

# 1 Bayesian Games

## 1.1 Swap/Retain Game

Consider the following two-player game.

- The players simultaneously and independently draw a number from the set  $\{10, 20, 30, 40, 50\}$ .
- After observing the value of her own sample, which is private information (that is, opponent does not observe it), players simultaneously and independently choose one of the following: SWAP, RETAIN.
- If both the players choose SWAP then they exchange their initially drawn numbers. Otherwise, if at least one person chooses RETAIN, both of them retain their numbers.
- A player earns as many Rupees as the number she is holding at the end of the game.

The Choice in this game will be a strategy  $s$ , such that;

$$s : \{10, 20, 30, 40, 50\} \rightarrow \{\text{SWAP}, \text{RETAIN}\}$$

Note that there are  $2^5$  number of such strategies for each of the players, and therefore there are  $(2^5)^2 = 32^2$  number of ways in which this game can be played assuming there are no mixed strategies that can be played!

Now we can choose the strategies in this way;

$$\begin{array}{ll} s(50) = \text{RETAIN} & s_{\text{other}}(50) = \text{RETAIN} \\ s(40) = \text{RETAIN} & s_{\text{other}}(40) = \text{RETAIN} \\ s(30) = \text{RETAIN} & s_{\text{other}}(30) = \text{RETAIN} \\ s(20) = \text{RETAIN} & s_{\text{other}}(20) = \text{RETAIN} \\ s(10) = \text{SWAP, RETAIN} & s_{\text{other}}(10) = \text{SWAP, RETAIN} \end{array}$$

Note that there are four Nash Equilibriums in this game!

More Formally,

A Bayesian Game consists of

- Set of Players:  $N = \{1, 2, \dots, n\}$

- Action Sets: Action set of player  $i$  is denoted by  $A_i$ . Set of all action profiles:  $A = A_1 \times \cdots \times A_n$
- Type Sets: Type set of player  $i$  is denoted by  $\Theta_i$ . Set of all type profiles:  $\Theta = \Theta_1 \times \cdots \times \Theta_n$
- Utility: Utility function of player  $i$  is:  $u_i : A \times \Theta \rightarrow \mathbb{R}$
- Belief:  $p_i(\theta_{-i} \mid \theta_i)$  is the conditional probability that  $i$  gives to others' types being  $\theta_{-i}$  given  $\theta_i$

Now we can formally define the Swap/Retain game;

- $N = \{1, 2\}$
- Action Sets:

$$A_1 = \{\text{SWAP}, \text{RETAIN}\}$$

$$A_2 = \{\text{SWAP}, \text{RETAIN}\}$$

$$A = A_1 \times A_2$$

- Type Sets:

$$\Theta_1 = \{10, 20, 30, 40, 50\}$$

$$\Theta_2 = \{10, 20, 30, 40, 50\}$$

$$\Theta = \Theta_1 \times \Theta_2$$

- Utility:  $u_i : A_1 \times A_2 \times \Theta_1 \times \Theta_2 \rightarrow \mathbb{R}$  is defined as follows

$$u_1(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_2 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_1 & \text{otherwise} \end{cases}$$

$$u_2(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_1 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_2 & \text{otherwise} \end{cases}$$

- $p_1(\theta_2 \mid \theta_1) = \frac{1}{5} \quad \forall \theta_2 \in \Theta_2, \quad \forall \theta_1 \in \Theta_1$   
 $p_2(\theta_1 \mid \theta_2) = \frac{1}{5} \quad \forall \theta_1 \in \Theta_1, \quad \forall \theta_2 \in \Theta_2$

Both players know the joint distribution of their types!

Now, Strategy of Player  $i$  in a Bayesian Game is a function from type set of the player  $\Theta_i$ ; to the set of actions  $A_i$  he can take.

$$s_i : \theta_i \rightarrow A_i$$

$s(\theta_i) \in A_i$  is the action specified by strategy  $s$ ; for type  $\hat{\theta}_i \in \Theta_i$ , i.e. if  $i$  plays according to  $s$ , then he takes an action  $s(\theta_i)$  when his type is  $\theta_i$ . Let  $S_i$  denote set of all such functions.

