TOPIC 3: UNDERSTANDING RISK*

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This lecture studies how to measure risk and assess its possible changes. It helps to understand why changes in risk lead to changes in the price of particular financial instruments and hence changes in the demand for those instruments. Every day we make decisions involving risk; making any decision that has more than one possible outcome is similar to gambling. Indeed, the tools used to measure risk were originally developed to analyze games of chance. Applying these rules of probability will help us understand the possibility of various occurrences and allow us to make better choices. While risk cannot be eliminated, in many cases it can be effectively managed. Risk also creates opportunities; people are compensated for bearing risk. In order to calculate a fair price for transferring risk from one person to another requires being able to measure risk.

1 DEFINING RISK

Risk is a measure of uncertainty about the future payoff to an investment, assessed over some time horizon and relative to a benchmark. Several remarks:

- Risk is a measure that can be quantified. Uncertainties that are not quantifiable cannot be priced.
- Risk arises from uncertainty about the future.
- Risk has to do with the future payoff of an investment, which is unknown.
- This definition of risk refers to an investment or group of investments.
- Risk must be assessed over some time horizon.
- Risk must be assessed relative to a benchmark rather than in isolation.

2 MEASURING RISK

We will use some elementary tools from probability theory to measure risk.

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Disclaimer: these are notes that I used by myself to lecture from and for educational purposes only. The material presented here is largely based upon the undergraduate textbook by Stephen Cecchetti and Kermit Schoenholtz (2014), *Money, Banking and Financial Markets*, 4th Edition, McGraw-Hill/Irwin. Please do NOT circulate.

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Possibilities, probabilities, and expected value. **Probability** is a measure, expressed as a number between zero and one, of the likelihood that an event will occur. For example, if we flip a fair coin, then the probabilities that it will come down head or tail are both one-half. The probabilities of all the possibilities always sum up to one. Now we can compute the **expected value** or **mean** of an investment with more than one possible returns or **payoffs**.

- Case 1: Suppose that for \$1,000 we can purchase a stock whose value is equally likely to fall to \$700 or rise to \$1,400. It means if we had purchased this stock one million times, about half million times the stock would pay off \$700 and the other half million times it would pay off \$1,400. So the average payoff from one million investments is

$$\begin{aligned}\underbrace{\$1,050}_{\text{EV}} &= \frac{500,000 \times \$700 + 500,000 \times \$1,400}{1,000,000} \\ &= \frac{1}{2} \times \$700 + \frac{1}{2} \times \$1,400 \end{aligned} \tag{2.1}$$

  which is also the expected value (EV) of the stock. It shows that the expected value is the probability-weighted sum of the possible payoffs.[1]

- Case 2: Suppose that the stock now can yield four possible payoffs, \$100 (0.1), \$700 (0.4), \$1,400 (0.4), and \$2,000 (0.1), with the associated probabilities indicated in the parentheses. Then the expected value of the stock is

$$\$1,050 = 0.1 \times \$100 + 0.4 \times \$700 + 0.4 \times \$1,400 + 0.1 \times \$2,000 \tag{2.2}$$

  Note that $0.1 + 0.4 + 0.4 + 0.1 = 1$.

- The two \$1,000 investments both yield an **expected return** of \$50, or an **expected return rate** of 5%, but they carry different levels of risk measured as follows.

Measures of risk. The two examples above suggest that we measure risk by quantifying the spread among an investment's possible outcomes. A **risk-free asset** is an investment whose future value is known with certainty (i.e. with probability one) and whose return is the **risk-free rate of return**. Two measures of risk:

- **Variance** (Var) is defined as the average of the the squared deviations of the possible outcomes from their expected value, weighted by their probabilities. For case 1,

$$\underbrace{122,500(\text{dollars}^2)}_{\text{Var}} = \frac{1}{2}(\$1,400 - \$1,050)^2 + \frac{1}{2}(\$700 - \$1,050)^2 \tag{2.3}$$

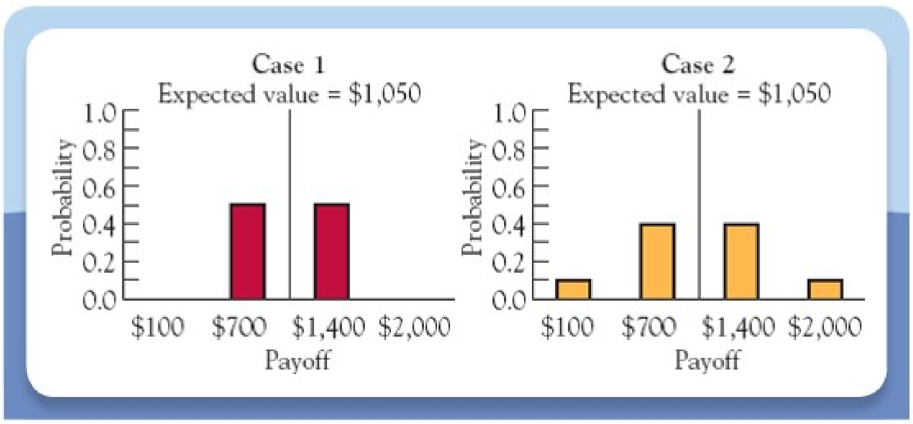
[1] But in reality, regardless of the number of times we purchase this stock, the payoff will never be \$1,050.

