1 Indirect Utility function

$$\max_{(x,y)\in\mathbb{R}^2_+} u(x,y)$$
$$\text{s.t. } p_x x + p_y y \leq M$$

Solving the above probelm gives us the demand functions,

$$x^d(p_x, p_y, M)$$
$$y^d(p_x, p_y, M)$$

The indirect utility function is defined as

$$V(p_x, p_y, M) = u(x^d(p_x, p_y, M), y^d(p_x, p_y, M))$$

It gives us the optimal utility level at the price-income vector (p_x, p_y, M)

1.1 Properties of Indirect Utility function

- Indirect utility function is homogenous of degree 0 because demand is homogenous of degree0.
- Indirect utility function is non-decreasing in income M

  and non-increasing in prices p_x, p_y because for $m' > m''$,

  $\mathcal{B}(p_x, p_y, M'') \subset \mathcal{B}(p_x, p_y, M')$

  and

  $v(p_x, p_y, M') \geq v(p_x, p_y, M'')$
- Indirect utility function is quasi-convex.

  we want to prove that $\{(p_x, p_y, M) \mid v(p_x, p_y, M) \leq \bar{v}\}$ is a convex set for all $\bar{v}$

  Proof

  Consider arbitrary $\bar{v}$ and consider arbitrary $\left(p'_x, p'_y, M'\right) \in A_{\bar{v}}$ and $\left(p''_x, p''_y, M''\right) \in A_{\bar{v}}$ and arbitray $\lambda \in [0,1]$ then we want to show

  $\lambda(p'_x, p'_y, M') + (1-\lambda)(p''_x, p''_y, M'') \in A_{\bar{v}}$

  In other words we want to show that

$v(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \leq \bar{v}$

we know that $v(p'_x, p'_y, M') \leq \bar{v}$ and $v(p''_x, p''_y, M'') \leq \bar{v}$

$\mathcal{B}(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \leq \bar{v}$ is our budget set.

and we know that this inequality holds,

$(\lambda p'_x + (1-\lambda)p''_x)x + (\lambda p'_y + (1-\lambda)p''_y)y \leq \lambda M' + (1-\lambda)M''$

This tells us that any choice from out budget set $\mathcal{B}$ that satisfy the above inequality also satisfies either

$p'_x x + p'_y y \leq M'$ or $p''_x x + p''_y y \leq M''$

this implies that

$$\begin{aligned} v(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \\ \leq \max\big(v(p'_x, p'_y, M'), v(p''_x, p''_y, M'')\big) \leq \bar{v} \end{aligned}$$

2 Expenditure Function

$$\begin{aligned} \min_{(x,y)\in\mathbb{R}^2_+} \ & p_x x + p_y y \\ \text{s.t. } & u(x,y) \geq \bar{u} \end{aligned}$$

Solving the above expenditure minimzation problem gives us the Hicksian demands,

$$\begin{aligned} x^h(p_x, p_y, \bar{u}) \\ y^h(p_x, p_y, \bar{u}) \end{aligned}$$

and the expenditure function is defined as follows

$e(p_x, p_y, \bar{u}) = p_x x^h(p_x, p_y, \bar{u}) + p_y y^h(p_x, p_y, \bar{u})$

2.1 Properties of the Expenditure function

- The expenditure function is homogeneous of degree 1 in prices,

  $e(\lambda p_x, \lambda p_y, \mu) = \lambda e(p_x, p_y, \mu)$

Note that the Hicksian demands are homogenous of degree 0 in prices because multiplying the objective in our expenditure minimzation problem by λ, (where $\lambda > 0$) does not change the solution.

- The expenditure function is non-decreasing in μ and it is also non-decreasing in prices p_x, p_y.

  we know that our expenditure minimization problem is

$$\min_{(x,y)\in\mathbb{R}^2_+} p_x x + p_y y$$
$$\text{s.t. } u(x,y) \geq \bar{\mu}'$$

  Now consider another satifaction level μ'' such that $\mu' > \mu''$

- The expenditure function is concave in prices.

2.1.1 Kuhn-Tucker Optimization Problems

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2_+} \quad & \sqrt{x} + \sqrt{y} \\ s.t. \quad & p_x x + p_y y \leq M \\ & x \geq 1 \\ & y \geq 1 \end{aligned}$$

Assume that $p_x + p_y < M$

$$\mathcal{L}(x,y) = \sqrt{x} + \sqrt{y} - \lambda(p_x x + p_y y - M) + \mu_x(x-1) + \mu_y(y-1)$$
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda p_x + \mu_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2\sqrt{y}} - \lambda p_y + \mu_y = 0$$
$$\lambda \geq 0, \quad p_x x + p_y y \leq M, \quad \lambda(p_x x + p_y y - M) = 0$$
$$\mu_x \geq 0, \quad x \geq 1, \quad \mu_x(x-1) = 0$$
$$\mu_y \geq 0, \quad y \geq 1, \quad \mu_y(y-1) = 0$$

Now if $p_x x + p_y y < M$ then $\lambda = 0$ and $\mu_x < 0$ as well as $\mu_y < 0$ this rules out four of the eight possible cases.

