Lecture 3: Advanced Strategic Form Games

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The Road Ahead

1. Probability Distributions
2. Mixed Strategy Nash Equilibrium
3. Comparative Statics
4. Rock-Paper-Scissors Game

Probability Distributions

A **probability distribution** is a set of events and the probability each event occurs

Examples:

- Coin flip: P(Heads) = 1/2, P(Tails) = 1/2
- Die roll: P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6
- Roulette: P(Red) = 18/38, P(Black) = 18/38, P(Green) = 2/38

Connection to game theory: Mixed strategies are probability distributions over pure strategies. Why This Matters?

- We'll work with complex probabilities like $\frac{x}{x+y+z}$
- Need to verify whether expressions form valid probability distributions
- Foundation for solving multi-strategy games

Golden Rules of Probability Distributions

Rule 1: All events occur with probability ≥ 0

Rule 2: The sum of all probabilities equals 1

Four key implications:

1. **No probability > 1**: If some probability exceeded 1, others would need to be negative to sum to 1
2. **Complete specification**: Cannot leave gaps (e.g., "world ends tomorrow with probability 1/100")
3. **Solving for unknowns**: If probabilities sum to 1, unknown probability = 1 - sum of known probabilities
4. **Pure strategies are special cases**: P(chosen strategy) = 1, P(all others) = 0

Example: Generalized Battle of Sexes

Payoff matrix with variables constraints: A > B > C and a > b > c

	Left	Right
Up	B, a	C, c
Down	C, c	A, b

Mixed strategy equilibrium:

- Player 1 plays Up with probability $\frac{b-c}{a+b-2c}$
- Player 1 plays Down with probability $\frac{a-c}{a+b-2c}$
- Player 2 plays Left with probability $\frac{A-C}{A+B-2C}$
- Player 2 plays Right with probability $\frac{B-C}{A+B-2C}$

Key insight: Each player's mixing probability depends on the opponent's payoffs!

Example: Generalized Prisoner's Dilemma

Payoff matrix with variable constraints: T > R > P > S and t > r > p > s

	Left (Cooperate)	Right (Defect)
Up (Cooperate)	R, r	S, t
Down (Defect)	T, s	P, p

Variable meanings:

- **T/t** = Temptation (defect when opponent cooperates)
- **R/r** = Reward (mutual cooperation)
- **P/p** = Punishment (mutual defection)
- **S/s** = Sucker (cooperate when opponent defects)

Result: Unique pure strategy Nash equilibrium at (Down, Right) = (Defect, Defect)

Why No Mixed Strategy Equilibrium?

Strict dominance analysis:

- Down strictly dominates Up for Player 1 (T > R and P > S)
- Right strictly dominates Left for Player 2 (t > r and p > s)

Mixed strategy algorithm:

Setting Player 2 indifferent → $\sigma_{up}(r + p - s - t) = p - s$

1. **Case 1:** $r + p - s - t = 0$ → $0 = p - s$ → Contradiction since $p > s$
2. **Case 2:** $r + p - s - t < 0$ → $\sigma_{up} < 0$ → Invalid probability
3. **Case 3:** $r + p - s - t > 0$ → $\sigma_{up} > 1$ → Invalid probability

Key insight: Strict dominance eliminates all mixed strategy possibilities

Support of Mixed Strategies

A strategy is in the support if it's played with positive probability (> 0)

Golden rule: All strategies in the support must yield equal expected utility in equilibrium.

- If one strategy was better, the player would use it exclusively (probability = 1)
- **Indifference principle**: Opponents make you indifferent among your support strategies

Important caveat: Equal expected utility is necessary but not sufficient

- Some strategies may yield equal expected utility but still not be played
- Multiple equilibria can exist with different support structures

Weak Dominance

If one player mixes among all strategies, the opponent **cannot** use weakly dominated strategies in equilibrium.

Take-or-Share Game:

	Take	Share
Take	0, 0	8, 4
Share	0, 8	4, 4

Analysis: If Player 2 mixes between Take and Share, Player 1 must choose Take:

- When P2 plays Take: P1 gets 0 from either Take or Share (tie)
- When P2 plays Share: P1 gets 8 from Take vs 4 from Share (Take wins)
- Take yields strictly higher expected utility → P1 never plays Share

Key insight: Mixing converts weak dominance into **strict dominance**, eliminating strategies and simplifying equilibrium calculations

Comparative Statics

Study of how changes in game parameters affect equilibrium outcomes

Four-step process:

1. Solve for the game's equilibria
2. Calculate the element of interest (probabilities, payoffs, outcomes)
3. Take the derivative with respect to the parameter
4. Analyze how parameter changes affect the element

Key insight: Game theory often produces counterintuitive results!

