

# 1 Utility Representation of Preferences

$$u : A \rightarrow \mathbb{R}$$

$$a \succsim b \text{ if and only if } u(a) \geq u(b)$$

When preferences are represented by utility functions, the analysis becomes easy.

A natural question to ask then is under what conditions preferences have a utility representation?

If  $A$  is a finite set then there exists a utility function  $u$  that represents the preference iff the preference is reflexive, complete and transitive, stated otherwise,

If  $A$  is a finite set then,  
 $\exists u : A \rightarrow \mathbb{R}$  that represent the  $\succsim$  iff  
 $\succsim$  is reflexive, complete and transitive.

## 2 Functions

Let  $A$  and  $B$  be any non-empty sets. A function from  $A$  to  $B$  is a rule that associates with each member of  $A$  a unique member of  $B$ .

The notation is  $f : A \rightarrow B$ , where input comes from the set  $A$  and output belongs to the set  $B$ .

If  $a \in A$ , we denote the unique element of  $B$  that the rule associates to  $a$  by  $f(a)$ . We refer to the element  $a$  of  $A$  as an argument of the function, and the corresponding element  $f(a)$  of  $B$  as the value of the function at that argument (or sometimes the image of the point  $a$  under  $f$ ).

Consider an example  $f(x) = x^2 + x + 1$ . The value of the function  $f$  at argument 2 is  $f(2)$ , which is further equal to 7.

### 2.1 Domain and Codomain

If  $f : A \rightarrow B$ , we refer the set  $A$  as the domain of  $f$  and the set  $B$  as the codomain.

### 2.2 Range

We say that  $f(A)$  is the range of  $f$  iff

$$f(A) = \{y \in B \mid \exists x \in A, f(x) = y\}$$

Note that  $f(a) \subset B$

- Pre-image: A pre-image of an element  $b \in B$  for a function  $f : A \rightarrow B$  is any element  $a$  of  $A$  for which  $f(a) = b$ .
- Range: Let  $f : A \rightarrow B$ . If  $X \subset A$ , the set  $\{f(x) \mid x \in X\}$  is the range of  $f$  on  $X$ , denoted by  $f(X)$ .
- Alternatively, we can define range of  $f$  on  $X$  as the set

$$\{b \in B \mid (\exists x \in X)(f(x) = b)\}$$

- We denote the set  $f(A)$  as the range of  $f$ .

## 2.3 Injective, or One-to-one

If a function  $f : A \rightarrow B$  is such that it never happens that different arguments lead to the same value, we say that  $f$  is injective.

- Mathematically,  $f : A \rightarrow B$  is injective iff  $(\forall a, b \in A)[a \neq b \Rightarrow f(a) \neq f(b)]$
- Alternatively, we may express this condition using contrapositive:  $f : A \rightarrow B$  is injective iff

$$(\forall a, b \in A)[f(a) = f(b) \Rightarrow a = b]$$

- The function  $f_1 : \mathbb{R} \rightarrow \mathbb{R}_+$  defined by  $f(x) = x^2$  and  $f_2 : \mathbb{R} \rightarrow \mathbb{R}$  also defined by  $f(x) = x^2$  are not injective but the function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $g(x) = x^2$  is injective.

## 2.4 Surjective or onto

If every member of  $B$  is the value of the function at some argument, we say  $f$  is surjective.

- Mathematically, a function  $f : A \rightarrow B$  is surjective iff  $(\forall b \in B)(\exists a \in A)[f(a) = b]$ .
- Note the order of the quantifiers in the above condition. For every  $b$  in  $B$  it must be possible to find an  $a$  in  $A$  such that  $f(a) = b$ .

- The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is not surjective but the function  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  defined by  $g(x) = x^2$  is surjective.

## 2.5 Inverses

Invertible: Let  $f : A \rightarrow B$ . We say  $f$  is invertible if there exists a function  $g : B \rightarrow A$  such that for all  $a \in A$  and all  $b \in B$

$$f(a) = b \iff g(b) = a$$

We call such a function  $g$  an inverse of  $f$ .

- Example: Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 2x + 3$ , there is an inverse function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $g(x) = \frac{x-3}{2}$
- Alternatively,  $g$  is an inverse of a function  $f : A \rightarrow B$  iff  $g : B \rightarrow A$  and

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

where  $I_A, I_B$  are the identity functions on  $A, B$ , respectively.

- Let  $f : A \rightarrow B$ . Then  $f$  is invertible iff it is a bijection. Moreover, if  $f$  is invertible, its inverse function is unique.
- The unique inverse of a bijective function  $f$  is denoted by  $f^{-1}$ .

## 3 Countability

Countability: The fundamental notion behind counting is that of pairing off.

