APPENDIX: A MATH REFRESHER*

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This appendix presents the mathematical tools and mathematical results that are used throughout our lectures.

Geometric Series A geometric series is a sum of numbers of the form

$$s_n = 1 + x + x^2 + \cdots + x^n \tag{0.1}$$

where x is a number that can be greater or smaller than one. For example, the present discounted value of a sequence of payments of one dollar each year for n years when the interest rate is equal to i:

$$V = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}$$

We look at what is the sum and under what conditions the sum reaches a finite limit.

- The sum is given by

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad \forall x \neq 1 \tag{0.2}$$

  Proof: since $(1-x)s_n = 1 - x^{n+1}$, the result follows trivially.

- If $|x| < 1$, the sum reaches a finite limit given by

$$s_n = \frac{1 - x^{n+1}}{1 - x} \to \frac{1}{1-x} = s, \quad \text{as } n \to \infty \tag{0.3}$$

  Proof: since $x^n \to 0$ as $n \to \infty$ with $|x| < 1$, the result follows trivially.

Useful Approximations The approximations we used throughout this course are most reliable when the variables x, y, and z below are small, say between 0 and 10%.

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Disclaimer: these are notes that I used by myself to lecture from and for educational purposes only. The material presented here is largely based upon the undergraduate textbook on intermediate macroeconomic theory by Blanchard and Johnson (2012), *Macroeconomics*, 6th Edition, Prentice Hall. Please do NOT circulate.

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- For small values of x and y

$$(1+x)(1+y) \approx 1+x+y \tag{0.4}$$
$$(1+x)^2 \approx 1+2x \tag{0.5}$$

Proof: note that

$$\begin{aligned}(1+x)(1+y) &= 1+x+y+xy \\ &\approx 1+x+y\end{aligned}$$

where xy is very small and can be ignored as an approximation. Letting $y = x$ gives the second approximation. For example, recall the arbitrage relation between one-year bonds and two-year bonds

$$(1+i_{2t})^2 = (1+i_{1t})(1+i^e_{1t+1})$$

Using the approximations for both sides of the above equation gives

$$1+2i_{2t} \approx 1+i_{1t}+i^e_{1t+1}$$

or reorganizing gives the term structure of interest rates

$$i^e_{1t+1} \approx 2i_{2t} - i_{1t}$$

- For small values of x

$$(1+x)^n \approx 1+nx \tag{0.6}$$

Proof: it follows by repeated application of the above approximations. The approximation, however, becomes worse as n increases.

- For small values of x and y

$$\frac{1+x}{1+y} \approx 1+x-y \tag{0.7}$$

Proof: note that

$$\begin{aligned}(1+x-y)(1+y) &= 1+x+xy-y^2 \\ &\approx 1+x\end{aligned}$$

Dividing both sides of the above approximation by $(1+y)$ gives the result. For example, recall the Fisher relation

$$1 + r_t = \frac{1 + i_t}{1 + \pi^e_{t+1}}$$

Using the approximation for the RHS of the above equation gives

$$1 + r_t \approx 1 + i_t - \pi^e_{t+1}$$

or reorganizing gives

$$r_t \approx i_t - \pi^e_{t+1}$$

- If $z = xy$, then

$$g_z \approx g_x + g_y \tag{0.8}$$

Proof: let Δz be the increase in z when x increases by Δx and y increases by Δy. Then, by definition

$$z + \Delta z = (x + \Delta x)(y + \Delta y)$$

Dividing both sides of the above equation by $z = xy$ and rearranging give

$$\begin{aligned} 1 + g_z &= 1 + \frac{\Delta z}{z} \\ &= \left(1 + \frac{\Delta x}{x}\right)\left(1 + \frac{\Delta y}{y}\right) \\ &= (1 + g_x)(1 + g_y) \\ &\approx 1 + g_x + g_y \end{aligned}$$

as desired. For example, recall the production function of the form $Y = AN$. Using the above approximation gives

$$g_Y \approx g_A + g_N$$

- If $z = x/y$, then

$$g_z \approx g_x - g_y \tag{0.9}$$

Proof: let Δz be the increase in z when x increases by Δx and y increases by Δy. Then, by definition

$$z + \Delta z = \frac{x + \Delta x}{y + \Delta y}$$

Dividing both sides of the above equation by $z = x/y$ and rearranging give

$$\begin{aligned} 1 + g_z &= 1 + \frac{\Delta z}{z} \\ &= \frac{1 + \Delta x/x}{1 + \Delta y/y} \\ &= \frac{1 + g_x}{1 + g_y} \\ &\approx 1 + g_x - g_y \end{aligned}$$

as desired. For example, the growth rate of the real money stock M/P is given by

$$g_{M/P} \approx g_M - g_P = g_M - \pi$$