# Lecture 6: Repeated Games

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## The Road Ahead

## Finite Repeated Games

**The Intuitive Hope**: If we play the Prisoner's Dilemma repeatedly, surely the shadow of the future will encourage cooperation?

**Classic Prisoner's Dilemma Payoffs**:

| | Cooperate | Defect |
|---|---|---|
| **Cooperate** | 3, 3 | 1, 4 |
| **Defect** | 4, 1 | 2, 2 |

**The Brutal Reality**: In any **finitely repeated** game where the end is known, backward induction destroys all hope of cooperation.

**Why?** Players can build trust, signal intentions, and seek rewards/punishments—but a cold, backward-flowing logic prevents cooperation when the end is certain.

## The Logic of the End

**Final Round Analysis**: In the last round, all previous payoffs are locked in. Players must play the one-shot Nash Equilibrium.

**In Prisoner's Dilemma**: The unique Nash Equilibrium is **(Defect, Defect)** with payoffs (2, 2).

**The First Domino**: Certainty of final-round betrayal triggers the unraveling process.

> "My actions in this round cannot sway what you will do in the final round. Since I cannot influence the future, my only rational choice is to maximize my payoff for today by defecting."

## The Domino Effect

**Second-to-Last Round**: Players know the final round outcome is (Defect, Defect).

1. **The Known Future**: Final round is predetermined as mutual defection
2. **The Empty Promise**: Cooperating now cannot influence future behavior
3. **The Inevitable Choice**: This round becomes strategically identical to a one-shot game

**The Unstoppable Collapse**: This logic continues backward through every round:

- Stage N → Stage N-1 → Stage N-2 → ... → Stage 1

> **Finite Repetition Result**: The only subgame perfect equilibrium is *(Defect, Defect)* in every single round.

## A Glimpse of Hope

**The Key Insight**: The entire unraveling depends on **certainty about the end**.

**Remove the certainty → Prevent backward induction**

**Two Ways to Eliminate the End**:

1. **Infinitely repeated games**: No final round exists
2. **Unknown termination**: Players don't know when the game ends

**The Result**: The incentive to build reputation and encourage future cooperation remains powerful because there's always a "next round" to consider.

**However**: This opens a complex world where almost anything can be justified as rational—the realm of the **Folk Theorem**.

## Discount Factors

**The Infinity Problem**: How do we compare infinite payoff streams?

Player 1 always defects: $4 + 4 + 4 + \ldots = \infty$
Player 1 always cooperates: $1 + 1 + 1 + \ldots = \infty$

**Paradox**: Best and worst outcomes both equal infinity!

**Solution**: The **discount factor** $\delta$ where $0 < \delta < 1$

**Two Interpretations**:

1. **Time Value**: \$3 today > \$3 tomorrow (can invest, consume sooner)
2. **Continuation Probability**: $\delta =$ probability the game continues to next period

## How Discount Factors Work

**Infinite Cooperation Payoff**: $3 + 3\delta + 3\delta^2 + 3\delta^3 + \ldots$

**Period-by-Period Breakdown**:

- **Period 1 (Today)**: Value = 3 (no discounting)
- **Period 2 (Tomorrow)**: Value = $3\delta$
- **Period 3**: Value = $3\delta^2$ (discounted twice)
- **Period t**: Value = $3\delta^{t-1}$

**Key Rule**: The exponent on $\delta$ is always one less than the period number.

**Interpretation of $\delta$ Values**:

- **High $\delta$ (close to 1)**: Patient player, future matters greatly
- **Low $\delta$ (close to 0)**: Impatient player, present dominates

