Precise Definition of limit

Let f be a function defined on some open interval that contains a, except possibly at a itself. Then we say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

and the precise definiton would be,

$$(\forall \epsilon > 0)\,(\exists \delta > 0)\,(\forall x)\,(0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

The negation of the above definiton would be

$$(\exists \epsilon > 0)\,(\forall \delta > 0)\,(\exists x)\,(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

The above statement tells us that if it is true then we have that,

$$\lim_{x \to a} f(x) \neq L$$

and we say that limit of $f(x)$ as x approaches a is not equal to L.

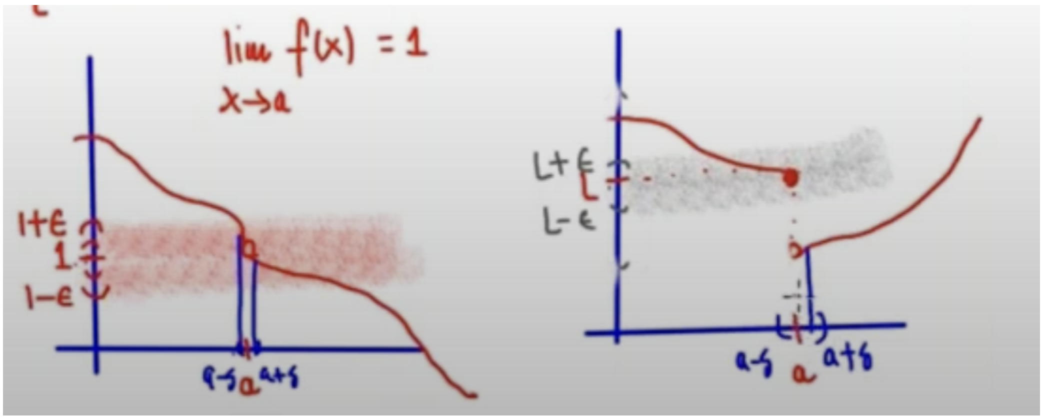


Figure 1: How to check for existence of limit graphically

Example

$$\lim_{x \to 0^+} \frac{1}{x} \quad \text{DNE}$$

1 Subsets of Real line

$\mathbb{R},\ \mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R}_+,\ \mathbb{Z}_+,\ \mathbb{Q}_+$

$$\{x \in \mathbb{R} | a < x < b\} = (a, b)$$
$$\{x \in \mathbb{R} | a \leq x \leq b\} = [a, b]$$
$$\{x \in \mathbb{R} | a \leq x < b\} = [a, b)$$
$$\{x \in \mathbb{R} | a < x \leq b\} = (a, b]$$

2 Bounds

If $X \subset \mathbb{R}$, then

$c \in \mathbb{R}$ is an upper bound of X if $c \geq x \quad \forall x \in X$

examples of the sets which are bounded above are $[0, 1],\ (0, 1)$ but $\mathbb{N}$ is not bounded above.

similiarly $c \in \mathbb{R}$ is a lower bound for X if $c \leq x \quad \forall x \in X$

examples of some sets bounded below are $\mathbb{N}, [0, 1], (0, 1)$.

We say that $X \subset \mathbb{R}$ is bounded if it is bounded above and bounded below as well. So $\mathbb{N}$ is not bounded but the sets $[0, 1]$ and $(0, 1)$ are bounded.

If $x \in X$ is an upperbound of X then $x = \max X$.

for example $1 = \max[0, 1]$ but $\max(0, 1)$ does not exsist and similiarly $\max \mathbb{N}$ does not exsist.

2.1 Supremum

say $X \subset \mathbb{R}$ and X is non empty then $\sup X$ is the lowest upperbound of X if the set is bounded above, and if the set is not bounded above then $\sup X = \infty$

for example, $\sup(0, 1) = 1$, $\sup[0, 1] = 1$, $\sup \mathbb{N} = \infty$ and $\sup \mathbb{Z} = \infty$.

But if the set X is empty then note that the every empty set is bounded above all $x \in \mathbb{R}$ therefore $\sup \phi = -\infty$.

2.2 Infimum

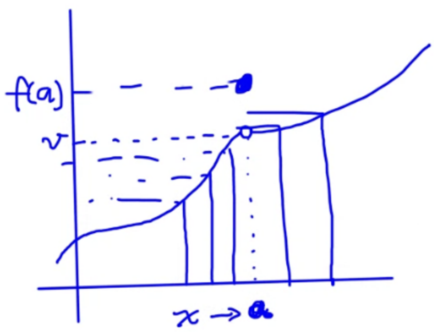
for $X \subset \mathbb{R}$ and $X \neq \phi$, $\inf X$ is the greatest lower bound if X is bounded below, and it is $-\infty$ if the set X is not bounded above.

