1 Convexity/Concavity of single variable functions

If we arbitrariliy pick any two points on the graph of a function and connect those two points with a line segment, if no part of this line segment lies above the graph of the function then this function is concave and if no part of this line segment lies below the graph of the function then the function is said to be convex.

examples of concave functions are $f(x) = \max(x, 1)$, $f(x) = x + 1$,

$f(x) = -x^2$, etc.

examples of convex functions are $f(x) = x^2$, $f(x) = x + 1$, etc.

Note: if we have a twice differentiable function then we can use the second derivatrive test for checking the concavity of the function.

Can there be a concave function which is discontinous?

If we have a function defined on an open interval of the real line then we can never find a discontinous concave function.

But we have a function defined on a closed interval of the real line then we can find examples of discontinous concave function.

2 Convexity/Concavity of multi variable functions

$f(x, y) = x^{0.5}y^{0.5}$ is a concave function but it's level curves are convex.

$f(x, y) = xy$ is neither concave nor convex but it's level curves are convex.

$f(x, y) = \min(x, y)$ is a concave function but it's level curves are convex.

$f(x, y) = x + 2y$ is both concave and convex and it's level curves are both concave and convex as well.

but $f(x, y) = (x + 2y)^2$ is a convex function and it's level curves are straight lines.

$u(x, y) = \max(\min(x, 2y), \min(2x, y))$ is neither convex nor concave and some holds for it's level curves.

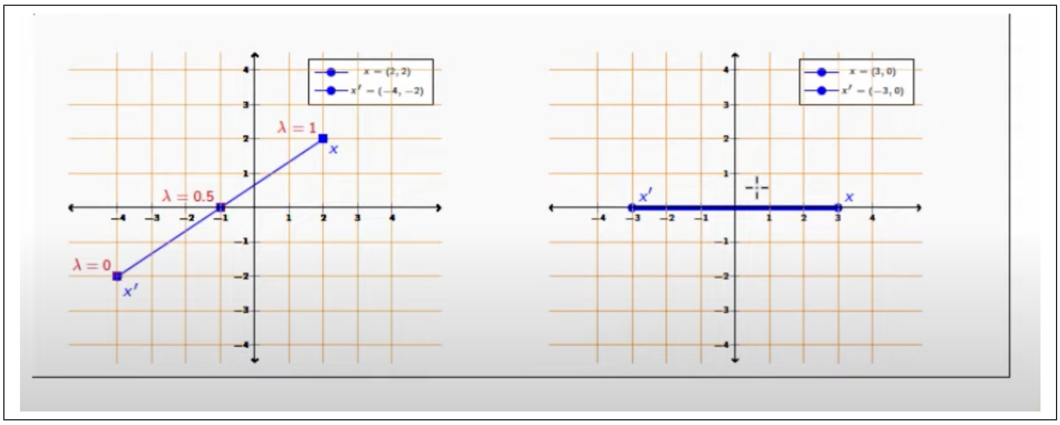
Note: Level curves are not functions.

3 Convex Sets

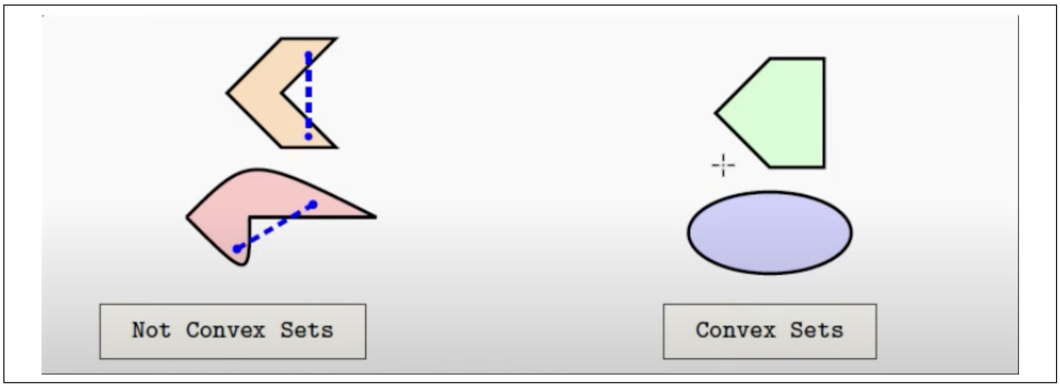
3.1 Convex Combination

Given two vectors $x, x' \in \mathbb{R}$ and $\lambda \in [0,1]$, a vector $\lambda x + (1-\lambda)x'$ is known as the convex combination of x and x'.

