1. Extensive Games with Perfect Information

An Extensive game with perfect information consists of the following ;

- Set of players
- Set of Sequences (terminal histories) with proper entry that no sequence is a proper subhistory of any other sequence.
- A player function that assigns a player to every sequence that is a proper subhistory of some terminal history.
- Prefernces for each player over the set of terminal histories.

Examples

Game 1: *Simultaneous move Duoploy*

An enterant firm decides to enter an Industry or not while the icumbent firm in that industry decides to wether to accomodate or fight the new enterant firm.

- Set of players:
  $N = \text{Enterant(E)}, \text{Incumbent(I)}$
- Action Set of Each Player:
  $T = \{\text{StayOut}, (\text{Enter}, \text{Fight}), (\text{Enter}, \text{Acommodate})\}$
- Player Function:
  $\mathcal{P}(\phi) = \text{E}$
  $\mathcal{P}(\text{Enter}) = \text{I}$
- Payoffs of both players:
  $U_E(\text{Stay out}) = 0$
  $U_E(\text{Enter,Fight}) = -50$
  $U_E(\text{Enter,Accomodate}) = 50$
  $U_I(\text{Stay out}) = 100$
  $U_I(\text{Stay out}) = 0$
  $U_I(\text{Stay out}) = 50$

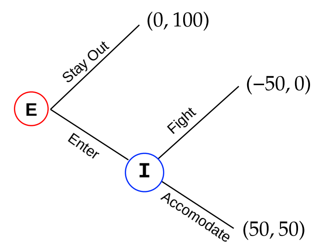

Figure 1: Simultaneous move Duopoly

Nash Equilibrium, $NE = \{(\text{Stayout}, \text{Fight}), (\text{Enter}, \text{Accomodate})\}$
Subgame Perfect Equilibrium, $SPE = \{(\text{Enter}, \text{Accomodate})\}$

Note that $(\text{Stayout}, \text{Fight})$ is a Nash equilbrium but not a subgame perfect equilibrium because it is not a Nash Equilibrium of every subgame in this game.

Game 2: @3866, *Piazza Contributing Efforts game*

Consider the following simultaneous-move effort-contributions' game of n players:

- Set of players: $\{1, 2, \ldots, n\}$
- Action set of each player : $A_i = [0, 1]$
- Payoff of player : $u_i(e_1, e_2, \ldots, e_n) = n \min(e_1, e_2, \ldots, e_n) - e_i$

Let us solve this game for $n = 3$ players;

- Strategy of player 3: $S_3 : [0, 1] \times [0, 1], \quad S_3(e_1, e_2)$
- Strategy of Player 2: $S_2 : \quad S_2(e_1)$
- Strategy of Player 1: $S_1 \in [0, 1]$

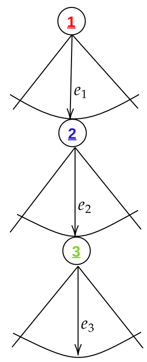

Figure 2: Efforts Contribution Game

Solving the game using backward induction, We solve for the player 3 first;
Player 3 will solve the following utility maximization problem given (e_1, e_2);

$$\max_{0 \leq e_3 \leq 1} 3 \min(e_1, e_2, e_3) - e_3$$

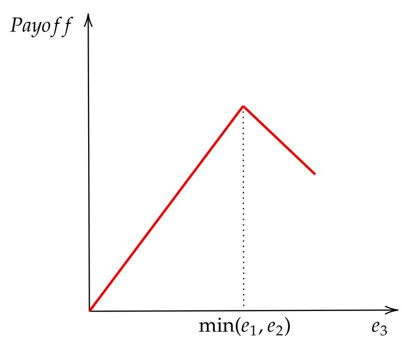

Figure 3: Payoff function of player 3

The solution to this problem is; $e_3 = S_3(e_1, e_2) = \min(e_1, e_2)$
Folding the game backward again, Player 2 will now solve his following utility

maximization problem given e_1;

$$\begin{aligned} \max_{0\le e_2 \le 1} \quad & 3\min(e_1, e_2, e_3) - e_2 \\ s.t. \quad & e_3 = S_3(e_1, e_2) = \min(e_1, e_2) \end{aligned}$$

