Prior Distribution

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We consider four types of commonly used prior distributions in the Bayesian estimation of DSGE model parameters. Typically one starts with some *a priori* knowledge, e.g. by running reduced-form regressions, about prior means and standard deviations (or variances). In what follows, we illustrate how these knowledge can be used to pin down the quantities that parameterize each prior distribution.

1. Let τ be the parameter characterizing the degree of risk aversion. Since $\tau > 0$, we can impose a Gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$. Its pdf is given by

$$p(\tau|a,b) = \frac{1}{\Gamma(a)b^a}\tau^{a-1}e^{-\frac{\tau}{b}}, \qquad \tau \in (0,\infty)$$

   Then we can obtain

$$\mathbb{E}(\tau) = ab \tag{1}$$

$$\mathbb{V}(\tau) = ab^2 \tag{2}$$

2. Let ρ be the autoregressive parameter characterizing the persistency of a structural shock. Since $0 < \tau < 1$, we can impose a Beta distribution with shape parameters $a > 0$ and $b > 0$. Its pdf is given by

$$p(\rho|a,b) = \frac{1}{B(a,b)}\rho^{a-1}(1-\rho)^{b-1}, \qquad \rho \in (0,1)$$

   Then we can obtain

$$\mathbb{E}(\rho) = \frac{a}{a+b} \tag{3}$$

$$\mathbb{V}(\rho) = \frac{ab}{(a+b)^2(a+b+1)} \tag{4}$$

3. Let γ^Q be the quarterly net growth rate. Since γ^Q is unrestricted, we can impose a Normal distribution with mean $a = \mathbb{E}(\gamma^Q)$ and standard deviation $b = \mathbb{V}(\gamma^Q)$. Its pdf is given by

$$p(\gamma^Q|a,b) = \frac{1}{b\sqrt{2\pi}}e^{-\frac{(\gamma^Q-a)^2}{2b^2}}, \qquad \tau \in (-\infty,\infty)$$

4. Let σ be the standard deviation parameter characterizing the dispersion of a structural shock. One typically imposes an Inverse-Gamma type-II (IG-2) distribution on σ^2 with shape parameter $a > 0$ and rate parameter $b > 0$. Its pdf is given by

$$p(\sigma^2|a,b) = \frac{b^a}{\Gamma(a)}(\sigma^2)^{-a-1}e^{-\frac{b}{\sigma^2}}, \qquad \sigma^2 \in (0,\infty)$$

Then we can obtain

$$\mathbb{E}(\sigma^2) = \frac{b}{a-1}, \qquad a > 1 \tag{5}$$

$$\mathbb{V}(\sigma^2) = \frac{b^2}{(a-1)^2(a-2)}, \qquad a > 2 \tag{6}$$

Sometimes we are interested in deriving the implied distribution for σ, which is also referred to as the Inverse-Gamma type-I (IG-1) distribution. This can be achieved by the usual transformation technique:

$$\begin{aligned} p(\sigma|a,b) &= \frac{2b^a}{\Gamma(a)}\sigma^{-2a-1}e^{-\frac{b}{\sigma^2}}, \qquad \sigma \in (0,\infty) \\ &\propto \sigma^{-\nu-1}e^{-\frac{\nu s^2}{2\sigma^2}} \end{aligned}$$

where $\nu = 2a$ and $s = \sqrt{b/a}$ as in An and Schorfheide (2007). Then we can obtain

$$\begin{aligned} \mathbb{E}(\sigma) &= \frac{2b^a}{\Gamma(a)}\int_0^\infty \sigma^{-2a}e^{-\frac{b}{\sigma^2}}d\sigma \\ &= \frac{b^a}{\Gamma(a)}\int_0^\infty t^{-a-\frac{1}{2}}e^{-\frac{b}{t}}dt \qquad (\text{set } t = \sigma^2) \\ &= \frac{b^a}{\Gamma(a)}\frac{\Gamma(\tilde{a})}{b^{\tilde{a}}}\underbrace{\frac{b^{\tilde{a}}}{\Gamma(\tilde{a})}\int_0^\infty t^{-\tilde{a}}e^{-\frac{b}{t}}dt}_{=\frac{b}{\tilde{a}-1}} \qquad (\text{set } \tilde{a} = a + 1/2) \\ &= \frac{\sqrt{b}\Gamma(a-0.5)}{\Gamma(a)}, \qquad a > 0.5 \end{aligned} \tag{7}$$

Combining (7) and (5) yields

$$\mathbb{V}(\sigma) = \mathbb{E}(\sigma^2) - [\mathbb{E}(\sigma)]^2 = \frac{b}{a-1} - \frac{b\Gamma(a-0.5)^2}{\Gamma(a)^2}, \qquad a > 1 \tag{8}$$

Observe that if $\sigma^2 \sim$ IG-2(a,b), then $1/\sigma^2 \sim$ Gamma$(a, 1/b)$. This means we can sample from IG-1(a,b) or its equivalent form, IG-1(ν, s), by taking the reciprocal of the positive square root for the random draws from Gamma$(a, 1/b)$.

In each case there are two equations in two unknowns (a,b) from which we may solve for (a,b) in terms of prior means and standard deviations (or variances).