1 Utility Representation of Preferences

$u : A \to \mathbb{R}$

$a \succsim b$ if and only if $u(a) \geq u(b)$

When preferences are represented by utility functions, the analysis becomes easy.

A natural question to ask then is under what conditions preferences have a utility representation?

If A is a finite set then there exists a utility function u that represents the preference iff the preference is reflexive, complete and transitive, stated oterwise,

$$\begin{aligned} &\text{If A is a finite set then,} \\ \exists u : A \to \mathbb{R}&\text{that represent the } \succsim \text{ iff} \\ \succsim &\text{ is reflexive, complete and transitive.} \end{aligned}$$

2 Functions

Let A and B be any non-empty sets. A function from A to B is a rule that associates with each member of A a unique member of B.

The notation is $f : A \to B$, where input comes from the set A and output belongs to the set B.

If $a \in A$, we denote the unique element of B that the rule associates to a by $f(a)$ We refer to the element a of A as an argument of the function, and the corresponding element $f(a)$ of B as the value of the function at that argument (or sometimes the image of the point a under f).

Consider an example $f(x) = x^2 + x + 1$. The value of the function f at argument 2 is $f(2)$, which is further equal to 7.

2.1 Domain and Codomain

If $f : A \to B$, we refer the set A as the domain of f and the set B as the codomain.

2.2 Range

We say that $f(a)$ is the range of f iff

$f(A) = \{y \in B | \exists x \in A, f(x) = y\}$

Note that $f(a) \subset B$

- Pre-image: A pre-image of an element $b \in B$ for a function $f : A \to B$ is any element a of A for which $f(a) = b$.
- Range: Let $f : A \to B$. If $X \subset A$, the set $\{f(x) \mid x \in X\}$ is the range of f on X, denoted by $f(X)$.
- Alternatively, we can define range of f on X as the set

$$\{b \in B \mid (\exists x \in X)(f(x) = b)\}$$

- We denote the set $f(A)$ as the range of f.

2.3 Injective, or One-to-one

If a function $f : A \to B$ is such that it never happens that different arguments lead to the same value, we say that f is injective.

- Mathematically, $f : A \to B$ is injective iff $(\forall a, b \in A)[a \neq b \Rightarrow f(a) \neq f(b)]$
- Alternatively, we may express this condition using contrapositive: $f : A \to B$ is injective iff

$$(\forall a, b \in A)[f(a) = f(b) \Rightarrow a = b]$$

- The function $f_1 : \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = x^2$ and $f_2 : \mathbb{R} \to \mathbb{R}$ also defined by $f(x) = x^2$ are not injective but the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(x) = x^2$ is injective.

2.4 Surjective or onto

If every member of B is the value of the function at some argument, we say f is surjective.

- Mathematically, a function $f : A \to B$ is surjective iff $(\forall b \in B)(\exists a \in A)[f(a) = b]$.
- Note the order of the quantifiers in the above condition. For every b in B it must be possible to find an a in A such that $f(a) = b$.

- The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective but the function $g : \mathbb{R} \to \mathbb{R}_+$defined by $g(x) = x^2$ is surjective.

Domain	CoDomain	Injective	Surjective
$\mathbb{R}$	$\mathbb{R}$	No	No
$\mathbb{R}$	$\mathbb{R}_+$	No	Yes
$\mathbb{R}_+$	$\mathbb{R}$	Yes	No
$\mathbb{R}_+$	$\mathbb{R}_+$	Yes	Yes

2.5 Inverses

Invertible: Let $f : A \to B$. We say f is invertible if there exists a function $g : B \to A$ such that for all $a \in A$ and all $b \in B$

$$f(a) = b \Longleftrightarrow g(b) = a$$

We call such a function g an inverse of f.

- Example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$, there is an inverse function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{x-3}{2}$
- Alternatively, g is an inverse of a function $f : A \to B$ iff $g : B \to A$ and

  $$g \circ f = I_A \text{ and } f \circ g = I_B$$

  where I_A, I_B are the identity functions on A, B, respectively.
- Let $f : A \to B$. Then f is invertible iff it is a bijection. Moreover, if f is invertible, its inverse function is unique.
- The unique inverse of a bijective function f is denoted by f^{-1}.

3 Countability

Countability: The fundamental notion behind counting is that of pairing off.