Note that if $X = \phi$ then $\inf X = \infty$.

for example, $\inf \mathbb{Z} = -\infty$, $\inf \mathbb{N} = 1$, $\inf[0, 1] = 0$ and $\inf(0, 1) = 0$.

3 Limits and Continuity of functions

The question we want to answer is what happens to the value of the function as x approaches or gets closer to a.



Note f must be defined on some open interval around a (except possibly at a itself). Here $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

Now the domain of $f(x)$is $\mathbb{R} - 1$

$$f(x) = \frac{x^2 - 1}{x - 1}$$
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = 2$$

So we say that $l \in \mathbb{R}$ is the limit of $f(x)$ as x approaches a and write $l = \lim_{x \to a} f(x)$ if the following holds;

- $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - a| < \delta \implies |f(x) - l| < \epsilon)$.

if we want to define $\lim_{x \to a} f(x) \neq l$ we can negate the above definiton;

$$\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - a| < \delta \implies |f(x) - l| < \epsilon)$$
$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in D)(0 < |x - a| < \delta \wedge |f(x) - l| \geq \epsilon)$$

In a smiliar way we can define the other related similiar concepts such as the left hand limit (LHL) and the right hand limit (RHL).

We say that $\lim_{x \to a^-} f(x) = l$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(a - \delta < x < a \implies |f(x) - l| < \epsilon)$$

and We say that $\lim_{x \to a^+} f(x) = l$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(a < x < a + \delta \implies |f(x) - l| < \epsilon)$$

again note that for the LHL to exist the fuction must be defined on some open interval to the left of a and similiary for the RHL to exist the function must be defined on some open interval to the right of a and not necessarily at a.

so for a limit to exist we need,

$$\lim_{x \to a} f(x) = l \text{ if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$$

more formally,

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)((a - \delta < x < a) \wedge (a < x < a + \delta) \implies |f(x) - l| < \epsilon)$$
$$= (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - a| < \delta \implies |f(x) - l| < \epsilon)$$

Now,

We say that $\lim_{x \to a} \frac{1}{x} = \infty$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - a| < \delta \implies f(x) > \epsilon)$$

We say that f is continous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Note that at the endpoints of an interval for a function to be continous we only need $\lim_{x \to a^+} f(x) = f(a)$ or $\lim_{x \to a^-} f(x) = f(a)$ given the left or right endpoint respectively.

4 Sequences of Real Numbers

We are always talking about infinte sequences when we are dealing with sequnces of real numbers becaue it contains countably infinite terms.

$x_1, x_2, x_3, x_4, \ldots$

Formally a sequence x_n or $x(n)$ is a function defined as $x : \mathbb{N} \to \mathbb{R}$, for example,

$x_n = \frac{1}{n}$, $x_n = (-1)^n$, etc,.

4.1 Limit of a sequence

We say that a sequence of reals (x_n) is convergent if there exists a number $l \in \mathbb{R}$ such that ,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \implies |x_n - l| < \epsilon)$$

and this number l is known as the limit of the sequence (x_n), which is written as $\lim_{n\to\infty} x_n = l$ or $x_n \to l$.

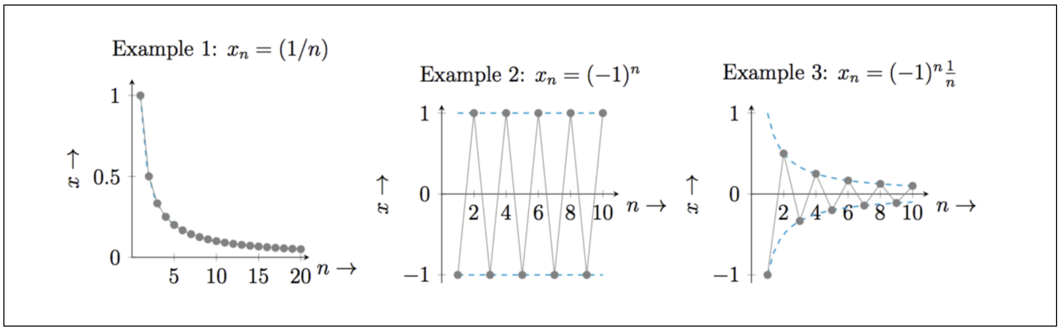


Figure 2: A few examples

Now to show that $1 \neq \lim_{n\to\infty}(-1)^n$ we can show,

$$(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N})(n > N \wedge |x_n - l| \geq \epsilon)$$

Propositon 1

- A sequence cannot have more than one limit.

