

## 1. Extensive Games with Perfect Information

An Extensive game with perfect information consists of the following ;

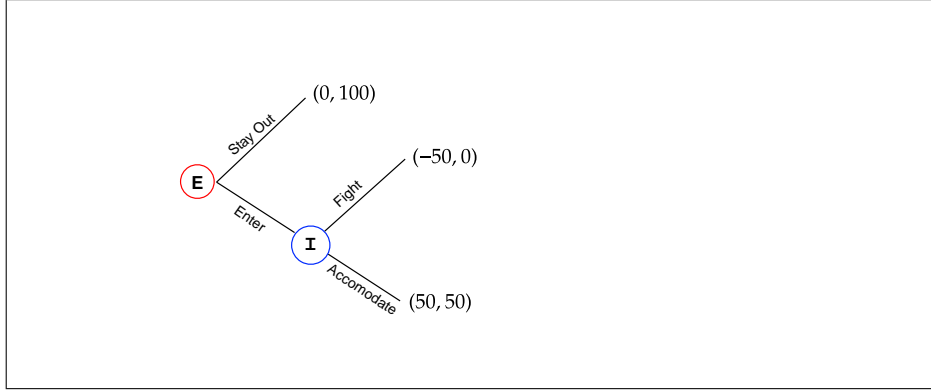
- Set of players
- Set of Sequences (terminal histories) with proper entry that no sequence is a proper subhistory of any other sequence.
- A player function that assigns a player to every sequence that is a proper subhistory of some terminal history.
- Preferences for each player over the set of terminal histories.

### Examples

#### Game 1: *Simultaneous move Duopoly*

An enterant firm decides to enter an Industry or not while the incumbent firm in that industry decides to whether to accommodate or fight the new enterant firm.

- Set of players:  
 $N = \text{Enterant}(E), \text{Incumbent}(I)$
- Action Set of Each Player:  
 $T = \{\text{StayOut}, (\text{Enter}, \text{Fight}), (\text{Enter}, \text{Accomodate})\}$
- Player Function:  
 $\mathcal{P}(\phi) = E$   
 $\mathcal{P}(\text{Enter}) = I$
- Payoffs of both players:  
 $U_E(\text{Stay out}) = 0$   
 $U_E(\text{Enter}, \text{Fight}) = -50$   
 $U_E(\text{Enter}, \text{Accomodate}) = 50$   
 $U_I(\text{Stay out}) = 100$   
 $U_I(\text{Stay out}) = 0$   
 $U_I(\text{Stay out}) = 50$



**Figure 1:** Simultaneous move Duopoly

Nash Equilibrium,  $NE = \{(Stayout, Fight), (Enter, Accomodate)\}$

Subgame Perfect Equilibrium,  $SPE = \{(Enter, Accomodate)\}$

Note that  $(Stayout, Fight)$  is a Nash equilibrium but not a subgame perfect equilibrium because it is not a Nash Equilibrium of every subgame in this game.

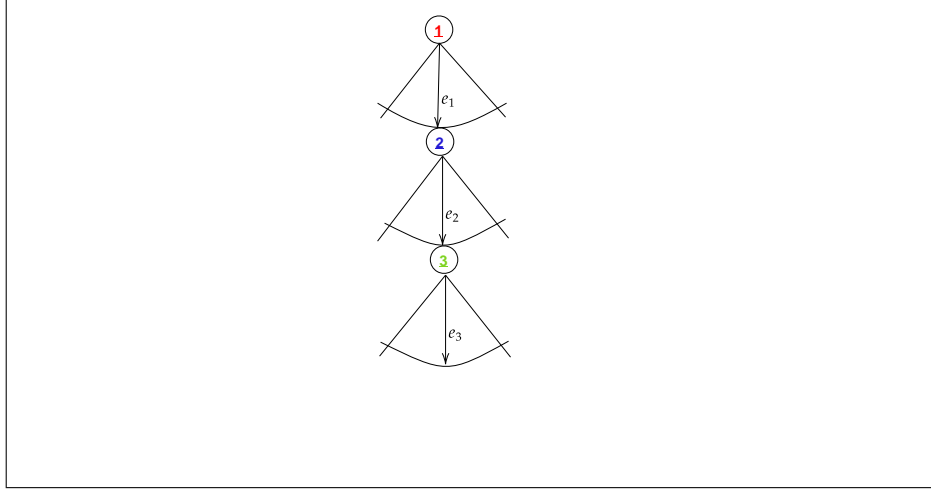
**Game 2:** @3866, *Piazza Contributing Efforts game*

Consider the following simultaneous-move effort-contributions' game of  $n$  players:

