Technical Appendix for Incomplete Exchange Rate Pass-Through, Imperfect Financial Market Integration and Optimal Monetary Policy

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Contents

This technical appendix contains full derivations for analytical results in the paper and additional supplementary derivations.

A Total Equilibrium Conditions

In this section, we list the necessary optimality conditions and aggregate conditions derived from the model outlined in the paper.

International Prices and CES Demands: We first display useful definitions of international prices: terms of trade $(\mathcal{T}_t, \mathcal{T}_t^*)$, the price index of imported goods relative to domestic goods (S_t, S_t^*), the measure of the deviation from the law of one price (M_t) which can be written out as Home currency misalignment (M_{Ht}) or Foreign currency misalignment (M_{Ft}), and export premium (Z_t) given by

$$\mathcal{T}_t \equiv \frac{P_{Ft}}{\mathcal{E}_t P^*_{Ht}} = \frac{S_t}{M_t Z_t} \quad \text{and} \quad \mathcal{T}_t^* \equiv \frac{\mathcal{E}_t P^*_{Ht}}{P_{Ft}} = S_t^* \frac{M_t}{Z_t},$$

$$S_t \equiv \frac{P_{Ft}}{P_{Ht}} \quad \text{and} \quad S_t^* \equiv \frac{P^*_{Ht}}{P^*_{Ft}},$$

$$M_t \equiv \left[\frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}} \cdot \frac{\mathcal{E}_t P^*_{Ft}}{P_{Ft}}\right]^{\frac{1}{2}} \quad \text{and} \quad Z_t \equiv \left[\frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}} \cdot \frac{\frac{1}{\mathcal{E}_t} P_{Ft}}{P^*_{Ft}}\right]^{\frac{1}{2}} = \left[\frac{P_{Ft}}{P_{Ht}} \cdot \frac{P^*_{Ht}}{P^*_{Ft}}\right]^{\frac{1}{2}} = \left[S_t S_t^*\right]^{\frac{1}{2}},$$

$$M_{Ht} \equiv M_t Z_t = \frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}} \quad \text{and} \quad M_{Ft} \equiv \frac{M_t}{Z_t} = \frac{\mathcal{E}_t P^*_{Ft}}{P_{Ft}}.$$

Standard expenditure minimization of consumers leads to expressions for price and consumption indexes in Home and Foreign given by

$$\frac{P_t}{P_{Ht}} = \left\{\frac{\nu}{2} + \left(1 - \frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}} \quad \text{and} \quad \frac{P_t}{P_{Ft}} = \left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}},$$

$$\frac{P_t^*}{P^*_{Ft}} = \left\{\frac{\nu}{2} + \left(1 - \frac{\nu}{2}\right) {S_t^*}^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}} \quad \text{and} \quad \frac{P_t^*}{P^*_{Ht}} = \left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}},$$

$$C_{Ht} = \left(\frac{\nu}{2}\right)\left\{\frac{\nu}{2} + \left(1 - \frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t \quad \text{and} \quad C_{Ft} = \left(1 - \frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t,$$

$$C^*_{Ft} = \left(\frac{\nu}{2}\right)\left\{\frac{\nu}{2} + \left(1 - \frac{\nu}{2}\right) {S_t^*}^{1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t^* \quad \text{and} \quad C^*_{Ht} = \left(1 - \frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t^*.$$

In addition, we can express real exchange rate (Q_t) and currency misalignments in terms of S_t and S_t^* as

$$
\begin{aligned}
Q_t &\equiv \frac{\mathcal{E}_t P_t^*}{P_t} = M_t Z_t \frac{\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}}}{\left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}}} = M_t \left(\begin{array}{cc} & \left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}\left(\frac{-1}{2}\right)} \\ \times & \left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}\left(\frac{-1}{2}\right)} \\ \times & \left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}\left(\frac{1}{2}\right)} \\ \times & \left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{*1-\epsilon}\right\}^{\frac{1}{1-\epsilon}\left(\frac{1}{2}\right)} \end{array}\right), \\
M_{Ht} &= M_t Z_t = Q_t \left(\frac{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}, \\
M_{Ft} &= \frac{M_t}{Z_t} = \frac{Q_t}{{Z_t}^2}\left(\frac{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}} = Q_t\left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)\left(S_t^*\right)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}}.
\end{aligned}
$$

Note that it is convenient to express prices and outputs in terms of the relative price of Home and Foreign goods (S_t, S_t^*) for log-linearization. We solve for 55 endogenous variables:

$$
\begin{array}{llllllllllllll}
\frac{W_t}{P_t}, & \frac{W_t^*}{P_t^*}, & C_t, & C_t^*, & N_t, & N_t^*, & C_{Ht}, & C_{Ft}, & C_{Ft}^*, & C_{Ht}^*, & S_t, & S_t^*, & \Pi_t, & \Pi_t^*, \\
\frac{P_{Ht}}{P_t}, & \frac{P_{Ft}}{P_t}, & \frac{P_{Ft}^*}{P_t^*}, & \frac{P_{Ht}^*}{P_t^*}, & \Pi_{Ht}, & \Pi_{Ft}, & \Pi_{Ft}^*, & \Pi_{Ht}^*, & \Pi_{Ht}^L, & \Pi_{Ht}^{L*}, & \Pi_{Ht}^P, & \Pi_{Ft}^{L*}, & \Pi_{Ft}^L, & \Pi_{Ft}^{P*}, \\
V_{Ht}^L, & V_{Ht}^{L*}, & V_{Ht}^P, & V_{Ht}^{P*}, & V_{Ft}^{L*}, & V_{Ft}^L, & V_{Ft}^{P*}, & V_{Ft}^P, & \Pi_{Ht}^{L,o}, & \Pi_{Ht}^{L,o*}, & \Pi_{Ht}^{P,o}, & \Pi_{Ft}^{L,o*}, & \Pi_{Ft}^{L,o}, & \Pi_{Ft}^{P,o*}, \\
\Theta_{Ht}^L, & \Psi_{Ht}^L, & \Theta_{Ht}^{L*}, & \Psi_{Ht}^{L*}, & \Theta_{Ht}^P, & \Psi_{Ht}^P, & \Theta_{Ft}^{L*}, & \Psi_{Ft}^{L*}, & \Theta_{Ft}^L, & \Psi_{Ft}^L, & \Theta_{Ft}^{P*}, & \Psi_{Ft}^{P*}, & Q_t, &
\end{array}
$$

while there are 53 equilibrium conditions. We can find two additional equations by characterizing the Ramsey problem.

Notations for Price Indexes: Define CPI and PPI inflation measures ($\Pi_t, \Pi_t^*, \Pi_{Ht}, \Pi_{Ft}, \Pi_{Ft}^*, \Pi_{Ht}^*$), PPI-deflated prices of LCP Home goods ($\Pi_{Ht}^L, \Pi_{Ht}^{L*}, \Pi_{Ht}^{L,o}, \Pi_{Ht}^{L,o*}$), PPI-deflated prices of PCP Home goods ($\Pi_{Ht}^P, \Pi_{Ht}^{P,o}$), PPI-deflated prices of LCP Foreign goods ($\Pi_{Ft}^{L*}, \Pi_{Ft}^L, \Pi_{Ft}^{L,o*}, \Pi_{Ft}^{L,o}$), and PPI-deflated prices of PCP Foreign goods ($\Pi_{Ft}^{P*}, \Pi_{Ft}^{P,o*}$) as

$$
\begin{array}{llllll}
\Pi_t \equiv \frac{P_t}{P_{t-1}}, & \Pi_t^* \equiv \frac{P_t^*}{P_{t-1}^*}, & \Pi_{Ht} \equiv \frac{P_{Ht}}{P_{H,t-1}}, & \Pi_{Ft} \equiv \frac{P_{Ft}}{P_{F,t-1}}, & \Pi_{Ft}^* \equiv \frac{P_{Ft}^*}{P_{F,t-1}^*}, & \Pi_{Ht}^* \equiv \frac{P_{Ht}^*}{P_{H,t-1}^*}, \\
\Pi_{Ht}^L \equiv \frac{P_{Ht}^L}{P_{Ht}}, & \Pi_{Ht}^{L*} \equiv \frac{P_{Ht}^{L*}}{P_{Ht}^*}, & \Pi_{Ht}^P \equiv \frac{P_{Ht}^P}{P_{Ht}}, & \Pi_{Ht}^{L,o} \equiv \frac{P_{Ht}^{L,o}}{P_{Ht}}, & \Pi_{Ht}^{L,o*} \equiv \frac{P_{Ht}^{L,o*}}{P_{Ht}^*}, & \Pi_{Ht}^{P,o} \equiv \frac{P_{Ht}^{P,o}}{P_{Ht}}, \\
\Pi_{Ft}^{L*} \equiv \frac{P_{Ft}^{L*}}{P_{Ft}^*}, & \Pi_{Ft}^L \equiv \frac{P_{Ft}^L}{P_{Ft}}, & \Pi_{Ft}^{P*} \equiv \frac{P_{Ft}^{P*}}{P_{Ft}^*}, & \Pi_{Ft}^{L,o*} \equiv \frac{P_{Ft}^{L,o*}}{P_{Ft}^*}, & \Pi_{Ft}^{L,o} \equiv \frac{P_{Ft}^{L,o}}{P_{Ft}}, & \Pi_{Ft}^{P,o*} \equiv \frac{P_{Ft}^{P,o*}}{P_{Ft}^*}.
\end{array}
$$

Here, those prices without superscript 'o' denote the LCP (PCP) price indexes deflated by relevant PPI. Prices with superscript 'o' represent PPI-deflated optimal prices reset by firms who are allowed to reoptimize.

Consumer Price Indexes and Nominal Domestic Interest Rates: Consumer price index (P_t, P_t^*) in each country can be derived in terms of producer price indexes ($P_{Ht}, P_{Ft}, P_{Ft}^*, P_{Ht}^*$) by

$$
{P_t}^{1-\epsilon} = \frac{\nu}{2}{P_{Ht}}^{1-\epsilon} + \left(1-\frac{\nu}{2}\right){P_{Ft}}^{1-\epsilon} \quad \text{and} \quad {P_t^*}^{1-\epsilon} = \frac{\nu}{2}{P_{Ft}^*}^{1-\epsilon} + \left(1-\frac{\nu}{2}\right){P_{Ht}^*}^{1-\epsilon}.
$$

The nominal interest rate (i_t, i_t^*) in each country can be found from the household's euler equation given by

$$1 = \beta(1+i_t)\mathbb{E}_t\left[\frac{\zeta_{C,t+1}C_{t+1}(h)^{-\sigma}}{\zeta_{C,t}C_t(h)^{-\sigma}}\cdot\frac{P_t}{P_{t+1}}\right] \quad \text{and} \quad 1 = \beta(1+i_t^*)\mathbb{E}_t\left[\frac{\zeta^*_{C,t+1}C^*_{t+1}(h^*)^{-\sigma}}{\zeta^*_{C,t}C^*_t(h^*)^{-\sigma}}\cdot\frac{P^*_t}{P^*_{t+1}}\right].$$

Home and Foreign Tradeable Goods: Total output in each country (Y_t, Y_t^*) is equal to total demand in the world economy and thus it can be expressed as

$$\begin{aligned}
Y_t &\equiv C_{Ht} + C^*_{Ht} = \tfrac{\nu}{2}\left(\tfrac{P_{Ht}}{P_t}\right)^{-\epsilon}C_t + \left(1-\tfrac{\nu}{2}\right)\left(\tfrac{P^*_{Ht}}{P^*_t}\right)^{-\epsilon}C^*_t \\
&= \tfrac{\nu}{2}\left\{\tfrac{\nu}{2} + \left(1-\tfrac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}}C_t + \left(1-\tfrac{\nu}{2}\right)\left\{\tfrac{\nu}{2}\left(\tfrac{1}{S^*_t}\right)^{1-\epsilon} + 1 - \tfrac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}}C^*_t, \\
Y^*_t &\equiv C^*_{Ft} + C_{Ft} = \tfrac{\nu}{2}\left(\tfrac{P^*_{Ft}}{P^*_t}\right)^{-\epsilon}C^*_t + \left(1-\tfrac{\nu}{2}\right)\left(\tfrac{P_{Ft}}{P_t}\right)^{-\epsilon}C_t \\
&= \tfrac{\nu}{2}\left\{\tfrac{\nu}{2} + \left(1-\tfrac{\nu}{2}\right)S_t^{*1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}}C^*_t + \left(1-\tfrac{\nu}{2}\right)\left\{\tfrac{\nu}{2}\left(\tfrac{1}{S_t}\right)^{1-\epsilon} + 1 - \tfrac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}}C_t.
\end{aligned}$$

Home and Foreign Monopolists' Pricing under Flexible Prices: Under flexible prices, the price of a firm's product does not depend on whether the firm sets prices in producer currency or consumer currency. Since there is no price dispersion, the optimal price of each firm is equal to producer price index:

$$\begin{aligned}
P_{Ht} &= P^L_{Ht}(f) = P^P_{Ht}(f) = \tfrac{\xi}{\xi-1}(1-\tau)\tfrac{W_t}{A_t} = \mathcal{E}_t P^{L*}_{Ht}(f) = \mathcal{E}_t P^*_{Ht}, \\
P^*_{Ft} &= P^{L*}_{Ft}(f^*) = P^{P*}_{Ft}(f^*) = \tfrac{\xi}{\xi-1}(1-\tau^*)\tfrac{W^*_t}{A^*_t} = \tfrac{1}{\mathcal{E}_t}P^L_{Ft}(f^*) = \tfrac{1}{\mathcal{E}_t}P_{Ft},
\end{aligned}$$

where $\frac{1}{\xi-1}$ denotes the monopoly markup and τ represents the government subsidy which offsets monopoly distortions in the steady-state equilibrium. In equilibrium under flexible prices, the law of one price holds for each variety.

Wage Setting of Households in Home and Foreign: Households are monopolistic labor suppliers since each labor is differentiated. They set wages given labor demand of firms and charge markup $\left(\frac{1}{\zeta_{N,t}-1}\right)$. The optimal wage is equal to the marginal utility cost of labor multiplied by the time-varying markup, implying

$$\frac{\dddot{W}_t}{P_t} = \frac{\zeta_{N,t}-1}{\zeta_{N,t}}\frac{W_t}{P_t} = {C_t}^{\sigma}{N_t}^{\phi}\frac{\kappa}{\zeta_{C,t}}, \qquad (1) \qquad\qquad \frac{\dddot{W}^*_t}{P^*_t} = \frac{\zeta^*_{N,t}-1}{\zeta^*_{N,t}}\frac{W^*_t}{P^*_t} = {C^*_t}^{\sigma}{N^*_t}^{\phi}\frac{\kappa}{\zeta^*_{C,t}}, \qquad (2)$$

where $\dddot{W}_t$ and $\dddot{W}^*_t$ denote markup-free nominal wages.

Financial Market Condition and Demand Imbalance: Under the complete asset market, the optimality condition for asset holdings equates shadow values of nominal wealth between Home and Foreign under the assumption of zero net bond supply in the steady state. Due to the effect of capital controls on external bond holdings, there is a wedge $(F_t = 1 + \varrho_t)$ which is interpreted as demand imbalance, i.e., the degree of deviation from perfect risk sharing. The first-order necessary condition for the complete set of assets implies

$$\left(\frac{C^*_t}{C_t}\right)^{-\sigma} = \left(\frac{\zeta_{C,t}}{\zeta^*_{C,t}}\right)Q_t F_t \quad \text{where} \quad F_t = 1 + \varrho_t = \left(\frac{\frac{P_{Ht}}{P_t}C_{Ht} + Q_t\frac{P^*_{Ht}}{P^*_t}C^*_{Ht}}{\frac{P_{Ht}}{P_t}C_{Ht} + \frac{P_{Ft}}{P_t}C_{Ft}}\right)^{\frac{\lambda}{(1-\lambda)}}. \qquad (3)$$

Household Demand, Real PPI, and Relative Prices between Local and Imported Goods: Household demands are derived from CES consumption baskets:

$$C_{Ht} = \frac{\nu}{2}\left(\frac{P_{Ht}}{P_t}\right)^{-\epsilon} C_t, \quad (4)$$

$$C_{Ft} = \left(1-\frac{\nu}{2}\right)\left(\frac{P_{Ft}}{P_t}\right)^{-\epsilon} C_t, \quad (5)$$

$$C^*_{Ft} = \frac{\nu}{2}\left(\frac{P^*_{Ft}}{P^*_t}\right)^{-\epsilon} C^*_t, \quad (6)$$

$$C^*_{Ht} = \left(1-\frac{\nu}{2}\right)\left(\frac{P^*_{Ht}}{P^*_t}\right)^{-\epsilon} C^*_t. \quad (7)$$

We can obtain real price indexes dual to these consumption baskets given by

$$\frac{P_{Ht}}{P_t} = \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}}, \quad (8)$$

$$\frac{P^*_{Ft}}{P^*_t} = \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{*1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}}, \quad (9)$$

The price index of imported goods relative to domestically-produced goods can be written in terms of real price indexes as

$$S_t = \frac{\left(\frac{P_{Ft}}{P_t}\right)}{\left(\frac{P_{Ht}}{P_t}\right)}, \quad (10)$$

$$S^*_t = \frac{\left(\frac{P^*_{Ht}}{P^*_t}\right)}{\left(\frac{P^*_{Ft}}{P^*_t}\right)}. \quad (11)$$

Inflation rates: PPI inflation can be expressed in terms of real price indexes and CPI inflation as

$$\Pi_{Ht} = \frac{\left(\frac{P_{Ht}}{P_t}\right)}{\left(\frac{P_{H,t-1}}{P_{t-1}}\right)}\Pi_t, \quad (12)$$

$$\Pi_{Ft} = \frac{\left(\frac{P_{Ft}}{P_t}\right)}{\left(\frac{P_{F,t-1}}{P_{t-1}}\right)}\Pi_t, \quad (13)$$

$$\Pi^*_{Ft} = \frac{\left(\frac{P^*_{Ft}}{P^*_t}\right)}{\left(\frac{P^*_{F,t-1}}{P^*_{t-1}}\right)}\Pi^*_t, \quad (14)$$

$$\Pi^*_{Ht} = \frac{\left(\frac{P^*_{Ht}}{P^*_t}\right)}{\left(\frac{P^*_{H,t-1}}{P^*_{t-1}}\right)}\Pi^*_t. \quad (15)$$

Price Dispersion Measures for Home Tradeables: In the paper, we define price dispersion terms for products of LCP and PCP firms in Home as

$$V^L_{Ht} \equiv \frac{1}{\chi}\int_0^\chi \left(\frac{P^L_{Ht}(f)}{P_{Ht}}\right)^{-\xi} df, \qquad V^{L*}_{Ht} \equiv \frac{1}{\chi}\int_0^\chi \left(\frac{P^{L*}_{Ht}(f)}{P^*_{Ht}}\right)^{-\xi} df,$$

$$V^P_{Ht} \equiv \frac{1}{1-\chi}\int_\chi^1 \left(\frac{P^P_{Ht}(f)}{P_{Ht}}\right)^{-\xi} df, \qquad V^{P*}_{Ht} \equiv \frac{1}{1-\chi}\int_\chi^1 \left(\frac{\frac{1}{\mathcal{E}_t}P^P_{Ht}(f)}{P^*_{Ht}}\right)^{-\xi} df,$$

where there are a fraction χ of LCP firms and a portion $1-\chi$ of PCP firms in Home. Using the standard formulation in the New-Keynesian literature, we can derive recursive equations for price dispersion terms by

$$V^L_{Ht} = (\Pi_{Ht})^\xi \theta V^L_{H,t-1} + (1-\theta)\left(\Pi^{L,o}_{Ht}\right)^{-\xi}, \quad (16)$$

$$V^{L*}_{Ht} = (\Pi^*_{Ht})^\xi \theta V^{L*}_{H,t-1} + (1-\theta)\left(\Pi^{L,o*}_{Ht}\right)^{-\xi}, \quad (17)$$

$$V^P_{Ht} = (\Pi_{Ht})^\xi \theta V^P_{H,t-1} + (1-\theta)\left(\Pi^{P,o}_{Ht}\right)^{-\xi}, \quad (18)$$

$$V^{P*}_{Ht} = \left(\frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}}\right)^\xi V^P_{Ht} = \left(Q_t\left(\frac{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S^*_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}\right)^\xi V^P_{Ht}, \quad (19)$$

where we rewrite the fourth price dispersion term (V^{P*}_{Ht}) by using Home currency misalignment: $M_{Ht} \equiv \frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}} = M_t Z_t = Q_t\left(\frac{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S^*_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}$.

Price Dispersion Measures for Foreign Tradeables: Analogously, price dispersion terms for products of

LCP and PCP firms in Foreign are defined as

$$V_{Ft}^{L*} \equiv \frac{1}{\chi}\int_0^{\chi}\left(\frac{P_{Ft}^{L*}(f^*)}{P_{Ft}^*}\right)^{-\xi} df^*, \qquad V_{Ft}^{L} \equiv \frac{1}{\chi}\int_0^{\chi}\left(\frac{P_{Ft}^{L}(f^*)}{P_{Ft}}\right)^{-\xi} df^*,$$
$$V_{Ft}^{P*} \equiv \frac{1}{1-\chi}\int_{\chi}^{1}\left(\frac{P_{Ft}^{P*}(f^*)}{P_{Ft}^*}\right)^{-\xi} df^*, \qquad V_{Ft}^{P} \equiv \frac{1}{1-\chi}\int_{\chi}^{1}\left(\frac{\mathcal{E}_t P_{Ft}^{P*}(f^*)}{P_{Ft}}\right)^{-\xi} df^*,$$

where there are also a fraction χ of LCP firms and a portion $1-\chi$ of PCP firms in Foreign. After making use of standard algebra, we can derive recursive equations for price dispersion terms by

$$V_{Ft}^{L*} = \left(\Pi_{Ft}^*\right)^{\xi}\theta V_{F,t-1}^{L*} + (1-\theta)\left(\Pi_{Ft}^{L,o*}\right)^{-\xi} \tag{20}$$
$$V_{Ft}^{L} = \left(\Pi_{Ft}\right)^{\xi}\theta V_{F,t-1}^{L} + (1-\theta)\left(\Pi_{Ft}^{L,o}\right)^{-\xi} \tag{21}$$
$$V_{Ft}^{P*} = \left(\Pi_{Ft}^*\right)^{\xi}\theta V_{F,t-1}^{P*} + (1-\theta)\left(\Pi_{Ft}^{P,o*}\right)^{-\xi} \tag{22}$$
$$V_{Ft}^{P} = \left(\frac{P_{Ft}}{\mathcal{E}_t P_{Ft}^*}\right)^{\xi} V_{Ft}^{P*} = \left(\frac{1}{Q_t}\left(\frac{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)\left(S_t^*\right)^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}\right)^{\xi} V_{Ft}^{P*} \tag{23}$$

where we rewrite the fourth price dispersion term (V_{Ft}^P) by using Foreign currency misalignment: $M_{Ft} \equiv \frac{\mathcal{E}_t P_{Ft}^*}{P_{Ft}} = \frac{M_t}{Z_t} = Q_t\left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)\left(S_t^*\right)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}}$.

Market Clearing Condition for Home and Foreign Tradeables: The market clearing condition for each Home tradeable good implies

$$A_t N_t = C_{Ht}\left(\chi V_{Ht}^{L} + (1-\chi)V_{Ht}^{P}\right) + C_{Ht}^*\left(\chi V_{Ht}^{L*} + (1-\chi)V_{Ht}^{P*}\right). \tag{24}$$

Given $Y_t \equiv C_{Ht}+C_{Ht}^*$, $A_t N_t$ is not equal to Y_t under sticky-price equilibrium. Likewise, the market clearing condition for each Foreign tradeable good leads to

$$A_t^* N_t^* = C_{Ft}^*\left(\chi V_{Ft}^{L*} + (1-\chi)V_{Ft}^{P*}\right) + C_{Ft}\left(\chi V_{Ft}^{L} + (1-\chi)V_{Ft}^{P}\right). \tag{25}$$

Given $Y_t^* \equiv C_{Ft}^* + C_{Ft}$, $A_t^* N_t^*$ is not equal to Y_t^* under sticky-price equilibrium. Complete derivations for (24) and (25) are available upon request.

Producer Price Indexes for Home Products: The price indexes dual to CES baskets for Home products in Home and Foreign can be derived by

$$P_{Ht}{}^{1-\xi} = \int_0^{\chi}\left(P_{Ht}^L(f)\right)^{1-\xi} df + \int_{\chi}^{1}\left(P_{Ht}^P(f)\right)^{1-\xi} df = \chi P_{Ht}^{L\ 1-\xi} + (1-\chi)P_{Ht}^{P\ 1-\xi},$$
$$P_{Ht}^*{}^{1-\xi} = \int_0^{\chi}\left(P_{Ht}^{L*}(f)\right)^{1-\xi} df + \left(\frac{1}{\mathcal{E}_t}\right)^{1-\xi}\int_{\chi}^{1}\left(P_{Ht}^P(f)\right)^{1-\xi} df = \chi P_{Ht}^{L*\,1-\xi} + \left(\frac{1}{\mathcal{E}_t}\right)^{1-\xi}(1-\chi)P_{Ht}^{P\ 1-\xi},$$

where we define price indexes for products of Home LCP and PCP firms as

$$P_{Ht}^{L\ 1-\xi} \equiv \frac{1}{\chi}\int_0^{\chi}\left(P_{Ht}^L(f)\right)^{1-\xi} df, \quad P_{Ht}^{L*\,1-\xi} \equiv \frac{1}{\chi}\int_0^{\chi}\left(P_{Ht}^{L*}(f)\right)^{1-\xi} df, \quad \text{and} \quad P_{Ht}^{P\ 1-\xi} \equiv \frac{1}{1-\chi}\int_{\chi}^{1}\left(P_{Ht}^P(f)\right)^{1-\xi} df.$$

Together these equations imply

$$1 = \chi {\Pi^{L}_{Ht}}^{1-\xi} + (1-\chi){\Pi^{P}_{Ht}}^{1-\xi}, \tag{26}$$

$$1 = \chi {\Pi^{L*}_{Ht}}^{1-\xi} + \left(\frac{1}{Q_t} \left(\frac{\frac{\nu}{2}\left(\frac{1}{S^*_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}}{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}} \right)^{\frac{1}{1-\epsilon}} \right)^{1-\xi} (1-\chi){\Pi^{P}_{Ht}}^{1-\xi}, \tag{27}$$

where we use $M_t Z_t = \frac{\mathcal{E}_t P^*_{Ht}}{P_{Ht}} = Q_t \left(\frac{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S^*_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}} \right)^{\frac{1}{1-\epsilon}}$ for the second equality. Standard derivation leads to recursive formula for PPI-deflated prices of LCP and PCP products:

$$ {\Pi^{L}_{Ht}}^{1-\xi} = \theta \left(\frac{\Pi^{L}_{H,t-1}}{\Pi_{Ht}} \right)^{1-\xi} + (1-\theta)\left(\Pi^{L,o}_{Ht}\right)^{1-\xi}, \tag{28}$$

$$ {\Pi^{L*}_{Ht}}^{1-\xi} = \theta \left(\frac{\Pi^{L*}_{H,t-1}}{\Pi^*_{Ht}} \right)^{1-\xi} + (1-\theta)\left(\Pi^{L,o*}_{Ht}\right)^{1-\xi}, \tag{29}$$

$$ {\Pi^{P}_{Ht}}^{1-\xi} = \theta \left(\frac{\Pi^{P}_{H,t-1}}{\Pi_{Ht}} \right)^{1-\xi} + (1-\theta)\left(\Pi^{P,o}_{Ht}\right)^{1-\xi}, \tag{30}$$

where prices with superscript 'o' represent optimal prices reset by firms who are allowed to reoptimize.

Producer Price Indexes for Foreign Products: The price indexes dual to CES baskets for Foreign products in Foreign and Home can be derived by

$$ {P^*_{Ft}}^{1-\xi} = \int_0^\chi \left(P^{L*}_{Ft}(f^*)\right)^{1-\xi} df^* + \int_\chi^1 \left(P^{P*}_{Ft}(f^*)\right)^{1-\xi} df^* = \chi {P^{L*}_{Ft}}^{1-\xi} + (1-\chi){P^{P*}_{Ft}}^{1-\xi},$$

$$ {P_{Ft}}^{1-\xi} = \int_0^\chi \left(P^{L}_{Ft}(f^*)\right)^{1-\xi} df^* + \int_\chi^1 \left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right)^{1-\xi} df^* = \chi {P^{L}_{Ft}}^{1-\xi} + (\mathcal{E}_t)^{1-\xi}(1-\chi){P^{P*}_{Ft}}^{1-\xi},$$

where we define price indexes for products of Foreign LCP and PCP firms as

$${P^{L*}_{Ft}}^{1-\xi} \equiv \frac{1}{\chi}\int_0^\chi \left(P^{L*}_{Ft}(f^*)\right)^{1-\xi} df^*, \quad {P^{L}_{Ft}}^{1-\xi} \equiv \frac{1}{\chi}\int_0^\chi \left(P^{L}_{Ft}(f^*)\right)^{1-\xi} df^*, \quad \text{and} \quad {P^{P*}_{Ft}}^{1-\xi} \equiv \frac{1}{1-\chi}\int_\chi^1 \left(P^{P*}_{Ft}(f^*)\right)^{1-\xi} df^*.$$

Together these equations imply

$$1 = \chi {\Pi^{L*}_{Ft}}^{1-\xi} + (1-\chi){\Pi^{P*}_{Ft}}^{1-\xi}, \tag{31}$$

$$1 = \chi {\Pi^{L}_{Ft}}^{1-\xi} + \left(Q_t \left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}}{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) (S^*_t)^{1-\epsilon}} \right)^{\frac{1}{1-\epsilon}} \right)^{1-\xi} (1-\chi){\Pi^{P*}_{Ft}}^{1-\xi}, \tag{32}$$

where we use $\frac{M_t}{Z_t} = \frac{\mathcal{E}_t P^*_{Ft}}{P_{Ft}} = Q_t \left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}}{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right)(S^*_t)^{1-\epsilon}} \right)^{\frac{1}{1-\epsilon}}$ for the second equality. Standard derivation leads to recursive formula for PPI-deflated prices of LCP and PCP products:

$$\Pi_{Ft}^{L*\,1-\xi} = \theta\left(\frac{\Pi_{F,t-1}^{L*}}{\Pi_{Ft}^{*}}\right)^{1-\xi} + (1-\theta)\left(\Pi_{Ft}^{L,o*}\right)^{1-\xi}, \tag{33}$$

$$\Pi_{Ft}^{L\;1-\xi} = \theta\left(\frac{\Pi_{F,t-1}^{L}}{\Pi_{Ft}}\right)^{1-\xi} + (1-\theta)\left(\Pi_{Ft}^{L,o}\right)^{1-\xi}, \tag{34}$$

$$\Pi_{Ft}^{P*\,1-\xi} = \theta\left(\frac{\Pi_{F,t-1}^{P*}}{\Pi_{Ft}^{*}}\right)^{1-\xi} + (1-\theta)\left(\Pi_{Ft}^{P,o*}\right)^{1-\xi}, \tag{35}$$

where prices with superscript 'o' represent optimal prices reset by firms who are allowed to reoptimize.

Home Monopolists' Pricing under Sticky Prices and Recursive Formulation: Optimal prices of Home LCP and PCP monopolists under sluggish price adjustment are given by

$$P_{Ht}^{L,o} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}}{A_{t+j}}\left(P_{H,t+j}\right)^{\xi} C_{H,t+j}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\left(P_{H,t+j}\right)^{\xi} C_{H,t+j}\right]},$$

$$P_{Ht}^{L,o*} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}}{A_{t+j}}\left(P_{H,t+j}^{*}\right)^{\xi} C_{H,t+j}^{*}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\mathcal{E}_{t+j}\left(P_{H,t+j}^{*}\right)^{\xi} C_{H,t+j}^{*}\right]},$$

$$P_{Ht}^{P,o} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}}{A_{t+j}}\left\{\left(P_{H,t+j}\right)^{\xi} C_{H,t+j}+\left(\mathcal{E}_{t+j}P_{H,t+j}^{*}\right)^{\xi} C_{H,t+j}^{*}\right\}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}\left\{\left(P_{H,t+j}\right)^{\xi} C_{H,t+j}+\left(\mathcal{E}_{t+j}P_{H,t+j}^{*}\right)^{\xi} C_{H,t+j}^{*}\right\}\right]},$$

where the Home stochastic discount factor is defined as $\Upsilon_{t,t+j} \equiv \beta^j\left(\frac{C_{t+j}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+j}}\right)\left(\frac{\zeta_{C,t+j}}{\zeta_{C,t}}\right)$. It is worth noting that LCP firms segment markets by country and set separate prices from the local demand of each market, whereas PCP firms set a single price from the world demand. These three prices differ under sticky prices and the coexistence of LCP and PCP firms captured by χ causes additional price dispersion terms in the sticky-price equilibrium which can be shown from the quadratic approximation to the loss function.

We use the standard derivation from the New-Keynesian literature and reformulate the above equations in a recursive form given by

$$\Pi_{Ht}^{L,o} = \frac{\Theta_{Ht}^{L}}{\Psi_{Ht}^{L}}, \tag{36}$$

$$\Theta_{Ht}^{L} = \frac{\xi(1-\tau)}{\xi-1}\left(\frac{W_t}{P_t A_t}\right) C_{H,t} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}/P_{t+1}}{P_{Ht}/P_t}\Pi_{t+1}\right)^{\xi}\Theta_{H,t+1}^{L}\right], \tag{37}$$

$$\Psi_{Ht}^{L} = \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} C_{H,t} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}/P_{t+1}}{P_{Ht}/P_t}\Pi_{t+1}\right)^{\xi-1}\Psi_{H,t+1}^{L}\right], \tag{38}$$

$$\Pi_{Ht}^{L,o*} = \frac{\Theta_{Ht}^{L*}}{\Psi_{Ht}^{L*}}, \tag{39}$$

$$\Theta_{Ht}^{L*} = \frac{\xi(1-\tau)}{\xi-1}\left(\frac{W_t}{P_t A_t}\right) C_{H,t}^{*} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}^{*}/P_{t+1}^{*}}{P_{Ht}^{*}/P_t^{*}}\Pi_{t+1}^{*}\right)^{\xi}\Theta_{H,t+1}^{L*}\right], \tag{40}$$

$$\Psi_{Ht}^{L*} = Q_t\left\{\frac{\nu}{2}\left(\frac{1}{S_t^{*}}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{-1}{1-\epsilon}} C_{H,t}^{*} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}^{*}/P_{t+1}^{*}}{P_{Ht}^{*}/P_t^{*}}\Pi_{t+1}^{*}\right)^{\xi-1}\Psi_{H,t+1}^{L*}\right], \tag{41}$$

$$\Pi_{Ht}^{P,o} = \frac{\Theta_{Ht}^{P}}{\Psi_{Ht}^{P}}, \tag{42}$$

$$\Theta_{Ht}^{P} = \left[\begin{array}{l} \frac{\xi(1-\tau)}{\xi-1}\left(\frac{W_t}{P_t A_t}\right)\left\{C_{H,t} + \left(Q_t\left(\frac{\frac{\nu}{2}+(1-\frac{\nu}{2})S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S_t^{*}}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}\right)^{\xi} C_{H,t}^{*}\right\} \\ +\ \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}/P_{t+1}}{P_{Ht}/P_t}\Pi_{t+1}\right)^{\xi}\Theta_{H,t+1}^{P}\right] \end{array}\right], \tag{43}$$

$$\Psi_{Ht}^{P} = \left[\begin{array}{l} \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} C_{H,t} + Q_t\left\{\frac{\nu}{2}\left(\frac{1}{S_t^{*}}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{-1}{1-\epsilon}}\left(Q_t\left(\frac{\frac{\nu}{2}+(1-\frac{\nu}{2})S_t^{1-\epsilon}}{\frac{\nu}{2}\left(\frac{1}{S_t^{*}}\right)^{1-\epsilon}+1-\frac{\nu}{2}}\right)^{\frac{1}{1-\epsilon}}\right)^{\xi-1} C_{H,t}^{*} \\ +\ \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}}{\zeta_{C,t}}\right)\left(\frac{P_{H,t+1}/P_{t+1}}{P_{Ht}/P_t}\Pi_{t+1}\right)^{\xi-1}\Psi_{H,t+1}^{P}\right] \end{array}\right], \tag{44}$$

where we use $\dddot{W}_t \equiv \frac{\zeta_{N,t}-1}{\zeta_{N,t}} W_t$, $\dddot{W}_t^{*} \equiv \frac{\zeta_{N,t}^{*}-1}{\zeta_{N,t}^{*}} W_t^{*}$, $(1-\tau)\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N}{\zeta_N-1}\right) = 1$, and $(1-\tau^{*})\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N^{*}}{\zeta_N^{*}-1}\right) = 1$. Note that equations, $(1-\tau)\frac{\xi}{\xi-1}\frac{\zeta_N}{\zeta_N-1} = 1$ and $(1-\tau^{*})\frac{\xi}{\xi-1}\frac{\zeta_N^{*}}{\zeta_N^{*}-1} = 1$, hold in the steady state. Optimal subsidies, τ and τ^{*}, are time invariant while labor markup shocks, $\frac{1}{\zeta_{N,t}-1}$ and $\left(\frac{1}{\zeta_{N,t}^{*}-1}\right)$, can be time varying over the business cycle.

Foreign Monopolists' Pricing under Sticky Prices and Recursive Formulation: Likewise, optimal prices of Foreign LCP and PCP monopolists under price stickiness are given by

$$P_{Ft}^{L,o*} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}^{*}}{A_{t+j}^{*}}\left(P_{F,t+j}^{*}\right)^{\xi} C_{F,t+j}^{*}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\left(P_{F,t+j}^{*}\right)^{\xi} C_{F,t+j}^{*}\right]},$$

$$P_{Ft}^{L,o} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}^{*}}{A_{t+j}^{*}}\left(P_{F,t+j}\right)^{\xi} C_{F,t+j}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\frac{1}{\mathcal{E}_{t+j}}\left(P_{F,t+j}\right)^{\xi} C_{F,t+j}\right]},$$

$$P_{Ft}^{P,o*} = \frac{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\left(\frac{\xi(1-\tau)}{\xi-1}\right)\frac{W_{t+j}^{*}}{A_{t+j}^{*}}\left\{\left(P_{F,t+j}^{*}\right)^{\xi} C_{F,t+j}^{*} + \left(\frac{P_{F,t+j}}{\mathcal{E}_{t+j}}\right)^{\xi} C_{F,t+j}\right\}\right]}{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j \Upsilon_{t,t+j}^{*}\left\{\left(P_{F,t+j}^{*}\right)^{\xi} C_{F,t+j}^{*} + \left(\frac{P_{F,t+j}}{\mathcal{E}_{t+j}}\right)^{\xi} C_{F,t+j}\right\}\right]},$$

where the Foreign stochastic discount factor is defined as $\Upsilon_{t,t+j}^{*} \equiv \beta^j\left(\frac{C_{t+j}^{*}}{C_t^{*}}\right)^{-\sigma}\left(\frac{P_t^{*}}{P_{t+j}^{*}}\right)\left(\frac{\zeta_{C,t+j}^{*}}{\zeta_{C,t}^{*}}\right)$. Foreign LCP firms also set separate prices by country while Foreign PCP firms set a single price from the world demand. The coexistence of LCP and PCP firms in Foreign augments price dispersion in the sticky-price equilibrium which will be shown from the quadratic approximation to the loss function.

We use the standard derivation from the New-Keynesian literature and reformulate the above equations in a recursive form given by

$$\Pi_{Ft}^{L,o*} = \frac{\Theta_{Ft}^{L*}}{\Psi_{Ft}^{L*}}, \tag{45}$$

$$\Theta_{Ft}^{L*} = \frac{\xi(1-\tau^*)}{\xi-1}\left(\frac{W_t^*}{P_t^* A_t^*}\right) C_{F,t}^* + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}^*/P_{t+1}^*}{P_{Ft}^*/P_t^*}\Pi_{t+1}^*\right)^{\xi}\Theta_{F,t+1}^{L*}\right], \tag{46}$$

$$\Psi_{Ft}^{L*} = \left\{\tfrac{\nu}{2} + \left(1-\tfrac{\nu}{2}\right) S_t^{*1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} C_{F,t}^* + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}^*/P_{t+1}^*}{P_{Ft}^*/P_t^*}\Pi_{t+1}^*\right)^{\xi-1}\Psi_{F,t+1}^{L*}\right], \tag{47}$$

$$\Pi_{Ft}^{L,o} = \frac{\Theta_{Ft}^{L}}{\Psi_{Ft}^{L}}, \tag{48}$$

$$\Theta_{Ft}^{L} = \frac{\xi(1-\tau^*)}{\xi-1}\left(\frac{W_t^*}{P_t^* A_t^*}\right) C_{F,t} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}/P_{t+1}}{P_{Ft}/P_t}\Pi_{t+1}\right)^{\xi}\Theta_{F,t+1}^{L}\right], \tag{49}$$

$$\Psi_{Ft}^{L} = \frac{1}{Q_t}\left\{\tfrac{\nu}{2}\left(\tfrac{1}{S_t}\right)^{1-\epsilon} + 1 - \tfrac{\nu}{2}\right\}^{\frac{-1}{1-\epsilon}} C_{F,t} + \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}/P_{t+1}}{P_{Ft}/P_t}\Pi_{t+1}\right)^{\xi-1}\Psi_{F,t+1}^{L}\right], \tag{50}$$

$$\Pi_{Ft}^{P,o*} = \frac{\Theta_{Ft}^{P*}}{\Psi_{Ft}^{P*}}, \tag{51}$$

$$\Theta_{Ft}^{P*} = \left[\begin{array}{l} \frac{\xi(1-\tau^*)}{\xi-1}\left(\frac{W_t^*}{P_t^* A_t^*}\right)\left\{C_{F,t}^* + \left(Q_t\left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}{\frac{\nu}{2}+(1-\frac{\nu}{2})(S_t^*)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}}\right)^{-\xi} C_{F,t}\right\} \\ + \ \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}^*/P_{t+1}^*}{P_{Ft}^*/P_t^*}\Pi_{t+1}^*\right)^{\xi}\Theta_{F,t+1}^{P*}\right] \end{array}\right], \tag{52}$$

$$\Psi_{Ft}^{P*} = \left[\begin{array}{l} \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{*1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} C_{Ft}^* + \frac{1}{Q_t}\left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{-1}{1-\epsilon}}\left(Q_t\left(\frac{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}}{\frac{\nu}{2}+(1-\frac{\nu}{2})(S_t^*)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}}\right)^{1-\xi} C_{F,t} \\ + \ \mathbb{E}_t\left[(\theta\beta)\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{\zeta_{C,t+1}^*}{\zeta_{C,t}^*}\right)\left(\frac{P_{F,t+1}^*/P_{t+1}^*}{P_{Ft}^*/P_t^*}\Pi_{t+1}^*\right)^{\xi-1}\Psi_{F,t+1}^{P*}\right] \end{array}\right], \tag{53}$$

where we use $\ddot{W}_t \equiv \frac{\zeta_{N,t}-1}{\zeta_{N,t}} W_t$, $\ddot{W}_t^* \equiv \frac{\zeta_{N,t}^*-1}{\zeta_{N,t}^*} W_t^*$, $(1-\tau)\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N}{\zeta_N-1}\right) = 1$, and $(1-\tau^*)\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N^*}{\zeta_N^*-1}\right) = 1$. Optimal subsidies are time invariant, but labor markup shocks fluctuate over the business cycle.

