Lecture 7: Bayesian Nash Equilibrium

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Date: November 6, 2025

The Road Ahead

1. Incomplete Information
2. Bayesian Nash Equilibrium
3. Solving for BNE
4. Dominance
5. Purification
6. Bayes' Rule

Why Information Matters

In strategic interactions, what you know—and what you don't know—is everything

Three types of information structures:

- Perfect and complete information: Know all past actions and all payoffs (e.g., sequential games with known payoffs)
- Imperfect information: Uncertainty about past actions, but all payoffs are known (e.g., simultaneous moves)
- Incomplete information: Know past actions, but uncertainty about payoffs, preferences, or "types"

Why it matters? Different information structures require completely different solution concepts

Imperfect Information

A player does not know the previous actions of other players

Example: Prisoner's Dilemma

- Both prisoners choose simultaneously
- Player 2 doesn't observe Player 1's choice, vice versa (imperfect information)
- But both know all possible payoffs (complete information)

	Cooperate	Defect
Cooperate	3, 3	1, 4
Defect	4, 1	2, 2

Incomplete Information

Uncertainty about another player's payoffs, preferences, or "type"

Examples:

Scenario	Hidden Characteristics
Soccer Penalty Kick	Striker's accuracy to left vs. right
Volunteer's Dilemma	Cost of calling for help (busy vs. leisurely)
Battle of the Sexes	Intensity of preference for ballet

Key point: You don't know what things are worth to your opponent

Elements of A Game

Complete information: Players, actions, payoffs

Incomplete information: Two additional elements

- **Player types**: Different variations of a player with private information about payoffs or capabilities
  - Each player knows their own type but opponents do not
  - Example: Striker knows if they're more accurate kicking left or right
- **Player beliefs**: Probabilities assigned to opponent types
  - Common prior assumption: All players know each other's beliefs at all levels
  - Example: Goalie believes 50% chance striker is left-accurate

Bayesian Nash Equilibrium (BNE)

A set of strategies—one for each type of player—such that no type has incentive to deviate, given beliefs about types and what other types are doing:

- Focus on types: Check for deviations by each type, not just each player
- Contingency plan: One strategy for each possible type
- Beliefs drive rationality: Player's optimal choice pivots on beliefs

BNE: Solution concept for simultaneous-move games with incomplete information

One-Sided Incomplete Information

Player 1 has two types with probabilities p and $1 - p$; Player 2 is uncertain which type they face

Type A Player 1:

Type A (prob = p)	Left	Right
Up	3, 4	1, 0
Down	4, 3	2, 0

Type B Player 1:

Type B (prob = 1-p)	Left	Right
Up	6, 2	0, 4
Down	5, 1	-1, 4

Finding Player 1's Strategy

Type A analysis: Compare Up vs. Down

- If Player 2 plays Left: Down (4) > Up (3)
- If Player 2 plays Right: Down (2) > Up (1)
- Down strictly dominates Up for Type A

Type B analysis: Compare Up vs. Down

- If Player 2 plays Left: Up (6) > Down (5)
- If Player 2 plays Right: Up (0) > Down (-1)
- Up strictly dominates Down for Type B

Player 1's equilibrium strategy: Type A plays Down; Type B plays Up

Finding Player 2's Strategy

Cannot use simple dominance because optimal action depends on Player 1's type

Expected utility:

- Left: $3p + 2(1-p) = p + 2$
- Right: $0p + 4(1-p) = 4 - 4p$

Player 2's best response:

- Play Left if $p + 2 > 4 - 4p \rightarrow p > \frac{2}{5}$
- Play Right if $p < \frac{2}{5}$
- Indifferent if $p = \frac{2}{5}$

Note: From Player 2's perspective, Player 1 appears to be "mixing" even though each type plays a pure strategy

Solving for BNE

Challenge: When games have multiple matrices (one per opponent type), how do we find the BNE?

