Lecture 8: Perfect Bayesian Equilibrium

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The Road Ahead

Introduction: Finding Our Place

Our journey so far:

Game Timing	Information	Solution Concept
Simultaneous	Complete	Nash Equilibrium
Sequential	Complete	Subgame Perfect Equilibrium
Simultaneous	Incomplete	Bayesian Nash Equilibrium
Sequential	**Incomplete**	**Perfect Bayesian Equilibrium**

PBE combines the core ideas of SPE (handling sequential moves) and BNE (handling private information)

The Definition: What is PBE?

A **Perfect Bayesian Equilibrium** is a set of strategies and beliefs such that:

1. Strategies are **sequentially rational** given players' beliefs
2. Players update beliefs via **Bayes' rule** wherever possible

Critical insight: A PBE specifies BOTH strategies AND beliefs. Forgetting one part means the entire solution is incorrect

The Three Pillars of PBE

Pillar 1: Strategies AND Beliefs (A New Partnership)

- The solution must specify what players do (strategies) AND what they think (beliefs)
- This is an inseparable pair—leave out one, your answer is wrong

Pillar 2: Sequential Rationality (Making Threats Believable)

- Players must actually want to follow through on their stated plans at every decision point
- A threat's credibility depends on what you believe about your opponent's type

Pillar 3: Updating Beliefs with Bayes' Rule (Learning as You Go)

- Players start with prior beliefs and update them as they observe actions
- Bayes' Rule provides the logical method for updating beliefs on the equilibrium path

The Big Caveat: "Wherever Possible"

Challenge: What happens when a player observes an action that should never happen according to the equilibrium strategies?

- These situations are known as being **"off the equilibrium path"**
- Bayes' Rule cannot be applied (probability = 0 → divide by zero error)
- Defining beliefs in these unexpected scenarios is a key challenge in solving PBE

Key insight: Off-the-path beliefs matter because the threat of what might happen off the path justifies players' actions on the path

PBE Summary Table

Component	Its Job in the Game
Strategies	Action plans for each player at every possible point in the game
Beliefs	Probabilistic assessment of opponent's type, updated throughout
Sequential Rationality	Ensures strategies are logical and threats are credible
Bayes' Rule	Logical method for updating beliefs on the equilibrium path

Screening Games

Definition: A game of incomplete information where the **uninformed player moves first**

Key characteristics:

- Uninformed player cannot signal information (they don't have any)
- Uninformed player's action can "screen" or "test" the informed player
- Informed player responds differently based on their type

Result: The uninformed player's action separates (filters) different types by eliciting different responses

Screening Game Example: Escalation

Setup:

- Nature chooses Player 2's type: Weak (probability p) or Strong (probability $1 - p$)
- Player 1 (uninformed) chooses: Escalate or Quit
- Player 2 (informed, knows own type) chooses: Fight or Concede

Payoffs (if Player 1 quits): $(0, 1)$ regardless of type

Payoffs (if Player 1 escalates):

- Player 2 is Weak: Concede $(1, 0)$ or Fight $(0.7, -0.1)$
- Player 2 is Strong: Concede $(1, 0)$ or Fight $(-0.2, 0.8)$

Solving Screening Games: Step-by-Step

Step 1: Solve for the informed player (Player 2)

- Weak type: Concede (0 > -0.1)
- Strong type: Fight (0.8 > 0)

Step 2: Solve for the uninformed player (Player 1)

- Expected payoff from Quit: 0
- Expected payoff from Escalate: $p \times 1 + (1 - p) \times (-0.2) = 1.2p - 0.2$

Step 3: Compare payoffs

- Player 1 escalates if: $1.2p - 0.2 > 0$ → $p > \frac{1}{6}$
- Player 1 quits if: $p < \frac{1}{6}$

The Complete PBE Solution

If $p > \frac{1}{6}$ (Player 1 is optimistic):

- Player 1: Escalate
- Player 2: Concede if Weak, Fight if Strong
- Belief: Player 2 is Weak with probability p

If $p < \frac{1}{6}$ (Player 1 is pessimistic):

- Player 1: Quit
- Player 2: Concede if Weak, Fight if Strong (off-path)
- Belief: Player 2 is Weak with probability p

Remark: Player 2's strategy is a complete plan of action, even for actions that don't occur in equilibrium

Adverse Selection

Core question: "If a person accepts your transaction at some price, does the very fact that they accepted it mean you no longer want to go through with it?"

