1 Proposition 22

- Let $f : X \to Y$ be a continuous function on a compact set X, then $f(X) = \{f(x) : x \in X\}$ is compact in Y.

Proof

- Suppose X is compact and $f : X \to Y$ be a continuous function.
- We claim that $f(X)$ is compact.
- Pick any arbitrary sequence (y_n) from $f(X)$.
- Consider a sequence (x_n) in X that satisfy the following condition: $y_n = f\left(x_n\right)$ for all $n \in \mathbb{N}$.
- Given that X is compact, $\left(\ x_n\ \right)$ has a convergent subsequence (x_{n_k}) that converges to some $x \in X$.
- By continuity of f, $f\left(x_{n_k}\right)$ (which is the subsequence of $\left(f\left(x_n\right)\right)$ or (y_n)) converges to $f(x)$.
- Thus, $f(X)$ is compact.

So we have the metric spaces (X, d_x) and (Y, d_y) and the function $f : X \to Y$ which is continous and we want to show that $f(X) = \{f(x) \in Y \mid x \in X\} \subset Y$ is compact.

Consider any sequence $(y_n) \subset f(X)$, note that $\exists$ a sequence (x_n) s.t. $y_n = f(x_n)\ \forall\ n$, by the definition of $f(X)$.

Now since X is compact, there exists a subsequence x_{n_k} of (x_n) such that $x_{n_k} \to l \in X$

Conisder (y_{n_k}) where $y_{n_k} = f(x_{n_k})$

Since f is continous at l, $y_{n_k} \to f(l) \in f(X)$

2 Proposition 23

Weierstrass Maximum Theorem :

- If X is a nonempty compact subset of $\mathbb{R}^m$ and $f : X \to \mathbb{R}$ is a real-valued continuous function on X, then there exists $x^* \in X$ such that

$$f\left(x^*\right) \geq f(x) \quad \forall x \in X$$

- Application: Consider the utility maximization problem:

$$\begin{aligned}
&\max_{(x_1,x_2)} u\left(x_1, x_2\right) \\
&\text{s.t. } \left(x_1, x_2\right) \in \mathcal{B}\left(p_1, p_2, I\right)=\left\{\left(y_1, y_2\right) \in \mathbb{R}_{+}^2 \mid p_1 y_1+p_2 y_2 \leq I\right\} \\
&\text{For } p_1>0, p_2>0, I \geq 0
\end{aligned}$$

and continuous u, the solution to this maximization problem exists.

Proof

- Since X is compact, $f(X)$ is compact (by proposition 22) and hence bounded (by proposition 19).
- There exists an increasing sequence (y_n) in $f(X)$ such that $y_n \rightarrow \sup f(X)$ (by proposition 16).
- $f(X)$ is also closed (by proposition 19), so we have $\sup f(X) \in f(X)$ (by proposition 15).
- Thus, there exists $x^* \in X$ such that

$$f\left(x^*\right) \geq f(x) \quad \forall x \in X$$

This result is applicalbe with Euclidean Metric.

So in other words if $X \neq \phi$ is a compact set of $\mathbb{R}^m$ and $f: X \rightarrow \mathbb{R}$ is a continous function on X, then the solution to the problem

$$\max_{x \in X} f(x)$$

exists.

notes that from the last result $f(X) \subset \mathbb{R}$ is compact, and since it is compact it is also bounded and we have a finite supremum, $\sup f(X)<\infty$.

Now we need to show that $\exists x^* \in X$ s.t. $f(x^*)=\sup f(X)$

Notice that by a previous result we proved earlier, $\exists(y_n)$ in $f(X)$ s.t. $y_n \rightarrow \sup f(X) \in f(X)$ because $f(X)$ is compact and hence closed the limit of the sequence y_n that is $\sup f(X)$ also lies in $f(X)$.

In a similar way the the solution to a minimization problem exitst if the conditons of this proposition holds, since mimizing $f(X)$ is the same thing as maximizing $-f(x)$.

3 Convexity

Our primary focus is to deal with the following type of problems,

$$\max_{x \in X} f(x)$$

or,

$$\min_{x \in X} f(x)$$

and we will primarily be focusing on the functions with convex domains, such that $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$

Theorems and Proofs on concavity and convexity of funcitons.

let's focus on the following optimization problems;

What can you say about the set of solutions to the problem;

$$\max_{x \in X} f(x)$$

where f is quasi-concave and X is a convex set.

Yes, the set of solutions to the above problem will be a convex set.

Proof

Suppose M denotes the set of solutions, i.e.,

$M = \{x^* \in X \mid f(x^*) \geq f(x) \quad \forall\, x \in X\}$

Suppose $x', x'' \in M$ then because f is quasiconcave,

$$\begin{aligned} f(\lambda x' + (1-\lambda)x'') &\geq \min(f(x'), f(x'')) \\ &= f(x') \geq f(x) \quad \forall\, x \in X \end{aligned}$$

Therefore $\lambda x' + (1-\lambda)x'' \in M$

Note that if f is strictly quasi concave then it can not have more than one solution that is M is either empty or singleton set, because when f is strictly quasi concave then the following holds,

$$f(\lambda x' + (1-\lambda)x'') > \min(f(x'), f(x''))$$

which contadicts the fact that x', x'' are maximizers.

3.1 The cost minimization probelm

$$
\begin{aligned}
C(w,q) = \max_{x\in\mathbb{R}^m_+} \quad & w\cdot x \\
s.t. \quad & f(x) \geq q
\end{aligned}
$$

where $x(w,q)$ is the solution to this problem and f is the production function and q is some target level of output and C is a concave function of w, then we want to prove that holding q fixed C is concave in input prices;

$$C(\lambda w' + (1-\lambda)w'', q) \geq \lambda C(w', q) + (1-\lambda)C(w'', q) \quad \forall\ q$$

Now $C(w', q) = w' \cdot x(w', q)$

$$C(w'', q) = w'' \cdot x(w'', q)$$

$$
\begin{aligned}
C(\lambda w' + (1-\lambda)w'', q) &= .(\lambda w' + (1-\lambda)w'') \cdot x(\lambda w' + (1-\lambda)w'', q) \\
&= \lambda w' \cdot x(\lambda w' + (1-\lambda)w'', q) + (1-\lambda)w'' \cdot x(\lambda w' + (1-\lambda)w'', q) \\
&\geq \lambda w' \cdot x(w', q) + (1-\lambda)w'' \cdot x(w'', q) \\
&= \lambda C(w', q) + (1-\lambda)C(w'', q)
\end{aligned}
$$

A few examples;

$f(l,k) = \min(l,k)$

$C(w,r,q) = (\omega + r)q$

$f(l,k) = l + k$

$C(\omega,r,q) = (\min(w,r))q$

$f(l,k) = lk$

$C(w,r,q) = (2\sqrt{wr})\sqrt{q}$

Holding q fixed all the cost functions are concave.