1 Logic

1.1 Proposition

A proposition is a statement that is either true or false. For a particular proposition p, the truth value of p is its truth (T) or falsity (F).

1.2 Negation

The proposition $\neg p$ ("not p", called the negation of the proposition p) is true when the proposition p is false, and is false when p is true.

p	$\neg p$
T	F
F	T

1.3 Conjunction (and), ($\wedge$)

The proposition $p \wedge q$ ("p and q", the conjunction of the propositions p and q) is true when both of the propositions p and q are true, and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.4 Disjunction (or), ($\vee$)

The proposition $p \vee q$ ("p or q", the disjunction of the propositions p and q) is false when both of the propositions p and q are false, and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

1.5 Implication, ($\Rightarrow$)

The proposition "p implies q" is false when p is true and q is false, and is true otherwise.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In the implication $p \Rightarrow q, p$ is known as the antecedent and q is known as the consequent.

Different ways of saying "p implies q":

- "if p, then q"
- "p only if q"
- "q whenever p"
- "q, if p"
- "q is necessary for p"
- "p is sufficient for q"

1.6 If and only if, ($\Leftrightarrow$)

The proposition $p \Leftrightarrow q$ ("p if ond only if q") is true when the propositions p or q have the same truth value (both p and q are true, or both p and q are false), and false otherwise.

- The reason that $\Leftrightarrow$ is read as "if and only if" is that $p \Leftrightarrow q$ means the same thing as the compound proposition $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
- It is also sometimes called the "biconditional".

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

1.7 Logical equivalence

Two propositions ϕ and ψ are logically equivalent, written $\phi \equiv \psi$, if they have exactly identical truth tables.

Example: $((p \wedge q) \Rightarrow \neg q) \equiv \neg(p \wedge q)$

p	q	$(p \wedge q) \Rightarrow \neg q$	$\neg(p \wedge q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

1.8 Converse, Contrapositive, Inverse

- The **Converse** of $p \Rightarrow q$ is the proposition $q \Rightarrow p$.
- The **Contrapositive** of $p \Rightarrow q$ is the proposition $\neg q \Rightarrow \neg p$.
- The **Inverse** of $p \Rightarrow q$ is the proposition $\neg p \Rightarrow \neg q$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\neg q \Rightarrow \neg p$	$\neg p \Rightarrow \neg q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Question 1:

what is the negation of $p \Rightarrow q$ i.e. $\neg(p \Rightarrow q)$?

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg(p \Rightarrow q)$	(a)	(b)	(c)	(d)
T	T	F	F	T	F	T	F	T	F
T	F	F	T	F	T	T	T	T	T
F	T	T	F	T	F	T	T	F	F
F	F	T	T	T	F	F	T	T	F

Question 2:

Prove the logical equivalence of the following; (De Morgan's laws)

$$\begin{aligned} \neg(p \vee q) &\equiv \neg p \wedge \neg q \\ \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ (p \Rightarrow q) &\equiv (\neg p) \vee q \\ \Rightarrow \neg(p \to q) &\equiv p \wedge \neg q \\ \Rightarrow \neg\neg(p \Rightarrow q) &\equiv \end{aligned}$$

1.9 Quantifiers

- Universal quantifier ("for all"), ($\forall$):

  The proposition $\forall x \in S, P(x)$ ("for all x in S, $P(x)$") is true when for every possible $x \in S, P(x)$ is true.

  for example 1 is an odd number or 2 is an odd number or 3 is an odd no. $\forall x \in \{1, 2, 3\}, x$ is an odd no. $\forall x \in S, P(x)$ where $P(x)$ is that x is an odd number.

- Existential quantifier ("there exists"), ($\exists$):

  The proposition $\exists x \in S, P(x)$ ("there exists an x in S such that $P(x)$)" is true when for at least one possible $x \in S$, we have $P(x)$ is true.

  for example 1 is an odd number and 2 is an odd number and 3 is an odd number, $\exists x \in \{1, 2, 3\}, x$ is an odd number or simply $\exists x \in S, P(x)$.

- De Morgan's Law (quantified form):

  $$\neg(\forall x \in S, P(x)) \Leftrightarrow (\exists x \in S, \neg P(x))$$
  $$\neg(\exists x \in S, P(x)) \Leftrightarrow (\forall x \in S, \neg P(x))$$

- Vacuous quantification: Consider the propositon "All even prime numbers greater than 12 have a 3 as their last digit". This proposition is vacuously true.

