Lecture 7 Linear Regression and Extensions

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Introduction to Bayesian Statistics

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Extending Linear Regressions (LR)

General setup

$$
\begin{aligned}
y_i^* &= x_i'\beta + u_i, \qquad u_i|x_i \sim_{i.i.d.} t_\nu(0,\sigma^2) \\
\mathbb{E}[y_i|x_i] &= G(x_i'\beta), \qquad i = 1,\ldots,n
\end{aligned}
$$

- Choice of link function $G(\cdot)$
  - standard LR

$$G(x_i'\beta) = x_i'\beta \quad \Rightarrow \quad y_i = y_i^*$$

  - tobit censored LR ($\nu = \infty$; $\mathcal{N}(0,1)$ p.d.f. ϕ, c.d.f. Φ)

$$G(x_i'\beta) = x_i'\beta + \frac{\phi(-x_i'\beta/\sigma)}{1-\Phi(-x_i'\beta/\sigma)}\sigma \quad \Rightarrow \quad y_i = \max\{y_i^*, 0\}$$

  - binary probit LR ($\nu = \infty$; $\sigma^2 = 1$)

$$G(x_i'\beta) = \Phi(x_i'\beta) \quad \Rightarrow \quad y_i = 1\{y_i^* > 0\}$$

The Road Ahead...

1. Continuous Dependent Variables
2. Limited Dependent Variables

LR with Gaussian Errors

- Conditionally conjugate prior

$$\beta \sim \mathcal{N}(\beta_0, B_0), \qquad \sigma^2 \sim \mathcal{IG}\text{-}2(\alpha_0/2, \delta_0/2)$$

- Gibbs algorithm
  - step 1: choose $\beta = \beta^{(0)}$, $\sigma^2 = \sigma^{2(0)}$, set $g = 0$
  - step 2: sample recursively

    $$\beta^{(g)} \sim \mathcal{N}(\beta_1^{(g+1)}, B_1^{(g+1)}), \qquad \sigma^{2(g+1)} \sim \mathcal{IG}\text{-}2(\alpha_1/2, \delta_1^{(g+1)}/2)$$

    where

    $$\begin{aligned} B_1^{(g+1)} &= (\sigma^{-2(g)} X'X + B_0^{-1})^{-1} \\ \beta_1^{(g+1)} &= B_1^{(g+1)}(\sigma^{-2(g)} X'y + B_0^{-1}\beta_0) \\ \alpha_1 &= \alpha_0 + n \\ \delta_1^{(g+1)} &= \delta_0 + (y - X\beta^{(g+1)})'(y - X\beta^{(g+1)}) \end{aligned}$$

  - step 3: set $g = g + 1$ and go to step 2

LR with Student-t Errors

- Conditional likelihood [see also Geweke (1993)]

$$f(y_i|\beta,\sigma^2,\lambda_i) = \mathcal{N}(x_i'\beta, \lambda_i^{-1}\sigma^2), \quad \lambda_i \sim \mathcal{G}(\nu/2, \nu/2) \text{ (latent)}$$

- Gibbs sampler for $\pi(\beta,\sigma^2,\lambda|y)$

$$\begin{aligned}
\beta|y,\lambda,\sigma^2 &\sim \mathcal{N}(\beta_1, B_1) \\
\sigma^2|y,\beta,\lambda &\sim \mathcal{IG}\text{-}2(\alpha_1/2, \delta_1/2) \\
\lambda_i|y,\beta,\sigma^2 &\sim \mathcal{G}(\nu_1/2, \nu_{2i}/2), \qquad i = 1,\ldots,n
\end{aligned}$$

where $\Lambda = \text{diag}(\lambda_i)$ and

$$\begin{aligned}
B_1 &= (\sigma^{-2}X'\Lambda X + B_0^{-1})^{-1} \\
\beta_1 &= B_1(\sigma^{-2}X'\Lambda y + B_0^{-1}\beta_0) \\
\alpha_1 &= \alpha_0 + n \\
\delta_1 &= \delta_0 + (y - X\beta)'\Lambda(y - X\beta) \\
\nu_1 &= \nu + 1 \\
\nu_{2i} &= \nu + \sigma^{-2}(y_i - x_i'\beta)^2
\end{aligned}$$

