Lecture 2: Extensive Form Games

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The Road Ahead

From Simultaneous to Sequential Games

Simultaneous Move Games: players move at the same time or cannot observe each other's moves

- Prisoner's dilemma: prisoners in separate interrogation rooms
- Football: offense and defense call plays simultaneously
- Meeting coordination: couple choosing locations without communication

Sequential Games: strategic interactions that flow over time in specific steps

- Military invasion followed by response decision
- Police officer requesting search permission, then deciding thoroughness
- Chess: white moves, black responds, cycle repeats
- Firm entry followed by incumbent's competitive response

Introducing Selten's Game

Firm 1 considers entering Firm 2's monopoly market. If Firm 1 enters, Firm 2 must decide whether to accommodate entry or wage a price war.

Payoff Structure:

- **Price war**: both firms earn 0 (profits eliminated)
- **Accommodation**: Firm 1 earns 3, Firm 2 earns 1
- **No entry**: Firm 1 earns 2 (saves investment), Firm 2 earns 2 (maintains monopoly)

	Firm 2: Accommodate	Firm 2: War
Firm 1: Enter	3, 1	0, 0
Firm 1: Stay Out	2, 2	2, 2

Extensive Form Games

Game Tree Representation of the firm entry game:

```
                 Firm 1
                /      \
           Enter      Stay Out
             /           \
         Firm 2          (2, 2)
          /      \
   Accommodate  War
        /         \
   (3, 1)       (0, 0)
```

Key Elements:

- **Decision nodes**: where players make choices
- **Terminal nodes**: end points with payoffs
- **Branches**: represent available strategies
- **Sequential structure**: order of play is explicit

Multiple Nash Equilibria in Strategic Form

Recall Selten's game has multiple Nash equilibria:

1. **Pure strategy equilibria**:
   - (Enter, Accommodate): (3, 1)
   - (Stay Out, War): (2, 2)
2. **Mixed strategy equilibria**:
   - Firm 1 plays Stay Out with probability 1
   - Firm 2 plays War with probability $\geq \frac{1}{3}$

Multiple equilibria make prediction difficult. Which one will actually occur?

Sequential structure helps us resolve this ambiguity!

Analyzing Sequential Play

If Firm 1 enters, Firm 2 faces this decision:

```
         Firm 2
        /      \
   Accommodate  War
      /           \
   (3, 1)      (0, 0)
```

Firm 2's analysis: 1 > 0, so Firm 2 prefers Accommodate over War

Firm 1's analysis (knowing Firm 2 will accommodate):

- Enter → Firm 2 accommodates → payoff = 3
- Stay Out → payoff = 2
- Since 3 > 2, Firm 1 should enter

Unique sequential solution: (Enter, Accommodate) with payoffs (3, 1)

Subgame Perfect Equilibrium

A **Subgame Perfect Equilibrium (SPE)** is a strategy profile where each player's strategy constitutes a Nash equilibrium in every subgame.

Key insight: SPE ensures that threats are **credible**

- Firm 2's threat to wage war is NOT credible
- Once Firm 1 enters, Firm 2 has no incentive to follow through
- Firm 1 recognizes this and enters anyway

Refinement: SPE eliminates Nash equilibria that rely on non-credible threats

- All SPE are Nash equilibria
- Not all Nash equilibria are SPE
- SPE is the gold standard for extensive form games

Games with Simultaneous Moves in Extensive Form

Some extensive form games include simultaneous moves, e.g. matching pennies

```
          Player 1
         /        \
     Heads        Tails
       /            \
  Player 2  - - -  Player 2    <- information set
  /      \         /      \
Heads   Tails    Heads   Tails
 /         \      /         \
(1,-1)  (-1,1)  (-1,1)   (1,-1)
```

Information set: dashed line shows Player 2 cannot observe Player 1's choice

Solution method: convert to matrix form and solve as simultaneous game

Constructing Games with Simultaneous Moves

Critical rules for information sets:

1. **Identical strategy sets**: same available actions at all nodes in the information set
2. **Irrelevance of player order**: whoever plays first, the resulting payoff matrix must be the same

Example of violation:

```
If Player 1 chooses Heads → Player 2 chooses {A, B}
If Player 1 chooses Tails → Player 2 chooses {C, D}
```

Problem: Player 2 can infer Player 1's move from available actions, violating simultaneity

Backward Induction

Method: solve extensive form games by working backwards from the end:

1. Start at the final decision nodes and determine optimal actions
2. Work backwards using optimal future play to determine current optimal actions
3. Continue until reaching the initial node

Backward induction always finds the subgame perfect equilibrium

Note: complete strategy must specify actions at ALL decision nodes, even those not reached in equilibrium

Example: Escalation Game

Two countries are on the brink of war:

```
Player 1: Accept ──── (0, 0)
     │
     └── Threaten ─ Player 2: Concede ──── (1, -2)
                         │
                         └── Escalate ─ Player 1: War ──── (-1, -1)
                                              │
                                              └── Back Down ──── (-2, 1)
```

Step 1: Player 1 chooses between War (-1) and Back Down (-2). Since -1 > -2, choose War.

