDF. **See python code file for histogram visualization of comparison to the true PDF.**

Problem 2: Generate samples from the distribution

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x} \tag{2}$$

using the finite mixture approach.

The given density is a finite mixture of two exponential distributions:

$$f(x) = \frac{2}{3} \cdot 2e^{-2x} + \frac{1}{3} \cdot 3e^{-3x}.$$

Here, the mixture weights are $w_1 = \frac{2}{3}$ and $w_2 = \frac{1}{3}$, and the component distributions are Exp(2) and Exp(3), respectively.

1. Simulate - Generate $U \sim \text{Uniform}(0, 1)$. - If $U \leq \frac{2}{3}$, draw X from Exp(2). - Otherwise, draw X from Exp(3).

2. Verify - The theoretical mean and variance of the mixture distribution are:

$$\text{Mean} = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{9} = \frac{4}{9} \approx 0.4444,$$
$$\text{Variance} = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{6} + \frac{1}{27} = \frac{11}{54} \approx 0.2037.$$

See python code file for histogram visualization of comparison to the true PDF.. For example, with $n = 1000$, the empirical mean should be close to 0.4444, and the empirical variance should be close to 0.2037.

Problem 3: Draw 500 observations from Beta$(3, 3)$ using the accept-reject algorithm. Compute the mean and variance of the sample and compare them to the true values.
1. Use Uniform$(0, 1)$ as the proposal distribution $g(x)$.
2. **Find the maximum ratio M:** The density of Beta$(3, 3)$ is:

$$f(x) = \frac{x^2(1-x)^2}{B(3,3)}, \quad \text{where } B(3,3) = \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{4!}{5!} = \frac{1}{30}.$$

The maximum of $f(x)$ occurs at $x = 0.5$, so:

$$M = \frac{f(0.5)}{g(0.5)} = \frac{(0.5)^2(0.5)^2/(1/30)}{1} = \frac{1/16}{1/30} = \frac{30}{16} = 1.875.$$

3. Accept-reject algorithm - Generate $U \sim$ Uniform$(0, 1)$ and $Y \sim$ Uniform$(0, 1)$. - Accept Y if $U \leq \frac{f(Y)}{M \cdot g(Y)} = \frac{f(Y)}{1.875}$.
4. - The theoretical mean and variance of Beta$(3, 3)$ are:

$$\text{Mean} = \frac{3}{3+3} = 0.5,$$

$$\text{Variance} = \frac{3 \cdot 3}{(3+3)^2(3+3+1)} = \frac{9}{6^2 \cdot 7} = \frac{1}{28} \approx 0.0357.$$

See python code file for histogram visualization of comparison to the true PDF. The histogram should closely match the theoretical density, confirming the correctness of the accept-reject algorithm. For example, with $n = 500$, the empirical mean should be close to 0.5, and the empirical variance should be close to 0.0357.