## 1 Bayesian Games

## 1.1 Swap/Retain Game

Consider the following two-player game.

- The players simultaneously and independently draw a number from the set  $\{10, 20, 30, 40, 50\}$ .
- After observing the value of her own sample, which is private information (that is, opponent does not observe it), players simultaneously and independently choose one of the following: SWAP,RETAIN.
- If both the players choose SWAP then they exchange their initially drawn numbers. Otherwise, if at least one person chooses RETAIN, both of them retain their numbers.
- A player earns as many Rupees as the number she is holding at the end of the game.

The Choice in this game will be a strategy s, such that;

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s: \{10, 20, 30, 40, 50\} \rightarrow \{\text{SWAP,RETAIN}\}\
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Note that there are  $2^5$  number of such strategies for each of the players, and therefore there are  $(2^5)^2 = 32^2$  number of ways in which this game can be played assuming there are no mixed strategies that can be played!

Now we can choose the strategies in this way;

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s(50) = \text{RETAIN} s_{\text{other}}(50) = \text{RETAIN} s(40) = \text{RETAIN} s_{\text{other}}(40) = \text{RETAIN} s(30) = \text{RETAIN} s_{\text{other}}(30) = \text{RETAIN} s(20) = \text{RETAIN} s_{\text{other}}(20) = \text{RETAIN} s_{\text{other}}(10) = \text{SWAP,RETAIN}
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Note that there are four Nash Equilibriums in this game!

More Formally,

A Bayesian Game consists of

• Set of Players:  $N = \{1, 2, ..., n\}$ 

- Action Sets: Action set of player i is denoted by  $A_i$ . Set of all action profiles:  $A = A_1 \times \cdots \times A_n$
- Type Sets: Type set of player i is denoted by  $\Theta_i$ . Set of all type profiles:  $\Theta = \Theta_1 \times \cdots \times \Theta_n$
- Utility: Utility function of player i is:  $u_i: A \times \Theta \to \mathbb{R}$
- Belief:  $p_i(\theta_{-i} \mid \theta_i)$  is the conditional probability that i gives to others' types being  $\theta_{-i}$  given  $\theta_i$

Now we can formally define the Swap/Retain game;

- $N = \{1, 2\}$
- Action Sets:

$$A_1 = \{SWAP, RETAIN\}$$
  
 $A_2 = \{SWAP, RETAIN\}$   
 $A = A_1 \times A_2$ 

• Type Sets:

$$\Theta_1 = \{10, 20, 30, 40, 50\}$$
  

$$\Theta_2 = \{10, 20, 30, 40, 50\}$$
  

$$\Theta = \Theta_1 \times \Theta_2$$

• Utility:  $u_i A_1 \times A_2 \times \Theta_1 \times \Theta_2 \to \mathbb{R}$  is defined as follows

$$u_1(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_2 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_1 & \text{otherwise} \end{cases}$$
$$u_2(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \theta_1 & \text{if } a_1 = a_2 = \text{SWAP} \\ \theta_2 & \text{otherwise} \end{cases}$$

• 
$$p_1(\theta_2|\theta_1) = \frac{1}{5}$$
  $\forall \theta_2 \in \Theta_2, \quad \forall \theta_1 \in \Theta_1$   
 $p_2(\theta_1|\theta_2) = \frac{1}{5}$   $\forall \theta_1 \in \Theta_1, \quad \forall \theta_2 \in \Theta_2$ 

Both players know the joint distribution of their types!

Now, Strategy of Player i in a Bayesian Game is a function from type set of the player  $\Theta_i$ ; to the set fo actions  $A_i$  he can take.

$$s_i:\theta_i\to A_i$$

 $s(\theta_i) \in A_1$  is the action specified by strategy s; for type  $\hat{\theta}_i \in \Theta_i$ , i.e. if i plays according to s, then he takes an action  $s_5(\theta_1)$  when his type his  $\theta_i$ . Let  $S_i$  denotes set of all such functions.

• A strategy profile is given by

$$s = (s_1, s_2, \dots, s_n) \in S_1 \times \dots \times S_n$$

• Let us call the set of all strategy profiles S i.e.

$$S = S_1 \times \cdots \times S_n$$

- Expected utility is a function  $U_i: A_i \times S_{-i} \times \Theta_i \to \mathbb{R}$ .
- Expected utility of type  $\theta_i$  of player i when players play according to s is given by

$$U_{i}\left(s_{i}\left(\theta_{i}\right), s_{-i}, \theta_{i}\right) = \sum_{\theta_{-i} \in \Theta_{-i}} p_{i}\left(\theta_{-i} \mid \theta_{i}\right) u_{i}\left(s_{i}\left(\theta_{i}\right), s_{-i}\left(\theta_{-i}\right), \theta_{i}, \theta_{-i}\right)$$

