Lecture 8: Perfect Bayesian Equilibrium

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The Road Ahead

1. Perfect Bayesian Equilibrium
2. Screening Games
3. Adverse Selection
4. Signaling Games
5. Separating Equilibrium
6. Pooling Equilibrium
7. Semi-Separating Equilibrium

Perfect Bayesian Equilibrium (PBE)

Finding our place in the game theory toolkit:

Game Timing	Information	Solution Concept
Simultaneous	Complete	Nash Equilibrium
Sequential	Complete	Subgame Perfect Equilibrium
Simultaneous	Incomplete	Bayesian Nash Equilibrium
Sequential	**Incomplete**	**Perfect Bayesian Equilibrium**

Key insight: PBE combines sequential rationality (from SPE) with beliefs about types (from BNE)

Definition of PBE

A Perfect Bayesian Equilibrium is a set of strategies and beliefs such that:

1. Strategies are sequentially rational given beliefs
2. Players update beliefs via Bayes' rule wherever possible

Critical point: A PBE solution requires BOTH strategies AND beliefs

- Leaving out one means your answer is wrong
- Common mistake: forgetting beliefs (like forgetting off-path strategies in SPE)

Why beliefs matter: In incomplete information, credibility of threats depends on what you believe about opponent types

Three Pillars of PBE

Pillar 1: Strategies AND Beliefs

- Solution must specify what players do (strategies) and what they think (beliefs)
- These are inseparable partners

Pillar 2: Sequential Rationality

- Strategy must be best choice at every point, given current beliefs
- Credibility filtered through beliefs: "Maybe I fight a weak type but not a strong type"

Pillar 3: Updating with Bayes' Rule

- Initial prior belief → Observe action → Updated posterior belief
- Example: If only strong types bully, seeing bully makes you certain opponent is strong
- Caveat: Only works "wherever possible" (when observed action has positive probability)

Off the Equilibrium Path

The problem: What if a player does something that should never happen?

- Proposed equilibrium assigns zero probability to this action
- Bayes' Rule breaks down (division by zero)
- Called "off the equilibrium path"

Example: In separating equilibrium where strong type Reveals and weak type Hides

- If we see Hide → Bayes' Rule tells us type is weak (on path)
- If we see unexpected action → Must specify belief without Bayes' Rule (off path)

Importance: Off-path beliefs justify on-path actions

- The threat of what would happen off path prevents players from deviating

Screening vs. Signaling Games

Two main categories of PBE applications:

Type	Who Moves First	Complexity	Signal Information
Screening	Uninformed player	Simpler	First move cannot signal private info
Signaling	Informed player	More complex	First move can signal type

Signaling game outcomes:

- Separating equilibrium: Different types choose different actions
- Pooling equilibrium: All types choose same action
- Semi-separating equilibrium: One type mixes, creating strategic bluffing

Screening Games

Definition: Incomplete information game where the uninformed player moves first

Example game structure:

1. Nature chooses Player 2's type: Weak (prob p) or Strong (prob $1 - p$)
2. Player 1 (uninformed) chooses: Escalate or Quit
3. Player 2 (informed) observes choice, then decides: Fight or Concede

Key feature: Dashed line in game tree shows Player 1's information set (uncertainty about type)

Solving a Screening Game

Step 1: Solve for informed player (Player 2)

- Weak type: Concede (0) > Fight (-0.1) → Always concede
- Strong type: Fight (0.8) > Concede (0) → Always fight

Step 2: Solve for uninformed player (Player 1)

- Payoff from Quit: 0 (certain)
- Expected payoff from Escalate: $p \times 1 + (1 - p) \times (-0.2) = 1.2p - 0.2$
- Escalate if $1.2p - 0.2 > 0$ → $p > \frac{1}{6}$

Equilibrium:

- If $p > \frac{1}{6}$: P1 escalates; P2 concedes if weak, fights if strong
- If $p < \frac{1}{6}$: P1 quits; (P2's strategy still specified but not used)

Why "Screening"?

Escalating serves as a test or "screen":

- Separates different types by eliciting different behaviors
- Weak type concedes, strong type fights
- Uninformed player can use this action to "filter" opponent types

Strategic choice: Whether to screen (escalate) or not (quit)

Contrast with signaling: In screening, first mover has no private information to reveal

Adverse Selection

"If a person accepts your transaction at some price, does the very fact that they accepted it mean that you no longer want to go through with it?"

