Lecture 3: Advanced Strategic Form Games

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The Road Ahead

Probability Distributions

A **probability distribution** is a set of events and the probability each event occurs

Examples:

- Coin flip: P(Heads) = 1/2, P(Tails) = 1/2
- Die roll: P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6
- Roulette: P(Red) = 18/38, P(Black) = 18/38, P(Green) = 2/38

Connection to game theory: Mixed strategies are probability distributions over pure strategies. Why This Matters?

- We'll work with complex probabilities like $\frac{x}{x+y+z}$
- Need to verify whether expressions form valid probability distributions
- Foundation for solving multi-strategy games

Golden Rules of Probability Distributions

Rule 1: All events occur with probability ≥ 0

Rule 2: The sum of all probabilities equals 1

Four key implications:

1. **No probability > 1**: If some probability exceeded 1, others would need to be negative to sum to 1
2. **Complete specification**: Cannot leave gaps (e.g., "world ends tomorrow with probability 1/100")
3. **Solving for unknowns**: If probabilities sum to 1, unknown probability = 1 - sum of known probabilities
4. **Pure strategies are special cases**: P(chosen strategy) = 1, P(all others) = 0

Generalized Battle of Sexes

Payoff matrix with variables constraints: A > B > C and a > b > c

	Left	Right
Up	B, a	C, c
Down	C, c	A, b

Mixed strategy equilibrium:

- Player 1 plays Up with probability $\frac{b-c}{a+b-2c}$
- Player 1 plays Down with probability $\frac{a-c}{a+b-2c}$
- Player 2 plays Left with probability $\frac{A-C}{A+B-2C}$
- Player 2 plays Right with probability $\frac{B-C}{A+B-2C}$

Key insight: Each player's mixing probability depends on the opponent's payoffs!

Generalized Prisoner's Dilemma

Payoff matrix with variable constraints: T > R > P > S and t > r > p > s

	Left (Cooperate)	Right (Defect)
Up (Cooperate)	R, r	S, t
Down (Defect)	T, s	P, p

Variable meanings:

- **T** = Temptation (defect when opponent cooperates)
- **R** = Reward (mutual cooperation)
- **P** = Punishment (mutual defection)
- **S** = Sucker (cooperate when opponent defects)

Result: Unique pure strategy Nash equilibrium at (Down, Right) = (Defect, Defect)

Why No Mixed Strategy Equilibrium?

Strict dominance analysis:

- Down strictly dominates Up for Player 1 (T > R and P > S)
- Right strictly dominates Left for Player 2 (t > r and p > s)

Mixed strategy algorithm:

Setting Player 2 indifferent → $\sigma_{up}(r + p - s - t) = p - s$

1. **Case 1:** $r + p - s - t = 0$ → $0 = p - s$ → Contradiction since $p > s$
2. **Case 2:** $r + p - s - t < 0$ → $\sigma_{up} < 0$ → Invalid probability
3. **Case 3:** $r + p - s - t > 0$ → $\sigma_{up} > 1$ → Invalid probability

Key insight: Strict dominance eliminates all mixed strategy possibilities

Comparative Statics

Study of how changes in game parameters affect equilibrium outcomes

Four-step process:

1. Solve for the game's equilibria
2. Calculate the element of interest (probabilities, payoffs, outcomes)
3. Take the derivative with respect to the parameter
4. Analyze how parameter changes affect the element

Key insight: Game theory often produces counterintuitive results!

Example: Soccer Penalty Kicks

Kicker has perfect accuracy right, accuracy x (where $0 < x < 1$) aiming left

	Goalie: Left	Goalie: Right
Kicker: Left	0, 0	x, -x
Kicker: Right	1, -1	0, 0

Mixed Strategy Nash Equilibrium:

- Goalie dives left with probability $\frac{x}{1+x}$
- Kicker aims left with probability $\frac{1}{1+x}$

Comparative static: $\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2} < 0$ → As kicker's left accuracy improves, he kicks left **less frequently**!

Strategic interaction: Goalie anticipates kicker's improved left accuracy and guards left more → kicker exploits the now less-defended right side → improved accuracy paradoxically shifts play toward the strong side.

