1 Subsequences of a Sequence

Subsequence of a sequence:

- Given a sequence (x_n), a subsequence (y_m) is formed by choosing an infinite collection of the entries of the original sequence in the order that these elements appear in the original sequence.

For example let us consider a sequence x such that $x : \mathbb{N} \to \mathbb{R}$,

now consider the subsequences of the above sequence $x_1, x_2, \ldots, x_n$, so a subsequence is a composite function $x \circ m : \mathbb{N} \to \mathbb{R}$ where $m : \mathbb{N} \to \mathbb{N}$ which is increasing and one to one, so $m_1 = 1, m_2 = 2, \ldots, m_n = n$.

the subsequnce can be denoted as $x(m(n))$ or x_{m_n}.

Proposition 5:

- if (x_n) is a convergent sequence with limit l_x, then every subsequence (x_{n_k}) of (x_n) converges to l_x.

Proof: Suppose $x_n \to l_x$, then we want to prove that $x_{n_k} \to l_x$.

Notice that $n_k \geq k$ because n_k is strictly increasing sequence such that $n_1 < n_2 < n_3 < \ldots$, we are basically saying that the nth term of a subsequence must have a subscript greater than or equal to n.

Now consider $\epsilon > 0$ and we drop some N terms of the original sequence then whatever is left from the original sequnce lies in the interal $(l_x - \epsilon, l_x + \epsilon)$ now instead of dropping those N terms from the original sequence we drop them from the subsequence then also the remaining terms will lie in $(l_x - \epsilon, l_x + \epsilon)$.

Proposition 6:

- Let (x_n) be a sequence in $\mathbb{R}$. We define

$$\limsup x_n = \lim_{N \to \infty} \sup \{x_n : n > N\}$$

and

$$\liminf x_n = \lim_{N \to \infty} \inf \{x_n : n > N\}$$

Then, $\lim x_n$ exists if and only if $\limsup x_n = \liminf x_n$.

Suppose $y_m = \sup_{n>M} x_n$ and consider $x_n = (-1)^n$ then notice that

$$y_1 = 1$$
$$y_2 = 1$$
$$y_M = 1$$

and suppose $z_M = \inf_{n>M} x_n$ then notice that,

$$z_M = -1$$

or if we consider the sequence $x_n = -1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \ldots$then,

$$\begin{aligned} y_1 &= \frac{1}{2} \\ y_2 &= \frac{1}{4} \\ y_3 &= \frac{1}{4} \\ y_M = \begin{cases} \frac{1}{m+2} & \text{if m is even} \\ \frac{1}{m+1} & \text{if m is odd} \end{cases} \end{aligned}$$

Proposition 7:

- Every convergent real sequence is bounded.

Proof:

- Take any (x_n) with $x_n \to x$ for some real number x.
- Then there must exist a natural number M such that $|x_m - x| < 1$, and hence $|x_m| < |x| + 1$, for all $m \geq M$.
- But then $|x_m| \leq \max\{|x| + 1, |x_1|, \ldots, |x_M|\}$ for all $m \in \mathbb{N}$.
- Hence, (x_n) is bounded.

Note that if we have a sequence x_n such that $x_n \to 0$ and we have another bounded sequence y_n then, $x_n \times y_n$ will also be a convergent infact, $x_n \times y_n \to 0$ as well.

Monotonic sequence :

- A real sequence (x_n) is said to be increasing if $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$
- A real sequence (x_n) is said to be strictly increasing if $x_n < x_{n+1}$ for each $n \in \mathbb{N}$.
- It is said to be (strictly) decreasing if $(-x_n)$ is (strictly) increasing.

A real sequence which is either increasing or decreasing is referred to as a monotonic sequence

Proposition 8 :

- Every increasing (decreasing) real sequence that is bounded from above (below) converges.

