1 Public Goods

1.1 Dolbear's Triangle

$$u_1(x_1, g)$$

$$u_2(x_2, g)$$

$$\mathcal{F} = \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid (10 - (x_1 + x_2)) = g\}$$

$$= \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid x_1 + x_2 + g = 10\}$$

$$\omega_1 = \omega_2 = 5$$

1.1.1 Question

$$u_1(x_1, g) = 8x_1 + 5g$$

$$u_2(x_2, g) = 8x_2 + 5g$$

$$\mathcal{F} = \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid 10 - (x_1 + x_2) = g\}$$

$$u_2^{ADJ}(x_1, g) = u_2(10 - x_1 - g, g)$$

$$= 80 - 8x_1 - 3g$$

$$PE = \{(x_1, x_2, g) \in \mathcal{F} \mid x_1 = 0 \lor x_2 = 0\}$$

Now we can solve the following problem

$$\max_{(x_1, x_2, g) \in \mathbb{R}_+^3} \alpha(8x_1 + 5g) + \beta(8x_2 + 5g)$$
s.t. $x_1 + x_2 + g = 10$

The above problem is equivalent to solving

$$\max_{(x_1, x_2, g) \in \mathbb{R}_+^3} 8\alpha x_1 + 8\beta x_2 + 5(\alpha + \beta)g$$
s.t. $x_1 + x_2 + g = 10$

Note that

- If $\alpha > \beta$ then $x_2 = 0$
- if $\alpha < \beta$ then $x_1 = 0$
- if $\alpha = \beta$ then $x_1 = x_2 = 0, g = 10$

Now

• if
$$\alpha > \beta$$
 and $8\alpha > 5(\alpha + \beta)$ then, $x_1 = 10, x_2 = 0, g = 0$

• if
$$\alpha > \beta$$
 and $8\alpha < 5(\alpha + \beta)$ then, $x_1 = 0, x_2 = 0, g = 10$

• if
$$\alpha > \beta$$
 and $8\alpha = 5(\alpha + \beta)$ then, $x_2 = 0, x_1 + g = 10$

Similarly;

• if
$$\alpha < \beta$$
 and $8\beta > 5(\alpha + \beta)$ then, $x_1 = 0, x_2 = 10, g = 0$

• if
$$\alpha < \beta$$
 and $8\beta < 5(\alpha + \beta)$ then, $x_1 = 0, x_2 = 0, g = 10$

• if
$$\alpha < \beta$$
 and $8\beta = 5(\alpha + \beta)$ then, $x_1 = 0, x_2 + g = 10$

1.2 Lindahl equilibrium

We have to find the equilibrium price ratio (p_1^*, p_2^*) ;

1.2.1 Question

Find the competitive equilibrium in the following economy;

$$u_1(x_1, g) = 8x_1 + 5g \qquad \omega_1 = 4$$

$$u_2(x_2, g) = 8x_2 + 5g \qquad \omega_2 = 6$$

$$g = f(x_0) = x_0$$

$$\mathcal{F} = \{(x_1, x_2, g) \in \mathbb{R}^3_+ | x_1 + x_2 + g = 10\}$$

$$(p_1^*, p_2^*) = \left(\frac{2}{5}, \frac{3}{5}\right)$$

How to check if it is correct?

To solve for the equilibrium price ratio, we can solve the following two problems;

$$\max_{(x_1,g)\in\mathbb{R}_+^2} 8x_1 + 5g$$
s.t. $x_1 + \frac{2}{5}g = 4$

Solution to the above problem is $(x_1, x_2, g) = (0, 0, 10)$ since $\frac{8}{5} < \frac{5}{2}$.

Similarly

$$\max_{(x_2,g)\in\mathbb{R}_+^2} 8x_2 + 5g$$

$$s.t. \quad x_2 + \frac{3}{5}g = 6$$

Solution to the above problem is $(x_1, x_2, g) = (0, 0, 10)$ since $\frac{8}{5} < \frac{5}{3}$.

1.2.2 First Welfare Theorem for Lindahl Equilibrium;

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\begin{aligned} u_1(x_1,g) \\ u_2(x_2,g) \\ g &= f(x_0) \\ (\omega_1,\omega_2) \\ (\theta_1,\theta_2) \\ \mathcal{F} &= \{(x_1,x_2,g) \in \mathbb{R}^3_+ | f(\omega_1+\omega_2-x_1-x_2) = g\} \\ u_1,u_2 \text{ are increasing by assumption} \end{aligned}
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is infact Pareto efficient.

