Lecture 1 Linear Classification

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E6930 Introduction to Neural Networks

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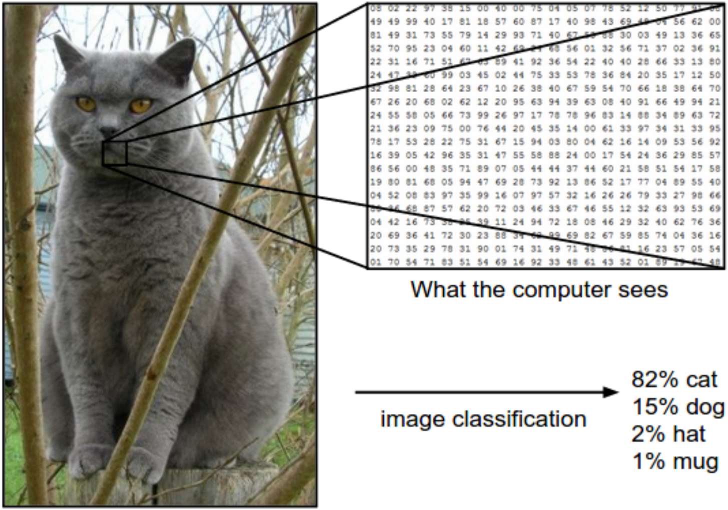
Image Classification



The Road Ahead...

1. Linear Classifiers
2. Gradient Descent
3. Backpropagation

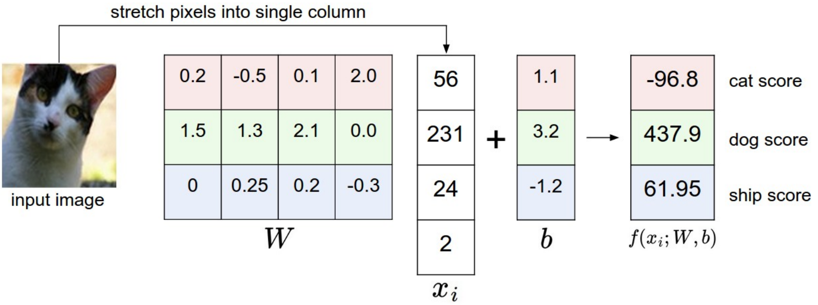
Score Function

Linear score

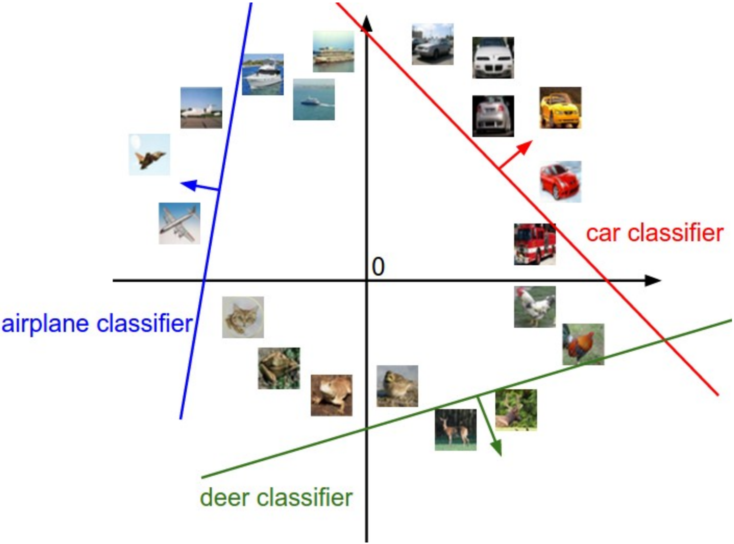
$$f(x_i, W, b) = Wx_i + b$$

- Notation
  - dataset of N examples $\{(x_i, y_i)\}_{i=1}^N$, D (normalized) features $x_i \in \mathbb{R}^D$, K categories of *single* attribute $y_i \in 1, \ldots, K$
  - $K \times D$ weights W, $K \times 1$ bias b
- Pipeline
  - split data into training, validation, and test sets
  - train parameters and (cross) validate hyperparameters
  - evaluate on test test only once at the end