Definition: A market problem arising when one person has important private information that the other lacks

Key consequence: Information imbalance can lead to market failure where mutually beneficial trades don't happen

Adverse Selection: Insurance Example

Metric	Healthy Customer	Unhealthy Customer
Probability	60%	40%
Cost to Insure	\$400	\$800
Willingness to Pay	Up to \$750	Up to \$1,250

Insurer's dilemma:

- Low price (\$500): Expected profit = \$60 - \$120 = -\$60
- High price (\$1,000): Expected profit = \$0 + \$80 = \$80

Equilibrium outcome: Only high price offered → Only unhealthy buy → Market failure for healthy customers

Real-World Adverse Selection

The "Market for Lemons" (Used Cars):

- Seller knows car's true quality; buyer doesn't
- Buyer offers "average quality" price
- Only sellers with below-average cars (lemons) accept
- George Akerlof won Nobel Prize for this insight

Housing Market:

- Seller knows about hidden defects; buyer doesn't
- Low offers more likely accepted by sellers with costly hidden problems

Solutions to Adverse Selection

Reputation: Long-term incentives create trust

- Car dealership vs. random Craigslist seller
- Repeat business depends on honest dealing

Third-Party Verification: Independent experts level the playing field

- Home inspectors, mechanics

Government Regulation: Lemon laws re-allocate risk

- Buyer can return defective purchases

Government Intervention: Universal healthcare

- Pools everyone together, avoiding selection problems

Signaling Games

Definition: A game where the **informed player moves first**

Key difference from screening: The informed player's action can "signal" information about their private type

Critical insight: The first mover must think carefully about what their action communicates to the uninformed player

Result: Leads to fascinating outcomes—pooling, separating, and semi-separating equilibria

Three Types of Equilibria

Equilibrium Type	What Players Do	What Is Learned
Separating	Different types choose different actions	Everything—action perfectly reveals type
Pooling	All types choose the same action	Nothing—action provides no new information
Semi-Separating	One type picks one action, other type mixes	Something—partial information revealed

Separating Equilibrium: Actions Speak Louder

Example: Job market with High/Low type applicant

- High type → College
- Low type → High School

Key insight: Perfect information revelation

- Employer sees "College" → 100% certain applicant is High type
- Employer sees "High School" → 100% certain applicant is Low type

Signal is crystal clear: Informed player's action completely reveals their private information

Pooling Equilibrium: Hiding in the Crowd

Example: Job market where both types get high school diploma

- High type → High School
- Low type → High School

Key insight: No new information

- Employer sees "High School" → learns nothing
- Belief remains at prior: 50/50

Signal is uninformative: Action provides no way to distinguish types

Challenge: Must define off-the-path beliefs (what if someone goes to college?)

Semi-Separating: Strategic Bluffing

Example: Job market with mixed strategies

- High type → always College (pure strategy)
- Low type → sometimes College (30%), sometimes High School (70%) (mixed strategy)

Key insight: Partial information

- "High School" signal → 100% certain it's Low type (perfectly revealing)
- "College" signal → ambiguous, could be High type or bluffing Low type
- Employer must use Bayes' rule to update beliefs

Most strategically rich: Involves true bluffing behavior

War Game: Testing Separating Equilibrium

Setup: State 1 knows if it's Strong (60%) or Weak (40%)

- State 1 chooses: Reveal (small cost) or Hide
- State 2 wants to Fight Weak but Quit against Strong

Test 1: Strong Reveals, Weak Hides

- State 2's belief: Hide → 100% Weak → Fight
- Strong's deviation check: Reveal (0.99) vs. Hide (0.5) ✓
- Weak's deviation check: Hide (-1) vs. Reveal (-1.01) ✓
- **Result**: This IS a stable equilibrium

War Game: Failed Separating Equilibrium

Test 2: Strong Hides, Weak Reveals

- State 2's belief: Hide → 100% Strong → Quit
- Weak's deviation check: Reveal (-1.01) vs. Hide & Bluff (1) ×
- **Fatal flaw**: Weak type has massive incentive to bluff

Key lesson: Signals must be credible. No type should have a profitable deviation to lie or mimic another type

Pooling Equilibrium: War Game

Strategy: Both Strong and Weak Hide

Analysis:

- State 2's belief: Hide → No new info → remains 60% Strong, 40% Weak
- State 2's best response: Expected payoff = $0.6 \times (-1) + 0.4 \times 0.5 = -0.4$
- Since -0.4 < 0, State 2 Quits

Deviation checks:

- Strong: Hide (1) vs. Reveal (0.99) ✓
- Weak: Hide (1) vs. Reveal (-1.01) ✓

Result: This IS a stable pooling equilibrium

Off-the-Path Beliefs

The problem: When a player makes a zero-probability move that "should never happen" according to equilibrium strategies