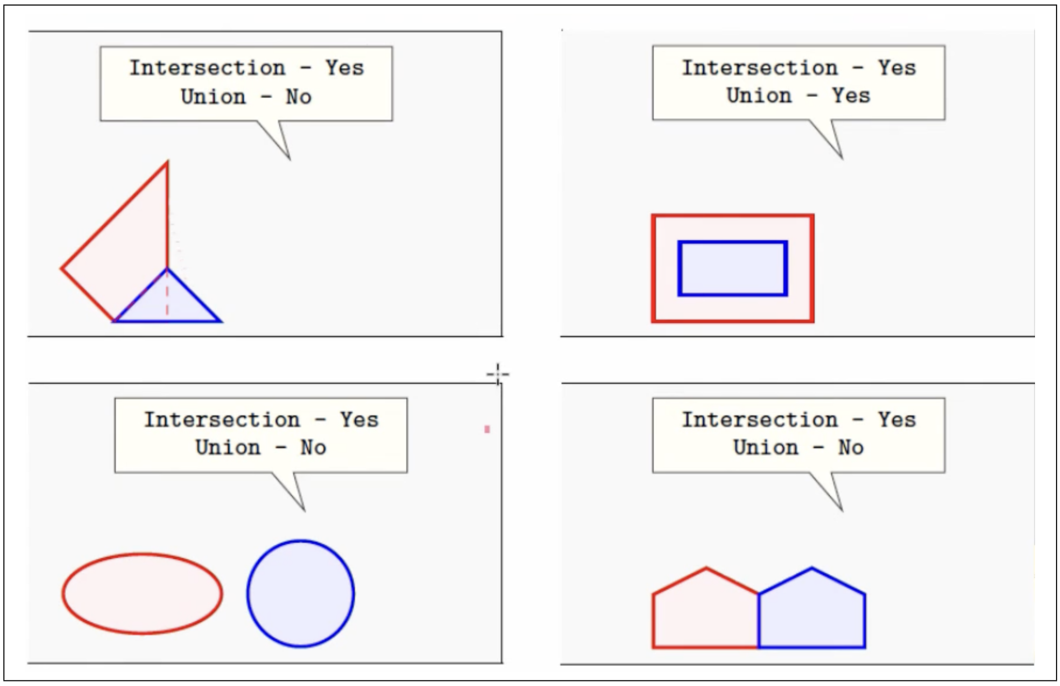
so set of all convex combinations of x and x' is the set of all the points lying on the line segment joining x and x'.



A set $S \subset \mathcal{R}^n$ is convex if $\lambda x + (1-\lambda)x' \in S$ whenever $x \in S$, $x' \in S$, and $\lambda \in [0,1]$.



Can we say anything about Union and Intersection of two Convex sets?



Note that Intersection of two convex sets is alaways a convex set but their union is not necessarily convex.

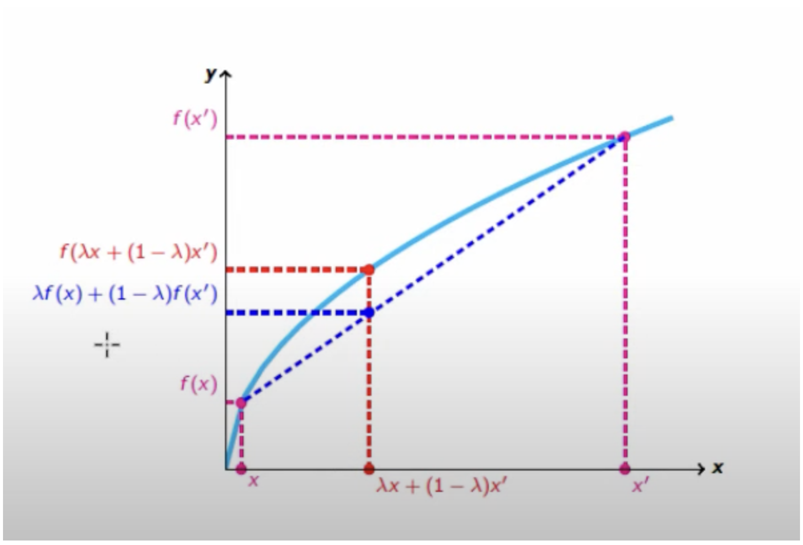
Proof:

- Let X adn Y be convex sets, Pick arbitray a and b from the set $X \cap Y$.
- Notice that $a, b \in X$ and $a, b \in Y$
- Consider any $\lambda \in [0, 1]$. Since X and Y are convex sets, we have $\lambda a + (1 - \lambda)b \in X$ and $\lambda a + (1 - \lambda)b \in Y$.
- therefore $\lambda a + (1 - \lambda)b \in X \cap Y$.

3.2 Concave Functions

Let $f : S \to \mathcal{R}$ be a function defined on the convex set $S \subset \mathcal{R}^n$. Then f is concave on the set S if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0, 1)$ we have

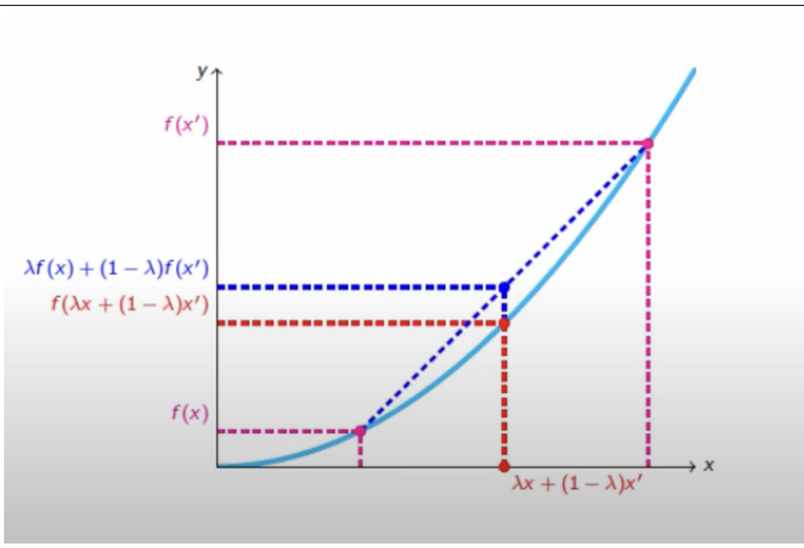
$$f(\lambda x + (1 - \lambda)x') \geq \lambda f(x) + (1 - \lambda)f(x')$$



3.3 Convex Funtions

Let $f : S \to \mathcal{R}$ be a function defined on the convex set $S \subset \mathcal{R}^n$. Then f is convex on the set S if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$



3.4 Sum Theorem

Sum of two concave functions is a concave function:

If $f : S \to \mathcal{R}$ and $g : S \to \mathcal{R}$ are two concave functions, defined on the convex set $S \subset \mathcal{R}^n$ then $t : S \to \mathcal{R}$ defined as

$$t(x) = f(x) + g(x)$$

will be a concave function.

