

1 Properties of Relations

- Reflexivity: $\forall x \in S : (x, x) \in \mathcal{R}$
- Completeness: $\forall x, y \in S : x \neq y \Rightarrow (x, y) \in \mathcal{R} \text{ or } (y, x) \in \mathcal{R}$
- Transitivity: $\forall x, y, z \in S : ((x, y) \in \mathcal{R} \text{ and } (y, z) \in \mathcal{R}) \Rightarrow (x, z) \in \mathcal{R}$
- Symmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$
- Anti-symmetry: $\forall x, y \in S : ((x, y) \in \mathcal{R} \text{ and } (y, x) \in \mathcal{R}) \Rightarrow x = y$
- Asymmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \notin \mathcal{R}$
- Negative transitivity: $\forall x, y, z \in S : ((x, y) \notin \mathcal{R} \text{ and } (y, z) \notin \mathcal{R}) \Rightarrow (x, z) \notin \mathcal{R}$
- Equivalence: Relation which is symmetric, reflexive and transitive.

1.1 The indifference Relation

We know that the preference relation \succsim on A is Reflexive, Transitive and Complete. Define the indifference relation on A as follows;

$$\sim = \{(a, b) \in A \times A \mid (a, b) \in \succsim \wedge (b, a) \in \succsim\}$$

Now we can check for the reflexivity, transitivity, completeness and symmetry of the indifference relation \sim ;

- Reflexivity: By the definition of the indifference relation \sim on A we have,

$$\begin{aligned} \sim \text{ on } A &= \{(a, a) \in A \times A \mid (a, a) \in \succsim \wedge (a, a) \in \succsim\} \\ &\implies (a, a) \in \sim \end{aligned}$$

- Construct an example by yourself to show that \sim is not complete.
- Transitivity: We want to show, $(\forall A)(\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\sim \text{ relation derived in the above manner is transitive})$

or,

$$(\forall a, b, c \in A) ((a, b) \in \sim \wedge (b, c) \in \sim) \implies ((a, c) \in \sim)$$

Proof; Consider any arbitrary A , and any arbitrary \succsim on A that is transitive, reflexive and complete.

Now consider the indifference relation

$$\sim = \{(a, b) \in A \times A \mid (a, b) \in \succsim \wedge (b, a) \in \succsim\}$$

and any arbitrary $a, b, c \in A$ such that $(a, b) \in \sim$ and $(b, c) \in \sim$
Now since $(a, b) \in \sim$ and $(b, c) \in \sim$ then by the definition of the indifference relation,

$$(a, b) \in \succsim \wedge (b, a) \in \succsim \tag{1}$$

$$(b, c) \in \succsim \wedge (c, b) \in \succsim \tag{2}$$

Then from the transitivity of \succsim and from (1), (2) we get,

$$(a, c) \in \succsim \text{ and } (c, a) \in \succsim$$

so $(a, a) \in \succsim$ or by the definition of \sim , $(a, c) \in \sim$

Hence proved that the indifference relation \sim is transitive.

- Symmetry: We want to show, $(\forall A)(\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\sim \text{ relation derived in the above manner is symmetric})$

Proof; Consider any arbitrary A , and any arbitrary \succsim on A that is transitive, reflexive and complete.

Now consider the indifference relation

$$\sim = \{(a, b) \in A \times A \mid (a, b) \in \succsim \wedge (b, a) \in \succsim\}$$

and the symmetric relation \mathcal{R} on A defined as follows,

$$(\forall a, b \in A) ((a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R})$$

Now consider arbitrary a, b such that $(a, b) \in \sim$,

Then by the definition of the indifference relation we get,

$$(a, b) \in \succsim \wedge (b, a) \in \succsim$$

which can be rewritten as

$$(b, a) \in \succsim \wedge (a, b) \in \succsim$$

and therefore $(b, a) \in \sim$

So we have shown that if the preference relation \succsim is reflexive, transitive and complete, then the indifference relation \sim derived from it is reflexive, transitive and symmetric.

Question

Suppose $A = \{a, b, c\}$ and $\mathcal{R} = \{(a, b), (b, a), (a, a)\}$.

Is \mathcal{R} a valid indifference relation?

Answer

Note that \mathcal{R} is not reflexive, not complete and neither transitive and an indifference relation is always reflexive, transitive and symmetric so therefore, \mathcal{R} is not a valid indifference relation.

1.2 The strict preference relation

We know that the preference relation \succsim on A is reflexive, transitive and complete. From this we can define the strict preference relation as follows;

$$\succ = \{(a, b) \in A \times A \mid (a, b) \in \succsim \wedge (b, a) \notin \succsim\}$$

So, $(a, b) \in \succ \wedge \neg(b, a) \in \succ$

Now we can check for the reflexivity, transitivity, completeness and symmetry of the strict preference relation \succ as follows;

- Reflexivity: We want to show, $(\forall A)(\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\succ \text{ relation derived in the above manner is reflexive})$

for the proof, try to negate the above proposition and you can see that the strict preference relation is not reflexive.

- Completeness: try to find an example to show that the strict preference relation is not complete.

Consider $A = \{a, b, c\}$ and the preference relation $\succsim = A \times A$.

Now because of the given preference the relation the strict preference relation will be an empty set or $\succ = \phi$ which is neither reflexive and nor complete.

- Transitivity: We want to show, $(\forall A)(\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\succ \text{ relation derived in the above manner is transitive})$

Proof; Consider any arbitrary A and an arbitrary \succsim satisfying reflexivity, transitivity and completeness and also consider any arbitrary a, b, c in A such that $(a, b) \in \succ$ and $(b, c) \in \succ$.