If we count the elements of some (finite) collection  $A$  of objects, “one, two, thre”, etc., we are explicitly defining a bijection between a subset of natural numbers and the elements of  $A$ . We may picture the counting process as follows:

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & \cdots & \cdots & a_{n-1} & a_n \\ \uparrow & \uparrow & \uparrow & & & \uparrow & \uparrow \\ 1 & 2 & 3 & \cdots & \cdots & n-1 & n \end{array}$$

The above counting process determines the bijection between

the set  $\{1, 2, \dots, n\}$  and  $A$ . Since the counting process stops when we get to the number  $n$ ,

we say that the set  $A$  has  $n$  elements, or that “the number of elements of  $A$  is  $n$ ”,

or “the cardinality of  $A$  is  $n$ ”. Notationally,  $|A| = n$ .

So  $A$  is finite if either

1.

$$(\exists n \in \mathbb{N}) () (\exists \text{a bijection } t : \{1, 2, \dots, n\} \rightarrow A)$$

or

2.  $A$  is empty.

There are two definitions of Countability;

- Definition 1

### 3.1 Finite and Infinite Sets

- Finite set: A non-empty set  $A$  is finite iff, for some natural number  $n$ , there is a bijection from the set  $\{1, 2, \dots, n-1, n\}$  to  $A$ . We also consider  $\emptyset$  as a finite set because it has 0 elements.
- Infinite set: If a set is not a finite set then, we call it an infinite set.

### 3.2 Countable Set

Countable set: An infinite set  $A$  is countable if there is a bijection  $f : \mathbb{N} \rightarrow A$ . If there does not exist such a bijection, we say that the infinite set  $A$  is uncountable.

Examples of Countable set:

1. The set of natural numbers  $\mathbb{N}$ : The identity function on  $\mathbb{N}$  is a suitable bijection.
2. The set of even natural numbers, let's denote it by  $\mathcal{E}$ . We can define  $f : \mathbb{N} \rightarrow \mathcal{E}$  defined by  $f(n) = 2n$  is a bijection. Now think of a bijection for the set of odd natural numbers to show that the set of odd natural numbers is countable.

3. The set  $\mathcal{R} = \{1/n \mid n \in \mathbb{N}\}$  is countable, the corresponding bijection is defined as  $f : \mathbb{N} \rightarrow \mathcal{R}$  such that  $f(n) = 1/n$ .
4. The set of all integers  $\mathbb{Z}$  is countable. To see this, define a function  $f : \mathbb{N} \rightarrow \mathbb{Z}$  by:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

5. The set  $\mathcal{P}$  of all prime numbers is also countable. We define a function  $f : \mathbb{N} \rightarrow \mathcal{P}$  by the following definition: let  $f(1) = 2$ , and for any  $n \geq 1$ , let  $f(n+1)$  be the least prime number bigger than  $f(n)$ . Convince yourself for two things: (a) that  $f$  is a function; and (b) that it is in fact a bijection.

Few examples of Countable sets are  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ .

Few examples of uncountable sets are  $\mathbb{R}, [0, 1], \mathbb{Q}^c = \frac{\mathbb{R}}{\mathbb{Q}}$

## 4 Increasing Transformation of a Utility function

**Theorem** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing function that is,  $\forall s, t \in \mathbb{R}, s > t \Rightarrow f(s) > f(t)$ . If  $u$  represents the preference relation  $\succsim$  on  $A$ , then so does the function  $w$  defined by  $w(a) = f(u(a))$  for all  $a \in A$ .

Consider any pair of alternatives  $a, b \in A$ . We have

$$\begin{aligned} w(a) &\geq w(b) \\ \Leftrightarrow f(u(a)) &\geq f(u(b)) \\ \Leftrightarrow u(a) &\geq u(b) \\ \Leftrightarrow a &\succsim b \end{aligned}$$

## 5 Representing preference relation by a utility function

**Theorem** Every preference relation on a finite set can be represented by a utility function.

- Let  $A$  be a finite set and let  $\succsim$  be a preference relation on  $A$ .
- For  $a \in A$ , define  $L_a = \{b \in A \mid a \succsim b\}$ . We'll call this the Lower Contour Set of  $a$  as it consists of alternatives in  $A$  that are at most as good as alternative  $a$ .
- Define  $u : A \rightarrow \mathbb{R}$  as follows: for  $a \in A$ ,  $u(a) := |L_a|$ , which is the number of alternatives in  $A$  that are at most as good as  $a$ . We'll show that  $u$  represents  $\succsim$ , that is,  $\forall a, a' \in A, a \succsim a' \Leftrightarrow u(a) \geq u(a')$
- First we'll show that  $\forall a, a' \in A, a \succsim a' \Rightarrow u(a) \geq u(a')$
- Consider any pair of alternatives  $a, a' \in A$  such that  $a \succsim a'$ . We'll now show that  $L_{a'} \subset L_a$ . If  $b \in L_{a'}$ , then we have  $a \succsim a'$  and  $a' \succsim b$ . By transitivity of  $\succsim$ , we get  $a \succsim b$ , and therefore  $b \in L_a$ . So,  $L_{a'} \subset L_a$  and  $u(a) \geq u(a')$ .
- Now we'll show that  $\forall a, a' \in A, u(a) \geq u(a') \Rightarrow a \succsim a'$ . Equivalently, we can show that  $\forall a, a' \in A, (\neg a \succsim a') \Rightarrow u(a) < u(a')$
- Consider any pair of alternatives  $a, a' \in A$  such that  $\neg a \succsim a'$ . By completeness of  $\succsim$ , we have  $a' \succ a$ . By the same argument as before,