and the **standard deviation** (s.d.) is simply the (positive) square root of the variance,

$$\text{s.d.} = \sqrt{\text{Var}} = \sqrt{122,500(\text{dollars}^2)} = \$350 \tag{2.4}$$

which is more useful than the variance because it is measured in the same unit as the payoffs. Similarly, s.d. = \$528 in case 2 (Check it!). Because the two investments have the same expected value, most people would prefer the first; the greater the standard deviation, the higher the risk. See Figure 5.1 below.

Figure 5.1 Investing \$1,000



- **Value at risk (VaR)** is the worst possible loss over a specific time horizon, at a given probability. It measures the riskiness of the worst outcome rather than the spread of all outcomes measured by standard deviation. For example, the VaR is \$300 with probability 0.5 in case 1, whereas it is \$900 with probability 0.1 in case 2.

3 Risk Aversion, the Risk Premium, and the Risk-Return Trade-off

A **risk-averse** investor will always prefer an investment with a certain return to one with the same expected return but any amount of uncertainty. A **risk-neutral** investor only cares about the expected return. Two remarks:

- Other than the risk-free return, a risky investment must offer an extra return, called **risk premium**, because investors require compensation for taking risk [Core Principle 2]. The riskier an investment, the higher the risk premium and hence the expected return.

- There is a tradeoff between risk and expected return; we cannot get a high return without taking considerable risk. See Figure 5.3 below.

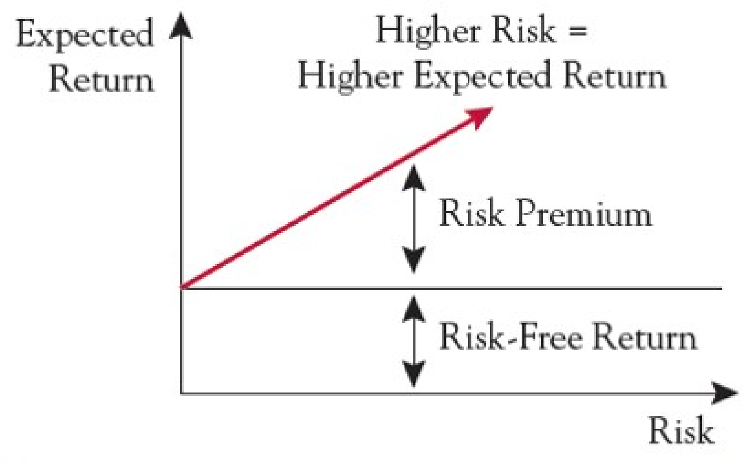


Figure 5.3. Tradeoff between risk and expected return

4 Sources of Risk: Idiosyncratic and Systematic Risk

We can classify all risks into one of two groups:

- **Idiosyncratic risks** or unique risks are those affecting a small number of people but no one else. For example, the risk that Ford's stock price may go up or down is idiosyncratic because it only affects Ford and its shareholders.

- **Systematic risks** or economy-wide risks are those affecting everyone. For example, macroeconomic factors, such as swings in consumer and business confidence, are sources of systematic risks that affect all firms and individuals.

5 Reducing Risk through Diversification

Risk can be reduced through diversification, the principle of holding more than one risk at a time, because holding several different investments can reduce the idiosyncratic risk an investor bears. There are two ways to diversify an investment as discussed below.

Hedging risk. **Hedging** is the strategy of reducing idiosyncratic risk by making two investments with opposing risks. For example, suppose that oil prices have an equal chance of rising or falling. When they rise, Texaco (an oil company) shareholders receive a payoff of $120 for each $100 invested; when oil prices fall, they just get $100 investment back. The reverse is true for GE.[2] Three strategies:

[2]This is because increases in the oil price are bad for most of the economy, but they are good for oil companies.

- Strategy 1: invest \$100 in GE. Then the expected payoff is

$$\$110 = \frac{1}{2} \times \$120 + \frac{1}{2} \times \$100 \tag{5.1}$$

with standard deviation

$$\$10 = \sqrt{\frac{1}{2} \times (\$120 - \$110)^2 + \frac{1}{2} \times (\$100 - \$110)^2} \tag{5.2}$$

- Strategy 2: invest \$100 in Texaco. Same EV and s.d. as strategy 1 (Check it!).

- Strategy 3: invest \$50 in GE and \$50 in Texaco. Then the expected payoff is

$$\$110 = \frac{1}{2} \times (\$60 + \$50) + \frac{1}{2} \times (\$60 + \$50) \tag{5.3}$$

with standard deviation

$$\$0 = \sqrt{\frac{1}{2} \times (\$60 + \$50 - \$110)^2 + \frac{1}{2} \times (\$60 + \$50 - \$110)^2} \tag{5.4}$$

So strategy 3 guarantees a payoff of \$110, regardless of whether oil prices go up or down. That is, hedging has eliminated the risk entirely in this example.