Bayesian Nash equilibrium

• Strategy profile  $s^* = (s_1^*, s_2^*, \dots, s_n^*) \in S$  constitutes the Bayesian Nash equilibrium if for all i, for all  $\theta_i, s_i^*(\theta_i)$  maximizes player i 's expected utility given that the other players play according to  $s_{-i}^*$  i.e.  $s^*$  is the Bayesian Nash equilibrium if  $\forall i \in N, \forall \theta_i \in \Theta_i$ ,

$$U_i\left(s_i^*\left(\theta_i\right), s_{-i}^*, \theta_i\right) \ge U_i\left(a_i, s_{-i}^*, \theta_i\right) \quad \forall a_i \in A_i$$

Suppose

$$s_2(10) = s_2(20) = SWAP$$

$$s_2(30) = s_2(40) = s_2(50) = RETAIN$$

Find the Best Response strategy for player 1?

$$u_1(\text{Retain}, \ s_2, 50) = 50$$

$$u_1(\text{Swap}, \ s_2, 50) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 50 = \frac{180}{5} = 36$$

$$u_1(\text{Retain}, \ s_2, 40) = 40$$

$$u_1(\text{Swap}, \ s_2, 40) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 40 = \frac{150}{5} = 30$$

$$u_1(\text{Retain}, \ s_2, 30) = 30$$

$$u_1(\text{Swap}, \ s_2, 30) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 30 = \frac{120}{5} = 24$$

$$u_1(\text{Retain}, \ s_2, 20) = 20$$

$$u_1(\text{Swap}, \ s_2, 20) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 20 = \frac{90}{5} = 18$$

$$u_1(\text{Retain}, \ s_2, 10) = 10$$

$$u_1(\text{Swap}, s_2, 10) = \frac{1}{5} \times 10 + \frac{1}{5} \times 20 + \frac{3}{5} \times 10 = \frac{60}{5} = 12$$

So we get the Best Response strategy of player 1 as;

$$s_1(50) = \text{RETAIN}$$
  
 $s_1(40) = \text{RETAIN}$   
 $s_1(30) = \text{RETAIN}$   
 $s_1(20) = \text{RETAIN}$   
 $s_1(10) = \text{SWAP}$ 

Given this best response of player 1, player 2 will also fix his own

## 1.2 Battle of Sexes

• There are two possible types of player 2 (column): "Meet" player 2 wishes to be at the same place as player 1, just as in the usual game (This type has probability p = 1/4). "Avoid" 2 wishes to avoid player 1 and go to the other place (This type has probability 1 - p = 3/4).

• 2 knows his type, and 1 does not. They simultaneously choose the place Movie (M) or Shopping (S). These payoffs are shown in the matrices below.

$p=rac{1}{4}$			$1-p=\tfrac{3}{4}$		
MEET	S	М	Avoid	S	М
S	(2,1) $(0,0)$	(0,0)	S	(2,0) $(0,1)$	(0,2)
M	(0,0)	(1, 2)	M	(0,1)	(1,0)

$$\begin{split} N &= \{1,2\} \\ A_1 &= \{S,M\} \\ A_2 &= \{S,M\} \\ \Theta_1 &= \{\text{Meet}\} \quad \Theta_2 = \{\text{Meet, Avoild}\} \\ u_1 &: A_1 \times A_2 \times \Theta_1 \times \Theta_2 \to \mathbb{R} \\ u_1(S,S,\text{Meet,Meet}) &= 2 \\ u_1(S,S,\text{Meet,Avoid}) &= 2 \\ P_2(\theta_1 \mid \theta_2) &= 1 \text{ for } \theta_1 = \text{ Meet } \forall \theta_2 \\ P_1(\theta_2 \mid \theta_1) &= \begin{cases} \frac{1}{4} & \text{for } \theta_2 = \text{ Meet} \\ \frac{3}{4} & \text{for } \theta_2 = \text{ Avoid} \end{cases} \end{split}$$

$$\begin{split} s_1 &= S \\ s_2(\text{Meet}) &= S \\ s_2(\text{Avoid}) &= M \\ u_1(S, s_2, \text{Meet}) &= 2\left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right) = \frac{1}{2} \\ u_1(M, s_2, \text{Meet}) &= 0\left(\frac{1}{4}\right) + 1\left(\frac{3}{4}\right) = \frac{3}{4} \end{split}$$

Now

$$\begin{split} s_1 &= M \\ s_2^{'}(\text{Meet}) &= M \\ s_2^{'}(\text{Avoid}) &= S \\ u_1(S, s_2^{'}, \text{Meet}) &= 0 \left(\frac{1}{4}\right) + 2\left(\frac{3}{4}\right) = \frac{3}{2} \\ u_1(M, s_2^{'}, \text{Meet}) &= 1 \left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right) = \frac{1}{4} \end{split}$$

$$M \to (M, S) \to S \to (S, M) \to M$$

Therefore there is no Bayesian Nash Equilibrium in Pure strategies of this game.