If we count the elements of some (finite) collection A of objects, "one, two, thre", etc., we are explicitly defining a bijection between a subset of natural numbers and the elements of A. We may picture the counting process as follows:

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & \cdots & \cdots & a_{n-1} & a_n \\ \uparrow & \uparrow & \uparrow & & & \uparrow & \uparrow \\ 1 & 2 & 3 & \cdots & \cdots & n-1 & n \end{array}$$

The above counting process determines the bijection between

the set $\{1, 2, \ldots, n\}$ and A. Since the counting process stops when we get to the number n,

we say that the set A has n elements, or that " the number of elements of A is n",

or "the cardinality of A is n". Notationally, $|A| = n$.

So A is finite if either

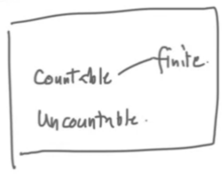
1. $(\exists n \in \mathbb{N})\,(\exists$ a bijection $t : \{1, 2, \ldots, n\} \to A)$

   or

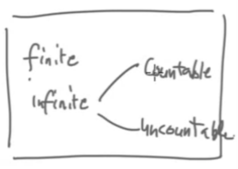
2. A is empty.

There are two definitions of Countability;

- Definition 1; A is Countable if either
  1. A is finite, or
  2. There exists a bijection $f : \mathbb{N} \to A$



- Definition 2; A is countable is there exists a bijection $f : \mathbb{N} \to A$.



3.1 Finite and Infinite Sets

- Finite set: A non-empty set A is finite iff, for some natural number n, there is a bijection from the set $\{1, 2, \ldots, n-1, n\}$ to A. We also consider $\emptyset$ as a finite set because it has 0 elements.
- Infinite set: If a set is not a finite set then, we call it an infinite set.

3.2 Countable Set

Countable set: An infinite set A is countable if there is a bijection $f : \mathbb{N} \to A$. If there does not exist such a bijection, we say that the infinite set A is uncountable.

Examples of Countable set:

1. The set of natural numbers $\mathbb{N}$: The identity function on $\mathbb{N}$ is a suitable bijection.
2. The set of even natural numbers, let's denote it by $\mathcal{E}$. We can define $f : \mathbb{N} \to \mathcal{E}$ defined by $f(n) = 2n$ is a bijection. Now think of a bijection for the set of odd natural numbers to show that the set of odd natural numbers is countable.
3. The set $\mathcal{R} = \{1/n \mid n \in \mathbb{N}\}$ is countable, the corresponding bijection is defined as $f : \mathbb{N} \to \mathcal{R}$ such that $f(n) = 1/n$.
4. The set of all integers $\mathbb{Z}$ is countable. To see this, define a function $f : \mathbb{N} \to \mathbb{Z}$ by:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

5. The set $\mathcal{P}$ of all prime numbers is also countable. We define a function $f : \mathbb{N} \to \mathcal{P}$ by the following definition: let $f(1) = 2$, and for any $n \geq 1$, let $f(n+1)$ be the least prime number bigger than $f(n)$. Convince

yourself for two things: (a) that f is a function; and (b) that it is in fact a bijection.

Few examples of Countable sets are $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

Few examples of uncountable sets are $\mathbb{R}, [0,1], \mathbb{Q}^c = \frac{\mathbb{R}}{\mathbb{Q}}$

4 Increasing Transformation of a Utility function

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function that is, $\forall s, t \in \mathbb{R}$, $s > t \Rightarrow f(s) > f(t)$. If u represents the preference relation $\succsim$ on A, then so does the function w defined by $w(a) = f(u(a))$ for all $a \in A$.

Consider any pair of alternatives $a, b \in A$. We have

$$
\begin{aligned}
& w(a) \geq w(b) \\
\Leftrightarrow & f(u(a)) \geq f(u(b)) \\
\Leftrightarrow & u(a) \geq u(b) \\
\Leftrightarrow & a \succsim b
\end{aligned}
$$

5 Representing preference realtion by a utility function

Theorem Every preference relation on a finite set can be represented by a utility function.

- Let A be a finite set and let $\succsim$ be a preference relation on A.
- For $a \in A$, define $L_a = \{b \in A \mid a \succsim b\}$. We'll call this the Lower Contour Set of a as it consists of alternatives in A that are at most as good as alternative a.
- Define $u : A \to \mathbb{R}$ as follows: for $a \in A, u(a) := |L_a|$, which is the number of alternatives in A that are at most as good as a. We'll show that u represents $\succsim$, that is, $\forall a, a' \in A, a \succsim a' \Leftrightarrow u(a) \geq u(a')$
- First we'll show that $\forall a, a' \in A, a \succsim a' \Rightarrow u(a) \geq u(a')$
- Consider any pair of alternatives $a, a' \in A$ such that $a \succsim a'$. We'll now show that $L_{a'} \subset L_a$. If $b \in L_{a'}$, then we have $a \succsim a'$ and $a' \succsim b$. By

transitivity of $\succsim$, we get $a \succsim b$, and therefore $b \in L_a$. So, $L_{a'} \subset L_a$ and $u(a) \geq u(a')$.

