Public Goods

Dolbear Triangle;

$$\begin{aligned}
&u_1(x_1, g)\\
&u_2(x_2, g)\\
&\mathcal{F} = \{(x_1, x_2, g)\\
&= \{(x_1, x_2, g) \in \mathbb{R}^3_+ | x_1 + x_2 + g = 10\}\\
&\omega_1 = \omega_2 = 5
\end{aligned}$$

Question 1

$$\begin{aligned}
u_1(x_1, g) &= 8x_1 + 5g\\
u_2(x_2, g) &= 8x_2 + 5g\\
\mathcal{F} &= \{(x_1, x_2, g) \in \mathbb{R}^3_+ | 10 - (x_1 + x_2) = g\}\\
u_2^{ADJ}(x_1, g) &= u_2(10 - x_1 - g, g)\\
&= 80 - 8x_1 - 3g\\
PE &= \{(x_1, x_2, g) \in \mathcal{F} | x_1 = 0 \vee x_2 = 0\}
\end{aligned}$$

$$\begin{aligned}
\max_{(x_1,x_2,g)\in\mathbb{R}^3_+} \quad & \alpha(8x_1 + 5g) + \beta(8x_2 + 5g)\\
s.t. \quad & x_1 + x_2 + g = 10
\end{aligned}$$

$$\begin{aligned}
\max_{(x_1,x_2,g)\in\mathbb{R}^3_+} \quad & 8\alpha x_1 + 8\beta x_2 + 5(\alpha + \beta)g\\
s.t. \quad & x_1 + x_2 + g = 10
\end{aligned}$$

Note that if $\alpha > \beta, x_2 = 0$ and if $\alpha < \beta, x_1 = 0$ and if $\alpha = \beta, x_1 = x_2 = 0, g = 10$

NOw if $\alpha > \beta$ then $8\alpha > 5(\alpha + \beta)$

Lindahl equilibrium

(p_1^*, p_2^*)

Question 1

Find the competitive equilibrium in the following economy;

$$
\begin{aligned}
u_1(x_1, g) &= 8x_1 + 5g \qquad \omega_1 = 4 \\
u_2(x_2, g) &= 8x_2 + 5g \qquad \omega_2 = 6 \\
g &= f(x_0) = x_0 \\
\mathcal{F} &= \{(x_1, x_2, g) \in \mathbb{R}_+^3 | x_1 + x_2 + g = 10\}
\end{aligned}
$$

$$(p_1^*, p_2^*) = \left(\frac{2}{5}, \frac{3}{5}\right)$$

How to check if it is correct?

$$\max_{(x_1, g) \in \mathbb{R}_+^2} \quad 8x_1$$
$$s.t.$$

First Welfare Theorem for Lindahl Equilibrium;

$$
\begin{aligned}
& u_1(x_1, g) \quad g = f(x_0) \quad (\omega_1, \omega_2) \\
& u_2(x_2, g)(\theta_1, \theta_2) \\
\mathcal{F} = \{9x_1, x_2, g) \in \mathbb{R}_+^3 | f(\omega_1 + \omega_2 - x_1 - x_2) = g\} & \\
& u_1, u_2 \text{are increasing}
\end{aligned}
$$

<u>Proof</u> Suppose $((p_1^*, p_2^*), (x_1^*, x_2^*, g^*))$ is the Lindahl equilibrium, and suppose (x_1^*, x_2^*, g^*) is not Pareto efficient.

So there must exist $(x_1', x_2', g') \in \mathcal{F}$ such that,

1. $u_1(x_1', g') \geq u_1(x_1^*, g^*) \text{and} u_2(x_2', g') \geq u_2(x_2^*, g^*)$
2. $u_1(x_1', g') > u_1(x_1^*, g^*) \text{and} u_2(x_2', g') > u_2(x_2^*, g^*)$

By 1, $x_1' + p_1^* g' \geq \omega_1 + \theta_1 \pi^*(p_1^* + p_2^*)$ and $x_2' + p_2^* g' \geq \omega_2 + \theta_2 \pi^*(p_1^* + p_2^*)$

By 2, $x_1' + p_1^* g' > \omega_1 + \theta_1 \pi^*(p_1^* + p_2^*)$ and $x_2' + p_2^* g' > \omega_2 + \theta_2 \pi^*(p_1^* + p_2^*)$

Question 2

Find the Lindahl Equilibrium;

$$
\begin{aligned}
u_1(x_1, g) &= x_1 + 2\sqrt{g} \qquad && \omega_1 = 2 \\
u_2(x_2, g) &= x_2 + 4\sqrt{g} \qquad && \omega_2 = 1 \\
u_3(x_3, g) &= x_2 + 4\sqrt{g} \qquad && \omega_3 = 1 \\
g &= f(x_0) = x_0
\end{aligned}
$$

First we solve the firm's profit maximization problem;

$$
\begin{gathered}
\max_{(g,x_0)} \quad (p_1 + p_2)g - x_0 \\
s.t.\ g \leq x_0
\end{gathered}
$$

$$
\max_{(g)} (p_1 + p_2 + p_3)g - g
$$

$$
g* \in \begin{cases} \phi & \text{if } p_1 + p_2 + p_3 > 1 \\ \{0\} & \text{if } p_1 + p_2 + p_3 < 1 \\ \mathbb{R}_+ & \text{if } p_1 + p_2 + p_3 = 1 \end{cases}
$$