Lecture 4: Games with Infinite Strategy Spaces

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The Road Ahead

Infinite Strategy Spaces

So far all games had **finite** number of pure strategies represented by matrices or game trees, but many strategic situations involve **infinite** strategy choices:

- Firms choosing production levels (any real number ≥ 0)
- Politicians choosing policy positions (continuum of possibilities)
- Auction bidders choosing bid amounts (any positive value)
- Pricing decisions (continuous price range)

New challenge: Matrices and trees become impractical with infinite strategies. How to find equilibria without drawing payoff matrices?

Games Without Payoff Matrices

Game definition remains the same:

- Set of players
- Strategy spaces (now possibly infinite)
- Order of moves
- Payoff functions

Solution method for infinite games:

1. Consider a single pure strategy from one player
2. Find the other player's best responses to that strategy
3. Check whether the original strategy is a best response to any of those best responses
4. If yes → equilibrium found; if no → try next strategy
5. Repeat for all strategies

Advantage: No need to draw massive (impossible) matrices!

Example: Simple Infinite Game

Players 1 and 2 simultaneously select whole numbers between 1 and 100. Each player's payoff = product of the two numbers

Step 1: Suppose Player 2 chooses 1

- Player 1's payoff = $x \times 1 = x$ → best response: choose 100
- But Player 2's best response to 100 is 100 (not 1) → No equilibrium

Step 2: Suppose Player 2 chooses any $k < 100$

- Player 1's payoff = $x \times k$ → best response: choose 100
- But Player 2's best response to 100 is 100 (not k) → No equilibrium

Step 3: Suppose Player 2 chooses 100

- Player 1's payoff = $x \times 100$ → best response: choose 100
- Player 2's best response to 100 is 100 ✓ → **Equilibrium: (100, 100)**

Equilibrium Existence in Infinite Games

Nash's theorem: Every **finite** game has at least one Nash equilibrium. **Infinite** games may have no equilibrium!

Counter-example: Players select any number > 0, payoff = product of numbers

- Suppose Player 1 chooses any value x > 0
- Player 2's best response: choose something slightly larger than x
- But then Player 1 wants to choose something even larger
- Result: No equilibrium exists (infinite regress)

Hotelling's Game

Harold Hotelling's 1929 model: Two ice cream vendors on a beach

- Beach stretches from position 0 to position 1
- Customers evenly distributed along the beach
- Two vendors sell identical ice cream for $2 per cone
- Customers buy from nearest vendor (travel cost consideration)
- Vendors simultaneously choose locations

Strategy space: Each vendor chooses position $x \in [0, 1]$

Key constraint: Each vendor must serve exactly half the customers in equilibrium:
$EU_1 \geq 1/2$, $EU_2 \geq 1/2$, and $EU_1 + EU_2 = 1$

Equilibrium Analysis

Why vendors can't be equidistant from center (1/2):

- Suppose vendors at positions 1/4 and 3/4
- Vendor at 1/4 could move to 1/2: keeps left customers + steals some right customers
- This gives more than half the business → profitable deviation

Why both vendors must choose the same location:

- If both choose same position ≠ 1/2 (say both at 1/4)
- One vendor could deviate to position 1/2 → get all customers on far side plus some customers near original position
- Again a profitable deviation

Unique equilibrium: Both vendors locate at position 1/2 (center)

- Any deviation from center gives less than half the business
- No profitable deviations exist

Median Voter Theorem

Hotelling's game in politics: Presidential candidates as "vendors," voters as "customers":

- Two candidates choose policy positions on left-right spectrum
- Voters distributed along this spectrum with single-peaked preferences
- Each voter supports candidate closest to their ideal position
- Candidates want to maximize vote share

Key insight: Both candidates will converge to the **median voter's position**

Cournot Duopolistic Competition

Antoine Augustin Cournot's model: Firms compete over **quantities** produced

- Two firms simultaneously choose production quantities: $q_1, q_2 \geq 0$
- Market price determined by demand: $P = 900 - (q_1 + q_2)$
- Firm 1's production cost: \$12 per unit
- Firm 2's production cost: \$24 per unit
- Each firm maximizes profit

Strategy space: Each firm chooses $q \in [0, \infty)$

Equilibrium Analysis

Solution algorithm:

1. Profit functions: $EU_1 = 888q_1 - q_1^2 - q_1q_2$, $EU_2 = 876q_2 - q_1q_2 - q_2^2$
2. Best responses (partial derivatives): $q_1 = 444 - q_2/2$, $q_2 = 438 - q_1/2$
3. Solve system: Substitute to get $q_2 = 438 - (444 - q_2/2)/2 = 216 + q_2/4$

Equilibrium outcome:

- Firm 1: 300 units, Firm 2: 288 units, Market price: $312
- Lower-cost firm (Firm 1) produces more and gains larger market share

Second Price Auctions

n bidders compete for a single item

- Each bidder i has private valuation v_i for the item
- Sealed bid auction: highest bidder wins, pays **second-highest bid**
- Strategy space: Each bidder chooses bid $b_i \in [0, \infty)$

Payoff function:

$$\text{Payoff}_i = \begin{cases} v_i - (\text{second highest bid}) & \text{if bidder } i \text{ wins} \\ 0 & \text{if bidder } i \text{ loses} \end{cases}$$

Example: Albert bids \$10, Barbara bids \$13, Charlie bids \$0.13, Fei bids \$30; Fei wins but pays \$13

Question: What is the optimal bidding strategy?

Equilibrium Analysis

Theorem: Bidding your true valuation ($b_i = v_i$) is a **dominant strategy**

Proof: Consider any honest bidder i with $b_i = v_i$

Case 1 - Your bid wins:

- You pay the second-highest bid
- Increase bid? Still win and pay same amount
- Decrease bid? Either still win and pay same amount or lose and get nothing

Case 2 - Your bid loses:

- Increase bid? Either still lose or win but pay more than valuation
- Decrease bid? Still lose and receive nothing

Conclusion: No profitable deviation → Truth-telling is optimal regardless of others' strategies