Lecture 8 Multivariate Responses

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Introduction to Bayesian Statistics

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System of Equations

General setup

$$y_{ij} = x_{ij}'\beta_i + u_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

- Two important examples
  - Zellner's (1962) seemingly unrelated regression (SUR): small # of units n, large # of observations m (e.g. time)
  - panel (longitudinal) data model: large # of units n, small # of periods $m = T$

The Road Ahead...

1. SUR Model
2. Panel Data Model

SUR Model

Setup

$$\underbrace{\begin{bmatrix} y_{1j} \\ \vdots \\ y_{nj} \end{bmatrix}}_{y_j} = \underbrace{\begin{bmatrix} x'_{1j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x'_{nj} \end{bmatrix}}_{X_j} \underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} u_{1j} \\ \vdots \\ u_{nj} \end{bmatrix}}_{u_j}, \qquad j = 1, \ldots, m$$

- Likelihood function under $u_j | X \sim_{i.i.d.} \mathcal{N}(0, \Sigma)$

$$f(y|\beta, \Sigma) \propto \frac{1}{|\Sigma|^{m/2}} \exp\left[-\frac{1}{2} \sum_{j=1}^{m} (y_j - X_j\beta)' \Sigma^{-1} (y_j - X_j\beta) \right]$$

- Equivalent to single-equation OLS when (i) $x_{ij} = x_j$ (same regressors) or (ii) $\text{Cov}(u_{sj}, u_{tj}) = 0$ for $s \neq t$ (truly unrelated)

Gibbs Algorithm

- Conditionally conjugate prior

$$\beta \sim \mathcal{N}(\beta_0, B_0), \qquad \Sigma^{-1} \sim \mathcal{W}(\nu_0, V_0) \text{ (Wishart distribution)}$$

- Gibbs sampler for $\pi(\beta, \Sigma^{-1}|y)$

$$\begin{aligned} \beta|y, \Sigma^{-1} &\sim \mathcal{N}(\beta_1, B_1) \\ \Sigma^{-1}|y, \beta &\sim \mathcal{W}(\nu_1, V_1) \end{aligned}$$

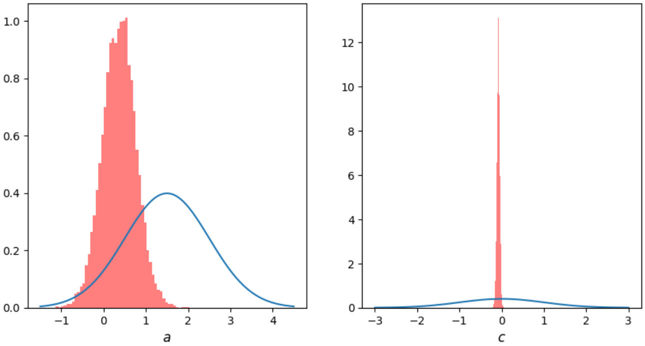
where (trick: $\mathrm{tr}(A_{p\times q}B_{q\times p}) = \mathrm{tr}(BA)$)

$$\begin{aligned} B_1 &= \left(\sum X_j'\Sigma^{-1}X_j + B_0^{-1}\right)^{-1} \\ \beta_1 &= B_1\left(\sum X_j'\Sigma^{-1}y_j + B_0^{-1}\beta_0\right) \\ \nu_1 &= \nu_0 + m \\ V_1 &= \left(V_0^{-1} + \sum (y_j - X_j\beta)(y_j - X_j\beta)'\right)^{-1} \end{aligned}$$

Python Code

```
def sur(y, x, n, b0, B0, nu0, V0):
    for i in range(1, n):
        B = inv(B0)
        for j in range(m):
            B += x[:, :, j].T @ s['inv_sig'][:, :,
                 i - 1] @ x[:, :, j]
        b = inv(B0) @ b0
        for j in range(m):
            b += x[:, :, j].T @ s['inv_sig'][:, :,
                 i - 1] @ y[j, :]
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=B1 @ b, cov=inv(B))
        V = inv(V0)
        for j in range(m):
            err = y[j, :].T - x[:, :, j] @ s['beta
                 '][i, :]
            V += err @ err.T
        s['inv_sig'][:, :, i] = wishart.rvs(df =
            nu1, size=1, scale=inv(V))
    return s
```

Application: Policy Interaction



- Monetary policy: $i_t = i^* + a(\pi_t - \pi^*) + b(y_t - y^*) + u_t$
- Fiscal policy: $s_t = s^* + c(b_{t-1} - b^*) + d(y_t - y^*) + v_t$
- $a \sim \mathcal{N}(1.5, 1)$, $c \sim \mathcal{N}(0, 1)$, 1990:Q1 - 2021:Q4
- Monetarist/Wicksellian vs. fiscal theory of price level

The Road Ahead...