Proof:

- Pick arbitrary $x, x' \in S$ and $\lambda \in [0, 1]$

$$\begin{array}{lll} & t\left(\lambda x+(1-\lambda) x'\right) & \\ = & f\left(\lambda x+(1-\lambda) x'\right)+g\left(\lambda x+(1-\lambda) x'\right) & [\text{ By definition of } t] \\ \geq & \lambda f(x)+(1-\lambda) f\left(x'\right)+\lambda g(x)+(1-\lambda) g\left(x'\right) & [\text{ By concavity of } f \text{ and } g] \\ = & \lambda(f(x)+g(x))+(1-\lambda)\left(f\left(x'\right)+g\left(x'\right)\right) & \\ = & \lambda t(x)+(1-\lambda) t\left(x'\right) & \end{array}$$

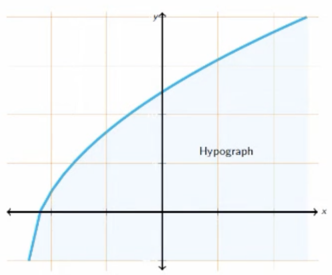
- Therefore, t is a concave function

A smiliar result holds for the sum of two convex functions which tells us that the sum of two convex functions will be a convex function.

3.5 Concave function and hypograph is a convex set

Let $f : S \to \mathbb{R}$ be a function defined on the convex set $S \subset \mathcal{R}^n$. Then f is concave if and only if the set of points below its graph (hypograph) is convex:

$$f \text{ is concave } \Leftrightarrow \{(x, y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq f(x)\} \text{ is convex}$$



To Show: f is concave $\implies \{(x,y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq f(x)\}$ is a convex set

1. Suppose f is concave.
2. Pick arbitrary (x', y') and (x'', y'') from the hypograph and $\lambda \in (0,1)$
3. Using (1), we get $f(\lambda x' + (1-\lambda)x'') \geq \lambda f(x') + (1-\lambda) f(x'')$
4. Using (2), we get $f(x') \geq y'$ and $f(x'') \geq y''$
5. Using (3) and (4), we get $f(\lambda x' + (1-\lambda)x'') \geq \lambda y' + (1-\lambda)y''$
6. Therefore, $(\lambda x' + (1-\lambda)x'', \lambda y' + (1-\lambda)y'')$ belongs to the hypograph

Hence, hypograph is a convex set.

To Show: $\{(x,y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq f(x)\}$ is a convex set $\implies f$ is concave

1. Suppose $\{(x,y) \in S \times \mathcal{R} : y \leq f(x)\}$ is convex
2. Pick any arbitrary $x', x'' \in S$, and any $\lambda \in (0,1)$
3. We have $(x', f(x'))$ and $(x'', f(x''))$ belongs to the hypograph i.e.,
$$(x', f(x')) \in \{(x,y) \in S \times \mathcal{R} : y \leq f(x)\}$$
$$(x'', f(x'')) \in \{(x,y) \in S \times \mathcal{R} : y \leq f(x)\}$$
4. Using (1), we get
$$\lambda (x', f(x')) + (1-\lambda)(x'', f(x'')) =$$
$$(\lambda x' + (1-\lambda)x'', \lambda f(x') + (1-\lambda) f(x''))$$
belongs to the hypograph
5. Therefore,
$$f(\lambda x' + (1-\lambda)x'') \geq \lambda f(x') + (1-\lambda) f(x'')$$

Hence, f is concave

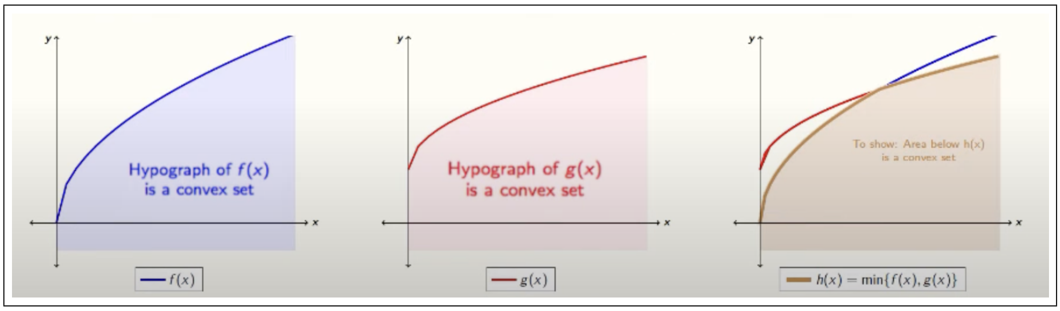
4 Min Theorem

Minimum of two concave functions is a concave function:

If $f: S \rightarrow \mathcal{R}$ and $g: S \rightarrow \mathcal{R}$ are two concave functions, defined on the convex set $S \subset \mathcal{R}^n$ then $h: S \rightarrow \mathcal{R}$ defined as

$$h(x) = \min\{f(x), g(x)\}$$

will be a concave function.