Python Code

```
def full_run(y, x, n, b0, B0, a0, d0, n0):
    for i in range(1, n):
        # Sample beta
        B1 = inv(x.T @ diag(s['lam'][i - 1, :]) @
             x / s['sig2'][i - 1] + inv(B0))
        b1 = B1 @ (x.T @ diag(s['lam'][i - 1, :])
             @ y / s['sig2'][i - 1] + inv(B0) @ b0)
        s['beta'][i, :] = multivariate_normal.rvs(
             size=1, mean=b1, cov=B1)
        # Sample sigma^2
        e = y - x @ s['beta'][i, :]
        d1 = d0 + e.T @ diag(s['lam'][i - 1, :]) @
              e
        s['sig2'][i] = invgamma.rvs((a0 + len(y))
             / 2, size=1, scale=d1 / 2)
        # Sample lambda
        for j in range(len(y)):
            n2 = n0 + e[j]**2 / s['sig2'][i]
            s['lam'][i, j] = gamma.rvs((nu0 + 1) /
                   2, size=1, scale=2 / n2)
    return s
```

Marginal Likelihood

Chib method

$$m(y) = \frac{\prod_{i=1}^{n} t_\nu(x_i'\beta^*, \sigma^{2*})\pi(\beta^*)\pi(\sigma^{2*})}{\pi(\beta^*, \sigma^{2*}|y)}, \quad \forall\theta^* = (\beta^*, \sigma^{2*}) \in \Theta$$

- Compute $\pi(\beta^*, \sigma^{2*}|y)$ (not involving λ) at high-density point θ^* from Gibbs output

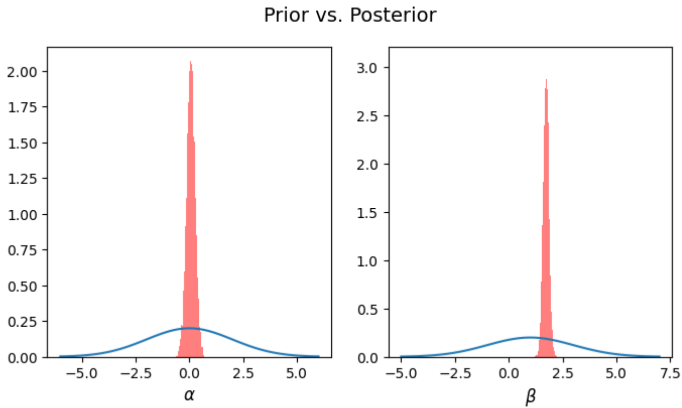
$$\pi(\beta^*, \sigma^{2*}|y) = \pi(\beta^*|y)\pi(\sigma^{2*}|\beta^*, y)$$

  - full run: $\hat{\pi}(\beta^*|y) = \frac{1}{G}\sum_{g=1}^{G} \pi(\beta^*|\sigma^{2(g)}, \lambda^{(g)}, y)$, where $(\theta^{(g)}, \lambda^{(g)}) \sim \pi(\theta, \lambda|y)$
  - reduced run: $\hat{\pi}(\sigma^{2*}|\beta^*, y) = \frac{1}{G}\sum_{g=1}^{G} \pi(\sigma^{2*}|\beta^*, \lambda^{(g)}, y)$, where $(\sigma^{2(g)}, \lambda^{(g)}) \sim \pi(\sigma^2, \lambda|\beta^*, y)$