$L_a \subset L_{a'}$ . By reflexivity,  $a' \in L_{a'}$ . Since  $\neg a \succsim a', a' \notin L_a$ . Therefore,  $L_a \subsetneq L_{a'}$  and we get  $u(a) < u(a')$ .

## 5.1 Lexicographic Preferences

Consider  $\mathbb{Z}_+ \times \mathbb{Z}_+$  and the lexicographic preference relation,  $(x_1, y_1) \succsim_L (x_2, y_2)$  if either  $x_1 > x_2$  or  $(x_1 = x_2 \text{ and } y_1 \geq y_2)$

and the strict lexicographic preference relation can be defined as follows;  $(x_1, y_1) \succ_L (x_2, y_2)$  if either  $x_1 > x_2$  or  $(x_1 = x_2 \text{ and } y_1 > y_2)$

and similarly the indifference relation can be defined as  $(x_1, y_1) \sim_L (x_2, y_2)$  if  $x_1 = x_2$  and  $y_1 = y_2$

Is there a utility representation of lexicographic preferences over  $\mathbb{Z}_+ \times \mathbb{Z}_+$ ?

we should try to find a function  $f : \mathbb{Z}_+ \rightarrow [0, 1)$

say  $f = \frac{y}{y+1}$  and then we can use the utility function  $u(x, y) = x + \frac{y}{y+1}$  to represent lexicographic preferences.

Is there a utility representation of lexicographic preferences over  $\mathbb{R}_+ \times \mathbb{R}_+$ ?

NO, There does not exist a utility function over this domain.

Below is a proof provided for the domain  $[0, 1] \times [0, 1]$  which is also uncountable and of the same cardinality as  $\mathbb{R}_+ \times \mathbb{R}_+$

**Theorem** The Lexicographic preference relation  $\succsim$  on  $[0, 1] \times [0, 1]$  defined as  $(x_1, y_1) \succsim (x_2, y_2)$  if and only if either (i)  $x_1 > x_2$  or (ii)  $x_1 = x_2$  and  $y_1 \geq y_2$  is not represented by any utility function.

- Suppose by contradiction that there existed a utility function  $u$  representing these preferences.
- For each  $x \in [0, 1]$ , we have  $(x, 1) \succ (x, 0)$ , and therefore,  $u(x, 1) > u(x, 0)$ . We can therefore assign to  $x$  a non-degenerate interval of values satisfying the above inequality  $I(x) = [u(x, 0), u(x, 1)]$ .
- For any  $1 \geq x' > x \geq 0$ , all commodity bundles generating utilities in the interval  $I(x')$  are strictly preferred to those in the disjoint interval  $I(x)$  and should therefore be assigned a greater utility level.
- Then from each of the interval  $I(x)$  we can pick a distinct rational number  $r_x \in I(x)$  which is increasing in  $x$ . Since  $x \in [0, 1]$ , there are uncountably many such intervals, but set of rational numbers are countable. This results in a contradiction.

Note that lexicographic preferences on  $[0, 1] \times [0, 1]$  are reflexive, transitive and complete.

let us show that lexicographic preferences are transitive;

We want to show that, if the following holds,

$$\begin{aligned} (x_1, y_1) &\succsim_L (x_2, y_2) \\ (x_2, y_2) &\succsim_L (x_3, y_3) \end{aligned}$$

then,

$$(x_1, y_1) \succsim_L (x_3, y_3)$$

## 6 Weak axiom of revealed preference (WARP)

Choice Structure: Let  $X$  be the consumption set. A choice structure  $(\mathcal{B}, C(\cdot))$  consists of two ingredients: (a)  $\mathcal{B}$ : It is a family (a set) of non-empty subsets of  $X$ ; that is, every element of  $\mathcal{B}$  is a set  $B \subset X$ . (b)  $C(\cdot)$ : It is a choice rule that assigns a nonempty set of chosen elements  $C(B) \subset B$  for every budget set  $B \in \mathcal{B}$

WARP- The choice structure  $(\mathcal{B}, C(\cdot))$  satisfies the weak axiom of revealed preference (WARP) if the following property holds:

If for some  $B \in \mathcal{B}$  with  $x, y \in B$  we have  $x \in C(B)$ , then for any  $B' \in \mathcal{B}$  with  $x, y \in B'$  and  $y \in C(B')$ , we must also have  $x \in C(B')$ .

