1 Bayesian Games

1.1 Swap/Retain Game

Consider the following two-player game.

- The players simultaneously and independently draw a number from the set $\{10, 20, 30, 40, 50\}$.
- After observing the value of her own sample, which is private information (that is, opponent does not observe it), players simultaneously and independently choose one of the following: SWAP,RETAIN.
- If both the players choose SWAP then they exchange their initially drawn numbers. Otherwise, if at least one person chooses RETAIN, both of them retain their numbers.
- A player earns as many Rupees as the number she is holding at the end of the game.

The Choice in this game will be a strategy s, such that;

$$s : \{10, 20, 30, 40, 50\} \rightarrow \{\text{SWAP,RETAIN}\}$$

Note that there are 2^5 number of such strategies for each of the players, and therefore there are $(2^5)^2 = 32^2$ number of ways in which this game can be played assuming there are no mixed strategies that can be played!

Now we can choose the strategies in this way;

$$\begin{aligned}
s(50) &= \text{RETAIN} & s_{\text{other}}(50) &= \text{RETAIN}\\
s(40) &= \text{RETAIN} & s_{\text{other}}(40) &= \text{RETAIN}\\
s(30) &= \text{RETAIN} & s_{\text{other}}(30) &= \text{RETAIN}\\
s(20) &= \text{RETAIN} & s_{\text{other}}(20) &= \text{RETAIN}\\
s(10) &= \text{SWAP,RETAIN} & s_{\text{other}}(10) &= \text{SWAP,RETAIN}
\end{aligned}$$

Note that there are four Nash Equilibriums in this game!

More Formally,

A Bayesian Game consists of

- Set of Players: $N = \{1, 2, \ldots, n\}$

- Action Sets: Action set of player i is denoted by A_i. Set of all action profiles: $A = A_1 \times \cdots \times A_n$
- Type Sets: Type set of player i is denoted by Θ_i. Set of all type profiles: $\Theta = \Theta_1 \times \cdots \times \Theta_n$
- Utility: Utility function of player i is: $u_i : A \times \Theta \to \mathbb{R}$
- Belief: $p_i\left(\theta_{-i} \mid \theta_i\right)$ is the conditional probability that i gives to others' types being θ_{-i} given θ_i

Now we can formally define the Swap/Retain game;

- $N = \{1, 2\}$
- Action Sets:

$$A_1 = \{\text{SWAP,RETAIN}\}$$
$$A_2 = \{\text{SWAP,RETAIN}\}$$
$$A = A_1 \times A_2$$

- Type Sets:

$$\Theta_1 = \{10, 20, 30, 40, 50\}$$
$$\Theta_2 = \{10, 20, 30, 40, 50\}$$
$$\Theta = \Theta_1 \times \Theta_2$$

- Utility: $u_i A_1 \times A_2 \times \Theta_1 \times \Theta_2 \to \mathbb{R}$ is defined as follows

$$u_1(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_2 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_1 & \text{otherwise} \end{cases}$$
$$u_2(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_1 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_2 & \text{otherwise} \end{cases}$$

- $p_1(\theta_2|\theta_1) = \frac{1}{5} \qquad \forall \theta_2 \in \Theta_2, \quad \forall \theta_1 \in \Theta_1$

  $p_2(\theta_1|\theta_2) = \frac{1}{5} \qquad \forall \theta_1 \in \Theta_1, \quad \forall \theta_2 \in \Theta_2$

Both players know the joint distribution of their types!

Now, Strategy of Player i in a Bayesian Game is a function from type set of the player Θ_i; to the set fo actions A_i he can take.

$$s_i : \theta_i \rightarrow A_i$$

$s(\theta_i) \in A_1$ is the action specified by strategy s; for type $\hat{\theta}_i \in \Theta_i$, i.e. if i plays according to s, then he takes an action $s_5(\theta_1)$ when his type his θ_i. Let S_i denotes set of all such functions.

- A strategy profile is given by

$$s = (s_1, s_2, \ldots, s_n) \in S_1 \times \cdots \times S_n$$

- Let us call the set of all strategy profiles S i.e.

$$S = S_1 \times \cdots \times S_n$$

- Expected utility is a function $U_i : A_i \times S_{-i} \times \Theta_i \rightarrow \mathbb{R}$.
- Expected utility of type θ_i of player i when players play according to s is given by

$$U_i\left(s_i\left(\theta_i\right), s_{-i}, \theta_i\right) = \sum_{\theta_{-i} \in \Theta_{-i}} p_i\left(\theta_{-i} \mid \theta_i\right) u_i\left(s_i\left(\theta_i\right), s_{-i}\left(\theta_{-i}\right), \theta_i, \theta_{-i}\right)$$

