1 Hypothesis Testing

1.1 Example 1

We have two hypotheses, a default (null) (H_0) hypothesis and some challenging (alternative) (H_1 or H_A) hypothesis. Say we have an experiment of tossing a coin 100 times and two hypotheses that the coin is fair H_0 and the coin is not fair H_A.

$$H_0 : \theta = \frac{1}{2}$$
$$H_A : \theta \neq \frac{1}{2}$$
$$\text{Test : Possible Observations} \rightarrow \{H_0, H_A\}$$

A test will classify the observations of an experiment according to our two possible hypotheses. As statisticians our job is to design a sensible test. A sensible test in our example could be

$$|X - 50| \leq t \rightarrow H_0$$

where X is the number of Heads.

Now under the null Hypothesis, $X \sim \text{Bin}(100, \frac{1}{2})$, and $X \mathrel{\dot{\sim}} \mathcal{N}(50, 25)$ and $\frac{X-50}{5} \mathrel{\dot{\sim}} \mathcal{N}(0,1)$ then under this a sensible type 1 error probability would be,

$$\Pr_{H_0}(|X - 50| > t) = 0.05$$
$$\Pr(|Z| > 2) = 0.05$$
$$\Pr\left(\left|\frac{X-50}{5}\right| > 2 = 0.05\right)$$
$$\Pr(|X - 50| > 10) = 0.05$$

So in this case we accept H_0 for $40, 41, \ldots, 50, \ldots, 60$ and for the other values we reject the null H_0.

1.2 Example 2

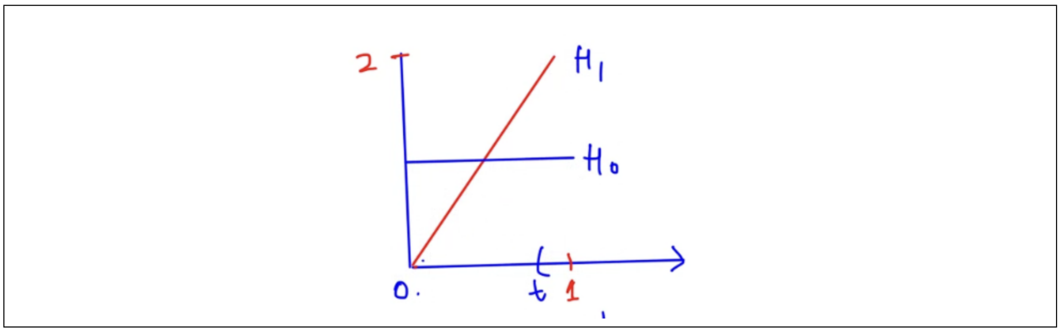
Suppose you draw a sample of size 1 and we have the following two hypotheses,

$$H_0 : X \sim \text{Unif}(0, 1)$$
$$H_1 : X \sim \text{Beta}(2, 1)$$

We need to find a sensible test for the $\alpha = 0.1$ level of significance, and also find the type 2 error probability;

$T : [0, 1] \to \{H_0, H_1\}$ will be the sensible test we are looking for;

$$\begin{aligned} X \leq t &\implies \text{Accept } H_0 \\ X > t &\implies \text{Accept } H_1 \end{aligned}$$



Since $\alpha = 0.1$ we know that $\Pr(\text{Type 1 Error}) = 0.1$ now we can solve for t;

$$\int_t^1 1 \mathrm{d}x = 0.1$$
$$\implies t = 0.9$$

And now we can solve for $\Pr(\text{Type 2 Error})$;

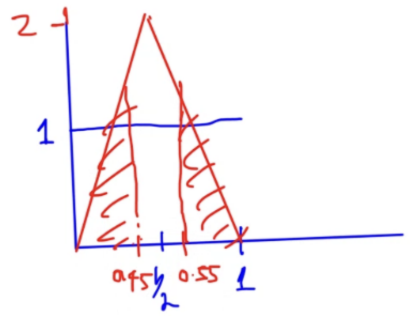
$$\begin{aligned} \Pr(\text{Type 2 Error}) &= \Pr_{H_1}(X \leq 0.9) \\ &= \int_0^{0.9} 2x \mathrm{d}x = 0.81 \end{aligned}$$

1.3 Example 3

Suppose X is one observation,

$$H_0 : X \sim \text{Unif}(0,1)$$
$$H_1 : f_X(x) = \begin{cases} 4x \text{ for } x \in [0, \frac{1}{2}] \\ 4 - 4x \text{ for } x \in (\frac{1}{2}, 1] \end{cases}$$
$$\alpha = 0.1$$

Find a sensible test for the α level of significance.



$\implies$ Reject if $0.45 < X < 0.55$, Accept H_0 otherwise.

And the probability of type 2 error would be,

$$\int_0^{0.45} 4x \mathrm{d}x + \int_{0.55}^{1} 4 - 4x \mathrm{d}x$$
$$= 2\int_0^{0.45} 4x \mathrm{d}x = \left[4x^2\right]_0^{0.45}$$
$$= 4(0.45)^2 = (0.9)^2 = 0.81$$

1.4 Example 4

Suppose $X_1, X_2, \ldots, X_n$ *iid* $\mathcal{N}(\theta, 1)$, $\quad n = 25$.

$$\begin{aligned}
H_0 &: \theta = 0 \\
H_1 &: \theta = 1 \\
\alpha &= 0.05
\end{aligned}$$

Note that,

$$\begin{aligned}
\overline{X}_{25} &\sim \mathcal{N}\left(0, \frac{1}{25}\right) \quad \text{Under } H_0 \\
\overline{X}_{25} &\sim \mathcal{N}\left(1, \frac{1}{25}\right) \quad \text{Under } H_1
\end{aligned}$$

A sensible test in this example would be $T : (-\infty, \infty) \to \{H_0, H_1\}$

$$\begin{aligned}
&\text{Accept } H_0 \text{ if } \ \overline{X}_{25} \leq 0.329 \\
&\text{Reject otherwise}
\end{aligned}$$

Because,

$$\begin{aligned}
\Pr_{H_0}\left(\overline{X}_{25} > t\right) &= 0.05 \\
\Pr_{H_0}\left(5\overline{X}_{25} > 5t\right) &= 0.05 \\
5t &= 1.645 \\
t = \frac{1.645}{5} &= 0.329
\end{aligned}$$