- Bayes' Rule breaks down (divide by zero)
- Must define beliefs for these unexpected scenarios

Why it matters: The threat of what might happen off the path justifies players' actions on the path

Solution approach: Test all possible beliefs (p from 0 to 1) to see if any can sustain the equilibrium

Testing with Off-Path Beliefs

Example: Both types Reveal (alleged equilibrium)

- On-path: Player 2 knows types after Reveal
- Off-path: What if Player 1 Hides?
- Let p = Player 2's belief that Hider is Strong

Player 2's optimal action:

- Expected utility of Fight: $0.5 - 1.5p$
- Fight if $p < \frac{1}{3}$, Quit if $p > \frac{1}{3}$

Deviation checks show: Weak type always wants to deviate regardless of p

- **Result**: This alleged equilibrium FAILS

The Beer-Quiche Game: Setup

Players and Types:

- Player 1: Real Man (60%) prefers Beer, or Wimp (40%) prefers Quiche
- Both types want to avoid a fight
- Player 2: Coward who wants to fight only a Wimp

Sequence:

1. Player 1 chooses: Beer or Quiche
2. Player 2 observes meal, chooses: Fight or Quit

Best possible outcomes:

- Real Man: Beer + Player 2 Quits (3 points)
- Wimp: Quiche + Player 2 Quits (3 points)
- Player 2: Fight a Wimp (1 point)

Beer-Quiche: Pooling on Beer

Strategy: Both Real Man and Wimp drink Beer

Player 2's on-path response:

- Sees Beer → Belief remains 60% Real Man, 40% Wimp
- Expected payoff from Fight: $(0.6)(-1) + (0.4)(1) = -0.2$
- Since -0.2 < 0, Player 2 Quits

Deviation check: Would Wimp deviate to Quiche?

- Current payoff: 2 (Beer + no fight)
- Deviation payoff: 3 if Player 2 Quits, or 1 if Player 2 Fights
- To prevent deviation, Player 2 must Fight after seeing Quiche

Beer-Quiche: Off-Path Beliefs

Two solution classes:

Class 1: Fighting is strictly better ($P < \frac{1}{2}$)

- Player 2 believes quiche-eater is Real Man with probability $P < \frac{1}{2}$
- Player 2 strictly prefers to Fight

Class 2: Indifference case ($P = \frac{1}{2}$)

- Player 2 is exactly indifferent between Fight and Quit
- Player 2 must Fight with probability $\sigma \geq \frac{1}{2}$ to deter Wimp's deviation
- Wimp must be kept indifferent: $2 \geq \sigma(1) + (1 - \sigma)(3)$ → $\sigma \geq \frac{1}{2}$

Semi-Separating: Terrorist Game Setup

Players: Terrorist group (Player 1) vs. Target (Player 2)

- Nature chooses group type: Robust (40%) or Vulnerable (60%)
- Robust type: Attack is dominant strategy (always profitable)
- Vulnerable type: What to do?

Payoffs:

- No attack: $(0, 0)$
- Attack + Ignore: $(1, -1)$
- Vulnerable attacks + Resist: $(-2, 2)$
- Robust attacks + Resist: $(3, -3)$

Question: Can the Vulnerable type sometimes bluff?

Semi-Separating: The Indifference Method

Key principle: For a player to mix strategies, they must be indifferent between their choices

Step 1: Make Vulnerable group indifferent

- Let R = Probability Target resists
- Vulnerable's expected payoff from Attack: $(-2)R + (1)(1-R) = 0$
- Solving: $1 - 3R = 0 \rightarrow R = \frac{1}{3}$

Step 2: Make Target indifferent

- Let P = Target's belief attacker is Robust
- Target's expected payoff from Resist: $(-3)P + (2)(1-P) = -1$
- Solving: $2 - 5P = -1 \rightarrow P = \frac{3}{5}$

Semi-Separating: Finding Bluff Frequency

Step 3: Use Bayes' rule to find bluffing frequency

- Need $P(Robust|Attack) = \frac{3}{5}$
- Let σ_b = Probability Vulnerable attacks

Using Bayes' rule: $\frac{3}{5} = \frac{0.4 \times 1}{0.4 \times 1 + 0.6 \times \sigma_b}$

Solving: $\sigma_b = \frac{4}{9}$

Complete equilibrium:

- Robust: Always attacks
- Vulnerable: Attacks with probability $\frac{4}{9}$
- Target: Resists with probability $\frac{1}{3}$ after observing attack

Single Raise Poker

Setup: Player 1 dealt Ace (50%) or Queen (50%); Player 2 has King

- Player 1: Bet or Fold
- Player 2 (if P1 bets): Call or Fold

Key insight: Ace has dominant strategy (always Bet)

- Folding: -1
- Betting: +1 (if P2 folds) or +2 (if P2 calls)

Question: What should Queen do? Always bet? Never bet? Sometimes bet?