- Now we'll show that $\forall a, a' \in A, u(a) \geq u(a') \Rightarrow a \succsim a'$. Equivalently, we can show that $\forall a, a' \in A, (\neg a \succsim a') \Rightarrow u(a) < u(a')$
- Consider any pair of alternatives $a, a' \in A$ such that $\neg a \succsim a'$. By completeness of $\succsim$, we have $a' \succsim a$. By the same argument as before, $L_a \subset L_{a'}$. By reflexivity, $a' \in L_{a'}$. Since $\neg a \succsim a', a' \notin L_a$. Therefore, $L_a \subsetneq L_{a'}$ and we get $u(a) < u(a')$.

5.1 Lexicographic Preferences

Consider $\mathbb{Z}_+ \times \mathbb{Z}_+$ and the lexicographic preference relation, $(x_1, y_1) \succsim_L (x_2, y_2)$ if either $x_1 > x_2$ or $(x_1 = x_2$ and $y_1 \geq y_2)$

and the strict lexicographic preference relation can be defined as follows; $(x_1, y_1) \succ_L (x_2, y_2)$ if either $x_1 > x_2$ or $(x_1 = x_2$ and $y_1 > y_2)$

and similiarly the indifference relation can be defined as $(x_1, y_1) \sim_L (x_2, y_2)$ if $x_1 = x_2$ and$y_1 = y_2$

Is there a utility representation of lexicographic preferences over $\mathbb{Z}_+ \times \mathbb{Z}_+$?

we should try to find a function $f : \mathbb{Z}_+ \to [0, 1)$

say $f = \frac{y}{y+1}$ and then we can use the utility function $u(x, y) = x + \frac{y}{y+1}$ to represent lexicographic preferences.

Is there a utility representation of lexicographic preferences over $\mathbb{R}_+ \times \mathbb{R}_+$?

NO, There does not exsist a utility fuction over this domain.

Below is a proof provided for the domain $[0, 1] \times [0, 1]$ which is also uncountable and of the same cardinality as $\mathbb{R}_{+\times\mathbb{R}_+}$

Theorem The Lexicographic preference relation $\succsim$ on $[0, 1] \times [0, 1]$ defined as $(x_1, y_1) \succsim (x_2, y_2)$ if and only if either (i) $x_1 > x_2$ or (ii) $x_1 = x_2$ and $y_1 \geq y_2$ is not represented by any utility function.

- Suppose by contradiction that there existed a utility function u representing these preferences.
- For each $x \in [0, 1]$, we have $(x, 1) \succ (x, 0)$, and therefore, $u(x, 1) > u(x, 0)$. We can therefore assign to x a non-degenerate interval of values satisfying the above inequality $I(x) = [u(x, 0), u(x, 1)]$.

- For any $1 \geq x' > x \geq 0$, all commodity bundles generating utilities in the interval $I\left(x'\right)$ are strictly preferred to those in the disjoint interval $I(x)$ and should therefore be assigned a greater utility level.
- Then from each of the interval $I(x)$ we can pick a distinct rational number $r_x \in I(x)$ which is increasing in x. Since $x \in [0,1]$, there are uncountably many such intervals, but set of rational numbers are countable. This results in a contradiction.

Note that lexicographic preferences on $[0,1] \times [0,1]$ are reflexive, transitive and complete.

let us show that lexicographic preferences are transitive;

We want to show that, if the following holds,

$$\begin{aligned} &(x_1, y_1) \succsim_L (x_2, y_2) \\ &(x_2, y_2) \succsim_L (x_3, y_3) \end{aligned}$$

then,

$$(x_1, y_1) \succsim_L (x_3, y_3)$$

6 Weak axiom of revealed preference (WARP)

Choice Structure: Let X be the consumption set. A choice structure $(\mathcal{B}, C(\cdot))$ consists of two ingredients:

(a) $\mathcal{B}$: It is a family (a set) of non-empty subsets of X; that is, every element of $\mathcal{B}$ is a set $B \subset X$.