The Volunteer's Dilemma

Two neighbors hear woman being attacked, must decide whether to call police

Payoffs:

- Woman's life worth 1, death worth 0
- Calling costs c where $0 < c < 1$
- If anyone calls, woman lives; if no one calls, she dies

	Call	Ignore
Call	1-c, 1-c	1-c, 1
Ignore	1, 1-c	0, 0

Mixed Strategy Nash Equilibrium:

- Each player ignores with probability c
- Each player calls with probability $1 - c$

Volunteer's Dilemma: Tragic Implications

Probability no one calls: c^2

Comparative static analysis:

$$\frac{d}{dc}(c^2) = 2c > 0$$

Result: As selfishness (c) increases, probability of woman's death increases

Bystander effect: Multiple potential helpers can lead to **less** help

- Each assumes someone else will act
- Coordination failure despite shared preferences
- Real-world relevance: Kitty Genovese case, public goods provision

Policy implication: Clear assignment of responsibility reduces coordination failure

Hawk-Dove Game: War and Peace

Crisis bargaining model: Two states decide whether to be aggressive (Hawk) or peaceful (Dove)

	Hawk	Dove
Hawk	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
Dove	$0, v$	$\frac{v}{2}, \frac{v}{2}$

Equilibrium depends on parameters:

- If $\frac{v}{2} > c$: Both play Hawk (war certain)
- If $\frac{v}{2} < c$: Mixed strategy with $P(\text{Hawk}) = \frac{v}{2c}$

Probability of war in mixed equilibrium: $\left(\frac{v}{2c}\right)^2 = \frac{v^2}{4c^2}$

Comparative static: $\frac{d}{dc}\left(\frac{v^2}{4c^2}\right) = -\frac{v^2}{2c^3} < 0$

Paradox of peace: Higher war costs → lower probability of war!

Baseball: Curveballs with Runner on Third

Setup: Pitcher chooses fastball/curveball, batter guesses. Curveball risk: ball might get past catcher

	Guess Fastball	Guess Curveball
Throw Fastball	-1, 1	0, 0
Throw Curveball	-x, x	-1-x, 1+x

Mixed Strategy Nash Equilibrium (when $0 < x < 1$):

- Batter guesses fastball with probability $\frac{1+x}{2}$
- Pitcher throws fastball with probability $\frac{1}{2}$

Comparative statics:

- Batter: $\frac{d}{dx}\left(\frac{1+x}{2}\right) = \frac{1}{2} > 0$ (more fastball guesses as risk increases)
- Pitcher: Strategy **independent** of x! Always throws 50-50 mix

Insight: Pitcher's strategy unaffected by wildness level (on relevant interval)

When Comparative Statics Don't Matter

Take or Share game: Some games have **trivial** comparative statics

	Take	Share
Take	$\frac{v}{2}, \frac{v}{2}$	$v, 0$
Share	$0, v$	$\frac{v}{2}, \frac{v}{2}$

Partially Mixed Strategy Nash Equilibria: $\langle \text{Take}, \sigma_{\text{take}} \rangle$ where $0 < \sigma_{\text{take}} < 1$

Comparative static with respect to v:

$$\frac{d\sigma_{\text{take}}}{dv} = 0$$

Result: Mixing probabilities completely **independent** of payoff magnitude

Lesson: Not all parameter changes affect strategic behavior!

Classic Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Observations:

- No pure strategy Nash equilibria (no cell with two positive payoffs)
- Each strategy beats one other and loses to one other
- Symmetric and zero-sum structure

No Pure Strategy Equilibria

Checking for pure strategy Nash equilibria:

Looking for cells where both players are playing best responses:

- (Rock, Rock): Player 1 gets 0, but could get 1 by switching to Paper
- (Paper, Paper): Player 1 gets 0, but could get 1 by switching to Scissors
- (Scissors, Scissors): Player 1 gets 0, but could get 1 by switching to Rock
- All off-diagonal cells: The losing player wants to deviate

Conclusion: No pure strategy Nash equilibria exist

Next step: Look for mixed strategy equilibria

Why Not Mix Between Only Two Strategies?

Consider Player 1 mixing between Rock and Paper only:

Player 2's analysis:

- If Player 2 plays Scissors, they beat both Rock (-1 to Player 1) and Paper (1 to Player 2)
- Scissors guarantees Player 2 a positive payoff regardless of Player 1's mixture

Player 1's expected payoff: Must be negative since Player 2 can guarantee positive payoff

Problem: This violates the zero-sum symmetry rule - Player 1 cannot have negative expected payoff in equilibrium

Same logic applies to all other two-strategy mixtures: Rock-Scissors, Paper-Scissors

The Mixed Strategy Nash Equilibrium

Since no pure strategy or two-strategy mixed equilibria exist, both players must mix among **all three strategies**.

