

# Lecture 9 Time Series

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Introduction to Bayesian Statistics

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# State Space Representation

## Gaussian linear model + Kalman filter

$$\text{Measurement: } y_t = D + Z\epsilon_t + v_t, \quad v_t \sim \mathcal{N}(0, \Sigma_v)$$

$$\text{Transition: } \epsilon_t = C + G\epsilon_{t-1} + u_t, \quad u_t \sim \mathcal{N}(0, \Sigma_u)$$

1. forecast  $\epsilon_t$  via transition:  $\epsilon_t | y_{1:t-1} \sim \mathcal{N}(\epsilon_{t|t-1}, P_{t|t-1})$

$$\epsilon_{t|t-1} = C + G\epsilon_{t-1|t-1}$$

$$P_{t|t-1} = GP_{t-1|t-1}G' + \Sigma_u$$

2. forecast  $y_t$  via measurement:  $y_t | y_{1:t-1} \sim \mathcal{N}(y_{t|t-1}, F_{t|t-1})$

$$y_{t|t-1} = D + Z\epsilon_{t|t-1}$$

$$F_{t|t-1} = ZP_{t|t-1}Z' + \Sigma_v$$

3. filter  $\epsilon_t$  via Bayes theorem:  $\epsilon_t | y_{1:t} \sim \mathcal{N}(\epsilon_{t|t}, P_{t|t})$

$$\epsilon_{t|t} = \epsilon_{t|t-1} + P_{t|t-1}Z'F_{t|t-1}^{-1}(y_t - y_{t|t-1})$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}Z'F_{t|t-1}^{-1}ZP_{t|t-1}$$

# The Road Ahead...

① Autoregressive Model

② Regime-Switching Model

# Autoregressive (AR) Model

Chib's (1993) regression with AR( $p$ ) error

$$\begin{aligned}y_t &= x_t' \beta + \epsilon_t, \quad t = 1, \dots, T \\ \phi(L) \epsilon_t &= u_t, \quad \phi(L) = 1 - \phi_1 L - \dots - \phi_p L^p, \quad L \epsilon_t = \epsilon_{t-1}\end{aligned}$$

- Likelihood function under  $u_t \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$

$$f(y_{p+1:T} | y_{1:p}, \beta, \sigma^2, \phi) \propto \sigma^{-(n-p)} \exp \left[ -\frac{1}{2\sigma^2} \sum_{t=p+1}^T (y_t - y_{t|t-1})^2 \right]$$

where  $y_{t|t-1} = (1 - \phi(L))y_t + \phi(L)x_t' \beta$

- Joint prior (stationarity:  $\Phi_s = \{\phi : \phi(z) \neq 0, \forall |z| < 1\}$ )

$$\pi(\beta, \sigma^2, \phi) = \underbrace{\mathcal{N}(\beta_0, \sigma^2 B_0)}_{\pi(\beta | \sigma^2)} \underbrace{\mathcal{IG}\text{-}2(\nu_0/2, \delta_0/2)}_{\pi(\sigma^2)} \underbrace{\mathcal{N}(\phi_0, \Phi_0) 1\{\phi \in \Phi_s\}}_{\pi(\phi)}$$

# Gibbs Algorithm

- Gibbs sampler for  $\pi(\beta, \sigma^2, \phi|y)$

$$\beta|y, \sigma^2, \phi \sim \mathcal{N}(\beta_1, \sigma^2 B_1)$$

$$\sigma^2|y, \beta, \phi \sim \mathcal{IG}\text{-}2(\nu_1/2, \delta_1/2)$$

$$\phi|y, \beta, \sigma^2 \sim \mathcal{N}(\phi_1, \Phi_1) 1\{\phi \in \Phi_s\}$$

where

$$B_1 = (X^{*'}X^* + B_0^{-1})^{-1}, \quad x_t^* = \phi(L)x_t$$

$$\beta_1 = B_1(X^{*'}y^* + B_0^{-1}\beta_0), \quad y_t^* = \phi(L)y_t$$

$$\nu_1 = \nu_0 + k + T - p, \quad k = \dim(x_t)$$

$$\delta_1 = \delta_0 + (\beta - \beta_0)'B_0^{-1}(\beta - \beta_0) + (y^* - X^*\beta)'(y^* - X^*\beta)$$

$$\Phi_1 = (\sigma^{-2}E'E + \Phi_0^{-1})^{-1}, \quad \epsilon = E\phi + u \text{ (known given } y, \beta, \sigma^2)$$

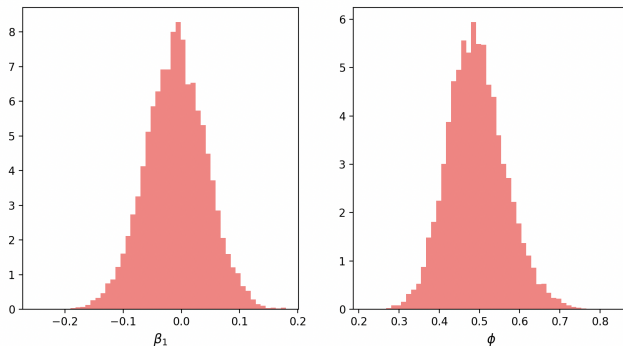
$$\phi_1 = \Phi_1(\sigma^{-2}E'\epsilon + \Phi_0^{-1}\phi_0)$$

- Extension to ARMA( $p, q$ ) error [Chib & Greenberg (1994)]

