

# Lecture 2: Extensive Form Games

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## The Road Ahead

1. From Simultaneous to Sequential Games
2. Subgame Perfect Equilibrium
3. Backward Induction
4. Multiple Subgame Perfect Equilibria
5. Making Threats Credible
6. Commitment Problems
7. Backward Induction with N Players
8. Forward Induction

## From Simultaneous to Sequential Games

**Simultaneous Move Games:** players move at the same time or cannot observe each other's moves

- Prisoner's dilemma: prisoners in separate interrogation rooms
- Football: offense and defense call plays simultaneously
- Meeting coordination: couple choosing locations without communication

**Sequential Games:** strategic interactions that flow over time in specific steps

- Military invasion followed by response decision
- Police officer requesting search permission, then deciding thoroughness
- Chess: white moves, black responds, cycle repeats
- Firm entry followed by incumbent's competitive response

## Introducing Selten's Game

Firm 1 considers entering Firm 2's monopoly market. If Firm 1 enters, Firm 2 must decide whether to accommodate entry or wage a price war.

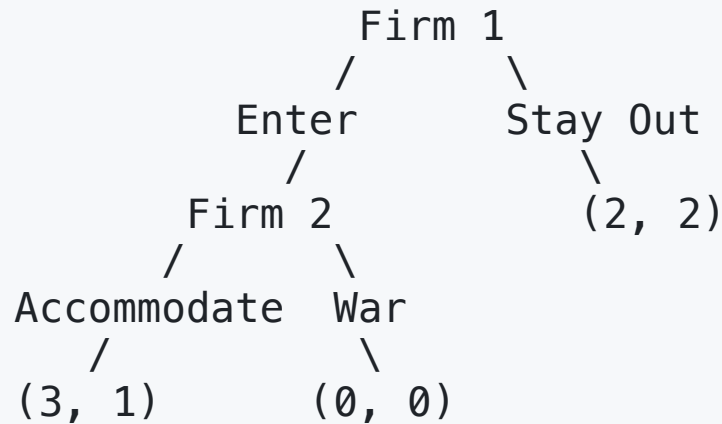
### Payoff Structure:

- **Price war:** both firms earn 0 (profits eliminated)
- **Accommodation:** Firm 1 earns 3, Firm 2 earns 1
- **No entry:** Firm 1 earns 2 (saves investment), Firm 2 earns 2 (maintains monopoly)

	Firm 2: Accommodate	Firm 2: War
Firm 1: Enter	3, 1	0, 0
Firm 1: Stay Out	2, 2	2, 2

## Extensive Form Games

Game Tree Representation of the firm entry game:



**Key Elements:**

- **Decision nodes:** where players make choices
- **Terminal nodes:** end points with payoffs
- **Branches:** represent available strategies
- **Sequential structure:** order of play is explicit

## Multiple Nash Equilibria in Strategic Form

Recall Selten's game has multiple Nash equilibria:

### 1. Pure strategy equilibria:

- (Enter, Accommodate): (3, 1)
- (Stay Out, War): (2, 2)

### 2. Mixed strategy equilibria:

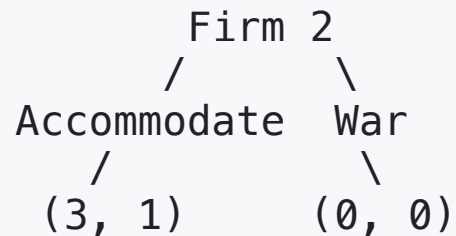
- Firm 1 plays Stay Out with probability 1
- Firm 2 plays War with probability  $\geq \frac{1}{3}$

Multiple equilibria make prediction difficult. Which one will actually occur?

**Sequential structure** helps us resolve this ambiguity!

## Analyzing Sequential Play

If Firm 1 enters, Firm 2 faces this decision:



**Firm 2's analysis:**  $1 > 0$ , so Firm 2 prefers Accommodate over War

**Firm 1's analysis** (knowing Firm 2 will accommodate):

- Enter  $\rightarrow$  Firm 2 accommodates  $\rightarrow$  payoff = 3
- Stay Out  $\rightarrow$  payoff = 2
- Since  $3 > 2$ , Firm 1 should enter

**Unique sequential solution:** (Enter, Accommodate) with payoffs (3, 1)

## Subgame Perfect Equilibrium

A **Subgame Perfect Equilibrium (SPE)** is a strategy profile where each player's strategy constitutes a Nash equilibrium in every subgame.