Intuition: Each player must make the opponent indifferent among all three pure strategies

Let Player 2 use probabilities: $(\sigma_{\text{rock}}, \sigma_{\text{paper}}, \sigma_{\text{scissors}})$

Player 1's expected utilities:

- Playing Rock: $0 \cdot \sigma_{\text{rock}} + (-1) \cdot \sigma_{\text{paper}} + 1 \cdot \sigma_{\text{scissors}}$
- Playing Paper: $1 \cdot \sigma_{\text{rock}} + 0 \cdot \sigma_{\text{paper}} + (-1) \cdot \sigma_{\text{scissors}}$
- Playing Scissors: $(-1) \cdot \sigma_{\text{rock}} + 1 \cdot \sigma_{\text{paper}} + 0 \cdot \sigma_{\text{scissors}}$

Solving for the Mixed Strategy Equilibrium

Indifference conditions: Player 1 must be indifferent among all strategies

$$EU_{\text{Rock}} = EU_{\text{Paper}} = EU_{\text{Scissors}}$$

$$-\sigma_{\text{paper}} + \sigma_{\text{scissors}} = \sigma_{\text{rock}} - \sigma_{\text{scissors}} = -\sigma_{\text{rock}} + \sigma_{\text{paper}}$$

Plus the constraint: $\sigma_{\text{rock}} + \sigma_{\text{paper}} + \sigma_{\text{scissors}} = 1$

Solution:

From the first equality: $\sigma_{\text{rock}} = \sigma_{\text{paper}} = \sigma_{\text{scissors}}$

Since they sum to 1: $\sigma_{\text{rock}} = \sigma_{\text{paper}} = \sigma_{\text{scissors}} = \frac{1}{3}$

Mixed Strategy Nash Equilibrium: Both players play each strategy with probability $\frac{1}{3}$

Generalized Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock	0, 0	-x, x	y, -y
Paper	x, -x	0, 0	-z, z
Scissors	-y, y	z, -z	0, 0

Constraints: $x > 0, y > 0, z > 0$ (maintains the cyclical dominance structure)

Interpretation: Different strategies have different "lethality" against each other

- Large x: Paper devastates Rock
- Large y: Rock crushes Scissors
- Large z: Scissors obliterate Paper

Solving the Generalized Game

Player 1's expected utilities (assuming Player 2 uses $\sigma_{\text{rock}}, \sigma_{\text{paper}}, \sigma_{\text{scissors}}$):

$$EU_{\text{Rock}} = -x\sigma_{\text{paper}} + y\sigma_{\text{scissors}}$$

$$EU_{\text{Paper}} = x\sigma_{\text{rock}} - z\sigma_{\text{scissors}}$$

$$EU_{\text{Scissors}} = -y\sigma_{\text{rock}} + z\sigma_{\text{paper}}$$

Setting equal for indifference:

$$EU_{\text{Rock}} = EU_{\text{Paper}} = EU_{\text{Scissors}}$$

Using constraint $\sigma_{\text{scissors}} = 1 - \sigma_{\text{rock}} - \sigma_{\text{paper}}$

Solution to Generalized Game

Solving the system of equations:

From $EU_{\text{Rock}} = EU_{\text{Scissors}}$:

$$\sigma_{\text{paper}} = \frac{y}{x+y+z}$$

From $EU_{\text{Paper}} = EU_{\text{Scissors}}$:

$$\sigma_{\text{rock}} = \frac{z}{x+y+z}$$

From the constraint:

$$\sigma_{\text{scissors}} = \frac{x}{x+y+z}$$

Mixed Strategy Equilibrium:

- Play Rock with probability $\frac{z}{x+y+z}$
- Play Paper with probability $\frac{y}{x+y+z}$
- Play Scissors with probability $\frac{x}{x+y+z}$

Counterintuitive Results

Surprising insight: The probability of playing each strategy depends on the **other strategies' effectiveness**!

- Probability of Scissors = $\frac{x}{x+y+z}$ where x is Paper's advantage over Rock
- As Paper gets better at beating Rock (x increases), players use Scissors **more often**
- This happens because opponents anticipate the increased Paper usage

Intuition:

1. Larger x makes Paper more attractive against Rock
2. Anticipating more Paper play, Scissors becomes more valuable
3. Equilibrium shifts toward more Scissors usage to counteract Paper's strength

Real-world application: Character selection in fighting video games

Example: Specific Values

Consider the game with $x = 2, y = 1, z = 1$:

	Rock	Paper	Scissors
Rock	0, 0	-2, 2	1, -1
Paper	2, -2	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Equilibrium probabilities:

- Rock: $\frac{1}{4}$, Paper: $\frac{1}{4}$, Scissors: $\frac{1}{2}$

Paper's doubled effectiveness against Rock leads to **doubling** the Scissors probability!

