1 Point Estimation

1.1 Evaluating Estimators

$$
\begin{aligned}
\mathrm{MSE}_{\hat{\Theta}}(\theta) &= \mathbb{E}_\theta(\hat{\Theta}-\theta)^2 \\
&= \mathbb{E}_\theta(\hat{\Theta}-\mathbb{E}_\theta(\hat{\Theta})+\mathbb{E}_\theta(\hat{\Theta})-\theta)^2 \\
&= \mathbb{E}_\theta\left(\hat{\Theta}-\mathbb{E}_\theta(\hat{\Theta})\right)^2+\left(\mathbb{E}_\theta(\hat{\Theta})-\theta\right)^2 \\
&= \mathbb{V}_\theta(\hat{\Theta})+(B_{\hat{\Theta}}(\theta))^2
\end{aligned}
$$

So Mean Squared Error (MSE) is equal to the Variance of the estimator $\hat{\Theta}$ plus the squared bias of the estimator $\hat{\Theta}$.

Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{Unif}[0,\theta]$, where $\theta > 0$ is the unkown parameter. Then find the following;

- $\hat{\Theta} = \max(X_1, X_2, \ldots, X_n)$.
- Check for the unbiasedness and consistency of $\hat{\Theta}$.
- Determine the $\mathrm{MSE}_{\hat{\Theta}}(\theta) \quad \forall\theta$.

We can easily see that the estimator is not unbiased because we are taking the maximum possible value of one particular sample, but $\max(X_1, \ldots, X_n) \leq \theta$ then it's expected value will always fall short of θ.

More Precisely, we want to find

$$\mathbb{E}_\theta(\max(X_1, \ldots, X_n))$$

Now,

$$
\begin{aligned}
F_{\hat{\Theta}}(x) &= Pr_\theta(\max(X_1, \ldots, X_n) \leq x) \\
&= (Pr_\theta\,(X_1 \leq x))^n \\
&= \left(\frac{x}{\theta}\right)^n \\
f_{\hat{\Theta}}(x) &= \frac{nx^{n-1}}{\theta^n}, \quad 0 \leq x \leq \theta
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbb{E}_\theta\left(\max\left(X_1, X_2, \ldots, X_n\right)\right) &= \int_0^\theta x \cdot \frac{nx^{n-1}}{\theta^n}\mathrm{d}x \\
&= \left(\frac{n}{n+1}\right)\theta
\end{aligned}
$$

Which is clearly less than θ and therefore our estimator $\hat{\Theta}$ is not unbiased.

Now to check for Consistency we want to show that,

$$\lim_{n\to\infty} \mathbb{P}_\theta \left(|\max\left(X_1, X_2, \ldots, X_n\right) - \theta| > \epsilon\right) = 0$$

Now,

$$\begin{aligned}\mathbb{P}_\theta \left(|\max\left(X_1, X_2, \ldots, X_n\right) - \theta| > \epsilon\right) &= \mathbb{P}_\theta \left(\theta - \max\left(X_1, X_2, \ldots, X_n\right) > \epsilon\right) \\ &= \mathbb{P}_\theta \left(\max\left(X_1, X_2, \ldots, X_n\right) < \theta - \epsilon\right) = \left(\frac{\theta - \epsilon}{\theta}\right)^n\end{aligned}$$

Then,

$$\lim_{n\to\infty} \left(\frac{\theta - \epsilon}{\theta}\right)^n = 0$$

and therefore are estimator $\hat{\Theta}$ is consistent.

Now to find the MSE we have to first find the variance of our esstimator, i.e., $\mathbb{V}_\theta(\max\left(X_1, X_2, \ldots, X_n\right))$ and then $\left(B_{\hat{\Theta}}(\theta)\right)^2$;

$$\begin{aligned}\mathbb{V}_\theta &= \mathbb{E}_\theta\left[\hat{\Theta}^2\right] - \left(\mathbb{E}_\theta\left[\hat{\Theta}\right]\right)^2 \\ &= \int_0^\theta x^2 \cdot \frac{nx^{n-1}}{\theta^n}\mathrm{d}x - \left[\int_0^\theta x \cdot \frac{nxn-1}{\theta^n}\mathrm{d}x\right]^2 \\ &= \frac{n\theta^2}{(n+1)^2(n+2)}\end{aligned}$$

and,

$$\begin{aligned}\left(B_{\hat{\Theta}}(\theta)\right)^2 &= (\mathbb{E}_\theta[\hat{\Theta} - \theta])^2 \\ &= (\mathbb{E}_\theta[\hat{\Theta}] - \mathbb{E}_\theta[\theta])^2 \\ &= \frac{\theta^2}{(n+1)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{MSE}_{\hat{\Theta}}(\theta) &= \mathbb{V}_\theta(\hat{\Theta}) + \left(B_{\hat{\Theta}}(\theta)\right)^2 \\ &= \frac{2\theta^2}{(n+1)(n+2)}\end{aligned}$$

When comparing two estimators such as $\hat{\Theta}_1$ and $\hat{\Theta}_2$, $\hat{\Theta}_1$ is better than $\hat{\Theta}_2$ if;