Algebraic Interpretation



Geometric Interpretation



Loss Function

Regularized loss

$$L = \frac{1}{N}\sum_i L_i + \lambda R(W), \quad \lambda > 0$$

- Examples of data loss
  - multiclass support vector machine (SVM) classifier uses hinge loss: $L_i = \sum_{j \neq y_i} \max(0, f_j - f_{y_i} + \Delta)$, $\Delta > 0$
  - Softmax classifier uses cross-entropy loss: $L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$
- Examples of regularization/prior
  - L^1 regularization: $R(W) = \sum_{i,j} |W_{ij}|$
  - L^2 regularization: $R(W) = \sum_{i,j} W_{ij}^2$

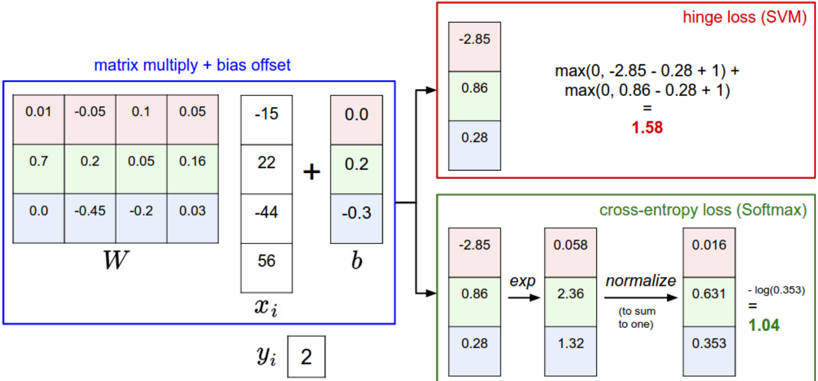
Loss Function (Cont'd)

```python
import numpy as np

# SVM classifier
def L1_i(x_i, y_i, W, b):
    delta = 1.0
    f = W.dot(x_i) + b
    margins = np.maximum(0, f - f[y_i] + delta)
    margins[y_i] = 0  # ignore true class
    return np.sum(margins)

# Softmax classifier
def L2_i(x_i, y_i, W, b):
    f = W.dot(x_i) + b
    f -= np.max(f)    # avoid potential blowup
    p = np.exp(f) / np.sum(np.exp(f))
    return -np.log(p[y_i])
```

SVM vs. Softmax



The Road Ahead...

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2. Gradient Descent

3. Backpropagation

Analytic Gradient

Derivative of 1-D function

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Gradient of multi-D function
  - vector of partial derivatives in each dimension
  - examples of 2-D function

$$\begin{aligned} f(x,y) = xy &\quad \to \quad \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right] = [y, x] \\ f(x,y) = x + y &\quad \to \quad \nabla f = [1, 1] \\ f(x,y) = \max(x,y) &\quad \to \quad \nabla f = [\mathbb{1}(x \geq y), \mathbb{1}(x \leq y)] \end{aligned}$$

Numerical Gradient

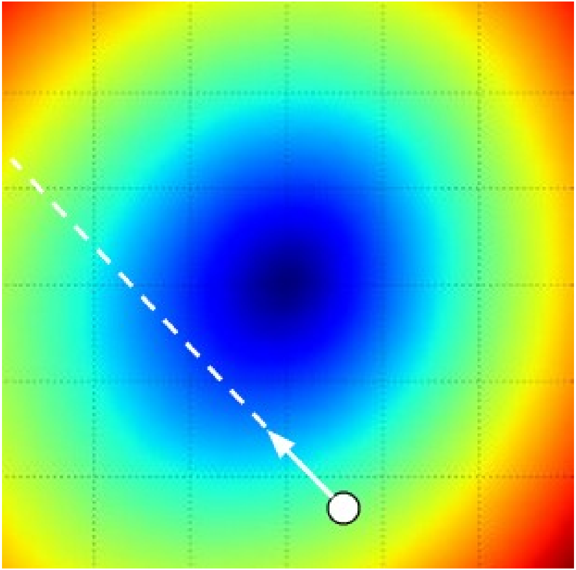
```python
import numpy as np

def num_grad(f, x):  # finite difference method
    fx = f(x)
    grad = np.zeros(x.shape)
    h = 0.00001
    it = np.nditer(x, flags=['multi_index'],
        op_flags=['readwrite'])
    while not it.finished:
        ix = it.multi_index
        old_value = x[ix]
        x[ix] = old_value + h
        fxh = f(x)
        x[ix] = old_value
        grad[ix] = (fxh - fx) / h # alternatively
            [f(x+h)-f(x-h)]/2h
        it.iternext()
    return grad
```