Government Budget Balance: Subsidies, Home government's capital controls, and lump-sum taxes are "budget-neutral", in that government budgets are exactly balanced in Home and Foreign.

$$(H) \qquad T_t + \tau W_t\left(\int_0^{\chi} N_t^L(f)df + \int_{\chi}^{1} N_t^P(f)df\right) + D\left(\nabla^t\right)\varrho_t = \sum_{\nabla^{t+1}\in\Omega_{t+1}} Z\left(\nabla^{t+1}|\nabla^t\right) D\left(\nabla^{t+1}\right)\varrho_t,$$

$$(F) \qquad T_t^* + \tau_t^* W_t^*\left(\int_0^{\chi} N_t^{L*}(f^*)df^* + \int_{\chi}^{1} N_t^{P*}(f^*)df^*\right) = 0,$$

where subsidies satisfy $(1-\tau)\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N}{\zeta_N-1}\right) = 1$ and $(1-\tau^*)\left(\frac{\xi}{\xi-1}\right)\left(\frac{\zeta_N^*}{\zeta_N^*-1}\right) = 1$ in the steady state.

Financial Market Clearing Conditions and Resource Constraints: Domestic bond markets are cleared by $B_t = 0$ for Home and $B_t^* = 0$ for Foreign. The international financial market is cleared by $D(\nabla_t) + D^*(\nabla_t) = 0$ for all states $\nabla_t \in \Omega_t$ in each period. The resource constraint in each country is given by

$$\begin{aligned} &(H) \quad P_{Ft}C_{Ft} + \textstyle\sum_{\nabla_{t+1}\in\Omega_{t+1}} Z\left(\nabla_{t+1}|\nabla_t\right) D\left(\nabla_{t+1}\right) &&= \mathcal{E}_t P_{Ht}^* C_{Ht}^* + D\left(\nabla_t\right), \\ &(F) \quad \mathcal{E}_t P_{Ht}^* C_{Ht}^* + \textstyle\sum_{\nabla_{t+1}\in\Omega_{t+1}} Z\left(\nabla_{t+1}|\nabla_t\right) D^*\left(\nabla_{t+1}\right) &&= P_{Ft}C_{Ft} + D^*\left(\nabla_t\right). \end{aligned}$$

Net Exports: Finally, net exports in Home and Foreign are defined as

$$(H) \quad NX_t \equiv \mathcal{E}_t P_{Ht}^* C_{Ht}^* - P_{Ft} C_{Ft} \quad = \textstyle\sum_{\nabla_{t+1} \in \Omega_{t+1}} Z\left(\nabla_{t+1} | \nabla_t\right) D\left(\nabla_{t+1}\right) - D\left(\nabla_t\right),$$

$$(F) \quad NX_t^* \equiv \tfrac{1}{\mathcal{E}_t} P_{Ft} C_{Ft} - P_{Ht}^* C_{Ht}^* \quad = \tfrac{1}{\mathcal{E}_t} \textstyle\sum_{\nabla_{t+1} \in \Omega_{t+1}} Z\left(\nabla_{t+1} | \nabla_t\right) D^*\left(\nabla_{t+1}\right) - \tfrac{1}{\mathcal{E}_t} D^*\left(\nabla_t\right).$$

Note that net exports are related to net capital outflows.

B Nonlinear Optimal Monetary Policy under Incomplete Exchange Rate Pass-Through and Imperfect Risk Sharing ($0 \leq \chi \leq 1$ and $0 \leq \lambda \leq 1$)

The System of Equilibrium: There are 55 endogenous variables

$$\begin{array}{cccccccccccccc}
\frac{W_t}{P_t}, & \frac{W_t^*}{P_t^*}, & C_t, & C_t^*, & N_t, & N_t^*, & C_{Ht}, & C_{Ft}, & C_{Ft}^*, & C_{Ht}^*, & S_t, & S_t^*, & \Pi_t, & \Pi_t^*, \\
\frac{P_{Ht}}{P_t}, & \frac{P_{Ft}}{P_t}, & \frac{P_{Ft}^*}{P_t^*}, & \frac{P_{Ht}^*}{P_t^*}, & \Pi_{Ht}, & \Pi_{Ft}, & \Pi_{Ft}^*, & \Pi_{Ht}^*, & \Pi_{Ht}^L, & \Pi_{Ht}^{L*}, & \Pi_{Ht}^P, & \Pi_{Ft}^{L*}, & \Pi_{Ft}^L, & \Pi_{Ft}^{P*}, \\
V_{Ht}^L, & V_{Ht}^{L*}, & V_{Ht}^P, & V_{Ht}^{P*}, & V_{Ft}^{L*}, & V_{Ft}^L, & V_{Ft}^{P*}, & V_{Ft}^P, & \Pi_{Ht}^{L,o}, & \Pi_{Ht}^{L,o*}, & \Pi_{Ht}^{P,o}, & \Pi_{Ft}^{L,o*}, & \Pi_{Ft}^{L,o}, & \Pi_{Ft}^{P,o*}, \\
\Theta_{Ht}^L, & \Psi_{Ht}^L, & \Theta_{Ht}^{L*}, & \Psi_{Ht}^{L*}, & \Theta_{Ht}^P, & \Psi_{Ht}^P, & \Theta_{Ft}^{L*}, & \Psi_{Ft}^{L*}, & \Theta_{Ft}^L, & \Psi_{Ft}^L, & \Theta_{Ft}^{P*}, & \Psi_{Ft}^{P*}, & Q_t, &
\end{array}$$

given 53 equilibrium conditions as shown in section A. By solving the Ramsey problem, we can find two additional equations and close the equilibrium under optimal monetary policy. The global planner maximizes the discounted sum of periodic world welfare which assigns the same weight to Home and Foreign utilities given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \zeta_{C,t} \frac{C_t^{1-\sigma}}{1-\sigma} + \zeta_{C,t}^* \frac{C_t^{*\,1-\sigma}}{1-\sigma} - \kappa \frac{N_t^{1+\phi}}{1+\phi} - \kappa \frac{N_t^{*\,1+\phi}}{1+\phi} \right\}$$

subject to 53 equilibrium conditions presented in section A. Then we obtain 55 first order necessary conditions. Combining 55 first order necessary conditions and 53 equilibrium equations, we set up a system of equilibrium under optimal monetary policy for 55 endogenous variables and 53 Lagrange multipliers. All the details of nonlinear characterization for optimal monetary policy are available upon request. We use Dynare software to implement the second-order approximation to solve for 108 endogenous variables and evaluate welfare costs. We conduct the first-order approximation to find impulse responses with respect to shocks to productivity, labor markup, and preference.

C Natural Allocations

As a benchmark, we characterize natural allocations which are the equilibrium allocations when all firms can adjust their prices in a flexible way. In a symmetric equilibrium, all PCP and LCP firms in each country set the common price for their products. Output depends only on real structural factors and stays at its natural level. Since international financial integration is not perfect under capital controls in our model, natural allocations are not efficient in general. The equilibrium under flexible prices is closed by ten equations (54)–(63) below. We solve for N_t, N_t^*, S_t, S_t^*, C_t, C_t^*, $\frac{\ddot{W}_t}{P_t}$, $\frac{\ddot{W}_t^*}{P_t^*}$, F_t, and Q_t.

Market Clearing Conditions for Home and Foreign Tradeables: Together market demands (Y_t, Y_t^*), (24), and (25) imply

$$A_t N_t = \frac{\nu}{2}\left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t + \left(1-\frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t^*, \tag{54}$$

$$A_t^* N_t^* = \frac{\nu}{2}\left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{*1-\epsilon}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t^* + \left(1-\frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}} C_t, \tag{55}$$

where all price dispersion terms are equal to unity under flexible prices.

Home and Foreign Monopolists' Pricing under Flexible Prices: In section A, we have shown that a firm under flexible prices equates its price to the marginal cost of labor multiplied by monopoly markup, government subsidy, and the reciprocal of productivity. Optimal prices in Home and Foreign can be written as

$$1 = \exp\left(\frac{1}{\delta}u_t\right)\frac{\ddot{W}_t}{P_{Ht}}\frac{1}{A_t} = \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}} \exp\left(\frac{1}{\delta}u_t\right)\frac{\ddot{W}_t}{P_t}\frac{1}{A_t}, \tag{56}$$

$$1 = \exp\left(\frac{1}{\delta}u_t^*\right)\frac{\ddot{W}_t^*}{P_{Ft}^*}\frac{1}{A_t^*} = \left\{\frac{\nu}{2} + \left(1-\frac{\nu}{2}\right) S_t^{*1-\epsilon}\right\}^{\frac{1}{1-\epsilon}} \exp\left(\frac{1}{\delta}u_t^*\right)\frac{\ddot{W}_t^*}{P_t^*}\frac{1}{A_t^*}, \tag{57}$$

where we define cost-push shocks from time-varying labor markups as

$$u_t \equiv \delta\left[\log\left(\frac{\zeta_{N,t}}{\zeta_{N,t}-1}\right) - \log\left(\frac{\zeta_N}{\zeta_N - 1}\right)\right] \quad \text{and} \quad u_t^* \equiv \delta\left[\log\left(\frac{\zeta_{N,t}^*}{\zeta_{N,t}^*-1}\right) - \log\left(\frac{\zeta_N^*}{\zeta_N^* - 1}\right)\right].$$

In (56) and (57), we use markup-free nominal wages, $\ddot{W}_t \equiv \frac{\zeta_{N,t}-1}{\zeta_{N,t}} W_t$ and $\ddot{W}_t^* \equiv \frac{\zeta_{N,t}^*-1}{\zeta_{N,t}^*} W_t^*$, with the steady-state relationship between monopoly markup and government subsidy, $(1-\tau)\frac{\xi}{\xi-1}\frac{\zeta_N}{\zeta_N-1} = 1$ and $(1-\tau^*)\frac{\xi}{\xi-1}\frac{\zeta_N^*}{\zeta_N^*-1} = 1$. Also we define $\delta \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$ as in Woodford (2003).

Wage Setting of Households in Home and Foreign: The optimal wage is equal to the marginal utility cost of labor multiplied by the time-varying markup, implying

$$\frac{\ddot{W}_t}{P_t} = C_t{}^{\sigma} N_t{}^{\phi}\frac{\kappa}{\zeta_{C,t}}, \tag{58}$$

$$\frac{\ddot{W}_t^*}{P_t^*} = C_t^{*\sigma} N_t^{*\phi}\frac{\kappa}{\zeta_{C,t}^*}, \tag{59}$$

Financial Market Condition, Demand Imbalance, and International Prices: Financial market condition and the definition for demand imbalance do not depend on price stickiness. They are given by

$$\left(\frac{C_t^*}{C_t}\right)^{-\sigma} = \left(\frac{\zeta_{C,t}}{\zeta_{C,t}^*}\right) Q_t F_t, \tag{60}$$

$$F_t = \left(\frac{\left(\frac{\nu}{2}\right)\left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{-1} C_t + Q_t\left(1-\frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{-1} C_t^*}{\left(\frac{\nu}{2}\right)\left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{-1} C_t + \left(1-\frac{\nu}{2}\right)\left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{-1} C_t}\right)^{\frac{\lambda}{(1-\lambda)}}. \tag{61}$$

Here we substituted for price and consumption indexes in (3). Real exchange rate and the relative price of Home and Foreign goods are given by

$$Q_t = \frac{\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon}+1-\frac{\nu}{2}\right\}^{\frac{1}{1-\epsilon}}}{\left\{\frac{\nu}{2}+\left(1-\frac{\nu}{2}\right)S_t^{1-\epsilon}\right\}^{\frac{1}{1-\epsilon}}}, \tag{62}$$

$$S_t^* \equiv \frac{P_{Ht}^*}{P_{Ft}^*} = \frac{1}{S_t}, \tag{63}$$

where currency misalignment and export premium are equal to unity under flexible prices: $M_t = Z_t = 1$.

D Globally Efficient Allocations

Natural allocations in section C achieve efficient outcome when there is no cost-push shock and risk sharing is perfect. Perfect risk sharing ($\lambda = 0$) implies $F_t = 1$ in (61). Given $\lambda = 0$, $F_t = 1$, and $u_t = u_t^* = 0$, equations (54)–(63) characterize globally efficient allocations.

E The Globally Efficient Steady-State

The economy at the initial period $t = 0$ is assumed to reside in the efficient steady state with zero net bond supply. In what follows, we keep the subscript t of each variable for convenience. However, in this section the subscript t indicates the initial time period ($t = 0$) or the long run ($t \to \infty$). In the steady state, optimal subsidies correct distortions from monopolies in labor and goods markets: $1 = (1-\tau)\frac{\xi}{\xi-1}\frac{\zeta_N}{\zeta_N-1} = (1-\tau^*)\frac{\xi}{\xi-1}\frac{\zeta_N^*}{\zeta_N^*-1}$. The price setting is flexible and risk sharing is perfect. National income accounting under zero net bond supply at $t = 0$ implies

$$P_{Ft}C_{Ft} = \mathcal{E}_t P_{Ht}^* C_{Ht}^* = P_{Ht}C_{Ht}^*,$$

where the second equality follows from the law of one price. Combining the above equation with (54) and (55), we derive the following relationship

$$\begin{aligned} A_t N_t &= C_{Ht} + C_{Ht}^* = C_{Ht} + \tfrac{P_{Ft}}{P_{Ht}} C_{Ft} = \tfrac{P_t C_t}{P_{Ht}}, \\ A_t^* N_t^* &= C_{Ft}^* + C_{Ft} = C_{Ft}^* + \tfrac{P_{Ht}}{P_{Ft}} C_{Ht}^* = \tfrac{\mathcal{E}_t P_{Ft} C_{Ft}^* + \mathcal{E}_t P_{Ht} C_{Ht}^*}{\mathcal{E}_t P_{Ft}} = \tfrac{P_t^* C_t^*}{P_{Ft}^*}, \end{aligned}$$

where we use the law of one price for the last equality. Recall that there is no cosh-push shock in the efficient equilibrium: $\zeta_{N,t} = \zeta_N$ and $\zeta_{N,t}^* = \zeta_N^*$. Together two equations above and four conditions (56) – (59) imply

$$\begin{aligned} &\tfrac{C_t}{N_t} = A_t \tfrac{P_{Ht}}{P_t} = \tfrac{\ddot{W}_t}{P_t} = C_t^{\sigma} N_t^{\phi} \kappa \quad \text{and} \quad \tfrac{C_t^*}{N_t^*} = A_t^* \tfrac{P_{Ft}^*}{P_t^*} = \tfrac{\ddot{W}_t^*}{P_t^*} = C_t^{*\sigma} N_t^{*\phi} \kappa, \\ \text{that is,} \quad &C_t^{1-\sigma} = \kappa N_t^{1+\phi} \qquad\qquad\qquad\quad \text{and} \quad C_t^{*1-\sigma} = \kappa N_t^{*1+\phi}. \end{aligned}$$

Since we focus on the symmetric steady-state equilibrium, $A_t = A_t^*$, $\zeta_{C,t} = \zeta_{C,t}^*$, and $C_t = C_t^*$ are imposed. Then we obtain

$$N_t = N_t^*, \quad \tfrac{P_{Ht}}{P_t} = \tfrac{P_{Ft}^*}{P_t^*}, \quad C_{Ht} = C_{Ft}^*, \quad C_{Ft} = C_{Ht}^*, \quad P_t = P_{Ht} = P_{Ft}, \quad P_t^* = P_{Ft}^* = P_{Ht}^*, \text{ and } \quad \tfrac{\mathcal{E}_t P_t^*}{P_t} = 1.$$

Therefore, the symmetric steady-state equilibrium with zero net bond supply implies purchasing power parity ($Q_t = 1$) even though there is consumption home bias.

Finally, we assume $\zeta_{C,t} = \zeta_{C,t}^* = A_t = A_t^* = 1$ in the steady state. Then the efficient allocations in the long run are

$$
\begin{aligned}
C &= N = C^* = N^* = \kappa^{\frac{-1}{\sigma+\phi}} && \text{and} & \Pi &= \Pi_H = \Pi_F = \Pi^* = \Pi_H^* = \Pi_F^* = 1, \\
Y &= C_H + C_H^* = \kappa^{\frac{-1}{\sigma+\phi}} && \text{and} & Y^* &= C_F^* + C_F = \kappa^{\frac{-1}{\sigma+\phi}}, \\
C_H &= C_F^* = \left(\tfrac{\nu}{2}\right)\kappa^{\frac{-1}{\sigma+\phi}} && \text{and} & C_F &= C_H^* = \left(1-\tfrac{\nu}{2}\right)\kappa^{\frac{-1}{\sigma+\phi}}, \\
\tfrac{\zeta_N-1}{\zeta_N}\tfrac{W}{P} &= \tfrac{\zeta_N-1}{\zeta_N}\tfrac{\xi-1}{\xi}\tfrac{1}{1-\tau} = 1 && \text{and} & \tfrac{\zeta_N^*-1}{\zeta_N^*}\tfrac{W^*}{P^*} &= \tfrac{\zeta_N^*-1}{\zeta_N^*}\tfrac{\xi-1}{\xi}\tfrac{1}{1-\tau^*} = 1, \\
S &= \tfrac{P_F}{P_H} = 1 && \text{and} & S^* &= \tfrac{P_H^*}{P_F^*} = 1, \\
Q &= \tfrac{\mathcal{E}P^*}{P} = 1 && \text{and} & \mathcal{T} &= \mathcal{T}^* = M = Z = 1.
\end{aligned}
$$

Real price indexes, inflation rates, and price dispersion terms are all unity:

$$
\begin{aligned}
1 &= \tfrac{P_H}{P} = \tfrac{P_F}{P} = \tfrac{P_F^*}{P^*} = \tfrac{P_H^*}{P^*} = \Pi_H = \Pi_F = \Pi_F^* = \Pi_H^* = \Pi_H^L = \Pi_H^{L*} = \Pi_H^P = \Pi_F^{L*} = \Pi_F^L = \Pi_F^{P*}, \\
1 &= V_H^L = V_H^{L*} = V_H^P = V_H^{P*} = V_F^{L*} = V_F^L = V_F^{P*} = V_F^P = \Pi_H^{L,o} = \Pi_H^{L,o*} = \Pi_H^{P,o} = \Pi_F^{L,o*} = \Pi_F^{L,o} = \Pi_F^{P,o*},
\end{aligned}
$$

thereby we can assign unity to all nominal price indexes in the steady state. Recursive pricing terms are given by

$$
\begin{aligned}
\left(\tfrac{\frac{\nu}{2}}{1-\theta\beta}\right)\kappa^{\frac{-1}{\sigma+\phi}} &= \Theta_H^L = \Psi_H^L = \Theta_F^{L*} = \Psi_F^{L*}, \\
\left(\tfrac{1-\frac{\nu}{2}}{1-\theta\beta}\right)\kappa^{\frac{-1}{\sigma+\phi}} &= \Theta_F^L = \Psi_F^L = \Theta_H^{L*} = \Psi_H^{L*}, \\
\left(\tfrac{1}{1-\theta\beta}\right)\kappa^{\frac{-1}{\sigma+\phi}} &= \Theta_H^P = \Psi_H^P = \Theta_F^{P*} = \Psi_F^{P*}.
\end{aligned}
$$

F Log-Linearized Model

F.1 Log-Linearization of Equilibrium Conditions

All small-letter variables stand for the log deviation from the globally efficient steady-state: $x_t = \log\left(\frac{X_t}{X}\right)$. We define shocks to preference in log as $\zeta_{c,t} \equiv \log\left(\zeta_{C,t}\right)$ and $\zeta^*_{c,t} \equiv \log\left(\zeta^*_{C,t}\right)$. Also cost-push shocks from time-varying labor markups are defined as

$$u_t \equiv \delta\left[\log\left(\frac{\zeta_{N,t}}{\zeta_{N,t}-1}\right) - \log\left(\frac{\zeta_N}{\zeta_N-1}\right)\right] \quad \text{and} \quad u^*_t \equiv \delta\left[\log\left(\frac{\zeta^*_{N,t}}{\zeta^*_{N,t}-1}\right) - \log\left(\frac{\zeta^*_N}{\zeta^*_N-1}\right)\right],$$

where $\delta \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Following a standard practice in international economics, we define relative and world values for any variables, x_t and x^*_t, as $x^R_t \equiv \frac{x_t - x^*_t}{2}$ and $x^W_t = \frac{x_t + x^*_t}{2}$. $\overline{x}_t$ stands for the first-order approximated allocation of the efficient equilibrium. Any variable denoted with a tilde represents the deviation from its first-best counterpart: $\widetilde{x}_t \equiv x_t - \overline{x}_t$. For later use, we define coefficients $D \equiv (\nu-1)^2 + \sigma\epsilon\nu(2-\nu)$ and $\delta \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

International Prices, CES Demands, CPI, and Interest Rates: Terms of trade (η_t, η^*_t), the relative price of imports (s_t, s^*_t), the measure of the deviation from the law of one price (m_t, m_{Ht}, m_{Ft}), and export premium (z_t) are

$$\eta_t \equiv \log\left(\frac{\mathcal{T}_t}{\mathcal{T}}\right) = s_t - m_t - z_t \quad \text{and} \quad \eta^*_t \equiv \log\left(\frac{\mathcal{T}^*_t}{\mathcal{T}^*}\right) = s^*_t + m_t - z_t,$$

$$s_t = p_{Ft} - p_{Ht} \quad \text{and} \quad s^*_t = p^*_{Ht} - p^*_{Ft},$$

$$m_t = \frac{1}{2}\left(e_t + p^*_{Ht} - p_{Ht} + e_t + p^*_{Ft} - p_{Ft}\right) \quad \text{and} \quad z_t = \frac{1}{2}\left(s_t + s^*_t\right),$$

$$m_{Ht} \equiv m_t + z_t = e_t + p^*_{Ht} - p_{Ht} \quad \text{and} \quad m_{Ft} \equiv m_t - z_t = e_t + p^*_{Ft} - p_{Ft}.$$

Log-linearized price and consumption indexes in Home and Foreign are given by

$$p_t - p_{Ht} = \left(1 - \frac{\nu}{2}\right) s_t \quad \text{and} \quad p_t - p_{Ft} = \left(-\frac{\nu}{2}\right) s_t,$$

$$p^*_t - p^*_{Ft} = \left(1 - \frac{\nu}{2}\right) s^*_t \quad \text{and} \quad p^*_t - p^*_{Ht} = \left(-\frac{\nu}{2}\right) s^*_t,$$

$$c_{Ht} = c_t + \epsilon\left(p_t - p_{Ht}\right) = c_t + \epsilon\left(1 - \frac{\nu}{2}\right)(s_t) \quad \text{and} \quad c_{Ft} = c_t + \epsilon\left(p_t - p_{Ft}\right) = c_t + \epsilon\left(\frac{\nu}{2}\right)(-s_t),$$

$$c^*_{Ft} = c^*_t + \epsilon\left(p^*_t - p^*_{Ft}\right) = c^*_t + \epsilon\left(1 - \frac{\nu}{2}\right)(s^*_t) \quad \text{and} \quad c^*_{Ht} = c^*_t + \epsilon\left(p^*_t - p^*_{Ht}\right) = c^*_t + \epsilon\left(\frac{\nu}{2}\right)(-s^*_t).$$

Note that it is convenient to express prices and outputs in terms of relative prices of Home and Foreign goods, s_t and s^*_t, for the subsequent analysis. Real exchange rate, consumer price indexes, and nominal interest rates are represented by

$$q_t = e_t + p^*_t - p_t = m_t + (\nu-1)\frac{s_t - s^*_t}{2},$$

$$\pi_t = \frac{\nu}{2}\pi_{Ht} + \left(1 - \frac{\nu}{2}\right)\pi_{Ft} \quad \text{and} \quad \pi^*_t = \frac{\nu}{2}\pi^*_{Ft} + \left(1 - \frac{\nu}{2}\right)\pi^*_{Ht},$$

$$i_t = \sigma\left(\mathbb{E}_t c_{t+1} - c_t\right) + \mathbb{E}_t\pi_{t+1} - \mathbb{E}_t\zeta_{c,t+1} + \zeta_{c,t} \quad \text{and} \quad i^*_t = \sigma\left(\mathbb{E}_t c^*_{t+1} - c^*_t\right) + \mathbb{E}_t\pi^*_{t+1} - \mathbb{E}_t\zeta^*_{c,t+1} + \zeta^*_{c,t}.$$

where i_t denotes the Home-currency Home nominal interest rate and i^*_t denotes the Foreign-currency Foreign nominal interest rate.

Home and Foreign Tradeable Goods: Total output in each country can be approximated up to the first order as

$$y_t = \left(\frac{\nu}{2}\right) c_{Ht} + \left(1 - \frac{\nu}{2}\right) c^*_{Ht} = \frac{\nu}{2} c_t + \left(1 - \frac{\nu}{2}\right) c^*_t + \epsilon \frac{\nu}{2} \left(1 - \frac{\nu}{2}\right) (s_t - s^*_t), \tag{64}$$

$$y^*_t = \left(\frac{\nu}{2}\right) c^*_{Ft} + \left(1 - \frac{\nu}{2}\right) c_{Ft} = \frac{\nu}{2} c^*_t + \left(1 - \frac{\nu}{2}\right) c_t - \epsilon \frac{\nu}{2} \left(1 - \frac{\nu}{2}\right) (s_t - s^*_t). \tag{65}$$

Home and Foreign Monopolists' Pricing under Flexible Prices: Under flexible prices, optimal pricing of firms can be written as

$$p_{Ht} = w_t - a_t = e_t + p^*_{Ht}$$

$$p^*_{Ft} = w^*_t - a^*_t = -e_t + p_{Ft}$$

Wage Setting of Households in Home and Foreign: Log-linearized wages in (1) and (2) are

$$w_t - p_{Ht} = \sigma c_t + \phi n_t + \left(1 - \frac{\nu}{2}\right) s_t + \frac{1}{\delta} u_t - \zeta_{c,t}, \tag{66}$$

$$w^*_t - p^*_{Ft} = \sigma c^*_t + \phi n^*_t + \left(1 - \frac{\nu}{2}\right) s^*_t + \frac{1}{\delta} u^*_t - \zeta^*_{c,t}. \tag{67}$$

Using (72), (74), and (84), we can rewrite these two wage equations as

$$w_t - p_{Ht} = \left(\frac{\sigma}{D} + \phi\right) y^R_t + (\sigma + \phi) y^W_t + \frac{D - \nu + 1}{2D} (m_t + f_t + 2\zeta^R_{c,t}) + \left(1 - \frac{\nu}{2}\right) z_t - \phi a_t + \frac{1}{\delta} u_t - \zeta_{c,t}, \tag{68}$$

$$w^*_t - p^*_{Ft} = \left(-\frac{\sigma}{D} - \phi\right) y^R_t + (\sigma + \phi) y^W_t + \frac{-D + \nu - 1}{2D} (m_t + f_t + 2\zeta^R_{c,t}) + \left(1 - \frac{\nu}{2}\right) z_t - \phi a^*_t + \frac{1}{\delta} u^*_t - \zeta^*_{c,t}. \tag{69}$$

Financial Market Condition and Demand Imbalance: Log-linearizing (3), we obtain

$$\sigma (c_t - c^*_t) = q_t + f_t + \zeta_{c,t} - \zeta^*_{c,t}, \tag{70}$$

$$f_t = \left(\frac{\lambda}{(1 - \lambda)}\right) \left(1 - \frac{\nu}{2}\right) \left[q_t - (c_t - c^*_t) + \left(\frac{\nu}{2}\right) (\epsilon - 1) (s_t - s^*_t)\right]. \tag{71}$$

Together (64), (65), (70), and $q_t = m_t + (\nu - 1) s^R_t$ imply

$$s^R_t = \frac{2\sigma}{D} y^R_t - \frac{\nu - 1}{D} (m_t + f_t + \zeta_{c,t} - \zeta^*_{c,t}), \tag{72}$$

while $s^W_t = z_t$ holds by definition. Here we use $D \equiv (\nu - 1)^2 + \sigma \epsilon \nu (2 - \nu)$. s^R_t and q_t can be expressed in terms of deviation from the efficient outcome as

$$\widetilde{s}^R_t = \tfrac{2\sigma}{D} \widetilde{y}^R_t - \tfrac{\nu - 1}{D} (m_t + f_t) \quad \text{and} \quad \widetilde{q}_t = m_t + (\nu - 1) \widetilde{s}^R_t. \tag{73}$$

Combining (64), (65), and (70), we obtain

$$
\begin{aligned}
c_t^R &= \left(\tfrac{1}{\nu-1}\right) y_t^R - \left(\tfrac{1}{\nu-1}\right) \tfrac{\epsilon\nu(2-\nu)}{2} s_t^R = \tfrac{\nu-1}{D} y_t^R + \tfrac{\epsilon\nu(2-\nu)}{2D}(m_t + f_t + 2\zeta_{c,t}^R), \\
c_t^W &= y_t^W, \\
c_t - y_t &= -\left(c_t^* - y_t^*\right) = \tfrac{-D+\nu-1}{D} y_t^R + \tfrac{\epsilon\nu(2-\nu)}{2D}(m_t + f_t + 2\zeta_{c,t}^R), \\
\tfrac{\sigma}{2}\left(c_t - c_t^*\right) &= \tfrac{\nu-1}{2} s_t^R + \tfrac{1}{2}\left(m_t + f_t + 2\zeta_{c,t}^R\right).
\end{aligned}
\tag{74}
$$

Together (71), (72), and (74) imply

$$
f_t = \tfrac{1}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \left(
\begin{array}{cl}
+ & 2(\lambda)(2-\nu)\left[\sigma(\epsilon\nu-1)+1-\nu\right] y_t^R \\
- & (\lambda)(2-\nu)\left(\epsilon\nu-\nu+1-D\right) m_t \\
- & (\lambda)(2-\nu)\left(\epsilon\nu-\nu+1\right) 2\zeta_{c,t}^R
\end{array}
\right).
\tag{75}
$$

Note that $\lambda = 0$ leads to perfect risk sharing, $f_t = 0$. For $\lambda \in (0, 1]$, f_t can be written out as

$$
f_t = \left(
\begin{array}{l}
+\widetilde{y}_t^R \tfrac{2(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(2-\nu)(\epsilon\nu-\nu+1)+2(\frac{1}{\lambda}-1)D} - m_t \tfrac{(2-\nu)(\epsilon\nu-\nu+1-D)}{(2-\nu)(\epsilon\nu-\nu+1)+2(\frac{1}{\lambda}-1)D} \\
+a_t^R \left(\tfrac{1+\phi}{\frac{\sigma}{D}+\phi}\right) \tfrac{2(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(2-\nu)(\epsilon\nu-\nu+1)+2(\frac{1}{\lambda}-1)D} \\
+2\zeta_{c,t}^R \left(\tfrac{(2-\nu)}{(2-\nu)(\epsilon\nu-\nu+1)+2(\frac{1}{\lambda}-1)D}\right) \left[\left(\tfrac{(\nu-1)[\sigma(\epsilon\nu-1)+1-\nu]}{\sigma+\phi D}\right) - \epsilon\nu + \nu - 1\right]
\end{array}
\right),
\tag{76}
$$

where we use $\overline{y}_t^R = \frac{1+\phi}{\frac{\sigma}{D}+\phi} a_t^R + \frac{\nu-1}{\sigma+\phi D} \zeta_{c,t}^R$ which will be derived later.

Price Dispersion Terms and PPI: Price dispersion measures (16)–(23) can be approximated as

$$
\begin{aligned}
v_{Ht}^L &= \theta v_{H,t-1}^L + \xi\left(\theta\pi_{Ht} - (1-\theta)\pi_{Ht}^{L,o}\right), & v_{Ht}^{L*} &= \theta v_{H,t-1}^{L*} + \xi\left(\theta\pi_{Ht}^* - (1-\theta)\pi_{Ht}^{L,o*}\right), \\
v_{Ht}^P &= \theta v_{H,t-1}^P + \xi\left(\theta\pi_{Ht} - (1-\theta)\pi_{Ht}^{P,o}\right), & v_{Ht}^{P*} &= \xi\left(e_t + p_{Ht}^* - p_{Ht}\right) + v_{Ht}^P, \\
v_{Ft}^{L*} &= \theta v_{F,t-1}^{L*} + \xi\left(\theta\pi_{Ft}^* - (1-\theta)\pi_{Ft}^{L,o*}\right), & v_{Ft}^L &= \theta v_{F,t-1}^L + \xi\left(\theta\pi_{Ft} - (1-\theta)\pi_{Ft}^{L,o}\right), \\
v_{Ft}^{P*} &= \theta v_{F,t-1}^{P*} + \xi\left(\theta\pi_{Ft}^* - (1-\theta)\pi_{Ft}^{P,o*}\right), & v_{Ft}^P &= \xi\left(p_{Ft} - e_t - p_{Ft}^*\right) + v_{Ft}^{P*}.
\end{aligned}
\tag{77}
$$

PPI equations (26), (27), (31), and (32) can be expressed in the log-linear form as

$$
\begin{aligned}
0 &= \chi\pi_{Ht}^L + (1-\chi)\pi_{Ht}^P, & 0 &= \chi\pi_{Ht}^{L*} + (1-\chi)\pi_{Ht}^P - (1-\chi)\left(e_t + p_{Ht}^* - p_{Ht}\right), \\
0 &= \chi\pi_{Ft}^{L*} + (1-\chi)\pi_{Ft}^{P*}, & 0 &= \chi\pi_{Ft}^L + (1-\chi)\pi_{Ft}^{P*} + (1-\chi)\left(e_t + p_{Ft}^* - p_{Ft}\right).
\end{aligned}
\tag{78}
$$

PPI recursive formula (28)–(30) and (33)–(35) reduce to

$$
\begin{aligned}
\pi_{Ht}^L &= \theta\left(\pi_{H,t-1}^L - \pi_{Ht}\right) + (1-\theta)\pi_{Ht}^{L,o}, & \pi_{Ht}^{L*} &= \theta\left(\pi_{H,t-1}^{L*} - \pi_{Ht}^*\right) + (1-\theta)\pi_{Ht}^{L,o*}, \\
\pi_{Ht}^P &= \theta\left(\pi_{H,t-1}^P - \pi_{Ht}\right) + (1-\theta)\pi_{Ht}^{P,o}, & & \\
\pi_{Ft}^{L*} &= \theta\left(\pi_{F,t-1}^{L*} - \pi_{Ft}^*\right) + (1-\theta)\pi_{Ft}^{L,o*}, & \pi_{Ft}^L &= \theta\left(\pi_{F,t-1}^L - \pi_{Ft}\right) + (1-\theta)\pi_{Ft}^{L,o}, \\
\pi_{Ft}^{P*} &= \theta\left(\pi_{F,t-1}^{P*} - \pi_{Ft}^*\right) + (1-\theta)\pi_{Ft}^{P,o*}. & &
\end{aligned}
\tag{79}
$$

Together (77), (78), (79), and the requirement that the economy resides in the efficient steady state at $t = 0$ lead to

$$\begin{aligned}
\chi v_{Ht}^{L} + (1-\chi) v_{Ht}^{P} &= \theta \left(\chi v_{H,t-1}^{L} + (1-\chi) v_{H,t-1}^{P} \right) = 0, \\
\chi v_{Ht}^{L*} + (1-\chi) v_{Ht}^{P*} &= \theta \left(\chi v_{H,t-1}^{L*} + (1-\chi) v_{H,t-1}^{P*} \right) = 0, \\
\chi v_{Ft}^{L*} + (1-\chi) v_{Ft}^{P*} &= \theta \left(\chi v_{F,t-1}^{L*} + (1-\chi) v_{F,t-1}^{P*} \right) = 0, \\
\chi v_{Ft}^{L} + (1-\chi) v_{Ft}^{P} &= \theta \left(\chi v_{F,t-1}^{L} + (1-\chi) v_{F,t-1}^{P} \right) = 0.
\end{aligned} \tag{80}$$

Therefore, the sum of LCP and PCP price dispersion terms weighted by χ and $(1-\chi)$ respectively amounts to zero. For the derivation of Phillips curves, we combine (78) and (79) to obtain

$$\begin{aligned}
&(1-\theta) \left[\chi \pi_{Ht}^{L,o} + (1-\chi) \pi_{Ht}^{P,o} \right] = \theta \pi_{Ht}, \\
&(1-\theta) \left[\chi \pi_{Ft}^{L,o*} + (1-\chi) \pi_{Ft}^{P,o*} \right] = \theta \pi_{Ft}^{*}, \\
&(1-\theta) \left[\chi \pi_{Ht}^{L,o*} + (1-\chi) \pi_{Ht}^{P,o} \right] = \theta \left(\chi \pi_{Ht}^{*} + (1-\chi) \pi_{Ht} \right) - \theta (1-\chi) \left(m_{t-1} + z_{t-1} \right) + (1-\chi) \left(m_t + z_t \right), \\
&(1-\theta) \left[\chi \pi_{Ft}^{L,o} + (1-\chi) \pi_{Ft}^{P,o*} \right] = \theta \left(\chi \pi_{Ft} + (1-\chi) \pi_{Ft}^{*} \right) - \theta (1-\chi) \left(-m_{t-1} + z_{t-1} \right) + (1-\chi) \left(-m_t + z_t \right),
\end{aligned} \tag{81}$$

where we use $m_t + z_t = e_t + p_{Ht}^{*} - p_{Ht}$ and $-m_t + z_t = p_{Ft} - e_t - p_{Ft}^{*}$ for the third and fourth equalities.

Market Clearing Conditions for Home and Foreign Tradeables: Since price dispersion terms are all zero up to the first order, (24) and (25) imply

$$a_t + n_t = y_t, \tag{82}$$

$$a_t^{*} + n_t^{*} = y_t^{*}, \tag{83}$$

which can be rewritten in terms of cross-country terms as

$$n_t^R = y_t^R - a_t^R \quad \text{and} \quad n_t^W = y_t^W - a_t^W. \tag{84}$$

Home Monopolists' Pricing under Sticky Prices: Recall the Home stochastic discount factor, $\Upsilon_{t,t+j} \equiv \beta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+j}} \right) \left(\frac{\zeta_{C,t+j}}{\zeta_{C,t}} \right)$. In log-linearization, common terms in numerator and denominator in the equation for optimal pricing are cancelled out. Optimal prices of Home LCP and PCP monopolists can be written out up to the first order as

$$\frac{p_{Ht}^{L,o}}{1-\beta\theta} = w_t - a_t + (\beta\theta) \mathbb{E}_t \left[\frac{p_{H,t+1}^{L,o}}{1-\beta\theta} \right], \quad \frac{p_{Ht}^{L,o*}}{1-\beta\theta} = w_t - a_t - e_t + (\beta\theta) \mathbb{E}_t \left[\frac{p_{H,t+1}^{L,o*}}{1-\beta\theta} \right], \quad \frac{p_{Ht}^{P,o}}{1-\beta\theta} = w_t - a_t + (\beta\theta) \mathbb{E}_t \left[\frac{p_{H,t+1}^{P,o}}{1-\beta\theta} \right],$$

which can be rewritten out as

$$\begin{aligned}
\pi_{Ht}^{L,o} &= (1-\beta\theta) \left(w_t - a_t - p_{Ht} \right) + (\beta\theta) \mathbb{E}_t \left[\pi_{H,t+1}^{L,o} + \pi_{H,t+1} \right], \\
\pi_{Ht}^{L,o*} &= (1-\beta\theta) \left(w_t - a_t - e_t - p_{Ht}^{*} \right) + (\beta\theta) \mathbb{E}_t \left[\pi_{H,t+1}^{L,o*} + \pi_{H,t+1}^{*} \right], \\
\pi_{Ht}^{P,o} &= (1-\beta\theta) \left(w_t - a_t - p_{Ht} \right) + (\beta\theta) \mathbb{E}_t \left[\pi_{H,t+1}^{P,o} + \pi_{H,t+1} \right].
\end{aligned} \tag{85}$$

Foreign Monopolists' Pricing under Sticky Prices: Recall the Foreign stochastic discount factor, $\Upsilon_{t,t+j}^{*} \equiv \beta^j \left(\frac{C_{t+j}^{*}}{C_t^{*}} \right)^{-\sigma} \left(\frac{P_t^{*}}{P_{t+j}^{*}} \right) \left(\frac{\zeta_{C,t+j}^{*}}{\zeta_{C,t}^{*}} \right)$. Analogously, optimal prices of Foreign LCP and PCP monopolists can be written out up to the first order as

$$\frac{p_{Ft}^{L,o*}}{1-\beta\theta}=w_t^*-a_t^*+(\beta\theta)\,\mathbb{E}_t\left[\frac{p_{F,t+1}^{L,o*}}{1-\beta\theta}\right],\quad \frac{p_{Ft}^{L,o}}{1-\beta\theta}=w_t^*-a_t^*+e_t+(\beta\theta)\,\mathbb{E}_t\left[\frac{p_{F,t+1}^{L,o}}{1-\beta\theta}\right],\quad \frac{p_{Ft}^{P,o*}}{1-\beta\theta}=w_t^*-a_t^*+(\beta\theta)\,\mathbb{E}_t\left[\frac{p_{F,t+1}^{P,o*}}{1-\beta\theta}\right],$$

which can be rewritten out as

$$\begin{aligned}\pi_{Ft}^{L,o*} &= (1-\beta\theta)\left(w_t^*-a_t^*-p_{Ft}^*\right)+(\beta\theta)\,\mathbb{E}_t\left[\pi_{F,t+1}^{L,o*}+\pi_{F,t+1}^*\right],\\ \pi_{Ft}^{L,o} &= (1-\beta\theta)\left(w_t^*-a_t^*+e_t-p_{Ft}\right)+(\beta\theta)\,\mathbb{E}_t\left[\pi_{F,t+1}^{L,o}+\pi_{F,t+1}\right],\\ \pi_{Ft}^{P,o*} &= (1-\beta\theta)\left(w_t^*-a_t^*-p_{Ft}^*\right)+(\beta\theta)\,\mathbb{E}_t\left[\pi_{F,t+1}^{P,o*}+\pi_{F,t+1}^*\right].\end{aligned}\tag{86}$$

Phillips Curves: Define $\delta\equiv\frac{(1-\theta)(1-\beta\theta)}{\theta}$ as in Woodford (2003). Combining (81), (85), and (86), we can derive the following four open-economy Phillips curves under imperfect financial market ($0\leq\lambda\leq1$) and pricing-to-market ($0\leq\chi\leq1$).