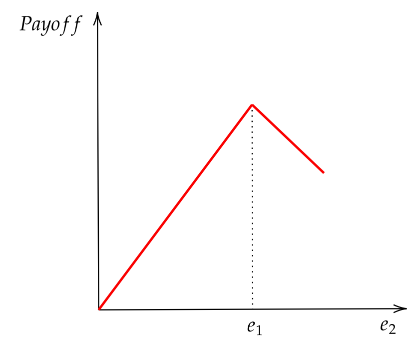

Figure 4: Payoff function of player 2

Solving the above problem by substituting for e_3 into the objective function we can rewrite our problem as

$$\max_{e_1, e_2} 3\min(e_1, e_2) - e_2$$

solving this we get, $e_2 = S_2(e_1) = e_1$
And then finally player 1 solves his utilty maximization problem;

$$\begin{aligned} \max_{0\le e_1 \le 1} \quad & 3\min(e_1, e_2, e_3) - e_1 \\ s.t. \quad & e_3 = S_3(e_1, e_2) = \min(e_1, e_2) \\ & e_2 = e_1 \end{aligned}$$

Solving the above gives $e_1 = 1$
Therefore $e_1^* = e_2^* = e_3^* = \min(e_1, e_2) = 1$
And the resulting terminal History where players move according to the equilibrium strategy $(1, 1, 1)$ is the Subgame Perfect Equilibrium outcome and the SPE stratrgy is;

$$e_1 = S_1 = 1$$
$$e_2 = S_2 = e_1$$
$$e_2 = S_3(e_1, e_2) = \min(e_1, e_2)$$

<u>Game 3:</u> *Stackelberg Duopoly Using Isoprofit Curves*

1. Set of Players;
   $N = \{1, 2\}$
2. Action set of players;
   $T = \{(q_1, q_2) \in \mathbb{R}_+ \times \mathbb{R}_+\}$
3. Player function;

   $$\mathcal{P}(\phi) = 1$$
   $$\mathcal{P}(q_1) = 2, \forall q_1 \in \mathbb{R}_+$$

4. Payoff of palyer i $\forall\ i \in \{1, 2\}$;
   $\pi_i(q_1, q_2) = q_i \max(15 - q_1 - q_2, 0)$

Solving this game Using Backward Induction;
First firm 2(follower) solves its profit maximization problem taking output of firm 1, q_1 as given;

$$\max_{q_2 \geq 0} q_2(\max(15 - q_1 - q_1, 0))$$

Solution to which gives us the best response function of player 2 given q_1, $BR_2(q_1)$

$$BR_2(q_1) \in \begin{cases} \{\frac{15-q_1}{2}\} & \text{if } \ q_1 < 15 \\ \mathbb{R}_+ & \text{if } \ q_1 \geq 15 \end{cases}$$

So,

$$q_2(q_1) = \begin{cases} \frac{15-q_1}{2} & \text{if } \ q_1 < 15 \\ 0 & \text{if } \ q_1 \geq 15 \end{cases}$$

Then the firm 1(leader) solve its profit maximization problem subject to the best response fucntion of the follower;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(15 - q_1 - q_2, 0)) \\ s.t. \quad & q_2(q_1) = \begin{cases} \frac{15-q_1}{2} & \text{if } \ q_1 < 15 \\ 0 & \text{if } \ q_1 \geq 15 \end{cases} \end{aligned}$$

The above problem can be rewritten as;

$$\max_{0 \leq q_1 \leq 15} q_1 \big(\frac{15 - q_1}{2} \big)$$

solving which gives us the following; $q_1^* = 7.5, q_2^* = \frac{7.5}{2}$

The above problem can also be solved using the isoprofit curves approach as follow;