Solution: The uninformed player needs a complete contingency plan specifying what to do for each possible type they might be:

1. Create compound strategies for the uncertain player
2. Calculate expected payoffs weighted by type probabilities
3. Solve for Nash equilibria in the combined payoff matrix → Bayesian Nash equilibria of the original game

Example: PD/SH Mixed Game

Player 1: Has one type (Stag Hunt preferences)

Player 2: Two possible types

- Prisoner's Dilemma type (probability 0.2)
- Stag Hunt type (probability 0.8)

P2: PD Type (0.2)	Left	Right
Up	3, 3	0, 4
Down	2, 1	1, 2

P2: SH Type (0.8)	Left	Right
Up	3, 3	0, 2
Down	2, 0	1, 1

Combined Payoff Matrix

Player 2's compound strategies: (Action if PD, Action if SH)

- (Left, Left): PD type plays Left and SH type plays Left
- (Left, Right): PD type plays Left and SH type plays Right
- (Right, Left): PD type plays Right and SH type plays Left
- (Right, Right): PD type plays Right and SH type plays Right

Expected payoffs: Weight outcomes by type probabilities

	(L, L)	(L, R)	(R, L)	(R, R)
Up	3, 3	0.6, 2.2	2.4, 3.2	0, 2.4
Down	2, 0.2	1.2, 1	1.8, 0.4	1, 1.2

Eliminating Dominated Strategies

PD type has dominant strategy to play Right (4 > 3, 2 > 1)

Eliminate (Left, Left): Dominated by (Right, Left)

- Compare Player 2's payoffs: 3.2 > 3 and 0.4 > 0.2
- Confirms PD type prefers Right when SH type plays Left

Eliminate (Left, Right): Dominated by (Right, Right)

- Compare Player 2's payoffs: 2.4 > 2.2 and 1.2 > 1
- Confirms PD type prefers Right when SH type plays Right

Simplified 2×2 game:

	(R, L)	(R, R)
Up	2.4, 3.2	0, 2.4
Down	1.8, 0.4	1, 1.2

Finding Nash Equilibria

Pure strategy equilibria:

1. (Up, Right-Left): Player 1 plays Up; P2's PD type plays Right, SH type plays Left
2. (Down, Right-Right): Player 1 plays Down; P2 plays Right regardless of type

Mixed strategy equilibrium:

- Player 1 mixes between Up (1/2) and Down (1/2)
- P2's PD type plays pure strategy Right (still dominant)
- P2's SH type mixes between Left (5/8) and Right (3/8)

Interpretation: What looks like Player 2 "mixing" might actually be different types playing different pure strategies

Information Paradox

Is more information always better? Intuitively, **YES**!

- In decision theory (one actor), this is true
- Can always ignore unhelpful information

But in game theory, **NO**!

- Extra information creates credible commitment problem
- You cannot credibly commit to ignore it
- Other players know you have it and adjust behavior

Lesson: In games, information value depends on strategic context

Coin Flip Game: Less Information

Two players, simultaneous moves

- Player 1: Play or Quit
- Player 2: Heads, Tails, or Pass
- Neither knows coin flip result yet

P1 Action	P2 Action	Payoff (P1, P2)
Quit	Any	(1, 1)
Play	Pass	(2, 2)
Play	H or T	(+3, -3) or (-3, +3), EV = (0, 0)

Equilibrium: P1 plays, P2 passes → (2, 2)

Coin Flip Game: More Information

Player 2 now sees coin flip before choosing

P2's new incentive:

- If P1 plays, P2 can guarantee herself 3 by calling correctly
- P2 will never Pass (getting 2) when she can get 3

P1's rational response:

- Knows P2 will use her information to win 3
- Means P1 gets -3 if he plays
- Better to Quit and get 1

Result: Only equilibrium is Quit with payoff (1, 1) → P2 is worse off with more information

Dominance Concepts

Two mindsets for evaluating strategies:

Mindset	Timing	Focus
Ex Ante	Before knowing your type	Overall player's grand strategy
Interim	After knowing your type	Specific type's optimal action

Poker analogy:

- Ex Ante: Planning what to do with Aces vs. Sevens before cards are dealt
- Interim: Choosing best move after seeing you have Aces

Interim Dominance

A strategy for a type is interim dominated if an alternative strategy for that type provides higher payoff, regardless of other players' strategies

Example: Player 2 as PD type

- If Player 1 plays Up: Right (4) > Left (3)
- If Player 1 plays Down: Right (2) > Left (1)
- Left is interim dominated for PD type

Question: "Given that I'm this type, is there a move that's always better for me?"

Ex Ante Dominance

A strategy for a player is ex ante dominated if an alternative strategy for that player provides higher payoff, regardless of other players' strategies

Example: Player 2's complete strategies

- Any plan where "PD type plays Left" is flawed
- Better plan: Switch PD type to Right, keep everything else same
- Eliminate (Left, Left) and (Left, Right)

Question: "Is there an overall plan that's always better for me, considering all types I could be?"