Mixed Strategies as Population Parameters

Alternative interpretation: Mixed strategies represent **population distributions** rather than individual randomization

Video game example:

- Players don't randomize between characters in each match
- Instead, they specialize in one character (pure strategy)
- The population contains different types of players

Population game:

- Individual players choose pure strategies (e.g., always Rock)
- Random matchmaking pairs players from large population
- Mixed strategy equilibrium tells us population distribution needed for individual indifference

Population Equilibrium Analysis

Setup: Large population where:

- Fraction $\frac{z}{x+y+z}$ specialize in Rock
- Fraction $\frac{y}{x+y+z}$ specialize in Paper
- Fraction $\frac{x}{x+y+z}$ specialize in Scissors

Individual optimality: A Rock specialist's expected payoff when randomly matched:

$$EU_{\text{Rock}} = 0 \cdot \frac{z}{x+y+z} + (-x) \cdot \frac{y}{x+y+z} + y \cdot \frac{x}{x+y+z} = 0$$

Key insight: All specialists earn the same expected payoff (zero), so no individual wants to switch specializations

Result: Everyone plays pure strategies, yet the population achieves mixed strategy equilibrium proportions

Applications and Implications

Real-world examples:

1. **Online gaming**: Character selection in multiplayer games
2. **Business strategy**: Product positioning in competitive markets
3. **Evolution**: Species adaptation and survival strategies
4. **Financial markets**: Trading strategy distributions

Why this matters:

- Explains diversity in competitive environments
- No central coordination needed - emerges from individual optimization
- Stable population distributions even with pure strategy players
- Provides foundation for evolutionary game theory

Key takeaway: Mixed strategy equilibria can represent **aggregate behavior** of purely strategic individuals

Computational Example

Given: $x = 3$, $y = 2$, $z = 1$ in generalized Rock-Paper-Scissors

Step 1: Calculate total $x + y + z = 6$

Step 2: Find equilibrium probabilities

- $\sigma_{\text{rock}} = \frac{z}{x+y+z} = \frac{1}{6}$
- $\sigma_{\text{paper}} = \frac{y}{x+y+z} = \frac{2}{6} = \frac{1}{3}$
- $\sigma_{\text{scissors}} = \frac{x}{x+y+z} = \frac{3}{6} = \frac{1}{2}$

Verification: Each player's expected payoff equals zero

- $EU_{\text{Rock}} = -3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = -1 + 1 = 0 \checkmark$

Interpretation: When Scissors devastates Paper ($z = 3$), half the population should specialize in Scissors

Strategy-Proofness and Robustness

Important properties of mixed strategy equilibria in symmetric zero-sum games:

1. **Individual rationality**: No player can improve by unilateral deviation
2. **Population stability**: No subset of players can coordinate profitable changes
3. **Robustness to information**: Equilibrium maintained even with limited knowledge of opponent strategies
4. **Scale invariance**: Results hold regardless of population size

Contrast with other games:

- Coordination games: Multiple equilibria, focal points matter
- Prisoner's dilemma: Unique equilibrium, but Pareto inefficient
- Battle of the sexes: Coordination problems, communication valuable

Unique feature: Symmetric zero-sum games have **conflict-free** mixed equilibria

Summary and Takeaways

Key insights from Chapter 3:

1. **Zero-sum symmetry principle**: Players earn zero expected payoff in equilibrium
2. **Indifference principle**: Mixed strategies make opponents indifferent among pure strategies
3. **Counterintuitive effects**: Strategy probabilities depend on **other** strategies' payoffs
4. **Population interpretation**: Mixed strategies as distributions of specialized players
5. **Computational methods**: System of indifference equations plus probability constraints

Next steps:

- Games with infinite strategy spaces (continuous choices)
- Incomplete information and Bayesian games
- Evolutionary stability and dynamics

Practice: Apply these methods to analyze competitive scenarios in your field of

Takeaway Points

1. In symmetric, zero-sum games, each player's payoff in equilibrium must equal 0.
2. Mixed strategies can be thought of as population parameters instead of single players randomizing over choices.
3. The indifference principle: In mixed strategy equilibria, players must be indifferent among all strategies in their support.
4. Counterintuitive result: The probability of playing a strategy often depends more on other strategies' payoffs than its own direct payoffs.