Proof:

- Let (x_n) be an increasing sequence which is bounded from above, and let $S = \{x_1, x_2, \ldots\}$.
- Let $x = \sup S$. We claim that $x_n \to x$.
- To show this, pick an arbitrary $\epsilon > 0$.
- Since x is the least upper boundnof $S, x - \epsilon$ cannot be an upper bound of S, so $x_M > x - \epsilon$ for some $M \in \mathbb{N}$.
- Since (x_n) is increasing, we must then have $x \geq x_m \geq x_M > x - \epsilon$, so $|x_m - x| < \epsilon$, for all $m \geq M$.
- Hence $x_n \to x$.
- The proof of the second claim is analogous.

Proposition 9:

- Every real sequence has a monotonic subsequence.

Proof:

- Take any (x_n) and define $S_m = \{x_m, x_{m+1}, \ldots\}$ for each $m \in \mathbb{N}$.
- If there is no maximum element in S_1, then it is easy to see that (x_n) has a monotonic subsequence.
- (Let $x_{n_1} = x_1$, let x_{n_2} be the first term in the sequence

  $(x_2, x_3, \ldots)$ greater than x_1, let x_{n_3} be the first term in the sequence $(x_{n_2+1}, x_{n_2+2}, \ldots)$ greater than x_{n_2}, and so on.)
- By the same logic, if, for any $m \in \mathbb{N}$, there is no maximum element in S_m, then we are done.
- Assume then $\max S_m$ exists for each $m \in \mathbb{N}$.
- Now define the subsequence (x_{n_k}) recursively as follows

$$x_{n_1} = \max S_1, x_{n_2} = \max S_{n_1+1}, x_{n_3} = \max S_{n_2+1}, \ldots$$

- Clearly, (x_{n_k}) is decreasing.

To prove: Every sequence has a monotonic subsequence.
Consider a real sequence x_n and the set $S_m = \{x_m, x_{m+1}, \ldots\}$ basically we are dropping $m-1$ terms from the sequence and the rest are in the set S_M.
So $S_1 = \{x_1, x_2, \ldots\}$, then either the maximum (max) of the set S_1 exists ir it does not exist.
Now suppoe the maximum of S_1 does not exist and then consider the subsequenc x_m such that

$$\begin{aligned} x_{m_1} &= x_1 \\ x_{m_2} &= x_{\min\{k|x_k > x_{m_1}\}} \\ x_{m_3} &= x_{\min\{k|x_k > x_{m_2}\}} \end{aligned}$$

and so on is our monotonic sequence,
note that the $x_{\min\{k|x_k > x_{m_1}\}}$ and $x_{\min\{k|x_k > x_{m_2}\}}$ exist since the $\max S_1$ does not exist.
Now suppose that $\max S_1 = x_{k_1}$ so the maximum exists then if we drop k_1 terms of the sequence then we are only left with the set S_{k+1} and now we again have only two possiblities either $\max S_{k+1}$ exist or it does not exist, if it does not exist then we can find a monotonic sequence in the manner discussed above and if it exist then suppose $x_{k_2} = \max S_{k+1}$ then again we have the same situation as above so either we have an increasing sequence S_n when the maximum does not exist or we have an decreasing sequence x_{k_m} both of which are monotonic sequences.

Proposition 10 (Bolzano Weierstrass Theorem):

- Every bounded real sequence has a convergent subsequence.

Proof:

- Putting the propositions 8 and 9 together, we get this result as an immediate corollary.