Home firms' domestic pricing combined with PPI for Home goods in Home implies

$$\pi_{Ht}=\delta\left(w_t-p_{Ht}-a_t\right)+\beta\mathbb{E}_t\left[\pi_{H,t+1}\right].\tag{87}$$

Home firms' export pricing combined with PPI for Home goods in Foreign implies

$$\left(\begin{array}{cc} & \chi\pi_{Ht}^*+(1-\chi)\pi_{Ht}\\ + & (1-\chi)\left(m_t+z_t-m_{t-1}-z_{t-1}\right)\end{array}\right)=\left[\begin{array}{cc} & \delta\left(w_t-p_{Ht}-a_t-m_t-z_t\right)\\ + & \beta\mathbb{E}_t\left(\begin{array}{cc} & \chi\pi_{H,t+1}^*+(1-\chi)\pi_{H,t+1}\\ + & (1-\chi)\left(m_{t+1}+z_{t+1}-m_t-z_t\right)\end{array}\right)\end{array}\right],\tag{88}$$

where $\chi\pi_{Ht}^*+(1-\chi)\pi_{Ht}+(1-\chi)\left(m_t+z_t-m_{t-1}-z_{t-1}\right)=\pi_{Ht}^*+(1-\chi)\left(e_t-e_{t-1}\right)=\pi_{Ht}^*+(1-\chi)\Delta e_t$.

Foreign firms' domestic pricing combined with PPI for Foreign goods in Foreign implies

$$\pi_{Ft}^*=\delta\left(w_t^*-p_{Ft}^*-a_t^*\right)+\beta\mathbb{E}_t\left[\pi_{F,t+1}^*\right].\tag{89}$$

Foreign firms' export pricing combined with PPI for Foreign goods in Home implies

$$\left(\begin{array}{cc} & \chi\pi_{Ft}+(1-\chi)\pi_{Ft}^*\\ + & (1-\chi)\left(-m_t+z_t+m_{t-1}-z_{t-1}\right)\end{array}\right)=\left[\begin{array}{cc} & \delta\left(w_t^*-p_{Ft}^*-a_t^*+m_t-z_t\right)\\ + & \beta\mathbb{E}_t\left(\begin{array}{cc} & \chi\pi_{F,t+1}+(1-\chi)\pi_{F,t+1}^*\\ + & (1-\chi)\left(-m_{t+1}+z_{t+1}+m_t-z_t\right)\end{array}\right)\end{array}\right],\tag{90}$$

where $\chi\pi_{Ft}+(1-\chi)\pi_{Ft}^*+(1-\chi)\left(-m_t+z_t+m_{t-1}-z_{t-1}\right)=\pi_{Ft}-(1-\chi)\left(e_t-e_{t-1}\right)=\pi_{Ft}-(1-\chi)\Delta e_t$.

Net Exports: In section A, we define net exports as

$$\begin{aligned}(H)\quad NX_t\equiv\mathcal{E}_tP_{Ht}^*C_{Ht}^*-P_{Ft}C_{Ft}\quad &=\textstyle\sum_{\nabla_{t+1}\in\Omega_{t+1}}Z\left(\nabla_{t+1}|\nabla_t\right)D\left(\nabla_{t+1}\right)-D\left(\nabla_t\right),\\ (F)\quad NX_t^*\equiv\frac{1}{\mathcal{E}_t}P_{Ft}C_{Ft}-P_{Ht}^*C_{Ht}^*\quad &=\frac{1}{\mathcal{E}_t}\textstyle\sum_{\nabla_{t+1}\in\Omega_{t+1}}Z\left(\nabla_{t+1}|\nabla_t\right)D^*\left(\nabla_{t+1}\right)-\frac{1}{\mathcal{E}_t}D^*\left(\nabla_t\right).\end{aligned}$$

Note that net exports are zero in the efficient steady state. Hence, we take linearization, not log-linearization for NX_t and NX_t^*. Applying Taylor approximation up to the first order and substituting out consumption indexes in Home and Foreign lead to

$$\begin{aligned}
(H) \quad \tfrac{NX_t}{P_t} &= \left(1 - \tfrac{\nu}{2}\right) C^* \left(q_t - c_t + c_t^* + \nu \left(\epsilon - 1\right) s_t\right), \\
(F) \quad \tfrac{NX_t^*}{P_t^*} &= \left(1 - \tfrac{\nu}{2}\right) C^* \left(-q_t + c_t - c_t^* - \nu \left(\epsilon - 1\right) s_t\right),
\end{aligned}$$

where $C^* = \kappa^{\frac{-1}{\sigma+\phi}}$. Therefore, we can define linearized net exports, nx_t and nx_t^*, as

$$\begin{aligned}
(H) \quad nx_t &\equiv \tfrac{1}{\left(1-\frac{\nu}{2}\right)C^*} \tfrac{NX_t}{P_t} = q_t - c_t + c_t^* + \nu \left(\epsilon - 1\right) s_t \\
&= \left(1 - \tfrac{1}{\sigma}\right) q_t - \tfrac{1}{\sigma} f_t + \nu \left(\epsilon - 1\right) s_t - \tfrac{1}{\sigma} \left(\zeta_{c,t} - \zeta_{c,t}^*\right), \qquad (91) \\
(F) \quad nx_t^* &\equiv \tfrac{1}{\left(1-\frac{\nu}{2}\right)C^*} \tfrac{NX_t^*}{P_t^*} = -nx_t,
\end{aligned}$$

where we make use of $\sigma \left(c_t - c_t^*\right) = q_t + f_t + \zeta_{c,t} - \zeta_{c,t}^*$.

Under the Cole and Obstfeld specification with $\sigma = \epsilon = 1$, the linearized net exports (91) translate into

$$\begin{aligned}
(H) \quad nx_t &= -f_t - \zeta_{c,t} + \zeta_{c,t}^*, \\
(F) \quad nx_t^* &= -nx_t.
\end{aligned} \qquad (92)$$

Therefore, the linearized net exports and hence net capital outflows depend only on the demand imbalance, f_t, and shocks to preferences in both countries in the Cole and Obstfeld specification.

F.2 Proof of Zero Export Premium ($z_t = 0 \; \forall t$) in Equilibrium

In this section, we extend Engel (2011) by showing that the log-linearized export premium z_t is zero for all time periods under the general framework of imperfect financial integration ($0 \leq \lambda \leq 1$) and incomplete exchange rate pass-through ($0 \leq \chi \leq 1$). Since the economy resides in the symmetrically efficient steady-state at time zero, we have $z_0 = s_0 = s_0^* = 0$. Subtraction of (88) from (87) and another subtraction of (90) from (89) imply

$$\begin{pmatrix} & \chi\pi_{Ht} - \chi\pi_{Ht}^* \\ - & (1-\chi)\left(m_t + z_t - m_{t-1} - z_{t-1}\right) \end{pmatrix} = \begin{bmatrix} & \delta\left(m_t + z_t\right) \\ + & \beta\mathbb{E}_t \begin{pmatrix} & \chi\pi_{H,t+1} - \chi\pi_{H,t+1}^* \\ - & (1-\chi)\left(m_{t+1} + z_{t+1} - m_t - z_t\right) \end{pmatrix} \end{bmatrix},$$

$$\begin{pmatrix} & \chi\pi_{Ft}^* - \chi\pi_{Ft} \\ - & (1-\chi)\left(-m_t + z_t + m_{t-1} - z_{t-1}\right) \end{pmatrix} = \begin{bmatrix} & \delta\left(-m_t + z_t\right) \\ + & \beta\mathbb{E}_t \begin{pmatrix} & \chi\pi_{F,t+1}^* - \chi\pi_{F,t+1} \\ - & (1-\chi)\left(-m_{t+1} + z_{t+1} + m_t - z_t\right) \end{pmatrix} \end{bmatrix}.$$

Summing up these two equations, we obtain

$$\begin{pmatrix} & \chi\left(-\pi_{Ft} + \pi_{Ht} - \pi_{Ht}^* + \pi_{Ft}^*\right) \\ - & (1-\chi)2\left(z_t - z_{t-1}\right) \end{pmatrix} = \begin{bmatrix} & 2\delta z_t \\ + & \beta\mathbb{E}_t \begin{pmatrix} & \chi\left(-\pi_{F,t+1} + \pi_{H,t+1} - \pi_{H,t+1}^* + \pi_{F,t+1}^*\right) \\ - & (1-\chi)2\left(z_{t+1} - z_t\right) \end{pmatrix} \end{bmatrix}.$$

Rewrite this in terms of $2z_t = s_t + s_t^* = p_{Ft} - p_{Ht} + p_{Ht}^* - p_{Ft}^*$ to derive

$$-2z_t + 2z_{t-1} = 2\delta z_t + \beta\mathbb{E}_t\left[-2z_{t+1} + 2z_t\right].$$

Therefore we solve for z_t from

$$z_t = \frac{\beta}{1+\beta+\delta}\mathbb{E}_t\left[z_{t+1}\right] + \frac{1}{1+\beta+\delta}z_{t-1}.$$

Since $z_0 = 0$ and $\frac{\beta}{1+\beta+\delta} < 1$, we conclude that $z_t = 0$ for all t by induction. This implies that $m_t = m_{Ht} = m_{Ft} = e_t + p_{Ht}^* - p_{Ht} = e_t + p_{Ft}^* - p_{Ft}$ hold for all t. Importantly, relative prices of Home and Foreign goods satisfy $s_t = -s_t^*$, which means the log-linearized terms of trade are given by $\eta_t = s_t - m_t = -\eta_t^*$.

F.3 Open-Economy Phillips Curves

In this section, we follow the analysis in Engel (2011) and show that four equations for Phillips curves (87)–(90) translate into three aggregate supply relations. Observe that wage equations (66), (67), (68), and (69) can be combined to derive

$$
\begin{aligned}
\sigma c_t + \phi \underbrace{(n_t + a_t)}_{=y_t} + \left(1 - \tfrac{\nu}{2}\right) s_t &= \left(\tfrac{\sigma}{D} + \phi\right) y_t^R + (\sigma + \phi) y_t^W + \tfrac{D-\nu+1}{2D}(m_t + f_t + 2\zeta_{c,t}^R) + \left(1 - \tfrac{\nu}{2}\right) z_t, \\
\sigma c_t^* + \phi \underbrace{(n_t^* + a_t^*)}_{=y_t^*} + \left(1 - \tfrac{\nu}{2}\right) s_t^* &= \left(-\tfrac{\sigma}{D} - \phi\right) y_t^R + (\sigma + \phi) y_t^W + \tfrac{-D+\nu-1}{2D}(m_t + f_t + 2\zeta_{c,t}^R) + \left(1 - \tfrac{\nu}{2}\right) z_t, \\
(1+\phi) a_t + \zeta_{c,t} &= \left(\tfrac{\sigma}{D} + \phi\right) \overline{y}_t^R + (\sigma + \phi) \overline{y}_t^W + \tfrac{D-\nu+1}{2D}(2\zeta_{c,t}^R), \\
(1+\phi) a_t^* + \zeta_{c,t}^* &= \left(-\tfrac{\sigma}{D} - \phi\right) \overline{y}_t^R + (\sigma + \phi) \overline{y}_t^W + \tfrac{-D+\nu-1}{2D}(2\zeta_{c,t}^R),
\end{aligned}
\tag{93}
$$

where the last two equalities come from wage setting and flexible pricing under globally efficient equilibrium in which $a_t = \overline{w}_t - \overline{p}_{Ht}$ and $a_t^* = \overline{w}_t^* - \overline{p}_{Ft}^*$ hold due to constant labor markups. Combine these four relations to obtain

$$
\begin{aligned}
\tfrac{\sigma}{2} c_t + \tfrac{\phi}{2} y_t + \left(1 - \tfrac{\nu}{2}\right) \tfrac{s_t}{2} - (1+\phi) a_t - \zeta_{c,t} &= \begin{pmatrix} + & \left(\tfrac{\sigma}{D} + \phi\right)\left(\tfrac{y_t^R}{2} - \overline{y}_t^R\right) + (\sigma + \phi)\left(\tfrac{y_t^W}{2} - \overline{y}_t^W\right) \\ + & \tfrac{D-\nu+1}{2D}\left(\tfrac{m_t + f_t}{2} - \zeta_{c,t}^R\right) + \left(1 - \tfrac{\nu}{2}\right)\tfrac{z_t}{2} \end{pmatrix}, \\
\tfrac{\sigma}{2} c_t^* + \tfrac{\phi}{2} y_t^* - \left(1 - \tfrac{\nu}{2}\right) \tfrac{s_t}{2} - (1+\phi) a_t^* - \zeta_{c,t}^* &= \begin{pmatrix} - & \left(\tfrac{\sigma}{D} + \phi\right)\left(\tfrac{y_t^R}{2} - \overline{y}_t^R\right) + (\sigma + \phi)\left(\tfrac{y_t^W}{2} - \overline{y}_t^W\right) \\ - & \tfrac{D-\nu+1}{2D}\left(\tfrac{m_t + f_t}{2} - \zeta_{c,t}^R\right) + \left(1 - \tfrac{\nu}{2}\right)\tfrac{z_t}{2} \end{pmatrix}.
\end{aligned}
\tag{94}
$$

We will use (94) for the second-order approximation to the periodic world welfare. Together (66), (67), and (93) imply

$$
\begin{aligned}
w_t - p_{Ht} - a_t &= \left(\tfrac{\sigma}{D} + \phi\right) \widetilde{y}_t^R + (\sigma + \phi) \widetilde{y}_t^W + \tfrac{D-\nu+1}{2D}(m_t + f_t) + \left(1 - \tfrac{\nu}{2}\right) z_t + \tfrac{1}{\delta} u_t, \\
w_t^* - p_{Ft}^* - a_t^* &= \left(-\tfrac{\sigma}{D} - \phi\right) \widetilde{y}_t^R + (\sigma + \phi) \widetilde{y}_t^W + \tfrac{-D+\nu-1}{2D}(m_t + f_t) + \left(1 - \tfrac{\nu}{2}\right) z_t + \tfrac{1}{\delta} u_t^*.
\end{aligned}
\tag{95}
$$

PPI Phillips Curves: Substituting for wages by (95), we can rewrite (87)–(90) as

$$
\begin{aligned}
\pi_{Ht} &= \delta \left(\begin{array}{cc} & \left(\frac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ + & \frac{D-\nu+1}{2D}(m_t+f_t) + \left(1-\frac{\nu}{2}\right) z_t \end{array} \right) + \beta \mathbb{E}_t\left[\pi_{H,t+1}\right] + u_t, \\
\left(\begin{array}{cc} & \pi_{Ft} \\ - & (1-\chi)(e_t - e_{t-1}) \end{array} \right) &= \left[\begin{array}{cc} & \delta \left(\begin{array}{cc} & \left(-\frac{\sigma}{D}-\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ + & \frac{-D+\nu-1}{2D}(m_t+f_t) + \left(1-\frac{\nu}{2}\right) z_t \end{array} \right) + \delta\left(m_{Ft}\right) \\ + & \beta \mathbb{E}_t \left(\begin{array}{cc} & \pi_{F,t+1} \\ - & (1-\chi)(e_{t+1}-e_t) \end{array} \right) + u_t^* \end{array} \right], \\
\pi_{Ft}^* &= \delta \left(\begin{array}{cc} & \left(-\frac{\sigma}{D}-\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ + & \frac{-D+\nu-1}{2D}(m_t+f_t) + \left(1-\frac{\nu}{2}\right) z_t \end{array} \right) + \beta \mathbb{E}_t\left[\pi_{F,t+1}^*\right] + u_t^*, \\
\left(\begin{array}{cc} & \pi_{Ht}^* \\ + & (1-\chi)(e_t - e_{t-1}) \end{array} \right) &= \left[\begin{array}{cc} & \delta \left(\begin{array}{cc} & \left(\frac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ + & \frac{D-\nu+1}{2D}(m_t+f_t) + \left(1-\frac{\nu}{2}\right) z_t \end{array} \right) - \delta\left(m_{Ht}\right) \\ + & \beta \mathbb{E}_t \left(\begin{array}{cc} & \pi_{H,t+1}^* \\ + & (1-\chi)(e_{t+1}-e_t) \end{array} \right) + u_t \end{array} \right],
\end{aligned} \tag{96}
$$

where we have $m_t = e_t + p_{Ht}^* - p_{Ht} = e_t + p_{Ft}^* - p_{Ft}$ implying $m_t = m_{Ht} = m_{Ft}$. Note that $z_t = 0$ leads to two equalities given by $\pi_{Ht}^* - \pi_{Ht} = \pi_{Ft}^* - \pi_{Ft}$ and $\Delta e_t = \Delta m_t - \pi_{Ht}^* + \pi_{Ht} = \Delta m_t - \pi_{Ft}^* + \pi_{Ft}$ where Δ denotes the first-difference operator. Therefore, one of the four equations in (96) can be omitted by the relation $\pi_{Ht}^* - \pi_{Ht} = \pi_{Ft}^* - \pi_{Ft}$ which is derived from $z_t = 0$.

CPI Phillips Curves: By definition, CPI inflation measures are $\pi_t = \frac{\nu}{2}\pi_{Ht} + \left(1-\frac{\nu}{2}\right)\pi_{Ft}$ and $\pi_t^* = \frac{\nu}{2}\pi_{Ft}^* + \left(1-\frac{\nu}{2}\right)\pi_{Ht}^*$. Together CPI inflation terms and (96) imply

$$
\begin{aligned}
\left(\begin{array}{cc} & \pi_t \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_t - e_{t-1}) \end{array} \right) &= \left[\begin{array}{cc} & \delta \left(\begin{array}{cc} & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ + & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t \end{array} \right) \\ + & \beta \mathbb{E}_t \left(\begin{array}{cc} & \pi_{t+1} \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_{t+1}-e_t) \end{array} \right) + \frac{\nu}{2}u_t + \left(1-\frac{\nu}{2}\right)u_t^* \end{array} \right], \\
\left(\begin{array}{cc} & \pi_t^* \\ + & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_t - e_{t-1}) \end{array} \right) &= \left[\begin{array}{cc} & \delta \left(\begin{array}{cc} - & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W \\ - & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t - \frac{D-(\nu-1)^2}{2D}m_t \end{array} \right) \\ + & \beta \mathbb{E}_t \left(\begin{array}{cc} & \pi_{t+1}^* \\ + & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_{t+1}-e_t) \end{array} \right) + \frac{\nu}{2}u_t^* + \left(1-\frac{\nu}{2}\right)u_t \end{array} \right].
\end{aligned} \tag{97}
$$

World Phillips Curves: We can further simplify (97) in terms of cross-country sum and difference of CPI inflation rates, $\pi_t^R \equiv \frac{\pi_t - \pi_t^*}{2}$ and $\pi_t^W \equiv \frac{\pi_t + \pi_t^*}{2}$, given by

$$\begin{aligned}\begin{pmatrix} & \pi_t^R \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_t - e_{t-1})\end{pmatrix} &= \left[\begin{array}{ll} & \delta\begin{pmatrix} & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R \\ + & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t\end{pmatrix} \\ + & \beta\mathbb{E}_t\begin{pmatrix} & \pi_{t+1}^R \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_{t+1}-e_t)\end{pmatrix} + (\nu-1)\frac{u_t - u_t^*}{2}\end{array}\right], \\ \pi_t^W &= \delta(\sigma+\phi)\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^W + \frac{u_t+u_t^*}{2}.\end{aligned} \tag{98}$$

Substituting for Δe_t by the equality $\Delta e_t = \Delta q_t + 2\pi_t^R = \Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R$, we can rewrite the world Phillips curves (98) as

$$\begin{aligned}\begin{pmatrix} & \pi_t^R\left[1-(2-\nu)(1-\chi)\right] \\ - & \Delta m_t\left(1-\frac{\nu}{2}\right)(1-\chi) \\ - & \Delta s_t(\nu-1)\left(1-\frac{\nu}{2}\right)(1-\chi)\end{pmatrix} &= \left[\begin{array}{ll} & \delta\begin{pmatrix} & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R \\ + & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t\end{pmatrix} \\ + & \beta\mathbb{E}_t\begin{pmatrix} & \pi_{t+1}^R\left[1-(2-\nu)(1-\chi)\right] \\ - & \Delta m_{t+1}\left(1-\frac{\nu}{2}\right)(1-\chi) \\ - & \Delta s_{t+1}(\nu-1)\left(1-\frac{\nu}{2}\right)(1-\chi)\end{pmatrix} + (\nu-1)\frac{u_t-u_t^*}{2}\end{array}\right], \\ \pi_t^W &= \beta\mathbb{E}_t\pi_{t+1}^W + \delta(\sigma+\phi)\widetilde{y}_t^W + \frac{u_t+u_t^*}{2}.\end{aligned} \tag{99}$$

Finally, the price growth of Foreign goods relative to Home goods, $\Delta s_t = \pi_{Ft} - \pi_{Ht}$, evolves by

$$\begin{aligned}\Delta s_t - (1-\chi)\Delta e_t &= \pi_{Ft} - (1-\chi)\Delta e_t - \pi_{Ht} \\ &= -\delta\left[2\left(\frac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + \frac{D-\nu+1}{D}f_t - \frac{\nu-1}{D}m_t\right] + \beta\mathbb{E}_t\left[\Delta s_{t+1} - (1-\chi)\Delta e_{t+1}\right] - u_t + u_t^* \\ &= -\delta\left[s_t - \overline{s}_t + 2\phi\widetilde{y}_t^R + f_t\right] + \beta\mathbb{E}_t\left[\Delta s_{t+1} - (1-\chi)\Delta e_{t+1}\right] - u_t + u_t^*,\end{aligned} \tag{100}$$

where we use (73) for the last equality. Applying $\Delta e_t = \Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R$, we can rewrite (100) as

$$\begin{pmatrix} & \Delta s_t \\ - & (1-\chi)\left(\Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R\right)\end{pmatrix} = \left[\begin{array}{ll} - & \delta\left[s_t - \overline{s}_t + 2\phi\widetilde{y}_t^R + f_t\right] - u_t + u_t^* \\ + & \beta\mathbb{E}_t\left[\begin{array}{ll} & \Delta s_{t+1} \\ - & (1-\chi)\left(\Delta m_{t+1} + (\nu-1)\Delta s_{t+1} + 2\pi_{t+1}^R\right)\end{array}\right]\end{array}\right]. \tag{101}$$

In sum, four equations for Phillips curves (87)–(90) reduce to three aggregate supply relations given by

$$\pi_t^W = \beta\mathbb{E}_t\pi_{t+1}^W + \delta(\sigma+\phi)\widetilde{y}_t^W + \frac{u_t+u_t^*}{2}, \tag{102}$$

$$\begin{pmatrix} & \pi_t^R \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_t - e_{t-1})\end{pmatrix} = \left[\begin{array}{ll} & \delta\begin{pmatrix} & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R \\ + & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t\end{pmatrix} \\ + & \beta\mathbb{E}_t\begin{pmatrix} & \pi_{t+1}^R \\ - & \left(1-\frac{\nu}{2}\right)(1-\chi)(e_{t+1}-e_t)\end{pmatrix} + (\nu-1)\frac{u_t - u_t^*}{2}\end{array}\right], \tag{103}$$

$$\Delta s_t - (1-\chi)\Delta e_t = -\delta\left[s_t - \overline{s}_t + 2\phi\widetilde{y}_t^R + f_t\right] + \beta\mathbb{E}_t\left[\Delta s_{t+1} - (1-\chi)\Delta e_{t+1}\right] - u_t + u_t^*, \tag{104}$$

where we can substitute for the nominal exchange rate growth by $\Delta e_t = \Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R$.

World Phillips Curves under PCP: Under the PCP case ($\chi = 0$), the law of one price holds and thus

$\pi_{Ht} = \pi^*_{Ht} + e_t - e_{t-1}$, $\pi^*_{Ft} = \pi_{Ft} - e_t + e_{t-1}$, and $m_t = m_{Ht} = m_{Ft} = 0$ are satisfied. Then four equations for Phillips curves (87)–(90) translate into only two evolutionary dynamics for domestic inflation. We define new cross-country inflation measures for the PCP case ($\chi = 0$) by

$$\pi_t^{P,R} \equiv \tfrac{\pi_{Ht}-\pi^*_{Ft}}{2} \quad \text{and} \quad \pi_t^{P,W} \equiv \tfrac{\pi_{Ht}+\pi^*_{Ft}}{2},$$

implying $\frac{1}{2}\left({\pi_{Ht}}^2 + {\pi^*_{Ft}}^2\right) = \left(\pi_t^{P,R}\right)^2 + \left(\pi_t^{P,W}\right)^2$. Then Phillips curves (87)–(90) are simplified as

$$\begin{aligned} \pi_t^{P,R} &= \delta\left[\left(\tfrac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + \tfrac{D-\nu+1}{2D}f_t\right] + \beta\mathbb{E}_t\pi_{t+1}^{P,R} + u_t^R, \\ \pi_t^{P,W} &= \delta(\sigma+\phi)\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^{P,W} + u_t^W, \end{aligned} \tag{105}$$

which hold only under the PCP case ($\chi = 0$).

In general, i.e., in both PCP ($\chi = 0$) and LCP ($\chi = 1$) or in the mixed case of $0 < \chi < 1$, the cross-country CPI inflation and the cross-country PPI inflation can be shown to be related through the following equations:

$$\begin{aligned} \tfrac{\pi_t+\pi_t^*}{2} &= \tfrac{\pi_{Ht}+\pi^*_{Ft}}{2}, \\ \tfrac{\pi_t-\pi_t^*}{2} &= \tfrac{\pi_{Ht}-\pi^*_{Ft}}{2} + \left(1-\tfrac{\nu}{2}\right)\Delta s_t, \end{aligned} \tag{106}$$

that is, $\pi_t^W = \pi_t^{P,W}$ and $\pi_t^R = \pi_t^{P,R} + \left(1-\frac{\nu}{2}\right)\Delta s_t$. We present the derivation for (106) in section F.4.

F.4 The Relationship between CPI and PPI Inflation Rates

So far, we have defined the cross-country inflation measures as

$$\begin{aligned} &\pi_t = \tfrac{\nu}{2}\pi_{Ht} + \left(1-\tfrac{\nu}{2}\right)\pi_{Ft} \quad &&\text{and} \quad \pi_t^* = \tfrac{\nu}{2}\pi^*_{Ft} + \left(1-\tfrac{\nu}{2}\right)\pi^*_{Ht}, \\ &\pi_t^R \equiv \tfrac{\pi_t-\pi_t^*}{2} &&\text{and} \quad \pi_t^W \equiv \tfrac{\pi_t+\pi_t^*}{2}, \\ &\pi_t^{P,R} \equiv \tfrac{\pi_{Ht}-\pi^*_{Ft}}{2} &&\text{and} \quad \pi_t^{P,W} \equiv \tfrac{\pi_{Ht}+\pi^*_{Ft}}{2}. \end{aligned}$$

Observe that we can derive the following equations in general (both PCP ($\chi = 0$) and LCP ($\chi = 1$), or the mixed case of $0 < \chi < 1$):

$$\begin{aligned} \pi_t^W &= \pi_t^{P,W} = \tfrac{\pi_{Ht}+\pi^*_{Ft}}{2} = \tfrac{\pi_{Ft}+\pi^*_{Ht}}{2}, \\ \pi_t^R &= (\nu-1)\pi_t^{P,R} + \left(1-\tfrac{\nu}{2}\right)\left(\tfrac{\pi_{Ft}-\pi^*_{Ht}+\pi_{Ht}-\pi^*_{Ft}}{2}\right) \\ &= (\nu-1)\pi_t^{P,R} + \left(1-\tfrac{\nu}{2}\right)(\pi_{Ht}-\pi^*_{Ht}) \\ &= (\nu-1)\pi_t^{P,R} + \left(1-\tfrac{\nu}{2}\right)(\Delta e_t - \Delta m_t) \\ &= \pi_t^{P,R} + \left(\tfrac{2-\nu}{2(\nu-1)}\right)(\Delta q_t - \Delta m_t) \\ &= \pi_t^{P,R} + \left(1-\tfrac{\nu}{2}\right)\Delta s_t, \end{aligned}$$

which completes the derivation for (106). Here $z_t = 0$ implies $\pi_{Ht} - \pi^*_{Ht} = \pi_{Ft} - \pi^*_{Ft}$ and $\Delta e_t = \Delta m_t + \pi_{Ht} - \pi^*_{Ht} = \Delta m_t + \pi_{Ft} - \pi^*_{Ft}$. In the last two equalities, we use $\Delta e_t = \Delta q_t + 2\pi_t^R$ and $q_t = m_t + (\nu-1)s_t$.

Similarly, we can express $\pi^*_{Ht} - \pi_{Ft}$ in terms of cross-country CPI inflation terms and the relative price between Home and Foreign goods by

$$\begin{aligned}
2\pi^R_t &= \pi_t - \pi^*_t = \tfrac{\nu}{2}\left(\pi_{Ht} - \pi^*_{Ft}\right) + \left(\tfrac{\nu}{2} - 1\right)\left(\pi^*_{Ht} - \pi_{Ft}\right) \\
&= \tfrac{\nu}{2}\left(\pi_{Ht} - \pi^*_{Ft} - \pi^*_{Ht} + \pi_{Ft} + \pi^*_{Ht} - \pi_{Ft}\right) + \left(\tfrac{\nu}{2} - 1\right)\left(\pi^*_{Ht} - \pi_{Ft}\right) \\
&= \tfrac{\nu}{2}\left(\pi_{Ht} - \pi^*_{Ft} - \pi^*_{Ht} + \pi_{Ft}\right) + (\nu - 1)\left(\pi^*_{Ht} - \pi_{Ft}\right) \\
&= \nu\left(\Delta e_t - \Delta m_t\right) + (\nu - 1)\left(\pi^*_{Ht} - \pi_{Ft}\right) \\
&= \nu\left(\Delta e_t - \Delta q_t + (\nu-1)\Delta s_t\right) + (\nu - 1)\left(\pi^*_{Ht} - \pi_{Ft}\right) \\
&= \nu\left(2\pi^R_t + (\nu-1)\Delta s_t\right) + (\nu - 1)\left(\pi^*_{Ht} - \pi_{Ft}\right),
\end{aligned}$$

which implies

$$\pi^*_{Ht} - \pi_{Ft} = -2\pi^R_t - \nu\Delta s_t. \tag{107}$$

In sum, PPI and CPI inflation measures are related through

$$\begin{aligned}
\pi_{Ht} &= \pi^W_t + \pi^R_t - \left(1 - \tfrac{\nu}{2}\right)\Delta s_t &&= \pi_t - \left(1 - \tfrac{\nu}{2}\right)\Delta s_t, \\
\pi^*_{Ft} &= \pi^W_t - \pi^R_t + \left(1 - \tfrac{\nu}{2}\right)\Delta s_t &&= \pi^*_t + \left(1 - \tfrac{\nu}{2}\right)\Delta s_t, \\
\pi_{Ft} &= \pi^W_t + \pi^R_t + \tfrac{\nu}{2}\Delta s_t &&= \pi_t + \tfrac{\nu}{2}\Delta s_t, \\
\pi^*_{Ht} &= \pi^W_t - \pi^R_t - \tfrac{\nu}{2}\Delta s_t &&= \pi^*_t - \tfrac{\nu}{2}\Delta s_t.
\end{aligned} \tag{108}$$

F.5 The Irrelevance Result of Financial Market Structure for Cross-Country Risk Sharing

Recall the model economy can be perturbed by shocks to preference or productivity or labor markups.

Proposition 6. *In response to asymmetric preference shocks* $\left(\zeta_{c,t} \neq \zeta^*_{c,t}\right)$, *the risk sharing is always imperfect if international financial markets are frictional* ($\lambda > 0$) *and goods markets in Home and Foreign are not completely separated* ($\nu \neq 2$).

Proof. Note that the demand imbalance in (75) is given by $f_t = \begin{pmatrix} + & \frac{2\lambda(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{\lambda(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} y^R_t \\ - & \frac{\lambda(2-\nu)(\epsilon\nu-\nu+1-D)}{\lambda(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} m_t \\ - & \frac{\lambda(2-\nu)(\epsilon\nu-\nu+1)}{\lambda(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} 2\zeta^R_{c,t} \end{pmatrix}$, where $D \equiv (\nu-1)^2 + \sigma\epsilon\nu(2-\nu) = 1 + \nu(2-\nu)(\epsilon\sigma - 1)$ and $0 \leq \nu \leq 2$. f_t is easily seen to be zero when $\lambda = 0$ or $\nu = 2$. Now suppose both $\lambda \neq 0$ and $\nu \neq 2$ hold. Observe that coefficients of y^R_t, m_t, and $\zeta^R_{c,t} = \frac{\zeta_{c,t} - \zeta^*_{c,t}}{2}$ cannot be simultaneously zero. This is because having both $\epsilon = \frac{\nu-1}{\nu}$ and $\sigma = \frac{\nu-1}{\epsilon\nu-1} = \frac{1-\nu}{2-\nu}$ violates the requirement of the positivity of both trade elasticity (ϵ) and risk aversion (σ). When $\sigma = \epsilon = 1$, coefficients of y^R_t and m_t become null and the Cole-Obstfeld(1991) result emerges if $\zeta_{c,t} = \zeta^*_{c,t}$. However, if shocks to preference are asymmetric between Home and Foreign, cross-country risk sharing is always nonzero by $f_t = -\left(\zeta_{c,t} - \zeta^*_{c,t}\right)\left(\frac{\lambda(2-\nu)}{2-\nu\lambda}\right)$ under $\sigma = \epsilon = 1$. $\square$

Proposition 7. *Suppose the cross-country trade elasticity is unity ($\epsilon = 1$) and consumption utility is in log ($\sigma = 1$). Then the degree of cross-country risk sharing is determined solely by the cross-country difference of preference shocks through $f_t = -\left(\zeta_{c,t} - \zeta^*_{c,t}\right)\left(\frac{\lambda(2-\nu)}{2-\nu\lambda}\right)$. If shocks to preference are symmetric between Home and Foreign, the cross-country risk sharing is perfect.*

Proof. When shocks to preference are asymmetric between Home and Foreign, the proof in claim 6 establishes the first statement. Now suppose preference shocks are symmetric across countries. Observe that the demand imbalance in (75) becomes null ($f_t = 0$) if both $\sigma(\epsilon\nu - 1) + 1 - \nu = 0$ and $\epsilon\nu - \nu + 1 - D = 0$ hold under $\zeta_{c,t} = \zeta^*_{c,t}$. These two relations can be simply written out as $\sigma(\epsilon\nu - 1) = \nu - 1$ and $(\epsilon - 1)(\epsilon\nu - \nu + 1) = 0$. Therefore, it is easily seen to be $f_t = 0$ under both $\epsilon = 1$ and $\sigma = 1$. If $\epsilon\nu - \nu + 1 = 0$ with $\epsilon \neq 1$, then zero demand imbalance requires both $\epsilon = \frac{\nu-1}{\nu}$ and $\sigma = \frac{1-\nu}{2-\nu}$, which violate the requirement of the positivity of both trade elasticity and risk aversion. Thus, it implies $\epsilon\nu - \nu + 1 \neq 0$. □

F.6 Log-Linearized Globally Efficient Allocations

In this section, we characterize log-linearized efficient allocations under flexible prices, optimal subsidies, and perfect risk sharing. $\overline{x}_t$ stands for the globally efficient allocation approximated up to the first order. Deviation from perfect risk sharing and currency misalignment are all zero. Also all the inflation terms and inefficient cost-push shocks in log are zero. Hence, efficient allocations fluctuate solely in response to productivity and preference shocks denoted by

$$\zeta^R_{c,t} = \frac{\zeta_{c,t} - \zeta^*_{c,t}}{2} \qquad \text{and} \qquad \zeta^W_{c,t} = \frac{\zeta_{c,t} + \zeta^*_{c,t}}{2},$$
$$a^R_t = \frac{a_t - a^*_t}{2} \qquad \text{and} \qquad a^W_t = \frac{a_t + a^*_t}{2}.$$

From the last two equations in (93), we can solve for cross-country outputs given by

$$\overline{y}^R_t = \frac{1+\phi}{\frac{\sigma}{D}+\phi} a^R_t + \frac{\nu-1}{\sigma+\phi D}\zeta^R_{c,t} \quad \text{and} \quad \overline{y}^W_t = \frac{1+\phi}{\sigma+\phi} a^W_t + \frac{1}{\sigma+\phi}\zeta^W_{c,t}. \tag{109}$$

Together (72), (74), and (84) imply

$$\begin{aligned}
&\overline{n}^R_t = \overline{y}^R_t - a^R_t && \text{and} \quad \overline{n}^W_t = \overline{y}^W_t - a^W_t, \\
&\overline{c}^R_t = \frac{(\nu-1)(1+\phi)}{\sigma+\phi D} a^R_t + \frac{1+\phi\epsilon\nu(2-\nu)}{\sigma+\phi D}\zeta^R_{c,t} && \text{and} \quad \overline{c}^W_t = \overline{y}^W_t, \\
&\overline{s}^R_t = \frac{2\sigma(1+\phi)}{\sigma+\phi D} a^R_t - \frac{2\phi(\nu-1)}{\sigma+\phi D}\zeta^R_{c,t},
\end{aligned} \tag{110}$$

where $\overline{s}_t^R = \overline{s}_t$ due to $z_t = 0$. Under the special parametrization of $\sigma = 1$ and $\phi = 0$, efficient allocations reduce to

$$\begin{aligned}
\overline{y}_t^R &= \tfrac{1+\phi}{\frac{\sigma}{D}+\phi} a_t^R + \tfrac{\nu-1}{\sigma+\phi D}\zeta_{c,t}^R &&= D a_t^R + (\nu-1)\zeta_{c,t}^R, \\
\overline{y}_t^W &= \tfrac{1+\phi}{\sigma+\phi} a_t^W + \tfrac{1}{\sigma+\phi}\zeta_{c,t}^W &&= a_t^W + \zeta_{c,t}^W, \\
\overline{s}_t &= \tfrac{2\sigma(1+\phi)}{\sigma+\phi D} a_t^R - \tfrac{2\phi(\nu-1)}{\sigma+\phi D}\zeta_{c,t}^R &&= 2a_t^R, \\
\overline{c}_t^R &= \tfrac{\nu-1}{D}\overline{y}_t^R + \tfrac{\epsilon\nu(2-\nu)}{D}\zeta_{c,t}^R &&= (\nu-1)a_t^R + \zeta_{c,t}^R, \\
\overline{c}_t^W &= \overline{y}_t^W &&= a_t^W + \zeta_{c,t}^W, \\
\overline{n}_t^R &= \tfrac{1-\frac{\sigma}{D}}{\frac{\sigma}{D}+\phi} a_t^R + \tfrac{\nu-1}{\sigma+\phi D}\zeta_{c,t}^R &&= (D-1)a_t^R + (\nu-1)\zeta_{c,t}^R, \\
\overline{n}_t^W &= \tfrac{1-\sigma}{\sigma+\phi} a_t^W + \tfrac{1}{\sigma+\phi}\zeta_{c,t}^W &&= \zeta_{c,t}^W.
\end{aligned} \tag{111}$$

We can rewrite each country-specific allocation as

$$\begin{aligned}
&\overline{y}_t = \tfrac{1+D}{2}a_t + \tfrac{1-D}{2}a_t^* + \tfrac{\nu}{2}\zeta_{c,t} + \left(1-\tfrac{\nu}{2}\right)\zeta_{c,t}^* &&\text{and}\quad \overline{y}_t^* = \tfrac{1-D}{2}a_t + \tfrac{1+D}{2}a_t^* + \left(1-\tfrac{\nu}{2}\right)\zeta_{c,t} + \tfrac{\nu}{2}\zeta_{c,t}^*, \\
&\overline{s}_t = a_t - a_t^* &&\text{and}\quad \overline{s}_t^* = -\overline{s}_t, \\
&\overline{c}_t = \tfrac{\nu}{2}a_t + \left(1-\tfrac{\nu}{2}\right)a_t^* + \zeta_{c,t} &&\text{and}\quad \overline{c}_t^* = \left(1-\tfrac{\nu}{2}\right)a_t + \tfrac{\nu}{2}a_t^* + \zeta_{c,t}^*, \\
&\overline{n}_t = \tfrac{D-1}{2}a_t - \tfrac{D-1}{2}a_t^* + \tfrac{\nu}{2}\zeta_{c,t} + \tfrac{2-\nu}{2}\zeta_{c,t}^* &&\text{and}\quad \overline{n}_t^* = \tfrac{1-D}{2}a_t - \tfrac{1-D}{2}a_t^* + \tfrac{2-\nu}{2}\zeta_{c,t} + \tfrac{\nu}{2}\zeta_{c,t}^*.
\end{aligned}$$

F.7 Log-Linearized Natural Allocations

In this section, we present log-linearized allocations under flexible prices, optimal subsidies, and imperfect risk sharing. In this equilibrium, all price dispersion terms and currency misalignment become null. However natural allocations are inefficient in general due to cost-push shocks and imperfect risk sharing. Together (72), (74), (75), and (84) imply

$$\begin{aligned}
&c_t^R = \tfrac{\nu-1}{D}y_t^R + \tfrac{\epsilon\nu(2-\nu)}{2D}(f_t + 2\zeta_{c,t}^R) &&\text{and}\quad c_t^W = y_t^W, \\
&n_t^R = y_t^R - a_t^R &&\text{and}\quad n_t^W = y_t^W - a_t^W, \\
&s_t = \tfrac{2\sigma}{D}y_t^R - \tfrac{\nu-1}{D}\left(f_t + 2\zeta_{c,t}^R\right) &&\text{and}\quad f_t = \begin{pmatrix} + & \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}y_t^R \\ + & \frac{(\lambda)(2-\nu)(-\epsilon\nu+\nu-1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}2\zeta_{c,t}^R \end{pmatrix}.
\end{aligned}$$

Two additional equations can be derived from the PCP Phillips curves (105) combined with zero inflation rates:

$$\begin{aligned}
0 &= \delta\left[\left(\tfrac{\sigma}{D}+\phi\right)(y_t^R - \overline{y}_t^R) + (\sigma+\phi)(y_t^W - \overline{y}_t^W) + \tfrac{D-\nu+1}{2D}f_t\right] + u_t, \\
0 &= \delta\left[\left(-\tfrac{\sigma}{D}-\phi\right)(y_t^R - \overline{y}_t^R) + (\sigma+\phi)(y_t^W - \overline{y}_t^W) + \tfrac{-D+\nu-1}{2D}f_t\right] + u_t^*.
\end{aligned}$$

Therefore, given eight equilibrium conditions, we solve for eight variables: c_t^R, c_t^W, y_t^R, y_t^W, n_t^R, n_t^W, s_t, and f_t.

In the paper, our study focuses on responses of allocations with respect to productivity and preference shocks.

