# Interval Estimation

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## Example 8.13

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a normal distribution $N(\theta, 1)$. Find a 95% confidence interval for $\theta$.

$\implies X_1, X_2 \ldots, X_n \sim \mathcal{N}(\theta, 1)$ find $(1-\alpha)100\%$ CI for $\theta$

More generally, we can find a $(1-\alpha)$ interval for the standard normal random variable. Assume $Z \sim N(0,1)$. Let us define a notation that is commonly used. For any $p \in [0,1]$, we define $z_p$ as the real value for which

$$P\left(Z > z_p\right) = p.$$

Therefore,

$$\Phi\left(z_p\right) = 1 - p, \quad z_p = \Phi^{-1}(1-p)$$

By symmetry of the normal distribution, we also conclude

$$z_{1-p} = -z_p$$

Note that,

$\sqrt{n}(\bar{X}_n - \theta) \sim \mathcal{N}(0,1)$

$\Pr_\theta(-z_{\frac{\alpha}{2}} \leq \sqrt{n}(\bar{X}_n - \theta) \leq z_{\frac{\alpha}{2}}) = 1 - \alpha \quad \forall\theta$

$\Pr_\theta\left(\frac{-z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq (\bar{X}_n - \theta) \leq \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1 - \alpha$

$\Pr_\theta\left(\frac{-z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq \theta - \bar{X}_n \leq \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1 - \alpha$

$\Pr_\theta\left(\bar{X}_n - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq \theta \leq \bar{X}_n + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1 - \alpha$

$\Pr_\theta\left(\theta \in \left[\bar{X}_n - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\ ,\ \bar{X}_n + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right]\right) = 1 - \alpha$

## Example 8.15

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with known variance $\operatorname{Var}(X_i) = \sigma^2$ , and unknown mean $EX_i = \theta$. Find a $(1-\alpha)$ confidence interval for $\theta$. Assume that $n$ is large.

$$
\begin{aligned}
&X_1, X_2, \ldots, X_n \quad \text{distributed with some} \quad \mathbb{V}(X_i) = \sigma^2(\text{known}) \\
&\mathbb{E}(X_i) = \theta(\text{unknown}) \text{ find } (1-\alpha)100\% \text{ CI for } \theta \\
&\text{Note That,} \\
&\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sigma} \dot{\sim} \mathcal{N}(0,1) \\
&\Pr_\theta\left(-z_{\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sigma} \leq z_{\frac{\alpha}{2}}\right) = 1-\alpha \\
&\Pr_\theta\left(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq (\bar{X}_n - \theta) \leq z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha \\
&\Pr_\theta\left(\bar{X}_n - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq \theta \leq \bar{X}_n + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha \\
&\Pr_\theta\left(\theta \in \left[\bar{X}_n - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \bar{X}_n + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right]\right) = 1-\alpha
\end{aligned}
$$

## Example 8.17

(Public Opinion Polling) We would like to estimate the portion of people who plan to vote for Candidate A in an upcoming election. It is assumed that the number of voters is large, and $\theta$ is the portion of voters who plan to vote for Candidate A . We define the random variable $X$ as follows. A voter is chosen uniformly at random among all voters and we ask her/him: "Do you plan to vote for Candidate $A$ ?" If she/he says "yes," then $X=1$, otherwise $X=0$. Then,

$$X \sim \text{Bernoulli}(\theta).$$

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from this distribution, which means that the $X_i$ 's are i.i.d. and $X_i \sim \text{Bernoulli}(\theta)$. In other words, we randomly select $n$ voters (with replacement) and we ask each of them if they plan to vote for Candidate A . Find a $(1-\alpha)100\%$ confidence interval for $\theta$ based on $X_1, X_2, X_3, \ldots, X_n$.

$X_1, X_2, \dots, X_n$ *iid* Bern$(\theta)$, $\quad \mathbb{E}(X_i) = \theta \quad , \mathbb{V}(X_i) = \theta(1-\theta) \le \frac{1}{4}$

Since,

$\max\limits_{\theta \in [0,1]} \theta(1-\theta) = \frac{1}{4}$

then if we find $\quad (1-\alpha)$ 100% CI for $\theta$ We get,

$\left[ \bar{X}_n - z_{\frac{\alpha}{2}} \frac{1}{2\sqrt{n}} \ , \ \bar{X}_n + z_{\frac{\alpha}{2}} \frac{1}{2\sqrt{n}} \right]$

OR

$\mathbb{V}(X_i) \approx \bar{X}_n(1-\bar{X}_n)$

$\left[ \bar{X}_n - z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}_n(1-\bar{X}_n}{n}} \ , \ \bar{X}_n + z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}_n(1-\bar{X}_n}{n}} \ \right]$