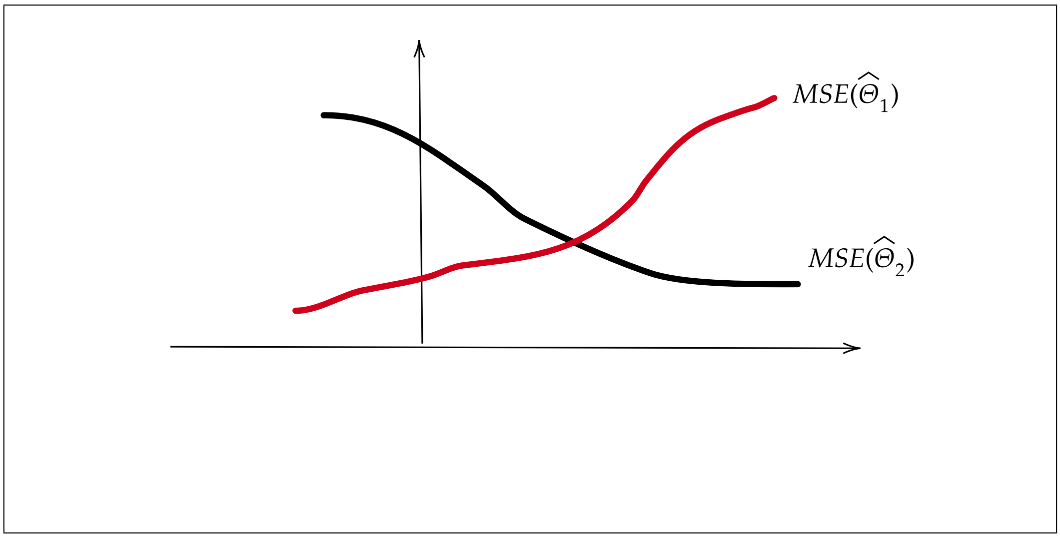

Figure 1: Comparing Mean Squared Errors

- $\text{MSE}_{\hat{\Theta}_1}(\theta) \leq \text{MSE}_{\hat{\Theta}_2}(\theta) \quad \forall\ \theta$
- $\text{MSE}_{\hat{\Theta}_1}(\theta) < \text{MSE}_{\hat{\Theta}_2}(\theta) \quad \text{for some } \theta$

Question: Suppose $X_1, X_2, \ldots, X_n$ are *iid* with $\mathbb{E}(X_i) = \mu(< \infty)$ and $Var(X_i) = \sigma^2(< \infty)$, Compare the MSE of these estimators for μ

- $m_1 = X_1$
- $m_2 = \frac{X_1+X_2}{2}$
- $m_n = \frac{X_1+X_2+\ldots+X_n}{n}$

Note that all of them have the same expected values, i.e., $\mathbb{E}(m_1) = \mathbb{E}(m_2) = \ldots = \mathbb{E}(m_n) = \mu$.

$$\begin{aligned} \text{MSE}(m_1) &= \sigma^2 \\ \text{MSE}(m_2) &= \frac{\sigma^2}{2} \\ \text{MSE}(m_n) &= \frac{\sigma^2}{n} \end{aligned}$$

so the MSE grows smaller as n increases.

1.2 Maximum Likelihood Estimator

$X_1, X_2, \ldots, X_n$ are *iid* random variables from some population $f(x; \theta)$ where θ is the unkown parameter we want to estimate.

Tak Obseravation $X_1, X_2, \ldots, X_n$ and find the joint density of them which will be the product of the marginals, we call this joint density a Likelihood function,

$$\mathcal{L}(\theta; x_1, x_2, \ldots, x_n) = f(x_1; \theta) f(x_2; \theta) \cdots f(x_n; \theta)$$

then we can solve the following;

$$\max_{\theta} \quad \mathcal{L}(\theta)$$

the solution to the above problem is $\hat{\Theta}_{MLE}$.

Now,

Suppose $X_1, X_2, \ldots, X_n$ are *iid* Bern(p),then estimate p using MLE;

$$\mathcal{L}(p; x_1, \ldots, x_n) = p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}$$

we want to maximise the above but we can take an increasing transformation and solve the following,

$$\begin{aligned}
\max_{p} \quad \ln(\mathcal{L}(p; x_1, \ldots, x_n)) &= (\sum_{i=1}^n x_i)(\ln(p)) + (n - \sum_{i=1}^n x_i)(\ln(1-p)) \\
\text{differentiating w.r.t p;} \quad \frac{\sum_{i=1}^n x_i}{p} - \frac{n - \sum_{i=1}^n x_i}{1-p} &= 0 \\
\implies \sum_{i=1}^n x_i - p\sum_{i=1}^n x_i &= np - p\sum_{i=1}^n x_i \\
= \hat{p}_{MLE} &= \frac{\sum_{i=1}^n x_i}{n}
\end{aligned}$$

So in this case it turns out to be jsut the sample mean.

Question: Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$, where μ is unknown, find $\hat{\mu}_{MLE}$.

$$
\begin{aligned}
\mathcal{L}(\mu; x_1, x_2, \ldots, x_n) &= f(x_1; \mu) f(x_2; \mu) \cdots f(x_n; \mu) \\
\mathcal{L}(\mu) &= \prod_{i=1}^{n} f_{X_i}(x_i; \mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \\
\ell(\mu) = \log \mathcal{L}(\mu) &= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2 \\
\frac{d\ell(\mu)}{d\mu} &= \frac{d}{d\mu}\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) = \sum_{i=1}^{n}(x_i - \mu) = 0 \\
\Rightarrow n\mu &= \sum_{i=1}^{n} x_i \\
\mu_{\hat{MLE}} &= \frac{1}{n}\sum_{i=1}^{n} X_i
\end{aligned}
$$

This is the sample mean, which makes intuitive sense as the best estimator of the population mean μ .