Spreading risk. **Spreading** is another strategy of reducing idiosyncratic risk by making several investments with unrelated payoffs. For example, suppose that GE and Microsoft's payoffs are independent of each other and both pay off either \$120 or \$100 with equal probability for each \$100 invested. Three strategies:

- Strategy 1: invest \$100 in GE. The expected payoff is \$110 with standard deviation \$10.

- Strategy 2: invest \$100 in Microsoft. Same EV and s.d. as strategy 1.

- Strategy 3: invest \$50 in GE and \$50 in Microsoft. Then the expected payoff is

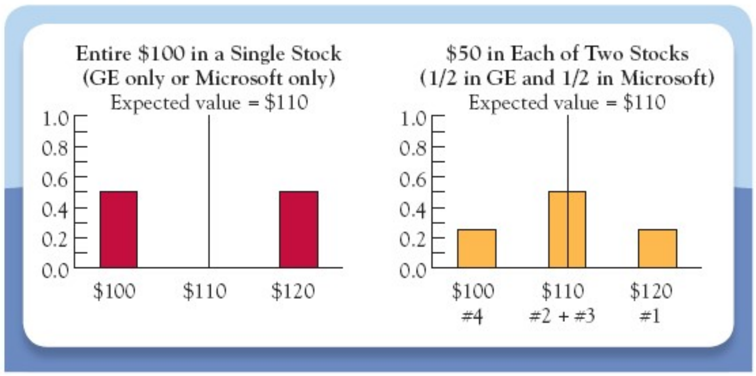
$$\$110 = \frac{1}{4} \times (\$60 + \$60) + \frac{1}{4} \times (\$60 + \$50) + \frac{1}{4} \times (\$50 + \$60) + \frac{1}{4} \times (\$50 + \$50) \tag{5.5}$$

with standard deviation

$$\$7.1 = \sqrt{\frac{1}{4} \times (\$60 + \$60 - \$110)^2 + \frac{1}{4} \times (\$50 + \$50 - \$110)^2} \tag{5.6}$$

So by spreading the investment among independently risky investments, strategy 3 lowers the spread of the outcomes and hence the risk.[3] See Figure 5.5 below.

Figure 5.5 Spreading Risk Payoffs from Two Investment Strategies



6 Math Appendix

We will show how diversification reduces risk in general. First, for risk hedging. Let (x, y) be the payoff pair to buying GE and Texaco stocks and p_i the probability associated with a particular outcome (x_i, y_i). Then the actual payoff on the investment is

$$\text{actual payoff} = ax + by \tag{6.1}$$

where $0 \leq a, b \leq 1$ are the proportions of the investment made in GE and Texaco, respectively, and satisfy $a + b = 1$. The variance is

$$\text{Var}(ax + by) = a^2\text{Var}(x) + b^2\text{Var}(y) + 2ab\text{Cov}(x, y) \tag{6.2}$$

where $\text{Cov}(x, y)$ is the covariance between x and y that measures the extent to which two risky assets move together. Symbolically, we have

$$\text{EV}(x) = \overline{x} = \sum_i p_i x_i \tag{6.3}$$

$$\text{Var}(x) = \sigma_x^2 = \sum_i p_i (x_i - \overline{x})^2 \tag{6.4}$$

and similarly for y. Moreover,

$$\text{Cov}(x, y) = \sigma_{x,y} = \sum_i p_i (x_i - \overline{x})(y_i - \overline{y}) \tag{6.5}$$

[3] It is easy to see that when the investment is split between the two stocks, the payoff becomes $110 or higher with probability 0.75 and $100 with only probability 0.25.

In the GE/Texaco example, $a = b = 1/2$, and we can show that (check it!)

$$\text{Var}(x) = \text{Var}(y) = 100, \quad \text{Cov}(x, y) = -100, \quad \text{Var}\left(\frac{1}{2}x + \frac{1}{2}y\right) = 0$$

That is, the two stocks behave as hedges for each other.

Second, for risk spreading. Assume that there are n independent investments $\{x^i\}_{i=1}^n$, each with the same expected payoff $\overline{x}$ and the same variance σ_x^2. If we hold $1/n$ of our portfolio in each stock, then the expected payoff is

$$\text{EV}\left(\sum_{i=1}^{n} \frac{x^i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} \text{EV}(x^i) = \overline{x} \tag{6.6}$$

Since the payoff on each stock is independent of all the rest, all the covariances are zero and hence the variance is

$$\text{Var}\left(\sum_{i=1}^{n} \frac{x^i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(x^i) = \frac{\sigma_x^2}{n} \tag{6.7}$$

Apparently, the variance declines as n increases; when the value of n is extremely large, the variance is essentially zero.