Convergence in Probability

Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of random variables (which can be dependent or independent). Let $a \in \mathbb{R}$, we say that Y_n converges to a in Probability, if

$$\forall \epsilon > 0, \quad \lim_{n \to \infty} Pr\left(|Y_n - a| > \epsilon\right) = 0$$

or

$$Pr\left(Y_n \notin (a - \epsilon, a + \epsilon)\right) = 0$$

0.1 Limit of a sequence

If $a_1, a_2, \ldots, a_{n+1}, \ldots$ is a sequence of reals where, $a : \mathbb{N} \to \mathbb{R}$, $a_n \in \mathbb{N}$

So (a_n) has a limit $l \in \mathbb{R}$ if the following is true;

$$(\forall \epsilon > 0)\,(\exists N \in \mathbb{N})\,(\forall n \in \mathbb{N})\,(n > \mathbb{N} \implies |a_n - l| < \epsilon)$$

We say that the sequence of Random variables $Y_1, Y_2, \ldots, Y_n, \ldots$ converges in probabilty to $a \epsilon \mathbb{R}$ if;

$$\forall \epsilon > 0 \quad \lim_{n \to \infty} Pr\left(|Y_n - a| > \epsilon\right) = 0$$

Consider a sequence of *iid* random variables $X_1, X_2, \ldots, X_n, \ldots$, with $\mathbb{E}[X_i] = \mu$ and $Var\left(X_i\right) = \sigma^2$.

We define Sample mean as follows;

$$M_n = \bar{X_n} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Note that $M_1, M_2, \ldots, M_n$ is also a sequnce of random variables with $\mathbb{E}[M_1] = \mathbb{E}[M_2] = \ldots = \mathbb{E}[M_n] = \mu$ and $Var[M_n] = \frac{\sigma^2}{n} = \mathbb{E}[M_n - \mu]^2$. So as n increases $Var(M_n)$ tends to μ as it gets smaller and smaller. Which gives us the Weak law of large numbers.

Weak Law of Large numbers

Let M_n be a sequence of sample means generated from *iid* sample X_n with $\mathbb{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2$ then M_n converges in probability to μ