Now we only check for the cases $p_x x + p_y y = M$

$x = 1$	$x = 1$	$x > 1$	$x > 1$
$y = 1$	$y > 1$	$y = 1$	$y > 1$
NP	$\mu_y = 0$ $y = \frac{M-p_x}{p_y} > 1$		

3 Compensated Law of Demand

Hicksian demand function is non-increasing in price.

Consider two sets of prices, $\left(p'_x, p'_y\right), \left(p''_x, p''_y\right)$ and then we focus on the following;

$$x^h(p'_x, p'_y, \bar{y})p'_x + y^h(p'_x, p'_y, \bar{u})p'_y$$

Is it comparable to

$$x^h(p''_x, p''_y, \bar{y})p'_x + y^h(p''_x, p''_y, \bar{u})p'_y$$

Note that because $(p'_x, p'_y, \bar{u})$ is expenditure minimizing therefore we have the following;

$$x^h(p'_x, p'_y, \bar{y})p'_x + y^h(p'_x, p'_y, \bar{u})p'_y \le x^h(p''_x, p''_y, \bar{y})p'_x + y^h(p''_x, p''_y, \bar{u})p'_y \tag{1}$$

and whe $(p''_x, p''_y, \bar{u})$ is expenditure minimizing we have,

$$x^h(p''_x, p''_y, \bar{y})p''_x + y^h(p''_x, p''_y, \bar{u})p''_y \le x^h(p'_x, p'_y, \bar{y})p''_x + y^h(p'_x, p'_y, \bar{u})p''_y \tag{2}$$

adding (1) and (2) we get,

$$x^h(p'_x, p'_y, \bar{u})(p'_x - p''_x) + y^h(p'_x, p'_y, \bar{u})(p'_y - p''_y) \le x^h(p''_x, p''_y, \bar{u})(p'_x - p''_x) + y^h(p''_x, p''_y, \bar{u})(p'_y - p''_y)$$

Now suppose we only change price of x and hold the price of y constant at $\bar{p_y}$ then the above inequality can be rewritten as,

$$x^h(p'_x, \bar{p_y}, \bar{u})(p'_x - p''_x) \le x^h(p''_x, \bar{p_y}, \bar{u})(p'_x - p''_x)$$

Rearranging the above we get

$$\left(x^h(p'_x, \bar{p_y}, \bar{u}) - x^h(p''_x, \bar{p_y}, \bar{u})\right)(p'_x - p''_x) \le 0$$

Now if we have $p'_x > p''_x$ then $x^h(p'_x, \bar{p_y}, \bar{u}) \le x^h(p''_x, \bar{y}, \bar{u})$

4 Envelope Theorem

Suppose we solve the following maximization problem

$$\max_x \; f(x,a)$$

then the solution to this problem $x^*(a)$ will satisfy $\frac{\partial f}{\partial x} = 0$

Now if we look at the function $v(a) = f(x^*(a), a)$

then

$$\frac{dv}{da} = \frac{\partial f}{\partial x}\bigg|_{x=x^*(a)} = \frac{dx^*(a)}{da} + \frac{\partial f}{\partial a}$$

and therefore

$$\frac{dv}{da} = \frac{\partial f}{\partial a}$$

5 Equivalent and Compensating variation

$EV = e(p^0, u^1) - w$

and

$CV = w - e(p^1, u^0)$

and if

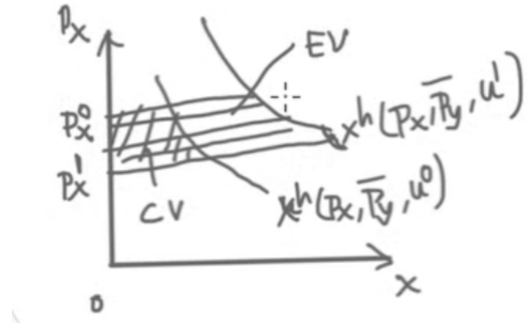
$w = e(p^0, u^0) = e(p^1, u^1)$

So

$$EV = e(p^0, u^1) - e(p^1, u^1) = \int_{p_x^1}^{p_x^0} h(p_x, \bar{p_y}, u^1) dp_x$$

and

$$CV = e(p^0, u^0) - e(p^1, u^0) = \int_{p_x^1}^{p_x^0} h(p_x, \bar{p_y}, u^0) dp_x$$