Definition: Market failure when one party has crucial private information

- Information asymmetry leads to dysfunctional outcomes
- Prevents mutually beneficial trades

Key examples:

- Insurance: Healthy vs. unhealthy customers
- Used cars: "Market for lemons"
- Real estate: Hidden defects

Insurance Game Example

Setup:

- Insurer wants profit, Customer wants protection
- Private information: Customer knows if healthy or unhealthy

Metric	Healthy (60%)	Unhealthy (40%)
Cost to Insure	$400	$800
Willingness to Pay	$750	$1,250

Insurer's calculation:

- High price ($1,000): Only unhealthy accept → $0.6(0) + 0.4(200) = 80$
- Low price ($500): Both accept → $0.6(100) + 0.4(-300) = -60$

Result: Insurer offers only high price → Healthy customers left uninsured

Market Failure in Action

Why this is a problem: Mutually beneficial trade is missed

- If insurer knew customer was healthy: Offer $500, customer accepts, both better off
- Insurer makes $100 profit, customer gets valued insurance

Information asymmetry prevents this:

- Cannot safely offer low price (risk of unhealthy customers)
- High price drives away profitable healthy customers
- Market failure: Inefficient outcome

Same principle applies to:

- Used cars: Low offers accepted mainly by "lemons"
- Houses: Low offers accepted by sellers hiding defects

Solutions to Adverse Selection

Four main approaches:

1. **Reputation**: Long-term incentives create trust
    - Dealership vs. Craigslist seller
2. **Third-party verification**: Independent experts level information
    - Home inspectors, mechanics
3. **Government regulation**: Re-allocate risk
    - Lemon laws give buyers right to return defective cars
4. **Government intervention**: Pool all types
    - Universal healthcare eliminates selection problem

Common thread: All solutions close the information gap

Signaling Games

Definition: Informed player moves first, action can signal their type

- Opposite of screening games
- First mover's action potentially reveals private information

Example: Job market

- Applicant (Player 1) knows own capability: High or Low type
- Employer (Player 2) has 50/50 prior belief
- Applicant chooses: College or High School
- Employer observes choice, then decides: Hire or Pass

Three possible equilibrium types: Separating, Pooling, Semi-separating

Separating Equilibrium

Definition: Different types choose different actions to distinguish themselves

Job market example:

- High type always goes to college
- Low type always gets high school diploma

Employer's inference:

- See College → 100% certain applicant is High type
- See High School → 100% certain applicant is Low type

Key insight: Signal completely reveals private information

- Uncertainty eliminated
- Action speaks louder than words

Pooling Equilibrium

Definition: All types choose same action, hiding their identity

Job market example:

- High type gets high school diploma
- Low type also gets high school diploma

Employer's inference:

- See High School → Learn nothing new
- Belief remains at prior: 50/50

Key insight: Signal reveals no information

- All types "pool" together
- Beliefs don't change

Complication: Must still define off-path beliefs (what if someone goes to college?)

Semi-Separating Equilibrium

Definition: One type uses pure strategy, other type mixes

- Most strategically interesting case
- Involves "true bluffing behavior"

Job market example:

- High type always goes to college (pure strategy)
- Low type sometimes goes to college (30%), sometimes high school (70%) (mixed)

Employer's inference:

- See High School → 100% certain Low type (perfectly revealing)
- See College → Ambiguous; use Bayes' rule to update belief

Key insight: Partial information revealed, requires belief updating

Summary: Three Equilibrium Types

Type	What Players Do	What Uninformed Learns
Separating	Different types → Different actions	Everything (perfect revelation)
Pooling	All types → Same action	Nothing (beliefs unchanged)
Semi-Separating	One type pure, other mixes	Something (partial revelation)

Understanding these patterns is key to analyzing signaling games

War Game: Finding Separating Equilibrium

Setup:

- State 1: Strong (60%) or Weak (40%)
- State 1 knows own type, State 2 doesn't
- State 1 chooses: Reveal (costly demonstration) or Hide
- State 2 wants to Fight Weak, Quit against Strong

Testing separating strategy: Strong Reveals, Weak Hides

Step 1: State 2 sees Hide → Believes Weak with 100% → Fights

Step 2: Check for profitable deviations

- Strong: Reveal (0.99) > Hide (0.5) ✓
- Weak: Hide (-1) > Reveal (-1.01) ✓

Result: This IS a stable separating equilibrium

Failed Separating Equilibrium

Testing opposite strategy: Strong Hides, Weak Reveals

Step 1: State 2 sees Hide → Believes Strong with 100% → Quits

Step 2: Check for profitable deviations

- Strong: Hide (1) > Reveal (0.99) ✓
- Weak: **Deviate!** Hide (1) >> Reveal (-1.01)

Fatal flaw: Weak type has huge incentive to bluff

- Can pretend to be strong and get much better payoff
- Signal not credible

Result: This is NOT a stable equilibrium

Pooling Equilibrium Example

Testing strategy: Both Strong and Weak Hide

Step 1: State 2 sees Hide → Learns nothing → Beliefs remain 60% Strong, 40% Weak

Step 2: State 2's expected utility

- Fight: $0.6(-1) + 0.4(0.5) = -0.4$
- Quit: 0
- Best response: Quit (since $0 > -0.4$)

