1 Indirect Utility function

$$\max_{(x,y)\in\mathbb{R}^2_+} u(x,y)$$
$$\text{s.t. } p_x x + p_y y \le M$$

Solving the above probelm gives us the demand functions,

$$x^d(p_x, p_y, M)$$
$$y^d(p_x, p_y, M)$$

The indirect utility function is defined as

$$V(p_x, p_y, M) = u(x^d(p_x, p_y, M), y^d(p_x, p_y, M))$$

It gives us the optimal utility level at the price-income vector (p_x, p_y, M)

1.1 Properties of Indirect Utility function

- Indirect utility function is homogenous of degree 0 because demand is homogenous of degree0.
- Indirect utility function is non-decreasing in income M

  and non-increasing in prices p_x, p_y because for $m' > m''$,

  $\mathcal{B}(p_x, p_y, M'') \subset \mathcal{B}(p_x, p_y, M')$

  and

  $v(p_x, p_y, M') \ge v(p_x, p_y, M'')$
- Indirect utility function is quasi-convex.

  we want to prove that $\{(p_x, p_y, M) \mid v(p_x, p_y, M) \le \bar{v}\}$ is a convex set for all $\bar{v}$

  Proof

  Consider arbitrary $\bar{v}$ and consider arbitrary $\left(p'_x, p'_y, M'\right) \in A_{\bar{v}}$ and $\left(p''_x, p''_y, M''\right) \in A_{\bar{v}}$ and arbitray $\lambda \in [0,1]$ then we want to show

  $\lambda(p'_x, p'_y, M') + (1-\lambda)(p''_x, p''_y, M'') \in A_{\bar{v}}$

  In other words we want to show that

$v(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \leq \bar{v}$

we know that $v(p'_x, p'_y, M') \leq \bar{v}$ and $v(p''_x, p''_y, M'') \leq \bar{v}$

$\mathcal{B}(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \leq \bar{v}$ is our budget set.

and we know that this inequality holds,

$(\lambda p'_x + (1-\lambda)p''_x)x + (\lambda p'_y + (1-\lambda)p''_y)y \leq \lambda M' + (1-\lambda)M''$

This tells us that any choice from out budget set $\mathcal{B}$ that satisfy the above inequality also satisfies either

$p'_x x + p'_y y \leq M'$ or $p''_x x + p''_y y \leq M''$

this implies that

$$\begin{aligned} v(\lambda p'_x + (1-\lambda)p''_x,\ \lambda p'_y + (1-\lambda)p''_y,\ \lambda M' + (1-\lambda)M'') \\ \leq \max\big(v(p'_x, p'_y, M'), v(p''_x, p''_y, M'')\big) \leq \bar{v} \end{aligned}$$

2 Expenditure Function

$$\min_{(x,y)\in\mathbb{R}^2_+} p_x x + p_y y$$
$$\text{s.t. } u(x,y) \geq \bar{u}$$

Solving the above expenditure minimzation problem gives us the Hicksian demands,

$$x^h(p_x, p_y, \bar{u})$$
$$y^h(p_x, p_y, \bar{u})$$

and the expenditure function is defined as follows

$e(p_x, p_y, \bar{u}) = p_x x^h(p_x, p_y, \bar{u}) + p_y y^h(p_x, p_y, \bar{u})$

2.1 Properties of the Expenditure function

- The expenditure function is homogeneous of degree 1 in prices,

  $e(\lambda p_x, \lambda p_y, \mu) = \lambda e(p_x, p_y, \mu)$

Note that the Hicksian demands are homogenous of degree 0 in prices because multiplying the objective in our expenditure minimzation problem by λ, (where $\lambda > 0$) does not change the solution.

- The expenditure function is non-decreasing in μ and it is also non-decreasing in prices p_x, p_y.

  we know that our expenditure minimization problem is

$$\min_{(x,y)\in\mathbb{R}^2_+} p_x x + p_y y$$
$$\text{s.t. } u(x,y) \geq \bar{\mu}'$$

  Now consider another satifaction level μ'' such that $\mu' > \mu''$

- The expenditure function is concave in prices.

2.1.1 Kuhn-Tucker Optimization Problems

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2_+} \quad & \sqrt{x} + \sqrt{y} \\ s.t. \quad & p_x x + p_y y \leq M \\ & x \geq 1 \\ & y \geq 1 \end{aligned}$$

Assume that $p_x + p_y < M$

$$\begin{aligned} &\mathcal{L}(x,y) = \sqrt{x} + \sqrt{y} - \lambda(p_x x + p_y y - M) + \mu_x(x-1) + \mu_y(y-1) \\ &\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda p_x + \mu_x = 0 \\ &\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2\sqrt{y}} - \lambda p_y + \mu_y = 0 \\ &\lambda \geq 0, \quad p_x x + p_y y \leq M, \quad \lambda(p_x x + p_y y - M) = 0 \\ &\mu_x \geq 0, \quad x \geq 1, \quad \mu_x(x-1) = 0 \\ &\mu_y \geq 0, \quad y \geq 1, \quad \mu_y(y-1) = 0 \end{aligned}$$

Now if $p_x x + p_y y < M$ then $\lambda = 0$ and $\mu_x < 0$ as well as $\mu_y < 0$ this rules out four of the eight possible cases.

Now we only check for the cases $p_x x + p_y y = M$

$x = 1$	$x = 1$	$x > 1$	$x > 1$
$y = 1$	$y > 1$	$y = 1$	$y > 1$
NP	$\mu_y = 0$ $y = \frac{M - p_x}{p_y} > 1$		