**Key insight:** SPE ensures that threats are **credible**

- Firm 2's threat to wage war is NOT credible
- Once Firm 1 enters, Firm 2 has no incentive to follow through
- Firm 1 recognizes this and enters anyway

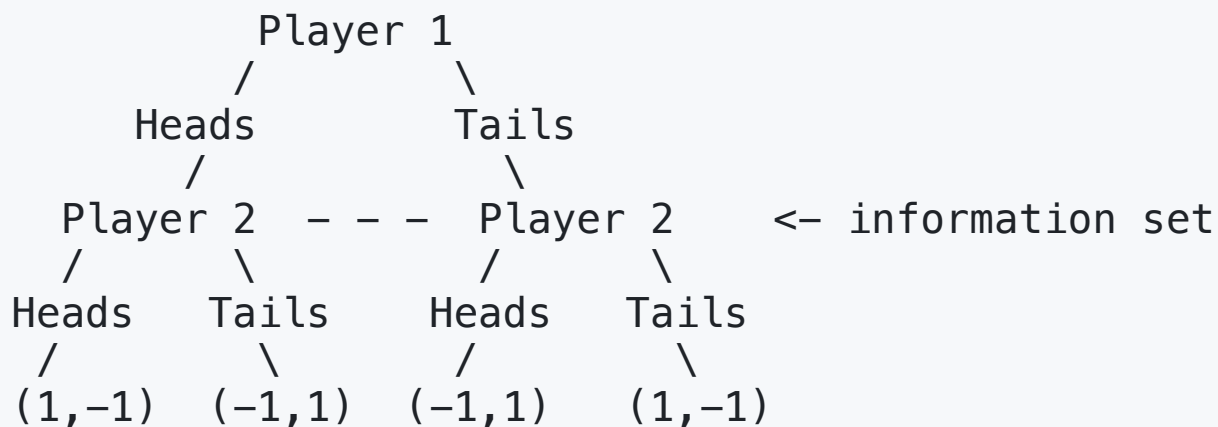
**Refinement:** SPE eliminates Nash equilibria that rely on non-credible threats

- All SPE are Nash equilibria
- Not all Nash equilibria are SPE
- SPE is the gold standard for extensive form games



## Games with Simultaneous Moves in Extensive Form

Some extensive form games include simultaneous moves, e.g. matching pennies



**Information set:** dashed line shows Player 2 cannot observe Player 1's choice

**Solution method:** convert to matrix form and solve as simultaneous game

## Constructing Games with Simultaneous Moves

Critical rules for information sets:

1. **Identical strategy sets:** same available actions at all nodes in the information set
2. **Irrelevance of player order:** whoever plays first, the resulting payoff matrix must be the same

**Example of violation:**

If Player 1 chooses Heads  $\rightarrow$  Player 2 chooses  $\{A, B\}$   
If Player 1 chooses Tails  $\rightarrow$  Player 2 chooses  $\{C, D\}$

**Problem:** Player 2 can infer Player 1's move from available actions, violating simultaneity

## Backward Induction

**Method:** solve extensive form games by working backwards from the end:

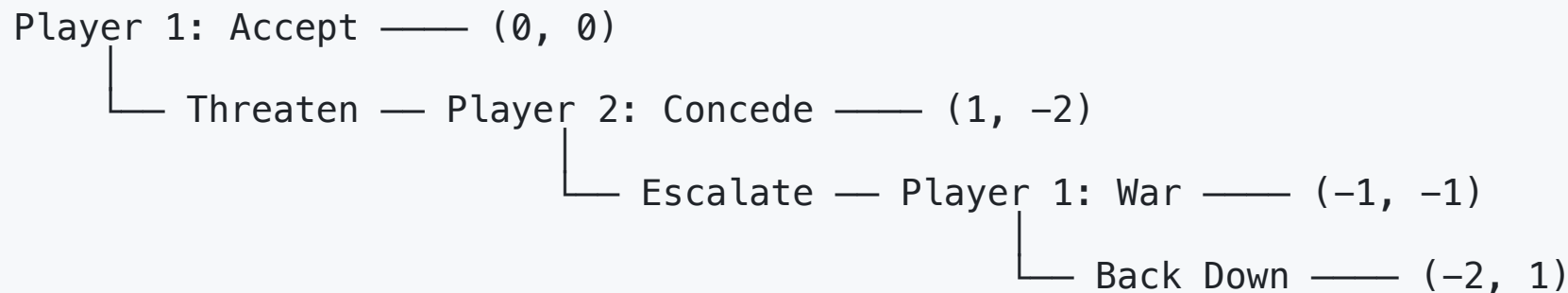
1. Start at the final decision nodes and determine optimal actions
2. Work backwards using optimal future play to determine current optimal actions
3. Continue until reaching the initial node