Relation Between Dominance Types

If a strategy is interim dominated for a type, then any complete strategy including that action is ex ante dominated for the player

- Interim dominance: "Playing Left is always bad for PD type"
- Ex ante implication: "Any plan that says 'if I'm PD type, play Left' must be suboptimal"

A strategy can be ex ante dominated without any individual type's action being interim dominated

- This can occur when probability of one type is overwhelming
- PD/SH mixed game: PD type 90%, SH type only 10%

Application: Use interim dominance to identify and eliminate ex ante dominated strategies in the combined payoff matrix

Example: Poker Without Antes

Simple poker game:

- Two players, each has own deck (2 through Ace)
- Each draws one card privately (their "type")
- Simultaneously choose: Bet $1 or Fold
- High card wins; ties split pot
- No initial pot (no ante)

The scale problem:

- 13 possible cards = 13 types per player
- Each type has 2 actions (Bet/Fold)
- Total strategies: $2^{13} = 8,192$ per player
- Payoff matrix would be 8,192 × 8,192 = 67 million cells!

Why Antes Matter?

Start with the Ace: Betting is weakly dominant

- Folding guarantees \$0
- Betting: Best case +\$1, worst case \$0 (tie)

The Two: Folding strictly dominates betting

- If opponent has Ace (which bets), you lose \$1
- Best case for betting is \$0 (tie)

The cascade begins: Two folds → Three has no one to beat → Folds → Four folds → ... → King folds → **BNE**:

- Only unbeatable hand (Ace) has incentive to bet
- Game collapses to "wait for perfect hand"

Solution: Create pot, make folding costly, force interaction across all hand strengths

Continuous Types

Challenge: When types aren't discrete, we face an infinite strategy space and cannot use finite payoff matrices

Solution: Cutpoint equilibrium C^* → All types below C^* take one action, all above take opposite

Example: Volunteer's Dilemma

- Two people, benefit = 1 for both, private cost $c \in [0, 1]$
- $F(C^*)$ = probability other's cost is below C^* (cumulative distribution function)
- If $c < C^*$: Volunteer; If $c > C^*$: Don't volunteer
- Every type plays pure strategy

Finding the Cutpoint

Indifference condition: At $c = C^*$, player is indifferent between actions

- Payoff from volunteering: $1 - C^*$
- Payoff from not volunteering: $F(C^*) \times 1 + (1 - F(C^*)) \times 0$

Equilibrium condition:

$$1 - C^* = F(C^*)$$

where a unique cutpoint C^* exists because $1 - C^*$ is strictly decreasing and $F(C^*)$ is increasing in C^*

Insight: Observed "mixing" is actually pure strategies

- Each type plays deterministically based on their cost
- Randomness comes from uncertainty about types

Interpreting Mixed Strategies

1. Players actually mix
   - Deliberate randomization to avoid exploitation
   - Poker, sports, competitive games
2. Population frequencies
   - Different players use different pure strategies
   - Games with random matching
3. Purification
   - Players have tiny private information differences
   - Each type plays pure strategy based on private info

Purification Theorem

Mixed strategies are pure strategies in disguise

- Add tiny private information (infinitesimal "nudge")
- Each type plays pure strategy based on their nudge
- As nudge → 0, game approaches original
- Probabilities match mixed strategy exactly

Example: Original game with complete information

	Left	Right
Up	0, 0	0, -1
Down	-1, 0	3, 5

Mixed strategy equilibrium: P1 plays Up with 5/6, P2 plays Left with 3/4

Adding Private Information

Perturbed game with types/shocks $\theta_1, \theta_2 \sim U[-1, 1]$ and small $\varepsilon > 0$

	Left	Right
Up	$\varepsilon\theta_1, \varepsilon\theta_2$	$\varepsilon\theta_1, -1$
Down	$-1, \varepsilon\theta_2$	$3, 5$

Pure strategy cutpoint equilibria:

- P1 plays Up if type θ_1 high enough (above cutpoint θ_1^*)
  → Probability P1 plays Up $= p = P(\theta_1 > \theta_1^*)$
- P2 plays Left if type θ_2 high enough (above cutpoint θ_2^*)
  → Probability P2 plays Left $= q = P(\theta_2 > \theta_2^*)$
- Note each player still uses pure strategy based on own type

Finding the Cutpoints

P1's indifference condition (when $\theta_1 = \theta_1^*$) → $\theta_1^* = \frac{3-4q}{\varepsilon}$