Step 3: Check for profitable deviations

- Strong: Hide (1) > Reveal (0.99) ✓
- Weak: Hide (1) > Reveal (-1.01) ✓

Result: This IS a stable pooling equilibrium

Off-the-Path Beliefs

The challenge: When unexpected action occurs with zero probability

- Bayes' Rule cannot be applied
- Must specify belief without logical constraint

Testing "Both Reveal" equilibrium:

- If both types supposed to Reveal, seeing Hide is unexpected
- Let p = belief that hider is Strong
- P2 fights if $0.5 - 1.5p > 0$ → $p < \frac{1}{3}$

Checking deviations:

- Strong: For $p \leq \frac{1}{3}$, deviation profitable only if P2 quits
- Weak: **Always has profitable deviation** (Hide always better than Reveal)

Result: This equilibrium fails regardless of off-path belief

Beer-Quiche Game

Setup:

- P1 type: Real Man (60%, prefers beer) or Wimp (40%, prefers quiche)
- P1 chooses meal: Beer or Quiche
- P2 (coward) observes, decides: Fight or Quit
- P2 wants to fight only Wimps

Payoffs:

- P1: 2 points for avoiding fight + 1 point for preferred meal
- P2: +1 for fighting Wimp, -1 for fighting Real Man, 0 for quitting

Question: Is there a pooling equilibrium where both types drink beer?

Solving Beer-Quiche Pooling

Step 1: Both types drink beer

Step 2: P2 sees beer → Beliefs unchanged (60% Real, 40% Wimp)

- Expected payoff for Fight: $0.6(-1) + 0.4(1) = -0.2$
- Payoff for Quit: 0
- Best response: Quit

Step 3: Check deviations

- Real Man: Beer gives 3 (maximum) → No deviation
- Wimp: Beer gives 2; Quiche could give 3 if P2 quits, or 1 if P2 fights

Key: To prevent Wimp's deviation, P2 must Fight if sees Quiche (off-path belief)

Beer-Quiche Solutions

Solution Class 1: $P(\text{Real Man}|\text{Quiche}) = p < \frac{1}{2}$

- P2 strictly prefers to Fight quiche-eater
- P1 strategy: Both drink beer
- P2 strategy: Quit if beer, Fight if quiche (with belief $p < 1/2$)

Solution Class 2: $P(\text{Real Man}|\text{Quiche}) = \frac{1}{2}$ exactly

- P2 indifferent between Fight and Quit
- P2 must fight with probability $\sigma \geq \frac{1}{2}$ to deter Wimp
- This keeps Wimp's expected payoff from deviating at most 2

Key lesson: Off-path beliefs hold equilibrium together

- Threat of fighting quiche-eater prevents deviation

Semi-Separating Equilibrium

When it arises: Pure strategies (always attack or never attack) both fail

Example: Terrorist game

- Robust type (40%): Always attacks
- Vulnerable type (60%): Mixes between attack and not attack
- Target: Doesn't know type, must respond to attack

Key concept: Strategic indifference

- For player to mix, must be indifferent between choices
- Opponent's mixing probability must make player exactly indifferent

Solving Semi-Separating: Indifference Conditions

Making Vulnerable type indifferent (to mix between attack/not):

- Payoff from not attacking: 0
- Expected payoff from attacking: $-2R + 1(1 - R) = 1 - 3R$
- Set equal: $1 - 3R = 0$ → $R = \frac{1}{3}$
- Target must resist with probability $\frac{1}{3}$

Making Target indifferent (to mix between resist/ignore):

- Let P = posterior belief attacker is Robust
- Expected payoff from resisting: $-3P + 2(1 - P) = 2 - 5P$
- Payoff from ignoring: -1
- Set equal: $2 - 5P = -1$ → $P = \frac{3}{5}$

Connecting Bluff to Belief

Using Bayes' Rule to find Vulnerable type's bluffing frequency:

$$P(\text{Robust}|\text{Attack}) = \frac{P(\text{Attack}|\text{Robust}) \cdot P(\text{Robust})}{P(\text{Attack})}$$

$$\frac{3}{5} = \frac{1 \times 0.4}{0.4 + \sigma_v \times 0.6}$$

Solving: $\sigma_v = \frac{4}{9}$

Equilibrium:

- Robust: Always attacks
- Vulnerable: Attacks with probability $\frac{4}{9}$
- Target: Resists with probability $\frac{1}{3}$ (given attack)
- Belief: $P(\text{Robust}|\text{Attack}) = \frac{3}{5}$

Single Raise Poker

Setup:

- P1 dealt: Ace (50%) or Queen (50%)
- P2 has: King
- P1 chooses: Bet or Fold
- If P1 bets, P2 chooses: Call or Fold

Payoffs (P1 perspective):

- Fold: -1
- Bet → P2 folds: +1
- Bet → P2 calls: Ace wins +2, Queen loses -2

Key insight: Ace always bets (dominant strategy)

Testing Pure Strategies in Poker

Pooling (both bet): Fails

- P2's expected payoff for calling: $0.5(2) + 0.5(-2) = 0 > -1$
- P2 always calls
- Queen gets -2 from betting vs. -1 from folding → Deviates

Separating (Queen folds): Fails

- P2 knows bet = Ace → P2 always folds
- Queen could bluff and win +1 vs. -1 from folding → Deviates

Conclusion: Must be semi-separating equilibrium

Solving Poker Semi-Separating

Making Queen indifferent (to mix):

- Payoff from folding: -1
- Expected from betting: $-2\sigma_c + 1(1 - \sigma_c) = 1 - 3\sigma_c$
- Set equal: $1 - 3\sigma_c = -1 \rightarrow \sigma_c = \frac{2}{3}$

Making P2 indifferent (to mix):

- Let p = belief facing Queen after seeing bet
- Expected from calling: $2p - 2(1 - p) = 4p - 2$
- Payoff from folding: -1
- Set equal: $4p - 2 = -1 \rightarrow p = \frac{1}{4}$

Finding bluff frequency (using Bayes' Rule): $\sigma_b = \frac{1}{3}$

Poker Equilibrium

Complete equilibrium strategy:

- Ace: Always bets (100%)
- Queen: Bets $\frac{1}{3}$ of time (bluffs), folds $\frac{2}{3}$ of time
- P2 (after seeing bet): Calls $\frac{2}{3}$ of time, folds $\frac{1}{3}$ of time
- Belief: $P(\text{Queen}|\text{Bet}) = \frac{1}{4}$

Strategic lessons:

- Strong hand: Always aggressive
- Weak hand: Bluff sometimes to remain unpredictable
- Opponent: Mix responses to prevent exploitation

Key insight: "Rich bluffing strategies" emerge naturally from rational play

Chain Store Paradox

Original setup (complete information):

- Chain store is weak (price war unprofitable)
- Faces potential competitor in Town 2
- Logic: Backward induction → Always acquiesce
- "Paradox": Real firms use aggressive price wars to deter entry

Why not really a paradox: Simplified model doesn't capture real-world uncertainty

Resolution: Introduce incomplete information

- Rival uncertain if chain store is weak or strong
- Strong store: Price war profitable
- Creates possibility for strategic bluffing

Chain Store with Incomplete Information

Setup:

- Prior belief: 90% weak, 10% strong
- Strong store: Always fights (dominant strategy)
- Question: What does weak store do?

Testing pure strategies:

- Separating (weak acquiesces): Fails—bluffing is profitable
- Pooling (weak fights): Fails—being challenged makes acquiescing better

Solution: Semi-separating equilibrium

- Weak store bluffs with probability $\frac{1}{9}$
- Rival challenges with probability $\frac{1}{2}$
- Creates strategic uncertainty

Lessons from Chain Store

Key insights:

1. Weak store cannot always acquiesce → Bluffing becomes too attractive
2. Weak store cannot always bluff → Skeptical rival would always challenge
3. Only stable solution: Occasional bluffing, occasional challenging

Resolving the "paradox":

- Incomplete information makes "irrational" price war rational
- Bluffing prevents perfect predictability
- Maintains strategic uncertainty
- Prevents exploitation

Bottom line: What seemed paradoxical under certainty becomes logical under uncertainty

Summary: Key Takeaways

Perfect Bayesian Equilibrium:

- Combines sequential rationality with beliefs about types
- Requires specifying both strategies AND beliefs
- Updates via Bayes' rule wherever possible

Three equilibrium types:

1. Separating: Actions reveal types completely
2. Pooling: Actions hide types completely
3. Semi-separating: Actions partially reveal through mixing

Applications:

- Screening: Uninformed moves first (escalation as test)
- Signaling: Informed moves first (education, military displays)
- Markets: Adverse selection and information asymmetry

Final Thoughts

Strategic bluffing is rational:

- Not random or irrational behavior
- Carefully calculated to maintain uncertainty
- Frequency determined by indifference conditions

Information is powerful:

- Can reveal types (separating)
- Can hide types (pooling)
- Can partially reveal (semi-separating)

Real-world relevance:

- Poker and sports (mixing strategies)
- Business competition (chain store)
- Labor markets (education signaling)
- Insurance and used cars (adverse selection)

Next: Applications and extensions of PBE