Hence, we abstract from cost-push shocks in what follows. The last three equations above imply

$$\widetilde{y}_t^W = 0,$$

$$\widetilde{y}_t^R = \begin{pmatrix} -a_t^R \left(\frac{(D-\nu+1)\Xi_1}{2(\sigma+\phi D)+\Xi_1(D-\nu+1)}\right)\left(\frac{D(1+\phi)}{\sigma+\phi D}\right) \\ -\zeta_{c,t}^R \left(\frac{(D-\nu+1)\left(\frac{2(\lambda)(2-\nu)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right)}{2(\sigma+\phi D)+\Xi_1(D-\nu+1)}\right)\left[\left(\frac{(\nu-1)[\sigma(\epsilon\nu-1)+1-\nu]}{\sigma+\phi D}\right)+(-\epsilon\nu+\nu-1)\right] \end{pmatrix},$$

$$f_t = \begin{pmatrix} +a_t^R \left(\frac{2(\sigma+\phi D)\Xi_1}{2(\sigma+\phi D)+\Xi_1(D-\nu+1)}\right)\left(\frac{(1+\phi)D}{\sigma+\phi D}\right) \\ +\zeta_{c,t}^R \left(\frac{2(\sigma+\phi D)\left(\frac{2(\lambda)(2-\nu)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right)}{2(\sigma+\phi D)+\Xi_1(D-\nu+1)}\right)\left[\left(\frac{(\nu-1)[\sigma(\epsilon\nu-1)+1-\nu]}{\sigma+\phi D}\right)+(-\epsilon\nu+\nu-1)\right] \end{pmatrix},$$

where $\overline{y}_t^R = \frac{1+\phi}{\frac{\sigma}{D}+\phi}a_t^R + \frac{\nu-1}{\sigma+\phi D}\zeta_{c,t}^R$, $D \equiv (\nu-1)^2 + \sigma\epsilon\nu(2-\nu)$, and $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$. Given $\widetilde{y}_t^W$, $\widetilde{y}_t^R$, and f_t, the other five equilibrium conditions pin down the rest of allocations.

Under the special parametrization of $\sigma = 1$ and $\phi = 0$, allocations are simplified as

$$\begin{aligned} \widetilde{y}_t^W &= 0, \\ \widetilde{y}_t^R &= -a_t^R\left(\frac{(D-\nu+1)D}{2+\Xi_1(D-\nu+1)}\right)\Xi_1 + \zeta_{c,t}^R\left(\frac{(D-\nu+1)D}{2+\Xi_1(D-\nu+1)}\right)\left(\frac{\Xi_1}{\nu(\epsilon-1)}\right), \\ f_t &= 2a_t^R\left(\frac{D}{2+\Xi_1(D-\nu+1)}\right)\Xi_1 - 2\zeta_{c,t}^R\left(\frac{D}{2+\Xi_1(D-\nu+1)}\right)\left(\frac{\Xi_1}{\nu(\epsilon-1)}\right). \end{aligned} \tag{112}$$

Given f_t, natural allocations under $\sigma = 1$ and $\phi = 0$ are determined by

$$\widetilde{y}_t^W = \widetilde{n}_t^W = \widetilde{c}_t^W = 0, \quad \widetilde{y}_t^R = \widetilde{n}_t^R = -\frac{D-\nu+1}{2}f_t, \quad \widetilde{c}_t^R = \left(1-\frac{\nu}{2}\right)f_t, \quad \widetilde{s}_t = -f_t, \quad \text{and} \quad \widetilde{q}_t = -(\nu-1)f_t, \tag{113}$$

where $\widetilde{y}_t^R = \widetilde{y}_t = -\widetilde{y}_t^*$ and $\widetilde{c}_t^R = \widetilde{c}_t = -\widetilde{c}_t^*$.

When we consider the Cole and Obstfeld (1991) economy by $\sigma = 1$, $\phi = 0$, and $\epsilon = 1$, coefficients become $D = 1$, $\Xi_1 = 0$, and $\left(\frac{\Xi_1}{\nu(\epsilon-1)}\right) = \frac{2\lambda(2-\nu)}{2-\lambda\nu}$ which lead to even simpler allocations given by

$$\widetilde{y}_t^W = 0, \quad \widetilde{y}_t^R = \zeta_{c,t}^R\left(\frac{\lambda(2-\nu)^2}{2-\lambda\nu}\right), \quad \text{and} \quad f_t = -\zeta_{c,t}^R\left(\frac{2\lambda(2-\nu)}{2-\lambda\nu}\right). \tag{114}$$

F.8 Log-Linearized Allocations under Financial Autarky

Suppose there is no international bond market so that the trade is balanced: $P_{Ft}C_{Ft} = \mathcal{E}_t P_{Ht}^* C_{Ht}^*$. Using $S_t = \frac{P_{Ft}}{P_{Ht}}$, $M_t Z_t = \frac{\mathcal{E}_t P_{Ht}^*}{P_{Ht}}$, and CES demands, we can rewrite the balanced-trade condition as

$$P_{Ht}S_t\left(1-\frac{\nu}{2}\right)\left(\frac{P_{Ft}}{P_t}\right)^{-\epsilon}C_t = P_{Ht}M_t Z_t\left(1-\frac{\nu}{2}\right)\left(\frac{P_{Ht}^*}{P_t^*}\right)^{-\epsilon}C_t^*.$$

Replace real price indexes with the relative price of Home and Foreign goods to obtain

$$S_t C_t\left\{\frac{\nu}{2}\left(\frac{1}{S_t}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}} = M_t Z_t C_t^*\left\{\frac{\nu}{2}\left(\frac{1}{S_t^*}\right)^{1-\epsilon} + 1 - \frac{\nu}{2}\right\}^{\frac{\epsilon}{1-\epsilon}}.$$

Log-linearizing this equation leads to $(1-\epsilon\nu)\, s_t^R = m_t - 2c_t^R$, where we use $z_t = \frac{s_t+s_t^*}{2}$. If we combine this with $s_t^R = \frac{2\sigma}{D} y_t^R - \frac{\nu-1}{D}(m_t + f_t + 2\zeta_{c,t}^R)$ and $c_t^R = \frac{\nu-1}{D} y_t^R + \frac{\epsilon\nu(2-\nu)}{2D}(m_t + f_t + 2\zeta_{c,t}^R)$, then the log-linearized financial-market equilibrium condition implies

$$f_t = \frac{2\left[\sigma(\epsilon\nu-1)+1-\nu\right]}{\epsilon\nu-\nu+1} y_t^R - \frac{\epsilon\nu-\nu+1-D}{\epsilon\nu-\nu+1} m_t - 2\zeta_{c,t}^R. \tag{115}$$

The demand imbalance f_t above is exactly consistent with (75) under $\lambda = 1$. Therefore, the equilibrium under financial autarky corresponds to the special case under the parametrization of $\lambda = 1$.

G The 2nd-order Approximation of the Loss Function

This section derives the quadratic loss function of the global policymaker. Its exposition closely follows derivation of the quadratic loss function in Corsetti et al. (2020) by using notations of Engel (2011). The new contribution of our analysis relative to Corsetti et al. (2020) is the derivation of price dispersion terms under generic degree of exchange rate pass-through given by (152).

G.1 The 2nd-order Approximation of the Periodic Utility

The idea for deriving a measure of global welfare is that we approximate equations for periodic utility, aggregate demand (Y_t, Y_t^*), and goods-market clearing condition (24, 25) up to the second order; then using second-order approximate aggregate demand and goods-market clearing condition, we substitute for consumption and labor in the utility function; finally we express the deviation of utility around its efficient level in terms of output gap, inflation, currency misalignment, and demand imbalance.

The periodic utility of the global planner is defined as

$$V_t \equiv \frac{\zeta_{C,t} C_t^{1-\sigma} + \zeta_{C,t}^* C_t^{*\,1-\sigma}}{1-\sigma} - \kappa \frac{N_t^{1+\phi} + N_t^{*\,1+\phi}}{1+\phi}.$$

We approximate this term up to the second order given by

$$\begin{aligned} v_t \quad &\equiv \quad \tfrac{V_t - V}{C^{1-\sigma}} = c_t + c_t^* - n_t - n_t^* + c_t\zeta_{c,t} + c_t^*\zeta_{c,t}^* + \tfrac{1-\sigma}{2}\left[c_t^2 + c_t^{*2}\right] - \tfrac{1+\phi}{2}\left[n_t^2 + n_t^{*2}\right] + t.i.p. + O(x_t^3) \\ &= \left(\begin{array}{ll} + & c_t + c_t^* - y_t - y_t^* + c_t\zeta_{c,t} + c_t^*\zeta_{c,t}^* \\ + & \left(\frac{1-\sigma}{2}\right)\left({c_t}^2 + {c_t^*}^2\right) \\ - & \left(\frac{1+\phi}{2}\right)\left({y_t}^2 - 2a_t y_t + {y_t^*}^2 - 2a_t^* y_t^*\right) \\ - & \left(\frac{\xi}{2}\right)\left[\left(\frac{\nu}{2}\right)\sigma_{P_H,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_H^*,t}{}^2 + \left(\frac{\nu}{2}\right)\sigma_{P_F^*,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_F,t}{}^2\right] \\ + & t.i.p. + O(x_t^3) \end{array} \right), \end{aligned} \tag{116}$$

where n_t and n_t^* are substituted out in the last equality by using (159) in section G.5. Thus, the expected sum of discounted loss functions is

$$
\begin{aligned}
& \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(-v_t\right) \\
= \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{ll}
- & c_t - c_t^* + y_t + y_t^* - c_t \zeta_{c,t} - c_t^* \zeta_{c,t}^* \\
- & \left(\frac{1-\sigma}{2}\right) \left({c_t}^2 + {c_t^*}^2\right) \\
+ & \left(\frac{1+\phi}{2}\right) \left({y_t}^2 - 2a_t y_t + {y_t^*}^2 - 2a_t^* y_t^*\right) \\
+ & \left(\frac{\xi}{2}\right) \left[\left(\frac{\nu}{2}\right) {\sigma_{P_H,t}}^2 + \left(1-\frac{\nu}{2}\right) {\sigma_{P_H^*,t}}^2 + \left(\frac{\nu}{2}\right) {\sigma_{P_F^*,t}}^2 + \left(1-\frac{\nu}{2}\right) {\sigma_{P_F,t}}^2\right] \\
+ & t.i.p. + O(x_t^3)
\end{array} \right),
\end{aligned}
\tag{117}
$$

where we use $C^{1-\sigma} = {C^*}^{1-\sigma} = \kappa N^{1+\phi} = \kappa {N^*}^{1+\phi}$ from the efficient steady state. We substitute for labor by using second-order approximated goods-market clearing conditions to derive (117). $\sigma_{P,t}^2$ terms represent price dispersion, which will be defined shortly. We define log of preference shocks as $\zeta_{c,t} = \log\left(\zeta_{C,t}\right)$ and $\zeta_{c,t}^* = \log\left(\zeta_{C,t}^*\right)$. $V = 2C^{1-\sigma}\left(\frac{1}{1-\sigma} - \frac{1}{1+\phi}\right)$ denotes the steady-state value of V_t. Note that $\zeta_{c,t}$, $\zeta_{c,t}^*$, a_t, a_t^*, and $\overline{x}_t$ are terms independent of policy $(t.i.p)$. $O(x_t^3)$ represents higher order terms.

Under generic degree of exchange rate pass-through ($0 \leq \chi \leq 1$), the loss function from (117) can be rewritten as

$$
\begin{aligned}
& \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t L_t \\
= \; & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{l}
+\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_t^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_t^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D}\left(m_t+f_t\right)^2 \\
+\left(\frac{\xi}{2\delta}\right) \left[\begin{array}{rl}
\multicolumn{2}{r}{+\left(\frac{\nu}{2}\right)\pi_{H,t}^2 + \left(1-\frac{\nu}{2}\right)\left(\pi_{H,t}^*\right)^2 + \left(\frac{\nu}{2}\right)\left(\pi_{F,t}^*\right)^2 + \left(1-\frac{\nu}{2}\right)\pi_{F,t}^2} \\
+\left(\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^L - \theta\pi_{H,t-1}^L\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P - \theta\pi_{H,t-1}^P\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^{L*} - \theta\pi_{H,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P - m_t - \theta(\pi_{H,t-1}^P - m_{t-1})\right\}^2\right) \\
+\left(\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^{L*} - \theta\pi_{F,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*} - \theta\pi_{F,t-1}^{P*}\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^L - \theta\pi_{F,t-1}^L\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*} + m_t - \theta(\pi_{F,t-1}^{P*} + m_{t-1})\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & (1-\chi) \left(\begin{array}{l}
\left(\Delta e_t\right)^2 \\
+ \left(\begin{array}{l}
\pi_{H,t}^P - \pi_{H,t-1}^P - \pi_{F,t}^{P*} + \pi_{F,t-1}^{P*} \\
+\pi_{H,t}^* - \pi_{F,t} + \Delta e_t - 2m_t + 2m_{t-1}
\end{array} \right)^2 \\
- \left(\begin{array}{l}
\pi_{H,t}^P - \pi_{H,t-1}^P - \pi_{F,t}^{P*} + \pi_{F,t-1}^{P*} \\
+\pi_{H,t}^* - \pi_{F,t} - 2m_t + 2m_{t-1}
\end{array} \right)^2
\end{array} \right)
\end{array} \right] \\
+t.i.p. + O(x_t^3)
\end{array} \right),
\end{aligned}
\tag{118}
$$

where $L_t \equiv -v_t$. The derivation for (118) will be presented in subsequent sections G.2, G.3, and G.5.

Loss Function under LCP: Under LCP ($\chi = 1$), the loss function (118) combined with (78) translates into

$$
\begin{aligned}
\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j L_{t+j}^{LCP} &= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} + \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} \left(m_{t+j} + f_{t+j}\right)^2 \\ + \left(\frac{\xi}{2\delta}\right) \left[\left(\frac{\nu}{2}\right) \pi_{Ht+j}{}^2 + \left(1 - \frac{\nu}{2}\right) \pi_{Ht+j}^*{}^2 + \left(\frac{\nu}{2}\right) \pi_{Ft+j}^*{}^2 + \left(1 - \frac{\nu}{2}\right) \pi_{Ft+j}{}^2\right] \\ + t.i.p. + O(x_{t+j}^3) \end{pmatrix} \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} + \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} \left(m_{t+j} + f_{t+j}\right)^2 \\ + \left(\frac{\xi}{\delta}\right) \left[\left(\pi_{t+j}^R\right)^2 + \left(\pi_{t+j}^W\right)^2 + \frac{\nu(2-\nu)}{4} \left(s_{t+j} - s_{t+j-1}\right)^2\right] \\ + t.i.p. + O(x_{t+j}^3) \end{pmatrix},
\end{aligned} \tag{119}
$$

where we use $\left[\frac{\nu}{2}\left(\pi_{Ht}{}^2 + \pi_{Ft}^*{}^2\right) + \left(1 - \frac{\nu}{2}\right)\left(\pi_{Ht}^*{}^2 + \pi_{Ft}{}^2\right)\right] = 2\left[\left(\pi_t^R\right)^2 + \left(\pi_t^W\right)^2 + \frac{\nu(2-\nu)}{4}\left(s_t - s_{t-1}\right)^2\right]$.

Loss Function under PCP: Under PCP ($\chi = 0$), the law of one price implies $\pi_{Ht} = \pi_{Ht}^* + e_t - e_{t-1}$ and $\pi_{Ft}^* = \pi_{Ft} - e_t + e_{t-1}$. Recall that we define the cross-country inflation measures under PCP as

$$
\pi_t^{P,R} \equiv \frac{\pi_{Ht} - \pi_{Ft}^*}{2} \quad \text{and} \quad \pi_t^{P,W} \equiv \frac{\pi_{Ht} + \pi_{Ft}^*}{2},
$$

which implies $\frac{1}{2}\left(\pi_{Ht}{}^2 + \pi_{Ft}^*{}^2\right) = \left(\pi_t^{P,R}\right)^2 + \left(\pi_t^{P,W}\right)^2$. Therefore, under PCP ($\chi = 0$) the loss function (118) reduces to

$$
\begin{aligned}
\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j L_{t+j}^{PCP} &= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} + \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} f_{t+j}{}^2 \\ + \left(\frac{\xi}{2\delta}\right) \begin{bmatrix} + & \left(\frac{\nu}{2}\right) \pi_{H,t+j}^2 + \left(1 - \frac{\nu}{2}\right) \left(\pi_{H,t+j}^* + \Delta e_{t+j}\right)^2 \\ + & \left(\frac{\nu}{2}\right) \left(\pi_{F,t+j}^*\right)^2 + \left(1 - \frac{\nu}{2}\right) \left(\pi_{F,t+j} - \Delta e_{t+j}\right)^2 \end{bmatrix} \\ + t.i.p. + O(x_{t+j}^3) \end{pmatrix} \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} + \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} f_{t+j}{}^2 \\ + \left(\frac{\xi}{2\delta}\right) \left(\pi_{Ht+j}{}^2 + \pi_{Ft+j}^*{}^2\right) + t.i.p. + O(x_{t+j}^3) \end{pmatrix} \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} + \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} f_{t+j}{}^2 \\ + \left(\frac{\xi}{\delta}\right) \left(\left(\pi_{t+j}^{P,R}\right)^2 + \left(\pi_{t+j}^W\right)^2\right) + t.i.p. + O(x_{t+j}^3) \end{pmatrix},
\end{aligned} \tag{120}
$$

where we use $\Delta e_t = m_t - m_{t-1} + (\nu - 1)\Delta s_t + 2\pi_t^R$, (78), (107), and (108) for the first equality. The second equality follows from the law of one price.

G.2 The 2nd-order Approximation of Price Dispersion Measures

The 2nd-order Approximation of Producer Price Indexes: Observe that PPI equations (26), (27), (31), and (32) are equivalent to

$$1=\int_0^{\chi}\left(\frac{P_{Ht}^L(f)}{P_{Ht}}\right)^{1-\xi}df+\int_{\chi}^{1}\left(\frac{P_{Ht}^P(f)}{P_{Ht}}\right)^{1-\xi}df \quad \text{and} \quad 1=\int_0^{\chi}\left(\frac{P_{Ft}^L(f^*)}{P_{Ft}}\right)^{1-\xi}df^*+\int_{\chi}^{1}\left(\frac{\mathcal{E}_tP_{Ft}^{P*}(f^*)}{P_{Ft}}\right)^{1-\xi}df^*,$$
$$1=\int_0^{\chi}\left(\frac{P_{Ft}^{L*}(f^*)}{P_{Ft}^*}\right)^{1-\xi}df^*+\int_{\chi}^{1}\left(\frac{P_{Ft}^{P*}(f^*)}{P_{Ft}^*}\right)^{1-\xi}df^* \quad \text{and} \quad 1=\int_0^{\chi}\left(\frac{P_{Ht}^{L*}(f)}{P_{Ht}^*}\right)^{1-\xi}df+\int_{\chi}^{1}\left(\frac{\frac{1}{\mathcal{E}_t}P_{Ht}^P(f)}{P_{Ht}^*}\right)^{1-\xi}df.$$

Claim 1. *The second-order approximated producer price indexes (26), (27), (31), and (32) are derived by*

$$\begin{aligned}
\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]df \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]df\end{array}\right) &= \frac{\xi-1}{2}\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]^2df \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]^2df\end{array}\right),\\
\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ft}^L(f^*)\right)-\log\left(P_{Ft}\right)\right]df^* \\ + & \int_{\chi}^{1}\left[\log\left(\mathcal{E}_tP_{Ft}^{P*}(f^*)\right)-\log\left(P_{Ft}\right)\right]df^*\end{array}\right) &= \frac{\xi-1}{2}\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ft}^L(f^*)\right)-\log\left(P_{Ft}\right)\right]^2df^* \\ + & \int_{\chi}^{1}\left[\log\left(\mathcal{E}_tP_{Ft}^{P*}(f^*)\right)-\log\left(P_{Ft}\right)\right]^2df^*\end{array}\right),\\
\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ft}^{L*}(f^*)\right)-\log\left(P_{Ft}^*\right)\right]df^* \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ft}^{P*}(f^*)\right)-\log\left(P_{Ft}^*\right)\right]df^*\end{array}\right) &= \frac{\xi-1}{2}\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ft}^{L*}(f^*)\right)-\log\left(P_{Ft}^*\right)\right]^2df^* \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ft}^{P*}(f^*)\right)-\log\left(P_{Ft}^*\right)\right]^2df^*\end{array}\right),\\
\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^{L*}(f)\right)-\log\left(P_{Ht}^*\right)\right]df \\ + & \int_{\chi}^{1}\left[\log\left(\frac{P_{Ht}^P(f)}{\mathcal{E}_t}\right)-\log\left(P_{Ht}^*\right)\right]df\end{array}\right) &= \frac{\xi-1}{2}\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^{L*}(f)\right)-\log\left(P_{Ht}^*\right)\right]^2df \\ + & \int_{\chi}^{1}\left[\log\left(\frac{P_{Ht}^P(f)}{\mathcal{E}_t}\right)-\log\left(P_{Ht}^*\right)\right]^2df\end{array}\right).
\end{aligned}\tag{121}$$

Proof. From the first price index, observe that

$$\begin{aligned}
1 &= \int_0^{\chi}\left(\frac{P_{Ht}^L(f)}{P_{Ht}}\right)^{1-\xi}df+\int_{\chi}^{1}\left(\frac{P_{Ht}^P(f)}{P_{Ht}}\right)^{1-\xi}df\\
&= \int_0^{\chi}\exp\left((1-\xi)\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]\right)df+\int_{\chi}^{1}\exp\left((1-\xi)\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]\right)df\\
&\approx \left(\begin{array}{cc} & \int_0^{\chi}\left(1+(1-\xi)\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]+\frac{(1-\xi)^2}{2}\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]^2\right)df \\ + & \int_{\chi}^{1}\left(1+(1-\xi)\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]+\frac{(1-\xi)^2}{2}\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]^2\right)df\end{array}\right),
\end{aligned}$$

where we use Taylor expansion up to the second order. Rearranging terms in the third line leads to

$$\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]df \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]df\end{array}\right)=\frac{\xi-1}{2}\left(\begin{array}{cc} & \int_0^{\chi}\left[\log\left(P_{Ht}^L(f)\right)-\log\left(P_{Ht}\right)\right]^2df \\ + & \int_{\chi}^{1}\left[\log\left(P_{Ht}^P(f)\right)-\log\left(P_{Ht}\right)\right]^2df\end{array}\right),$$

which completes the proof. Derivations for the rest follow the same procedure. □

Note that equations in (121) can be rewritten as

$$\begin{aligned}
p_{Ht} &= \hat{p}_{Ht}+\frac{(1-\xi)}{2}\int_0^{\chi}\left[\log\left(P_{Ht}^L(f)\right)-p_{Ht}\right]^2df+\frac{(1-\xi)}{2}\int_{\chi}^{1}\left[\log\left(P_{Ht}^P(f)\right)-p_{Ht}\right]^2df,\\
p_{Ft} &= \hat{p}_{Ft}+\frac{(1-\xi)}{2}\int_0^{\chi}\left[\log\left(P_{Ft}^L(f^*)\right)-p_{Ft}\right]^2df^*+\frac{(1-\xi)}{2}\int_{\chi}^{1}\left[\log\left(\mathcal{E}_tP_{Ft}^{P*}(f^*)\right)-p_{Ft}\right]^2df^*,\\
p_{Ft}^* &= \hat{p}_{Ft}^*+\frac{(1-\xi)}{2}\int_0^{\chi}\left[\log\left(P_{Ft}^{L*}(f^*)\right)-p_{Ft}^*\right]^2df^*+\frac{(1-\xi)}{2}\int_{\chi}^{1}\left[\log\left(P_{Ft}^{P*}(f^*)\right)-p_{Ft}^*\right]^2df^*,\\
p_{Ht}^* &= \hat{p}_{Ht}^*+\frac{(1-\xi)}{2}\int_0^{\chi}\left[\log\left(P_{Ht}^{L*}(f)\right)-p_{Ht}^*\right]^2df+\frac{(1-\xi)}{2}\int_{\chi}^{1}\left[\log\left(\frac{P_{Ht}^P(f)}{\mathcal{E}_t}\right)-p_{Ht}^*\right]^2df,
\end{aligned}\tag{122}$$

where average prices $(\hat{p}_{Ht}, \hat{p}_{Ft}, \hat{p}^*_{Ft}, \hat{p}^*_{Ht})$ are defined in (127). (122) implies, for example, $(\hat{p}_{Ht} - x_t)^2 = (p_{Ht} - x_t)^2$ holds for some variable x_t up to the second order.

Price Dispersion Measures: We define price dispersion terms as

$$\begin{aligned}
\sigma^2_{P_H,t} &\equiv \begin{pmatrix} & \int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \int_0^\chi \left[\log\left(P^L_{Ht}(f)\right)\right] df - \int_\chi^1 \left[\log\left(P^P_{Ht}(f)\right)\right] df\right\}^2 df \\ + & \int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \int_0^\chi \left[\log\left(P^L_{Ht}(f)\right)\right] df - \int_\chi^1 \left[\log\left(P^P_{Ht}(f)\right)\right] df\right\}^2 df \end{pmatrix}, \\
\sigma^2_{P_F,t} &\equiv \begin{pmatrix} & \int_0^\chi \left\{\log\left(P^L_{Ft}(f^*)\right) - \int_0^\chi \left[\log\left(P^L_{Ft}(f^*)\right)\right] df^* - \int_\chi^1 \left[\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right)\right] df^*\right\}^2 df^* \\ + & \int_\chi^1 \left\{\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - \int_0^\chi \left[\log\left(P^L_{Ft}(f^*)\right)\right] df^* - \int_\chi^1 \left[\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right)\right] df^*\right\}^2 df^* \end{pmatrix}, \\
\sigma^2_{P^*_F,t} &\equiv \begin{pmatrix} & \int_0^\chi \left\{\log\left(P^{L*}_{Ft}(f^*)\right) - \int_0^\chi \left[\log\left(P^{L*}_{Ft}(f^*)\right)\right] df^* - \int_\chi^1 \left[\log\left(P^{P*}_{Ft}(f^*)\right)\right] df^*\right\}^2 df^* \\ + & \int_\chi^1 \left\{\log\left(P^{P*}_{Ft}(f^*)\right) - \int_0^\chi \left[\log\left(P^{L*}_{Ft}(f^*)\right)\right] df^* - \int_\chi^1 \left[\log\left(P^{P*}_{Ft}(f^*)\right)\right] df^*\right\}^2 df^* \end{pmatrix}, \\
\sigma^2_{P^*_H,t} &\equiv \begin{pmatrix} & \int_0^\chi \left\{\log\left(P^{L*}_{Ht}(f)\right) - \int_0^\chi \left[\log\left(P^{L*}_{Ht}(f)\right)\right] df - \int_\chi^1 \left[\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right)\right] df\right\}^2 df \\ + & \int_\chi^1 \left\{\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) - \int_0^\chi \left[\log\left(P^{L*}_{Ht}(f)\right)\right] df - \int_\chi^1 \left[\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right)\right] df\right\}^2 df \end{pmatrix}.
\end{aligned} \tag{123}$$

Feeding (122) into (123), we can rewrite price dispersion terms as

$$\begin{aligned}
\sigma^2_{P_H,t} &= \int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \log\left(P_{Ht}\right)\right\}^2 df + \int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \log\left(P_{Ht}\right)\right\}^2 df, \\
\sigma^2_{P_F,t} &= \int_0^\chi \left\{\log\left(P^L_{Ft}(f^*)\right) - \log\left(P_{Ft}\right)\right\}^2 df^* + \int_\chi^1 \left\{\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - \log\left(P_{Ft}\right)\right\}^2 df^*, \\
\sigma^2_{P^*_F,t} &= \int_0^\chi \left\{\log\left(P^{L*}_{Ft}(f^*)\right) - \log\left(P^*_{Ft}\right)\right\}^2 df^* + \int_\chi^1 \left\{\log\left(P^{P*}_{Ft}(f^*)\right) - \log\left(P^*_{Ft}\right)\right\}^2 df^*, \\
\sigma^2_{P^*_H,t} &= \int_0^\chi \left\{\log\left(P^{L*}_{Ht}(f)\right) - \log\left(P^*_{Ht}\right)\right\}^2 df^* + \int_\chi^1 \left\{\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) - \log\left(P^*_{Ht}\right)\right\}^2 df^*,
\end{aligned} \tag{124}$$

which are also approximated up to the second order. Under PCP ($\chi = 0$), the law of one price implies $P_{Ht}(f) = \mathcal{E}_t P^*_{Ht}(f)$, $P^*_{Ft}(f^*) = \frac{P_{Ft}(f^*)}{\mathcal{E}_t}$, $P_{Ht} = \mathcal{E}_t P^*_{Ht}$, and $P^*_{Ft} = \frac{P_{Ft}}{\mathcal{E}_t}$. Therefore, under PCP those four relations in (124) reduce to two equations due to $\sigma^2_{P^*_H,t} = \sigma^2_{P_H,t}$ and $\sigma^2_{P_F,t} = \sigma^2_{P^*_F,t}$.

Note that we will conduct the second-order approximation to market clearing conditions for Home and Foreign tradeable goods (24, 25) given by

$$\begin{aligned}
A_t N_t &= C_{Ht}\left(\chi V^L_{Ht} + (1-\chi) V^P_{Ht}\right) + C^*_{Ht}\left(\chi V^{L*}_{Ht} + (1-\chi) V^{P*}_{Ht}\right), \\
A^*_t N^*_t &= C^*_{Ft}\left(\chi V^{L*}_{Ft} + (1-\chi) V^{P*}_{Ft}\right) + C_{Ft}\left(\chi V^L_{Ft} + (1-\chi) V^P_{Ft}\right),
\end{aligned}$$

where the weighted sum of price dispersion terms are

$$\begin{aligned}
\chi V^L_{Ht} + (1-\chi) V^P_{Ht} &= \int_0^\chi \left(\frac{P^L_{Ht}(f)}{P_{Ht}}\right)^{-\xi} df + \int_\chi^1 \left(\frac{P^P_{Ht}(f)}{P_{Ht}}\right)^{-\xi} df, \\
\chi V^{L*}_{Ht} + (1-\chi) V^{P*}_{Ht} &= \int_0^\chi \left(\frac{P^{L*}_{Ht}(f)}{P^*_{Ht}}\right)^{-\xi} df + \int_\chi^1 \left(\frac{\frac{1}{\mathcal{E}_t} P^P_{Ht}(f)}{P^*_{Ht}}\right)^{-\xi} df, \\
\chi V^{L*}_{Ft} + (1-\chi) V^{P*}_{Ft} &= \int_0^\chi \left(\frac{P^{L*}_{Ft}(f^*)}{P^*_{Ft}}\right)^{-\xi} df^* + \int_\chi^1 \left(\frac{P^{P*}_{Ft}(f^*)}{P^*_{Ft}}\right)^{-\xi} df^*, \\
\chi V^L_{Ft} + (1-\chi) V^P_{Ft} &= \int_0^\chi \left(\frac{P^L_{Ft}(f^*)}{P_{Ft}}\right)^{-\xi} df^* + \int_\chi^1 \left(\frac{\mathcal{E}_t P^{P*}_{Ft}(f^*)}{P_{Ft}}\right)^{-\xi} df^*.
\end{aligned} \tag{125}$$

For this reason, claim 2 displays the second-order approximation to the weighted sum of price dispersion terms (125).

Claim 2.

$$\log\left(\chi V^{L}_{Ht}+(1-\chi)V^{P}_{Ht}\right)=\tfrac{\xi}{2}\sigma^2_{P_H,t}\quad and\quad \log\left(\chi V^{L*}_{Ht}+(1-\chi)V^{P*}_{Ht}\right)=\tfrac{\xi}{2}\sigma^2_{P^*_H,t},$$
$$\log\left(\chi V^{L*}_{Ft}+(1-\chi)V^{P*}_{Ft}\right)=\tfrac{\xi}{2}\sigma^2_{P^*_F,t}\quad and\quad \log\left(\chi V^{L}_{Ft}+(1-\chi)V^{P}_{Ft}\right)=\tfrac{\xi}{2}\sigma^2_{P_F,t}. \tag{126}$$

Proof. For the proof of the first one, we have

$$\log\left(\chi V^{L}_{Ht}+(1-\chi)V^{P}_{Ht}\right)=\log\left(\int_0^{\chi}\left(\frac{P^L_{Ht}(f)}{P_{Ht}}\right)^{-\xi}df+\int_{\chi}^{1}\left(\frac{P^P_{Ht}(f)}{P_{Ht}}\right)^{-\xi}df\right)$$

from (125). Taylor expansion of the term in the right-hand side up to the second order implies

$$\begin{aligned}
&\textstyle\int_0^{\chi}\left(\frac{P^L_{Ht}(f)}{P_{Ht}}\right)^{-\xi}df+\int_{\chi}^{1}\left(\frac{P^P_{Ht}(f)}{P_{Ht}}\right)^{-\xi}df\\
=\;&\textstyle\int_0^{\chi}\exp\left((-\xi)\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]\right)df+\int_{\chi}^{1}\exp\left((-\xi)\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]\right)df\\
\approx\;&\left(\begin{array}{cl} & \int_0^{\chi}\left(1-\xi\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]+\frac{\xi^2}{2}\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2\right)df\\ + & \int_{\chi}^{1}\left(1-\xi\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]+\frac{\xi^2}{2}\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2\right)df\end{array}\right).
\end{aligned}$$

Therefore, using $\log(1+x)\approx x$ for $x\approx 0$ we obtain

$$\begin{aligned}
\log\left(\chi V^{L}_{Ht}+(1-\chi)V^{P}_{Ht}\right)\;&\approx\left(\begin{array}{l}-\xi\left(\int_0^{\chi}\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]df+\int_{\chi}^{1}\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]df\right)\\ +\frac{\xi^2}{2}\left(\int_0^{\chi}\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df+\int_{\chi}^{1}\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df\right)\end{array}\right)\\
&\approx\left(\begin{array}{l}\frac{-\xi^2+\xi}{2}\left(\int_0^{\chi}\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df+\int_{\chi}^{1}\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df\right)\\ +\frac{\xi^2}{2}\left(\int_0^{\chi}\left[\log\left(P^L_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df+\int_{\chi}^{1}\left[\log\left(P^P_{Ht}(f)\right)-\log\left(P_{Ht}\right)\right]^2df\right)\end{array}\right)\\
&\approx\tfrac{\xi}{2}\sigma^2_{P_H,t},
\end{aligned}$$

where the second and third equalities follow from (121) and (124), respectively. Derivations for the rest follow the same procedure. □

G.3 Price Dispersion and Inflation

In this section, we show steps to transform price dispersion terms into inflation following Woodford (2003). Recall in section A we define inflation measures ($\Pi_{Ht},\Pi_{Ft},\Pi^*_{Ft},\Pi^*_{Ht}$), PPI-deflated prices of LCP Home goods ($\Pi^L_{Ht},\Pi^{L*}_{Ht},\Pi^{L,o}_{Ht},\Pi^{L,o*}_{Ht}$), PPI-deflated prices of PCP Home goods ($\Pi^P_{Ht},\Pi^{P,o}_{Ht}$), PPI-deflated prices of LCP Foreign goods ($\Pi^{L*}_{Ft},\Pi^L_{Ft},\Pi^{L,o*}_{Ft},\Pi^{L,o}_{Ft}$), and PPI-deflated prices of PCP Foreign goods ($\Pi^{P*}_{Ft},\Pi^{P,o*}_{Ft}$) as

$$\begin{array}{llllll}
\Pi_{Ht}\equiv\frac{P_{Ht}}{P_{H,t-1}}, & \Pi_{Ft}\equiv\frac{P_{Ft}}{P_{F,t-1}}, & \Pi^*_{Ft}\equiv\frac{P^*_{Ft}}{P^*_{F,t-1}}, & \Pi^*_{Ht}\equiv\frac{P^*_{Ht}}{P^*_{H,t-1}}, & & \\
\Pi^L_{Ht}\equiv\frac{P^L_{Ht}}{P_{Ht}}, & \Pi^{L*}_{Ht}\equiv\frac{P^{L*}_{Ht}}{P^*_{Ht}}, & \Pi^P_{Ht}\equiv\frac{P^P_{Ht}}{P_{Ht}}, & \Pi^{L,o}_{Ht}\equiv\frac{P^{L,o}_{Ht}}{P_{Ht}}, & \Pi^{L,o*}_{Ht}\equiv\frac{P^{L,o*}_{Ht}}{P^*_{Ht}}, & \Pi^{P,o}_{Ht}\equiv\frac{P^{P,o}_{Ht}}{P_{Ht}},\\
\Pi^{L*}_{Ft}\equiv\frac{P^{L*}_{Ft}}{P^*_{Ft}}, & \Pi^L_{Ft}\equiv\frac{P^L_{Ft}}{P_{Ft}}, & \Pi^{P*}_{Ft}\equiv\frac{P^{P*}_{Ft}}{P^*_{Ft}}, & \Pi^{L,o*}_{Ft}\equiv\frac{P^{L,o*}_{Ft}}{P^*_{Ft}}, & \Pi^{L,o}_{Ft}\equiv\frac{P^{L,o}_{Ft}}{P_{Ft}}, & \Pi^{P,o*}_{Ft}\equiv\frac{P^{P,o*}_{Ft}}{P^*_{Ft}}.
\end{array}$$

Average Prices of Differentiated Varieties: We define average prices of differentiated goods in (127). There are four average prices: the average price of Home goods in Home market ($\hat{p}_{Ht}$), the average price of Home goods in Foreign market ($\hat{p}^*_{Ht}$), the average price of Foreign goods in Foreign market ($\hat{p}^*_{Ft}$), and the average price of Foreign goods in Home market ($\hat{p}_{Ft}$), given by

$$
\begin{aligned}
\hat{p}_{Ht} &\equiv \int_0^\chi \log\left(P^L_{Ht}(f)\right) df + \int_\chi^1 \log\left(P^P_{Ht}(f)\right) df, \\
\hat{p}^*_{Ht} &\equiv \int_0^\chi \log\left(P^{L*}_{Ht}(f)\right) df + \int_\chi^1 \log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) df, \\
\hat{p}^*_{Ft} &\equiv \int_0^\chi \log\left(P^{L*}_{Ft}(f^*)\right) df^* + \int_\chi^1 \log\left(P^{P*}_{Ft}(f^*)\right) df^*, \\
\hat{p}_{Ft} &\equiv \int_0^\chi \log\left(P^L_{Ft}(f^*)\right) df^* + \int_\chi^1 \log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) df^*.
\end{aligned}
\tag{127}
$$

Under Calvo sticky prices, a fraction θ of firms is not allowed to change prices while a fraction $(1-\theta)$ of firms resets their prices. Therefore, we can derive recursive equations for these average prices. The recursive formula for the average price of Home goods in Home market can be derived by

$$
\hat{p}_{Ht} = \theta \hat{p}_{H,t-1} + (1-\theta)\chi p^{L,o}_{H,t} + (1-\theta)(1-\chi) p^{P,o}_{H,t}. \tag{128}
$$

Likewise, the recursive formula for the average price of Home goods in Foreign market can be derived by

$$
\hat{p}^*_{Ht} = \theta \hat{p}^*_{H,t-1} - \theta(1-\chi)(e_t - e_{t-1}) + (1-\theta)\chi p^{L,o*}_{H,t} + (1-\theta)(1-\chi)\left(p^{P,o}_{H,t} - e_t\right). \tag{129}
$$

For the average price of Foreign goods in Foreign market, we obtain

$$
\hat{p}^*_{Ft} = \theta \hat{p}^*_{F,t-1} + (1-\theta)\chi p^{L,o*}_{F,t} + (1-\theta)(1-\chi) p^{P,o*}_{F,t}. \tag{130}
$$

Finally the recursive equation for the average price of Foreign goods in Home market is given by

$$
\hat{p}_{Ft} = \theta \hat{p}_{F,t-1} + \theta(1-\chi)(e_t - e_{t-1}) + (1-\theta)\chi p^{L,o}_{F,t} + (1-\theta)(1-\chi)\left(e_t + p^{P,o*}_{F,t}\right). \tag{131}
$$

Producer Price Index for Home goods in Home, P_{Ht}: From (26), (28), and (30), recall producer price indexes for LCP and PCP Home products in Home given by

$$
\begin{aligned}
1 &= \chi\left(\frac{P^L_{Ht}}{P_{Ht}}\right)^{1-\xi} + (1-\chi)\left(\frac{P^P_{Ht}}{P_{Ht}}\right)^{1-\xi} &&= \int_0^\chi \left(\frac{P^L_{Ht}(f)}{P_{Ht}}\right)^{1-\xi} df + \int_\chi^1 \left(\frac{P^P_{Ht}(f)}{P_{Ht}}\right)^{1-\xi} df, \\
1 &\equiv \frac{1}{\chi}\int_0^\chi \left(\frac{P^L_{Ht}(f)}{P^L_{Ht}}\right)^{1-\xi} df &&= \frac{\theta}{\chi}\int_0^\chi \left(\frac{P^L_{H,t-1}(f)}{P^L_{Ht}}\right)^{1-\xi} df + (1-\theta)\left(\frac{P^{L,o}_{Ht}}{P^L_{Ht}}\right)^{1-\xi}, \\
1 &\equiv \frac{1}{1-\chi}\int_\chi^1 \left(\frac{P^P_{Ht}(f)}{P^P_{Ht}}\right)^{1-\xi} df &&= \frac{\theta}{1-\chi}\int_\chi^1 \left(\frac{P^P_{H,t-1}(f)}{P^P_{Ht}}\right)^{1-\xi} df + (1-\theta)\left(\frac{P^{P,o}_{Ht}}{P^P_{Ht}}\right)^{1-\xi}.
\end{aligned}
$$

The second-order approximation of these three equations implies

$$\begin{aligned} p_{Ht} &= \chi p^L_{Ht} + (1-\chi) p^P_{Ht} + \tfrac{\chi(1-\xi)}{2}\left(p^L_{Ht} - p_{Ht}\right)^2 + \tfrac{(1-\chi)(1-\xi)}{2}\left(p^P_{Ht} - p_{Ht}\right)^2 \\ &= \hat{p}_{Ht} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{Ht}(f)\right) - p_{Ht}\right)^2 df + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^P_{Ht}(f)\right) - p_{Ht}\right)^2 df, \end{aligned} \tag{132}$$

$$\begin{aligned} p^L_{Ht} &= \tfrac{1}{\chi}\int_0^\chi \log\left(P^L_{Ht}(f)\right) df + \tfrac{(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{Ht}(f)\right) - p^L_{Ht}\right)^2 df \\ &= \left(\begin{array}{ll} & \frac{\theta}{\chi}\int_0^\chi \left(\log\left(P^L_{H,t-1}(f)\right)\right) df + (1-\theta) p^{L,o}_{Ht} \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{H,t-1}(f)\right) - p^L_{Ht}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o}_{Ht} - p^L_{Ht}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p^L_{H,t-1} + (1-\theta) p^{L,o}_{Ht} \\ - & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{H,t-1}(f)\right) - p^L_{H,t-1}\right)^2 df \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{H,t-1}(f)\right) - p^L_{Ht}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o}_{Ht} - p^L_{Ht}\right)^2 \end{array}\right), \end{aligned} \tag{133}$$

$$\begin{aligned} p^P_{Ht} &= \tfrac{1}{1-\chi}\int_\chi^1 \left(\log\left(P^P_{Ht}(f)\right)\right) df + \tfrac{(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^P_{Ht}(f)\right) - p^P_{Ht}\right)^2 df \\ &= \left(\begin{array}{ll} & \frac{\theta}{1-\chi}\int_\chi^1 \left(\log\left(P^P_{H,t-1}(f)\right)\right) df + (1-\theta) p^{P,o}_{Ht} \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^P_{H,t-1}(f)\right) - p^P_{Ht}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o}_{Ht} - p^P_{Ht}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p^P_{H,t-1} + (1-\theta) p^{P,o}_{Ht} \\ - & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^P_{H,t-1}(f)\right) - p^P_{H,t-1}\right)^2 df \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^P_{H,t-1}(f)\right) - p^P_{Ht}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o}_{Ht} - p^P_{Ht}\right)^2 \end{array}\right). \end{aligned} \tag{134}$$