Step 2: Player 2 chooses between Concede (-2) and Escalate (leading to War: -1). Since -1 > -2, choose Escalate.

Step 3: Player 1 chooses between Accept (0) and Threaten (leading to Escalate→War: -1). Since 0 > -1, choose Accept.

SPE: ⟨(Accept, War), Escalate⟩ with outcome (0, 0)

Example: Ultimatum Game

Player 1 has a good worth 2 and must bargain with Player 2 over division

- **Split**: Offer equal division (1, 1)
- **Take**: Attempt to take everything (2, 0)
- Player 2 can **Accept** or **Reject** any proposal
- Rejection leads to (0, 0) for both

```
Player 1: Split — Player 2: Accept ——— (1, 1)
      │                  │
      │                  └── Reject ——— (0, 0)
      │
      └── Take — Player 2: Accept ——— (2, 0)
                         │
                         └── Reject ——— (0, 0)
```

Multiple SPE in Ultimatum Game

After **Split**: Player 2 prefers Accept (payoff 1) over Reject (payoff 0)

After **Take**: Player 2 is indifferent between Accept (payoff 0) and Reject (payoff 0)

Multiple equilibria arise from Player 2's indifference:

1. **Player 2 always Accepts**: Player 1 takes (payoff 2 > 1), so SPE is ⟨Take, (Accept, Accept)⟩
2. **Player 2 always Rejects**: Player 1 splits (payoff 1 > 0), so SPE is ⟨Split, (Accept, Reject)⟩
3. **Player 2 mixes**: Any probability p of accepting after "Take"
    - If p > 1/2: Player 1 takes
    - If p < 1/2: Player 1 splits
    - If p = 1/2: Player 1 indifferent, can mix with any probability q

Example: Weighted Matching Pennies

```
         Player 1
        /        \
     Stay        Go
     /            \
(-1/3, 4)      Player 2
              /        \
           Left        Right
           /              \
     Player 1 - - - - - Player 1
     /      \            /    \
    Up     Down        Up     Down
   /         \         /        \
(3,-3)     (-1,1)   (-2,2)    (0,0)
```

Why backward induction fails? It requires every decision node to have unique history, but Player 1's last decision (Up/Down) violates this requirement.

Valid subgame: Only the simultaneous portion after "Go" forms a proper subgame since Player 2's choice has unique history.

Multiple SPE in Weighted Matching Pennies

Solution method:

1. Solve the simultaneous subgame first: Player 1 plays Up with prob 1/6; Player 2 plays Left with prob 1/3
2. Replace subgame with expected payoffs (-1/3, 1/3)
3. Player 1 compares Stay(-1/3) vs Go(-1/3) → Indifferent!

Multiple SPE: Player 1 can mix with any probability between Stay and Go

Key insight: Simultaneous moves within extensive form games can create multiple SPE even when all payoffs are unique

Making Threats Credible

Two armies fight over an island. Each has a bridge for access:

- Island is valuable but not worth fighting for
- Each army prefers to concede rather than fight
- First army occupies island, second decides whether to invade

```
        Army 1: Burn Bridge?
         /              \
      Burn               Don't Burn
       /                    \
    Army 2                Army 2
     /    \               /      \
 Invade   Concede       Invade    Concede
   /         \           /           \
(-1,-1)     (1,0)     Army 1        (1,0)
                      /      \
                  Fight    Retreat
                   /          \
               (-1,-1)       (0,1)
```

Bridge Burning Analysis

Backward Induction:

Step 1 - If bridge not burned and Army 2 invades: Army 1 chooses Fight (-1) vs Retreat (0) → Choose Retreat

Step 2 - Army 2's decision if bridge not burned: Concede (0) vs Invade (1, since Army 1 retreats) → Choose Invade

Step 3 - Army 2's decision if bridge burned: Concede (0) vs Invade (-1, since Army 1 must fight) → Choose Concede

Step 4 - Army 1's initial decision: Burn (1) vs Not Burn (0) → Choose Burn

SPE: ⟨(Burn, Concede), (Not Burn, Retreat, Invade)⟩

Tying Hands: Deliberately limiting your future options to make a threat or commitment **credible**!

Commitment Problems

Definition: Situations where the ability to commit to future actions provides strategic advantage

Examples:

1. **Burning Bridges**: Remove option to retreat, making fight threat credible
2. **Tying Hands**: Make deviation from threat extremely costly
3. **Reputation Building**: Invest in credibility of future threats

Paradox: Reducing your own options can **improve** your payoff

Applications:

- Military strategy and deterrence
- Business competition and market entry
- Labor negotiations and strikes
- International relations and sanctions

The Pirate Game (5 Pirates, 10 Gold Coins)

Rules:

- Hierarchy: Nash > Pirate 2 > Pirate 3 > Pirate 4 > Pirate 5
- Senior-most pirate proposes allocation
- Majority vote required (including proposer's vote)
- If proposal fails, proposer walks the plank, next pirate proposes
- All pirates are rational, greedy, but prefer life over death

Question: What allocation will Nash (most senior) propose?

