Hypothesis Testing

We have two hypotheses, a default (null) (H_0) hypothesis and some challenging (alternative) (H_1 or H_A) hypothesis. Say we have an experiment of tossing a coin 100 times and two hypotheses that the coin is fair H_0 and the coin is not fair H_A.

$$\begin{aligned} H_0 : \theta &= \frac{1}{2} \\ H_A : \theta &\neq \frac{1}{2} \\ \text{Test : Possible Observations} &\rightarrow \{H_0, H_A\} \end{aligned}$$

A test will classify the observations of an experiment according to our two possible hypotheses. As statisticians our job is to design a sensible test. A sensible test in our example could be

$$|X - 50| \leq t \rightarrow H_0$$

where X is the number of Heads.

Now under the null Hypothesis, $X \sim \text{Bin}(100, \frac{1}{2})$, and $X \dot{\sim} \mathcal{N}(50, 25)$ and $\frac{X-50}{5} \dot{\sim} \mathcal{N}(0, 1)$ then under this a sensible type 1 error probability would be,

$$\begin{aligned} \Pr_{H_0}(|X - 50| > t) &= 0.05 \\ \Pr(|Z| > 2) &= 0.05 \\ \Pr\left(\left|\frac{X-50}{5}\right| > 2 = 0.05\right) \\ \Pr(|X - 50| > 10) &= 0.05 \end{aligned}$$

So in this case we accept H_0 for $40, 41, \ldots, 50, \ldots, 60$ and for the other values we reject the null H_0.

Suppose you draw a sample of size 1 and we have the following two hypotheses,

$$H_0 : X \sim \text{Unif}(0, 1)$$
$$H_1 : X \sim \text{Beta}(2, 1)$$

We need to find a sensible test for the $\alpha = 0.1$ level of significance, and also find the type 2 error probability;

$T : [0, 1] \to \{H_0, H_1\}$ will be the sensible test we are looking for,

Suppose X is one observation,

$$H_0 : X \sim \text{Unif}(0, 1)$$
$$H_1 : f_X(x) = \begin{cases} 4x \text{ for } x \in [0, \frac{1}{2}] \\ 4 - 4x \text{ for } x \in (\frac{1}{2}, 1] \end{cases} \quad \alpha = 0.1$$

Find a sensible test for the α level of significance.

$\implies$ Reject if $0.45 < X < 0.55$, Accept H_0 otherwise.

And the probability of type 2 error would be, Suppose $X_1, X_2, \ldots, X_n$ iid $\mathcal{N}(\theta, 1)$, $\quad n = 25$.

$$H_0 : \theta = 0$$
$$H_1 : \theta = 1$$
$$\alpha = 0.05$$

Note that,

$$\overline{X}_{25} \sim \mathcal{N}\left(0, \frac{1}{25}\right) \quad \text{Under } H_0$$
$$\overline{X}_{25} \sim \mathcal{N}\left(1, \frac{1}{25}\right) \quad \text{Under } H_1$$

A sensible test in this example would be $T : (-\infty, \infty) \to \{H_0, H_1\}$