Together (133) and (134) imply

$$\begin{aligned} & \chi p^L_{Ht} + (1-\chi) p^P_{Ht} \\ = \; & \hat{p}_{Ht} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{Ht}(f)\right) - p^L_{Ht}\right)^2 df + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^P_{Ht}(f)\right) - p^P_{Ht}\right)^2 df \\ = \; & \left(\begin{array}{ll} & \theta \hat{p}_{H,t-1} + (1-\theta)\left(\chi p^{L,o}_{Ht} + (1-\chi) p^{P,o}_{Ht}\right) \\ + & \frac{\theta(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{H,t-1}(f)\right) - p^L_{Ht}\right)^2 df + \frac{\theta(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^P_{H,t-1}(f)\right) - p^P_{Ht}\right)^2 df \\ + & \frac{(1-\theta)(1-\xi)}{2}\chi\left(p^{L,o}_{Ht} - p^L_{Ht}\right)^2 + \frac{(1-\theta)(1-\xi)}{2}(1-\chi)\left(p^{P,o}_{Ht} - p^P_{Ht}\right)^2 \end{array}\right) \end{aligned} \tag{135}$$

Producer Price Index for Home goods in Foreign, P^*_{Ht}: From (27), (29), and (30), recall producer price indexes for LCP and PCP Home products in Foreign given by

$$\begin{aligned} 1 &= \chi\left(\tfrac{P^{L*}_{Ht}}{P^*_{Ht}}\right)^{1-\xi} + (1-\chi)\left(\tfrac{P^P_{Ht}}{\mathcal{E}_t P^*_{Ht}}\right)^{1-\xi} && = \int_0^\chi \left(\tfrac{P^{L*}_{Ht}(f)}{P^*_{Ht}}\right)^{1-\xi} df + \int_\chi^1 \left(\tfrac{P^P_{Ht}(f)}{\mathcal{E}_t P^*_{Ht}}\right)^{1-\xi} df, \\ 1 &\equiv \tfrac{1}{\chi}\int_0^\chi \left(\tfrac{P^{L*}_{Ht}(f)}{P^{L*}_{Ht}}\right)^{1-\xi} df && = \tfrac{\theta}{\chi}\int_0^\chi \left(\tfrac{P^{L*}_{H,t-1}(f)}{P^{L*}_{Ht}}\right)^{1-\xi} df + (1-\theta)\left(\tfrac{P^{L,o*}_{Ht}}{P^{L*}_{Ht}}\right)^{1-\xi}, \\ 1 &\equiv \tfrac{1}{1-\chi}\int_\chi^1 \left(\tfrac{P^P_{Ht}(f)}{P^P_{Ht}}\right)^{1-\xi} df && = \tfrac{\theta}{1-\chi}\int_\chi^1 \left(\tfrac{P^P_{H,t-1}(f)}{P^P_{Ht}}\right)^{1-\xi} df + (1-\theta)\left(\tfrac{P^{P,o}_{Ht}}{P^P_{Ht}}\right)^{1-\xi}. \end{aligned}$$

The second-order approximation of these three equations implies

$$\begin{aligned} p_{Ht}^{*} &= \chi p_{Ht}^{L*} + (1-\chi)\left(p_{Ht}^{P} - e_t\right) + \tfrac{\chi(1-\xi)}{2}\left(p_{Ht}^{L*} - p_{Ht}^{*}\right)^2 + \tfrac{(1-\chi)(1-\xi)}{2}\left(p_{Ht}^{P} - e_t - p_{Ht}^{*}\right)^2 \\ &= \hat{p}_{Ht}^{*} + \tfrac{(1-\xi)}{2}\int_0^{\chi}\left(\log\left(P_{Ht}^{L*}(f)\right) - p_{Ht}^{*}\right)^2 df + \tfrac{(1-\xi)}{2}\int_{\chi}^{1}\left(\log\left(P_{Ht}^{P}(f)\right) - e_t - p_{Ht}^{*}\right)^2 df, \end{aligned} \tag{136}$$

$$\begin{aligned} p_{Ht}^{L*} &= \tfrac{1}{\chi}\int_0^{\chi}\log\left(P_{Ht}^{L*}(f)\right) df + \tfrac{(1-\xi)}{2\chi}\int_0^{\chi}\left(\log\left(P_{Ht}^{L*}(f)\right) - p_{Ht}^{L*}\right)^2 df \\ &= \left(\begin{array}{ll} & \frac{\theta}{\chi}\int_0^{\chi}\left(\log\left(P_{H,t-1}^{L*}(f)\right)\right) df + (1-\theta)p_{Ht}^{L,o*} \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^{\chi}\left(\log\left(P_{H,t-1}^{L*}(f)\right) - p_{Ht}^{L*}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p_{Ht}^{L,o*} - p_{Ht}^{L*}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p_{H,t-1}^{L*} + (1-\theta)p_{Ht}^{L,o*} \\ - & \frac{\theta(1-\xi)}{2\chi}\int_0^{\chi}\left(\log\left(P_{H,t-1}^{L*}(f)\right) - p_{H,t-1}^{L*}\right)^2 df \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^{\chi}\left(\log\left(P_{H,t-1}^{L*}(f)\right) - p_{Ht}^{L*}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p_{Ht}^{L,o*} - p_{Ht}^{L*}\right)^2 \end{array}\right), \end{aligned} \tag{137}$$

$$\begin{aligned} p_{Ht}^{P} &= \tfrac{1}{1-\chi}\int_{\chi}^{1}\left(\log\left(P_{Ht}^{P}(f)\right)\right) df + \tfrac{(1-\xi)}{2(1-\chi)}\int_{\chi}^{1}\left(\log\left(P_{Ht}^{P}(f)\right) - p_{Ht}^{P}\right)^2 df \\ &= \left(\begin{array}{ll} & \frac{\theta}{1-\chi}\int_{\chi}^{1}\left(\log\left(P_{H,t-1}^{P}(f)\right)\right) df + (1-\theta)p_{Ht}^{P,o} \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_{\chi}^{1}\left(\log\left(P_{H,t-1}^{P}(f)\right) - p_{Ht}^{P}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p_{Ht}^{P,o} - p_{Ht}^{P}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p_{H,t-1}^{P} + (1-\theta)p_{Ht}^{P,o} \\ - & \frac{\theta(1-\xi)}{2(1-\chi)}\int_{\chi}^{1}\left(\log\left(P_{H,t-1}^{P}(f)\right) - p_{H,t-1}^{P}\right)^2 df \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_{\chi}^{1}\left(\log\left(P_{H,t-1}^{P}(f)\right) - p_{Ht}^{P}\right)^2 df + \frac{(1-\theta)(1-\xi)}{2}\left(p_{Ht}^{P,o} - p_{Ht}^{P}\right)^2 \end{array}\right). \end{aligned} \tag{138}$$

Together (137) and (138) imply

$$\begin{aligned} & \chi p_{Ht}^{L*} + (1-\chi)(p_{Ht}^{P} - e_t) \\ = \; & \hat{p}_{Ht}^{*} + \tfrac{(1-\xi)}{2}\int_0^{\chi}\left(\log\left(P_{Ht}^{L*}(f)\right) - p_{Ht}^{L*}\right)^2 df + \tfrac{(1-\xi)}{2}\int_{\chi}^{1}\left(\log\left(P_{Ht}^{P}(f)\right) - p_{Ht}^{P}\right)^2 df \\ = \; & \left(\begin{array}{ll} & \theta\hat{p}_{H,t-1}^{*} + (1-\theta)\left(\chi p_{Ht}^{L,o*} + (1-\chi)(p_{Ht}^{P,o} - e_t)\right) - \theta(1-\chi)\left(e_t - e_{t-1}\right) \\ + & \frac{\theta(1-\xi)}{2}\int_0^{\chi}\left(\log\left(P_{H,t-1}^{L*}(f)\right) - p_{Ht}^{L*}\right)^2 df + \frac{\theta(1-\xi)}{2}\int_{\chi}^{1}\left(\log\left(P_{H,t-1}^{P}(f)\right) - p_{Ht}^{P}\right)^2 df \\ + & \frac{(1-\theta)(1-\xi)}{2}\chi\left(p_{Ht}^{L,o*} - p_{Ht}^{L*}\right)^2 + \frac{(1-\theta)(1-\xi)}{2}(1-\chi)\left(p_{Ht}^{P,o} - p_{Ht}^{P}\right)^2 \end{array}\right) \end{aligned} \tag{139}$$

Producer Price Index for Foreign goods in Foreign, P_{Ft}^{*}: From (31), (33), and (35), recall producer price indexes for LCP and PCP Foreign products in Foreign given by

$$\begin{aligned} 1 &= \chi\left(\frac{P_{Ft}^{L*}}{P_{Ft}^{*}}\right)^{1-\xi} + (1-\chi)\left(\frac{P_{Ft}^{P*}}{P_{Ft}^{*}}\right)^{1-\xi} && = \int_0^{\chi}\left(\frac{P_{Ft}^{L*}(f^*)}{P_{Ft}^{*}}\right)^{1-\xi} df^* + \int_{\chi}^{1}\left(\frac{P_{Ft}^{P*}(f^*)}{P_{Ft}^{*}}\right)^{1-\xi} df^*, \\ 1 &\equiv \frac{1}{\chi}\int_0^{\chi}\left(\frac{P_{Ft}^{L*}(f^*)}{P_{Ft}^{L*}}\right)^{1-\xi} df^* && = \frac{\theta}{\chi}\int_0^{\chi}\left(\frac{P_{F,t-1}^{L*}(f^*)}{P_{Ft}^{L*}}\right)^{1-\xi} df^* + (1-\theta)\left(\frac{P_{Ft}^{L,o*}}{P_{Ft}^{L*}}\right)^{1-\xi}, \\ 1 &\equiv \frac{1}{1-\chi}\int_{\chi}^{1}\left(\frac{P_{Ft}^{P*}(f^*)}{P_{Ft}^{P*}}\right)^{1-\xi} df^* && = \frac{\theta}{1-\chi}\int_{\chi}^{1}\left(\frac{P_{F,t-1}^{P*}(f^*)}{P_{Ft}^{P*}}\right)^{1-\xi} df^* + (1-\theta)\left(\frac{P_{Ft}^{P,o*}}{P_{Ft}^{P*}}\right)^{1-\xi}. \end{aligned}$$

The second-order approximation of these three equations implies

$$\begin{aligned}
p^*_{Ft} &= \chi p^{L*}_{Ft} + (1-\chi)p^{P*}_{Ft} + \tfrac{\chi(1-\xi)}{2}\left(p^{L*}_{Ft} - p^*_{Ft}\right)^2 + \tfrac{(1-\chi)(1-\xi)}{2}\left(p^{P*}_{Ft} - p^*_{Ft}\right)^2 \\
&= \hat{p}^*_{Ft} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^{L*}_{Ft}(f^*)\right) - p^*_{Ft}\right)^2 df^* + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right) - p^*_{Ft}\right)^2 df^*,
\end{aligned} \tag{140}$$

$$\begin{aligned}
p^{L*}_{Ft} &= \tfrac{1}{\chi}\int_0^\chi \left(\log\left(P^{L*}_{Ft}(f^*)\right)\right) df^* + \tfrac{(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^{L*}_{Ft}(f^*)\right) - p^{L*}_{Ft}\right)^2 df^* \\
&= \left(\begin{array}{ll} & \frac{\theta}{\chi}\int_0^\chi \left(\log\left(P^{L*}_{F,t-1}(f^*)\right)\right) df^* + (1-\theta)p^{L,o*}_{Ft} \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^{L*}_{F,t-1}(f^*)\right) - p^{L*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o*}_{Ft} - p^{L*}_{Ft}\right)^2 \end{array}\right) \\
&= \left(\begin{array}{ll} & \theta p^{L*}_{F,t-1} + (1-\theta)p^{L,o*}_{Ft} \\ - & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^{L*}_{F,t-1}(f^*)\right) - p^{L*}_{F,t-1}\right)^2 df^* \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^{L*}_{F,t-1}(f^*)\right) - p^{L*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o*}_{Ft} - p^{L*}_{Ft}\right)^2 \end{array}\right),
\end{aligned} \tag{141}$$

$$\begin{aligned}
p^{P*}_{Ft} &= \tfrac{1}{1-\chi}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right)\right) df^* + \tfrac{(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\
&= \left(\begin{array}{ll} & \frac{\theta}{1-\chi}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right)\right) df^* + (1-\theta)p^{P,o*}_{Ft} \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right) \\
&= \left(\begin{array}{ll} & \theta p^{P*}_{F,t-1} + (1-\theta)p^{P,o*}_{Ft} \\ - & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{F,t-1}\right)^2 df^* \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right).
\end{aligned} \tag{142}$$

Together (141) and (142) imply

$$\begin{aligned}
& \chi p^{L*}_{Ft} + (1-\chi)p^{P*}_{Ft} \\
= \ & \hat{p}^*_{Ft} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^{L*}_{Ft}(f^*)\right) - p^{L*}_{Ft}\right)^2 df^* + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\
= \ & \left(\begin{array}{ll} & \theta\hat{p}^*_{F,t-1} + (1-\theta)\left(\chi p^{L,o*}_{Ft} + (1-\chi)p^{P,o*}_{Ft}\right) \\ + & \frac{\theta(1-\xi)}{2}\int_0^\chi \left(\log\left(P^{L*}_{F,t-1}(f^*)\right) - p^{L*}_{Ft}\right)^2 df^* + \frac{\theta(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\ + & \frac{(1-\theta)(1-\xi)}{2}\chi\left(p^{L,o*}_{Ft} - p^{L*}_{Ft}\right)^2 + \frac{(1-\theta)(1-\xi)}{2}(1-\chi)\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right)
\end{aligned} \tag{143}$$

Producer Price Index for Foreign goods in Home, P_{Ft}: From (32), (34), and (35), recall producer price indexes for LCP and PCP Foreign products in Home given by

$$\begin{aligned}
1 &= \chi\left(\frac{P^L_{Ft}}{P_{Ft}}\right)^{1-\xi} + (1-\chi)\left(\frac{\mathcal{E}_t P^{P*}_{Ft}}{P_{Ft}}\right)^{1-\xi} && = \int_0^\chi \left(\frac{P^L_{Ft}(f^*)}{P_{Ft}}\right)^{1-\xi} df^* + \int_\chi^1 \left(\frac{\mathcal{E}_t P^{P*}_{Ft}(f^*)}{P_{Ft}}\right)^{1-\xi} df^*, \\
1 &\equiv \frac{1}{\chi}\int_0^\chi \left(\frac{P^L_{Ft}(f^*)}{P^L_{Ft}}\right)^{1-\xi} df^* && = \frac{\theta}{\chi}\int_0^\chi \left(\frac{P^L_{F,t-1}(f^*)}{P^L_{Ft}}\right)^{1-\xi} df^* + (1-\theta)\left(\frac{P^{L,o}_{Ft}}{P^L_{Ft}}\right)^{1-\xi}, \\
1 &\equiv \frac{1}{1-\chi}\int_\chi^1 \left(\frac{P^{P*}_{Ft}(f^*)}{P^{P*}_{Ft}}\right)^{1-\xi} df^* && = \frac{\theta}{1-\chi}\int_\chi^1 \left(\frac{P^{P*}_{F,t-1}(f^*)}{P^{P*}_{Ft}}\right)^{1-\xi} df^* + (1-\theta)\left(\frac{P^{P,o*}_{Ft}}{P^{P*}_{Ft}}\right)^{1-\xi}.
\end{aligned}$$

The second-order approximation of these three equations implies

$$\begin{aligned} p_{Ft} &= \chi p^L_{Ft} + (1-\chi)\left(e_t + p^{P*}_{Ft}\right) + \tfrac{\chi(1-\xi)}{2}\left(p^L_{Ft} - p_{Ft}\right)^2 + \tfrac{(1-\chi)(1-\xi)}{2}\left(e_t + p^{P*}_{Ft} - p_{Ft}\right)^2 \\ &= \hat{p}_{Ft} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{Ft}(f^*)\right) - p_{Ft}\right)^2 df^* + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - p_{Ft}\right)^2 df^*, \end{aligned} \tag{144}$$

$$\begin{aligned} p^L_{Ft} &= \tfrac{1}{\chi}\int_0^\chi \left(\log\left(P^L_{Ft}(f^*)\right)\right) df^* + \tfrac{(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{Ft}(f^*)\right) - p^L_{Ft}\right)^2 df^* \\ &= \left(\begin{array}{ll} & \frac{\theta}{\chi}\int_0^\chi \left(\log\left(P^L_{F,t-1}(f^*)\right)\right) df^* + (1-\theta)p^{L,o}_{Ft} \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{F,t-1}(f^*)\right) - p^L_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o}_{Ft} - p^L_{Ft}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p^L_{F,t-1} + (1-\theta)p^{L,o}_{Ft} \\ - & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{F,t-1}(f^*)\right) - p^L_{F,t-1}\right)^2 df^* \\ + & \frac{\theta(1-\xi)}{2\chi}\int_0^\chi \left(\log\left(P^L_{F,t-1}(f^*)\right) - p^L_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{L,o}_{Ft} - p^L_{Ft}\right)^2 \end{array}\right), \end{aligned} \tag{145}$$

$$\begin{aligned} p^{P*}_{Ft} &= \tfrac{1}{1-\chi}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right)\right) df^* + \tfrac{(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\ &= \left(\begin{array}{ll} & \frac{\theta}{1-\chi}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right)\right) df^* + (1-\theta)p^{P,o*}_{Ft} \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right) \\ &= \left(\begin{array}{ll} & \theta p^{P*}_{F,t-1} + (1-\theta)p^{P,o*}_{Ft} \\ - & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{F,t-1}\right)^2 df^* \\ + & \frac{\theta(1-\xi)}{2(1-\chi)}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* + \frac{(1-\theta)(1-\xi)}{2}\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right) \end{aligned} \tag{146}$$

Together (145) and (146) imply

$$\begin{aligned} & \chi p^L_{Ft} + (1-\chi)(e_t + p^{P*}_{Ft}) \\ = \;& \hat{p}_{Ft} + \tfrac{(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{Ft}(f^*)\right) - p^L_{Ft}\right)^2 df + \tfrac{(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^{P*}_{Ft}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\ = \;& \left(\begin{array}{ll} & \theta\hat{p}_{F,t-1} + (1-\theta)\left(\chi p^{L,o}_{Ft} + (1-\chi)(e_t + p^{P,o*}_{Ft})\right) + \theta(1-\chi)(e_t - e_{t-1}) \\ + & \frac{\theta(1-\xi)}{2}\int_0^\chi \left(\log\left(P^L_{F,t-1}(f^*)\right) - p^L_{Ft}\right)^2 df^* + \frac{\theta(1-\xi)}{2}\int_\chi^1 \left(\log\left(P^{P*}_{F,t-1}(f^*)\right) - p^{P*}_{Ft}\right)^2 df^* \\ + & \frac{(1-\theta)(1-\xi)}{2}\chi\left(p^{L,o}_{Ft} - p^L_{Ft}\right)^2 + \frac{(1-\theta)(1-\xi)}{2}(1-\chi)\left(p^{P,o*}_{Ft} - p^{P*}_{Ft}\right)^2 \end{array}\right) \end{aligned} \tag{147}$$

Recursive Formula for Price Dispersion of Home Goods in Home: From (123), recall price dispersion of Home goods in Home is defined as

$$\sigma^2_{P_H,t} \equiv \int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \hat{p}_{Ht}\right\}^2 df + \int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \hat{p}_{Ht}\right\}^2 df.$$

Claim 3. *The price dispersion of Home products in Home market is related to squared PPI inflation for Home products and squared PPI-deflated prices of Home LCP and PCP products.*

$$\sum_{t=0}^\infty \beta^t\sigma^2_{P_H,t} = \left(\begin{array}{ll} + & \frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\pi^2_{H,t} + \frac{\theta}{1-\beta\theta}\sigma^2_{P_H,-1} \\ + & \frac{\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\left\{\pi^L_{H,t} - \theta\pi^L_{H,t-1}\right\}^2 + \frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\left\{\pi^P_{H,t} - \theta\pi^P_{H,t-1}\right\}^2 \end{array}\right) \tag{148}$$

Proof. Rewrite $\sigma^2_{P_H,t}$ as

$$\begin{aligned}
\sigma^2_{P_H,t} &= \int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \hat{p}_{H,t-1} + \hat{p}_{H,t-1} - \hat{p}_{Ht}\right\}^2 df + \int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \hat{p}_{H,t-1} + \hat{p}_{H,t-1} - \hat{p}_{Ht}\right\}^2 df \\
&= \left(\begin{array}{l} +\int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df + \int_0^\chi \left\{\hat{p}_{H,t-1} - \hat{p}_{Ht}\right\}^2 df - \int_0^\chi 2\left\{\log\left(P^L_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}\left\{\hat{p}_{H,t} - \hat{p}_{H,t-1}\right\} df \\ +\int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df + \int_\chi^1 \left\{\hat{p}_{H,t-1} - \hat{p}_{Ht}\right\}^2 df - \int_\chi^1 2\left\{\log\left(P^P_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}\left\{\hat{p}_{H,t} - \hat{p}_{H,t-1}\right\} df \end{array}\right) \\
&= \int_0^\chi \left\{\log\left(P^L_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df + \int_\chi^1 \left\{\log\left(P^P_{Ht}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df - \left\{\hat{p}_{Ht} - \hat{p}_{H,t-1}\right\}^2,
\end{aligned}$$

where we use the definition of $\hat{p}_{Ht}$ in (127) for the last equality. Under Calvo sticky prices, a portion θ of firms is not allowed to change prices while a fraction $(1-\theta)$ of firms reprices their products. Therefore, we can derive the recursive formula for $\sigma^2_{P_H,t}$ by

$$\begin{aligned}
\sigma^2_{P_H,t} &= \left(\begin{array}{l} +\theta\int_0^\chi \left\{\log\left(P^L_{H,t-1}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df + (1-\theta)\chi\left\{\log\left(P^{L,o}_{Ht}\right) - \hat{p}_{H,t-1}\right\}^2 \\ +\theta\int_\chi^1 \left\{\log\left(P^P_{H,t-1}(f)\right) - \hat{p}_{H,t-1}\right\}^2 df + (1-\theta)(1-\chi)\left\{\log\left(P^{P,o}_{Ht}\right) - \hat{p}_{H,t-1}\right\}^2 - \left\{\hat{p}_{Ht} - \hat{p}_{H,t-1}\right\}^2 \end{array}\right) \\
&= \theta\sigma^2_{P_H,t-1} + (1-\theta)\chi\left\{p^{L,o}_{Ht} - \hat{p}_{H,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{p^{P,o}_{Ht} - \hat{p}_{H,t-1}\right\}^2 - \left\{\hat{p}_{Ht} - \hat{p}_{H,t-1}\right\}^2 \\
&= \left(\begin{array}{l} \theta\sigma^2_{P_H,t-1} - \left\{p_{Ht} - p_{H,t-1}\right\}^2 \\ +(1-\theta)\chi\left\{\frac{p^L_{Ht} - \theta p^L_{H,t-1}}{1-\theta} - p_{H,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{p^P_{Ht} - \theta p^P_{H,t-1}}{1-\theta} - p_{H,t-1}\right\}^2 + O(x_t^3) \end{array}\right),
\end{aligned}$$

where the second and third equalities follow from (122), (123), (133), and (134). We replace nominal prices with inflation and relative price terms:

$$\begin{aligned}
\sigma^2_{P_H,t} &= \theta\sigma^2_{P_H,t-1} - \pi^2_{Ht} + (1-\theta)\chi\left\{\frac{\pi^L_{Ht} + \pi_{Ht} - \theta\pi^L_{H,t-1}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{\pi^P_{Ht} + \pi_{Ht} - \theta\pi^P_{H,t-1}}{1-\theta}\right\}^2 + O(x_t^3) \\
&= \theta\sigma^2_{P_H,t-1} + \frac{\theta}{1-\theta}\pi^2_{Ht} + \frac{\chi}{1-\theta}\left\{\pi^L_{Ht} - \theta\pi^L_{H,t-1}\right\}^2 + \frac{1-\chi}{1-\theta}\left\{\pi^P_{Ht} - \theta\pi^P_{H,t-1}\right\}^2 + O(x_t^3),
\end{aligned}$$

where the second equality follows from $\chi\pi^L_{Ht} + (1-\chi)\pi^P_{Ht} = 0$. Iterating over time, we obtain

$$\sigma^2_{P_H,t} = \frac{\theta}{1-\theta}\sum_{j=0}^{t}\theta^{t-j}\pi^2_{H,j} + \frac{\chi}{1-\theta}\sum_{j=0}^{t}\theta^{t-j}\left\{\pi^L_{H,j} - \theta\pi^L_{H,j-1}\right\}^2 + \frac{1-\chi}{1-\theta}\sum_{j=0}^{t}\theta^{t-j}\left\{\pi^P_{H,j} - \theta\pi^P_{H,j-1}\right\}^2 + \theta^{t+1}\sigma^2_{P_H,-1} + O(x_t^3).$$

Therefore, the life-time discounted price dispersion term, $\sum_{t=0}^\infty \beta^t\sigma^2_{P_H,t}$, can be represented by

$$\begin{aligned}
\sum_{t=0}^\infty \beta^t\sigma^2_{P_H,t} &= \left(\begin{array}{ll} + & \frac{\theta}{1-\theta}\sum_{t=0}^\infty \beta^t\sum_{j=0}^t \theta^{t-j}\pi^2_{H,j} + \sum_{t=0}^\infty \beta^t\theta^{t+1}\sigma^2_{P_H,-1} \\ + & \frac{\chi}{1-\theta}\sum_{t=0}^\infty \beta^t\sum_{j=0}^t \theta^{t-j}\left\{\pi^L_{H,j} - \theta\pi^L_{H,j-1}\right\}^2 + \frac{1-\chi}{1-\theta}\sum_{t=0}^\infty \beta^t\sum_{j=0}^t \theta^{t-j}\left\{\pi^P_{H,j} - \theta\pi^P_{H,j-1}\right\}^2 + O(x_t^3) \end{array}\right) \\
&= \left(\begin{array}{ll} + & \frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\pi^2_{H,t} + \frac{\theta}{1-\beta\theta}\sigma^2_{P_H,-1} \\ + & \frac{\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\left\{\pi^L_{H,t} - \theta\pi^L_{H,t-1}\right\}^2 + \frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty \beta^t\left\{\pi^P_{H,t} - \theta\pi^P_{H,t-1}\right\}^2 + O(x_t^3) \end{array}\right).
\end{aligned}$$

□

Recursive Formula for Price Dispersion of Foreign Goods in Home: From (123), recall price dispersion of

Foreign goods in Home is defined as

$$\sigma^2_{P_F,t} \equiv \int_0^{\chi} \left\{\log\left(P^L_{Ft}(f^*)\right) - \hat{p}_{Ft}\right\}^2 df^* + \int_{\chi}^{1} \left\{\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - \hat{p}_{Ft}\right\}^2 df^*.$$

Claim 4. *The price dispersion of Foreign products in Home market is related to squared PPI inflation for Foreign products and squared currency-adjusted prices of Foreign LCP and PCP products, deflated by PPI. The price dispersion also includes interaction terms with nominal exchange rate growth since there are PCP products of no price change from the previous period under exchange rate fluctuation.*

$$\sum_{t=0}^{\infty}\beta^t\sigma^2_{P_F,t} = \left(\begin{array}{l} +\frac{\theta}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\pi^2_{F,t} + \frac{\chi}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\left[\pi^L_{F,t} - \theta\pi^L_{F,t-1}\right]^2 \\ +\frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\left[\pi^{P*}_{F,t} + m_t - \theta(\pi^{P*}_{F,t-1} + m_{t-1})\right]^2 \\ -\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\Delta e_t\left[\pi^{P*}_{F,t} + m_t - \theta(\pi^{P*}_{F,t-1} + m_{t-1})\right] \\ -\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\pi_{F,t}\Delta e_t + \frac{(1-\chi)\theta^2}{(1-\theta)(1-\beta\theta)}\sum\limits_{t=0}^{\infty}\beta^t\left(\Delta e_t\right)^2 \\ +\frac{\theta(1-\chi)}{1-\beta\theta}\sum\limits_{t=0}^{\infty}\beta^t\left(\pi^{P*}_{F,t-1} + m_{t-1} + \Delta e_t\right)^2 - \frac{\theta(1-\chi)}{1-\beta\theta}\sum\limits_{t=0}^{\infty}\beta^t\left(\pi^{P*}_{F,t-1} + m_{t-1}\right)^2 + \frac{\theta}{1-\beta\theta}\sigma^2_{P_F,-1} \end{array}\right) \tag{149}$$

Proof. Rewrite $\sigma^2_{P_F,t}$ as

$$\begin{aligned} &\sigma^2_{P_F,t} = \int_0^{\chi}\left\{\log\left(P^L_{Ft}(f^*)\right) - \hat{p}_{F,t-1} + \hat{p}_{F,t-1} - \hat{p}_{Ft}\right\}^2 df^* + \int_{\chi}^{1}\left\{\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - \hat{p}_{F,t-1} + \hat{p}_{F,t-1} - \hat{p}_{Ft}\right\}^2 df^* \\ &= \int_0^{\chi}\left\{\log\left(P^L_{Ft}(f^*)\right) - \hat{p}_{F,t-1}\right\}^2 df^* + \int_{\chi}^{1}\left\{\log\left(\mathcal{E}_t P^{P*}_{Ft}(f^*)\right) - \hat{p}_{F,t-1}\right\}^2 df^* - \left\{\hat{p}_{Ft} - \hat{p}_{F,t-1}\right\}^2 \\ &= \left(\begin{array}{l} +\theta\int_0^{\chi}\left\{\log\left(P^L_{F,t-1}(f^*)\right) - \hat{p}_{F,t-1}\right\}^2 df^* + (1-\theta)\chi\left\{\log\left(P^{L,o}_{Ft}\right) - \hat{p}_{F,t-1}\right\}^2 \\ +\theta\int_{\chi}^{1}\left\{\log\left(\mathcal{E}_t P^{P*}_{F,t-1}(f^*)\right) - \hat{p}_{F,t-1}\right\}^2 df^* + (1-\theta)(1-\chi)\left\{\log\left(\mathcal{E}_t P^{P,o*}_{Ft}\right) - \hat{p}_{F,t-1}\right\}^2 - \left\{\hat{p}_{Ft} - \hat{p}_{F,t-1}\right\}^2 \end{array}\right), \end{aligned}$$

where we use the definition of $\hat{p}_{Ft}$ in (127) for the second equality. The third equality follows from the standard recursive formulation under Calvo sticky prices. Rewriting terms with nominal exchange rate, we can derive the recursive formula for $\sigma^2_{P_F,t}$ by

$$\begin{aligned} \sigma^2_{P_F,t} &= \left(\begin{array}{l} +\theta\int_0^{\chi}\left\{\log\left(P^L_{F,t-1}(f^*)\right) - \hat{p}_{F,t-1}\right\}^2 df^* + (1-\theta)\chi\left\{\log\left(P^{L,o}_{Ft}\right) - \hat{p}_{F,t-1}\right\}^2 - \left\{\hat{p}_{Ft} - \hat{p}_{F,t-1}\right\}^2 \\ +\theta\int_{\chi}^{1}\left\{\log\left(\mathcal{E}_{t-1} P^{P*}_{F,t-1}(f^*)\right) - \hat{p}_{F,t-1} + \log\left(\frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}\right)\right\}^2 df^* + (1-\theta)(1-\chi)\left\{\log\left(\mathcal{E}_t P^{P,o*}_{Ft}\right) - \hat{p}_{F,t-1}\right\}^2 \end{array}\right) \\ &= \left(\begin{array}{l} +\theta\sigma^2_{P_F,t-1} + (1-\theta)\chi\left\{p^{L,o}_{Ft} - \hat{p}_{F,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{p^{P,o*}_{Ft} + e_t - \hat{p}_{F,t-1}\right\}^2 \\ +2\theta\left(e_t - e_{t-1}\right)\int_{\chi}^{1}\left\{\log\left(\mathcal{E}_{t-1}P^{P*}_{F,t-1}(f^*)\right) - \hat{p}_{F,t-1}\right\} df^* + \theta\left(1-\chi\right)\left(e_t - e_{t-1}\right)^2 - \left\{\hat{p}_{Ft} - \hat{p}_{F,t-1}\right\}^2 \end{array}\right) \\ &= \left(\begin{array}{l} +\theta\sigma^2_{P_F,t-1} + (1-\theta)\chi\left\{\frac{p^L_{Ft} - \theta p^L_{F,t-1}}{1-\theta} - p_{F,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{p^{P*}_{Ft} - \theta p^{P*}_{F,t-1}}{1-\theta} + e_t - p_{F,t-1}\right\}^2 \\ +2\theta\left(e_t - e_{t-1}\right)\int_{\chi}^{1}\left\{\log\left(\mathcal{E}_{t-1}P^{P*}_{F,t-1}(f^*)\right) - p_{F,t-1}\right\} df^* + \theta\left(1-\chi\right)\left(e_t - e_{t-1}\right)^2 - \left\{p_{Ft} - p_{F,t-1}\right\}^2 + O(x_t^3) \end{array}\right), \end{aligned}$$

where the second and third equalities follow from (122), (123), (145), and (146). We replace nominal prices with inflation and relative price terms:

$$\begin{aligned}
\sigma^2_{P_F,t} &= \left(\begin{array}{l} +\theta\sigma^2_{P_F,t-1} + (1-\theta)\chi\left\{\frac{\pi^L_{Ft}-\theta\pi^L_{F,t-1}+\pi_{Ft}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{(\pi^{P*}_{Ft}+m_t)-\theta(\pi^{P*}_{F,t-1}+m_{t-1})+\pi_{Ft}-\theta\Delta e_t}{1-\theta}\right\}^2 \\ +2\theta(1-\chi)\left(e_t-e_{t-1}\right)\left(\pi^{P*}_{F,t-1}+m_{t-1}\right) + \theta\left(1-\chi\right)\left(e_t-e_{t-1}\right)^2 - \pi^2_{Ft} + O(x_t^3) \end{array}\right) \\
&= \left(\begin{array}{l} +\theta\sigma^2_{P_F,t-1} + (1-\theta)\chi\left\{\frac{\pi^L_{Ft}-\theta\pi^L_{F,t-1}+\pi_{Ft}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{(\pi^{P*}_{Ft}+m_t)-\theta(\pi^{P*}_{F,t-1}+m_{t-1})+\pi_{Ft}-\theta\Delta e_t}{1-\theta}\right\}^2 \\ +\theta(1-\chi)\left(\pi^{P*}_{F,t-1}+m_{t-1}+\Delta e_t\right)^2 - \theta(1-\chi)\left(\pi^{P*}_{F,t-1}+m_{t-1}\right)^2 - \pi^2_{Ft} + O(x_t^3) \end{array}\right) \\
&= \left(\begin{array}{l} +\theta\sigma^2_{P_F,t-1} + \frac{\theta}{1-\theta}\pi^2_{Ft} + \frac{\chi}{1-\theta}\left[\pi^L_{Ft}-\theta\pi^L_{F,t-1}\right]^2 + \frac{1-\chi}{1-\theta}\left[\pi^{P*}_{Ft}+m_t-\theta(\pi^{P*}_{F,t-1}+m_{t-1})\right]^2 \\ -\frac{2(1-\chi)\theta}{1-\theta}\Delta e_t\left[\pi^{P*}_{Ft}+m_t-\theta(\pi^{P*}_{F,t-1}+m_{t-1})\right] - \frac{2(1-\chi)\theta}{1-\theta}\pi_{Ft}\Delta e_t + \frac{(1-\chi)\theta^2}{1-\theta}\left(\Delta e_t\right)^2 \\ +\theta(1-\chi)\left(\pi^{P*}_{F,t-1}+m_{t-1}+\Delta e_t\right)^2 - \theta(1-\chi)\left(\pi^{P*}_{F,t-1}+m_{t-1}\right)^2 + O(x_t^3) \end{array}\right),
\end{aligned}$$

where the last equality follows from $0 = \chi\pi^L_{Ft} + (1-\chi)(\pi^{P*}_{Ft}+m_t)$. We also use $\Delta e_t = \Delta m_t + \pi_{Ht} - \pi^*_{Ht} = \Delta m_t + \pi_{Ft} - \pi^*_{Ft}$ and $m_t = e_t + p^*_{Ht} - p_{Ht} = e_t + p^*_{Ft} - p_{Ft}$. Iterating over time, we obtain

$$\sigma^2_{P_F,t} = \left(\begin{array}{l} +\frac{\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}\pi^2_{F,j} + \frac{\chi}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left[\pi^L_{F,j}-\theta\pi^L_{F,j-1}\right]^2 \\ +\frac{1-\chi}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left[\pi^{P*}_{F,j}+m_j-\theta(\pi^{P*}_{F,j-1}+m_{j-1})\right]^2 \\ -\frac{2(1-\chi)\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}\Delta e_j\left[\pi^{P*}_{F,j}+m_j-\theta(\pi^{P*}_{F,j-1}+m_{j-1})\right] \\ -\frac{2(1-\chi)\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}\pi_{F,j}\Delta e_j + \frac{(1-\chi)\theta^2}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left(\Delta e_j\right)^2 \\ +\theta(1-\chi)\sum_{j=0}^t\theta^{t-j}\left(\pi^{P*}_{F,j-1}+m_{j-1}+\Delta e_j\right)^2 - \theta(1-\chi)\sum_{j=0}^t\theta^{t-j}\left(\pi^{P*}_{F,j-1}+m_{j-1}\right)^2 \\ +\theta^{t+1}\sigma^2_{P_F,-1} \end{array}\right).$$

Therefore, the life-time discounted price dispersion term, $\sum_{t=0}^\infty\beta^t\sigma^2_{P_F,t}$, can be represented by

$$\begin{aligned}
\sum_{t=0}^\infty\beta^t\sigma^2_{P_F,t} &= \left(\begin{array}{l} +\frac{\theta}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\pi^2_{F,j} + \frac{\chi}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\left[\pi^L_{F,j}-\theta\pi^L_{F,j-1}\right]^2 \\ +\frac{1-\chi}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\left[\pi^{P*}_{F,j}+m_j-\theta(\pi^{P*}_{F,j-1}+m_{j-1})\right]^2 \\ -\frac{2(1-\chi)\theta}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\Delta e_j\left[\pi^{P*}_{F,j}+m_j-\theta(\pi^{P*}_{F,j-1}+m_{j-1})\right] \\ -\frac{2(1-\chi)\theta}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\pi_{F,j}\Delta e_j + \frac{(1-\chi)\theta^2}{1-\theta}\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\left(\Delta e_j\right)^2 \\ +\theta(1-\chi)\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\left(\pi^{P*}_{F,j-1}+m_{j-1}+\Delta e_j\right)^2 - \theta(1-\chi)\sum_{t=0}^\infty\beta^t\sum_{j=0}^t\theta^{t-j}\left(\pi^{P*}_{F,j-1}+m_{j-1}\right)^2 \\ +\sum_{t=0}^\infty\beta^t\theta^{t+1}\sigma^2_{P_F,-1} \end{array}\right) \\
&= \left(\begin{array}{l} +\frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\pi^2_{F,t} + \frac{\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left[\pi^L_{F,t}-\theta\pi^L_{F,t-1}\right]^2 \\ +\frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left[\pi^{P*}_{F,t}+m_t-\theta(\pi^{P*}_{F,t-1}+m_{t-1})\right]^2 \\ -\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\Delta e_t\left[\pi^{P*}_{F,t}+m_t-\theta(\pi^{P*}_{F,t-1}+m_{t-1})\right] \\ -\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\pi_{F,t}\Delta e_t + \frac{(1-\chi)\theta^2}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left(\Delta e_t\right)^2 \\ +\frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^\infty\beta^t\left(\pi^{P*}_{F,t-1}+m_{t-1}+\Delta e_t\right)^2 - \frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^\infty\beta^t\left(\pi^{P*}_{F,t-1}+m_{t-1}\right)^2 + \frac{\theta}{1-\beta\theta}\sigma^2_{P_F,-1} \end{array}\right)
\end{aligned}$$

□

Recursive Formula for Price Dispersion of Foreign Goods in Foreign: From (123), recall price dispersion of Foreign goods in Foreign is defined as

$$\sigma^2_{P^*_F,t} \equiv \int_0^\chi \left\{\log\left(P^{L*}_{Ft}(f^*)\right) - \hat{p}^*_{Ft}\right\}^2 df^* + \int_\chi^1 \left\{\log\left(P^{P*}_{Ft}(f^*)\right) - \hat{p}^*_{Ft}\right\}^2 df^*.$$

Claim 5. *The price dispersion of Foreign products in Foreign market is related to squared PPI inflation for Foreign products and squared PPI-deflated prices of Foreign LCP and PCP products.*

$$\sum_{t=0}^\infty \beta^t \sigma^2_{P^*_F,t} = \begin{pmatrix} + & \frac{\theta}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t (\pi^*_{F,t})^2 + \frac{\theta}{1-\beta\theta}\sigma^2_{P^*_F,-1} \\ + & \frac{\chi}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t \left\{\pi^{L*}_{F,t} - \theta\pi^{L*}_{F,t-1}\right\}^2 + \frac{1-\chi}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t \left\{\pi^{P*}_{F,t} - \theta\pi^{P*}_{F,t-1}\right\}^2 \end{pmatrix} \tag{150}$$

Proof. We follow the same procedure as in claim 3 to derive

$$\begin{aligned} \sigma^2_{P^*_F,t} &= \theta\sigma^2_{P^*_F,t-1} - (\pi^*_{Ft})^2 + (1-\theta)\chi\left\{\frac{\pi^{L*}_{Ft} - \theta\pi^{L*}_{F,t-1} + \pi^*_{Ft}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{\pi^{P*}_{Ft} - \theta\pi^{P*}_{F,t-1} + \pi^*_{Ft}}{1-\theta}\right\}^2 + O(x_t^3) \\ &= \theta\sigma^2_{P^*_F,t-1} + \frac{\theta}{1-\theta}(\pi^*_{Ft})^2 + \frac{\chi}{1-\theta}\left\{\pi^{L*}_{Ft} - \theta\pi^{L*}_{F,t-1}\right\}^2 + \frac{1-\chi}{1-\theta}\left\{\pi^{P*}_{Ft} - \theta\pi^{P*}_{F,t-1}\right\}^2 + O(x_t^3). \end{aligned}$$

where the last equality follows from $\chi\pi^{L*}_{Ft} + (1-\chi)\pi^{P*}_{Ft} = 0$. Iterating over time, we obtain

$$\sigma^2_{P^*_F,t} = \frac{\theta}{1-\theta}\sum_{j=0}^t \theta^{t-j}(\pi^*_{F,j})^2 + \frac{\chi}{1-\theta}\sum_{j=0}^t \theta^{t-j}\left\{\pi^{L*}_{F,j} - \theta\pi^{L*}_{F,j-1}\right\}^2 + \frac{1-\chi}{1-\theta}\sum_{j=0}^t \theta^{t-j}\left\{\pi^{P*}_{F,j} - \theta\pi^{P*}_{F,j-1}\right\}^2 + \theta^{t+1}\sigma^2_{P^*_F,-1}.$$

