1 Metric Spaces Contd.

1.0.1 Proposition 14

- A set S in a metric space X is closed if, and only if, it contains all its limit points.

To prove: If A set S in a metric space X is closed then it contains all its limit points.

- Suppose S does not contain all its limit points.
- Then there exists $x \in X$ which is the limit point of S but is not in S.
- Since x is the limit point of $S, \mathcal{N}_\epsilon(x) \setminus \{x\} \cap S \neq \emptyset$ for all $\epsilon > 0$ which implies that S^c not open and hence S is not closed.

To prove: If A set S in a metric space contains all its limit points then it is a closed set.

- Conversely, assume that S contain all its limit points but S is not closed.
- Then S^c is not open.
- Hence there is $x \in S^c$ such that for every $\epsilon > 0, \mathcal{N}_\epsilon(x) \cap S \neq \emptyset$.
- This together with $x \notin S$ implies that for every $\epsilon > 0, \mathcal{N}_\epsilon(x) \setminus \{x\} \cap S \neq \emptyset$ implying further that x is the limit point of S but $x \notin S$, contradicting that S contains all its lim points.

To Prove; If a set $A \subset X$ is closed in (X, d) implies Set A contains all it's limit points.

The contrapositive of the above statement is, if a set $A \subset X$ does not contain all it's limit points then the set A is not closed.

let's prove the contrapositive

- Suppose A does not contain all it's limit points.
- Now we want to show that $X \setminus A$ is not open.
- since A does not contain all it's limit points the following must be true $\exists l \in X$ such that l is the limit point of $A \subset X$ but $l \notin A$.
- but this means, $(\forall \epsilon > 0)(\mathcal{N}_\epsilon(l) \setminus \{l\} \cap A \neq \phi$
- but this implies $(\forall \epsilon > 0)\mathcal{N}_\epsilon(l) \cap A \neq \phi$

- $\implies (\forall \epsilon > 0)(\mathcal{N}_\epsilon(l) \not\subset X \setminus A$.
- Therefore the set $X \setminus A$ is not open and hence set A is not closed.

1.0.2 Propositon 15

- A set S in a metric space X is closed if, and only if, every sequence in S that converges in X converges to a point in S

To prove: If A set S in a metric space X is closed then every sequence in S that converges in X converges to a point in S

- Assume that S is closed and that (x_n) is a sequence of points belonging to S converging to $x \in X$.
- We claim that $x \in S$.
- Arguing by contradiction, we assume that $x \notin S$.
- Since S^c is open, there is $r_x > 0$ such that $\mathcal{N}_{r_\varepsilon}(x) \subseteq S^c$.
- Then, since $x_n \in S, d(x_n, x) \geq r_x$, contradicting that $d(x_n, x) \to 0$.

To prove: If every sequence in S that converges in X converges to a point in S then S in a metric space X is closed

- Conversely, assume that every convergent sequence (x_n) such that $x_n \in S$ converges to a point in S but S is not closed. Then S^c is not open.
- Hence there is $x \in S^c$ such that for every $r > 0, \mathcal{N}_r(x) \cap S \neq \emptyset$.
- In particular, for numbers $1/n$ where $n \geq 1$, we find points $x_n \in S$ such that $d(x_n, x) < 1/n$.
- Hence the sequence (x_n) converges to x and $x \notin S$, contradiction.

To Prove; A set $A \subset X$ is closed or it contains all it's limit points if $(\forall (x_n) \subset A)$(if there is $l \in X$ such that $x_n \to l \implies l \in A$).

- The contrapositive of the above statement is; If a set does not contain all it's limit points then it contains a sequence that converges to a point that does not lie in the set or lies outside of it.
- Suppose set A does not contain all it's limit points
- $\implies (\exists l \in X \setminus A)(\forall \epsilon > 0)(\mathcal{N}_\epsilon(l) \setminus \{l\} \cap A \neq \phi$
- $\implies (\exists l \in X \setminus A)(\forall n \in \mathbb{N})(\mathcal{N}_{\frac{1}{n}}(l) \setminus \{l\} \cap A \neq \phi$

- now if we pick a point form this non-empty set $x_n \in \mathcal{N}_{\frac{1}{n}} \setminus \{l\} \cap A$
- Noitce that $0 \leq d(x_n, l) < \frac{1}{n} \quad \forall n \in \mathbb{N}$ is true and also $x_n \in A$
- so by squeeze theorem $d(x_n, l) \to 0$ and $l \in X \setminus A$

1.0.3 Propositon 16

- Let S be a nonempty bounded subset of $\mathbb{R}$. Show that there is an increasing sequence (x_n) in S such that $x_n \to \sup S$ and a decreasing sequence (y_n) in S such that $y_n \to \inf S$.

To prove: There is an increasing sequence (x_n) in S such that $x_n \to \sup S$

- - Let S be a nonempty bounded subset of $\mathbb{R}$ and $x = \sup S$. - For numbers $1/n$ where $n \geq 1$, we find points $x_n \in S$ such that $d(x_n, x) < 1/n$. - Hence, the sequence (x_n) converges to x. - Consider the sequence (y_n) obtained from (x_n) in the following way:

$$y_n = \max\{x_1, x_2, \ldots, x_n\}$$

- Clearly, (y_n) is an increasing sequence that converges to x.
- The proof of the second claim is analogous.

