1 Compact Sets Contd.

1.1 Proposition 18

- Any compact subset of a metric space X is closed and totally bounded.
- We will first show that any compact subset of a metric space X is closed:
- Let S be the compact subset of X.
- Take any (x_n) in S with $x_n \to I_x$ for some $I_x \in X$.
- We just need to show $I_x \in S$ (See Proposition 15).
- Since S is compact there is a subsequence (x_{n_*}) of (x_n) that converges to a point in S.
- Since (x_n) converges to I_x, so (x_{n_k}) also converges to I_x. Thus, $I_x \in S$.

Now let us show that any compact subset of a metric space X is totally bounded:

- Suppose the claim is not true, that is, there is a compact subset S of X with the following property: there exists an $\epsilon > 0$ such that $\{\mathcal{N}_e(x) : x \in T\}$ does not cover S for any finite $T \subseteq S$.
- To derive a contradiction, we wish to construct a sequence in S with no convergent subsequence.
- Begin by picking an $x_1 \in S$ arbitrarily.
- By hypothesis, we cannot have $S \subseteq \mathcal{N}_\varepsilon (x_1)$ so there must exist an $x_2 \in S$ such that $d(x_1, x_2) \geq \epsilon$.
- Again, $S \subseteq \mathcal{N}_\epsilon (x_1) \cup \mathcal{N}_\epsilon (x_2)$ cannot hold, so we can find an $x_3 \in S$ such that $d(x_i, x_3) \geq \epsilon, i = 1, 2$.
- Proceeding inductively, we obtain a sequence (x_n) such that $d(x_i, x_j) \geq \epsilon$ for any distinct $i, j \in \mathbb{N}$.
- Since S is compact, there must exist a convergent subsequence, say (x_{n_k}), of (x_n).
- But this is impossible since $\lim x_{n_k} = x$ would imply that

  $$d\left(x_{n_k}, x_{n_j}\right) \leq d\left(x_{n_k}, x\right) + d\left(x, x_{n_j}\right) < \epsilon$$

  for large enough (distinct) k and I.

Let (X,d) be our metric space then we want to show that $Y \subset X$, Y is compact, implies that Y is closed and Y is totally bounded.

1. Y is compact $\rightarrow$ Y is closed. Since Y is compact, so every sequence in Y has a subsequence that converges to a point in Y.

   Proof Consider an arbitray sequence $(x_n) \subset Y$ that converges to $l \in X$. Since Y is compact, $l \in Y$, and therefore Y is closed.

2. Y is compact $\implies$ Y is totally bounded.

   Proof The contrapositive of the above statement is;

   Y is not totally bounded $\implies$ Y is not compact.

   Y is not totally bounded means $\exists \epsilon > 0$ such that

   $$\left(\text{for no finite subset } T \text{ of } X, Y \subset \bigcup_{x \in T} \mathcal{N}_\epsilon(x) \right)$$

   or,

   $$\left(\text{for any finte subset } T \text{ of } X, Y \not\subset \bigcup_{x \in T} \mathcal{N}_\epsilon(x) \right)$$

   Y is not compact means that there exsits a sequence in Y that has no subsequence that converges to a point in Y.

   Since Y in not totally bounded we know that Y is not an empty set because an empty set is totally bounded, therefore we start by picking a point $x_1 \in Y$ then it is true $Y \not\subset \mathcal{N}_\epsilon(x_1)$ since Y is not totally bounded,

   Now we pick any $x_2 \in Y \setminus \mathcal{N}_\epsilon(x_1)$, we can do this because we know the first n-ball does not contain the set Y completely, then it is true that;

   $$Y \not\subset \mathcal{N}_\epsilon(x_1) \cup \mathcal{N}_\epsilon(x_2)$$

   similiarly we now pick $x_3 \in Y \setminus \mathcal{N}_\epsilon(x_1) \cup \mathcal{N}_\epsilon(x_2)$ and then we know

   $$Y \not\subset \mathcal{N}_\epsilon(x_1) \cup \mathcal{N}_\epsilon(x_2) \cup \mathcal{N}_\epsilon(x_3)$$

   In this manner we get our sequence (x_n) which has the property

   $$d(x_i, x_j) \geq \epsilon \quad \text{for } i \neq j$$

Now suppose that $x_n \to l$

then $d(x_n, l) \to 0$

and $\exists\ N$ s.t. $n > N, d(x_n, l) < \frac{\epsilon}{4}$

and by triangle's inequality

$$d(x_i, x_j) \leq d(x_i, l) + d(x_j, l) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \text{ for } i \neq j$$

which is a contradiction!

1.2 Proposition 19

- Given any $m \in \mathbb{N}$, a subset of $\mathbb{R}^m$ (with Euclidean metric) is compact if, and only if, it is closed and bounded.

If compact then closed and bounded:

- By proposition 17 and 18, every compact set is both closed

  and bounded.