Therefore, the life-time discounted price dispersion term, $\sum_{t=0}^\infty \beta^t\sigma^2_{P^*_F,t}$, can be represented by

$$\sum_{t=0}^\infty \beta^t \sigma^2_{P^*_F,t} = \begin{pmatrix} + & \frac{\theta}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t (\pi^*_{F,t})^2 + \frac{\theta}{1-\beta\theta}\sigma^2_{P^*_F,-1} \\ + & \frac{\chi}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t \left\{\pi^{L*}_{F,t} - \theta\pi^{L*}_{F,t-1}\right\}^2 + \frac{1-\chi}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^\infty \beta^t \left\{\pi^{P*}_{F,t} - \theta\pi^{P*}_{F,t-1}\right\}^2 \end{pmatrix}.$$

□

Recursive Formula for Price Dispersion of Home Goods in Foreign: From (123), recall price dispersion of Home goods in Foreign is defined as

$$\sigma^2_{P^*_H,t} \equiv \int_0^\chi \left\{\log\left(P^{L*}_{Ht}(f)\right) - \hat{p}^*_{Ht}\right\}^2 df + \int_\chi^1 \left\{\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) - \hat{p}^*_{Ht}\right\}^2 df.$$

Claim 6. *The price dispersion of Home products in Foreign market is related to squared PPI inflation for Home products and squared currency-adjusted prices of Home LCP and PCP products, deflated by PPI. The price dispersion also includes interaction terms with nominal exchange rate growth since there are PCP products of no price change* *from the previous period under exchange rate fluctuation.*

$$\sum_{t=0}^{\infty}\beta^t\sigma^2_{P^*_H,t} = \left(\begin{array}{l} +\frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t(\pi^*_{H,t})^2 + \frac{\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\left[\pi^{L*}_{H,t} - \theta\pi^{L*}_{H,t-1}\right]^2 \\ +\frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\left[\pi^P_{H,t} - m_t - \theta(\pi^P_{H,t-1} - m_{t-1})\right]^2 \\ +\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\Delta e_t\left[\pi^P_{H,t} - m_t - \theta(\pi^P_{H,t-1} - m_{t-1})\right] \\ +\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\pi^*_{H,t}\Delta e_t + \frac{(1-\chi)\theta^2}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\left(\Delta e_t\right)^2 \\ +\frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^{\infty}\beta^t\left(\pi^P_{H,t-1} - m_{t-1} - \Delta e_t\right)^2 - \frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^{\infty}\beta^t\left(\pi^P_{H,t-1} - m_{t-1}\right)^2 + \frac{\theta}{1-\beta\theta}\sigma^2_{P^*_H,-1} \end{array}\right) \tag{151}$$

Proof. Rewrite $\sigma^2_{P^*_H,t}$ as

$$\begin{aligned} \sigma^2_{P^*_H,t} &= \int_0^\chi\left\{\log\left(P^{L*}_{Ht}(f)\right) - \hat{p}^*_{H,t-1} + \hat{p}^*_{H,t-1} - \hat{p}^*_{Ht}\right\}^2 df + \int_\chi^1\left\{\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) - \hat{p}^*_{H,t-1} + \hat{p}^*_{H,t-1} - \hat{p}^*_{Ht}\right\}^2 df \\ &= \int_0^\chi\left\{\log\left(P^{L*}_{Ht}(f)\right) - \hat{p}^*_{H,t-1}\right\}^2 df + \int_\chi^1\left\{\log\left(\frac{P^P_{Ht}(f)}{\mathcal{E}_t}\right) - \hat{p}^*_{H,t-1}\right\}^2 df - \left\{\hat{p}^*_{Ht} - \hat{p}^*_{H,t-1}\right\}^2 \\ &= \left(\begin{array}{l} +\theta\int_0^\chi\left\{\log\left(P^{L*}_{H,t-1}(f)\right) - \hat{p}^*_{H,t-1}\right\}^2 df + (1-\theta)\chi\left\{\log\left(P^{L,o*}_{Ht}\right) - \hat{p}^*_{H,t-1}\right\}^2 \\ +\theta\int_\chi^1\left\{\log\left(\frac{P^P_{H,t-1}(f)}{\mathcal{E}_t}\right) - \hat{p}^*_{H,t-1}\right\}^2 df + (1-\theta)(1-\chi)\left\{\log\left(\frac{P^{P,o}_{Ht}}{\mathcal{E}_t}\right) - \hat{p}^*_{H,t-1}\right\}^2 - \left\{\hat{p}^*_{Ht} - \hat{p}^*_{H,t-1}\right\}^2 \end{array}\right), \end{aligned}$$

where we use the definition of $\hat{p}^*_{Ht}$ in (127) for the second equality. The third equality follows from the standard recursive formulation under Calvo sticky prices. Rewriting terms with nominal exchange rate, we can derive the recursive formula for $\sigma^2_{P^*_H,t}$ by

$$\begin{aligned} \sigma^2_{P^*_H,t} &= \left(\begin{array}{l} +\theta\int_0^\chi\left\{\log\left(P^{L*}_{H,t-1}(f)\right) - \hat{p}^*_{H,t-1}\right\}^2 df + (1-\theta)\chi\left\{\log\left(P^{L,o*}_{Ht}\right) - \hat{p}^*_{H,t-1}\right\}^2 - \left\{\hat{p}^*_{Ht} - \hat{p}^*_{H,t-1}\right\}^2 \\ +\theta\int_\chi^1\left\{\log\left(\frac{P^P_{H,t-1}(f)}{\mathcal{E}_{t-1}}\right) - \hat{p}^*_{H,t-1} - \log\left(\frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}\right)\right\}^2 df + (1-\theta)(1-\chi)\left\{\log\left(\frac{P^{P,o}_{Ht}}{\mathcal{E}_t}\right) - \hat{p}^*_{H,t-1}\right\}^2 \end{array}\right) \\ &= \left(\begin{array}{l} +\theta\sigma^2_{P^*_H,t-1} + (1-\theta)\chi\left\{p^{L,o*}_{Ht} - \hat{p}^*_{H,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{p^{P,o}_{Ht} - e_t - \hat{p}^*_{H,t-1}\right\}^2 \\ -2\theta\left(e_t - e_{t-1}\right)\int_\chi^1\left\{\log\left(\frac{P^P_{H,t-1}(f)}{\mathcal{E}_{t-1}}\right) - \hat{p}^*_{H,t-1}\right\} df + \theta\left(1-\chi\right)\left(e_t - e_{t-1}\right)^2 - \left\{\hat{p}^*_{Ht} - \hat{p}^*_{H,t-1}\right\}^2 \end{array}\right) \\ &= \left(\begin{array}{l} +\theta\sigma^2_{P^*_H,t-1} + (1-\theta)\chi\left\{\frac{p^{L*}_{Ht} - \theta p^{L*}_{H,t-1}}{1-\theta} - p^*_{H,t-1}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{p^P_{Ht} - \theta p^P_{H,t-1}}{1-\theta} - e_t - p^*_{H,t-1}\right\}^2 \\ -2\theta\left(e_t - e_{t-1}\right)\int_\chi^1\left\{\log\left(\frac{P^P_{H,t-1}(f)}{\mathcal{E}_{t-1}}\right) - p^*_{H,t-1}\right\} df + \theta\left(1-\chi\right)\left(e_t - e_{t-1}\right)^2 - \left\{p^*_{Ht} - p^*_{H,t-1}\right\}^2 + O(x_t^3) \end{array}\right), \end{aligned}$$

where the second and third equalities follow from (122), (123), (137), and (138). We replace nominal prices with inflation and relative price terms:

$$\begin{aligned}
\sigma^2_{P^*_H,t} &= \left(\begin{array}{l} +\theta\sigma^2_{P^*_H,t-1} + (1-\theta)\chi\left\{\frac{\pi^{L*}_{Ht}-\theta\pi^{L*}_{H,t-1}+\pi^*_{Ht}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{(\pi^P_{Ht}-m_t)-\theta(\pi^P_{H,t-1}-m_{t-1})+\pi^*_{Ht}+\theta\Delta e_t}{1-\theta}\right\}^2 \\ -2\theta(1-\chi)\left(e_t-e_{t-1}\right)\left(\pi^P_{H,t-1}-m_{t-1}\right)+\theta(1-\chi)\left(e_t-e_{t-1}\right)^2-(\pi^*_{Ht})^2+O(x^3_t) \end{array} \right) \\
&= \left(\begin{array}{l} +\theta\sigma^2_{P^*_H,t-1} + (1-\theta)\chi\left\{\frac{\pi^{L*}_{Ht}-\theta\pi^{L*}_{H,t-1}+\pi^*_{Ht}}{1-\theta}\right\}^2 + (1-\theta)(1-\chi)\left\{\frac{(\pi^P_{Ht}-m_t)-\theta(\pi^P_{H,t-1}-m_{t-1})+\pi^*_{Ht}+\theta\Delta e_t}{1-\theta}\right\}^2 \\ +\theta(1-\chi)\left(\pi^P_{H,t-1}-m_{t-1}-\Delta e_t\right)^2-\theta(1-\chi)\left(\pi^P_{H,t-1}-m_{t-1}\right)^2-(\pi^*_{Ht})^2+O(x^3_t) \end{array} \right) \\
&= \left(\begin{array}{l} +\theta\sigma^2_{P^*_H,t-1}+\frac{\theta}{1-\theta}(\pi^*_{Ht})^2+\frac{\chi}{1-\theta}\left[\pi^{L*}_{Ht}-\theta\pi^{L*}_{H,t-1}\right]^2+\frac{1-\chi}{1-\theta}\left[\pi^P_{Ht}-m_t-\theta(\pi^P_{H,t-1}-m_{t-1})\right]^2 \\ +\frac{2(1-\chi)\theta}{1-\theta}\Delta e_t\left[\pi^P_{Ht}-m_t-\theta(\pi^P_{H,t-1}-m_{t-1})\right]+\frac{2(1-\chi)\theta}{1-\theta}\pi^*_{Ht}\Delta e_t+\frac{(1-\chi)\theta^2}{1-\theta}\left(\Delta e_t\right)^2 \\ +\theta(1-\chi)\left(\pi^P_{H,t-1}-m_{t-1}-\Delta e_t\right)^2-\theta(1-\chi)\left(\pi^P_{H,t-1}-m_{t-1}\right)^2+O(x^3_t) \end{array} \right),
\end{aligned}$$

where the last equality follows from $0 = \chi\pi^{L*}_{Ht} + (1-\chi)(\pi^P_{Ht} - m_t)$. We also use $\Delta e_t = \Delta m_t + \pi_{Ht} - \pi^*_{Ht} = \Delta m_t + \pi_{Ft} - \pi^*_{Ft}$ and $m_t = e_t + p^*_{Ht} - p_{Ht} = e_t + p^*_{Ft} - p_{Ft}$. Iterating over time, we obtain

$$\sigma^2_{P^*_H,t} = \left(\begin{array}{l} +\frac{\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}(\pi^*_{H,j})^2+\frac{\chi}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left[\pi^{L*}_{H,j}-\theta\pi^{L*}_{H,j-1}\right]^2 \\ +\frac{1-\chi}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left[\pi^P_{H,j}-m_j-\theta(\pi^P_{H,j-1}-m_{j-1})\right]^2 \\ +\frac{2(1-\chi)\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}\Delta e_j\left[\pi^P_{H,j}-m_j-\theta(\pi^P_{H,j-1}-m_{j-1})\right] \\ +\frac{2(1-\chi)\theta}{1-\theta}\sum_{j=0}^t\theta^{t-j}\pi^*_{H,j}\Delta e_j+\frac{(1-\chi)\theta^2}{1-\theta}\sum_{j=0}^t\theta^{t-j}\left(\Delta e_j\right)^2 \\ +\theta(1-\chi)\sum_{j=0}^t\theta^{t-j}\left(\pi^P_{H,j-1}-m_{j-1}-\Delta e_j\right)^2 \\ -\theta(1-\chi)\sum_{j=0}^t\theta^{t-j}\left(\pi^P_{H,j-1}-m_{j-1}\right)^2 \\ +\theta^{t+1}\sigma^2_{P^*_F,-1} \end{array} \right).$$

Therefore, the life-time discounted price dispersion term, $\sum_{t=0}^\infty\beta^t\sigma^2_{P^*_H,t}$, can be represented by

$$\sum_{t=0}^\infty\beta^t\sigma^2_{P^*_H,t} = \left(\begin{array}{l} +\frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t(\pi^*_{H,t})^2+\frac{\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left[\pi^{L*}_{H,t}-\theta\pi^{L*}_{H,t-1}\right]^2 \\ +\frac{1-\chi}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left[\pi^P_{H,t}-m_t-\theta(\pi^P_{H,t-1}-m_{t-1})\right]^2 \\ +\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\Delta e_t\left[\pi^P_{H,t}-m_t-\theta(\pi^P_{H,t-1}-m_{t-1})\right] \\ +\frac{2(1-\chi)\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\pi^*_{H,t}\Delta e_t+\frac{(1-\chi)\theta^2}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^\infty\beta^t\left(\Delta e_t\right)^2 \\ +\frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^\infty\beta^t\left(\pi^P_{H,t-1}-m_{t-1}-\Delta e_t\right)^2 \\ -\frac{\theta(1-\chi)}{1-\beta\theta}\sum_{t=0}^\infty\beta^t\left(\pi^P_{H,t-1}-m_{t-1}\right)^2 \\ +\frac{\theta}{1-\beta\theta}\sigma^2_{P^*_H,-1} \end{array} \right)$$

□

Transform Price Dispersion into Inflation: After rearranging and cancelling out terms, together (148), (149), (150), and (151) imply

$$
\begin{aligned}
& \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[\left(\tfrac{\nu}{2}\right) \sigma_{P_H,t+j}{}^2 + \left(1-\tfrac{\nu}{2}\right) \sigma_{P_H^*,t+j}{}^2 + \left(\tfrac{\nu}{2}\right) \sigma_{P_F^*,t+j}{}^2 + \left(1-\tfrac{\nu}{2}\right) \sigma_{P_F,t+j}{}^2 \right] \\
= \; & \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{\beta^j}{\delta}\right) \left[\begin{array}{rl}
 & +\left(\tfrac{\nu}{2}\right) \pi_{H,t+j}^2 + \left(1-\tfrac{\nu}{2}\right) (\pi_{H,t+j}^*)^2 + \left(\tfrac{\nu}{2}\right) (\pi_{F,t+j}^*)^2 + \left(1-\tfrac{\nu}{2}\right) \pi_{F,t+j}^2 \\
+\left(\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{H,t+j}^L - \theta \pi_{H,t+j-1}^L \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{H,t+j}^P - \theta \pi_{H,t+j-1}^P \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{H,t+j}^{L*} - \theta \pi_{H,t+j-1}^{L*} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{H,t+j}^P - m_{t+j} - \theta (\pi_{H,t+j-1}^P - m_{t+j-1}) \right\}^2 \right) \\
+\left(\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{F,t+j}^{L*} - \theta \pi_{F,t+j-1}^{L*} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{F,t+j}^{P*} - \theta \pi_{F,t+j-1}^{P*} \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{F,t+j}^{L} - \theta \pi_{F,t+j-1}^{L} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{F,t+j}^{P*} + m_{t+j} - \theta (\pi_{F,t+j-1}^{P*} + m_{t+j-1}) \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & \left(\begin{array}{l} +2(1-\chi)(\Delta e_{t+j}) \left(\begin{array}{l} +\pi_{H,t+j}^P - \pi_{H,t+j-1}^P - \left(\pi_{F,t+j}^{P*} - \pi_{F,t+j-1}^{P*}\right) \\ -2\left(m_{t+j} - m_{t+j-1}\right) + \pi_{H,t+j}^* - \pi_{F,t+j} \end{array} \right) \\ +2(1-\chi)\left(\Delta e_{t+j}\right)^2 \end{array} \right)
\end{array} \right] \\
= \; & \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{\beta^j}{\delta}\right) \left[\begin{array}{rl}
 & +\left(\tfrac{\nu}{2}\right) \pi_{H,t+j}^2 + \left(1-\tfrac{\nu}{2}\right) (\pi_{H,t+j}^*)^2 + \left(\tfrac{\nu}{2}\right) (\pi_{F,t+j}^*)^2 + \left(1-\tfrac{\nu}{2}\right) \pi_{F,t+j}^2 \\
+\left(\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{H,t+j}^L - \theta \pi_{H,t+j-1}^L \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{H,t+j}^P - \theta \pi_{H,t+j-1}^P \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{H,t+j}^{L*} - \theta \pi_{H,t+j-1}^{L*} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{H,t+j}^P - m_{t+j} - \theta (\pi_{H,t+j-1}^P - m_{t+j-1}) \right\}^2 \right) \\
+\left(\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{F,t+j}^{L*} - \theta \pi_{F,t+j-1}^{L*} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{F,t+j}^{P*} - \theta \pi_{F,t+j-1}^{P*} \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & \left(+\tfrac{\chi}{\theta} \left\{ \pi_{F,t+j}^{L} - \theta \pi_{F,t+j-1}^{L} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi_{F,t+j}^{P*} + m_{t+j} - \theta (\pi_{F,t+j-1}^{P*} + m_{t+j-1}) \right\}^2 \right) \\
+\left(1-\tfrac{\nu}{2}\right) & (1-\chi) \left(\begin{array}{l} (\Delta e_{t+j})^2 \\ + \left(\begin{array}{l} +\pi_{H,t+j}^P - \pi_{H,t+j-1}^P - \pi_{F,t+j}^{P*} + \pi_{F,t+j-1}^{P*} \\ +\pi_{H,t+j}^* - \pi_{F,t+j} - 2m_{t+j} + 2m_{t+j-1} + \Delta e_{t+j} \end{array} \right)^2 \\ - \left(\begin{array}{l} +\pi_{H,t+j}^P - \pi_{H,t+j-1}^P - \pi_{F,t+j}^{P*} + \pi_{F,t+j-1}^{P*} \\ +\pi_{H,t+j}^* - \pi_{F,t+j} - 2m_{t+j} + 2m_{t+j-1} \end{array} \right)^2 \end{array} \right)
\end{array} \right]
\end{aligned} \tag{152}
$$

where we use $\Delta e_t = \Delta m_t + (\nu - 1)\Delta s_t + 2\pi_t^R$ and $\pi_{Ht}^* - \pi_{Ft} = -2\pi_t^R - \nu \Delta s_t$. Here we define $\delta \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. β is the household discount factor and θ is the Calvo-staggered pricing parameter.

G.4 Price Dispersion among PCP and LCP Products

In this section, we provide derivations for the reformulation of the weighted sum of price dispersion terms (152). The periodic weighted sum of price dispersion terms in (152) can be rewritten out as

$$
\begin{aligned}
& \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\tfrac{\nu}{2}\right) \sigma_{P_H,t}{}^2 + \left(1-\tfrac{\nu}{2}\right) \sigma_{P_H^*,t}{}^2 + \left(\tfrac{\nu}{2}\right) \sigma_{P_F^*,t}{}^2 + \left(1-\tfrac{\nu}{2}\right) \sigma_{P_F,t}{}^2 \right] \\
= \; & \mathbb{E}_0 \sum_{t=0}^{\infty} \left(\frac{\beta^t}{\delta}\right)
\left[\begin{array}{lll}
 & & + \left(\tfrac{\nu}{2}\right) (\pi_{H,t})^2 + \left(1-\tfrac{\nu}{2}\right) (\pi^*_{H,t})^2 + \left(\tfrac{\nu}{2}\right) (\pi^*_{F,t})^2 + \left(1-\tfrac{\nu}{2}\right) (\pi_{F,t})^2 \\
 + \left(\tfrac{\nu}{2}\right) & \frac{\chi(1-\chi)(1-\theta)^2}{\theta} & \left[\pi^{L,o}_{Ht} - \pi^{P,o}_{Ht}\right]^2 \\
 + \left(1-\tfrac{\nu}{2}\right) & \frac{\chi(1-\chi)(1-\theta)^2}{\theta} & \left((1-\chi)\left[\pi^{L,o*}_{Ht} - \pi^{P,o}_{Ht} + m_t\right]^2 + \chi \left[\pi^{L,o}_{Ht} - \pi^{P,o}_{Ht}\right]^2 \right) \\
 + \left(\tfrac{\nu}{2}\right) & \frac{\chi(1-\chi)(1-\theta)^2}{\theta} & \left[\pi^{L,o*}_{Ft} - \pi^{P,o*}_{Ft}\right]^2 \\
 + \left(1-\tfrac{\nu}{2}\right) & \frac{\chi(1-\chi)(1-\theta)^2}{\theta} & \left((1-\chi)\left[\pi^{L,o}_{Ft} - \pi^{P,o*}_{Ft} - m_t\right]^2 + \chi \left[\pi^{L,o*}_{Ft} - \pi^{P,o*}_{Ft}\right]^2 \right) \\
 + \left(1-\tfrac{\nu}{2}\right) & (1-\chi) & \left(\begin{array}{l}
 + \left[1 + 2\theta\chi(1-\chi)\right](e_t - e_{t-1})^2 + \frac{1}{\theta}(m_t - \theta m_{t-1})^2 \\
 + \left(\begin{array}{l} \chi(1-\chi)(1-\theta)\left(\pi^{L,o*}_{Ht} - \pi^{P,o}_{Ht} - \pi^{L,o}_{Ft} + \pi^{P,o*}_{Ft} + 2m_t\right) \\ + \pi^P_{H,t} - \pi^P_{H,t-1} - \pi^{P*}_{F,t} + \pi^{P*}_{F,t-1} + \pi^*_{H,t} - \pi_{F,t} \\ -2m_t + 2m_{t-1} + e_t - e_{t-1} \end{array} \right)^2 \\
 - \left(\begin{array}{l} \chi(1-\chi)(1-\theta)\left(\pi^{L,o*}_{Ht} - \pi^{P,o}_{Ht} - \pi^{L,o}_{Ft} + \pi^{P,o*}_{Ft} + 2m_t\right) \\ + \pi^P_{H,t} - \pi^P_{H,t-1} - \pi^{P*}_{F,t} + \pi^{P*}_{F,t-1} + \pi^*_{H,t} - \pi_{F,t} \\ -2m_t + 2m_{t-1} \end{array} \right)^2 \\
 + \frac{1}{\theta}\left[(1-\theta)\chi\left(\pi^{L,o}_{Ht} - \pi^{P,o}_{Ht} - \pi^{L,o*}_{Ft} + \pi^{P,o*}_{Ft}\right) + m_t - \theta m_{t-1}\right]^2 \\
 - \frac{1}{\theta}\left[(1-\theta)\chi\left(\pi^{L,o}_{Ht} - \pi^{P,o}_{Ht} - \pi^{L,o*}_{Ft} + \pi^{P,o*}_{Ft}\right)\right]^2
 \end{array} \right)
\end{array} \right] .
\end{aligned}
\tag{153}
$$

The rest of this subsection proves the equality in (153).

The Price Dispersion of Home LCP and PCP Products in Home: Note that quadratic terms of PPI-deflated prices of Home LCP and PCP products in Home can be rewritten as

$$
\begin{aligned}
& \left(\tfrac{\chi}{\theta} \left\{ \pi^L_{H,t} - \theta \pi^L_{H,t-1} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ \pi^P_{H,t} - \theta \pi^P_{H,t-1} \right\}^2 \right) \\
= \; & \left(\tfrac{\chi}{\theta} \left\{ (1-\theta)\pi^{L,o}_{Ht} - \theta \pi_{Ht} \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ (1-\theta)\pi^{P,o}_{Ht} - \theta \pi_{Ht} \right\}^2 \right) \\
= \; & \left(\tfrac{\chi}{\theta} \left\{ (1-\theta)\pi^{L,o}_{Ht} - (1-\theta)\left[\chi\pi^{L,o}_{Ht} + (1-\chi)\pi^{P,o}_{Ht}\right] \right\}^2 + \tfrac{1-\chi}{\theta} \left\{ (1-\theta)\pi^{P,o}_{Ht} - (1-\theta)\left[\chi\pi^{L,o}_{Ht} + (1-\chi)\pi^{P,o}_{Ht}\right] \right\}^2 \right) \\
= \; & \tfrac{\chi(1-\chi)(1-\theta)^2}{\theta} \left\{ \pi^{L,o}_{Ht} - \pi^{P,o}_{Ht} \right\}^2 .
\end{aligned}
\tag{154}
$$

where for the second and third equalities, we used (78) and (81).

The Price Dispersion of Home LCP and PCP Products in Foreign: Note that quadratic terms of

PPI-deflated prices of Home LCP and PCP products in Foreign can be rewritten as

$$\begin{aligned}
&\left(\tfrac{\chi}{\theta}\left\{\pi^{L*}_{H,t}-\theta\pi^{L*}_{H,t-1}\right\}^2+\tfrac{1-\chi}{\theta}\left\{\pi^{P}_{H,t}-\theta\pi^{P}_{H,t-1}-m_t+\theta m_{t-1}\right\}^2\right)\\
=\;&\left(\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o*}_{Ht}-\theta\pi^{*}_{Ht}\right\}^2+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o}_{Ht}-\theta\pi_{Ht}-m_t+\theta m_{t-1}\right\}^2\right)\\
=\;&\left(\begin{array}{l}+\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o*}_{Ht}-(1-\theta)\left[\chi\pi^{L,o*}_{Ht}+(1-\chi)\pi^{P,o}_{Ht}\right]+(1-\theta)(1-\chi)m_t+\theta(1-\chi)(e_t-e_{t-1})\right\}^2\\
+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o}_{Ht}-(1-\theta)\left[\chi\pi^{L,o}_{Ht}+(1-\chi)\pi^{P,o}_{Ht}\right]-m_t+\theta m_{t-1}\right\}^2\end{array}\right)\\
=\;&\left(\begin{array}{l}+\tfrac{\chi}{\theta}\left\{(1-\theta)(1-\chi)\left[\pi^{L,o*}_{Ht}-\pi^{P,o}_{Ht}+m_t\right]+\theta(1-\chi)(e_t-e_{t-1})\right\}^2\\
+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\chi\left[\pi^{L,o}_{Ht}-\pi^{P,o}_{Ht}\right]+m_t-\theta m_{t-1}\right\}^2\end{array}\right).
\end{aligned} \tag{155}$$

where for the second and third equalities, we used (78) and (81).

The Price Dispersion of Foreign LCP and PCP Products in Foreign: Note that quadratic terms of PPI-deflated prices of Foreign LCP and PCP products in Foreign can be rewritten as

$$\begin{aligned}
&\left(\tfrac{\chi}{\theta}\left\{\pi^{L*}_{F,t}-\theta\pi^{L*}_{F,t-1}\right\}^2+\tfrac{1-\chi}{\theta}\left\{\pi^{P*}_{F,t}-\theta\pi^{P*}_{F,t-1}\right\}^2\right)\\
=\;&\left(\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o*}_{Ft}-\theta\pi^{*}_{Ft}\right\}^2+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o*}_{Ft}-\theta\pi^{*}_{Ft}\right\}^2\right)\\
=\;&\left(\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o*}_{Ft}-(1-\theta)\left[\chi\pi^{L,o*}_{Ft}+(1-\chi)\pi^{P,o*}_{Ft}\right]\right\}^2+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o*}_{Ft}-(1-\theta)\left[\chi\pi^{L,o*}_{Ft}+(1-\chi)\pi^{P,o*}_{Ft}\right]\right\}^2\right)\\
=\;&\tfrac{\chi(1-\chi)(1-\theta)^2}{\theta}\left\{\pi^{L,o*}_{Ft}-\pi^{P,o*}_{Ft}\right\}^2.
\end{aligned} \tag{156}$$

where for the second and third equalities, we used (78) and (81).

The Price Dispersion of Foreign LCP and PCP Products in Home: Note that quadratic terms of PPI-deflated prices of Foreign LCP and PCP products in Home can be rewritten as

$$\begin{aligned}
&\left(\tfrac{\chi}{\theta}\left\{\pi^{L}_{F,t}-\theta\pi^{L}_{F,t-1}\right\}^2+\tfrac{1-\chi}{\theta}\left\{\pi^{P*}_{F,t}-\theta\pi^{P*}_{F,t-1}+m_t-\theta m_{t-1}\right\}^2\right)\\
=\;&\left(\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o}_{Ft}-\theta\pi_{Ft}\right\}^2+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o*}_{Ft}-\theta\pi^{*}_{Ft}+m_t-\theta m_{t-1}\right\}^2\right)\\
=\;&\left(\begin{array}{l}+\tfrac{\chi}{\theta}\left\{(1-\theta)\pi^{L,o}_{Ft}-(1-\theta)\left[\chi\pi^{L,o}_{Ft}+(1-\chi)\pi^{P,o*}_{Ft}\right]-(1-\theta)(1-\chi)m_t-\theta(1-\chi)(e_t-e_{t-1})\right\}^2\\
+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\pi^{P,o*}_{Ft}-(1-\theta)\left[\chi\pi^{L,o*}_{Ft}+(1-\chi)\pi^{P,o*}_{Ft}\right]+m_t-\theta m_{t-1}\right\}^2\end{array}\right)\\
=\;&\left(\begin{array}{l}+\tfrac{\chi}{\theta}\left\{(1-\theta)(1-\chi)\left[\pi^{L,o}_{Ft}-\pi^{P,o*}_{Ft}-m_t\right]-\theta(1-\chi)(e_t-e_{t-1})\right\}^2\\
+\tfrac{1-\chi}{\theta}\left\{(1-\theta)\chi\left[\pi^{L,o*}_{Ft}-\pi^{P,o*}_{Ft}\right]-m_t+\theta m_{t-1}\right\}^2\end{array}\right).
\end{aligned} \tag{157}$$

where for the second and third equalities, we used (78) and (81). Feeding (154), (155), (156), and (157) into (152), rearranging and cancelling out relevant terms lead to the equation (153).

G.5 Derivation of the Quadratic World Welfare Function

The section G.5 closely follows derivation of the quadratic loss function in Corsetti et al. (2020). What follows is the replication of their results by using notations of Engel (2011). The new contribution of our analysis relative to Corsetti et al. (2020) is the derivation of price dispersion terms under generic degree of exchange rate pass-through given by (152).

The Second-Order Approximation of Goods-Market Clearing Conditions: Goods-market clearing conditions in Home and Foreign, (24) and (25), can be approximated as

$$\begin{aligned} a_t + n_t &= \left(\tfrac{\nu}{2}\right)\left(c_{Ht} + \chi v_{Ht}^{L} + (1-\chi) v_{Ht}^{P}\right) + \left(1 - \tfrac{\nu}{2}\right)\left(c_{Ht}^* + \chi v_{Ht}^{L*} + (1-\chi) v_{Ht}^{P*}\right), \\ a_t^* + n_t^* &= \left(\tfrac{\nu}{2}\right)\left(c_{Ft}^* + \chi v_{Ft}^{L*} + (1-\chi) v_{Ft}^{P*}\right) + \left(1 - \tfrac{\nu}{2}\right)\left(c_{Ft} + \chi v_{Ft}^{L} + (1-\chi) v_{Ft}^{P}\right). \end{aligned} \tag{158}$$

Feeding (126) into (158), we obtain

$$\begin{aligned} n_t &= y_t - a_t + \left(\tfrac{\nu}{2}\right)\left(\tfrac{\xi}{2}\right)\sigma_{P_H,t}^2 + \left(1-\tfrac{\nu}{2}\right)\left(\tfrac{\xi}{2}\right)\sigma_{P_H^*,t}^2, \\ n_t^* &= y_t^* - a_t^* + \left(\tfrac{\nu}{2}\right)\left(\tfrac{\xi}{2}\right)\sigma_{P_F^*,t}^2 + \left(1-\tfrac{\nu}{2}\right)\left(\tfrac{\xi}{2}\right)\sigma_{P_F,t}^2, \end{aligned} \tag{159}$$

where we substitute for c_{Ht}, c_{Ht}^*, c_{Ft}^*, and c_{Ft} by using log-linearized aggregate demands which are

$$y_t = \tfrac{\nu}{2} c_{Ht} + \left(1 - \tfrac{\nu}{2}\right) c_{Ht}^* \quad \text{and} \quad y_t^* = \tfrac{\nu}{2} c_{Ft}^* + \left(1-\tfrac{\nu}{2}\right) c_{Ft}.$$

The Second-Order Approximation of Aggregate Demand: Following Corsetti et al. (2020), we derive the second-order approximation to the world aggregate demand in terms of Home consumption index given by

$$\tfrac{P_{Ht}}{P_t} Y_t + \tfrac{\mathcal{E}_t P_{Ft}^*}{P_t} Y_t^* = \tfrac{P_{Ht}}{P_t}\left(C_{Ht} + C_{Ht}^*\right) + \tfrac{\mathcal{E}_t P_{Ft}^*}{P_t}\left(C_{Ft}^* + C_{Ft}\right), \tag{160}$$

where we convert different units of Home and Foreign tradeable goods into the same unit, Home consumption index (C_t), to derive the world demand. We rewrite the right-hand side of (160) as

$$\tfrac{P_{Ht}}{P_t}\left(C_{Ht} + C_{Ht}^*\right) + \tfrac{\mathcal{E}_t P_{Ft}^*}{P_t}\left(C_{Ft}^* + C_{Ft}\right) = \tfrac{1}{P_t}\left\{P_t C_t + \mathcal{E}_t P_t^* C_t^* + P_{Ht} C_{Ht}^*\left(1 - \tfrac{\mathcal{E}_t P_{Ht}^*}{P_{Ht}}\right) + P_{Ft} C_{Ft}\left(\tfrac{\mathcal{E}_t P_{Ft}^*}{P_{Ft}} - 1\right)\right\}.$$

Therefore, (160) imply

$$\begin{aligned} &\left\{\tfrac{\nu}{2} + \left(1-\tfrac{\nu}{2}\right) S_t^{1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} Y_t + \left\{\tfrac{\nu}{2} + \left(1-\tfrac{\nu}{2}\right) {S_t^*}^{1-\epsilon}\right\}^{\frac{-1}{1-\epsilon}} Q_t Y_t^* \\ &= \begin{pmatrix} + & C_t + Q_t C_t^* \\ + & \left(1-\tfrac{\nu}{2}\right)\left\{\tfrac{\nu}{2}\left(\tfrac{1}{S_t}\right)^{1-\epsilon} + 1 - \tfrac{\nu}{2}\right\}^{(-1)} C_t \left(\tfrac{M_t}{Z_t} - 1\right) \\ + & \left(1-\tfrac{\nu}{2}\right)\left\{\tfrac{\nu}{2}\left(\tfrac{1}{S_t^*}\right)^{1-\epsilon} + 1 - \tfrac{\nu}{2}\right\}^{(-1)} Q_t C_t^* \left(\tfrac{1}{M_t Z_t} - 1\right) \end{pmatrix}. \end{aligned}$$

Approximating this equation up to the second order and using relations of $z_t = 0$ and $s_t = -s_t^*$, we obtain[1]

$$c_t + c_t^* - y_t - y_t^* = \begin{pmatrix} + & \frac{1}{2}\left({y_t}^2 + {y_t^*}^2 - {c_t}^2 - {c_t^*}^2\right) \\ + & q_t\left(y_t^* - c_t^*\right) \\ - & \left(1-\frac{\nu}{2}\right) s_t \left(y_t - y_t^*\right) - \left(1-\frac{\nu}{2}\right) m_t \left(c_t - c_t^*\right) \\ + & \epsilon\left(\frac{\nu}{2}\right)\left(1-\frac{\nu}{2}\right)\left(s_t^2 + 2 s_t m_t\right) \end{pmatrix}. \tag{161}$$

The derivation of the quadratic world utility function: Together (116), (159), and (161) imply

$$\begin{aligned} v_t &= \begin{pmatrix} + & c_t + c_t^* - y_t - y_t^* + c_t\zeta_{c,t} + c_t^*\zeta_{c,t}^* \\ + & \left(\frac{1-\sigma}{2}\right)\left({c_t}^2 + {c_t^*}^2\right) - \left(\frac{1+\phi}{2}\right)\left({y_t}^2 - 2a_t y_t + {y_t^*}^2 - 2a_t^* y_t^*\right) \\ - & \left(\frac{\xi}{2}\right)\left[\left(\frac{\nu}{2}\right)\sigma_{P_H,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_H^*,t}{}^2 + \left(\frac{\nu}{2}\right)\sigma_{P_F^*,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_F,t}{}^2\right] + t.i.p. + O(x_t^3) \end{pmatrix} \\ &= \begin{pmatrix} + & \begin{pmatrix} + & \frac{1}{2}\left({y_t}^2 + {y_t^*}^2 - {c_t}^2 - {c_t^*}^2\right) \\ + & q_t\left(y_t^* - c_t^*\right) \\ - & \left(1-\frac{\nu}{2}\right) s_t \left(y_t - y_t^*\right) - \left(1-\frac{\nu}{2}\right) m_t \left(c_t - c_t^*\right) \\ + & \epsilon\left(\frac{\nu}{2}\right)\left(1-\frac{\nu}{2}\right)\left(s_t^2 + 2 s_t m_t\right) \end{pmatrix} + c_t\zeta_{c,t} + c_t^*\zeta_{c,t}^* \\ + & \left(\frac{1-\sigma}{2}\right)\left({c_t}^2 + {c_t^*}^2\right) - \left(\frac{1+\phi}{2}\right)\left({y_t}^2 - 2a_t y_t + {y_t^*}^2 - 2a_t^* y_t^*\right) \\ - & \left(\frac{\xi}{2}\right)\left[\left(\frac{\nu}{2}\right)\sigma_{P_H,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_H^*,t}{}^2 + \left(\frac{\nu}{2}\right)\sigma_{P_F^*,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_F,t}{}^2\right] + t.i.p. + O(x_t^3) \end{pmatrix}. \end{aligned} \tag{162}$$

We feed (72), (74), and (94) into (162) and use relations of $z_t = 0$, $s_t = -s_t^*$, and $q_t = m_t + (\nu - 1)s_t$ to rewrite terms. After rearranging and cancelling out terms accordingly, we obtain[2]

$$v_t = \begin{pmatrix} - & \left(\frac{\epsilon}{D}\right)\left(\frac{\nu}{2}\right)\left(\frac{2-\nu}{2}\right)\left(m_t + f_t\right)^2 \\ - & \left(\frac{\sigma}{D} + \phi\right)\left\{\widetilde{y}_t^R\right\}^2 - \left(\sigma + \phi\right)\left\{\widetilde{y}_t^W\right\}^2 \\ - & \left(\frac{\xi}{2}\right)\left[\left(\frac{\nu}{2}\right)\sigma_{P_H,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_H^*,t}{}^2 + \left(\frac{\nu}{2}\right)\sigma_{P_F^*,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_F,t}{}^2\right] \\ + & t.i.p. + O(x_t^3) \end{pmatrix}. \tag{163}$$

[1]Derivations in detail are available upon request.
[2]Derivations in detail are available upon request.

Therefore, combining (152) and (163), we derive the life-time discounted loss function represented by

$$
\begin{aligned}
&\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(-v_t\right) \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{ll}
+ & \left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_t^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_t^W\right)^2 + \left(\frac{\epsilon\nu(2-\nu)}{4D}\right)(m_t+f_t)^2 \\
+ & \left(\frac{\xi}{2}\right)\left[\left(\frac{\nu}{2}\right)\sigma_{P_H,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_H^*,t}{}^2 + \left(\frac{\nu}{2}\right)\sigma_{P_F^*,t}{}^2 + \left(1-\frac{\nu}{2}\right)\sigma_{P_F,t}{}^2\right] \\
+ & t.i.p. + O(x_{t+j}^3)
\end{array} \right) \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{l}
+\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_t^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_t^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D}(m_t+f_t)^2 \\
+\left(\frac{\xi}{2\delta}\right) \left[\begin{array}{rl}
& +\left(\frac{\nu}{2}\right)\pi_{H,t}^2 + \left(1-\frac{\nu}{2}\right)(\pi_{H,t}^*)^2 + \left(\frac{\nu}{2}\right)(\pi_{F,t}^*)^2 + \left(1-\frac{\nu}{2}\right)\pi_{F,t}^2 \\
+\left(\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^L - \theta\pi_{H,t-1}^L\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P - \theta\pi_{H,t-1}^P\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^{L*} - \theta\pi_{H,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P - m_t - \theta(\pi_{H,t-1}^P - m_{t-1})\right\}^2\right) \\
+\left(\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^{L*} - \theta\pi_{F,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*} - \theta\pi_{F,t-1}^{P*}\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & \left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^L - \theta\pi_{F,t-1}^L\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*} + m_t - \theta(\pi_{F,t-1}^{P*} + m_{t-1})\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right) & (1-\chi)\left(\begin{array}{l}
(\Delta e_t)^2 \\
+\left(\begin{array}{l} \pi_{H,t}^P - \pi_{H,t-1}^P - \pi_{F,t}^{P*} + \pi_{F,t-1}^{P*} \\ +\pi_{H,t}^* - \pi_{F,t} + \Delta e_t - 2m_t + 2m_{t-1} \end{array} \right)^2 \\
-\left(\begin{array}{l} \pi_{H,t}^P - \pi_{H,t-1}^P - \pi_{F,t}^{P*} + \pi_{F,t-1}^{P*} \\ +\pi_{H,t}^* - \pi_{F,t} - 2m_t + 2m_{t-1} \end{array} \right)^2
\end{array} \right)
\end{array} \right] \\
+t.i.p. + O(x_t^3)
\end{array} \right),
\end{aligned}
$$

where we have $\Delta e_t = \Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R$ and $\pi_{Ht}^* - \pi_{Ft} = -2\pi_t^R - \nu\Delta s_t$ from section F.4. This completes the derivation of the quadratic loss function given by (118).

H Linearized Optimal Monetary Policy under Incomplete Exchange Rate Pass-Through and Imperfect Risk Sharing ($0 \leq \chi \leq 1$ and $0 \leq \lambda \leq 1$)

Section H solves the linear-quadratic problem characterized in section F and G, and finds a local linear approximation to the actual optimal monetary policy presented in section B. Due to the complexity of the quadratic loss function (118), there are no simple targeting rules available under the generic parametrization for incomplete exchange rate pass-through ($0 \leq \chi \leq 1$) and imperfect risk sharing ($0 \leq \lambda \leq 1$). Thus in what follows we first describes the linear-quadratic Ramsey problem under the generic parametrization. Then we restrict our focus on two polar cases of PCP ($\chi = 0$) and LCP ($\chi = 1$) to derive analytical decision rules and closed-form allocations.