6 Linear Algebra

6.1 Vector spaces

A vector space is a set V along with an addition operation on V and a scalar multiplication operation on V such that the following properties hold;

- Commutativity; $u + v = v + u \quad \forall\, u, v \in V$
- Associativity; $(u + v) + w = u + (v + w) \quad \forall\, u, v \in V$

  $(a, b) \cdot v = a \cdot (b \cdot v)\ \forall\, a, b \in \mathbb{R},\ \forall\, v \in V$
- Additive identity; $\exists\, 0 \in V$ s.t. $v + 0 = v\ \forall\, v \in V$
- Additive inverse; $(\forall u \in V)\,(\exists w \in V)\,(v + w = 0)$
- 1 is the multiplicative identity ; $1 \cdot v = v\ \forall\, v \in V$
- Distributive Properties; $a(u + V) = a \cdot u + a \cdot v$

  $(a + b) \cdot u = a \cdot u + b \cdot u\ \forall\, a, b \in \mathbb{R}\ \forall\, u, v \in V$

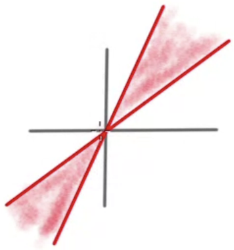
6.2 Supspaces

A subset U of V is called a subspace of V if it is also a vector space. To check if U is a subspace or not we only need to check the following;

- Additive identity; $0 \in U$
- Closed under addition; $(\forall\, u, v \in U)(u + v \in U)$
- Closed under scalar multiplication; $(\forall\, a \in \mathbb{R})(\forall\, u \in U)(a \cdot u \in U)$

6.2.1 Examples

1. $(\mathbb{R}^2, +, \cdot)$



   Note that it is not closed under additon;

2. $(\mathbb{R}^2, +, \cdot)$, $\{0, 0\}$ is a subspace.
3. $(\mathbb{R}^2, +, \cdot)$, a line passing through the origin is also a subspace.

6.3 Linear Independence

Suppose we have the following vectors $v_1, v_2, \ldots, v_n \in V$

Then a linear combination of all these vectors is $a_1 \cdot v_1 + a_2 \cdot v_2 + \ldots + a_n \cdot v_n$ where $a_1, a_2, \ldots, a_n \in \mathbb{R}$

and the span of these vectors is

$\text{Span}(v_1, v_2, \ldots, v_n) = \{a_1 \cdot v_1 + a_2 \cdot v_2 + \ldots + a_n \cdot v_n \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$

Note that the span of these vectors is also a vector space.

Now say for some v we can write v as linear combination of $v_1, v_2, \ldots, v_n$ in more than way that is $v = a_1 \cdot v_1 + a_2 \cdot v_2 + \ldots + a_n \cdot v_n$

and $v = a'_1 \cdot v_1 + a'_2 \cdot v_2 + \ldots + a'_n \cdot v_n$

then we can also write 0 as, $0 = (a'_1 - a_1)v_1 + \ldots + (a'_n - a_n)v_n$ then if we have a way of writing the two non zero vectors as an all zero vector where the scalars are not all zeros then we have that those two vectors are not linearly independent.

6.3.1 Examples

$$\begin{aligned}
&\left(\mathbb{R}^2, +, \cdot\right) \\
&(1, 2), (2, 4) \\
&\operatorname{span}((1, 2), (2, 4)) = \left\{(x, y) \in \mathbb{R}^2 \mid y = 2x\right\}
\end{aligned}$$

Are $(1, 2), (2, 4)$ linearly independence?

$$2 \cdot (1, 2) - 1 \cdot (2, 4) = (0, 0)$$

$(1, 2)$ and $(2, 4)$ are not linearly independent: $\operatorname{span}((0, 0), (1, 1)) = \{(x, y) \in \mathbb{R}^2 \mid y = x\}$

$$1 \cdot (0, 0) + 0 \cdot (1, 1) = (0, 0)$$

A basis of V is a list of vectors that is linearly independent and spans V

6.3.2 Examples

Is $\{(1, 0), (0, 1), (2, 3)\}$ the basic for $\mathbb{R}^2$?

$$2(1, 0) + 3(0, 1) - 1(2, 3) = (0, 0)$$

Is $\{(1, 0)\}$ the basis for $\mathbb{R}^2$?

$$\operatorname{span}(\{(1, 0)\}) \neq \mathbb{R}^2$$