If closed and bounded then compact :

- Now suppose S is closed and bounded.
- To show S is compact, pick any sequence (x_n).
- Since (x_n) is bounded, there exist a convergent subsequence (x_{n_k}) (by proposition 13).
- Limit of (x_{n_k}) will lie in S since S is closed (by proposition 15).

Now let's prove that In (X, d) - Euclidean metric space $(X \subset \mathbb{R}^n)$, if $Y \subset X$ is closed and bounded then Y is compact.

Proof

Suppose Y is closed and bounded, consider an arbitrary sequence $(x_n \subset Y)$, then by Bolzano-Weierstarass theorem, since (x_n) is bounded in $\mathbb{R}^n$, it has a convergent subsequence $x_{n_k} \to l$, since Y is closed, it follows that $l \in Y$, therefore Y is compact.

We can find examples of metric spaces in which a set is compact and bounded that is not compact.

$X = \mathbb{R}$ and $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Note that in this discrete metric space $\mathbb{R}$ closed and bounded.

say we now pick the sequence $\left(\frac{1}{n}\right)$ or (n) and it is easy to observe that neither of these sequences are not convergent and none of their subsequences convergent to a point in the set X.

2 Continuity of a function

2.1 Continuity of a function at a point:

A function $f : (X, d_x) \to (Y, d_y)$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f\left(\mathcal{N}_\delta\left(x_0\right)\right) \subseteq \mathcal{N}_c\left(f\left(x_0\right)\right)$ (where for any subset A of X, we define $f(A) = \{f(x) : x \in A\}$).

We say that $f : X \to Y$ is continuous at $x_0 \in X$ if $(\forall\epsilon > 0)(\exists\delta > 0)(f(\mathcal{N}_\delta(x_0)) \subseteq \mathcal{N}_\epsilon(f(x_0)))$

Examples;

1. Suppose $X = \mathbb{N}$ and $d_x(m, n) = |m - n|$

   $Y = \mathbb{R}$ and $d_y(a, b) = |a - b|$

   Is $f(x) = x$ where $f : X \to Y$ continuous?

   Yes! because $\mathcal{N}_\delta(2) = \{2\} \quad$ for $\delta \leq 1$

2. Suppose $X = \mathbb{N}$ and $d_x(m, n) = |m - n|$

   $Y = \mathbb{R}$ and $d_y(a, b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$

   Is $f(x) = x$ where $f : X \to Y$ continuous? Yes in this case

   $f\left(\mathcal{N}_\delta\left(x_0\right)\right) = \mathcal{N}_c\left(f\left(x_0\right)\right)$

3. Suppose $X = \mathbb{R}$ and $d_x(m, n) = |m - n|$

   $Y = \mathbb{R}$ and $d_y(a, b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$

   Is $f(x) = x$ where $f : X \to Y$ continuous at $x = 2$? NO!

4. Suppose $X = \mathbb{R}$ and $d_x(a, b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$

   $Y = \mathbb{R}$ and $d_y(a, b) = |a - b|$

Is $f(x) = \lfloor x \rfloor$ where $f : X \to Y$ continuous at $x = 2$? Yes! take $\delta \leq 1$.

2.2 Proposition 20

- A function $f : X \to Y$ is continuous at $x_0 \in X$ if, and only if, for every sequence (x_n) in X that converges to x_0, the sequence $f(x_n)$ in Y converges to $f(x_0)$.

f is continous at $x_0 \in X$ iff $(\forall (x_n) \subset X)\,(x_n \to x_0 \in X \implies f(x_n) \to f(x_0))$

To prove : If f is continuous at x_0 then for every sequence (x_n) in X that converges to x_0, we have the sequence $f(x_n)$ in Y converges to $f(x_0)$:

- Suppose $f : X \to Y$ is continuous at $x_0 \in X$.
- Now pick any sequence (x_n) in X that converges to x_0.
- We claim that $f(x_n)$ converges to $f(x_0)$.
- To prove this, consider an arbitrary $\epsilon > 0$.
- Since f is continuous at x_0, there exists $\delta > 0$ such that $f(\mathcal{N}_\delta(x_0)) \subseteq \mathcal{N}_\epsilon(f(x_0))$.
- Now (x_n) converges to x_0, thus there exists N such that $x_n \in \mathcal{N}_\delta(x_0)$ for all $n \geq N$.
- So, we have $f(x_n) \in f(\mathcal{N}_\delta(x_0)) \subseteq \mathcal{N}_\epsilon(f(x_0))$ for all $n \geq N$.
- Hence, $f(x_n)$ converges to $f(x_0)$.