Hypograph of $h(x)$:

$$\{(x, y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq h(x)\}$$

or, $\{(x, y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq \min\{f(x), g(x)\}\}$

or, $\{(x, y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq f(x)\} \cap \{(x, y) \in S \times \mathcal{R} : x \in S \text{ and } y \leq g(x)\}$

or, Intersection of hypograph of $f(x)$ and hypograph of $g(x)$

5 Transformations of Concave functions

If

- $f: S \rightarrow \mathcal{R}$ is concave, where S is a convex subset of $\mathcal{R}^n$ and,
- g : $\mathcal{R} \rightarrow \mathcal{R}$ is increasing and concave

Then

$g \circ f : S \rightarrow \mathbb{R}$ defined as $g \circ f(x) = g(f(x))$ is also concave.

Pick arbitrary $x', x'' \in S$ and $\lambda \in (0, 1)$

$$\begin{aligned} g \circ f\left(\lambda x' + (1-\lambda)x''\right) &= g\left(f\left(\lambda x' + (1-\lambda)x''\right)\right) \ldots (\text{ by definition of } g \circ f) \\ &\geq g\left(\lambda f\left(x'\right) + (1-\lambda) f\left(x''\right)\right) \ldots (\because f \text{ is concave and } g \text{ is increasing }) \\ &\geq \lambda g\left(f\left(x'\right)\right) + (1-\lambda) g\left(f\left(x''\right)\right) \ldots (\because g \text{ is concave }) \\ &= \lambda g \circ f\left(x'\right) + (1-\lambda) g \circ f\left(x''\right) \ldots (\text{ by definition of } g \circ f) \end{aligned}$$

Therefore, $g \circ f$ is a concave function.

6 Domain Extension

If $f : S \to \mathcal{R}$ is a concave function, defined on the convex set $S \subset \mathcal{R}^n$ then $h : S \times T \to \mathcal{R}$ defined as

$$h(x, y) = f(x)$$

will be a concave function, where T is a convex subset of $\mathcal{R}$

Proof

- Pick arbitrary $(x, y), (x', y') \in S \times T$ and arbitrary $\lambda \in [0, 1]$

$$\begin{aligned}
& h\left(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'\right) \\
= \quad & f\left(\lambda x + (1-\lambda)x'\right) \qquad [\text{ By definition of } h] \\
\geq \quad & \lambda f(x) + (1-\lambda) f\left(x'\right) \qquad [\text{ By concavity of } f] \\
= \quad & \lambda h(x, y) + (1-\lambda) h\left(x', y'\right) \qquad [\text{ By definition of } h]
\end{aligned}$$

- Therefore, h is a concave function

7 Connection b/w Concave and Convex functions

Suppose a function $f : S \to \mathcal{R}$ defined on the convex set $S \subset \mathcal{R}^n$, we say f is a concave function if and only if $-f$ is a convex function.

Proof

If f is a concave function, then the following statements are equivalent;

- $f(\lambda x + (1-\lambda)x') \geq \lambda f(x) + (1-\lambda) f(x')$
- $-f(\lambda x + (1-\lambda)x') \geq -(\lambda f(x) + (1-\lambda) f(x'))$
- $-f(\lambda x + (1-\lambda)x') \geq \lambda(-f(x)) + (1-\lambda)(-f(x'))$
- $-f$ is a convex function.