Backward induction always finds the subgame perfect equilibrium

**Note:** complete strategy must specify actions at ALL decision nodes, even those not reached in equilibrium

## Example: Escalation Game

Two countries are on the brink of war:



**Step 1:** Player 1 chooses between War (-1) and Back Down (-2). Since  $-1 > -2$ , choose War.

**Step 2:** Player 2 chooses between Concede (-2) and Escalate (leading to War: -1). Since  $-1 > -2$ , choose Escalate.

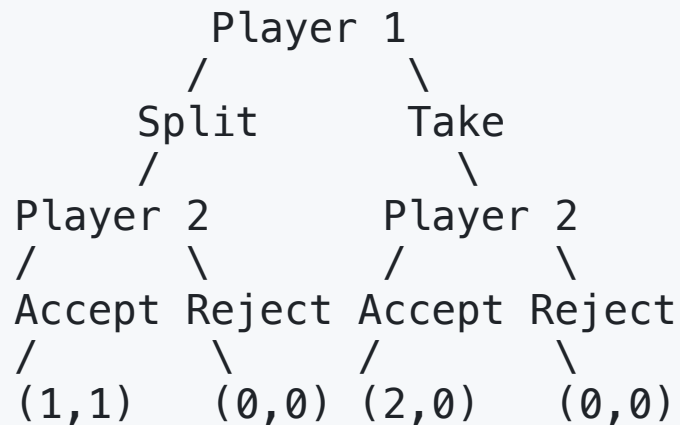
**Step 3:** Player 1 chooses between Accept (0) and Threaten (leading to Escalate→War: -1). Since  $0 > -1$ , choose Accept.

**SPE:**  $\langle (\text{Accept}, \text{War}), \text{Escalate} \rangle$  with outcome (0, 0)

## Example: Ultimatum Game

**Setup:** Player 1 has a good worth 2 and must bargain with Player 2 over division

- **Split:** Offer equal division (1, 1)
- **Take:** Attempt to take everything (2, 0)
- Player 2 can **Accept** or **Reject** any proposal
- Rejection leads to (0, 0) for both



## Multiple SPE in Ultimatum Game

**After Split:** Player 2 prefers Accept (payoff 1) over Reject (payoff 0)

**After Take:** Player 2 is **indifferent** between Accept (payoff 0) and Reject (payoff 0)

**Multiple Equilibria Arise** from Player 2's indifference:

1. **Player 2 always Accepts:** Player 1 takes (payoff  $2 > 1$ ), so SPE is  $\langle \text{Take}, (\text{Accept}, \text{Accept}) \rangle$
2. **Player 2 always Rejects:** Player 1 splits (payoff  $1 > 0$ ), so SPE is  $\langle \text{Split}, (\text{Accept}, \text{Reject}) \rangle$
3. **Player 2 mixes:** Any probability  $p$  of accepting after "Take"
  - If  $p > 1/2$ : Player 1 takes
  - If  $p < 1/2$ : Player 1 splits
  - If  $p = 1/2$ : Player 1 indifferent, can mix