To prove : If for every sequence (x_n) in X that converges to x_0, we have the sequence $f(x_n)$ in Y converges to $f(x_0)$ then f is continuous at x_0 :

- Conversely, suppose f is not continuous at x_0.
- That is, there exists $\epsilon > 0$ for which $f(\mathcal{N}_\delta(x_0)) \nsubseteq \mathcal{N}_\epsilon(f(x_0))$ for all $\delta > 0$.
- In particular, $f\left(\mathcal{N}_{\frac{1}{n}}(x_0)\right) \nsubseteq \mathcal{N}_\epsilon(f(x_0))$ for all $n \in \mathbb{N}$.
- So we can pick a sequence (x_n) in such a way that the following is satisfied: $x_n \in \mathcal{N}_{\frac{1}{n}}(x_0)$ and $f(x_n) \notin \mathcal{N}_e(f(x_0))$ for all $n \in \mathbb{N}$.
- Thus, $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$.

Let's prove the contrapositve of $B \implies A$

$$(\exists (x_n) \subset X)\,(x_n \to x_0 \vee f(x_n \not\longrightarrow f(x_0))$$

suppose f is not continous at $x_0 \in X$ means that there exists $\epsilon > 0$ for which $f\left(\mathcal{N}_\delta\left(x_0\right)\right) \nsubseteq \mathcal{N}_\epsilon\left(f\left(x_0\right)\right)$ for all $\delta > 0$.
then the following is also true

$$(\exists \epsilon > 0)(\forall n \in \mathbb{N})(f(\mathcal{N}_{\frac{1}{n}}(x_0)) \not\subset \mathcal{N}_\epsilon(f(x_0)))$$

Pick $x_n \in \mathcal{N}_{\frac{1}{n}}(x_0)$ such that $f(x_n) \notin \mathcal{N}_\epsilon(f(x_0))$
$0 \leq d_x(x_n, x_0) \leq \frac{1}{n}$
$(x_n) \to x_0$ and
$d_y(f(x_n), f(x0)) \geq \epsilon$
So $f(x_n)$ does not converge to $f(x_0)$

2.3 Continuity of a function over a set

A function $f : X \to Y$ is said to be continuous over $D \subset X$ if it is continuous at every $x \in D$.

2.4 Continuity of a function

- A function $f : X \to Y$ is said to be continuous if it is continuous at every $x \in X$.

2.5 Proposition 21

- A function $f : X \to Y$ is continuous if, and only if, for each open set V of $Y, f^{-1}(V)$ is open in X (where for any subset B of Y, we define $f^{-1}(B) = \{x : f(x) \in B\}$).

To prove: If a function $f : X \to Y$ is continuous then for each open set V of $Y, f^{-1}(V)$ is open in X :

- - Suppose $f : X \to Y$ is continuous.
- Consider any open set V of Y. We claim that $f^{-1}(V)$ is open in X.
- To prove this, pick an arbitrary $x \in f^{-1}(V)$.

- Since V is an open subset of Y and $f(x) \in V$, there exists $\epsilon > 0$ such that $\mathcal{N}_\epsilon(f(x)) \subseteq V$.
- By continuity of f at x, there exists $\delta > 0$ such that $f\left(\mathcal{N}_\delta(x)\right) \subseteq \mathcal{N}_\epsilon(f(x)) \subseteq V$.
- Thus we have $\mathcal{N}_\delta(x) \subseteq f^{-1}(V)$.

To prove: If for each open set V of $Y, f^{-1}(V)$ is open in X then $f : X \to Y$ is continuous:

- - Conversely, suppose $f^{-1}(V)$ is open in X for every V open in Y.
- We claim that f is continuous.
- To prove this, pick an arbitrary $x \in X$ and an arbitrary $\epsilon > 0$.
- Since $\mathcal{N}_\epsilon(f(x))$ is open in Y, so $f^{-1}\left(\mathcal{N}_e(f(x))\right)$ is open in X which along with the fact $x \in f^{-1}\left(\mathcal{N}_\epsilon(f(x))\right)$ implies that there exists $\delta > 0$ such that $\mathcal{N}_\delta(x) \subseteq f^{-1}\left(\mathcal{N}_\epsilon(f(x))\right)$ or equivalently, $f\left(\mathcal{N}_\delta(x)\right) \subseteq \mathcal{N}_e(f(x))$.

Let's prove that if a function $f : X \to Y$ is continuous implies

$(\forall$ open set $V \subset Y)(f^{-1}(v)$ is open in X $)$

<u>Proof</u>

Consider any open set $V \subset Y$

Consider $f^{-1}(v)$, if $f^{-1}(v)$ is empty then it is open. If it is non-empty, then consider any $x_0 \in f^{-1}(v)$

Since we know that f is contionus, f is continuous at x_0 and since $f(x_0) \in V$ and V is open, so $\exists \epsilon > 0$ such that $\mathcal{N}_\epsilon(f(x_0)) \subset V$ and since f is continous at x_0 $\exists \delta > 0$such that $f(\mathcal{N}_\delta(x_0)) \subset \mathcal{N}_\epsilon(f(x_0)) \subset V$

So $\mathcal{N}_\delta(x_0) \subset f^{-1}(V)$