Appendix: A Math Refresher*

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This appendix presents the mathematical tools and mathematical results that are used throughout our lectures.

1 Geometric Series

A geometric series is a sum of numbers of the form

$$s_n = 1 + x + x^2 + \cdots + x^n \tag{1.1}$$

where x is a number that can be greater or smaller than one. For example, the present discounted value of a sequence of payments of one dollar each year for n years when the interest rate is equal to i:

$$V = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}$$

We look at what is the sum and under what conditions the sum reaches a finite limit.

- The sum is given by

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}, \quad \forall x \neq 1 \tag{1.2}$$

  Proof: since $(1-x)s_n = 1 - x^{n+1}$, the result follows trivially.

- If $|x| < 1$, the sum reaches a finite limit given by

$$s_n = \frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x} = s, \quad \text{as } n \to \infty \tag{1.3}$$

  Proof: since $x^n \to 0$ as $n \to \infty$ with $|x| < 1$, the result follows trivially.

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These are notes that I used by myself to lecture from and for educational purposes only. The material presented here is largely based upon the undergraduate textbook by Blanchard and Johnson (2012), *Macroeconomics*, 6th Edition, Prentice Hall. Please do NOT circulate.

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2 USEFUL APPROXIMATIONS

The approximations we used throughout this course are most reliable when the variables x, y, and z below are small, say between 0 and 10%.

- For small values of x and y

$$(1+x)(1+y) \approx 1+x+y \tag{2.1}$$
$$(1+x)^2 \approx 1+2x \tag{2.2}$$

Proof: note that

$$\begin{aligned}(1+x)(1+y) &= 1+x+y+xy \\ &\approx 1+x+y\end{aligned}$$

where xy is very small and can be ignored as an approximation. Letting $y = x$ gives the second approximation. For example, recall the arbitrage relation between one-year bonds and two-year bonds

$$(1+i_{2t})^2 = (1+i_{1t})(1+i^e_{1t+1})$$

Using the approximations for both sides of the above equation gives

$$1+2i_{2t} \approx 1+i_{1t}+i^e_{1t+1}$$

or reorganizing gives the term structure of interest rates

$$i^e_{1t+1} \approx 2i_{2t} - i_{1t}$$

- For small values of x

$$(1+x)^n \approx 1+nx \tag{2.3}$$

Proof: it follows by repeated application of the above approximations. The approximation, however, becomes worse as n increases.

- For small values of x and y

$$\frac{1+x}{1+y} \approx 1+x-y \tag{2.4}$$

Proof: note that

$$\begin{aligned}(1+x-y)(1+y) &= 1+x+xy-y^2 \\ &\approx 1+x\end{aligned}$$

Dividing both sides of the above approximation by $(1+y)$ gives the result. For example, recall the Fisher relation

$$1+r_t = \frac{1+i_t}{1+\pi^e_{t+1}}$$

Using the approximation for the RHS of the above equation gives

$$1+r_t \approx 1+i_t-\pi^e_{t+1}$$

or reorganizing gives

$$r_t \approx i_t - \pi^e_{t+1}$$

- If $z = xy$, then

$$g_z \approx g_x + g_y \tag{2.5}$$

Proof: let Δz be the increase in z when x increases by Δx and y increases by Δy. Then, by definition

$$z + \Delta z = (x+\Delta x)(y+\Delta y)$$

Dividing both sides of the above equation by $z = xy$ and rearranging give

$$\begin{aligned}1+g_z &= 1+\frac{\Delta z}{z} \\ &= \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right) \\ &= (1+g_x)(1+g_y) \\ &\approx 1+g_x+g_y\end{aligned}$$

as desired. For example, recall the production function of the form $Y = AN$. Using the above approximation gives

$$g_Y \approx g_A + g_N$$

- If $z = x/y$, then

$$g_z \approx g_x - g_y \tag{2.6}$$

Proof: let Δz be the increase in z when x increases by Δx and y increases by Δy. Then, by definition

$$z + \Delta z = \frac{x + \Delta x}{y + \Delta y}$$

Dividing both sides of the above equation by $z = x/y$ and rearranging give

$$\begin{aligned} 1 + g_z &= 1 + \frac{\Delta z}{z} \\ &= \frac{1 + \Delta x/x}{1 + \Delta y/y} \\ &= \frac{1 + g_x}{1 + g_y} \\ &\approx 1 + g_x - g_y \end{aligned}$$

as desired. For example, the growth rate of the real money stock M/P is given by

$$g_{M/P} \approx g_M - g_P = g_M - \pi$$