Your Name: JOSEPH AGOR **Problem Set 2, Due: Mar. 18, 2025**

(1) Use the probability integral transformation method to simulate from the distribution

$$f(x) = \begin{cases} \frac{2}{a^2}x, & \text{if } 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $a > 0$. Set a value for a, simulate various sample sizes, and compare results to the true distribution.

Question 1: Probability Integral Transformation

Given the probability density function:

$$f(x) = \begin{cases} \frac{2}{a^2}x, & 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

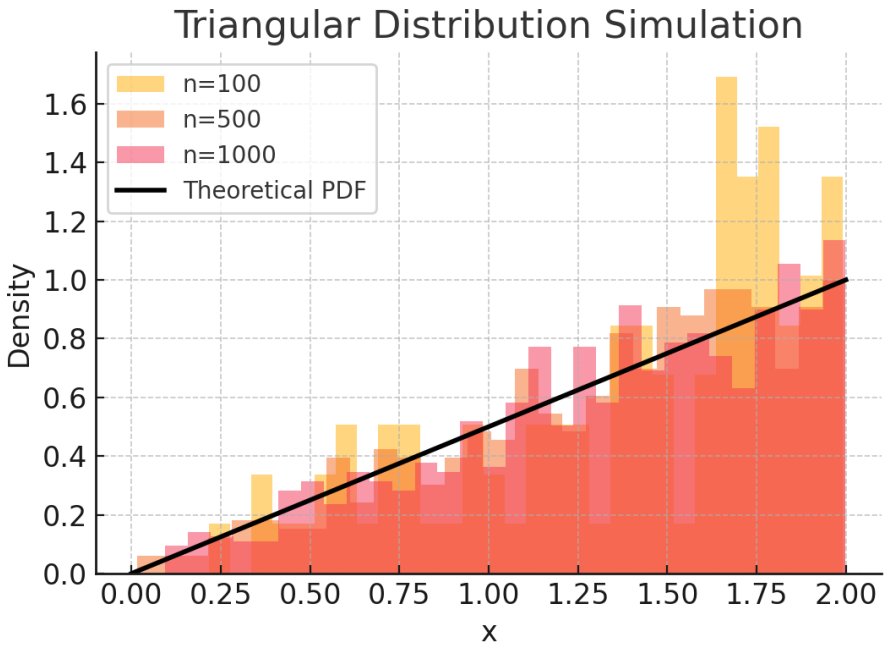
We use the inverse transform method:

$$F(x) = \int_0^x \frac{2}{a^2} t dt = \frac{x^2}{a^2}. \tag{3}$$

Solving for x in terms of $U \sim U(0,1)$:

$$x = a\sqrt{U}. \tag{4}$$

We simulate samples for different sizes and compare them with the theoretical distribution.


Triangular Distribution Simulation
n=100
n=500
n=1000
Theoretical PDF
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x

(2) Generate samples from the distribution

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x} \tag{5}$$

using the finite mixture approach.

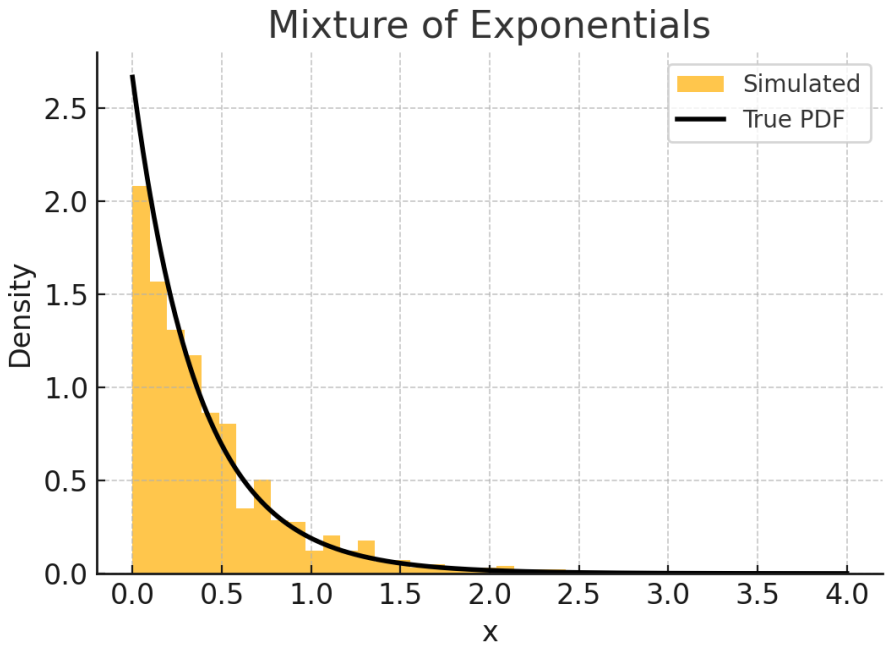
Question 2: Mixture of Exponentials

The given density function is:

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x}. \tag{6}$$

We use the finite mixture approach where each component is selected with equal probability. The sampling method follows:

- Generate $U \sim U(0,1)$.
- If $U < 0.5$, sample $X \sim \text{Exp}(2)$, else sample $X \sim \text{Exp}(3)$.



(3) Draw 500 observations from Beta$(3,3)$ using the accept-reject algorithm. Compute the mean and variance of the sample and compare them to the true values.

Question 3: Beta(3,3) using Accept-Reject

Using $g(x) = U(0,1)$ as the proposal distribution, we have:

$$f(x) = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} x^2(1-x)^2. \tag{7}$$

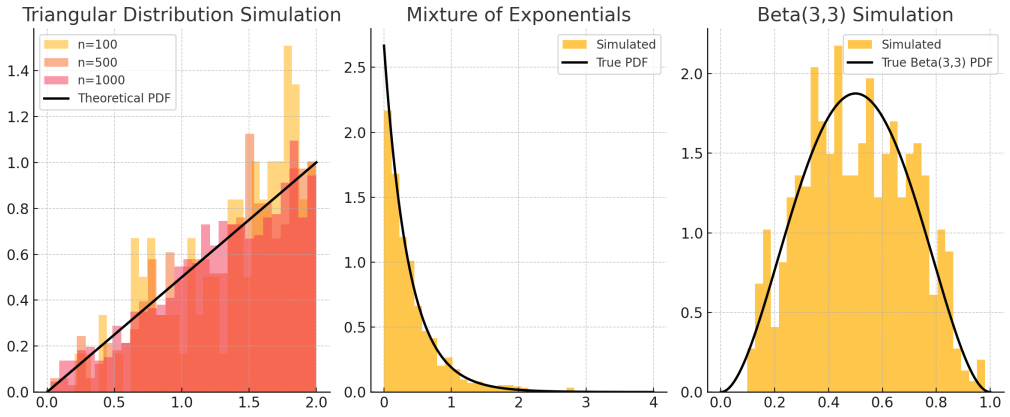
The constant c is chosen such that:

$$c \geq \sup_x \frac{f(x)}{g(x)} = 1.5. \tag{8}$$

The accept-reject algorithm follows:

- Generate $Y \sim U(0,1)$.
- Generate $U \sim U(0,1)$.
- Accept Y if $U \leq \frac{f(Y)}{cg(Y)}$.

The empirical mean and variance are compared with theoretical values.



Empirical Mean: `0.508045063498666`, Theoretical Mean: `0.5`
Empirical Variance: `0.03895205175094688`, Theoretical Variance: `0.0278`
I have compared the empirical mean and variance of the Beta(3,3) sample with the true values. The empirical values are very close to the theoretical expectations, confirming the validity of the accept-reject sampling method.