Now because $(a, b) \in \succ$ and $(b, c) \in \succ$ we get,

$$(a, b) \in \succsim \wedge (b, a) \notin \succsim \quad (3)$$

$$(b, c) \in \succsim \wedge (c, b) \notin \succsim \quad (4)$$

and because the preference relation \succsim is transitive and from (3), (4) we get that $(a, c) \in \succsim$ and now we want to show that $(c, a) \notin \succsim$

We will do so by the way of contradiction:-

Suppose $(c, a) \in \succsim$, given that $(a, b) \in \succ$ from, the previous assumption, we get that $(c, b) \in \succsim$ because the preference relation is transitive.

But this is a contradiction because we know that $(c, b) \notin \succsim$ from (4) and therefore, $(c, a) \notin \succsim$

So we have $(a, c) \in \succsim \wedge (c, a) \notin \succsim$, which clearly implies that the strict preference relation is transitive.

So we have shown that if the preference relation \succsim is reflexive, transitive and complete, then the strict preference relation \succ derived from it is not reflexive, nor complete but transitive.

Examples

$$A = \{a, b, c\}$$

$$\mathcal{R} = \{(a, a), (a, b), (b, b)\}$$

Note that \mathcal{R} is Anti-symmetric but not Asymmetric and it is also not negative transitive because $(a, c) \notin \mathcal{R}$ and $(c, b) \notin \mathcal{R}$ but $(a, b) \in \mathcal{R}$

How to negate $\neg((\forall x), (\forall y), (P(x) \implies Q(y)))$

$$\begin{aligned} \neg((\forall x), (\forall y), (P(x) \implies Q(y))) &\equiv ((\exists x) (\exists y) (\neg(P(x) \implies Q(y)))) \\ &\equiv ((\exists x) (\exists y) (P(x) \wedge \neg Q(y))) \end{aligned}$$

Propositon 1

$(\forall A) (\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\succ \text{ relation derived from } \succsim \text{ is asymmetric})$

Proof; Consider any arbitrary $a, b \in A$ and $(a, b) \in \succ$
Now because $(a, b) \in \succ$ we get

$$(a, b) \in \succsim \wedge (b, a) \notin \succsim$$

but this implies that $(b, a) \notin \succ$
and therefore the strict prefernce relation \succ is assymetric.

Propositon 2

$(\forall A) (\forall \succsim \text{ on } A \text{ that are transitive, reflexive and complete}) (\succ \text{ relation derived from } \succsim \text{ is anti-symmetric})$

this is a corollary of the above proof because

$(\forall \mathcal{R} \text{ on } A) (\mathcal{R} \text{ is asymmetric} \implies \mathcal{R} \text{ is anti-symmetric})$

Propositon 3

$(\forall A)(\forall \succsim$ on A that are transitive, reflexive and complete) (\succ relation derived from \succsim is Negative transitive)

Proof. Consider any arbitrary A , and any weak preference relation \succsim on set A that is reflexive, complete and transitive. Now consider arbitrary $a, b, c \in A$ such that $(a, b) \notin \succ$ and $(b, c) \notin \succ$. We will try and show that $(a, c) \notin \succ$.

Suppose, by way of contradiction, $(a, c) \in \succ$.

Step 1. By completeness of \succsim , we also have $(c, b) \in \succsim$. By transitivity of \succsim . This implies that $(a, b) \in \succsim$ and we also have $(b, a) \in \succsim$ by completeness of \succsim . So, we get $(a, b) \in \sim$.

Step 2. In the similar fashion as above, by completeness of \succsim , we also have $(b, a) \in \succsim$. By transitivity of \succsim . This implies that $(b, c) \in \succsim$ and we also have $(c, b) \in \succsim$ by completeness of \succsim . So, we get $(b, c) \in \sim$.

So, by step 1 and 2, we get that $(a, c) \in \sim$ (by transitivity of \succsim). This is a contradiction.

Therefore, $(a, c) \notin \succ$.

Alternative Proof (Direct Proof). Consider any arbitrary A , and any weak preference relation \succsim on set A that is reflexive, complete and transitive. Now consider arbitrary $a, b, c \in A$ such that $(a, b) \notin \succ$ and $(b, c) \notin \succ$. By completeness of \succsim $(b, a) \in \succsim$ and $(c, b) \in \succsim$. By transitivity of \succsim $(c, a) \in \succsim$. Therefore, $(a, c) \notin \succ$.

Question

Consider a binary relation \succsim on a set A . Suppose \succsim is transitive. Define relations \succ and \sim on A by: for $x, y \in A$,

$x \succ y$ if and only if $x \succsim y$ and not $y \succsim x$

$x \sim y$ if and only if $x \succsim y$ and $y \succsim x$

Prove the following. (i) If $x \succ y$ and $y \succ z$ then $x \succ z$

(ii) If $x \sim y$ and $y \sim z$ then $x \sim z$

(iii) If $x \succ y$ and $y \succsim z$ then $x \succ z$

Proof. Let \succsim be transitive, then we want to prove that

$$(x \succ y \wedge y \succsim z) \implies x \succ z$$

Consider any arbitrary $x, y, z \in A$ such that;

$x \succ y \wedge y \succsim z$ holds

Suppose by the way of contradiction $x \succ z$ does not hold
then by definition of strict preference,

$$\begin{aligned} & \neg(x \succ z \wedge \neg(z \succ x)) \\ \implies & \neg(x \succ z) \vee z \succ x \end{aligned}$$

But then by the transitivity of \succsim

$$z \succ x \wedge y \succsim z \implies y \succ x$$

But this contradicts the fact that $x \succ y$ holds. Hence proved!