Proof Suppose $((p_1^*, p_2^*), (x_1^*, x_2^*, g^*))$ is the Lindahl equilibrium, and suppose (x_1^*, x_2^*, g^*) is not Pareto efficient. So there must exist $(x_1', x_2', g') \in \mathcal{F}$ such that,

1. $u_1(x_1', g') \ge u_1(x_1^*, g^*)$ and $u_2(x_2', g') \ge u_2(x_2^*, g^*)$ 2. $u_1(x_1', g') > u_1(x_1^*, g^*)$ or $u_2(x_2', g') > u_2(x_2^*, g^*)$ By $1, x_1' + p_1^* g' \ge \omega_1 + \theta_1 \pi^* (p_1^* + p_2^*)$ and $x_2' + p_2^* g' \ge \omega_2 + \theta_2 \pi^* (p_1^* + p_2^*)$ or
By $2, x_1' + p_1^* g' > \omega_1 + \theta_1 \pi^* (p_1^* + p_2^*)$ and $x_2' + p_2^* g' > \omega_2 + \theta_2 \pi^* (p_1^* + p_2^*)$ So, adding the above two conditions which we got form 2, we get $x_1' + x_2' + (p_1^* + p_2^*)g' > \omega_1 + \omega_2 + \pi^* (p_1^* + p_2^*)$ $\Rightarrow \pi^* (p_1^* + p_2^*) < (p_1^* + p_2^*)g' - (\omega_1 + \omega_2 - x_1' - x_2')$ But this is a contradiction, hence it must be the case that (x_1^*, x_2^*, g^*)

1.2.3 Question 2

Find the Lindahl Equilibrium;

$$u_1(x_1, g) = x_1 + 2\sqrt{g}$$
 $\omega_1 = 2$
 $u_2(x_2, g) = x_2 + 4\sqrt{g}$ $\omega_2 = 1$
 $u_3(x_3, g) = x_2 + 4\sqrt{g}$ $\omega_3 = 1$
 $g = f(x_0) = x_0$

First we solve the firm's profit maximization problem;

$$\max_{(g,x_0)} (p_1 + p_2 + p_3)g - x_0$$
s.t. $g \le x_0$

Which is equivalent to solving,

$$\max_{(g)} (p_1 + p_2 + p_3)g - g$$

So the solution to the above problem is,

$$g* \in \begin{cases} \phi & \text{if } p_1 + p_2 + p_3 > 1\\ \{0\} & \text{if } p_1 + p_2 + p_3 < 1\\ \mathbb{R}_+ & \text{if } p_1 + p_2 + p_3 = 1 \end{cases}$$

Now we can solve the utility maximization probelms of the agents;

$$\max_{(x_1,g)\in\mathbb{R}_+^2} \quad x_1 + 2\sqrt{g}$$

$$s.t. \quad x_1 + p_1^*g = 2$$

Which is equivalent to solving;

$$\max_{0 \le g_1 \le \frac{2}{p_1^*}} \quad 2 - p_1^* g_1 + 2\sqrt{g_1}$$

Differentiating w.r.t g_1 we get $-p_1^* + \frac{1}{\sqrt{g_1}}$

So the solution to the above problem is,

$$g_1 = \begin{cases} \left(\frac{1}{p_1^*}\right)^2 & \text{if } p_1^* > \frac{1}{2} \\ \frac{2}{p_1^*} & \text{if } p_1^* \le \frac{1}{2} \end{cases}$$

$$\max_{(x_2,g)\in\mathbb{R}_+^2} x_2 + 4\sqrt{g}$$
s.t. $x_2 + p_2^*g = 1$

Which is equivalent to solving;

$$\max_{0 \le g_2 \le \frac{1}{p_2^*}} \quad 1 - p_2^* g_2 + 4\sqrt{g_2}$$

Differentiatig w.r.t g_2 we get $-p_2^* + \frac{2}{\sqrt{g_2}}$

So the solution to the above problem is,

$$g_2 = \begin{cases} \left(\frac{2}{p_2^*}\right)^2 & \text{if } p_2^* > 4\\ \frac{1}{p_2^*} & \text{if } p_2^* \le 4 \end{cases}$$

$$\max_{(x_3,g)\in\mathbb{R}_+^2} x_3 + 4\sqrt{g}$$
s.t. $x_3 + p_3^*g = 1$

Which is equivalent to solving;

$$\max_{0 \le g_3 \le \frac{1}{p_3^*}} \quad 1 - p_3^* g_3 + 4\sqrt{g_3}$$

Differentiatig w.r.t g_3 we get $-p_3^* + \frac{2}{\sqrt{g_3}}$

So the solution to the above problem is,

$$g_3 = \begin{cases} \left(\frac{2}{p_3^*}\right)^2 & \text{if } p_3^* > 4\\ \frac{1}{p_3^*} & \text{if } p_3^* \le 4 \end{cases}$$

Now we can equate the quantity demanded and quantity supplied of the public good equal to solve for the equilibrium price ratio;

We know that for supply of the public good to be positive we need $p_1^* + p_2^* + p_3^* = 1$ or simply because of the preferences and endowments of agent 2 and $p_1^* + 2p_2^* = 1 \dots (i)$.

So if $g_1^* = \frac{2}{p_1^*}$ and $g_2 = g_3 = \frac{1}{p_2^*} = \frac{1}{p_3^*}$ then equating $g_1 = g_2 = g_3$ we get $\frac{2}{p_1^*} = \frac{1}{p_2^*} = \frac{1}{p_3^*}$ or $p_1^* = 2p_2^* \dots (ii)$

now using (i) and (ii) we get $p_2^*=\frac{1}{4}$ and therfore $p_1^*=\frac{1}{2}$ and since $p_2^*=p_3^*$ we also get $p_3^*=\frac{1}{4}$.

substituing for p_1^*, p_2^*, p_3^* in g_1, g_2, g_3 we get that $g_1^* = g_2^* = g_3^* = 4$