2 Metric Space:

- Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a metric (distance function) if, for any a, b, and c in X, it satisfies the following three conditions:

  1. Properness: $d(a, b) = 0$ if and only if $a = b$,

2. Symmetry: $d(a,b) = d(b,a)$, and
3. Triangle Inequality: $d(a,b) \leq d(a,c) + d(c,b)$.

A nonempty set X equipped with a metric d constitutes a metric space (X, d).

let $X = \mathbb{R}$ then $d(x,y) = |x - y|$ is a valid distance function.

let $X \subseteq \mathbb{R}^2$ then $d((x_1, y_1)(x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the Euclidean distance and also a valid metric, also another example of a metric is the taxicab metric or the Manhattan distance, $d((x_1, y_1)(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

Another valid mertic example is, Let $X \subseteq \mathbb{R}^2$

$d((x_1, y_1)(x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$

Another example the Discrete metric is defined as, $X \neq \phi$

$$d(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

Similarly the Euclidean metric for $\mathbb{R}^n$ is defined as, $X = \mathbb{R}^n$

$$d((x_1, y_1)(x_2, y_2) \dots (x_n, y_n)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

let (X, d) be a metric space, then for any $a \in X$, and any $\epsilon > 0$, ϵ-neighbourhood is defined as

$$\mathcal{N}_\epsilon(a) = \{b \in X \mid d(b,a) < \epsilon\}$$

Examples;

1. $X = \mathbb{R}$ and $d(x,y) = |x - y|$ then $\mathcal{N}_{\frac{1}{2}}(0) = \left(\frac{-1}{2}, \frac{1}{2}\right)$.
2. $X = [0,1]$ and $d(x,y) = |x - y|$ then $\mathcal{N}_{\frac{1}{2}}(0) = [0, \frac{1}{2})$.
3. $X = \mathbb{Z}$ and $d(x,y) = |x - y|$ then $\mathcal{N}_{\frac{1}{2}}(0) = \{0\}$.
4. $X = \mathbb{R}^2$ and

   $d((x_1, y_1)(x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ then $\mathcal{N}_1(0,0)$ will be an open circle at $(0,0)$ of radius 1 and similiarly $\mathcal{N}_2(0,0)$ will be an open circle at $(0,0)$ of radius 2. More genreally if $r \leq s$ if $\mathcal{N}_r(a) \subset \mathcal{N}_s(a)$.

5. $X = \mathbb{R}$ and metric is the discrete metic then, $\mathcal{N}_2(0) = \mathbb{R}$ and $\mathcal{N}_3(0) = \mathbb{R}$ and note that $\mathcal{N}_3(0) \subset \mathcal{N}_2(0)$
6. $X = \mathbb{R}^2$ and

   $d((x_1, y_1)(x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$ then $\mathcal{N}_1(0, 0)$ will be an open rhombus with the origin at $(0, 0)$.
7. $X = \mathbb{R}^2$ and

   $d((x_1, y_1)(x_2, y_2)) = \max(|x_2 - x_1|, |y_2 - y_1|)$ then $\mathcal{N}_1(0, 0)$ will be an open square at the origin.

Let (X, d) be a metric space then $Y \subset X$ is said to be open in (X, d) if, $(\forall y \in Y)(\exists \epsilon > 0)(\mathcal{N}_\epsilon(y) \subset Y)$, where $\mathcal{N}_\epsilon(y) = \{x \in X \mid d(x, y) < \epsilon\}$

Examples;

1. $X = \mathbb{R}$ and $d(x, y) = |x - y|$ then $[0, 1)$ is not open because $(-\epsilon, \epsilon) \not\subset [0, 1) \forall \epsilon > 0$
2. $X = \mathbb{R}_+$ and $d(x, y) = |x - y|$ then $[0, 1)$ is an open set because now 0 does not create a problem like before.

Let (X, d) be a metric space then $Z \subset X$ is said to be closed in (X, d) if, X/Z is open in (X, d) where $X/Z = \{x \in X \mid x \notin Z\}$

Examples;

1. $X = \mathbb{R}$ and $d(x, y) = |x - y|$ then $Z = [0, 1)$ is not closed because it's complement $(-\infty, 0) \cup [1, \infty)$ is not open.

(X, d) be a metric space then

$Y \subset X$ is said to be bounded if

$(\exists \in > 0, x \in X)\,(Y \subset N_\epsilon(x))$