H.1 Linear-Quadratic Ramsey Problem for Optimal Monetary Policy

The goal of policy is to minimize a discounted loss function (118) by choosing thirteen variables: π_t^W, $\widetilde{y}_t^W$, π_t^R, $\widetilde{y}_t^R$, f_t, m_t, s_t, $\pi_{H,t}^L$, $\pi_{H,t}^{L*}$, $\pi_{H,t}^P$, $\pi_{F,t}^{L*}$, $\pi_{F,t}^L$, and $\pi_{F,t}^{P*}$ under eleven constraints. Here we reproduce the policy objective given by

$$
\begin{aligned}
\min \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t L_t \\
s.t. \quad L_t = & \left(
\begin{array}{l}
+\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_t^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_t^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D}(m_t+f_t)^2 \\
+\left(\frac{\xi}{2\delta}\right)\left[
\begin{array}{l}
2\left(\pi_t^R\right)^2 + 2\left(\pi_t^W\right)^2 + \frac{\nu(2-\nu)}{2}(s_t-s_{t-1})^2 \\
+\left(\frac{\nu}{2}\right)\left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^L-\theta\pi_{H,t-1}^L\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P-\theta\pi_{H,t-1}^P\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right)\left(+\frac{\chi}{\theta}\left\{\pi_{H,t}^{L*}-\theta\pi_{H,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{H,t}^P-m_t-\theta(\pi_{H,t-1}^P-m_{t-1})\right\}^2\right) \\
+\left(\frac{\nu}{2}\right)\left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^{L*}-\theta\pi_{F,t-1}^{L*}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*}-\theta\pi_{F,t-1}^{P*}\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right)\left(+\frac{\chi}{\theta}\left\{\pi_{F,t}^{L}-\theta\pi_{F,t-1}^{L}\right\}^2 + \frac{1-\chi}{\theta}\left\{\pi_{F,t}^{P*}+m_t-\theta(\pi_{F,t-1}^{P*}+m_{t-1})\right\}^2\right) \\
+\left(1-\frac{\nu}{2}\right)\quad(1-\chi)\left(
\begin{array}{l}
(\Delta e_t)^2 \\
+\left(\begin{array}{l}\pi_{H,t}^P-\pi_{H,t-1}^P-\pi_{F,t}^{P*}+\pi_{F,t-1}^{P*} \\ +\pi_{H,t}^*-\pi_{F,t}+\Delta e_t-2m_t+2m_{t-1}\end{array}\right)^2 \\
-\left(\begin{array}{l}\pi_{H,t}^P-\pi_{H,t-1}^P-\pi_{F,t}^{P*}+\pi_{F,t-1}^{P*} \\ +\pi_{H,t}^*-\pi_{F,t}-2m_t+2m_{t-1}\end{array}\right)^2
\end{array}\right)
\end{array}\right] \\
+t.i.p.+O(x_t^3)
\end{array}
\right),
\end{aligned}
$$

where PPI inflation terms in (118) are converted into cross-country CPI inflation measures by

$$
\left(\frac{\nu}{2}\right)\pi_{H,t}^2 + \left(1-\frac{\nu}{2}\right)(\pi_{H,t}^*)^2 + \left(\frac{\nu}{2}\right)(\pi_{F,t}^*)^2 + \left(1-\frac{\nu}{2}\right)\pi_{F,t}^2 = 2\left(\pi_t^R\right)^2 + 2\left(\pi_t^W\right)^2 + \frac{\nu(2-\nu)}{2}(s_t-s_{t-1})^2 .
$$

Note that nominal exchange rate growth and additional PPI inflation terms in (118) can be substituted out by

$$
\Delta e_t = m_t - m_{t-1} + (\nu-1)(s_t-s_{t-1}) + 2\pi_t^R \quad \text{and} \quad \pi_{Ht}^* - \pi_{Ft} = -2\pi_t^R - \nu(s_t-s_{t-1}).
$$

We assign a Lagrange multiplier to each constraint and there are eleven multipliers denoted by γ_t, γ_t^f, $\gamma_t^{H,L}$, $\gamma_t^{H,L*}$, $\gamma_t^{F,L*}$, $\gamma_t^{F,L}$, $\gamma_t^{H,P}$, $\gamma_t^{F,P*}$, γ_t^W, γ_t^R, and γ_t^s. Aggregate demand and financial market condition (73) and (75) are

$$(\gamma_t) \quad 0 = s_t - \frac{2\sigma}{D}\widetilde{y}_t^R + \left(\frac{\nu-1}{D}\right)(m_t + f_t) - \overline{s}_t, \tag{164}$$

$$(\gamma_t^f) \quad 0 = f_t - \Xi_1\left(\widetilde{y}_t^R + \overline{y}_t^R\right) + \Xi_2 m_t + \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}2\zeta_{c,t}^R, \tag{165}$$

where $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, $\Xi_2 \equiv \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1-D)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, and $D \equiv (\nu-1)^2 + \epsilon\sigma\nu(2-\nu)$. PPI-deflated price indexes for LCP and PCP products (78) are

$$(\gamma_t^{H,L}) \quad 0 = \chi\pi_{Ht}^L + (1-\chi)\pi_{Ht}^P, \tag{166}$$

$$(\gamma_t^{H,L*}) \quad 0 = \chi\pi_{Ht}^{L*} + (1-\chi)\left(\pi_{Ht}^P - m_t\right), \tag{167}$$

$$(\gamma_t^{F,L*}) \quad 0 = \chi\pi_{Ft}^{L*} + (1-\chi)\pi_{Ft}^{P*}, \tag{168}$$

$$(\gamma_t^{F,L}) \quad 0 = \chi\pi_{Ft}^L + (1-\chi)\left(\pi_{Ft}^{P*} + m_t\right). \tag{169}$$

Applying (95), pricing equations for PCP products (85) and (86) can be rewritten as

$$(\gamma_t^{H,P}) \quad \pi_{Ht}^{P,o} = (1-\beta\theta)\left(\left(\tfrac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W + \tfrac{D-\nu+1}{2D}(m_t+f_t) + \tfrac{1}{\delta}u_t\right) + (\beta\theta)\,\mathbb{E}_t\left[\pi_{H,t+1}^{P,o} + \pi_{H,t+1}\right], \tag{170}$$

$$(\gamma_t^{F,P*}) \quad \pi_{Ft}^{P,o*} = (1-\beta\theta)\left(\left(-\tfrac{\sigma}{D}-\phi\right)\widetilde{y}_t^R + (\sigma+\phi)\widetilde{y}_t^W + \tfrac{-D+\nu-1}{2D}(m_t+f_t) + \tfrac{1}{\delta}u_t^*\right) + (\beta\theta)\,\mathbb{E}_t\left[\pi_{F,t+1}^{P,o*} + \pi_{F,t+1}^*\right], \tag{171}$$

where we should substitute π_{Ht}^P and π_{Ft}^{P*} for $\pi_{Ht}^{P,o}$ and $\pi_{Ft}^{P,o*}$ by (79):

$$\pi_{Ht}^P = \theta\left(\pi_{H,t-1}^P - \pi_{Ht}\right) + (1-\theta)\pi_{Ht}^{P,o} \quad \text{and} \quad \pi_{Ft}^{P*} = \theta\left(\pi_{F,t-1}^{P*} - \pi_{Ft}^*\right) + (1-\theta)\pi_{Ft}^{P,o*}.$$

Here PPI inflation terms, π_{Ht} and π_{Ft}^*, should be also replaced with cross-country CPI inflation measures and the relative price of Foreign goods through (108):

$$\pi_{Ht} = \pi_t^W + \pi_t^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_t \quad \text{and} \quad \pi_{Ft}^* = \pi_t^W - \pi_t^R + \left(1-\tfrac{\nu}{2}\right)\Delta s_t.$$

The world aggregate supply curves (102), (103), and (104) are

$$(\gamma_t^W) \quad \pi_t^W = \delta(\sigma+\phi)\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^W + \tfrac{u_t+u_t^*}{2}, \tag{172}$$

$$(\gamma_t^R) \quad \begin{pmatrix} \pi_t^R\left[1-(2-\nu)(1-\chi)\right] \\ -\Delta m_t\left(1-\frac{\nu}{2}\right)(1-\chi) \\ -\Delta s_t(\nu-1)\left(1-\frac{\nu}{2}\right)(1-\chi) \end{pmatrix} = \begin{bmatrix} & \delta\begin{pmatrix} & \left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R \\ + & \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t \end{pmatrix} \\ + & \beta\mathbb{E}_t\begin{pmatrix} \pi_{t+1}^R\left[1-(2-\nu)(1-\chi)\right] \\ -\Delta m_{t+1}\left(1-\frac{\nu}{2}\right)(1-\chi) \\ -\Delta s_{t+1}(\nu-1)\left(1-\frac{\nu}{2}\right)(1-\chi) \end{pmatrix} + (\nu-1)\,\frac{u_t-u_t^*}{2} \end{bmatrix}, \tag{173}$$

$$(\gamma_t^s) \quad \begin{pmatrix} \Delta s_t \\ -(1-\chi)\left(\Delta m_t + (\nu-1)\Delta s_t + 2\pi_t^R\right) \end{pmatrix} = \begin{bmatrix} - & \delta\left(s_t - \overline{s}_t + 2\phi\widetilde{y}_t^R + f_t\right) - u_t + u_t^* \\ + & \beta\mathbb{E}_t\begin{pmatrix} \Delta s_{t+1} \\ -(1-\chi)\left(\Delta m_{t+1} + (\nu-1)\Delta s_{t+1} + 2\pi_{t+1}^R\right) \end{pmatrix} \end{bmatrix}. \tag{174}$$

Therefore, we can derive thirteen first-order necessary conditions. Given thirteen FONCs and eleven constraints, we solve for thirteen endogenous variables:

$$\pi_t^W,\ \widetilde{y}_t^W,\ \pi_t^R,\ \widetilde{y}_t^R,\ f_t,\ m_t,\ s_t,\ \pi_{H,t}^L,\ \pi_{H,t}^{L*},\ \pi_{H,t}^P,\ \pi_{F,t}^{L*},\ \pi_{F,t}^L,\ \text{and}\ \pi_{F,t}^{P*},$$

and eleven Lagrange multipliers:

$$\gamma_t,\ \gamma_t^f,\ \gamma_t^{H,L},\ \gamma_t^{H,L*},\ \gamma_t^{F,L*},\ \gamma_t^{F,L},\ \gamma_t^{H,P},\ \gamma_t^{F,P*},\ \gamma_t^W,\ \gamma_t^R,\ \text{and}\ \gamma_t^s.$$

Complete description for the system of 24 equilibrium conditions is available upon request.[3]

H.2 Linear-Quadratic Ramsey Problem for Optimal Monetary Policy under PCP ($\chi = 0$)

Recall cross-country inflation measures defined by

$$\pi_t^{P,R} \equiv \tfrac{\pi_{Ht}-\pi_{Ft}^*}{2},\ \pi_t^{P,W} \equiv \tfrac{\pi_{Ht}+\pi_{Ft}^*}{2},\ \pi_t^R \equiv \tfrac{\pi_t-\pi_t^*}{2},\ \text{and}\ \pi_t^W \equiv \tfrac{\pi_t+\pi_t^*}{2}.$$

CPI inflation terms are $\pi_t = \frac{\nu}{2}\pi_{Ht} + \left(1-\frac{\nu}{2}\right)\pi_{Ft}$ and $\pi_t^* = \frac{\nu}{2}\pi_{Ft}^* + \left(1-\frac{\nu}{2}\right)\pi_{Ht}^*$. These terms have relations (106) and (108) which are reproduced here:

$$\begin{aligned}
\pi_t^R &= \tfrac{\pi_{Ht}-\pi_{Ft}^*}{2} + \left(1-\tfrac{\nu}{2}\right)\Delta s_t && \text{and} && \pi_t^W = \tfrac{\pi_t+\pi_t^*}{2} = \tfrac{\pi_{Ht}+\pi_{Ft}^*}{2},\\
\pi_{Ht} &= \pi_t^W + \pi_t^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_t && \text{and} && \pi_{Ft}^* = \pi_t^W - \pi_t^R + \left(1-\tfrac{\nu}{2}\right)\Delta s_t,\\
\pi_{Ft} &= \pi_t^W + \pi_t^R + \tfrac{\nu}{2}\Delta s_t && \text{and} && \pi_{Ht}^* = \pi_t^W - \pi_t^R - \tfrac{\nu}{2}\Delta s_t,
\end{aligned}$$

with $\pi_{Ht} + \pi_{Ft}^* = \pi_{Ft} + \pi_{Ht}^*$ from $z_t = 0$. Using these equations, we reformulate the policy problem for the case of PCP ($\chi = 0$). Under PCP ($\chi = 0$), together (166)–(169) imply that relative price terms for PCP products and currency misalignment are all zero: $0 = \pi_{Ht}^P = \pi_{Ft}^{P*} = m_t$. This leads to a simple loss function (120) given by

$$\begin{aligned}
\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j L_{t+j}^{PCP} &= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} +\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} {f_{t+j}}^2 \\ +\left(\frac{\xi}{2\delta}\right)\left({\pi_{Ht+j}}^2 + {\pi_{Ft+j}^*}^2\right) + t.i.p. + O(x_{t+j}^3) \end{pmatrix}\\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} +\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_{t+j}^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_{t+j}^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} {f_{t+j}}^2 \\ +\left(\frac{\xi}{\delta}\right)\left(\left(\pi_{t+j}^{P,R}\right)^2 + \left(\pi_{t+j}^W\right)^2\right) + t.i.p. + O(x_{t+j}^3) \end{pmatrix}.
\end{aligned}$$

The financial market condition (165) reduces to

$$(\gamma_t^f) \quad f_t = \Xi_1\left(\widetilde{y}_t^R + \overline{y}_t^R\right) - 2\zeta_{c,t}^R \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D},$$

[3] We verified that impulse responses from linear-quadratic Ramsey equilibrium exactly matched those from non-linear Ramsey equilibrium in section B.

where $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$ and $D \equiv (\nu-1)^2 + \epsilon\sigma\nu(2-\nu)$. Substituting π_{Ht} and π^*_{Ft} for $\pi^{P,o}_{Ht}$ and $\pi^{P,o*}_{Ft}$ by $\pi^{P,o}_{Ht} = \frac{\theta}{1-\theta}\pi_{Ht}$ and $\pi^{P,o*}_{Ft} = \frac{\theta}{1-\theta}\pi^*_{Ft}$, pricing equations for PCP products (170) and (171) are given by

$$\frac{\theta}{1-\theta}\pi_{Ht} = (1-\beta\theta)\left(\left(\frac{\sigma}{D}+\phi\right)\widetilde{y}^R_t + (\sigma+\phi)\widetilde{y}^W_t + \frac{D-\nu+1}{2D}f_t + \frac{1}{\delta}u_t\right) + (\beta\theta)\,\mathbb{E}_t\left[\frac{1}{1-\theta}\pi_{H,t+1}\right],$$

$$\frac{\theta}{1-\theta}\pi^*_{Ft} = (1-\beta\theta)\left(\left(-\frac{\sigma}{D}-\phi\right)\widetilde{y}^R_t + (\sigma+\phi)\widetilde{y}^W_t + \frac{-D+\nu-1}{2D}f_t + \frac{1}{\delta}u^*_t\right) + (\beta\theta)\,\mathbb{E}_t\left[\frac{1}{1-\theta}\pi^*_{F,t-1}\right],$$

which can be rewritten as the world aggregate supply curves given by

$$\begin{aligned}
&(\gamma^W_t) \quad \pi^W_t = \delta(\sigma+\phi)\widetilde{y}^W_t + \beta\mathbb{E}_t\left[\pi^W_{t+1}\right] + \frac{u_t+u^*_t}{2},\\
&(\gamma^R_t) \quad \pi^{P,R}_t = \delta\left[\left(\frac{\sigma}{D}+\phi\right)\widetilde{y}^R_t + \frac{D-\nu+1}{2D}f_t\right] + \beta\mathbb{E}_t\left[\pi^{P,R}_{t+1}\right] + \frac{u_t-u^*_t}{2},
\end{aligned}$$

where $\delta \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Characterization of the Ramsey problem under PCP is presented in section I.

H.3 Linear-Quadratic Ramsey Problem for Optimal Monetary Policy under LCP $(\chi = 1)$

In this subsection, we reformulate the policy problem for the case of LCP ($\chi = 1$). Under LCP ($\chi = 1$), together (166)–(169) imply that relative price terms for LCP products are all zero: $0 = \pi^L_{Ht} = \pi^{L*}_{Ht} = \pi^{L*}_{Ft} = \pi^L_{Ft}$. This leads to a simple loss function (119) given by

$$\mathbb{E}_t\sum_{j=0}^{\infty}\beta^j L^{LCP}_{t+j} = \mathbb{E}_t\sum_{j=0}^{\infty}\beta^j\left(\begin{array}{l}+\left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}^R_{t+j}\right)^2 + (\sigma+\phi)\left(\widetilde{y}^W_{t+j}\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D}\left(m_{t+j}+f_{t+j}\right)^2\\ +\left(\frac{\xi}{\delta}\right)\left[\left(\pi^R_{t+j}\right)^2 + \left(\pi^W_{t+j}\right)^2 + \frac{\nu(2-\nu)}{4}\left(s_{t+j}-s_{t+j-1}\right)^2\right] + t.i.p. + O(x^3_{t+j})\end{array}\right).$$

The relevant constraints reduce to

$$\begin{aligned}
&(\gamma_t) \quad 0 = s_t - \frac{2\sigma}{D}\widetilde{y}^R_t + \left(\frac{\nu-1}{D}\right)(m_t+f_t) - \overline{s}_t,\\
&(\gamma^f_t) \quad 0 = f_t - \Xi_1\left(\widetilde{y}^R_t + \overline{y}^R_t\right) + \Xi_2 m_t + 2\zeta^R_{c,t}\frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D},\\
&(\gamma^W_t) \quad \pi^W_t = \delta(\sigma+\phi)\widetilde{y}^W_t + \beta\mathbb{E}_t\left[\pi^W_{t+1}\right] + \frac{u_t+u^*_t}{2},\\
&(\gamma^R_t) \quad \pi^R_t = \delta\left[\left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}^R_t + \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \frac{D-(\nu-1)^2}{2D}m_t\right] + \beta\mathbb{E}_t\left[\pi^R_{t+1}\right] + (\nu-1)\frac{u_t-u^*_t}{2},\\
&(\gamma^s_t) \quad \Delta s_t = -\delta\left(s_t - \overline{s}_t + 2\phi\widetilde{y}^R_t + f_t\right) - u_t + u^*_t + \beta\mathbb{E}_t\left[\Delta s_{t+1}\right],
\end{aligned}$$

where $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, $\Xi_2 \equiv \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1-D)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, and $D \equiv (\nu-1)^2 + \epsilon\sigma\nu(2-\nu)$. We characterize LCP allocations under optimal monetary policy in section J.

I Optimal Monetary Policy under PCP and Imperfect Risk Sharing

In this section we derive optimal targeting rules and characterize allocations under PCP ($\chi = 0$) and imperfect financial integration ($0 \leq \lambda \leq 1$). Since the law of one price holds under PCP, there is no currency misalignment,

$m_t = 0$.

I.1 Optimal Targeting Rules under PCP

As shown in section H.2, the cooperative monetary authority chooses output gap, inflation, and demand imbalance $\left(\widetilde{y}_t^R, \widetilde{y}_t^W, \pi_t^{P,R}, \pi_t^W, f_t\right)$ by solving the constrained minimization problem given by

$$
\begin{array}{ll}
\min & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t L_t^{PCP} \\
s.t. & L_t^{PCP} \propto \left(\frac{\sigma}{D} + \phi\right) \left(\widetilde{y}_t^R\right)^2 + (\sigma + \phi) \left(\widetilde{y}_t^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D} \left(f_t\right)^2 + \frac{\xi}{\delta} \left(\left(\pi_t^{P,R}\right)^2 + \left(\pi_t^W\right)^2\right) \\
(\gamma_t^R) & \pi_t^{P,R} = \delta \left[\left(\frac{\sigma}{D} + \phi\right) \widetilde{y}_t^R + \frac{D-\nu+1}{2D} f_t\right] + \beta \mathbb{E}_t \pi_{t+1}^{P,R} + u_t^R \\
(\gamma_t^W) & \pi_t^W = \delta(\sigma + \phi)\widetilde{y}_t^W + \beta \mathbb{E}_t \pi_{t+1}^W + u_t^W \\
(\gamma_t^f) & f_t = \Xi_1 \left(\widetilde{y}_t^R + \overline{y}_t^R\right) - 2\zeta_{c,t}^R \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}
\end{array}
\tag{175}
$$

$\left(\gamma_t^R, \gamma_t^W, \gamma_t^f\right)$ are the Lagrange multipliers associated with the constraints. The first-order necessary conditions are easily seen to be

$$
\begin{array}{cl}
\left(\partial \widetilde{y}_t^W\right) & 2(\sigma+\phi)\widetilde{y}_t^W - \gamma_t^W \delta(\sigma+\phi) = 0, \\
\left(\partial \pi_t^W\right) & \frac{2\xi}{\delta} \pi_t^W + \gamma_t^W - \gamma_{t-1}^W = 0, \\
\left(\partial \widetilde{y}_t^R\right) & 2\left(\frac{\sigma}{D} + \phi\right) \widetilde{y}_t^R - \gamma_t^R \delta \left(\frac{\sigma}{D} + \phi\right) - \gamma_t^f \Xi_1 = 0, \\
\left(\partial f_t\right) & \frac{\epsilon\nu(2-\nu)}{2D} f_t - \gamma_t^R \frac{\delta(D-\nu+1)}{2D} + \gamma_t^f = 0, \\
\left(\partial \pi_t^{P,R}\right) & \frac{2\xi}{\delta} \pi_t^{P,R} + \gamma_t^R - \gamma_{t-1}^R = 0.
\end{array}
$$

where $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$ and $D \equiv (\nu-1)^2 + \epsilon\sigma\nu(2-\nu)$. By substituting for all Lagrange multipliers, linear targeting criteria can be derived by

$$
\begin{array}{rcl}
0 & = & \xi \pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\
0 & = & \left(1 + \Xi_1 \frac{(D-\nu+1)}{2(\sigma+\phi D)}\right) \xi \pi_t^{P,R} + \widetilde{y}_t^R - \widetilde{y}_{t-1}^R + \Xi_1 \frac{\epsilon\nu(2-\nu)}{4(\sigma+\phi D)} \left(f_t - f_{t-1}\right).
\end{array}
\tag{176}
$$

If financial market integration is perfect ($\lambda = 0$) or goods markets are completely separated ($\nu = 2$) or parametrization follows Cole-Obstfeld(1991) by $\sigma = \epsilon = 1$, then the coefficient Ξ_1 becomes zero and the cross-country difference targeting rule translates into

$$
0 = \xi \pi_t^{P,R} + \widetilde{y}_t^R - \widetilde{y}_{t-1}^R, \quad \text{so that} \quad 0 = \xi \pi_{Ht} + \widetilde{y}_t - \widetilde{y}_{t-1} \quad \text{and} \quad 0 = \xi \pi_{Ft}^* + \widetilde{y}_t^* - \widetilde{y}_{t-1}^* \quad \text{hold.}
$$

This implies that under $\sigma = \epsilon = 1$, optimal targeting rules are inward-looking policies: targeting rules do not take into account the cross-country demand imbalance and PPI inflation stabilization leads to efficient output given zero output gap in the previous period ($\widetilde{y}_{t-1} = \widetilde{y}_{t-1}^* = 0$).

Given the policymaker's optimal choices for output gap, inflation, and demand imbalance, together aggregate demands and the Euler equations determine the setting for nominal interest rates, i_t and i_t^*, necessary to achieve the desired values of target variables. For that reason, we can treat output gap, inflation, and demand imbalance as if they were the policy instruments.

I.2 Dependence of s_t and f_t on Policy under PCP with $\sigma = 1$ and $\phi = 0$

In section J.2, we will show that the relative price of Foreign goods (s_t) and demand imbalance (f_t) are independent of policy under LCP with log consumption utility and linear labor disutility ($\sigma = 1$ and $\phi = 0$). The natural question is whether s_t and f_t are also independent of policy under PCP with $\sigma = 1$ and $\phi = 0$. The answer is no. With $\chi = 0$, $\sigma = 1$, and $\phi = 0$, the two aggregate supply relations (173) and (174) are

$$(\gamma_t^R) \quad (\nu-1)\left[\pi_t^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_t\right] = \begin{bmatrix} \delta\left(\frac{\nu-1}{D}\widetilde{y}_t^R + \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t\right) \\ +\quad \beta(\nu-1)\mathbb{E}_t\left[\pi_{t+1}^R - \Delta s_{t+1}\left(1-\frac{\nu}{2}\right)\right] + (\nu-1)\,u_t^R \end{bmatrix},$$

$$(\gamma_t^s) \quad -2\left[\pi_t^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_t\right] = -\delta\left(s_t - \overline{s}_t + f_t\right) - 2u_t^R - 2\beta\mathbb{E}_t\left[\pi_{t+1}^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_{t+1}\right].$$

(73) implies these two equations collapse into one identical equation given by

$$\pi_t^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_t = \tfrac{\delta}{2}\left(s_t - \overline{s}_t + f_t\right) + \beta\mathbb{E}_t\left[\pi_{t+1}^R - \left(1-\tfrac{\nu}{2}\right)\Delta s_{t+1}\right] + u_t^R,$$

$$\text{that is,}\quad \pi_t^{P,R} = \tfrac{\delta}{2}\left(s_t - \overline{s}_t + f_t\right) + \beta\mathbb{E}_t\left[\pi_{t+1}^{P,R}\right] + u_t^R,$$

where we make use of $\pi_t^{P,R} = \pi_t^R - \left(1-\frac{\nu}{2}\right)\Delta s_t$ in (106). Since the cross-country PPI inflation ($\pi_t^{P,R}$) is not independent of policy, both the relative price of Foreign goods (s_t) and demand imbalance (f_t) are not autonomous.

I.3 Allocations under PCP

Our goal is to describe allocations in terms of exogenous shocks by solving equilibrium conditions:

$$\begin{aligned}
&(i) && 0 = \xi\pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\
&(ii) && \pi_t^W = \delta(\sigma+\phi)\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^W + u_t^W, \\
&(iii) && 0 = \left(1 + \Xi_1\tfrac{(D-\nu+1)}{2(\sigma+\phi D)}\right)\xi\pi_t^{P,R} + \widetilde{y}_t^R - \widetilde{y}_{t-1}^R + \Xi_1\tfrac{\epsilon\nu(2-\nu)}{4(\sigma+\phi D)}\left(f_t - f_{t-1}\right), \\
&(iv) && \pi_t^{P,R} = \delta\left[\left(\tfrac{\sigma}{D}+\phi\right)\widetilde{y}_t^R + \tfrac{D-\nu+1}{2D}f_t\right] + \beta\mathbb{E}_t\pi_{t+1}^{P,R} + u_t^R, \\
&(v) && f_t = \Xi_1\left(\widetilde{y}_t^R + \overline{y}_t^R\right) - 2\zeta_{c,t}^R\tfrac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D},
\end{aligned} \tag{177}$$

where $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)[1+\nu(\epsilon-1)]+2(1-\lambda)D}$ and $D \equiv (\nu-1)^2 + \sigma\epsilon\nu(2-\nu) = 1 + \nu(2-\nu)(\epsilon\sigma - 1)$.

Characterize $\widetilde{y}_t^W$: Together (i) and (ii) imply

$$\mathbb{E}_t\left[\beta\widetilde{y}_{t+1}^W - \left(1+\beta+\xi\delta(\sigma+\phi)\right)\widetilde{y}_t^W + \widetilde{y}_{t-1}^W\right] = \xi u_t^W.$$

Using eigenvalues defined by $x^W \equiv \frac{[1+\beta+\xi\delta(\sigma+\phi)]\pm\sqrt{[1+\beta+\xi\delta(\sigma+\phi)]^2-4\beta}}{2\beta}$ with $|x_1^W| < 1 < |x_2^W|$, we rewrite the above equation as

$$\mathbb{E}_t \widetilde{y}_t^W \beta L \left[\left(\tfrac{1}{L} - x_1^W \right) \left(\tfrac{1}{L} - x_2^W \right) \right] = \xi u_t^W,$$

where L denotes the lag operator. Therefore, we solve for the cross-country sum of output gap given by

$$\widetilde{y}_t^W = x_1^W \widetilde{y}_{t-1}^W + \tfrac{1}{\beta} \left(\tfrac{-1}{x_2^W} \right) \sum_{h=0}^{\infty} \left(\tfrac{1}{x_2^W} \right)^h \mathbb{E}_t \xi u_{t+h}^W = \left(x_1^W \right)^t \widetilde{y}_0^W + \sum_{l=0}^{t-1} \left(x_1^W \right)^l \left[\tfrac{1}{\beta} \left(\tfrac{-1}{x_2^W} \right) \sum_{h=0}^{\infty} \left(\tfrac{1}{x_2^W} \right)^h \mathbb{E}_{t-l} \xi u_{t-l+h}^W \right], \tag{178}$$

which leads to

$$\begin{aligned} \widetilde{y}_{t+j}^W &= \left(x_1^W \right)^j \widetilde{y}_t^W + \sum_{l=0}^{j-1} \left(x_1^W \right)^l \left[\tfrac{1}{\beta} \left(\tfrac{-1}{x_2^W} \right) \sum_{h=0}^{\infty} \left(\tfrac{1}{x_2^W} \right)^h \mathbb{E}_{t+j-l} \xi u_{t+j-l+h}^W \right], \\ \text{that is,}\ \mathbb{E}_t \widetilde{y}_{t+j}^W &= \left(x_1^W \right)^j \widetilde{y}_t^W + \sum_{l=0}^{j-1} \left(x_1^W \right)^l \left[\tfrac{1}{\beta} \left(\tfrac{-1}{x_2^W} \right) \sum_{h=0}^{\infty} \left(\tfrac{1}{x_2^W} \right)^h \mathbb{E}_t \xi u_{t+j-l+h}^W \right]. \end{aligned}$$

The solution for the cross-country sum of output gap applies commonly for both PCP and LCP. (178) implies under no cost-push shock and zero output gap in the initial time period, the cross-country sum of output is efficient: $\widetilde{y}_t^W = 0\ \forall t$ given $\widetilde{y}_0^W = 0$.

Characterize $\widetilde{y}_t^R$: Using (v) to substitute for f_t in (iii) and (iv), we obtain

$$\xi \pi_t^{P,R} = \left[\begin{array}{l} - \left(\frac{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)} \right) \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R \right) - \left(\frac{(\Xi_1)^2 \epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)} \right) \left(\frac{1+\phi}{\frac{\sigma}{D}+\phi} \right) \left(a_t^R - a_{t-1}^R \right) \\ - \left(\frac{\Xi_1 \epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)} \right) \left[\Xi_1 \left(\frac{\nu-1}{\sigma+\phi D} \right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \right] \left(\zeta_{c,t}^R - \zeta_{c,t-1}^R \right) \end{array} \right],$$

$$\xi \pi_t^{P,R} = \left[\begin{array}{l} \beta \mathbb{E}_t \xi \pi_{t+1}^{P,R} + \xi\delta \left[\left(\frac{\sigma}{D} + \phi \right) + \frac{(D-\nu+1)}{2D} \Xi_1 \right] \widetilde{y}_t^R \\ + \frac{\xi\delta(D-\nu+1)}{2D} \Xi_1 \left(\frac{1+\phi}{\frac{\sigma}{D}+\phi} \right) a_t^R + \frac{\xi\delta(D-\nu+1)}{2D} \left[\Xi_1 \left(\frac{\nu-1}{\sigma+\phi D} \right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \right] \zeta_{c,t}^R + \xi u_t^R \end{array} \right],$$

where we use $\overline{y}_t^R = \frac{1+\phi}{\frac{\sigma}{D}+\phi} a_t^R + \frac{\nu-1}{\sigma+\phi D} \zeta_{c,t}^R$. Substituting for $\pi_t^{P,R}$ in the second equation, we derive

$$\mathbb{E}_t \left[\beta \widetilde{y}_{t+1}^R - \left(1 + \beta + \left(\tfrac{\xi\delta}{4D} \right) \left(\tfrac{[4(\sigma+\phi D)+2\Xi_1(D-\nu+1)]^2}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \right) \widetilde{y}_t^R + \widetilde{y}_{t-1}^R \right] = \Phi_{P,t},$$

where $\Phi_{P,t}$ is defined as

$$\Phi_{P,t} \equiv \left(\begin{array}{l} + \left(\frac{(\Xi_1)^2 \epsilon\nu(2-\nu)}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \left(\frac{1+\phi}{\frac{\sigma}{D}+\phi} \right) \left(-\beta \mathbb{E}_t a_{t+1}^R + (1+\beta) a_t^R - a_{t-1}^R \right) \\ + \left(\frac{\Xi_1 \epsilon\nu(2-\nu)}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \left[\Xi_1 \left(\frac{\nu-1}{\sigma+\phi D} \right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \right] \left(-\beta \mathbb{E}_t \zeta_{c,t+1}^R + (1+\beta) \zeta_{c,t}^R - \zeta_{c,t-1}^R \right) \\ + \left(\frac{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \frac{\xi\delta(D-\nu+1)}{2D} \Xi_1 \left(\frac{1+\phi}{\frac{\sigma}{D}+\phi} \right) a_t^R \\ + \left(\frac{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \frac{\xi\delta(D-\nu+1)}{2D} \left[\Xi_1 \left(\frac{\nu-1}{\sigma+\phi D} \right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \right] \zeta_{c,t}^R \\ + \left(\frac{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}{4(\sigma+\phi D)+(\Xi_1)^2 \epsilon\nu(2-\nu)} \right) \xi u_t^R \end{array} \right), \tag{179}$$

which consists of all exogenous shocks. Define eigenvalues as

$$x^P = \frac{\left[1+\beta+\left(\frac{\xi\delta}{4D}\right)\left(\frac{[4(\sigma+\phi D)+2\Xi_1(D-\nu+1)]^2}{4(\sigma+\phi D)+(\Xi_1)^2\epsilon\nu(2-\nu)}\right)\right]\pm\sqrt{\left[1+\beta+\left(\frac{\xi\delta}{4D}\right)\left(\frac{[4(\sigma+\phi D)+2\Xi_1(D-\nu+1)]^2}{4(\sigma+\phi D)+(\Xi_1)^2\epsilon\nu(2-\nu)}\right)\right]^2-4\beta}}{2\beta},$$

with $|x_1^P| < 1 < |x_2^P|$. Therefore, the solution for $\widetilde{y}_t^R$ is given by

$$\widetilde{y}_t^R = x_1^P \widetilde{y}_{t-1}^R + \frac{1}{\beta}\left(\frac{-1}{x_2^P}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^P}\right)^h \mathbb{E}_t\Phi_{P,t+h} = \left(x_1^P\right)^t \widetilde{y}_0^R + \sum_{l=0}^{t-1}\left(x_1^P\right)^l\left[\frac{1}{\beta}\left(\frac{-1}{x_2^P}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^P}\right)^h \mathbb{E}_{t-l}\Phi_{P,t-l+h}\right], \tag{180}$$

which can be rewritten as

$$\begin{aligned}
\widetilde{y}_{t+j}^R &= \left(x_1^P\right)^j \widetilde{y}_t^R + \sum_{l=0}^{j-1}\left(x_1^P\right)^l\left[\frac{1}{\beta}\left(\frac{-1}{x_2^P}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^P}\right)^h \mathbb{E}_{t+j-l}\Phi_{P,t+j-l+h}\right], \\
\text{that is, } \mathbb{E}_t\widetilde{y}_{t+j}^R &= \left(x_1^P\right)^j \widetilde{y}_t^R + \sum_{l=0}^{j-1}\left(x_1^P\right)^l\left[\frac{1}{\beta}\left(\frac{-1}{x_2^P}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^P}\right)^h \mathbb{E}_t\Phi_{P,t+j-l+h}\right].
\end{aligned}$$

Characterize $f_t, \pi_t^W, \pi_t^{P,R}$: Solutions for $\widetilde{y}_t^W$ and $\widetilde{y}_t^R$ pin down

$$\begin{aligned}
\xi\pi_t^W &= -\left(\widetilde{y}_t^W - \widetilde{y}_{t-1}^W\right), \\
f_t &= \Xi_1\widetilde{y}_t^R + \Xi_1\left(\frac{1+\phi}{\frac{\sigma}{D}+\phi}\right)a_t^R + \left[\Xi_1\left(\frac{\nu-1}{\sigma+\phi D}\right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right]\zeta_{c,t}^R, \\
\xi\pi_t^{P,R} &= \left[\begin{array}{l}
-\left(\frac{4(\sigma+\phi D)+(\Xi_1)^2\epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}\right)\left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right) - \left(\frac{(\Xi_1)^2\epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}\right)\left(\frac{1+\phi}{\frac{\sigma}{D}+\phi}\right)\left(a_t^R - a_{t-1}^R\right) \\
-\left(\frac{\Xi_1\epsilon\nu(2-\nu)}{4(\sigma+\phi D)+2\Xi_1(D-\nu+1)}\right)\left[\Xi_1\left(\frac{\nu-1}{\sigma+\phi D}\right) - \frac{2(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right]\left(\zeta_{c,t}^R - \zeta_{c,t-1}^R\right)
\end{array}\right].
\end{aligned} \tag{181}$$

I.4 Comparison with PCP Allocations under Uncontingent Bonds when $\sigma = \epsilon = 1$ and $\phi = 0$

In this subsection, we establish the equivalence result between optimal monetary policy responses of our economy with capital controls and the economy with uncontingent bonds when producers price in their own currency under the Cole and Obstfeld specification: $\sigma = \epsilon = 1$ and $\phi = 0$.

Under the economy with uncontingent bonds, resource constraints are replaced with

$$\begin{aligned}
(H) \quad & P_{Ft}C_{Ft} + \frac{B_t}{R_t} = \mathcal{E}_t P_{Ht}^* C_{Ht}^* + B_{t-1}, \\
(F) \quad & \mathcal{E}_t P_{Ht}^* C_{Ht}^* + \frac{B_t^*}{R_t} = P_{Ft}C_{Ft} + B_{t-1}^*,
\end{aligned}$$

where $B_t + B_t^* = 0$ for all t. Here B_t and B_t^* denote nominal uncontingent bond holdings in units of Home currency traded in international financial markets.

Following derivations in Engel (2014) and Corsetti et al. (2020), we can obtain targeting rules and equilibrium conditions under the economy with uncontingent bonds as

$$\begin{array}{llll} (i) & 0 & = & \xi\pi_t^{P,W} + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W \\ (ii) & \pi_t^{P,W} & = & \beta\mathbb{E}_t\pi_{t+1}^{P,W} + \delta\widetilde{y}_t^W + u_t^W \\ (iii) & 0 & = & \xi\pi_t^{P,R} + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right) \\ (iv) & \pi_t^{P,R} & = & \beta\mathbb{E}_t\pi_{t+1}^{P,R} + \delta\widetilde{y}_t^R + \delta\left(\frac{2-\nu}{2}\right)f_t + u_t^R \\ (v) & f_t & = & \frac{-2}{(2-\nu)(1+\nu\epsilon-\nu)}\left(\beta b_t - b_{t-1}\right) - 2\zeta_{c,t}^R \\ (vi) & f_t & = & \mathbb{E}_t f_{t+1} \end{array} \tag{182}$$

where we define the linearized bond holdings as $b_t \equiv \frac{B_t}{PC}$ and we make use of $\beta R = 1$.

On the other hand, with $\sigma = \epsilon = 1$ and $\phi = 0$, PCP equilibrium conditions in (177) are reduced to

$$\begin{array}{llll} (i) & 0 & = & \xi\pi_t^{P,W} + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W \\ (ii) & \pi_t^{P,W} & = & \beta\mathbb{E}_t\pi_{t+1}^{P,W} + \delta\widetilde{y}_t^W + u_t^W \\ (iii) & 0 & = & \xi\pi_t^{P,R} + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right) \\ (iv) & \pi_t^{P,R} & = & \beta\mathbb{E}_t\pi_{t+1}^{P,R} + \delta\widetilde{y}_t^R + \delta\left(\frac{2-\nu}{2}\right)f_t + u_t^R \\ (v) & f_t & = & -2\zeta_{c,t}^R\frac{(\lambda)(2-\nu)}{2-\lambda\nu} \end{array} \tag{183}$$

Therefore, comparing the equilibrium system of our economy in (183) with that of the uncontingent-bond economy in (182), we can observe that the only difference is the equation for demand imbalance (f_t). Since demand imbalance is exogenous to policy in both economies under the Cole and Obstfeld specification ($\sigma = \epsilon = 1$, $\phi = 0$), policy responses to f_t are identical between these two economies. This also holds true for policy responses to net capital flows which are independent of policy by equation (92).

J Optimal Monetary Policy under LCP and Imperfect Risk Sharing

In this section we derive optimal targeting rules and characterize allocations under LCP ($\chi = 1$) and imperfect financial integration ($0 \leq \lambda \leq 1$).