Let A be an non-empty bounded subset of $\mathbb{R}$, this means it has finite infimum and supremum, and A contains an increasing sequence that converges to $\sup A$ and A also contains a decreasing sequence that converges to $\inf A$.

let's prove the first part that it contains an increasing sequence...

There are two possiblities that is

- if $\sup A \in A$, then $x_n = \sup$ is the required sequence since a constant sequence is an increasing sequence.
- if $\sup A \notin A$, then pick $x_1 \in (\sup A - 1, \sup A) \cap A$ and let $d_2 = \min(\frac{1}{2}, |x_1 - \sup A|$
- then we pick $x_2 \in \sup A - d_2, \sup A) \cap A$ and let $d_3 = \min(\frac{1}{3}, |x_2 - \sup A|$, and so on pick $x_3 \in (\sup A - d_3, \sup A) \cap A$,...
- note that since $d_n \to 0$ the aboslute value of the distance between x_n and $\sup A$ goes to 0.
- therefore we get an increasing sequence in this manner that converges to $\sup A$.

1.1 Compact Sets and Totally Bounded Sets

1.1.1 Compact Set

- - Let (X, d) be a metric space. A subset S of a metric space X is compact if every sequence in S has a subsequence that converges to a point in S.

Exapmles;

- Is $A_1 = \mathbb{R}$ a compact set? No! there are numerous sequences in $\mathbb{R}$ which have non convergent subsequences.
- Is $A_2 = \phi$ a compact set? Yes, vacously true!
- Is $A_3 = (0, 1)$ a compact set? No, since the sequence $x_n = \frac{1}{n+1} \in A_3$ has no subsequence that converges to a point in A_3. Basically in other words we also need a set to be closed and bounded for it to be compact in euclidean metric spaces but is not necessarily true for other metric spaces.
- Consider $X = [0, 1]$ and

  $$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  Then is $\frac{1}{n+1}$ a convergent sequence in (X, d)?

  No! because we say that $x_n \to l$ in (X, d) if $d(x_n, l) \to 0$

  A convergent sequence in discrete metric space has the property that all of it's terms except possibly finitely many terms are same, for example $x_n = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$
- is $A = (0, 1) \in (X, d)$ compact? where $X = [0, 1]$ and $d(x, y)$ is the discrete metric? No it is not compact.
- is $A_2 = [0, 1] \in (X, d)$ compact? No
- is $A_3 = \{0, 1\} \in (X, d)$ compact? Yes
- Any finite subset of the set X will be compact.
- let $X = [0, 1]$ and $d(x, y) = |x - y|$ then is $A = [0, 1] \in (X, d)$ compact? Yes, it is easier to see that this set is closed and bounded than to find a sequence that has all it's subsequences coverging to a point in A.

1.1.2 Totally Bounded Set

- A set S in X is said to be totally bounded (or precompact) if, for any $\epsilon > 0$, there exists a finite subset T of X such that

$$S \subseteq \bigcup_{x \in T} \mathcal{N}_\epsilon(x)$$

Examples;

- say $\mathbb{R}, d(x,y)$ is our metric space where d(x,y) is the discrete metric, Is the set $(0,1)$ bounded in $(\mathbb{R}, d)$? yes(any epsilon greater than works for the neighbourhood) but is it totally bounded? no (it fails for $\epsilon = \frac{1}{2}$).

1.1.3 Proposiotion 17

- Every totally bounded subset of a metric space is bounded.

Proof:

- Consider a totally bounded subset S of X.
- Thus, $S \subseteq \cup_{i=1}^{n} \mathcal{N}_1(t_i)$ for some $T = \{t_1, \ldots, t_n\} \subseteq X$.
- S is bounded because for $r = \max\{d(t_i, t_j) \mid t_i, t_j \in T\} + 1, S \subseteq \mathcal{N}_r(t_1)$.

To Prove that a totally bounded set is bounded;

Proof:

- Suppoe a set A is totally bounded.
- WTS; A is bounded.
- Note that it is vacously true that if A is empty then it is bounded. so we only need to consider a non-empty set A that is totally bounded.
- Since A is totally bounded, for $\epsilon = 1$, there exists finite set $T \subset X$ such that

$$A \subset \bigcup_{x \in T} \mathcal{N}_1(x)$$

- Then we can take the max $d(x_i, x_{j)}$ where $x_i, x_j \in T$ for the bigger ball or
- for $x_1 \in T$, take $l = \max_{x_i \in T} d(x_i, x_1)$ then we wnat to show that $A \subset \mathcal{N}_{l+1}(x_1)$

- we can prove this as follows;

$$\begin{aligned}
&p \in A \implies p \in \mathcal{N}_{l+1}(x_1) \\
&d(p, x_1) \leq d(p, x_{k_p}) + d(x_{k_p}, x_1) \qquad \text{[By triangle's inequality]}
\end{aligned}$$

  and therefore $p \in \mathcal{N}_{l+1}(x_1)$ must be true.