$q_1(15 - q_1 - q_2) = \overline{\pi}$

differntiating both sides w.r.t q_1

$\implies \dfrac{\mathrm{d}q_1}{\mathrm{d}q_2} = \dfrac{15 - 2q_1 - q_2}{q_1}$

But, $\dfrac{\mathrm{d}q_2}{\mathrm{d}q_1} = \dfrac{-1}{2}$

$\implies \dfrac{15 - 2q_1 - q_2}{q_1} = \dfrac{-1}{2}$

$\implies q_1 = 7.5, q_2 = 3.75$

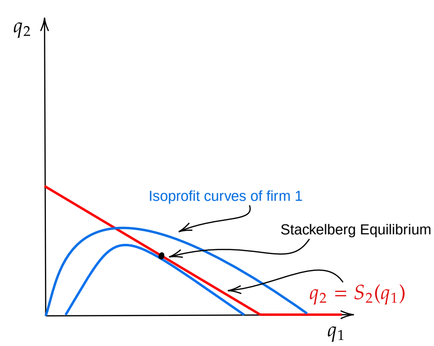

Figure 5: Stackelberg Equilibrium

Game 4: *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2,) - F_i$; if $q_i > 0$
- $F_i(q_i) = \begin{cases} 25 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking q_1 as given;

$$\max_{q_2 > 0} \; q_2(\max(16 - q_1 - q_2, 0)) - 25$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{\frac{16-q_1}{2}\} & \text{if } \; q_1 < 6 \\ \mathbb{R}_+ & \text{if } \; q_1 \geq 6 \end{cases}$$

and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } \; q_1 < 6 \\ 0 & \text{if } \; q_1 \geq 6 \end{cases}$$

Then the leader firm 1 solves it's profit maximization problem subject to followers's best response;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(16 - q_1 - q_2, 0)) - 25 \\ s.t. \quad & q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } \; q_1 < 6 \\ 0 & \text{if } \; q_1 \geq 6 \end{cases} \end{aligned}$$

The solution to this problem is that firm 1 will produce the monoply outcome of $q_1 = 8$ given follower's strategy and $q_2 = 0$. This game is also known as Natural Monopoly since the monopoly output is enough to deter the entry of the follower firm.

Game 5: *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2) - F_i$; if $q_i > 0$
- $F_i(q_i) = \begin{cases} 9 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking q_1 as given;

$$\max_{q_2 > 0} \; q_2(\max(16 - q_1 - q_2, 0)) - 9$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{\frac{16-q_1}{2}\} & \text{if } \; q_1 < 10 \\ \mathbb{R}_+ & \text{if } \; q_1 \geq 10 \end{cases}$$

and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } \; q_1 < 10 \\ 0 & \text{if } \; q_1 \geq 10 \end{cases}$$

Then the leader firm 1 solves it's profit maximization problem subject to followers's best response;

$$\begin{aligned} &\max_{q_1 \geq 0} \quad q_1(\max(16 - q_1 - q_2, 0)) - 9 \\ &s.t. \quad q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } \; q_1 < 10 \\ 0 & \text{if } \; q_1 \geq 10 \end{cases} \end{aligned}$$

Unlike the previous game the monopoly outcome $q_1 = 8$ is not enough to deter the entry and hence the leader firm produces $q_1 = 10$ now and $q_2 = 0$. This case is known as the Entry deterrence case.

<u>Game 6:</u> *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2) - F_i \; ; \text{ if } q_i > 0$
- $F_i(q_i) = \begin{cases} 1 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking q_1 as given;

$$\max_{q_2 > 0} \; q_2(\max(16 - q_1 - q_2, 0)) - 1$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{\frac{16-q_1}{2}\} & \text{if } \; q_1 < 14 \\ \mathbb{R}_+ & \text{if } \; q_1 \geq 14 \end{cases}$$

and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 14 \\ 0 & \text{if } q_1 \geq 14 \end{cases}$$

Then the leader firm 1 solves it's profit maximization problem subject to followers's best response;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(16 - q_1 - q_2, 0)) - 1 \\ s.t. \quad & q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 14 \\ 0 & \text{if } q_1 \geq 14 \end{cases} \end{aligned}$$

In this case the fixed costs are so low that leader firm decides to accomodate the entry and produces $q_1 = 8$ and the follower produces $q_2 = 8$.