J.1 Optimal Targeting Rules under LCP

As shown in section H.3, the goal of policy is to minimize a discounted loss function by choosing seven variables: $\widetilde{y}_t^R$, $\widetilde{y}_t^W$, π_t^R, π_t^W, s_t, m_t, and f_t subject to five constraints. The policymaker solves

$$
\begin{array}{ll}
\min & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t L_t^{LCP} \\
s.t. & L_t^{LCP} \propto \left(\frac{\sigma}{D}+\phi\right)\left(\widetilde{y}_t^R\right)^2 + (\sigma+\phi)\left(\widetilde{y}_t^W\right)^2 + \frac{\epsilon\nu(2-\nu)}{4D}(m_t+f_t)^2 + \frac{\xi}{\delta}\left[\left(\pi_t^R\right)^2 + \left(\pi_t^W\right)^2 + \frac{\nu(2-\nu)}{4}(\Delta s_t)^2\right] \\
(\gamma_t) & s_t = \overline{s}_t + \frac{2\sigma}{D}\widetilde{y}_t^R - \frac{\nu-1}{D}(m_t+f_t) \\
(\gamma_t^f) & f_t = \Xi_1\left(\widetilde{y}_t^R + \overline{y}_t^R\right) - \Xi_2 m_t - 2\zeta_{c,t}^R \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D} \\
(\gamma_t^W) & \pi_t^W = \delta(\sigma+\phi)\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^W + u_t^W \\
(\gamma_t^R) & \pi_t^R = \delta\left[\left(\frac{\sigma}{D}+\phi\right)(\nu-1)\widetilde{y}_t^R + \left(\frac{D-\nu+1}{2D}\right)(\nu-1)f_t + \left(\frac{D-(\nu-1)^2}{2D}\right)m_t\right] + \beta\mathbb{E}_t\pi_{t+1}^R + (\nu-1)u_t^R \\
(\gamma_t^s) & \Delta s_t = -\delta\left[s_t - \overline{s}_t + 2\phi\widetilde{y}_t^R + f_t\right] + \beta\mathbb{E}_t\left[\Delta s_{t+1}\right] - 2u_t^R
\end{array}
\tag{184}
$$

where coefficients are defined as $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$ and $\Xi_2 \equiv \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1-D)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$. γ_t, γ_t^f, γ_t^W, γ_t^R, and γ_t^s are relevant Lagrange multipliers. The last constraint implies that even in case that the inverse of the labor-supply elasticity is zero ($\phi = 0$), the dynamic of the relative price of Foreign goods (s_t) does not become autonomous due to the presence of demand imbalance (f_t). This extends the result in Engel (2011) and the relative price of Foreign goods must show up as a choice variable in characterizing the optimal monetary policy. Differentiation of the Lagrangian then yields first-order necessary conditions given by

$$
\begin{array}{ll}
\left(\partial\widetilde{y}_t^W\right) & 2\widetilde{y}_t^W - \gamma_t^W\delta = 0, \\
\left(\partial\pi_t^W\right) & \frac{2\xi}{\delta}\pi_t^W + \gamma_t^W - \gamma_{t-1}^W = 0, \\
\left(\partial\widetilde{y}_t^R\right) & \frac{2(\sigma+\phi D)}{D}\widetilde{y}_t^R - \gamma_t^R\frac{\delta(\sigma+\phi D)(\nu-1)}{D} + \gamma_t^s 2\delta\phi - \gamma_t\frac{2\sigma}{D} - \gamma_t^f\Xi_1 = 0, \\
(\partial m_t) & \frac{\epsilon\nu(2-\nu)}{2D}(m_t+f_t) - \gamma_t^R\frac{\delta\left[D-(\nu-1)^2\right]}{2D} + \gamma_t\frac{\nu-1}{D} + \gamma_t^f\Xi_2 = 0, \\
(\partial s_t) & \frac{\xi}{\delta}\frac{\nu(2-\nu)}{2}\left[(1+\beta)s_t - s_{t-1} - \beta\mathbb{E}_t s_{t+1}\right] + \gamma_t^s(1+\beta+\delta) - \beta\mathbb{E}_t\gamma_{t+1}^s - \gamma_{t-1}^s + \gamma_t = 0, \\
(\partial f_t) & \frac{\epsilon\nu(2-\nu)}{2D}(m_t+f_t) - \gamma_t^R\frac{\delta(D-\nu+1)(\nu-1)}{2D} + \gamma_t^s\delta + \gamma_t\frac{\nu-1}{D} + \gamma_t^f = 0, \\
\left(\partial\pi_t^R\right) & \frac{2\xi}{\delta}\pi_t^R + \gamma_t^R - \gamma_{t-1}^R = 0.
\end{array}
$$

After substituting for Lagrange multipliers, we obtain optimal targeting rules represented by three equations:

$$
\begin{aligned}
0 &= \xi\pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\
0 &= \left[\begin{array}{ll} & \frac{2\xi}{\delta}\pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right)\frac{4(\sigma+\phi D)}{\delta\Xi_4} \\ + & (m_t - m_{t-1} + f_t - f_{t-1})\frac{2\epsilon\sigma\nu(2-\nu)}{\delta\Xi_4(\nu-1)} \\ + & (\gamma_t^s - \gamma_{t-1}^s)\frac{2D}{\Xi_4}\left[\frac{1}{1-\Xi_2}\left(\Xi_1 - \frac{2\sigma\Xi_2}{\nu-1}\right) + 2\phi\right] \end{array}\right], \\
0 &= \left[\begin{array}{ll} & \gamma_t^s\left(1+\beta+\delta+\frac{\delta\Xi_2 D}{(\nu-1)(1-\Xi_2)} - 2D\frac{\Xi_3}{\Xi_4}\left[\frac{1}{1-\Xi_2}\left(\Xi_1 - \frac{2\sigma\Xi_2}{\nu-1}\right)+2\phi\right]\right) - \beta\mathbb{E}_t\gamma_{t+1}^s - \gamma_{t-1}^s \\ + & \frac{\xi}{\delta}\frac{\nu(2-\nu)}{2}\left[(1+\beta)s_t - s_{t-1} - \beta\mathbb{E}_t s_{t+1}\right] \\ - & \widetilde{y}_t^R\frac{\Xi_3}{\Xi_4}\frac{4(\sigma+\phi D)}{\delta} \\ - & (m_t+f_t)\frac{\epsilon\nu(2-\nu)}{\nu-1}\left[\frac{1}{2}+\frac{2\sigma}{\delta}\frac{\Xi_3}{\Xi_4}\right] \end{array}\right].
\end{aligned} \tag{185}
$$

In addition to $\Xi_1 \equiv \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$ and $\Xi_2 \equiv \frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1-D)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$[4], we define Ξ_3 and Ξ_4 as

$$
\begin{aligned}
\Xi_3 &\equiv \frac{\delta\left[\Xi_2(D-\nu+1)(\nu-1)-D+(\nu-1)^2\right]}{2(1-\Xi_2)(\nu-1)} = -\delta(2-\nu)\nu\left(\frac{(\lambda)(2-\nu)[\epsilon+(\nu-1)(\epsilon\sigma-1)]+2(1-\lambda)\sigma\epsilon}{2(\nu-1)[(\lambda)(2-\nu)+2(1-\lambda)]}\right), \\
\Xi_4 &\equiv \left[\begin{array}{ll} & 2(\sigma+\phi D)(\nu-1) + \Xi_1(D-\nu+1)(\nu-1) \\ + & \frac{\Xi_1(\nu-1)-2\sigma}{(1-\Xi_2)(\nu-1)}\left[\Xi_2(D-\nu+1)(\nu-1) - D + (\nu-1)^2\right] \end{array}\right] \\
&= 2(\sigma+\phi D)(\nu-1) + 2(D-\nu+1)\sigma + \frac{D(\nu-2)[\Xi_1(\nu-1)-2\sigma]}{(\nu-1)(1-\Xi_2)},
\end{aligned}
$$

where $D \equiv (\nu-1)^2 + \sigma\epsilon\nu(2-\nu) = 1 + \nu(2-\nu)(\epsilon\sigma-1)$. Since the lagged relative price (s_{t-1}) enters the structural equations, the targeting criterion involves forecasts for the relative price as well as the Lagrange multiplier associated with the dynamic of the relative price, $\mathbb{E}_t[s_{t+1}]$ and $\mathbb{E}_t[\gamma_{t+1}^s]$, respectively. In what follows, we try special parametrization to investigate whether (185) can be simplified further.

The case of $\epsilon = 1$: If the elasticity of substitution between Home and Foreign consumption baskets is unity by $\epsilon = 1$, coefficients are rewritten out as

$$
\begin{aligned}
&\Xi_1 = \frac{2(\lambda)(2-\nu)[\sigma(\nu-1)+1-\nu]}{(\lambda)(2-\nu)+2(1-\lambda)D}, \quad \Xi_2 = \frac{(\lambda)(2-\nu)(1-D)}{(\lambda)(2-\nu)+2(1-\lambda)D}, \\
&\Xi_3 = -\delta(2-\nu)\nu\left(\frac{(\lambda)(2-\nu)[1+(\nu-1)(\sigma-1)]+2(1-\lambda)\sigma}{2(\nu-1)[(\lambda)(2-\nu)+2(1-\lambda)]}\right), \\
&\Xi_4 = 2(\sigma+\phi D)(\nu-1) + 2(D-\nu+1)\sigma + \frac{D(\nu-2)[\Xi_1(\nu-1)-2\sigma]}{(\nu-1)(1-\Xi_2)},
\end{aligned}
$$

where $D = (\nu-1)^2 + \sigma\nu(2-\nu) = 1 + \nu(2-\nu)(\sigma-1)$ and we have

$$
\left(\Xi_1 - \frac{2\sigma}{\nu-1}\Xi_2\right) = \frac{2(\lambda)(2-\nu)(\sigma-1)}{\nu-1}\left(\frac{D}{(\lambda)(2-\nu)+2(1-\lambda)D}\right).
$$

[4] Also observe that

$$
\left(\Xi_1 - \frac{2\sigma}{\nu-1}\Xi_2\right) = \frac{2(\lambda)(2-\nu)}{\nu-1}\left(\frac{\sigma\epsilon\nu(2-\nu)(\sigma-1)+(\sigma-1)(\nu-1)^2}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right) = \frac{2(\lambda)(2-\nu)(\sigma-1)}{\nu-1}\left(\frac{D}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}\right).
$$

With $\sigma = 1$, this equation implies $\Xi_1 = \frac{2}{\nu-1}\Xi_2$.

Then the optimal targeting rules are described by three equations:

$$
\begin{aligned}
0 &= \xi \pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\
0 &= \left[\begin{array}{ll} & \frac{2\xi}{\delta} \pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R \right) \frac{4(\sigma + \phi D)}{\delta \Xi_4} \\ + & (m_t - m_{t-1} + f_t - f_{t-1}) \frac{2\sigma\nu(2-\nu)}{\delta \Xi_4 (\nu - 1)} \\ + & \left(\gamma_t^s - \gamma_{t-1}^s \right) \frac{2D}{\Xi_4} \left[\frac{1}{1-\Xi_2} \left(\Xi_1 - \frac{2\sigma \Xi_2}{\nu - 1} \right) + 2\phi \right] \end{array} \right], \\
0 &= \left[\begin{array}{ll} & \gamma_t^s \left(1 + \beta + \delta + \frac{\delta \Xi_2 D}{(\nu-1)(1-\Xi_2)} - 2D \frac{\Xi_3}{\Xi_4} \left[\frac{1}{1-\Xi_2} \left(\Xi_1 - \frac{2\sigma\Xi_2}{\nu-1} \right) + 2\phi \right] \right) - \beta \mathbb{E}_t \gamma_{t+1}^s - \gamma_{t-1}^s \\ + & \frac{\xi}{\delta} \frac{\nu(2-\nu)}{2} \left[(1+\beta) s_t - s_{t-1} - \beta \mathbb{E}_t s_{t+1} \right] \\ - & \widetilde{y}_t^R \frac{\Xi_3}{\Xi_4} \frac{4(\sigma + \phi D)}{\delta} \\ - & (m_t + f_t) \frac{\nu(2-\nu)}{\nu - 1} \left[\frac{1}{2} + \frac{2\sigma}{\delta} \frac{\Xi_3}{\Xi_4} \right] \end{array} \right].
\end{aligned}
$$

Thus, simple targeting rules are not available under $\epsilon = 1$.

The case of $\sigma = 1$: If the consumption utility is in log by $\sigma = 1$, coefficients are

$$
\begin{aligned}
\Xi_1 &= \frac{2(\lambda)(2-\nu)\nu(\epsilon - 1)}{(\lambda)(2-\nu)(\epsilon\nu - \nu + 1) + 2(1-\lambda)D}, & \Xi_2 &= \frac{(\lambda)(2-\nu)\nu(\epsilon-1)(\nu-1)}{(\lambda)(2-\nu)(\epsilon\nu - \nu + 1) + 2(1-\lambda)D}, \\
\Xi_3 &= -\delta(2-\nu)\nu \left(\frac{(\lambda)(2-\nu)[1 - \nu + \epsilon\nu] + 2(1-\lambda)\epsilon}{2(\nu-1)[(\lambda)(2-\nu) + 2(1-\lambda)]} \right), & \Xi_4 &= 2D \left(\phi(\nu - 1) + \frac{1}{\nu - 1} \right),
\end{aligned}
$$

where $D = (\nu-1)^2 + \epsilon\nu(2-\nu) = 1 + \nu(2-\nu)(\epsilon - 1)$ and we have $\Xi_1 = \frac{2}{\nu-1}\Xi_2$. The optimal targeting rules are described by three equations:

$$
\begin{aligned}
0 &= \xi \pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\
0 &= \left[\begin{array}{ll} & \frac{2\xi}{\delta} \pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R \right) \frac{4(1+\phi D)}{\delta \Xi_4} + (m_t - m_{t-1} + f_t - f_{t-1}) \frac{2\epsilon\nu(2-\nu)}{\delta\Xi_4(\nu-1)} \\ + & \left(\gamma_t^s - \gamma_{t-1}^s \right) \frac{4D\phi}{\Xi_4} \end{array} \right], \\
0 &= \left[\begin{array}{ll} & \gamma_t^s \left(1 + \beta + \delta + \frac{\delta \Xi_2 D}{(\nu-1)(1-\Xi_2)} - 4D\phi \frac{\Xi_3}{\Xi_4} \right) - \beta \mathbb{E}_t \gamma_{t+1}^s - \gamma_{t-1}^s \\ + & \frac{\xi}{\delta} \frac{\nu(2-\nu)}{2} \left[(1+\beta)s_t - s_{t-1} - \beta \mathbb{E}_t s_{t+1} \right] - \widetilde{y}_t^R \frac{\Xi_3}{\Xi_4} \frac{4(1+\phi D)}{\delta} - (m_t + f_t) \frac{\epsilon\nu(2-\nu)}{\nu-1} \left[\frac{1}{2} + \frac{2}{\delta} \frac{\Xi_3}{\Xi_4} \right] \end{array} \right].
\end{aligned}
$$

Hence simple targeting rules are not available under $\sigma = 1$, either.

The case of $\phi = 0$: If the inverse of the Frisch labor-supply elasticity is zero ($\phi = 0$), coefficients collapse into

$$
\begin{aligned}
\Xi_1 &= \frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu - 1) + 1 - \nu]}{(\lambda)(2-\nu)(\epsilon\nu - \nu + 1) + 2(1-\lambda)D}, & \Xi_2 &= \frac{(\lambda)(2-\nu)(\epsilon\nu - \nu + 1 - D)}{(\lambda)(2-\nu)(\epsilon\nu - \nu + 1) + 2(1-\lambda)D}, \\
\Xi_3 &= -\delta(2-\nu)\nu \left(\frac{(\lambda)(2-\nu)[\epsilon + (\nu-1)(\epsilon\sigma - 1)] + 2(1-\lambda)\sigma\epsilon}{2(\nu-1)[(\lambda)(2-\nu) + 2(1-\lambda)]} \right), & \Xi_4 &= 2\sigma(\nu - 1) + 2(D - \nu + 1)\sigma + \frac{D(\nu-2)[\Xi_1(\nu-1) - 2\sigma]}{(\nu-1)(1-\Xi_2)},
\end{aligned}
$$

where $D=(\nu-1)^2+\sigma\epsilon\nu(2-\nu)=1+\nu(2-\nu)(\epsilon\sigma-1)$. The optimal targeting rules are described by three equations:

$$
\begin{aligned}
0 &= \xi\pi_t^W+\widetilde{y}_t^W-\widetilde{y}_{t-1}^W, \\
0 &= \left[\begin{array}{ll} & \frac{2\xi}{\delta}\pi_t^R+\left(\widetilde{y}_t^R-\widetilde{y}_{t-1}^R\right)\frac{4\sigma}{\delta\Xi_4} \\ + & (m_t-m_{t-1}+f_t-f_{t-1})\frac{2\epsilon\sigma\nu(2-\nu)}{\delta\Xi_4(\nu-1)} \\ + & \left(\gamma_t^s-\gamma_{t-1}^s\right)\frac{2D}{\Xi_4}\left[\frac{1}{1-\Xi_2}\left(\Xi_1-\frac{2\sigma\Xi_2}{\nu-1}\right)\right]\end{array}\right], \\
0 &= \left[\begin{array}{ll} & \gamma_t^s\left(1+\beta+\delta+\frac{\delta\Xi_2 D}{(\nu-1)(1-\Xi_2)}-2D\frac{\Xi_3}{\Xi_4}\left[\frac{1}{1-\Xi_2}\left(\Xi_1-\frac{2\sigma\Xi_2}{\nu-1}\right)\right]\right)-\beta\mathbb{E}_t\gamma_{t+1}^s-\gamma_{t-1}^s \\ + & \frac{\xi}{\delta}\frac{\nu(2-\nu)}{2}\left[(1+\beta)s_t-s_{t-1}-\beta\mathbb{E}_t s_{t+1}\right] \\ - & \widetilde{y}_t^R\frac{\Xi_3}{\Xi_4}\frac{4\sigma}{\delta} \\ - & (m_t+f_t)\frac{\epsilon\nu(2-\nu)}{\nu-1}\left[\frac{1}{2}+\frac{2\sigma}{\delta}\frac{\Xi_3}{\Xi_4}\right]\end{array}\right].
\end{aligned}
$$

Hence there are no simple targeting rules available under $\phi=0$.

The case of $\sigma=1$ and $\phi=0$: If we set the relative risk aversion to one ($\sigma=1$) and restrict the utility to be quasi-linear in labor by setting ϕ to zero, coefficients translate into

$$
\begin{aligned}
\Xi_1 &= \frac{2(\lambda)(2-\nu)\nu(\epsilon-1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}, & \Xi_2 &= \frac{(\lambda)(2-\nu)\nu(\epsilon-1)(\nu-1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}, \\
\Xi_3 &= -\delta(2-\nu)\left(\frac{(\lambda)(2-\nu)\nu[1-\nu+\epsilon\nu]+2(1-\lambda)\epsilon\nu}{2(\nu-1)[(\lambda)(2-\nu)+2(1-\lambda)]}\right), & \Xi_4 &= \frac{2D}{\nu-1},
\end{aligned}
$$

where $D=(\nu-1)^2+\epsilon\nu(2-\nu)=1+\nu(2-\nu)(\epsilon-1)$ and we have $\Xi_1(\nu-1)=2\Xi_2$. Then relatively simple expression for the targeting rules emerges as follows:

$$
\begin{aligned}
0 &= \xi\pi_t^W+\widetilde{y}_t^W-\widetilde{y}_{t-1}^W, \\
0 &= \xi\pi_t^R+\left(\widetilde{y}_t^R-\widetilde{y}_{t-1}^R\right)\frac{\nu-1}{D}+(m_t-m_{t-1}+f_t-f_{t-1})\frac{\epsilon\nu(2-\nu)}{2D}.
\end{aligned}
\tag{186}
$$

In the following section J.2, we show that the evolution of s_t and f_t becomes independent of policy under LCP with $\sigma=1$ and $\phi=0$.

J.2 Independence of s_t and f_t on Policy under LCP with $\sigma=1$ and $\phi=0$

Claim 7. *If the relative risk aversion is unity ($\sigma=1$) and the inverse of the Frisch labor-supply elasticity is zero ($\phi=0$), the relative price between Home and Foreign goods (s_t) and demand imbalance (f_t) are independent of policy under LCP ($\chi=1$).*

Proof. From (184), demand imbalance (f_t) and the dynamic for the price of Foreign goods relative to Home goods (s_t) are given by

$$
\begin{aligned}
&(\gamma_t^s) \quad \Delta s_t=-\delta\left[s_t-\overline{s}_t+2\phi\widetilde{y}_t^R+f_t\right]+\beta\mathbb{E}_t\left[\Delta s_{t+1}\right]-2u_t^R, \\
&(\gamma_t) \quad s_t=\overline{s}_t+\frac{2\sigma}{D}\widetilde{y}_t^R-\frac{\nu-1}{D}(m_t+f_t), \\
&(\gamma_t^f) \quad f_t=\Xi_1\left(\widetilde{y}_t^R+\overline{y}_t^R\right)-\Xi_2 m_t-2\zeta_{c,t}^R\frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D},
\end{aligned}
$$

where coefficients are $\Xi_1\equiv\frac{2(\lambda)(2-\nu)[\sigma(\epsilon\nu-1)+1-\nu]}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, $\Xi_2\equiv\frac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1-D)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, and $D\equiv(\nu-1)^2+\sigma\epsilon\nu(2-\nu)=$

$1+\nu(2-\nu)(\epsilon\sigma-1)$. Assume $\epsilon \neq 1$, so that $\Xi_1 \neq 0$ under $\sigma = 1$. Rewrite the third equation as

$$\widetilde{y}_t^R = \tfrac{1}{\Xi_1} f_t - \overline{y}_t^R + \tfrac{\Xi_2}{\Xi_1} m_t + 2\zeta_{c,t}^R \left(\tfrac{(\epsilon\nu-\nu+1)}{2[\sigma\epsilon\nu-\nu+1-\sigma]} \right) \tag{187}$$

and feed it into the second equation to obtain

$$\begin{aligned} s_t &= \bar{s}_t + \tfrac{2\sigma}{D}\widetilde{y}_t^R - \tfrac{\nu-1}{D}(m_t+f_t), \\ &= \bar{s}_t + \left(\tfrac{2\sigma}{D}\tfrac{1}{\Xi_1} - \tfrac{\nu-1}{D}\right) f_t + \left(\tfrac{2\sigma}{D}\tfrac{\Xi_2}{\Xi_1} - \tfrac{\nu-1}{D}\right) m_t + \tfrac{2\sigma}{D}\left(\tfrac{(\epsilon\nu-\nu+1)}{[\sigma\epsilon\nu-\nu+1-\sigma]}\zeta_{c,t}^R - \overline{y}_t^R \right), \end{aligned}$$

which can be written out for f_t as

$$f_t = \frac{(s_t-\bar{s}_t)}{\left(\frac{2\sigma}{D}\frac{1}{\Xi_1} - \frac{\nu-1}{D}\right)} - \frac{\left(\frac{2\sigma}{D}\frac{\Xi_2}{\Xi_1} - \frac{\nu-1}{D}\right)}{\left(\frac{2\sigma}{D}\frac{1}{\Xi_1} - \frac{\nu-1}{D}\right)} m_t - \frac{\frac{2\sigma}{D}}{\left(\frac{2\sigma}{D}\frac{1}{\Xi_1} - \frac{\nu-1}{D}\right)} \left(\tfrac{(\epsilon\nu-\nu+1)}{[\sigma\epsilon\nu-\nu+1-\sigma]}\zeta_{c,t}^R - \overline{y}_t^R \right). \tag{188}$$

Applying $\sigma = 1$ and $\phi = 0$, coefficients translate into $\Xi_1 = \frac{2(\lambda)(2-\nu)\nu(\epsilon-1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$ and $\frac{2\Xi_2}{\Xi_1} = \nu - 1$ with $\epsilon \neq 1$. Therefore m_t disappears in (188), and f_t and the dynamic for s_t are characterized by the system of two equations given by

$$\begin{aligned} \Delta s_t &= -\delta\left[s_t - \bar{s}_t + f_t\right] + \beta\mathbb{E}_t\left[\Delta s_{t+1}\right] - 2u_t^R, \\ f_t &= \tfrac{D\Xi_1}{(2-\Xi_1(\nu-1))}(s_t - \bar{s}_t) + \tfrac{2\Xi_1}{(2-\Xi_1(\nu-1))}\overline{y}_t^R - \left(\tfrac{\Xi_1}{\nu(\epsilon-1)}\right)\left(\tfrac{2(\epsilon\nu-\nu+1)}{(2-\Xi_1(\nu-1))}\right)\zeta_{c,t}^R, \end{aligned}$$

where $\frac{\Xi_1}{\nu(\epsilon-1)} = \frac{2(\lambda)(2-\nu)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$. Since shocks, u_t^R and $\zeta_{c,t}^R$, and efficient allocations, $\bar{s}_t$ and $\overline{y}_t^R$, are all exogenous, these two equations completely govern the evolution of s_t and f_t. Therefore, s_t and f_t are autonomous. □

J.3 Allocations under LCP with $\sigma = 1$ and $\phi = 0$

Since optimal monetary policy under LCP allows for simple targeting rules only with the special parametrization of $\sigma = 1$ and $\phi = 0$ as we show in section J.1, we search for allocations in their closed form to obtain intuition under that parametrization. In this subsection, our goal is to describe allocations in terms of exogenous shocks by solving constraints in (184) together with simple targeting rules (186):

$$\begin{aligned} &(i) && (1+\delta+\beta)s_t = s_{t-1} + \beta\mathbb{E}_t s_{t+1} - \delta f_t + \delta\bar{s}_t - 2u_t^R, \\ &(ii) && f_t = \tfrac{D\Xi_1}{(2-\Xi_1(\nu-1))}s_t + \tfrac{2}{(2-\Xi_1(\nu-1))}\left[\Xi_1(\nu-2) - \left(\tfrac{\Xi_1}{\nu(\epsilon-1)}\right)\right]\zeta_{c,t}^R, \\ &(iii) && 0 = \xi\pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\ &(iv) && \pi_t^W = \delta\widetilde{y}_t^W + \beta\mathbb{E}_t\pi_{t+1}^W + u_t^W, \\ &(v) && 0 = \xi\pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right)\tfrac{\nu-1}{D} + (m_t - m_{t-1} + f_t - f_{t-1})\tfrac{\epsilon\nu(2-\nu)}{2D}, \\ &(vi) && \pi_t^R = \tfrac{\delta(\nu-1)}{D}\widetilde{y}_t^R + \tfrac{\delta(D-\nu+1)(\nu-1)}{2D}f_t + \tfrac{\delta\left(D-(\nu-1)^2\right)}{2D}m_t + \beta\mathbb{E}_t\pi_{t+1}^R + (\nu-1)u_t^R, \\ &(vii) && f_t = \Xi_1\left(\widetilde{y}_t^R + \overline{y}_t^R\right) - \Xi_2 m_t - 2\zeta_{c,t}^R\tfrac{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}. \end{aligned} \tag{189}$$

Here we replace the condition associated with γ_t in (184) with (188), given by (ii). Efficient allocations and coefficients are $\overline{y}_t^R = Da_t^R + (\nu-1)\zeta_{c,t}^R$, $\bar{s}_t = 2a_t^R$, $\Xi_1 = \frac{2(\lambda)(2-\nu)\nu(\epsilon-1)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, $\frac{\Xi_1}{\nu(\epsilon-1)} = \frac{2(\lambda)(2-\nu)}{(\lambda)(2-\nu)(\epsilon\nu-\nu+1)+2(1-\lambda)D}$, $\Xi_2 = \left(\frac{\nu-1}{2}\right)\Xi_1$, and $D = (\nu-1)^2 + \epsilon\nu(2-\nu) = 1 + \nu(2-\nu)(\epsilon-1)$.

Characterize s_t and f_t: As we discuss in section J.2, (i) and (ii) determine s_t and f_t independently from other endogenous variables. Combining (i) and (ii), we obtain

$$\left(1+\delta+\beta+\tfrac{\delta D\Xi_1}{(2-\Xi_1(\nu-1))}\right)s_t = s_{t-1}+\beta\mathbb{E}_t s_{t+1}+\underbrace{\left[\delta 2a_t^R - 2u_t^R - \frac{2\delta\left(\Xi_1(\nu-2)-\frac{\Xi_1}{\nu(\epsilon-1)}\right)}{(2-\Xi_1(\nu-1))}\zeta_{c,t}^R\right]}_{-\Phi_{s,t}},$$

$$\text{that is,}\quad \mathbb{E}_t\left[\beta s_{t+1}-(1+\delta+\beta+\tfrac{\delta D\Xi_1}{(2-\Xi_1(\nu-1))})s_t+s_{t-1}\right]=\Phi_{s,t},$$

where we take into account all exogenous shocks by

$$\Phi_{s,t}\equiv -\delta 2a_t^R+2u_t^R+\tfrac{2\delta\left(\Xi_1(\nu-2)-\frac{\Xi_1}{\nu(\epsilon-1)}\right)}{(2-\Xi_1(\nu-1))}\zeta_{c,t}^R.$$

Define eigenvalues as

$$x^s=\tfrac{\left[1+\delta+\beta+\frac{\delta D\Xi_1}{(2-\Xi_1(\nu-1))}\right]\pm\sqrt{\left[1+\delta+\beta+\frac{\delta D\Xi_1}{(2-\Xi_1(\nu-1))}\right]^2-4\beta}}{2\beta}\quad\text{with}\quad |x_1^s|<1<|x_2^s|.$$

Then we solve $\mathbb{E}_t\left[s_t\beta L\left(\frac{1}{L}-x_1^s\right)\left(\frac{1}{L}-x_2^s\right)\right]=\Phi_{s,t}$ for s_t:

$$\begin{aligned} s_t &= x_1^s s_{t-1}-\tfrac{1}{\beta x_2^s}\textstyle\sum_{h=0}^{\infty}\left(\frac{1}{x_2^s}\right)^h\mathbb{E}_t\Phi_{s,t+h}\\ &= x_1^s s_{t-1}-\tfrac{1}{\beta x_2^s}\left[\Phi_{s,t}+\left(\tfrac{1}{x_2^s}\right)\mathbb{E}_t\Phi_{s,t+1}+\left(\tfrac{1}{x_2^s}\right)^2\mathbb{E}_t\Phi_{s,t+2}+\left(\tfrac{1}{x_2^s}\right)^3\mathbb{E}_t\Phi_{s,t+3}+\cdots\right], \end{aligned}\tag{190}$$

which can be rewritten as

$$s_t = (x_1^s)^t s_0-\textstyle\sum_{l=0}^{t-1}(x_1^s)^l\left[\frac{1}{\beta x_2^s}\sum_{h=0}^{\infty}\left(\frac{1}{x_2^s}\right)^h\mathbb{E}_{t-l}\Phi_{s,t-l+h}\right].$$

Therefore, given s_t, f_t is pinned down by (ii):

$$f_t = \tfrac{D\Xi_1}{(2-\Xi_1(\nu-1))}s_t+\tfrac{2}{(2-\Xi_1(\nu-1))}\left[\Xi_1(\nu-2)-\left(\tfrac{\Xi_1}{\nu(\epsilon-1)}\right)\right]\zeta_{c,t}^R.\tag{191}$$

Characterize $\widetilde{y}_t^W$: (iii) and (iv) imply that characterization of $\widetilde{y}_t^W$ under LCP is identical to that under PCP. The solution is given by (178) under $\sigma=1$ and $\phi=0$.

Characterize $\widetilde{y}_t^R$: Since f_t is autonomous under $\sigma=1$ and $\phi=0$, we solve for $\widetilde{y}_t^R$ in terms of f_t and other exogenous shocks. Note that $\epsilon=1$ implies $\Xi_1=\Xi_2=0$ and claim 7 shows (vii) reduces to $f_t=-\left(\zeta_{c,t}-\zeta_{c,t}^*\right)\left(\frac{\lambda(2-\nu)}{2-\nu\lambda}\right)$. Therefore (v) and (vi) together with $s_t=\bar{s}_t+\frac{2\sigma}{D}\widetilde{y}_t^R-\frac{\nu-1}{D}(m_t+f_t)$ in (184) pin down $\widetilde{y}_t^R$ given f_t and s_t after substituting for π_t^R and m_t. In what follows, we proceed with $\epsilon\neq 1$, so that $\Xi_1\neq 0$ and $\Xi_2\neq 0$.

Rewriting (vii) for m_t, we obtain

$$m_t=\tfrac{2}{\nu-1}\widetilde{y}_t^R-\tfrac{2}{\nu-1}\tfrac{1}{\Xi_1}f_t+\tfrac{2}{\nu-1}Da_t^R+\tfrac{2}{\nu-1}\left(\nu-2-\tfrac{1}{\nu(\epsilon-1)}\right)\zeta_{c,t}^R,$$

where $\Xi_2 = \left(\frac{\nu-1}{2}\right)\Xi_1$. Plugging m_t in (v) and (vi), we can derive two equations for π_t^R and $\widetilde{y}_t^R$ given f_t:

$$\xi\pi_t^R = \begin{pmatrix} -\left(\frac{1}{\nu-1}\right)\left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right) + \left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right)\left(\frac{2}{\Xi_1} - (\nu-1)\right)(f_t - f_{t-1}) \\ -\left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right)2D\left(a_t^R - a_{t-1}^R\right) - \left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right)\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right)2\left(\zeta_{c,t}^R - \zeta_{c,t-1}^R\right) \end{pmatrix},$$

$$\xi\pi_t^R = \begin{pmatrix} \frac{\xi\delta}{\nu-1}\widetilde{y}_t^R + \xi\delta\left[\frac{(D-\nu+1)(\nu-1)}{2D} - \frac{\left(D-(\nu-1)^2\right)}{D(\nu-1)\Xi_1}\right]f_t \\ +\beta\mathbb{E}_t\left[\xi\pi_{t+1}^R\right] + \frac{\xi\delta\left(D-(\nu-1)^2\right)}{\nu-1}a_t^R + \frac{\xi\delta\left(D-(\nu-1)^2\right)}{D(\nu-1)}\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right)\zeta_{c,t}^R + \xi(\nu-1)u_t^R \end{pmatrix}.$$

Equating these two to remove π_t^R and rearranging terms, we obtain

$$\mathbb{E}_t\left[\beta\widetilde{y}_{t+1}^R - (1+\beta+\xi\delta)\widetilde{y}_t^R + \widetilde{y}_{t-1}^R\right] = \underbrace{\begin{pmatrix} +\frac{\xi\delta}{D}\left[\frac{(D-\nu+1)(\nu-1)^2}{2} - \frac{\left(D-(\nu-1)^2\right)}{\Xi_1}\right]f_t \\ +\left(\frac{\epsilon\nu(2-\nu)}{2D}\right)\left(\frac{2}{\Xi_1} - (\nu-1)\right)\left(\beta\mathbb{E}_t f_{t+1} - (1+\beta)f_t + f_{t-1}\right) \\ -\left(\frac{\epsilon\nu(2-\nu)}{2D}\right)2D\left(\beta\mathbb{E}_t a_{t+1}^R - (1+\beta)a_t^R + a_{t-1}^R\right) \\ -\left(\frac{\epsilon\nu(2-\nu)\left(\nu-2-\frac{1}{\nu(\epsilon-1)}\right)}{2D}\right)2\left(\beta\mathbb{E}_t\zeta_{c,t+1}^R - (1+\beta)\zeta_{c,t}^R + \zeta_{c,t-1}^R\right) \\ +\xi\delta\left(D-(\nu-1)^2\right)a_t^R \\ +\frac{\xi\delta\left(D-(\nu-1)^2\right)}{D}\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right)\zeta_{c,t}^R + \xi(\nu-1)^2u_t^R \end{pmatrix}}_{\equiv\Phi_{L,t}}.$$

Here all exogenous variations are captured by

$$\Phi_{L,t} \equiv \begin{pmatrix} +\frac{\xi\delta}{D}\left[\frac{(D-\nu+1)(\nu-1)^2}{2} - \frac{\left(D-(\nu-1)^2\right)}{\Xi_1}\right]f_t \\ +\left(\frac{\epsilon\nu(2-\nu)}{2D}\right)\left(\frac{2}{\Xi_1} - (\nu-1)\right)\left(\beta\mathbb{E}_t f_{t+1} - (1+\beta)f_t + f_{t-1}\right) \\ -\left(\frac{\epsilon\nu(2-\nu)}{2D}\right)2D\left(\beta\mathbb{E}_t a_{t+1}^R - (1+\beta)a_t^R + a_{t-1}^R\right) \\ -\left(\frac{\epsilon\nu(2-\nu)\left(\nu-2-\frac{1}{\nu(\epsilon-1)}\right)}{2D}\right)2\left(\beta\mathbb{E}_t\zeta_{c,t+1}^R - (1+\beta)\zeta_{c,t}^R + \zeta_{c,t-1}^R\right) \\ +\xi\delta\left(D-(\nu-1)^2\right)a_t^R \\ +\frac{\xi\delta\left(D-(\nu-1)^2\right)}{D}\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right)\zeta_{c,t}^R + \xi(\nu-1)^2u_t^R \end{pmatrix}. \tag{192}$$

Define eigenvalues as

$$x^L = \frac{[1+\beta+\xi\delta] \pm \sqrt{[1+\beta+\xi\delta]^2 - 4\beta}}{2\beta} \quad \text{with} \quad |x_1^L| < 1 < |x_2^L|.$$

Therefore we solve for

$$\begin{aligned} \widetilde{y}_t^R &= x_1^L\widetilde{y}_{t-1}^R + \frac{1}{\beta}\left(\frac{-1}{x_2^L}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^L}\right)^h\mathbb{E}_t\Phi_{L,t+h} \\ &= \left(x_1^L\right)^t\widetilde{y}_0^R - \sum_{l=0}^{t-1}\left(x_1^L\right)^l\left[\frac{1}{\beta}\left(\frac{1}{x_2^L}\right)\sum_{h=0}^{\infty}\left(\frac{1}{x_2^L}\right)^h\mathbb{E}_{t-l}\Phi_{L,t-l+h}\right], \end{aligned} \tag{193}$$

which can be rewritten as

$$\begin{aligned}
\widetilde{y}^R_{t+j} &= \left(x^L_1\right)^j \widetilde{y}^R_t - \sum_{l=0}^{j-1} \left(x^L_1\right)^l \left[\frac{1}{\beta}\left(\frac{1}{x^L_2}\right) \sum_{h=0}^{\infty} \left(\frac{1}{x^L_2}\right)^h \mathbb{E}_{t+j-l} \Phi_{L,t+j-l+h}\right], \\
\text{that is, } \mathbb{E}_t \widetilde{y}^R_{t+j} &= \left(x^L_1\right)^j \widetilde{y}^R_t - \sum_{l=0}^{j-1} \left(x^L_1\right)^l \left[\frac{1}{\beta}\left(\frac{1}{x^L_2}\right) \sum_{h=0}^{\infty} \left(\frac{1}{x^L_2}\right)^h \mathbb{E}_t \Phi_{L,t+j-l+h}\right].
\end{aligned}$$

Characterize π^W_t, m_t, π^R_t: Given solutions for s_t, f_t, $\widetilde{y}^W_t$, and $\widetilde{y}^R_t$, other variables are determined by

$$\begin{aligned}
\xi \pi^W_t &= -\left(\widetilde{y}^W_t - \widetilde{y}^W_{t-1}\right), \\
m_t &= \frac{2}{\nu-1}\widetilde{y}^R_t - \frac{2}{(\nu-1)\Xi_1} f_t + \frac{2D}{\nu-1} a^R_t + \frac{2}{\nu-1}\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right)\zeta^R_{c,t}, \\
\xi \pi^R_t &= \begin{pmatrix} -\left(\frac{1}{\nu-1}\right)\left(\widetilde{y}^R_t - \widetilde{y}^R_{t-1}\right) + \left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right)\left(\frac{2}{\Xi_1} - (\nu-1)\right)(f_t - f_{t-1}) \\ -\left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right) 2D\left(a^R_t - a^R_{t-1}\right) - \left(\frac{\epsilon\nu(2-\nu)}{2D(\nu-1)}\right)\left(\nu - 2 - \frac{1}{\nu(\epsilon-1)}\right) 2\left(\zeta^R_{c,t} - \zeta^R_{c,t-1}\right) \end{pmatrix},
\end{aligned} \tag{194}$$

where $\left(\frac{2}{\Xi_1} - \nu + 1\right) = \frac{(\lambda)(2-\nu)[(\epsilon\nu-\nu+1)-\nu(\nu-1)(\epsilon-1)]+2(1-\lambda)D}{(\lambda)(2-\nu)\nu(\epsilon-1)}$ and $D = 1 + \nu(2-\nu)(\epsilon-1)$.

J.4 Comparison with LCP Allocations under Uncontingent Bonds when $\sigma = \epsilon = 1$ and $\phi = 0$

In this subsection, we establish the equivalence result between optimal monetary policy responses of our economy with capital controls and the economy with uncontingent bonds when producers price in the consumer's currency under the Cole and Obstfeld specification: $\sigma = \epsilon = 1$ and $\phi = 0$.

Under the economy with uncontingent bonds, resource constraints are replaced with

$$\begin{aligned}
(H) \quad & P_{Ft}C_{Ft} + \frac{B_t}{R_t} = \mathcal{E}_t P^*_{Ht} C^*_{Ht} + B_{t-1}, \\
(F) \quad & \mathcal{E}_t P^*_{Ht} C^*_{Ht} + \frac{B^*_t}{R_t} = P_{Ft}C_{Ft} + B^*_{t-1},
\end{aligned}$$

where $B_t + B^*_t = 0$ for all t. Here B_t and B^*_t denote nominal uncontingent bond holdings in units of Home currency traded in international financial markets.

Following derivations in Engel (2014) and Corsetti et al. (2020), we can obtain targeting rules and equilibrium conditions under the economy with uncontingent bonds as

$$\begin{array}{llcl} (i) & 0 & = & \xi\pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\ (ii) & \pi_t^W & = & \beta\mathbb{E}_t\pi_{t+1}^W + \delta\widetilde{y}_t^W + u_t^W, \\ (iii) & 0 & = & \xi\pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right)(\nu-1) + (m_t + f_t - m_{t-1} - f_{t-1})\left(\frac{\nu(2-\nu)}{2}\right), \\ (iv) & \pi_t^R & = & \beta\mathbb{E}_t\pi_{t+1}^R + \delta(\nu-1)\widetilde{y}_t^R + \frac{\delta(2-\nu)(\nu-1)}{2}f_t + \frac{\delta\nu(2-\nu)}{2}m_t + (\nu-1)u_t^R, \\ (v) & 0 & = & (1+\beta+\delta)s_t - s_{t-1} - \beta\mathbb{E}_t s_{t+1} + \delta f_t - \delta\overline{s}_t + 2u_t^R, \\ (vi) & \widetilde{s}_t & = & 2\widetilde{y}_t^R - (\nu-1)(m_t + f_t), \\ (vii) & f_t & = & \frac{-2}{(2-\nu)}(\beta b_t - b_{t-1}) - 2\zeta_{c,t}^R, \\ (viii) & f_t & = & \mathbb{E}_t f_{t+1}, \end{array} \tag{195}$$

where we define the linearized bond holdings as $b_t \equiv \frac{B_t}{PC}$ and we make use of $\beta R = 1$.

On the other hand, with $\sigma = \epsilon = 1$ and $\phi = 0$, LCP equilibrium conditions in (189) are reduced to

$$\begin{array}{llcl} (i) & 0 & = & \xi\pi_t^W + \widetilde{y}_t^W - \widetilde{y}_{t-1}^W, \\ (ii) & \pi_t^W & = & \beta\mathbb{E}_t\pi_{t+1}^W + \delta\widetilde{y}_t^W + u_t^W, \\ (iii) & 0 & = & \xi\pi_t^R + \left(\widetilde{y}_t^R - \widetilde{y}_{t-1}^R\right)(\nu-1) + (m_t + f_t - m_{t-1} - f_{t-1})\left(\frac{\nu(2-\nu)}{2}\right), \\ (iv) & \pi_t^R & = & \beta\mathbb{E}_t\pi_{t+1}^R + \delta(\nu-1)\widetilde{y}_t^R + \frac{\delta(2-\nu)(\nu-1)}{2}f_t + \frac{\delta\nu(2-\nu)}{2}m_t + (\nu-1)u_t^R, \\ (v) & 0 & = & (1+\beta+\delta)s_t - s_{t-1} - \beta\mathbb{E}_t s_{t+1} + \delta f_t - \delta\overline{s}_t + 2u_t^R, \\ (vi) & \widetilde{s}_t & = & 2\widetilde{y}_t^R - (\nu-1)(m_t + f_t), \\ (vii) & f_t & = & -2\zeta_{c,t}^R\frac{(\lambda)(2-\nu)}{2-\lambda\nu}. \end{array} \tag{196}$$

Here the equation (vi) is from the first constraint in (184). Therefore, comparing the equilibrium system of our economy in (196) with that of the uncontingent-bond economy in (195), we can observe that the only difference is the equation for demand imbalance (f_t). Since demand imbalance (f_t) and the relative price of Foreign goods (s_t) are exogenous to policy in both economies under the Cole and Obstfeld specification ($\sigma = \epsilon = 1$, $\phi = 0$), policy responses to f_t are identical between these two economies. This also holds true for policy responses to net capital flows which are independent of policy by equation (92).