Single Raise Poker: Testing Pooling

Strategy: Both Ace and Queen always Bet

Player 2's response:

- Sees Bet → No new info → 50% Ace, 50% Queen
- Expected payoff from Call: $(0.5)(-2) + (0.5)(2) = 0$
- Since 0 > -1, Player 2 Calls

Queen's dilemma:

- Betting (with P2 calling): -2
- Folding: -1
- **Result**: Pooling fails—Queen wants to deviate to Fold

Single Raise Poker: Testing Separating

Strategy: Ace Bets, Queen always Folds

Player 2's belief: Bet → 100% Ace

Player 2's response: Fold (payoff -1 vs. -2 from calling)

Queen's temptation:

- Sticking with Fold: -1
- Deviating to Bet (P2 will fold): +1
- **Result**: Separating fails—Queen wants to bluff

Conclusion: Must have semi-separating equilibrium

Single Raise Poker: Semi-Separating Solution

Making Queen indifferent:

- Let σ_c = Probability P2 calls
- Queen's expected payoff from Bet: $\sigma_c(-2) + (1 - \sigma_c)(1) = -1$
- Solving: $1 - 3\sigma_c = -1$ → $\sigma_c = \frac{2}{3}$

Making Player 2 indifferent:

- Let p = Belief attacker is Queen given Bet
- P2's expected payoff from Call: $p(2) + (1 - p)(-2) = -1$
- Solving: $4p - 2 = -1$ → $p = \frac{1}{4}$

Finding bluff frequency: Using Bayes' rule with $p = \frac{1}{4}$ → Queen bets with probability $\frac{1}{3}$

Chain Store Paradox: Setup

Complete information version:

- Chain Store faces competitor in Town 1, then Rival in Town 2
- Weak store payoffs: 0 (Price War), 1 (Acquiesce), 3 (Rival quits)
- Backward induction → Always acquiesce

The "paradox": Logic says always be passive, but real firms fight price wars to build reputation

Resolution: The paradox arises from assuming complete information. Real world has uncertainty!

Chain Store: Incomplete Information

Two types of chain stores:

- Weak (90%): Price war payoff = 0
- Strong (10%): Price war payoff = 2.5

Game structure:

- Strong type: Always fights (dominant strategy)
- Weak type: What to do?

Rival's belief: Initially 90% chance store is Weak

Key insight: Uncertainty transforms price war into a credible signal, allowing strategic bluffing

Chain Store: Semi-Separating Equilibrium

Testing pure strategies:

- Separating (Weak acquiesces): Fails—Weak wants to bluff (payoff 3 > 2)
- Pooling (Weak always fights): Fails—Rival still enters, Weak gets payoff 1 < 2

Solution: Weak store bluffs sometimes

- Weak fights with probability $\frac{1}{9}$ to keep Rival indifferent
- Rival challenges with probability $\frac{1}{2}$ to keep Weak indifferent

Bottom line: Bluffing is rational! It prevents the weak store from being perfectly predictable and exploitable

Key Takeaways

Perfect Bayesian Equilibrium: Essential tool for sequential games with incomplete information

- Must specify BOTH strategies AND beliefs
- Sequential rationality ensures credible threats given beliefs
- Bayes' rule updates beliefs wherever possible

Screening vs. Signaling: Who moves first matters

- Screening: Uninformed moves first → action tests/filters types
- Signaling: Informed moves first → action reveals information

Three equilibrium types: Separating, Pooling, Semi-Separating

- Semi-separating involves strategic bluffing and mixed strategies
- Indifference conditions are key to solving mixed strategy equilibria

Applications Summary

Adverse Selection: Information asymmetry causes market failures

- Insurance, used cars, housing
- Solutions: reputation, verification, regulation

Beer-Quiche Game: Off-path beliefs enforce pooling equilibria

- Multiple equilibrium classes depending on beliefs

Poker & Chain Store: Semi-separating equilibria in action

- Bluffing emerges naturally from rational play
- Prevents predictability and exploitation

Common thread: Uncertainty about types creates strategic complexity and rich behavior