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Interval Estimation

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Example 8.13

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a normal distribution $N(\theta, 1)$. Find a 95% confidence interval for θ.

$\implies X_1, X_2 \ldots, X_n \sim \mathcal{N}(\theta, 1)$ find $(1 - \alpha)100\%$ CI for θ

More generally, we can find a $(1-\alpha)$ interval for the standard normal random variable. Assume $Z \sim N(0, 1)$. Let us define a notation that is commonly used. For any $p \in [0, 1]$, we define z_p as the real value for which

$$P\left(Z > z_p\right) = p.$$

Therefore,

$$\Phi\left(z_p\right) = 1 - p, \quad z_p = \Phi^{-1}(1 - p)$$

By symmetry of the normal distribution, we also conclude

$$z_{1-p} = -z_p$$

Note that,

$\sqrt{n}(\bar{X}_n - \theta) \sim \mathcal{N}(0, 1)$

$\Pr_\theta(-z_{\frac{\alpha}{2}} \leq \sqrt{n}(\bar{X}_n - \theta) \leq z_{\frac{\alpha}{2}}) = 1 - \alpha \quad \forall \theta$

$\Pr_\theta \left(\frac{-z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq (\bar{X}_n - \theta) \leq \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \right) = 1 - \alpha$

$\Pr_\theta \left(\frac{-z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq \theta - \bar{X}_n \leq \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \right) = 1 - \alpha$

$\Pr_\theta \left(\bar{X}_n - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq \theta \leq \bar{X}_n + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \right) = 1 - \alpha$

$\Pr_\theta \left(\theta \in \left[\bar{X}_n - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} , \ \bar{X}_n + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \right] \right) = 1 - \alpha$

Example 8.15

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with known variance $\operatorname{Var}(X_i) = \sigma^2$, and unknown mean $EX_i = \theta$. Find a $(1 - \alpha)$ confidence interval for θ. Assume that n is large.

$X_1, X_2, \ldots, X_n \quad$ distributed with some $\quad \mathbb{V}(X_i) = \sigma^2 (\text{known})$

$\mathbb{E}(X_i) = \theta (\text{unknown})$ find $(1 - \alpha)100\%$ CI for θ

Note That,

$$\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sigma} \dot{\sim} \mathcal{N}(0, 1)$$

$$\Pr_\theta \left(-z_{\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sigma} \leq z_{\frac{\alpha}{2}} \right) = 1 - \alpha$$

$$\Pr_\theta \left(-z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq (\bar{X}_n - \theta) \leq z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 1 - \alpha$$

$$\Pr_\theta \left(\bar{X}_n - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \theta \leq \bar{X}_n + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 1 - \alpha$$

$$\Pr_\theta \left(\theta \in \left[\bar{X}_n - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \ , \ \bar{X}_n + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right] \right) = 1 - \alpha$$

Example 8.17

(Public Opinion Polling) We would like to estimate the portion of people who plan to vote for Candidate A in an upcoming election. It is assumed that the number of voters is large, and θ is the portion of voters who plan to vote for Candidate A . We define the random variable X as follows. A voter is chosen uniformly at random among all voters and we ask her/him: "Do you plan to vote for Candidate A ?" If she/he says "yes," then $X = 1$, otherwise $X = 0$. Then,

$$X \sim \text{Bernoulli}(\theta).$$

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from this distribution, which means that the X_i 's are i.i.d. and $X_i \sim \text{Bernoulli}(\theta)$. In other words, we randomly select n voters (with replacement) and we ask each of them if they plan to vote for Candidate A . Find a $(1 - \alpha)100\%$ confidence interval for θ based on $X_1, X_2, X_3, \ldots, X_n$.

$$X_1, X_2, \ldots, X_n \ iid \ \text{Bern}(\theta), \quad \mathbb{E}(X_i) = \theta \quad , \mathbb{V}(X_i) = \theta(1-\theta) \leq \frac{1}{4}$$

Since,

$$\max_{\theta \in [0,1]} \theta(1-\theta) = \frac{1}{4}$$

then if we find $(1-\alpha)\ 100\%$ CI for θ We get,

$$\left[\bar{X}_n - z_{\frac{\alpha}{2}} \frac{1}{2\sqrt{n}} \ , \ \bar{X}_n + z_{\frac{\alpha}{2}} \frac{1}{2\sqrt{n}} \right]$$

OR

$$\mathbb{V}(X_i) \approx \bar{X}_n(1-\bar{X}_n)$$

$$\left[\bar{X}_n - z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}} \ , \ \bar{X}_n + z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}} \right]$$