1 Public Goods

1.1 Dolbear's Triangle

$$\begin{aligned}
&u_1(x_1, g)\\
&u_2(x_2, g)\\
&\mathcal{F} = \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid (10 - (x_1 + x_2)) = g\}\\
&\quad = \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid x_1 + x_2 + g = 10\}\\
&\omega_1 = \omega_2 = 5
\end{aligned}$$

1.1.1 Question

$$\begin{aligned}
u_1(x_1, g) &= 8x_1 + 5g\\
u_2(x_2, g) &= 8x_2 + 5g\\
\mathcal{F} &= \{(x_1, x_2, g) \in \mathbb{R}^3_+ \mid 10 - (x_1 + x_2) = g\}\\
u_2^{ADJ}(x_1, g) &= u_2(10 - x_1 - g, g)\\
&= 80 - 8x_1 - 3g\\
PE &= \{(x_1, x_2, g) \in \mathcal{F} \mid x_1 = 0 \vee x_2 = 0\}
\end{aligned}$$

Now we can solve the following problem

$$\begin{aligned}
\max_{(x_1,x_2,g)\in\mathbb{R}^3_+} \quad & \alpha(8x_1 + 5g) + \beta(8x_2 + 5g)\\
s.t. \quad & x_1 + x_2 + g = 10
\end{aligned}$$

The above problem is equivalent to solving

$$\begin{aligned}
\max_{(x_1,x_2,g)\in\mathbb{R}^3_+} \quad & 8\alpha x_1 + 8\beta x_2 + 5(\alpha + \beta)g\\
s.t. \quad & x_1 + x_2 + g = 10
\end{aligned}$$

Note that

- If $\alpha > \beta$ then $x_2 = 0$
- if $\alpha < \beta$ then $x_1 = 0$
- if $\alpha = \beta$ then $x_1 = x_2 = 0, g = 10$

Now

- if $\alpha > \beta$ and $8\alpha > 5(\alpha+\beta)$ then,$x_1 = 10, x_2 = 0, g = 0$
- if $\alpha > \beta$ and $8\alpha < 5(\alpha+\beta)$ then, $x_1 = 0, x_2 = 0, g = 10$
- if $\alpha > \beta$ and $8\alpha = 5(\alpha+\beta)$ then, $x_2 = 0, x_1 + g = 10$

Similiarly;

- if $\alpha < \beta$ and $8\beta > 5(\alpha+\beta)$ then,$x_1 = 0, x_2 = 10, g = 0$
- if $\alpha < \beta$ and $8\beta < 5(\alpha+\beta)$ then, $x_1 = 0, x_2 = 0, g = 10$
- if $\alpha < \beta$ and $8\beta = 5(\alpha+\beta)$ then, $x_1 = 0, x_2 + g = 10$

1.2 Lindahl equilibrium

We have to find the equilibrium price ratio (p_1^*, p_2^*);

1.2.1 Question

Find the competitive equilibrium in the following economy;

$$\begin{aligned}
u_1(x_1, g) &= 8x_1 + 5g \qquad \omega_1 = 4 \\
u_2(x_2, g) &= 8x_2 + 5g \qquad \omega_2 = 6 \\
g &= f(x_0) = x_0 \\
\mathcal{F} &= \{(x_1, x_2, g) \in \mathbb{R}^3_+ | x_1 + x_2 + g = 10\}
\end{aligned}$$

$$(p_1^*, p_2^*) = \left(\frac{2}{5}, \frac{3}{5}\right)$$

How to check if it is correct?

To solve for the equilibrium price ratio, we can solve the following two problems;

$$\begin{aligned}
\max_{(x_1, g) \in \mathbb{R}^2_+} \quad & 8x_1 + 5g \\
s.t. \quad & x_1 + \frac{2}{5}g = 4
\end{aligned}$$

Solution to the above problem is $(x_1, x_2, g) = (0, 0, 10)$ since $\frac{8}{5} < \frac{5}{2}$.

Similiarly

$$\max_{(x_2,g)\in\mathbb{R}^2_+} \quad 8x_2 + 5g$$
$$s.t. \quad x_2 + \frac{3}{5}g = 6$$

Solution to the above problem is $(x_1, x_2, g) = (0, 0, 10)$ since $\frac{8}{5} < \frac{5}{3}$.

1.2.2 First Welfare Theorem for Lindahl Equilibrium;

$u_1(x_1, g)$
$u_2(x_2, g)$
$g = f(x_0)$
(ω_1, ω_2)
(θ_1, θ_2)
$\mathcal{F} = \{(x_1, x_2, g) \in \mathbb{R}^3_+ | f(\omega_1 + \omega_2 - x_1 - x_2) = g\}$
u_1, u_2 are increasing by assumption

Proof Suppose $((p_1^*, p_2^*), (x_1^*, x_2^*, g^*))$ is the Lindahl equilibrium, and suppose (x_1^*, x_2^*, g^*) is not Pareto efficient.
So there must exist $(x_1', x_2', g') \in \mathcal{F}$ such that,

1. $u_1(x_1', g') \geq u_1(x_1^*, g^*)$ and $u_2(x_2', g') \geq u_2(x_2^*, g^*)$
2. $u_1(x_1', g') > u_1(x_1^*, g^*)$ or $u_2(x_2', g') > u_2(x_2^*, g^*)$