7.1 Properties of a convex function:

- Sum Theorem:

  Sum of two convex functions is a convex function

- Max Theorem:

  Max of two convex functions is a convex function

- Transformation:

  Suppose $f : S \to \mathcal{R}$ is convex, where S is a convex subset of $\mathcal{R}^n$ and $g : \mathcal{R} \to \mathcal{R}$ is increasing and convex then $g \circ f : S \to \mathbb{R}$ defined as $g \circ f(x) = g(f(x))$ is also a convex function

- Epigraph is a convex set:

  f is convex iff the set of points above its graph(epigraph) is a convex set

8 Quasiconcavity

Let $f : S \to \mathcal{R}$ be a function defined on the convex set $S \subset \mathcal{R}^n$. Then f is Quasiconcave if every upper contour/level set of f is convex i.e

$$P_a = \{x \in S : f(x) \geq a\} \text{ is a convex set, } \forall\ a \in \mathcal{R}$$

Examples

- $f : \mathcal{R} \to \mathcal{R}$ given by $f(x) = x$

  Suppose, $a = -2$

  $$P_{-2} = \{x \in \mathcal{R} : f(x) \geq -2\}$$
  $$P_{-2} = \{x \in \mathcal{R} : x \geq -2\} = [-2, \infty)$$

  Similiarly for any arbitrary a, $P_a = [a, \infty)$ which is a convex set and therefore f is a quasiconcave function.

- $g : \mathcal{R}^2_+ \to \mathcal{R}$ given by $g(x, y) = xy$ Suppose, $a = 4$

  $$P_4 = \left\{(x, y) \in \mathcal{R}^2_+ : g(x, y) \geq 4\right\}$$
  $$P_4 = \left\{(x, y) \in \mathcal{R}^2_+ : xy \geq 4\right\} \text{ is a convex set}$$

  Similarly for any arbitrary a, $P_a = \{(x, y) \in \mathbb{R}^2_+ : xy \geq a\}$ is a convext set and hence the function g is also quasiconcave.

  Other examples of quasiconcave functions are,

- $u(x, y) = \min(x, y)$

- $h(x, y) = \min(x, y) + \max(x, y) = x + y$

8.1 Averages are better than extremes

Suppose a function $f : S \to \mathcal{R}$ defined on the convex set $S \subset \mathcal{R}^n$;

> f is quasiconcave $\Leftrightarrow \forall x, x' \in S$, and $\forall \lambda \in [0,1]$,
> we have $f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\}$

To Show: f is quasiconcave $\Longrightarrow f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\}$

1. Suppose f is quasiconcave i.e $P_a = \{y \in S : f(y) \geq a\}$ is a convex set
2. Pick arbitrary $x, x' \in S$ and $\lambda \in [0,1]$
3. We know, $f(x) \geq \min\{f(x), f(x')\}$ and $f(x') \geq \min\{f(x), f(x')\}$
4. Using (3), we get

$$x \in P_{\min\{f(x), f(x')\}} \text{ and } x' \in P_{\min\{f(x), f(x')\}}$$

5. Using (1) and (4), we get

$$\lambda x + (1-\lambda)x' \in P_{\min\{f(x), f(x')\}}$$

6. Using definition of P_a, we get $f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\}$

   Therefore, f satisfies "Averages Better than extremes" property.

To Show: $f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\} \Longrightarrow f$ is quasiconcave i.e P_a is a convex set.

1. Suppose f satisfies "Averages Better than extremes" property i.e

$$f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\}$$

2. Pick any arbitrary $x, x' \in P_a$, and arbitrary $a \in \mathcal{R}$
3. Using (2), we get $f(x) \geq a$ and $f(x') \geq a$
4. Using (3), we get $\min\{f(x), f(x')\} \geq a$
5. Using (1) and (4), we get

$$f(\lambda x + (1-\lambda)x') \geq a$$

6. Using (5) and definition of P_a we get $\lambda x + (1-\lambda)x' \in P_a$ i.e., P_a is a convex set.

8.2 Domain Restriction

$f : S \to \mathbb{R}$ is concave $\Leftrightarrow f : S_{\text{rest}} \to \mathbb{R}$ is also concave where $S_{\text{rest}} \subset S$.

If we consider any subset of S which is also convex, then restricting the function to this subset of S will give us a concave function.

Examples

- Is $u = xy$ concave?

  if we restrict our domain to $x = y$ then $u|_{\text{rests}} = x^2$ which is not a concave function and hence u can not be concave.

A similar result holds for convex functions.