- Set of players:  $\{1, 2, \dots, n\}$
- Action set of each player :  $A_i = [0, 1]$
- Payoff of player :  $u_i(e_1, e_2, \dots, e_n) = n \min(e_1, e_2, \dots, e_n) - e_i$

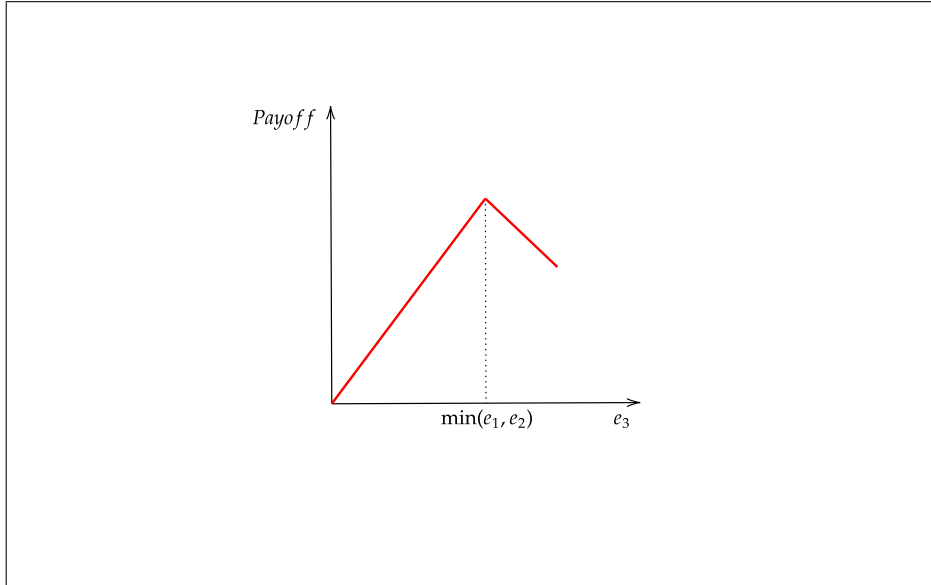
Let us solve this game for  $n = 3$  players;

- Strategy of player 3:  $S_3 : [0, 1] \times [0, 1], S_3(e_1, e_2)$
- Strategy of Player 2:  $S_2 : S_2(e_1)$
- Strategy of Player 1:  $S_1 \in [0, 1]$

**Figure 2:** Efforts Contribution Game

Solving the game using backward induction, We solve for the player 3 first; Player 3 will solve the following utility maximization problem given  $(e_1, e_2)$ ;

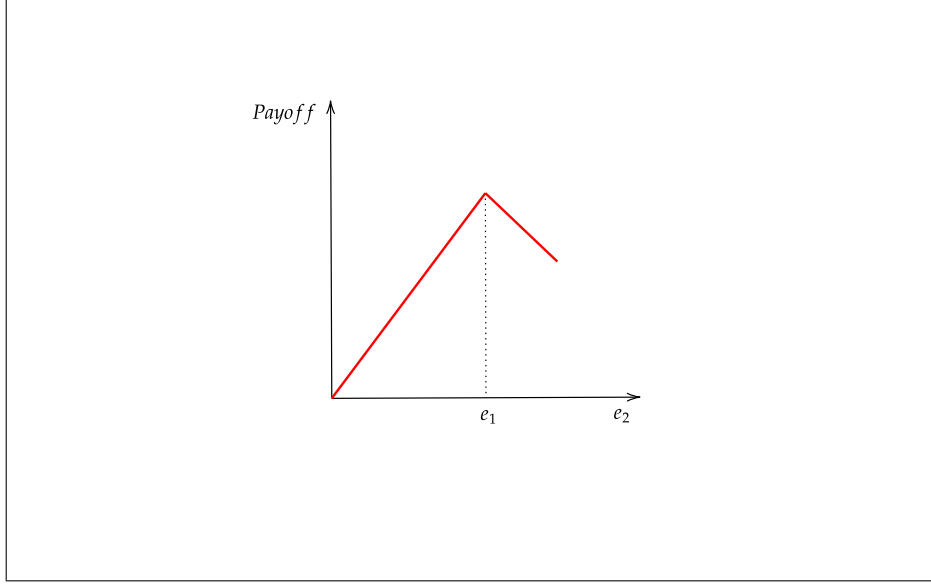
$$\max_{0 \leq e_3 \leq 1} 3 \min(e_1, e_2, e_3) - e_3$$

**Figure 3:** Payoff function of player 3

The solution to this problem is;  $e_3 = S_3(e_1, e_2) = \min(e_1, e_2)$   
Folding the game backward again, Player 2 will now solve his following utility

maximization problem given  $e_1$ ;

$$\begin{aligned} \max_{0 \leq e_2 \leq 1} \quad & 3 \min(e_1, e_2, e_3) - e_2 \\ \text{s.t.} \quad & e_3 = S_3(e_1, e_2) = \min(e_1, e_2) \end{aligned}$$



