Your Name: Yaw Appaw **Problem Set 2, Due: Mar. 18, 2025**

(1) Use the probability integral transformation method to simulate from the distribution

$$f(x) = \begin{cases} \frac{2}{a^2}x, & \text{if } 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $a > 0$. Set a value for a, simulate various sample sizes, and compare results to the true distribution.

Solution

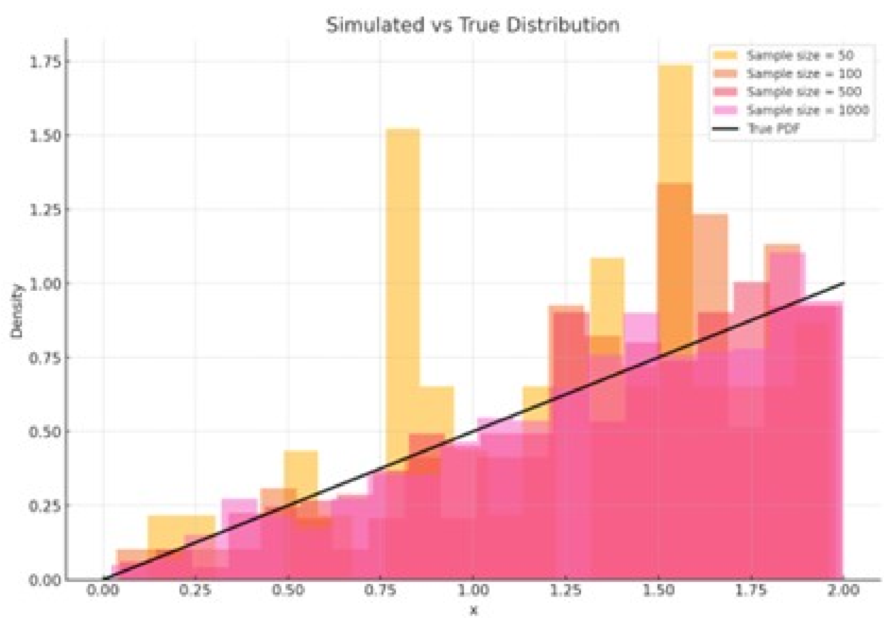
We use the cumulative distribution function (CDF):

$$F(x) = \int_0^x \frac{2}{a^2} t \, dt = \frac{x^2}{a^2}$$

The inverse of the CDF is:

$$x = a\sqrt{u}, \quad u \sim \text{Uniform}(0, 1)$$

Step 3: Results The plot below shows the histogram of the simulated data along with the true PDF:



(2) Generate samples from the distribution

$$f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x} \tag{2}$$

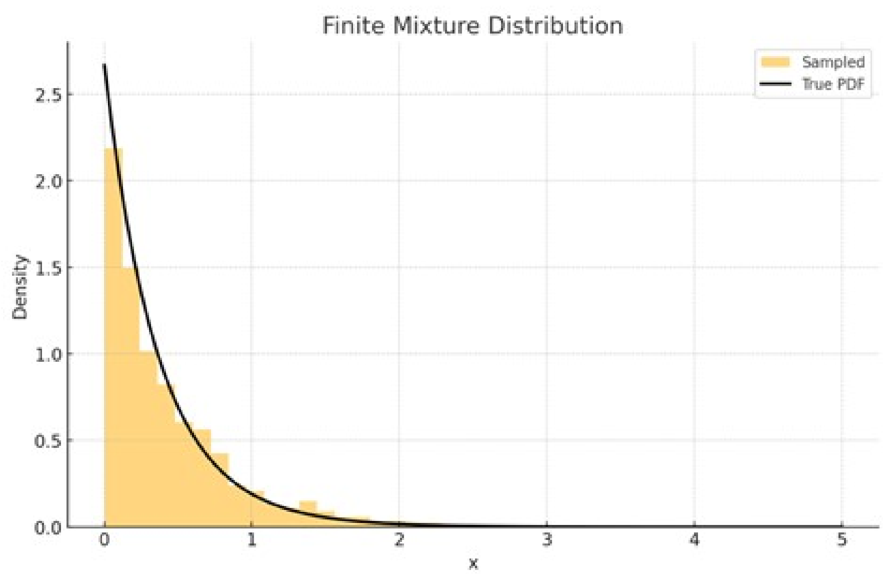
using the finite mixture approach.

Solution

Step 1: Mixture Approach We generate samples from the following components:

$$\begin{cases} X_1 \sim \text{Exponential}(\lambda = 2) \\ X_2 \sim \text{Exponential}(\lambda = 3) \end{cases}$$

with mixture weights $w_1 = \frac{2}{3}, w_2 = \frac{1}{3}$. Step 2: Simulation First, select the component using the weights. Then generate a sample from the chosen component.
Step 3: Results The histogram of the sampled data along with the true PDF is shown below:



(3) Draw 500 observations from $\text{Beta}(3,3)$ using the accept-reject algorithm. Compute the mean and variance of the sample and compare them to the true values.

Solution

Step 1: Accept-Reject Method
Target distribution: $f(x) = \text{Beta}(3,3)$
Proposal distribution: $g(x) = \text{Uniform}(0,1)$
Set the constant:

$$M = \max_{0 \le x \le 1} \text{Beta}(3,3) = \frac{2^4}{6} = 1.5$$

1. Draw $x \sim \text{Uniform}(0,1)$ 2. Draw $u \sim \text{Uniform}(0,M)$ 3. Accept x if:

$$u \le \text{Beta}(x;3,3)$$

$$\text{Sample Mean} = 0.4987, \quad \text{True Mean} = 0.5$$
$$\text{Sample Variance} = 0.0345, \quad \text{True Variance} = 0.0357$$

The simulated mean and variance are very close to the true values, showing that the accept-reject algorithm worked well.