Revealed Preference: Given a choice structure  $(\mathcal{B}, C(\cdot))$  the revealed preference relation  $\succsim^*$  is defined by

$$x \succsim^* y \Leftrightarrow \exists B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B)$$

- We read  $x \succsim^* y$  as “ $x$  is revealed at least as good as  $y$ ”
- With this terminology we can restate the weak axiom as follows: “if  $x$  is revealed at least as good as  $y$ , then  $y$  cannot be revealed preferred to  $x$ ”

If for some  $B \in \mathcal{B}$  with  $x, y \in B$  we have  $x \in C(B)$  then for any  $B' \in \mathcal{B}$  with  $x, y \in B'$  and  $y \in C(B')$ , we must also have that  $x \in C(B')$

$$(\forall B, B' \in \mathcal{B}) (\forall x, y \in B \cap B') ((x \in C(B) \wedge y \in C(B')) \implies x \in C(B'))$$



revealed strict preference relation;

$$x \succ^{**} y \implies \exists B \in \mathcal{B}$$

### Example 1

$$\begin{aligned} A &= \{x, y, z\} \\ C(\{x, y\}) &= \{x\} \\ C(\{y, z\}) &= \{y\} \\ C(\{x, z\}) &= \{z\} \end{aligned}$$

Yes the WARP is satisfied of the above.

now,  $\succ^* = \{(x, x), (x, y), (y, y), (y, z), (z, z), (z, x)\}$  and

$$\succ^{**} = \{(x, y), (y, z), (x, z)\}$$

### Example 2

$$\begin{aligned} A &= \{a, b, c\} \\ C(\{a, b\}) &= \{a, b\} \\ C(\{b, c\}) &= \{b\} \\ C(\{a, c\}) &= \{a\} \\ C(\{a, b, c\}) &= \{a\} \end{aligned}$$

The WARP is not satisfied here because first we are saying that  $b \succ^* a$  or  $u(a) = u(b)$  but at last we are saying that  $a \succ^{**} b$  or  $u(a) > u(b)$ .

## Precise Definition of limit

Let  $f$  be a function defined on some open interval that contains  $a$ , except possibly at  $a$  itself. Then we say that limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$ , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

and the precise definition would be,

$$(\forall \epsilon > 0) (\exists \delta > 0) (\forall x) (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

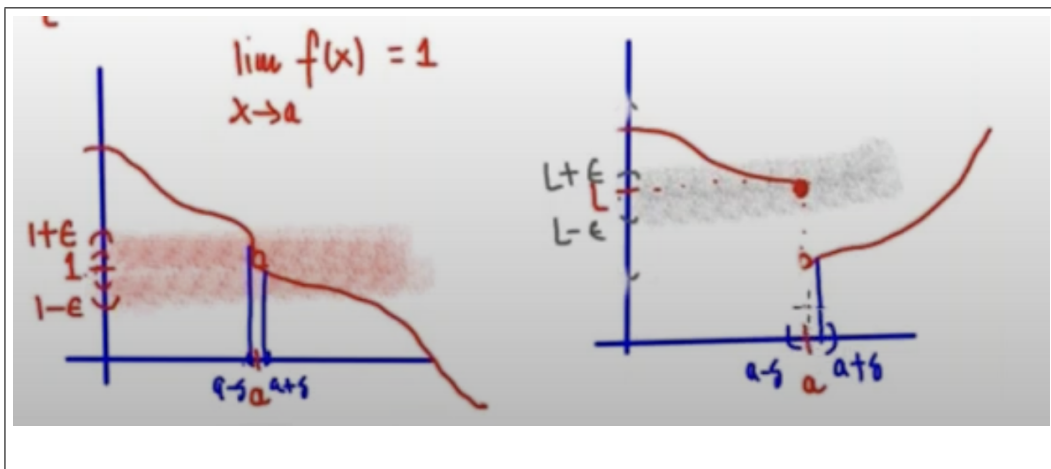
The negation of the above definition would be

$$(\exists \epsilon > 0) (\forall \delta > 0) (\exists x) (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

The above statement tells us that if it is true then we have that,

$$\lim_{x \rightarrow a} f(x) \neq L$$

and we say that limit of  $f(x)$  as  $x$  approaches  $a$  is not equal to  $L$ .



**Figure 1:** How to check for existence of limit graphically

**Example**

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \quad \text{DNE}$$