**Figure 4:** Payoff function of player 2

Solving the above problem by substituting for  $e_3$  into the objective function we can rewrite our problem as

$$\max_{e_1, e_2} 3 \min(e_1, e_2) - e_2$$

solving this we get,  $e_2 = S_2(e_1) = e_1$

And then finally player 1 solves his utility maximization problem;

$$\begin{aligned} \max_{0 \leq e_1 \leq 1} \quad & 3 \min(e_1, e_2, e_3) - e_1 \\ \text{s.t.} \quad & e_3 = S_3(e_1, e_2) = \min(e_1, e_2) \\ & e_2 = e_1 \end{aligned}$$

Solving the above gives  $e_1 = 1$

Therefore  $e_1^* = e_2^* = e_3^* = \min(e_1, e_2) = 1$

And the resulting terminal History where players move according to the equilibrium strategy  $(1, 1, 1)$  is the Subgame Perfect Equilibrium outcome and the SPE strategy is;

$$\begin{aligned}
e_1 &= S_1 = 1 \\
e_2 &= S_2 = e_1 \\
e_2 &= S_3(e_1, e_2) = \min(e_1, e_2)
\end{aligned}$$

**Game 3:** *Stackelberg Duopoly Using Isoprofit Curves*

1. Set of Players;  
 $N = \{1, 2\}$
2. Action set of players;  
 $T = \{(q_1, q_2) \in \mathbb{R}_+ \times \mathbb{R}_+\}$
3. Player function;  
 $\mathcal{P}(\phi) = 1$   
 $\mathcal{P}(q_1) = 2, \forall q_1 \in \mathbb{R}_+$
4. Payoff of palyer  $i \ \forall i \in \{1, 2\}$ ;  
 $\pi_i(q_1, q_2) = q_i \max(15 - q_1 - q_2, 0)$

Solving this game Using Backward Induction;

First firm 2(follower) solves its profit maximization problem taking output of firm 1,  $q_1$  as given;

$$\max_{q_2 \geq 0} q_2(\max(15 - q_1 - q_2, 0))$$

Solution to which gives us the best response function of player 2 given  $q_1$ ,  $BR_2(q_1)$

$$BR_2(q_1) \in \begin{cases} \{\frac{15-q_1}{2}\} & \text{if } q_1 < 15 \\ \mathbb{R}_+ & \text{if } q_1 \geq 15 \end{cases}$$

So,

$$q_2(q_1) = \begin{cases} \frac{15-q_1}{2} & \text{if } q_1 < 15 \\ 0 & \text{if } q_1 \geq 15 \end{cases}$$

Then the firm 1(leader) solve its profit maximization problem subject to the best response fuction of the follower;

$$\begin{aligned}
&\max_{q_1 \geq 0} \quad q_1(\max(15 - q_1 - q_2, 0)) \\
&s.t. \quad q_2(q_1) = \begin{cases} \frac{15-q_1}{2} & \text{if } q_1 < 15 \\ 0 & \text{if } q_1 \geq 15 \end{cases}
\end{aligned}$$

The above problem can be rewritten as;

$$\max_{0 \leq q_1 \leq 15} q_1 \left( \frac{15 - q_1}{2} \right)$$

solving which gives us the following;  $q_1^* = 7.5, q_2^* = \frac{7.5}{2}$

The above problem can also be solved using the isoprofit curves approach as follow;