Python Code

```
def marg_lik(y, x, s1, s2, beta, sig2, b0, B0, a0,
    d0, n0):
    # s1, s2 samples from full & reduced runs
    for i in range(n):
        B1 = inv(x.T @ diag(s1['lam'][i, :]) @ x /
            s1['sig2'][i] + inv(B0))
        b1 = B1 @ (x.T @ diag(s1['lam'][i, :]) @ y
            / s1['sig2'][i] + inv(B0) @ b0)
        pd1[i] = multivariate_normal.pdf(beta,
            mean=b1, cov=B1)
        e = y - x @ beta
        d1 = d0 + e.T @ diag(s2['lam'][i, :]) @ e
        pd2[i] = invgamma.pdf(sig2, a1 / 2, scale=
            d1 / 2)
    ll = sum(t.logpdf(e, n0, scale=sqrt(sig2)))
    lp = multivariate_normal.logpdf(beta, mean=b0,
        cov=B0) + invgamma.logpdf(sig2, a0 / 2,
        scale=d0 / 2)
    return ll + lp - log(mean(pd1)) - log(mean(pd2
        ))
```

Application: Stock Return Risk



- $R_{\mathsf{TSLA}} = \alpha + \beta R_{\mathsf{SPY}} + u$, 02/25/2022 - 02/25/2023
- $\alpha \sim \mathcal{N}(0, 2^2)$, $\beta \sim \mathcal{N}(1, 2^2)$, $\sigma^2 \sim \mathcal{IG}\text{-}2(\frac{5}{2}, \frac{5}{2})$
- $\ln m(y) = -661.0163$ for $\nu = 5$

The Road Ahead...

1. Continuous Dependent Variables
2. Limited Dependent Variables

Tobit Censored LR

Model

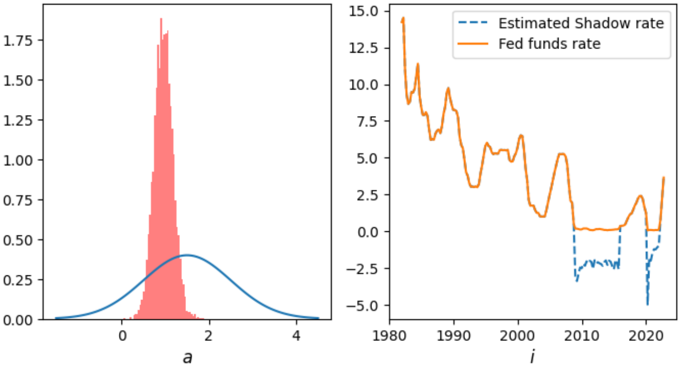
$$
\begin{aligned}
y_i^* &= x_i'\beta + u_i, \qquad u_i|x_i \sim_{i.i.d.} \mathcal{N}(0,\sigma^2) \\
y_i &= \max\{y_i^*, 0\}, \qquad i = 1, \ldots, n
\end{aligned}
$$

- Chib (1992) introduces latent variables z for censored observations and Gibbs sampler for $\pi(\beta, \sigma^2, z|y)$
  - conditionally conjugate prior for (β, σ^2) as before
  - sample $\beta|y_z, \sigma^2 \sim \mathcal{N}(\beta_1, B_1)$, where y_z replaces $y_i = 0$ by $z_i < 0$
  - sample $\sigma^2|y_z, \beta \sim \mathcal{IG}\text{-}2(\alpha_1/2, \delta_1/2)$
  - sample $z_i|y, \beta, \sigma^2 \sim \mathcal{TN}_{(-\infty,0)}(x_i'\beta, \sigma^2)$ (truncated normal)
  - exercise: Student-t version
- Data augmentation technique [Tanner & Wong (1987)]

Python Code

```
def tobit(y, x, n, b0, B0, a0, d0, c):
    ind = where(y < c)[0]
    for i in range(1, n):
        y[ind] = s['z'][i - 1, :]
        B1 = inv(x.T @ x / s['sig2'][i - 1] + inv(
            B0))
        b1 = B1 @ (x.T @ y / s['sig2'][i - 1] +
            inv(B0) @ b0)
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=b1, cov=B1)
        m = x @ s['beta'][i, :]
        d1 = d0 + (y - m).T @ (y - m)
        s['sig2'][i] = invgamma.rvs((a0 + len(y))
            / 2, size=1, scale=d1 / 2)
        for j in range(len(y)):
            s['z'][i, j] = truncnorm.rvs(-1e3, (c
                - m[ind[j]]) / sqrt(s['sig2'][i]),
                 loc=m[ind[j]], scale=sqrt(s['sig2
                '][i]), size=1)
    return s
```