## Geometric Series

**The Challenge**: Calculate $X + X\delta + X\delta^2 + X\delta^3 + \ldots$

**Step 1 - Finite Series**: $S = X + X\delta + X\delta^2 + \ldots + X\delta^{n-1}$

**Step 2 - The Magic Trick**:

- Original: $S = X + X\delta + X\delta^2 + \ldots + X\delta^{n-1}$
- Multiply by $\delta$: $\delta S = X\delta + X\delta^2 + \ldots + X\delta^{n-1} + X\delta^n$
- Subtract: $S - \delta S = X - X\delta^n$
- Factor: $S(1-\delta) = X(1-\delta^n)$
- Solve: $S = \frac{X(1-\delta^n)}{1-\delta}$

# From Finite to Infinite

**Taking the Limit**: As $n \to \infty$, what happens to $\delta^n$?

**Example with $\delta = 1/2$**:

- $(1/2)^1 = 1/2 = 0.5$
- $(1/2)^2 = 1/4 = 0.25$
- $(1/2)^3 = 1/8 = 0.125$
- $(1/2)^{10} = 1/1024 \approx 0.001$
- As $n \to \infty$: $\delta^n \to 0$

**Infinite Geometric Series Formula**: $X + X\delta + X\delta^2 + ... = \frac{X}{1-\delta}$

**We've tamed infinity!** The "dot dot dot" becomes a simple, finite value.

## One-Shot Deviation Principle

**The Exponential Strategy Problem**: In a 16-decision game:

- 1 decision → 2 strategies
- 2 decisions → 4 strategies
- 3 decisions → 8 strategies
- 16 decisions → 65,536 strategies!

**Traditional Approach**: To prove a strategy is optimal, check it against all 65,535 alternatives. **Impossible!**

**One-Shot Deviation Principle**: A strategy is a subgame perfect equilibrium **if and only if** no player can profitably deviate at a single stage while maintaining their strategy everywhere else.

## How the Principle Works

**The "Obvious" Part**: If a strategy is truly optimal, it must beat any single-change alternative.

**The "Powerful" Part**: If you can't improve by changing one decision, you can't improve by changing multiple decisions.

**Why?** Any complex, profitable multi-stage deviation must contain at least one "pivotal" stage that alone provides improvement.

**Practical Benefit**:

- **Without principle**: Check 65,535 alternatives
- **With principle**: Check only 16 one-shot deviations

**Result**: Transform an impossible task into a manageable one!

## Grim Trigger Strategy

**The Unforgiving Rule**:
*"If anyone has defected at any point previously, defect forever. Otherwise, cooperate."*

**Two Simple Rules**:

1. **Start by cooperating** (offer initial trust)
2. **Permanent punishment** for any defection (no forgiveness, ever)

**Two Possible Paths**:

- **Cooperation Path**: Both cooperate → steady payoff of 3 each round
- **Punishment Path**: Someone defects → mutual defection forever (payoff 2)

**Why "Bloodthirsty"?** A single mistake triggers permanent conflict.

# The Trade-Off in Grim Trigger

**The Choice**: Stick to cooperation or grab a short-term gain?

**Option A - Cooperate**: Steady payoff stream of $3, 3, 3, 3, \ldots$

**Option B - Defect Once**:

- **Today**: Get 4 instead of 3 (temptation payoff)
- **Forever After**: Stuck with 2 in all future rounds (punishment)
- **Payoff Stream**: $4, 2, 2, 2, \ldots$

**When Does Cooperation Work?**

- Present value of cooperation: $\frac{3}{1-\delta}$
- Present value of defection: $4 + \frac{2\delta}{1-\delta}$
- **Cooperation condition**: $\frac{3}{1-\delta} \geq 4 + \frac{2\delta}{1-\delta}$
- **Simplifies to**: $\delta \geq \frac{1}{2}$

## Tit-for-Tat Strategy

**The "Nice" Alternative**: A forgiving strategy that doesn't hold grudges forever.

**Two Simple Rules**:

1. **Start by cooperating** (be nice first)
2. **Copy opponent's last move** (forgive quickly, but retaliate immediately)

**On Equilibrium Path**:

- Round 1: Both cooperate
- Round 2: Both cooperate (copying each other)
- Forever: Mutual cooperation with payoff (3, 3)

**Off Equilibrium Path** (after one defection):

- Creates alternating pattern: Cooperate, Defect, Cooperate, Defect, ...
- **Payoff streams**: $4, 1, 4, 1, \ldots$ vs. steady $3, 3, 3, \ldots$

## Tit-for-Tat's Success

**Robert Axelrod's Computer Tournaments**: Tit-for-Tat consistently performed well against complex strategies submitted by experts.