Gradient Descent

- Repeated local search to minimize loss function
  - update in *negative* gradient direction
  - validate learning rate (step size)
- Mini-batch/stochastic gradient descent

```
while True:
    data_batch = sample_data(training_data,
        256)
    weights_grad = eval_grad(loss_fun,
        data_batch, weights)
    weights += - step_size * weights_grad
```

Gradient Descent (Cont'd)



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Backpropagation

Chain rule

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

- Example of composite function

$$f(v(p(x,y),q(z,w))) = 2\ \underbrace{\left[\ \underbrace{xy}_{p\ (*\ \text{gate})} + \underbrace{\max(z,w)}_{q\ (\text{max gate})}\ \right]}_{v\ (+\ \text{gate})}$$

  - forward pass: $[x,y,z,w] = [3,-4,2,-1]$, $f = -20$
  - backward pass: $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right] = [-8,6,2,0]$

Backpropagation (Cont'd)

```
# Forward pass
x = 3; y = -4; z = 2; w = -1
p = x * y       # -12
q = max(z, w)   # 2
v = p + q       # -10
f = 2 * v       # -20

# Backward pass
dfdv = 2
dvdp = 1
dpdx = y
dpdy = x
dvdq = 1
dqdz = (z > w)
dqdw = (w > z)
dfdx = dfdv * dvdp * dpdx  # -8
dfdy = dfdv * dvdp * dpdy  # 6
dfdz = dfdv * dvdq * dqdz  # 2
dfdw = dfdv * dvdq * dqdw  # 0
```

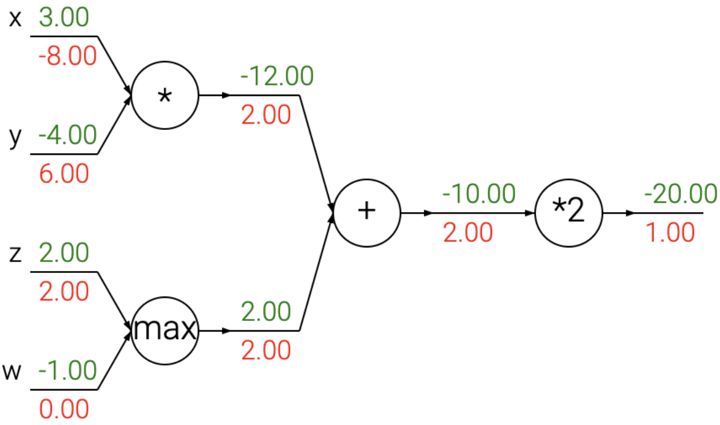
PyTorch Implementation

```
import torch

# Forward pass
x = torch.tensor(3., requires_grad=True)
y = torch.tensor(-4., requires_grad=True)
z = torch.tensor(2., requires_grad=True)
w = torch.tensor(-1., requires_grad=True)
p = x * y    # tensor(-12., grad_fn=<MulBackward0>)
q = max(z, w) # tensor(2., grad_fn=<MaxBackward0>)
v = p + q    # tensor(-10., grad_fn=<AddBackward0>)
f = 2 * v    # tensor(-20., grad_fn=<MulBackward0>)

# Backward pass
f.backward()   # compute gradients
print(x.grad)  # tensor(-8.)
print(y.grad)  # tensor(6.)
print(z.grad)  # tensor(2.)
print(w.grad)  # tensor(0.)
```

PyTorch Computation Graph



References

- cs231n.stanford.edu – CS231n: Deep Learning for Computer Vision, by Stanford University
- Tang (2013), "Deep Learning using Linear Support Vector Machines", arXiv:1306.0239
- github.com/karpathy/micrograd – micrograd, by Andrej Karpathy