Application: Taylor Rule with ZLB



- $i = i^* + a(\pi - \pi^*) + b(y - y^*) + u$, 1982:Q1 - 2022:Q4
- $i^* \sim \mathcal{N}(4, 1)$, $a \sim \mathcal{N}(1.5, 1)$, $b \sim \mathcal{N}(0.5, 1)$, $\sigma^2 \sim \mathcal{IG}\text{-}2(\frac{5}{2}, \frac{5}{2})$
- Effective lower bound $= 0.25\%$

Binary Probit LR

Model

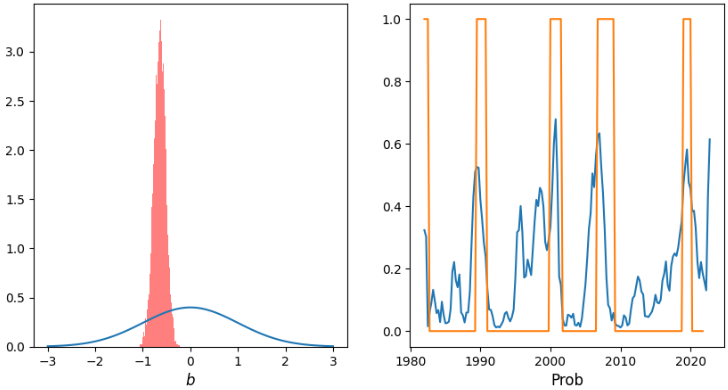
$$
\begin{aligned}
y_i^* &= x_i'\beta + u_i, \qquad u_i|x_i \sim_{i.i.d.} \mathcal{N}(0,1) \\
y_i &= 1\{y_i^* > 0\}, \qquad i = 1, \ldots, n
\end{aligned}
$$

- Albert and Chib (1993) introduce latent variables $z = y^*$ and Gibbs sampler for $\pi(\beta, z|y)$
  - $\beta \sim \mathcal{N}(\beta_0, B_0)$ as before; $\sigma^2 = 1$ for identification
  - sample $\beta|z \sim \mathcal{N}(\beta_1, B_1)$ (B_1 not updated)
  - sample $z_i|y, \beta \sim \mathcal{TN}_{(-\infty,0]}(x_i'\beta, 1)$ if $y_i = 0$ or $\mathcal{TN}_{(0,\infty)}(x_i'\beta, 1)$ if $y_i = 1$
  - exercise: Student-t version
- Binary logit LR: $u_i|x_i \sim_{i.i.d.} \mathcal{L}(0,1)$ (logistic distribution)

Python Code

```
def probit(y, x, n, b0, B0):
    for i in range(1, n):
        b1 = B1 @ (x.T @ s['z'][i - 1, :] + inv(B0
            ) @ b0)
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=b1, cov=B1)
        m = x @ s['beta'][i, :]
        for j in range(len(y)):
            if y[j] == 0:
                s['z'][i, j] = truncnorm.rvs(-1e3,
                     -m[j], loc=m[j], scale=1,
                    size=1)
            else:
                s['z'][i, j] = truncnorm.rvs(-m[j
                    ], 1e3, loc=m[j], scale=1,
                    size=1)
    return s
```

Application: Forecasting Recession



- $\mathbb{P}(\text{NBER}_{t+1,t+4} = 1) = \Phi(a + b \times \text{Spread}_t)$, 1982:Q1 - 2022:Q4
- $a \sim \mathcal{N}(0,1)$, $b \sim \mathcal{N}(0,1)$

Readings

- Albert & Chib (1993), "Bayesian Analysis of Binary and Polychotomous Response Data," *Journal of the American Statistical Association*
- Chib (1992), "Bayes Inference in the Tobit Censored Regression Model," *Journal of Econometrics*
- Geweke (1993), "Bayesian Treatment of the Independent Student-t Linear Model," *Journal of Applied Econometrics*
- Tanner & Wong (1987), "The Calculation of Posterior Distributions by Data Augmentation (with Discussion)," *Journal of the American Statistical Association*