# Python Code

```
def ar_err(y, x, s, b0, B0, nu0, d0, p0, P0):
    for i in range(1, s):
        # Sample beta
        y2 = filter(y, s['phi'][i - 1, :])
        x2 = filter(x, s['phi'][i - 1, :])
        ...
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=b1, cov=B1)
        # Sample sig2
        ...
        s['sig2'][i] = invgamma.rvs(nu1 / 2, size
            =1, scale=d1 / 2)
        # Sample phi
        for t in range(T - p):
            for j in range(p):
                E[t, j] = err[t + p - j - 1]
        ...
        s['phi'][i, :] = multivariate_normal.rvs(
            size=1, mean=p1, cov=P1)
    return s
```

# Application: Phillips Curve



- ▶  $\Delta\pi_t = \beta_0 + \beta_1 u_t + \epsilon_t$ ,  $\epsilon_t = \phi\epsilon_{t-1} + u_t$ , 2000–2022 monthly
- ▶  $\beta_0, \beta_1 \sim \mathcal{N}(0, \sigma^2)$ ,  $\sigma^2 \sim \text{IG-2}(\frac{5}{2}, \frac{5}{2})$ ,  $\phi \sim \mathcal{N}(0.5, 0.15)$
- ▶ Phillips curve has flattened considerably since 2000

# The Road Ahead...

① Autoregressive Model

② Regime-Switching Model



# Regime-Switching Model

## Chib's (1996) Markov mixture model

$$\begin{aligned}f(y_t|y_{1:t-1}, s_{t-1}, \theta) &= \sum_{k=1}^m f(y_t|y_{1:t-1}, \theta_k) p(s_t = k | s_{t-1}) \\p(s_t = j | s_{t-1} = i) &= p_{ij}, \quad s_t \in \{1, \dots, m\}, \quad t = 1, \dots, T\end{aligned}$$

### ► Chib's (1998) change-point model

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} & 0 & \cdots & 0 & 0 \\ 0 & p_{22} & 1 - p_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{mm} & 1 - p_{mm} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

### ► Which specification is more general?

# Gibbs Algorithm

- ▶ Sample  $s_{1:T}$  in one block

$$\begin{aligned} p(s_{1:T}|y_{1:T}) &= p(s_T|y_{1:T}) \cdots p(s_t|y_{1:T}, s_{t+1:T}) \cdots p(s_1|y_{1:T}, s_{2:T}) \\ p(s_t|y_{1:T}, s_{t+1:T}) &\propto p(s_t|y_{1:t})p(s_{t+1}|s_t), \quad \text{where for } t = 1, \dots, T \end{aligned}$$

- ▶ prediction step: for  $j = 1, \dots, m$

$$p(s_t = j|y_{1:t-1}) = \sum_{k=1}^m p(s_t = j|s_{t-1} = k)p(s_{t-1} = k|y_{1:t-1})$$

- ▶ update step: for  $j = 1, \dots, m$

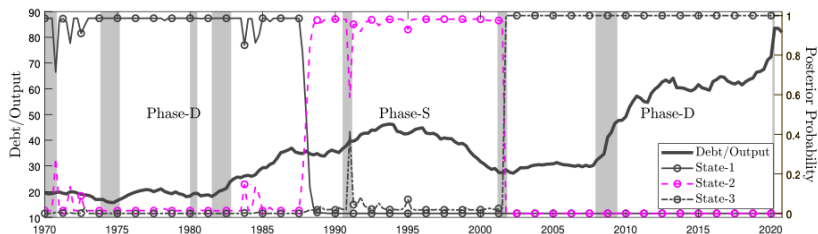
$$p(s_t|y_{1:t}) \propto p(s_t = j|y_{1:t-1})f(y_t|y_{1:t-1}, \theta_j)$$

- ▶ Sample each row of  $P$  from Dirichlet distribution

$$p_i \sim \mathcal{D}(\alpha_{i1}, \dots, \alpha_{im}) \quad \Rightarrow \quad p_i|s_{1:T} \sim \mathcal{D}(\alpha_{i1} + n_{i1}, \dots, \alpha_{im} + n_{im})$$

where  $n_{ij} = \#$  of one-step transitions  $i \rightarrow j$  in  $s_{1:T}$

# Application: Debt Cycles



►  $y_t = \Phi_{0,s_t} + \Phi_{1,s_t}y_{t-1} + \Phi_{2,s_t}y_{t-2} + u_t, u_t \sim \mathcal{N}(0, \Sigma_{s_t}),$

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} & 0 \\ 0 & p_{22} & 1 - p_{22} \\ 0 & 0 & 1 \end{bmatrix}$$

- 1970–2020 quarterly data: GDP growth rate, inflation rate, nominal interest rate, surplus-debt ratio, credit spread
- Tan (2022), “Appetite for Treasuries, Debt Cycles, and Fiscal Inflation,” manuscript

## Readings

- ▶ Chib (1993), "Bayes Regression with Autoregressive Errors: A Gibbs Sampling Approach," *Journal of Econometrics*
- ▶ Chib & Greenberg (1994), "Bayes Inference in Regression Models with ARMA( $p, q$ ) Errors," *Journal of Econometrics*
- ▶ Chib (1996), "Calculating Posterior Distributions and Modal Estimates in Markov Mixture Models," *Journal of Econometrics*
- ▶ Chib (1998), "Estimation and Comparison of Multiple Change-Point Models," *Journal of Econometrics*