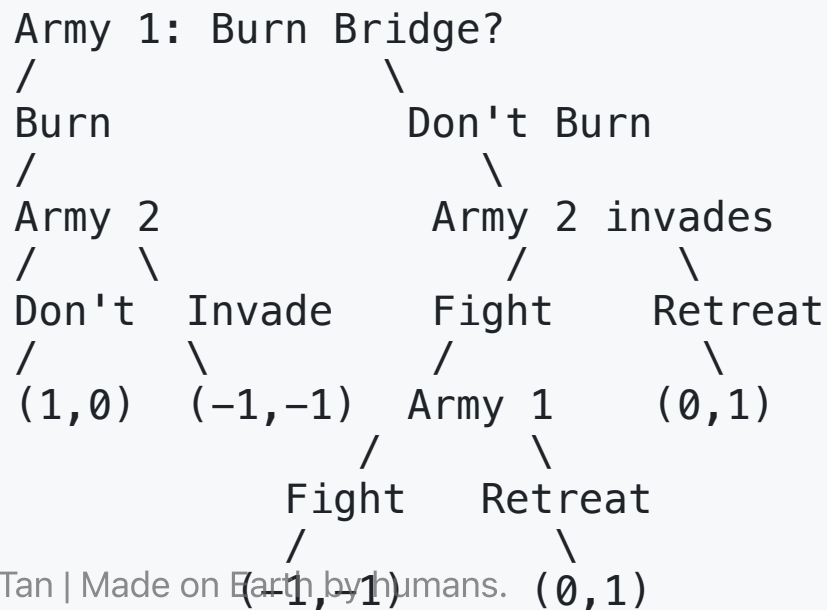
## Making Threats Credible: Burning Bridges

**The Classic Story:** Two armies fight over an island. Each has a bridge for access.

**Strategic Problem:**

- Island is valuable but not worth fighting for
- Each army prefers to concede rather than fight
- First army occupies island, second decides whether to invade

**Game Tree:**



## Bridge Burning Analysis

### Backward Induction:

**Step 1** - If bridge not burned and Army 2 invades:

- Army 1 chooses Fight(-1) vs Retreat(0) → Choose Retreat

**Step 2** - Army 2's decision if bridge not burned:

- Don't Invade(0) vs Invade(1, since Army 1 retreats) → Choose Invade

**Step 3** - Army 2's decision if bridge burned:

- Don't Invade(0) vs Invade(-1, since Army 1 must fight) → Choose Don't Invade

**Step 4** - Army 1's initial decision:

- Burn(1) vs Don't Burn(0) → Choose Burn

**SPE:**  $\langle (\text{Burn}, \text{Retreat}), (\text{Don't Invade}, \text{Invade}) \rangle$

**Key Insight:** Burning bridges makes the threat to fight **credible**!



## Tying Hands

**Story:** Boss wants to stop employee theft but prefers keeping employee over firing

**Without Warning:**

Employee steals → Boss: Fire(-1) vs Ignore(0) → Boss ignores → Employee steals

**With Warning** (Boss values reputation):

Employee steals after warning → Boss: Fire(-1) vs Ignore(-2) → Boss fires → Employee doesn't steal

**Analysis:**

- Warning changes Boss's payoffs (reputation cost for not following through)
- Makes threat to fire **credible**
- Employee responds by not stealing

**SPE:**  $\langle (\text{Warning, Ignore, Follow Through}), (\text{Steal, Don't Steal}) \rangle$

## Commitment Problems

**Definition:** Situations where the ability to commit to future actions provides strategic advantage

**Examples:**

1. **Burning Bridges:** Remove option to retreat, making fight threat credible
2. **Tying Hands:** Make deviation from threat extremely costly
3. **Reputation Building:** Invest in credibility of future threats

**Paradox:** Reducing your own options can **improve** your payoff

**Applications:**

- Military strategy and deterrence
- Business competition and market entry
- Labor negotiations and strikes
- International relations and sanctions

## The Pirate Game (5 Pirates, 10 Gold Coins)

### Rules:

- Hierarchy: Nash > Pirate 2 > Pirate 3 > Pirate 4 > Pirate 5
- Senior-most pirate proposes allocation
- Majority vote required (including proposer's vote)
- If proposal fails, proposer walks the plank, next pirate proposes
- All pirates are rational, greedy, but prefer life over death

**Question:** What allocation will Nash (most senior) propose?

## Pirate Game: Backward Induction

**If only Pirate 4 and 5 remain:** Pirate 4 proposes  $(10, 0)$  and wins with his own vote

**If Pirates 3, 4, 5 remain:**

- Pirate 3 needs 2 votes total (including his own)
- Pirate 4 rejects any offer (can get 10 coins if Pirate 3 dies)
- Pirate 5 gets 0 if Pirate 3 dies, so accepts any positive offer
- **Pirate 3's optimal proposal:**  $(9, 0, 1)$

