TOPIC 2: FUTURE VALUE, PRESENT VALUE, AND INTEREST RATES*

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This lecture introduces the concepts of present value, future value, and interest rates, which are then applied to the valuation of bonds. The relationship between inflation and interest rates is also discussed. Interest rates are of enormous importance to virtually everyone; they link the present to the future, allowing us to compare payments made on different dates. They also tell us the future reward for lending today, as well as the cost of borrowing now and repaying in the future. But to make sound financial decisions we must learn how to calculate and compare different interest rates on various financial instruments.

1 VALUING MONETARY PAYMENTS NOW AND IN THE FUTURE

We need a set of tools, called present value and future value, in order to compare the value of payments realized on different dates. For now we assume away the possibility of default, but we will relax such assumption in the next lecture.

Future value and compound interest. **Future value** is the value on some future date of an investment made today. For example, the future value of \$100 one year from now at an annual interest rate of 5% is \$105 [Core Principle 1]. In general, the future value (FV) of an investment with a present value (PV) invested at interest rate (i) is

$$\mathrm{FV} = \mathrm{PV} + \mathrm{PV} \times i = \mathrm{PV} \times (1 + i) \tag{1.1}$$

which suggests that the higher the interest rate or the amount invested, the higher the future value. Three remarks:

- The future value of \$100 two years from now at an annual interest rate of 5% is

$$\begin{aligned} \$110.25 &= \underbrace{\$100}_{\text{PV}} + \underbrace{\$100 \times 5\%}_{\text{1st year interest}} + \underbrace{\$100 \times 5\%}_{\text{2nd year interest}} + \underbrace{\$5 \times 5\%}_{\text{compound interest}} \\ &= \$100 \times (1 + 5\%)^2 \end{aligned} \tag{1.2}$$

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Disclaimer: these are notes that I used by myself to lecture from and for educational purposes only. The material presented here is largely based upon the undergraduate textbook by Stephen Cecchetti and Kermit Schoenholtz (2014), *Money, Banking and Financial Markets*, 4th Edition, McGraw-Hill/Irwin. Please do NOT circulate.

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where **compound interest** is the interest on interest.

- More generally, future value of an investment n years from now at an annual interest rate of i is

$$\text{FV}_n = \text{PV} \times (1+i)^n \tag{1.3}$$

where n is not necessarily a whole number. For example, $n = 1/2$ means half a year.

- Caution: in computing future value, both the interest rate and n must be measured in the same time units. For example, let i^m be the monthly interest rate and n the number of months. Then the future value of \$100 one year from now is

$$\$100 \times (1+i^m)^{12} = \$100 \times (1+5\%) \tag{1.4}$$

from which we can solve for i^m, which is 0.41%, or 41 **basis points** (one basis point equals one-hundredth of a percentage point).

Present value. **Present value** is the value today (in the present) of a payment that is promised to be made in the future. For example, the present value of \$105 one year from now at an annual interest rate of 5% is \$100 [Core Principle 1]. In general, the present value of an investment with an interest rate (i) is

$$\text{PV} = \frac{\text{FV}}{1+i} \tag{1.5}$$

which tells us how much we must invest today to realize the future value one year from now. Also, present value falls as the interest rate rises. Two remarks:

- The present value of \$105 in two years at an annual interest rate of 5% is

$$\$95.24 = \frac{\$105}{(1+5\%)^2} \tag{1.6}$$

- More generally, present value of an investment in n years at an annual interest rate of i is

$$\text{PV} = \frac{\text{FV}_n}{(1+i)^n} \tag{1.7}$$

which suggests that the present value is higher when the future value is higher, or the time until payment is shorter, or the interest rate is lower.

- Note again that n is not necessarily a whole number, and in computing present value, both the interest rate and n must be measured in the same time units.

2 Applying Present Value

We can compute the present value not just of a single payment but also of any group of payments made on any number of dates.

Internal rate of return. Suppose we agree to make a \$225 loan and the borrower offers to repay either \$100 a year for three years or \$125 a year for two years. Which option should we take? To figure this out, we need to compute the **internal rate of return**, which is the interest rate that equates the present value of an investment with its cost, of the two payment streams. Three steps:

- For the first option, we need to find the interest rate i that solves

$$\$225 = \frac{\$100}{1+i} + \frac{\$100}{(1+i)^2} + \frac{\$100}{(1+i)^3} \tag{2.1}$$

  The answer is $i = 15.9\%$.

- For the second option, we need to find the interest rate i that solves

$$\$225 = \frac{\$125}{1+i} + \frac{\$125}{(1+i)^2} \tag{2.2}$$

  The answer is $i = 7.3\% < 15.9\%$.

- Clearly, the three payments are more profitable for us as the lender.

Bonds: the basics. A **bond** is a promise to make a series of payments on specific future dates. Among many different types of bonds, we will focus on the most common kind, the **coupon bond**. Some definitions:

- **Coupon payments** are the annual payments made by the bond issuer.
- **Coupon rate** is the annual payments expressed as a percentage of the amount borrowed. For example, if the coupon rate is 5%, then the borrower/issuer pays the lender/bondholder \$5 per year per \$100 borrowed.
- **Principal**, **face value**, or **par value** of the bond is the final repayment of the initial loan.
- **Maturity date** or **term to maturity** is the date when the issuer is going to repay the initial loan and the payments will stop.

We consider a coupon bond that promises a string of yearly coupon payment $\$C$ made over n years and a principal payment of $\$F$ on its maturity date. There are three steps for pricing such a bond:

- Step 1: valuing the principal payment. The present value, or price, of the bond principal (BP) is

$$\$\mathrm{P}_{\mathrm{BP}} = \frac{\$\mathrm{F}}{(1+i)^n} \tag{2.3}$$

  For example, with an interest rate of 6% and a final payment of \$1000 to be made in 30 years, we have

$$\$\mathrm{P}_{\mathrm{BP}} = \frac{\$1000}{(1+6\%)^{30}} = \$174.11 \tag{2.4}$$

  a value much less than its principal payment.