- Payoff from Up: $q \cdot \varepsilon\theta_1^* + (1-q) \cdot \varepsilon\theta_1^*$
- Payoff from Down: $q \cdot (-1) + (1-q) \cdot 3$

P2's indifference condition (when $\theta_2 = \theta_2^*$) → $\theta_2^* = \frac{5-6p}{\varepsilon}$

- Payoff from Left: $p \cdot \varepsilon\theta_2^* + (1-p) \cdot \varepsilon\theta_2^*$
- Payoff from Right: $p \cdot (-1) + (1-p) \cdot 5$

Probabilities depend on cutpoints

- $p = P(\theta_1 > \frac{3-4q}{\varepsilon}) = \frac{1-\frac{3-4q}{\varepsilon}}{2}$
- $q = P(\theta_2 > \frac{5-6p}{\varepsilon}) = \frac{1-\frac{5-6p}{\varepsilon}}{2}$

Upshot: As $\varepsilon \to 0$, $(p, q) = (5/6, 3/4)$ exactly matches mixed strategy equilibrium! Apparent randomness stems from uncertainty about private info

Bayes' Rule

Update beliefs based on new information

- Start with prior belief
- Observe signal/evidence
- Calculate posterior belief

Formula:

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

Intuition:

$$P(\text{State}|\text{Signal}) = \frac{P(\text{State}) \cdot P(\text{Signal}|\text{State})}{\text{Probability of observing this signal}}$$

Example: Cancer Test

60% chance cancer, 40% clean. 90% positive if cancer, 10% negative if cancer; 100% negative if clean. **Question**: What's cancer probability given negative test?

Probability Tree:

```
                              ┌─ Positive (0.90) → 0.60 × 0.90 = 0.54
          Cancer ─────────────┤
         (0.60)               └─ Negative (0.10) → 0.60 × 0.10 = 0.06
Start ──┤
          Clean ──────────────┬─ Positive (0.00) → 0.40 × 0.00 = 0.00
         (0.40)               └─ Negative (1.00) → 0.40 × 1.00 = 0.40
```

Updated belief: $P(\text{Cancer}|\text{Negative}) = \frac{0.06}{0.06+0.40} \approx 13\%$

Winner's Curse

Common value auction

- Item has single true value (same for everyone), but unknown
- Example: Oil deposits, sports free agent
- Contrast with independent value auction (autographed book)

Paradox: In auctions with uncertainty, winning is bad news

- Winner is person who most overestimated the value
- Lab experiments show people overbid and fall victim to curse; Professional bidders understand curse and bid conservatively

Key lesson: Must bid conditional on what winning tells you

Example: Oil Field Auction

Two companies bid for drilling rights:

- True value: $0 (prob 1/4), $25M (prob 1/2), or $50M (prob 1/4)
- Second-price sealed-bid auction
- Each gets private signal: "Low" or "High"

Signal structure:

True Value	Bidder 1 Signal	Bidder 2 Signal
$0	Low	Low
$25M	Low	High
$25M	High	Low
$50M	High	High

Question: How much should you bid based on your signal?

Naive Strategy

Observe low signal:

- $P(\$0|Low) = \frac{P(Low|\$0) \cdot P(\$0)}{P(Low)} = 0.5$
- $P(\$25M|Low) = \frac{P(Low|\$25M) \cdot P(\$25M)}{P(Low)} = 0.5$
- Expected value: $0.5 \times \$0 + 0.5 \times \$25M = \$12.5M$

Observe high signal:

- $P(\$25M|High) = \frac{P(High|\$25M) \cdot P(\$25M)}{P(High)} = 0.5$
- $P(\$50M|High) = \frac{P(High|\$50M) \cdot P(\$50M)}{P(High)} = 0.5$
- Expected value: $0.5 \times \$25M + 0.5 \times \$50M = \$37.5M$

Naive strategy: Bid your expected value (Low → \$12.5M, High → \$37.5M). Why this fails? For example, low bidder gets expected payoff of -\$3.125M (verify)! Better to bid \$0

Rational Strategy

Bayesian Nash equilibrium (verify):

- Low signal → Bid $0 (expected payoff $0)
- High signal → Bid $50M (expected payoff $12.5M)

What happens when the number of bidders increases?

- Standard first-price auctions: more bidders → higher bids due to competition
- Common value auctions: If you win against 999 other bidders, your estimate was higher than all 999 others → Overwhelmingly likely you overestimated → "Curse" becomes stronger with more competition