Measurement Equation

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To understand how to specify measurement equations, we need to go through some preliminaries. Let the productivity (A) be given by a unit root process

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln Z_t \quad \text{where} \quad \ln Z_t = \rho \ln Z_{t-1} + \epsilon_t$$

This implies that technology (and hence output, consumption, etc. on a balanced growth path) grows at a gross rate of γ.[1] Moreover, the steady-state gross real interest rate can be written as $r^* = \gamma/\beta$ where β is the discount factor. Thus, the steady-state net real interest rate can be approximated as

$$r^* - 1 \approx (\gamma - 1) + (\frac{1}{\beta} - 1)$$

where we applied the property that if $a, b > 1$ and $a, b \approx 1$, then $ab - 1 \approx (a-1) + (b-1)$.

Let π^* denote the steady-state gross inflation rate. Then the steady-state gross nominal interest rate is $R^* = \pi^* r^*$, whose net value can be approximated similarly as

$$\begin{aligned} R^* - 1 &\approx (\pi^* - 1) + (\gamma - 1) + (\frac{1}{\beta} - 1) \\ &= \frac{1}{400}\pi^{(A)} + \frac{1}{100}\gamma^{(Q)} + \frac{1}{400}r^{(A)} \end{aligned}$$

where $\pi^{(A)}$ is annualized net inflation rate, $\gamma^{(Q)}$ is quarterly net growth rate, and $r^{(A)}$ is annualized net real interest rate. Caution: these parameters are the numbers in front of % that have to be estimated jointly with the DSGE model parameters. For example, $\pi^{(A)} = 12$ means a quarterly net inflation rate of 3%.

Now we are ready to link observable economic data (all in net values) to unobservable model variables (all quarterly) via specifying measurement equations. Following the convention, we always assume that the model variable $\hat{x}_t$ denotes the log-deviation of X_t from its steady-state X^*.[2] Three examples:

1. Suppose that we have the data on quarterly per capita GDP growth rate, YGR_t.[3] YGR_t can be obtained as follows: [i.] divide the nominal GDP by population 16 years

[1] Note that in steady state, $Z^* = 1$.

[2] Here is an important subtlety. Since $\hat{x}_t$ has a very small value (it approximates the percentage deviation of X_t from X^*), it is convenient to estimate all standard deviation parameters scaled by 100 as in An and Schorfheide (2007). Smets and Wouters (2007), however, implicitly defines $\hat{x}_t$ as log-deviation scaled by 100 and directly estimates the standard deviation parameters. This difference can be seen from the slightly different measurement equations in the aforementioned papers. It also matters for how we interpret the scale of impulse response functions.

[3] For instance, $\text{YGR}_t = 5$ means a quarterly per capita net GDP growth rate of 5% in period t.

and older; [ii.] deflate using GDP deflator; [iii.] take log difference; and [iv.] multiply by 100. Then the measurement equation can be derived as

$$\begin{aligned}\mathrm{YGR}_t &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} \times 100 \quad \text{(definition)} \\ &= \left(\frac{Y_t/A_t}{Y_{t-1}/A_{t-1}}\frac{A_t}{A_{t-1}} - 1\right) \times 100 \quad \text{(detrending)} \\ &\approx \ln\left(\frac{y_t}{y_{t-1}}\gamma Z_t\right) \times 100\end{aligned}$$

where $y_t = Y_t/A_t$ and we used the property that $c-1 \approx \ln c$ for $c > 1$ and $c \approx 1$. With some simple manipulations, we can easily show that

$$\mathrm{YGR}_t \approx \gamma^{(Q)} + 100(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) \tag{1}$$

2. Suppose that we have the data on annual inflation rate, INF_t. INF_t can be obtained as follows: [i.] take log difference of GDP deflator; and [ii.] multiply by 400. Then the measurement equation can be derived as

$$\begin{aligned}\mathrm{INF}_t &= 400(\pi_t - 1) \quad \text{(definition)} \\ &= 400(\pi^* - 1) + 400[(\pi_t - 1) - (\pi^* - 1)] \\ &\approx \pi^{(A)} + 400\hat{\pi}_t \end{aligned} \tag{2}$$

where $\pi_t - 1 \approx \ln \pi_t$ and $\pi^* - 1 \approx \ln \pi^*$.

3. Suppose that we have the data on annual nominal interest rate, INT_t. INT_t corresponds to the effective federal funds rate multiplied by 100. Then the measurement equation can be derived as

$$\begin{aligned}\mathrm{INT}_t &= 400(R_t - 1) \quad \text{(definition)} \\ &= 400(R^* - 1) + 400[(R_t - 1) - (R^* - 1)] \\ &\approx 400[(\gamma - 1) + (\frac{1}{\beta} - 1) + (\pi^* - 1)] + 400\hat{R}_t \\ &= 4\gamma^{(Q)} + r^{(A)} + \pi^{(A)} + 400\hat{R}_t \end{aligned} \tag{3}$$

where $R_t - 1 \approx \ln R_t$ and $R^* - 1 \approx \ln R^*$.

In estimation, equations (1)–(3) can be taken as equalities. In case the number of structural disturbances is less than that of observables, some measurement errors may be added. Last but not least, because output growth, inflation, and interest rate, etc. are themselves very small numbers in absolute value, we fit the model to the *scaled* data (numbers in front of %) in the above examples.