Example: Soccer Penalty Kicks

Kicker has perfect accuracy right, accuracy x (where $0 < x < 1$) aiming left

	Goalie: Left	Goalie: Right
Kicker: Left	0, 0	x, -x
Kicker: Right	1, -1	0, 0

Mixed Strategy Nash Equilibrium:

- Goalie dives left with probability $\frac{x}{1+x}$
- Kicker aims left with probability $\frac{1}{1+x}$

Comparative static: $\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2} < 0$ → As kicker's left accuracy improves, he kicks left **less frequently**!

Strategic interaction: Goalie anticipates kicker's improved left accuracy and guards left more → kicker exploits the now less-defended right side → improved accuracy paradoxically shifts play toward the strong side

Example: Volunteer's Dilemma

Two neighbors hear woman being attacked, must decide whether to call police. Woman's life worth 1, death worth 0; calling costs c where $0 < c < 1$

	Call	Ignore
Call	1-c, 1-c	1-c, 1
Ignore	1, 1-c	0, 0

Mixed Strategy Nash Equilibrium: Each player calls with probability $1 - c$

Comparative static: Probability no one calls = c^2 → $\frac{d}{dc}(c^2) = 2c > 0$

Bystander effect: More potential helpers → less help! Each assumes someone else will act, creating coordination failure (e.g. public goods provision) → need clear assignment of responsibility

Example: Hawk-Dove Game

Two states decide whether to be aggressive (Hawk) or peaceful (Dove); v is the prize value and c is the cost of fighting.

	Hawk	Dove
Hawk	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
Dove	$0, v$	$\frac{v}{2}, \frac{v}{2}$

Equilibrium depends on parameters:

- If $\frac{v}{2} > c$: Both play Hawk (war certain)
- If $\frac{v}{2} < c$: Pure strategy (Hawk, Dove) and (Dove, Hawk); Mixed strategy with $P(\text{Hawk}) = \frac{v}{2c}$

Comparative static: Probability of war = $\frac{v^2}{4c^2} \rightarrow \frac{d}{dc}\left(\frac{v^2}{4c^2}\right) = -\frac{v^2}{2c^3} < 0$

Paradox of peace: Higher war costs → lower probability of war!

Rock-Paper-Scissors Game

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Observations:

- Cyclical dominance: No pure strategy Nash equilibria
- Any two-strategy support is exploitable: Omitted third strategy beats both members of the support
- Exploitation gives the opponent a guaranteed positive payoff and forces the mixer to a negative expected payoff—impossible in symmetric zero-sum equilibrium
- Conclusion: Both players must mix over all three strategies with probability $\frac{1}{3}$ for each strategy

Generalized Rock-Paper-Scissors Game

	Rock	Paper	Scissors
Rock	0, 0	-x, x	y, -y
Paper	x, -x	0, 0	-z, z
Scissors	-y, y	z, -z	0, 0

Player 1's expected utilities (Player 2 uses $\sigma_{\text{rock}} + \sigma_{\text{paper}} + \sigma_{\text{scissors}} = 1$):

- $EU_{\text{Rock}} = -x\sigma_{\text{paper}} + y\sigma_{\text{scissors}}$
- $EU_{\text{Paper}} = x\sigma_{\text{rock}} - z\sigma_{\text{scissors}}$
- $EU_{\text{Scissors}} = -y\sigma_{\text{rock}} + z\sigma_{\text{paper}}$

Mixed Strategy Equilibrium:

- Play Rock with probability $\frac{z}{x+y+z}$ (from $EU_{\text{Paper}} = EU_{\text{Scissors}}$)
- Play Paper with probability $\frac{y}{x+y+z}$ (from $EU_{\text{Rock}} = EU_{\text{Scissors}}$)
- Play Scissors with probability $\frac{x}{x+y+z}$ (from $EU_{\text{Rock}} = EU_{\text{Paper}}$)

Counterintuitive Results

The probability of playing each strategy depends on the **other strategies' effectiveness**, e.g.

- Probability of Scissors = $\frac{x}{x+y+z}$ where x is Paper's advantage over Rock
- As Paper gets better at beating Rock (x increases), players use Scissors **more often** because opponents anticipate the increased Paper usage
- Numeric check (x=2, y=1, z=1): equilibrium weights = $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$ — Paper's doubled effectiveness corresponds to doubling Scissors' weight.

Real-world application: Character selection in fighting video games

Mixed Strategies as Population Parameters

Alternative interpretation: Mixed strategies represent **population distributions** rather than individual randomization:

- Individual players choose pure strategies (e.g., always Rock)
- Random matchmaking pairs players from large population
- Mixed strategy equilibrium tells us population distribution needed for individual indifference, e.g. a Rock specialist's expected payoff when randomly matched: $EU_{\text{Rock}} = 0 \cdot \frac{z}{x+y+z} + (-x) \cdot \frac{y}{x+y+z} + y \cdot \frac{x}{x+y+z} = 0$

Key insight: All specialists earn the same expected payoff (zero), so no individual wants to switch specializations

Result: Everyone plays pure strategies, yet the population achieves mixed strategy equilibrium proportions