Pirate Game: Backward Induction

If only Pirate 4 and 5 remain: Pirate 4 proposes (10, 0) and wins with his own vote

If Pirates 3, 4, 5 remain:

- Pirate 3 needs 2 votes total (including his own)
- Pirate 4 rejects any offer (can get 10 coins if Pirate 3 dies)
- Pirate 5 gets 0 if Pirate 3 dies, so accepts any positive offer
- **Pirate 3's optimal proposal**: (9, 0, 1)

If Pirates 2, 3, 4, 5 remain:

- Pirate 2 needs 2 votes total
- Pirate 3 gets 9 if Pirate 2 dies (expensive to buy)
- Pirate 4 gets 0 if Pirate 2 dies (cheap to buy - needs only 1 coin)
- **Pirate 2's optimal proposal**: (9, 0, 1, 0)

Pirate Game Solution

Nash's Decision (all 5 pirates):

- Needs 3 votes total (including his own)
- Pirate 2 gets 9 if Nash dies (too expensive)
- Pirates 3 and 5 get 0 if Nash dies (each needs only 1 coin)
- Pirate 4 gets 1 if Nash dies (needs 2 coins)

Nash's Optimal Strategy: Buy Pirates 3 and 5 with 1 coin each

Final Allocation: Nash(8), Pirate 2(0), Pirate 3(1), Pirate 4(0), Pirate 5(1)

Votes: Nash ✓, Pirate 2 ✕, Pirate 3 ✓, Pirate 4 ✕, Pirate 5 ✓

Key Insight: Even the most senior player cannot extract everything - must share just enough to create winning coalition

Nim: Game of Perfect Information

Setup: 21 chips, players alternate taking 1 or 2 chips, last player to take chips wins

Backward Induction Logic:

- **1 chip left**: Player must take it and wins
- **2 chips left**: Player takes both and wins
- **3 chips left**: Player loses (opponent will win regardless)
- **4 chips left**: Player takes 1, leaving 3 for opponent → Player wins
- **5 chips left**: Player takes 2, leaving 3 for opponent → Player wins
- **6 chips left**: Player loses (opponent can force win)

Pattern: Player loses with 3, 6, 9, 12, 15, 18, 21 chips remaining

21-chip game: Player 1 starts with 21 chips and is in a losing position!

Optimal strategy: Leave opponent with multiple of 3 chips

Forward Induction

Definition: Forward induction uses the assumption that all previous play was **rational** to make inferences about opponents' private information or strategies

Key Difference from Backward Induction:

- **Backward Induction**: Assumes optimal play in the future
- **Forward Induction**: Assumes rational play in the past

Application: When players observe actions that seem suboptimal, forward induction helps explain why those actions might actually be rational

Requirement: Extremely sophisticated thinking and strong rationality assumptions

Forward Induction Example: Battle of Sexes with Burning Money

Modified Battle of the Sexes: Player 1 can burn money before choosing

Standard Battle of the Sexes:

	Ballet	Fight
Ballet	1, 2	0, 0
Fight	0, 0	2, 1

With Money Burning: Player 1 can burn $1 before the game, changing his payoffs to (0,1) and (1,0)

Forward Induction Logic:

1. If Player 1 burns money, he must be planning to go to Fight (only way to get positive payoff)
2. Knowing this, Player 2 should also choose Fight
3. This gives Player 1 payoff of 1 (better than mixing in original game)
4. Therefore, Player 1 burns money and both go to Fight

Takeaway Points

1. **Subgame Perfect Equilibrium** requires credible threats - eliminates Nash equilibria based on non-credible threats
2. **Backward Induction** systematically finds SPE by analyzing optimal play from the end of the game backwards
3. **Multiple SPE** can exist when players are indifferent between actions at some decision nodes
4. **Commitment Strategies** (burning bridges, tying hands) can improve payoffs by making threats credible
5. **Sequential games** often have unique predictions even when the corresponding simultaneous game has multiple equilibria
6. **Forward induction** uses rationality assumptions about past play to make inferences about opponents' strategies
7. **Complete strategies** must specify actions at all decision nodes, not just those reached in equilibrium

Applications and Extensions

Real-World Applications:

- **Business Strategy**: Entry deterrence, capacity investment, price wars
- **Military Strategy**: Nuclear deterrence, alliance formation
- **Political Science**: Legislative bargaining, coalition formation
- **Economics**: Auction design, contract theory, mechanism design

Advanced Topics (beyond this course):

- Perfect Bayesian Equilibrium (incomplete information)
- Repeated games and reputation
- Bargaining theory
- Evolutionary game theory

Next Lecture: Advanced Strategic Form Games - dominance, rationalizability, and equilibrium refinements