By 1, $x_1' + p_1^* g' \geq \omega_1 + \theta_1 \pi^*(p_1^* + p_2^*)$ and $x_2' + p_2^* g' \geq \omega_2 + \theta_2 \pi^*(p_1^* + p_2^*)$
or
By 2, $x_1' + p_1^* g' > \omega_1 + \theta_1 \pi^*(p_1^* + p_2^*)$ and $x_2' + p_2^* g' > \omega_2 + \theta_2 \pi^*(p_1^* + p_2^*)$
So, adding the above two conditions which we got form 2, we get
$x_1' + x_2' + (p_1^* + p_2^*)g' > \omega_1 + \omega_2 + \pi^*(p_1^* + p_2^*)$
$\implies \pi^*(p_1^* + p_2^*) < (p_1^* + p_2^*)g' - (\omega_1 + \omega_2 - x_1' - x_2')$
But this is a contradiction, hence it must be the case that (x_1^*, x_2^*, g^*) is infact Pareto efficient.

1.2.3 Question 2

Find the Lindahl Equilibrium;

$$\begin{aligned} u_1(x_1, g) &= x_1 + 2\sqrt{g} \qquad \omega_1 = 2 \\ u_2(x_2, g) &= x_2 + 4\sqrt{g} \qquad \omega_2 = 1 \\ u_3(x_3, g) &= x_2 + 4\sqrt{g} \qquad \omega_3 = 1 \\ g &= f(x_0) = x_0 \end{aligned}$$

First we solve the firm's profit maximization problem;

$$\max_{(g,x_0)} \quad (p_1 + p_2 + p_3)g - x_0$$
$$s.t.\ g \leq x_0$$

Which is equivalent to solving,

$$\max_{(g)} (p_1 + p_2 + p_3)g - g$$

So the solution to the above problem is,

$$g* \in \begin{cases} \phi & \text{if } p_1 + p_2 + p_3 > 1 \\ \{0\} & \text{if } p_1 + p_2 + p_3 < 1 \\ \mathbb{R}_+ & \text{if } p_1 + p_2 + p_3 = 1 \end{cases}$$

Now we can solve the utility maximization probelms of the agents;

$$\max_{(x_1,g)\in\mathbb{R}^2_+} \quad x_1 + 2\sqrt{g}$$
$$s.t. \quad x_1 + p_1^* g = 2$$

Which is equivalent to solving;

$$\max_{0\leq g_1 \leq \frac{2}{p_1^*}} \quad 2 - p_1^* g_1 + 2\sqrt{g_1}$$

Differentiatig w.r.t g_1 we get $-p_1^* + \frac{1}{\sqrt{g_1}}$

So the solution to the above problem is,

$$g_1 = \begin{cases} \left(\frac{1}{p_1^*}\right)^2 & \text{if } p_1^* > \frac{1}{2} \\ \frac{2}{p_1^*} & \text{if } p_1^* \leq \frac{1}{2} \end{cases}$$

$$\begin{aligned} \max_{(x_2,g)\in\mathbb{R}^2_+} \quad & x_2 + 4\sqrt{g} \\ s.t. \quad & x_2 + p_2^* g = 1 \end{aligned}$$

Which is equivalent to solving;

$$\max_{0\leq g_2 \leq \frac{1}{p_2^*}} \quad 1 - p_2^* g_2 + 4\sqrt{g_2}$$

Differentiatig w.r.t g_2 we get $-p_2^* + \frac{2}{\sqrt{g_2}}$

So the solution to the above problem is,

$$g_2 = \begin{cases} \left(\frac{2}{p_2^*}\right)^2 & \text{if } p_2^* > 4 \\ \frac{1}{p_2^*} & \text{if } p_2^* \leq 4 \end{cases}$$

$$\begin{aligned} \max_{(x_3,g)\in\mathbb{R}^2_+} \quad & x_3 + 4\sqrt{g} \\ s.t. \quad & x_3 + p_3^* g = 1 \end{aligned}$$

Which is equivalent to solving;

$$\max_{0\leq g_3 \leq \frac{1}{p_3^*}} \quad 1 - p_3^* g_3 + 4\sqrt{g_3}$$

Differentiatig w.r.t g_3 we get $-p_3^* + \frac{2}{\sqrt{g_3}}$

So the solution to the above problem is,

$$g_3 = \begin{cases} \left(\frac{2}{p_3^*}\right)^2 & \text{if } p_3^* > 4 \\ \frac{1}{p_3^*} & \text{if } p_3^* \leq 4 \end{cases}$$

Now we can equate the quantity demanded and quantity supplied of the public good equal to solve for the equilibrium price ratio;

We know that for supply of the public good to be positive we need $p_1^* + p_2^* + p_3^* = 1$ or simply because of the preferences and endowments of agent 2 and 3 $p_1^* + 2p_2^* = 1 \ldots (i)$.

So if $g_1^* = \frac{2}{p_1^*}$ and $g_2 = g_3 = \frac{1}{p_2^*} = \frac{1}{p_3^*}$ then equating $g_1 = g_2 = g_3$ we get $\frac{2}{p_1^*} = \frac{1}{p_2^*} = \frac{1}{p_3^*}$ or $p_1^* = 2p_2^* \ldots (ii)$

now using (i) and (ii) we get $p_2^* = \frac{1}{4}$ and therfore $p_1^* = \frac{1}{2}$ and since $p_2^* = p_3^*$ we also get $p_3^* = \frac{1}{4}$.

substituing for p_1^*, p_2^*, p_3^* in g_1, g_2, g_3 we get that $g_1^* = g_2^* = g_3^* = 4$