- A strategy profile is given by

$$s = (s_1, s_2, \dots, s_n) \in S_1 \times \dots \times S_n$$

- Let us call the set of all strategy profiles  $S$  i.e.

$$S = S_1 \times \dots \times S_n$$

- Expected utility is a function  $U_i : A_i \times S_{-i} \times \Theta_i \rightarrow \mathbb{R}$ . - Expected utility of type  $\theta_i$  of player  $i$  when players play according to  $s$  is given by

$$U_i(s_i(\theta_i), s_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i} | \theta_i) u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

Bayesian Nash equilibrium

- Strategy profile  $s^* = (s_1^*, s_2^*, \dots, s_n^*) \in S$  constitutes the Bayesian Nash equilibrium if for all  $i$ , for all  $\theta_i$ ,  $s_i^*(\theta_i)$  maximizes player  $i$ 's expected utility given that the other players play according to  $s_{-i}^*$  i.e.  $s^*$  is the Bayesian Nash equilibrium if  $\forall i \in N, \forall \theta_i \in \Theta_i$ ,

$$U_i(s_i^*(\theta_i), s_{-i}^*, \theta_i) \geq U_i(a_i, s_{-i}^*, \theta_i) \quad \forall a_i \in A_i$$

Suppose

$$s_2(10) = s_2(20) = \text{SWAP}$$

$$s_2(30) = s_2(40) = s_2(50) = \text{RETAIN}$$

Find the Best Response strategy for player 1?

$$s_1(50) = \text{RETAIN}$$

$$s_1(50) = \text{RETAIN}$$

$$s_1(50) = \text{RETAIN}$$

$$s_1(50) = \text{RETAIN}$$

$$s_1(50) = \text{SWAP}$$

Given this best response of player 1, player 2 will also fix his own

## 1.2 Battle of Sexes

- There are two possible types of player 2 (column): "Meet" player 2 wishes to be at the same place as player 1, just as in the usual game (This type has probability  $p = 1/4$ ). "Avoid" 2 wishes to avoid player 1 and go to the other place (This type has probability  $1 - p = 3/4$ ).
- 2 knows his type, and 1 does not. They simultaneously choose the place Movie ( $M$ ) or Shopping ( $S$ ). These payoffs are shown in the matrices below.

$$p = \frac{1}{4}$$

Meet	S	M
S	(2, 1)	(0, 0)
M	(0, 0)	(1, 2)

$$1 - p = \frac{3}{4}$$

Avoid	S	M
S	(2, 0)	(0, 2)
M	(0, 1)	(1, 0)

$$N = \{1, 2\}$$

$$A_1 = \{S, M\}$$

$$A_2 = \{S, M\}$$

$$\Theta_1 = \{\text{Meet}\} \quad \Theta_2 = \{\text{Meet}, \text{Avoid}\}$$

$$u_1 : A_1 \times A_2 \times \Theta_1 \times \Theta_2 \rightarrow \mathbb{R}$$

$$u_1(S, S, \text{Meet}, \text{Meet}) = 2$$

$$u_1(S, S, \text{Meet}, \text{Avoid}) = 2$$

$$s_1 = S \quad u_1(S, s_2, \text{Meet}) = 2 \left( \frac{1}{4} \right) + 0 \left( \frac{3}{4} \right) = \frac{1}{2}$$

$$s_2(\text{Meet}) = S \quad u_1(M, s_2, \text{Meet}) = 0 \left( \frac{1}{4} \right) + 1 \left( \frac{3}{4} \right) = \frac{3}{4}$$

$$s_2(\text{Avoid}) = M$$

Now

$$s_1 = M \quad u_1(S, s_2', \text{Meet}) = 0 \left( \frac{1}{4} \right) + 2 \left( \frac{3}{4} \right) = \frac{3}{2}$$

$$s_2'(\text{Meet}) = M \quad u_1(M, s_2', \text{Meet}) = 1 \left( \frac{1}{4} \right) + 0 \left( \frac{3}{4} \right) = \frac{1}{4}$$

$$s_2'(\text{Avoid}) = S$$

$$M \rightarrow (M, S) \rightarrow S \rightarrow (S, M) \rightarrow M$$

Therefore there is no Bayesian Nash Equilibrium in Pure strategies of this game.