1. SUR Model
2. Panel Data Model

Panel Data Model

Setup

$$\underbrace{\begin{bmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{bmatrix}}_{y_i} = \underbrace{\begin{bmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{bmatrix}}_{X_i} \beta + \underbrace{\begin{bmatrix} w'_{i1} \\ \vdots \\ w'_{iT} \end{bmatrix}}_{W_i} b_i + \underbrace{\begin{bmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{bmatrix}}_{u_i}, \qquad i = 1, \ldots, n$$

- Likelihood function under $u_i | X, W \sim_{i.i.d.} \mathcal{N}(0, h^{-1} I_T)$

  $$f(y|\beta, b, h) \propto h^{nT/2} \exp\left[-\frac{h}{2} \sum_{i=1}^{n} (y_i - X_i\beta - W_i b_i)'(y_i - X_i\beta - W_i b_i) \right]$$

  where β = fixed effect, b_i = random effect/heterogeneity

- Conditionally conjugate prior

  $$\beta \sim \mathcal{N}(\beta_0, B_0),\ h \sim \mathcal{G}(\alpha_0/2, \delta_0/2),\ b_i | D \sim \mathcal{N}(0, D),\ D^{-1} \sim \mathcal{W}(\nu_0, D_0)$$

Gibbs Algorithm

- Gibbs sampler for $\pi(h, D, (\beta, b)|y)$

$$h|y,\beta,b,D \sim \mathcal{G}(\alpha_1/2, \delta_1/2), \quad D^{-1}|y,\beta,h,b =_d D^{-1}|b \sim \mathcal{W}(\nu_1, D_1)$$

$$\beta, b|y,h,D: \quad b_i|\beta, y, D, h \sim \mathcal{N}(b_{1i}, D_{1i}), \quad \beta|y,D,h \sim \mathcal{N}(\beta_1, B_1)$$

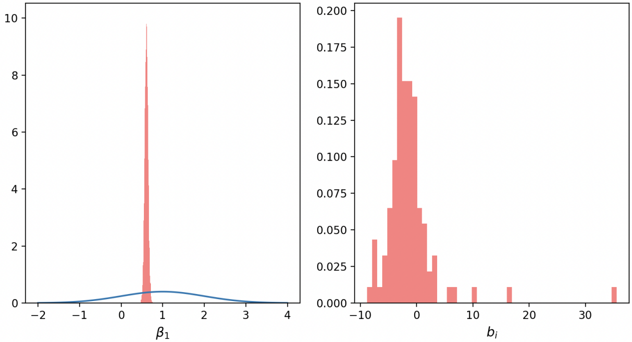
where (composition: (β, b) in one block)

$$\begin{aligned}
\alpha_1 &= \alpha_0 + nT,\ \delta_1 = \delta_0 + \sum (y_i - X_i\beta - W_i b_i)'(y_i - X_i\beta - W_i b_i) \\
\nu_1 &= \nu_0 + \mathrm{dim}(b),\ D_1 = \left(D_0^{-1} + \sum b_i b_i'\right)^{-1} \\
D_{1i} &= \left(hW_i'W_i + D^{-1}\right)^{-1} \\
b_{1i} &= D_{1i}[hW_i'(y_i - X_i\beta)] \\
B_{1i} &= W_i D W_i' + h^{-1} I_T \\
B_1 &= \left(\sum X_i' B_{1i}^{-1} X_i + B_0^{-1}\right)^{-1} \\
\beta_1 &= B_1 \left(\sum X_i' B_{1i}^{-1} y_i + B_0^{-1}\beta_0\right)
\end{aligned}$$

Python Code

```
def panel(y, x, w, m, b0, B0, a0, d0, nu0, D0):
    for i in range(1, m):
        # Sample h
        ...
        s['h'][i] = gamma.rvs(a1 / 2, size=1,
            scale=2 / d1)
        # Sample D^{-1}
        ...
        inv_D = wishart.rvs(df = nu1, size=1,
            scale=D1)
        # Sample b_i
        for j in range(n):
            ...
            s['b'][i, :, j] = multivariate_normal.
                rvs(size=1, mean=b1j, cov=D1j)
        # Sample beta
        ...
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=b1, cov=B1)
    return s
```

Application: Money Growth and Inflation



- $\pi_{it} = \beta_0 + \beta_1 m_{it} + b_i + u_{it}$, 104 countries, 2018–2021
- $\beta_0 \sim \mathcal{N}(0,1)$, $\beta_1 \sim \mathcal{N}(1,1)$, $h \sim \mathcal{G}(\frac{5}{2}, \frac{5}{2})$, $D^{-1} \sim \mathcal{W}(5,1)$
- Countries with higher money growth often experienced higher inflation

Readings

- Chib (2008), "Panel Data Modeling and Inference: A Bayesian Primer," *The Econometrics of Panel Data*
- Zellner (1962), "An Efficient Method of Estimating Seemingly Unrelated Regression Equations and Tests for Aggregation Bias," *Journal of the American Statistical Association*