(b) $C(\cdot)$: It is a choice rule that assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathcal{B}$

WARP: The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WARP) if the following property holds:

If for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C\left(B'\right)$, we must also have $x \in C\left(B'\right)$.

$\forall x, y \in A, x \succsim^* y \implies \neg\left(y \succ^{**} x\right)$

Revealed Preference: Given a choice structure $(\mathcal{B}, C(\cdot))$ the revealed preference relation $\succsim^*$ is defined by

$$x \succsim^* y \Leftrightarrow \exists B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B)$$

- We read $x \succsim^* y$ as "x is revealed at least as good as y"
- With this terminology we can restate the weak axiom as follows: "if x is revealed at least as good as y, then y cannot be revealed preferred to x".

Revealed at least as good as relation;

$(\forall B, B' \in \mathcal{B})\,(\forall x, y \in B \cap B')\,((x \in C(B) \wedge y \in C(B')) \implies x \in C(B'))$

revealed strictly preferred to relation;

$x \succ^{**} y \Leftrightarrow \exists B \in \mathcal{B}$ such that $x, y \in B$ and $x \in C(B)$ but $y \notin C(B)$

Example 1

$$\begin{aligned} A &= \{x, y, z\} \\ C(\{x, y\}) &= \{x\} \\ C(\{y, z\}) &= \{y\} \\ C(\{x, z\}) &= \{z\} \end{aligned}$$

Yes the WARP is satisfied of the above.

now, $\succsim^* = \{(x, x), (x, y), (y, y), (y, z), (z, z), (z, x)\}$ and

$\succ^{**} = \{(x, y), (y, z), (x, z)\}$

Example 2

$$\begin{aligned} A &= \{a, b, c\} \\ C(\{a, b\}) &= \{a, b\} \\ C(\{b, c\}) &= \{b\} \\ C(\{a, c\}) &= \{a\} \\ C(\{a, b, c\}) &= \{a\} \end{aligned}$$

The WARP is not satisfied here because first we are saying that $b \succsim^* a$ or $u(a) = u(b)$ but at last we are saying that $a \succ^{**} b$ or $u(a) > u(b)$.

Precise Definition of limit

Let f be a function defined on some open interval that contains a, except possibly at a itself. Then we say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

and the precise definiton would be,

$$(\forall \epsilon > 0)\,(\exists \delta > 0)\,(\forall x)\,(0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

The negation of the above definiton would be

$$(\exists \epsilon > 0)\,(\forall \delta > 0)\,(\exists x)\,(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

The above statement tells us that if it is true then we have that,

$$\lim_{x \to a} f(x) \neq L$$

and we say that limit of $f(x)$ as x approaches a is not equal to L.

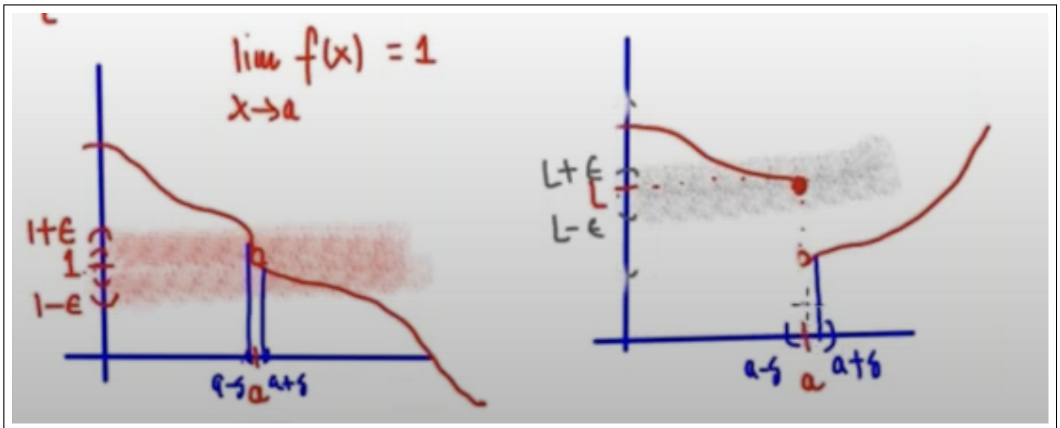


Figure 1: How to check for existence of limit graphically

Example

$$\lim_{x \to 0^+} \frac{1}{x} \quad \text{DNE}$$