Lecture 4: Games with Infinite Strategy Spaces

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Date: September 25, 2025

The Road Ahead

Introduction to Infinite Strategy Spaces

So far: All games had **finite** number of pure strategies → could represent with matrices or game trees

Reality: Many strategic situations involve **infinite** strategy choices:

- Firms choosing production levels (any real number ≥ 0)
- Politicians choosing policy positions (continuum of possibilities)
- Auction bidders choosing bid amounts (any positive value)
- Pricing decisions (continuous price range)

Key insight: Matrices and trees become **impractical** or **impossible** with infinite strategies

New challenge: How to find equilibria without drawing payoff matrices?

Games Without Payoff Matrices

Game definition remains the same:

- Set of players
- Strategy spaces (now possibly infinite)
- Order of moves
- Payoff functions

New solution method for infinite games:

1. Consider a single pure strategy from one player
2. Find the other player's best responses to that strategy
3. Check whether the original strategy is a best response to any of those best responses
4. If yes → equilibrium found; if no → try next strategy
5. Repeat for all strategies

Advantage: No need to draw massive (impossible) matrices!

Example: Simple Infinite Game

Setup: Players 1 and 2 simultaneously select whole numbers between 1 and 100. Each player's payoff = product of the two numbers.

Analysis using new method:

Step 1: Suppose Player 2 chooses 1

- Player 1's payoff = $x \times 1 = x$ → best response: choose 100
- But Player 2's best response to 100 is 100 (not 1) → No equilibrium

Step 2: Suppose Player 2 chooses any $k < 100$

- Player 1's payoff = $x \times k$ → best response: choose 100
- But Player 2's best response to 100 is 100 (not k) → No equilibrium

Step 3: Suppose Player 2 chooses 100

- Player 1's payoff = $x \times 100$ → best response: choose 100
- Player 2's best response to 100 is 100 ✓ → **Equilibrium: (100, 100)**

Equilibrium Existence in Infinite Games

Nash's theorem: Every **finite** game has at least one Nash equilibrium

Warning: **Infinite** games may have **no equilibrium**!

Counter-example: Players select any number > 0, payoff = product of numbers

Analysis:

- Suppose Player 1 chooses any value x > 0
- Player 2's best response: choose something slightly larger than x
- But then Player 1 wants to choose something even larger
- **Result**: No equilibrium exists (infinite regress)

Key lesson: Existence theorems don't apply to infinite games

When Do Infinite Games Have Equilibria?

Sufficient conditions for equilibrium existence:

1. **Bounded strategy spaces** (e.g., [0, 100] instead of all positive numbers)
2. **Continuous payoff functions**
3. **Compact strategy sets**
4. **Quasi-concave payoff functions**

Example of bounded game: Choose number in [1, 100]

- Upper bound prevents infinite regress
- Guarantees equilibrium exists

Example without bounds: Choose any positive number

- No upper limit → players keep trying to choose larger numbers
- No equilibrium possible

Applications: Second Price Auctions

Setup:

- n bidders compete for single item
- Each bidder has private valuation v_i for the item
- Sealed bid auction: highest bidder wins, pays **second-highest bid**

Strategy space: Each bidder chooses bid $b_i \in [0, \infty)$

Payoff function:

- Winner: v_i - (second highest bid)
- Losers: 0

Key question: What is the optimal bidding strategy?

Second Price Auction Analysis

Claim: Bidding your true valuation ($b_i = v_i$) is a **dominant strategy**

Proof idea:

Case 1: Your bid wins

- You pay second-highest bid (independent of your exact bid)
- Bidding higher than v_i: same outcome, but risk paying more than value
- Bidding lower than v_i: might lose auction you could have won profitably

Case 2: Your bid loses

- You pay 0 regardless of exact bid amount
- No incentive to deviate from truth-telling

Result: Truth-telling is optimal regardless of others' strategies → **dominant strategy equilibrium**

Applications: The Median Voter Theorem

Setup:

- Two candidates choose policy positions on a line (e.g., left-right spectrum)
- Voters distributed along this line with single-peaked preferences
- Each voter supports candidate closest to their ideal position
- Candidates want to maximize vote share

Strategy space: Choose any position $x \in \mathbb{R}$

Key insight: Both candidates will converge to the **median voter's position**

Intuition:

- If you're away from median, opponent can move slightly toward median
- Opponent captures majority of voters between your positions
- Only stable outcome: both at median position

Median Voter Theorem: Formal Analysis

Assumptions:

1. Voters have single-peaked preferences
2. Candidates care only about winning (not ideology)
3. Voters vote for candidate closest to their ideal point
4. Odd number of voters (to avoid ties)

Equilibrium prediction: Both candidates locate at median voter's ideal point

Real-world implications:

- Explains convergence to political center
- Why extreme candidates often lose general elections
- Strategic moderation in two-party systems

Extensions: Result may fail with multiple dimensions, strategic voting, or candidate ideology

Summary: Key Insights

Main takeaways from infinite strategy games:

1. **Solution method**: Use best-response analysis instead of matrices
2. **Existence**: Infinite games may have no equilibrium (unlike finite games)
3. **Bounded sets**: Help guarantee equilibrium existence
4. **Dominant strategies**: Still powerful solution concept (second-price auctions)
5. **Convergence**: Competition can lead to clustering (median voter theorem)

Applications beyond this lecture:

- Cournot competition (firms choosing quantities)
- Bertrand competition (firms choosing prices)
- Public goods provision
- Arms races and conflict models

Practice Problems

Problem 1: Two firms choose production levels $x_1, x_2 \in [0, 10]$. Profit functions: $\pi_1 = x_1(12 - x_1 - x_2)$, $\pi_2 = x_2(12 - x_1 - x_2)$. Find Nash equilibrium.

Problem 2: In a first-price auction (highest bidder wins and pays their bid), is truth-telling still optimal? Why or why not?

Problem 3: Consider median voter theorem with three candidates instead of two. What happens to the equilibrium prediction?

Problem 4: Players choose numbers in [0, 1]. Player 1's payoff = $x_1 - x_1^2$, Player 2's payoff = $x_2 - 2x_1x_2$. Find best responses and equilibrium.

Takeaway Points

1. **Infinite strategy spaces require new solution methods beyond payoff matrices.**
2. **Nash's existence theorem doesn't apply to infinite games - equilibria may not exist.**
3. **Bounded strategy sets and continuous payoffs help ensure equilibrium existence.**
4. **Truth-telling is dominant in second-price auctions but not in first-price auctions.**
5. **The median voter theorem explains convergence to the center in two-candidate elections.**