- If the set S is nonempty then, $(\forall x \in S, P(x)) \Rightarrow (\exists x \in S, P(x))$

1.10 Order of Quantification

The order of the quantification matters! One of the following propositions is true; the other is false. Classify them;

$$\begin{aligned}
&\text{Proposition 1:} \quad \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x < y \\
&\text{Proposition 2:} \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y
\end{aligned}$$

Clearly the proposition 2 is false and 1 is true because there does not exist a real number bigger than all the other real numbers but for all real numbers we can find a real number bigger than that.

2 Sets

- Set: Any well-defined collection of objects is a set.
- Element of a Set: If A is a set, then the objects in the collection A are called elements of A. If x is an element of A, it is denoted by $x \in A$.
- Subset: Let U be the universal set. We say set A is a subset of set B, denoted by $A \subset B$ when the following is true:

  $(\forall x \in U)(x \in A \Rightarrow x \in B)$.

2.1 Operartions on Sets

Suppose U is the universal set. Let $A, B \subset U$.

- Complement of set $A : A' = \{x \in U \mid x \notin A\}$.
- Union $(A \cup B)$: It is the set of elements of U which are members of either A or B (or both). Formally, $\{x \in U \mid (x \in A) \vee (x \in B)\}$.
- Intersection $(A \cap B)$: It is the set of all members which A and B have in common.Formally, $\{x \in U \mid (x \in A) \wedge (x \in B)\}$.
- Disjoint sets: Two sets A, B are disjoint if they have no elements in common, that is, if $A \cap B = \emptyset$.

2.2 Laws on set operations

Suppose U is the universal set, Let $A, B, C \subset U$.

Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Commutative laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$
De Morgan's laws	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
Copmlementation laws	$A \cup A' = U$ $A \cap A' = \phi$
Self-inverse law	$(A')' = A$

2.3 Arbitraty Unions and Intersections

We refer to the collection $\{A_i \mid i \in I\}$ as an indexed family of sets and the set I as the index set.

- The union of an indexed family $\{A_i \mid i \in I\}$ is defined as

$$\bigcup_{i \in I} A_i = \{a \mid (\exists i \in I)\,[a \in A_i]\}$$

- The intersection of an indexed family $\{A_i \mid i \in I\}$ is defined as

$$\bigcap_{i \in I} A_i = \{a \mid (\forall i \in I)\,[a \in A_i]\}$$

Consider for example, if $I = \{1, 2\}$, then

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \quad \text{and} \quad \bigcap_{i \in I} A_i = A_1 \cap A_2$$

2.4 Relations

- Power Set: The Power set of A is the set of all subsets of A, denoted by $\mathcal{P}(A)$ and 2^A. Note that the cardinality of the power set is $2^{|A|}$.
- Cartesian product: Given two sets A and B, the Cartesian product of A, B is defined to be the set:

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

- Relation: Relation between a set A and a set B is a subset of the Cartesian product $A \times B$.
- Relation on a set: A relation on the set S is formally defined as a subset of $S \times S$ i.e $\mathcal{R} \subseteq S \times S$

2.5 Properties of Relations

- Reflexivity: $\forall x \in S : (x, x) \in \mathcal{R}$
- Completeness: $\forall x, y \in S : x \neq y \Rightarrow (x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$

- Transitivity: $\forall x, y, z \in S : ((x, y) \in \mathcal{R} \text{ and } (y, z) \in \mathcal{R}) \Rightarrow (x, z) \in \mathcal{R}$
- Symmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$
- Anti-symmetry: $\forall x, y \in S : ((x, y) \in \mathcal{R} \text{ and } (y, x) \in \mathcal{R}) \Rightarrow x = y$
- Asymmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \notin \mathcal{R}$
- Negative transitivity: $\forall x, y, z \in S : ((x, y) \notin \mathcal{R} \text{ and } (y, z) \notin \mathcal{R}) \Rightarrow (x, z) \notin \mathcal{R}$
- Equivalence: Relation which is symmetric, reflexive and transitive.

Question:

Consider a binary relation $\succeq$ on a set A. Suppose $\succeq$ is transitive. Define relations $\succ$ and $\sim$ on A by: for $x, y \in A$,

$x \succ y$ if and only if $x \succeq y$ and not $y \succeq x$

$x \sim y$ if and only if $x \succeq y$ and $y \succeq x$

Prove the following. (i) If $x \succ y$ and $y \succ z$ then $x \succ z$

(ii) If $x \sim y$ and $y \sim z$ then $x \sim z$

(iii) If $x \succ y$ and $y \succeq z$ then $x \succ z$