Bayesian Nash equilibrium

- Strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*) \in S$ constitutes the Bayesian Nash equilibrium if for all i, for all $\theta_i, s_i^*\left(\theta_i\right)$ maximizes player i 's expected utility given that the other players play according to s_{-i}^* i.e. s^* is the Bayesian Nash equilibrium if $\forall i \in N, \forall \theta_i \in \Theta_i$,

$$U_i\left(s_i^*\left(\theta_i\right), s_{-i}^*, \theta_i\right) \geq U_i\left(a_i, s_{-i}^*, \theta_i\right) \quad \forall a_i \in A_i$$

Suppose

$s_2(10) = s_2(20) = \text{SWAP}$

$s_2(30) = s_2(40) = s_2(50) = \text{RETAIN}$

Find the Best Response strategy for player 1?

$$u_1(\text{Retain},\ s_2, 50) = 50$$
$$u_1(\text{Swap},\ s_2, 50) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 50 = \frac{180}{5} = 36$$

$$u_1(\text{Retain},\ s_2, 40) = 40$$
$$u_1(\text{Swap},\ s_2, 40) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 40 = \frac{150}{5} = 30$$

$$u_1(\text{Retain},\ s_2, 30) = 30$$
$$u_1(\text{Swap},\ s_2, 30) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 30 = \frac{120}{5} = 24$$

$$u_1(\text{Retain},\ s_2, 20) = 20$$
$$u_1(\text{Swap},\ s_2, 20) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 20 = \frac{90}{5} = 18$$

$$u_1(\text{Retain},\ s_2, 10) = 10$$
$$u_1(\text{Swap},\ s_2, 10) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 10 = \frac{60}{5} = 12$$

So we get the Best Response strategy of player 1 as;

$$s_1(50) = \text{RETAIN}$$
$$s_1(40) = \text{RETAIN}$$
$$s_1(30) = \text{RETAIN}$$
$$s_1(20) = \text{RETAIN}$$
$$s_1(10) = \text{SWAP}$$

Given this best response of player 1, player 2 will also fix his own

1.2 Battle of Sexes

- There are two possible types of player 2 (column): "Meet" player 2 wishes to be at the same place as player 1, just as in the usual game (This type has probability $p = 1/4$). "Avoid" 2 wishes to avoid player 1 and go to the other place (This type has probability $1 - p = 3/4$).

- 2 knows his type, and 1 does not. They simultaneously choose the place Movie (M) or Shopping (S). These payoffs are shown in the matrices below.

$p = \frac{1}{4}$

MEET	S	M
S	$(2,1)$	$(0,0)$
M	$(0,0)$	$(1,2)$

$1 - p = \frac{3}{4}$

AVOID	S	M
S	$(2,0)$	$(0,2)$
M	$(0,1)$	$(1,0)$

$N = \{1,2\}$
$A_1 = \{S, M\}$
$A_2 = \{S, M\}$
$\Theta_1 = \{\text{Meet}\} \quad \Theta_2 = \{\text{Meet, Avoild}\}$
$u_1 : A_1 \times A_2 \times \Theta_1 \times \Theta_2 \to \mathbb{R}$
$u_1(S, S, \text{Meet,Meet}) = 2$
$u_1(S, S, \text{Meet,Avoid}) = 2$
$P_2(\theta_1 \mid \theta_2) = 1 \text{ for } \theta_1 = \text{ Meet } \forall \theta_2$

$$P_1(\theta_2 \mid \theta_1) = \begin{cases} \frac{1}{4} & \text{for } \theta_2 = \text{ Meet} \\ \frac{3}{4} & \text{for } \theta_2 = \text{ Avoid} \end{cases}$$

$s_1 = S$
$s_2(\text{Meet}) = S$
$s_2(\text{Avoid}) = M$

$$u_1(S, s_2, \text{Meet}) = 2\left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right) = \frac{1}{2}$$

$$u_1(M, s_2, \text{Meet}) = 0\left(\frac{1}{4}\right) + 1\left(\frac{3}{4}\right) = \frac{3}{4}$$

Now

$$
\begin{aligned}
&s_1 = M \\
&s_2^{'}(\text{Meet}) = M \\
&s_2^{'}(\text{Avoid}) = S \\
&u_1(S, s_2^{'}, \text{Meet}) = 0\left(\frac{1}{4}\right) + 2\left(\frac{3}{4}\right) = \frac{3}{2} \\
&u_1(M, s_2^{'}, \text{Meet}) = 1\left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right) = \frac{1}{4}
\end{aligned}
$$

$M \to (M, S) \to S \to (S, M) \to M$

Therefore there is no Bayesian Nash Equilibrium in Pure strategies of this game.