$$q_1(15 - q_1 - q_2) = \bar{\pi}$$

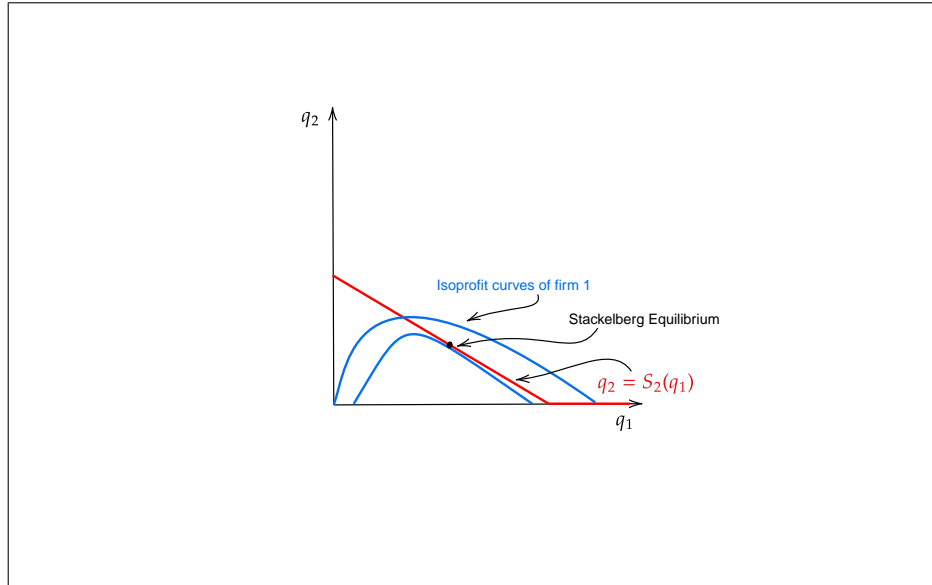
differentiating both sides w.r.t  $q_1$

$$\Rightarrow \frac{dq_1}{dq_2} = \frac{15 - 2q_1 - q_2}{q_1}$$

$$\text{But, } \frac{dq_2}{dq_1} = \frac{-1}{2}$$

$$\Rightarrow \frac{15 - 2q_1 - q_2}{q_1} = \frac{-1}{2}$$

$$\Rightarrow q_1 = 7.5, q_2 = 3.75$$



**Figure 5:** Stackelberg Equilibrium

**Game 4:** *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2, 0) - F_i$  ; if  $q_i > 0$
- $F_i(q_i) = \begin{cases} 25 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking  $q_1$  as given;

$$\max_{q_2 > 0} q_2(\max(16 - q_1 - q_2, 0)) - 25$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{0\} & \text{if } q_1 > 6 \\ \{\frac{16-q_1}{2}\} & \text{if } q_1 < 6 \\ \{0, \frac{16-q_1}{2}\} & \text{if } q_1 = 6 \end{cases}$$

and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 6 \\ 0 & \text{if } q_1 \geq 6 \end{cases}$$

Then the leader firm 1 solves its profit maximization problem subject to followers's best response;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(16 - q_1 - q_2, 0)) - 25 \\ \text{s.t.} \quad & q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 6 \\ 0 & \text{if } q_1 \geq 6 \end{cases} \end{aligned}$$

The solution to this problem is that firm 1 will produce the monopoly outcome of  $q_1 = 8$  given follower's strategy and  $q_2 = 0$ . This game is also known as Natural Monopoly since the monopoly output is enough to deter the entry of the follower firm.

**Game 5:** *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2) - F_i$  ; if  $q_i > 0$
- $F_i(q_i) = \begin{cases} 9 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking  $q_1$  as given;

$$\max_{q_2 > 0} q_2(\max(16 - q_1 - q_2, 0)) - 9$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{\frac{16-q_1}{2}\} & \text{if } q_1 < 10 \\ \mathbb{R}_+ & \text{if } q_1 \geq 10 \end{cases}$$

and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 10 \\ 0 & \text{if } q_1 \geq 10 \end{cases}$$

Then the leader firm 1 solves its profit maximization problem subject to followers's best response;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(16 - q_1 - q_2, 0)) - 9 \\ \text{s.t.} \quad & q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 10 \\ 0 & \text{if } q_1 \geq 10 \end{cases} \end{aligned}$$

Unlike the previous game the monopoly outcome  $q_1 = 8$  is not enough to deter the entry and hence the leader firm produces  $q_1 = 10$  now and  $q_2 = 0$ . This case is known as the Entry deterrence case.

**Game 6:** *Stackelberg Duopoly with fixed costs*

- $\pi_i = q_i \max(16 - q_1 - q_2) - F_i$  ; if  $q_i > 0$
- $F_i(q_i) = \begin{cases} 1 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$

Solving using backward induction follower firm 2 solves its profit maximization problem first taking  $q_1$  as given;

$$\max_{q_2 > 0} q_2(\max(16 - q_1 - q_2, 0)) - 1$$

Solving which gives us,

$$BR_2(q_1) \in \begin{cases} \{\frac{16-q_1}{2}\} & \text{if } q_1 < 14 \\ \mathbb{R}_+ & \text{if } q_1 \geq 14 \end{cases}$$



and,

$$q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 14 \\ 0 & \text{if } q_1 \geq 14 \end{cases}$$

Then the leader firm 1 solves its profit maximization problem subject to followers's best response;

$$\begin{aligned} \max_{q_1 \geq 0} \quad & q_1(\max(16 - q_1 - q_2, 0)) - 1 \\ \text{s.t.} \quad & q_2(q_1) = \begin{cases} \frac{16-q_1}{2} & \text{if } q_1 < 14 \\ 0 & \text{if } q_1 \geq 14 \end{cases} \end{aligned}$$

In this case the fixed costs are so low that leader firm decides to accomodate the entry and produces  $q_1 = 8$  and the follower produces  $q_2 = 8$ .