**If Pirates 2, 3, 4, 5 remain:**

- Pirate 2 needs 2 votes total
- Pirate 3 gets 9 if Pirate 2 dies (expensive to buy)
- Pirate 4 gets 0 if Pirate 2 dies (cheap to buy - needs only 1 coin)
- **Pirate 2's optimal proposal:**  $(9, 0, 1, 0)$

## Pirate Game Solution

**Nash's Decision** (all 5 pirates):

- Needs 3 votes total (including his own)
- Pirate 2 gets 9 if Nash dies (too expensive)
- Pirates 3 and 5 get 0 if Nash dies (each needs only 1 coin)
- Pirate 4 gets 1 if Nash dies (needs 2 coins)

**Nash's Optimal Strategy:** Buy Pirates 3 and 5 with 1 coin each

**Final Allocation:** Nash(8), Pirate 2(0), Pirate 3(1), Pirate 4(0), Pirate 5(1)

**Votes:** Nash ✓, Pirate 2 ✗, Pirate 3 ✓, Pirate 4 ✗, Pirate 5 ✓

**Key Insight:** Even the most senior player cannot extract everything - must share just enough to create winning coalition

## Nim: Game of Perfect Information

**Setup:** 21 chips, players alternate taking 1 or 2 chips, last player to take chips wins

**Backward Induction Logic:**

- **1 chip left:** Player must take it and wins
- **2 chips left:** Player takes both and wins
- **3 chips left:** Player loses (opponent will win regardless)
- **4 chips left:** Player takes 1, leaving 3 for opponent → Player wins
- **5 chips left:** Player takes 2, leaving 3 for opponent → Player wins
- **6 chips left:** Player loses (opponent can force win)

**Pattern:** Player loses with 3, 6, 9, 12, 15, 18, 21 chips remaining

**21-chip game:** Player 1 starts with 21 chips and is in a losing position!

**Optimal strategy:** Leave opponent with multiple of 3 chips

## Forward Induction

**Definition:** Forward induction uses the assumption that all previous play was **rational** to make inferences about opponents' private information or strategies

**Key Difference from Backward Induction:**

- **Backward Induction:** Assumes optimal play in the future
- **Forward Induction:** Assumes rational play in the past

**Application:** When players observe actions that seem suboptimal, forward induction helps explain why those actions might actually be rational

**Requirement:** Extremely sophisticated thinking and strong rationality assumptions

## Forward Induction Example: Battle of Sexes with Burning Money

**Modified Battle of the Sexes:** Player 1 can burn money before choosing

**Standard Battle of the Sexes:**

	Ballet	Fight
Ballet	1, 2	0, 0
Fight	0, 0	2, 1

**With Money Burning:** Player 1 can burn \$1 before the game, changing his payoffs to (0,1) and (1,0)

**Forward Induction Logic:**

1. If Player 1 burns money, he must be planning to go to Fight (only way to get positive payoff)
2. Knowing this, Player 2 should also choose Fight
3. This gives Player 1 payoff of 1 (better than mixing in original game)
4. Therefore, Player 1 burns money and both go to Fight



## Takeaway Points

1. **Subgame Perfect Equilibrium** requires credible threats - eliminates Nash equilibria based on non-credible threats
2. **Backward Induction** systematically finds SPE by analyzing optimal play from the end of the game backwards
3. **Multiple SPE** can exist when players are indifferent between actions at some decision nodes
4. **Commitment Strategies** (burning bridges, tying hands) can improve payoffs by making threats credible
5. **Sequential games** often have unique predictions even when the corresponding simultaneous game has multiple equilibria
6. **Forward induction** uses rationality assumptions about past play to make inferences about opponents' strategies
7. **Complete strategies** must specify actions at all decision nodes, not just those reached in equilibrium

## Applications and Extensions

### Real-World Applications:

- **Business Strategy:** Entry deterrence, capacity investment, price wars
- **Military Strategy:** Nuclear deterrence, alliance formation
- **Political Science:** Legislative bargaining, coalition formation
- **Economics:** Auction design, contract theory, mechanism design

### Advanced Topics (beyond this course):

- Perfect Bayesian Equilibrium (incomplete information)
- Repeated games and reputation
- Bargaining theory
- Evolutionary game theory

**Next Lecture:** Advanced Strategic Form Games - dominance, rationalizability, and equilibrium refinements