Proof: Suppose a sequence has two differnet limits a and b and it is approaching both a and b,

Now by definiton of the limit of a sequence,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \implies |x_n - l| < \epsilon)$$

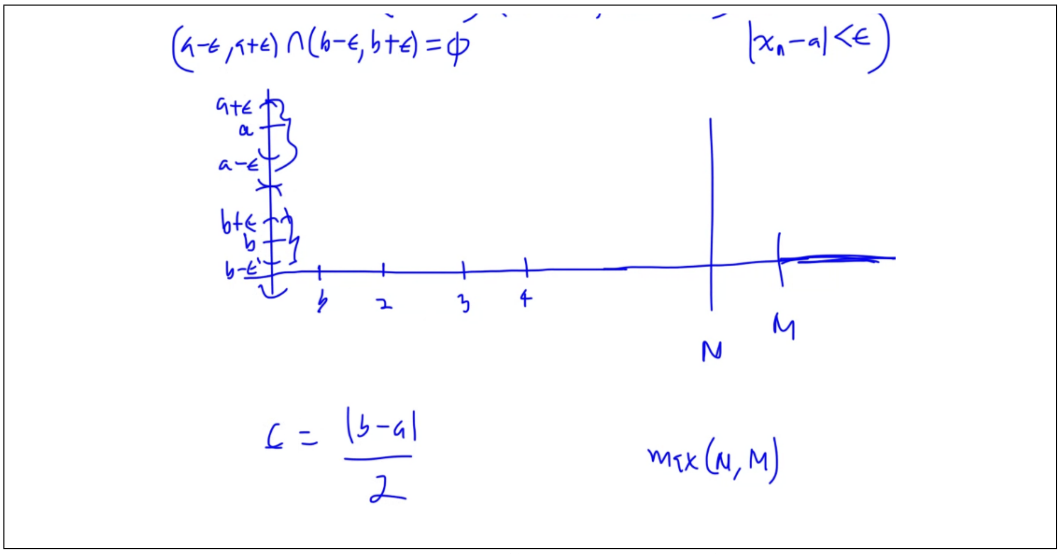


Figure 3: Graphical Proof

Proposition 3

- Let (x_n) and (y_n) be convergent sequences in $\mathbb{R}$ and $x_n \leq y_n$ for infinitely many n. Then $\lim x_n \leq \lim y_n$

Propositon 4 (Squeeze Theorem

- Let (x_n) and (y_n) and (z_n) be sequences in $\mathbb{R}$ and $x_n \leq y_n \leq z_n$ for almost all n. If $\lim x_n = \lim z_n = a$, then (y_n) converges to a.

4.2 Bounded Sequences

Bounded sequence of real numbers:

- We say that a real sequence (x_n) is bounded from above if there exists a real number K with $x_n \leq K$ for all $n = 1, 2, \ldots$.
- This is equivalent to saying that

$$\sup \{x_n \mid n \in \mathbb{N}\} < \infty$$

- Dually, (x_n) is said to be bounded from below if $\inf \{x_n : n \in \mathbb{N}\} > -\infty$

(x_n) is called bounded if it is bounded from both above and below, that is,

$$\sup\{|x_n| \mid n \in \mathbb{N}\} < \infty$$

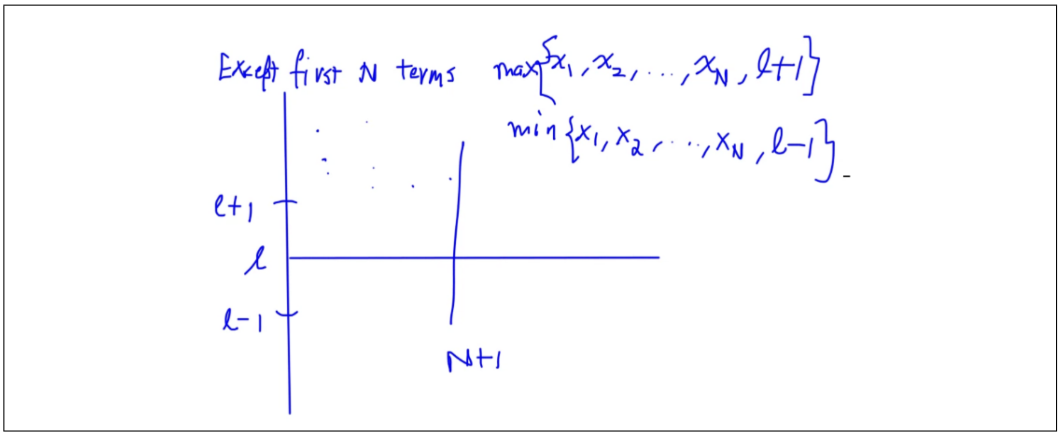


Figure 4: Proof of Boundedness