- Step 2: valuing the coupon payments. The present value, or price, of the coupon payments (CP) is simply the sum of the present value of the payments for each year from one to n years:

$$\$\mathrm{P}_{\mathrm{CP}} = \frac{\$\mathrm{C}}{1+i} + \frac{\$\mathrm{C}}{(1+i)^2} + \cdots + \frac{\$\mathrm{C}}{(1+i)^n} = \sum_{k=1}^{n} \frac{\$\mathrm{C}}{(1+i)^k} \tag{2.5}$$

  See the appendix below for a simplified version of the above formula. For example, with an interest rate of 6% and a coupon payment of \$10 to be made over 2 years, we have

$$\$\mathrm{P}_{\mathrm{CP}} = \frac{\$10}{1+6\%} + \frac{\$10}{(1+6\%)^2} = \$9.43 + \$8.90 = \$18.33 \tag{2.6}$$

  In general, the higher the coupon payment, or the lower the interest rate, or the longer the payments last for, the higher the present value.

- Step 3: valuing the coupon payments plus principal. The price of this coupon bond (CB) is simply the sum of the above two prices:

$$\$\mathrm{P}_{\mathrm{CB}} = \$\mathrm{P}_{\mathrm{BP}} + \$\mathrm{P}_{\mathrm{CP}} = \frac{\$\mathrm{F}}{(1+i)^n} + \sum_{k=1}^{n} \frac{\$\mathrm{C}}{(1+i)^k} \tag{2.7}$$

  which suggests that the value of a bond varies inversely with the interest rate used to calculate the present value of the promised payment.

3 Real and Nominal Interest Rates

When we borrow, what we care about is how many goods–rather than dollars–we have to give up in the future in exchange for the goods we get today. Likewise, when we lend, what we care about is how many goods–rather than dollars–we get in the future for the goods we give up today. The presence of inflation makes the distinction between nominal interest rates and real interest rates important:

- Interest rates expressed in terms of units of the national currency are called **nominal interest rates**. If the nominal interest rate for year t is i_t, borrowing one dollar this year requires you to pay $1 + i_t$ dollars next year.

- Interest rates expressed in terms of a basket of goods are called **real interest rates**. If the real interest rate for year t is r_t, borrowing the equivalent of one basket of goods this year requires you to pay the equivalent of $1 + r_t$ baskets of goods next year.

Now we relate nominal interest rate, which we do observe, and real interest rate, which we typically do not observe. To do this, we must adjust the nominal interest rate to take into account expected inflation.

- Suppose we want to consume one more basket of goods this year. If the price level (the price of a basket of goods) this year is P_t, we must borrow P_t dollars and repay $(1 + i_t)P_t$ dollars next year.

- Let P^e_{t+1} be the expected price level (the expected price of a basket of goods) next year.[1] Then how much we expect to repay next year, in terms of the equivalent of baskets of goods, is equal to $(1 + i_t)P_t/P^e_{t+1}$.

- Putting together the definition and derivation of the real interest rate, it follows that the one-year real interest rate is given by

$$1 + r_t = (1 + i_t)\frac{P_t}{P^e_{t+1}} \tag{3.1}$$

Denote expected inflation between t and $t + 1$ by π^e_{t+1}. Then we have

$$\pi^e_{t+1} \equiv \frac{P^e_{t+1} - P_t}{P_t} \tag{3.2}$$

[1] If we use CPI to measure the price level, the real interest rate tells us how much consumption we must give up next year to consume more this year.

and thus equation (3.1) can be rewritten as

$$1 + r_t = \frac{1 + i_t}{1 + \pi^e_{t+1}} \tag{3.3}$$

which gives us the exact relation among the real interest rate, the nominal interest rate, and expected inflation.

- When the nominal interest rate and expected inflation are not too large, a close approximation to equation (3.3) is given by the simpler relation

$$i_t \approx r_t + \pi^e_{t+1} \tag{3.4}$$

which is also called the **Fisher relation** after Irving Fisher.[2] It suggests that in general, the nominal interest rate is positively related to expected inflation. In what follows, we treat the above relation as if it were an equality.

4 Math Appendix

We will derive a simpler version of the following present value formula:

$$\$\mathrm{PV} = \frac{\$\mathrm{C}}{1+i} + \frac{\$\mathrm{C}}{(1+i)^2} + \cdots + \frac{\$\mathrm{C}}{(1+i)^n}$$

First, multiplying both sides by $\frac{1}{1+i}$ gives

$$\frac{\$\mathrm{PV}}{1+i} = \frac{\$\mathrm{C}}{(1+i)^2} + \frac{\$\mathrm{C}}{(1+i)^3} + \cdots + \frac{\$\mathrm{C}}{(1+i)^{n+1}}$$

Second, subtracting the second equation from the first and simplifying give

$$\frac{i}{1+i} \times \$\mathrm{PV} = \frac{\$\mathrm{C}}{1+i} - \frac{\$\mathrm{C}}{(1+i)^{n+1}}$$

Finally, solving for \$PV yields the following formula

$$\$\mathrm{PV} = \frac{\$C}{i}\left[1 - \frac{1}{(1+i)^n}\right] \tag{4.1}$$

[2] More precisely, r_t in (3.4) is the *ex ante* real interest rate, meaning "before the fact", which cannot be directly observed and hence has to be estimated. This is quite different from the *ex post* real interest rate, meaning "after the fact", which can always be computed.