**Why It Works**:

- **Simple**: Easy to understand and implement
- **Nice**: Starts with cooperation and forgives quickly
- **Retaliatory**: Punishes defection immediately
- **Forgiving**: Returns to cooperation after one round of punishment

**The Trade-Off**: Cooperation $(3, 3, 3, \ldots)$ vs. Alternating $(4, 1, 4, 1, \ldots)$

**Condition for Cooperation**: Same as Grim Trigger in this example: $\delta \geq \frac{1}{2}$

## The Credibility Problem

**The Fatal Flaw**: Tit-for-Tat's threat isn't actually credible!

**After Opponent Defects, You Face**:

- **Option A (Punish)**: Defect → payoff stream $4, 1, 4, 1, \ldots$
- **Option B (Forgive)**: Cooperate → return to payoff stream $3, 3, 3, \ldots$

**The Contradiction**:

- **To cooperate initially**: Need $\delta \geq \frac{1}{2}$ (future matters enough)
- **To punish credibly**: Need $\delta \leq \frac{1}{2}$ (prefer alternating to steady 3s)

**Knife-Edge Condition**: Only works if $\delta = \frac{1}{2}$ exactly!

**Result**: Tit-for-Tat is a Nash Equilibrium but *not* a Subgame Perfect Equilibrium (except at the knife-edge).

## Folk Theorem

**The Big Idea**: In infinitely repeated games with patient players, almost any reasonable outcome can be sustained as an equilibrium.

**Formal Statement**: Any outcome that gives all players payoffs **strictly better** than their punishment payoff can be supported as a subgame perfect equilibrium.

**Why "Folk"?** The theorem emerged simultaneously from the "folklore" of game theory—many theorists discovered it around the same time.

**The Mechanism**:

1. **Agreement**: Players commit to a specific strategy profile
2. **Temptation**: Players always face short-term gains from deviating
3. **Punishment**: Any deviation triggers permanent reversion to Nash equilibrium

## Beyond Simple Cooperation

**Example of Complex Equilibrium**:

- Player 1: Cooperate 100% of the time
- Player 2: Cooperate 95% of the time, defect 5% of the time

**Why This Works**:

- Both players still get expected payoffs > 2 (punishment level)
- Any deviation triggers permanent mutual defection (payoff = 2 forever)
- As long as players are patient enough, they won't risk losing the good deal

**The Explosion**: There are **infinitely many equilibria** in infinitely repeated games!

**Key Condition**: Players must be "sufficiently patient" (high enough $\delta$).

## The Prediction Problem

**The Challenge**: When everything is possible, nothing is predictable.

**Example**: Imagine observing this 8-period sequence:

1. Mutual Cooperation
2. Mutual Defection
3. Mutual Defection
4. (Defect, Cooperate)
5. Mutual Cooperation
6. Mutual Cooperation
7. (Defect, Cooperate)
8. (Cooperate, Defect)

**Seemingly Random?** The Folk Theorem shows this can be a rational equilibrium!

## The Chaotic Equilibrium Explained

**The Strategy**:

1. **Periods 1-8**: Follow the exact prescribed sequence
2. **Period 9 onward**: Cooperate forever
3. **Punishment**: Any deviation triggers defection forever

**Why It Works**:

- The value of infinite future cooperation (payoff 3 forever) vastly outweighs any finite sequence of strange payoffs
- For patient players, any 8-period "cost" becomes irrelevant compared to infinite future benefits

**The Problem**: If **any** observed behavior can be explained as rational, our theory "predicts everything and therefore predicts nothing."