1 Proposition 22

• Let $f: X \to Y$ be a continuous function on a compact set X, then $f(X) = \{f(x) : x \in X\}$ is compact in Y.

Proof

- Suppose X is compact and $f: X \to Y$ be a continuous function.
- We claim that f(X) is compact.
- Pick any arbitrary sequence (y_n) from f(X).
- Consider a sequence (x_n) in X that satisfy the following condition: $y_n = f(x_n)$ for all $n \in \mathbb{N}$.
- Given that X is compact, (x_n) has a convergent subsequence (x_{n_k}) that converges to some $x \in X$.
- By continuity of f, $f(x_{n_k})$ (which is the subsequence of $(f(x_n))$ or (y_n)) converges to f(x).
- Thus, f(X) is compact.

So we have the metric spaces (X, d_x) and (Y, d_y) and the function $f: X \to Y$ which is continuous and we want to show that $f(X) = \{f(x) \in Y \mid x \in X\} \subset Y$ is compact.

Consider any sequence $(y_n) \subset f(X)$, note that \exists a sequence (x_n) s.t. $y_n = f(x_n) \forall n$, by the definition of f(X).

Now since X is compact, there exists a subsequence x_{n_k} of (x_n) such that $x_{n_k} \to l \in X$

Conisder (y_{n_k}) where $y_{n_k} = f(x_{n_k})$

Since f is continous at $l, y_{n_k} \to f(l) \in f(X)$

2 Proposition 23

Weierstrass Maximum Theorem:

• If X is a nonempty compact subset of \mathbb{R}^m and $f: X \to \mathbb{R}$ is a real-valued continuous function on X, then there exists $x^* \in X$ such that

$$f(x^*) \ge f(x) \quad \forall x \in X$$

• Application: Consider the utility maximization problem:

$$\max_{(x_1,x_2)} u(x_1,x_2)$$
s.t. $(x_1,x_2) \in \mathcal{B}(p_1,p_2,I) = \{(y_1,y_2) \in \mathbb{R}^2_+ \mid p_1y_1 + p_2y_2 \leq I\}$
For $p_1 > 0, p_2 > 0, I \geq 0$

and continuous u, the solution to this maximization problem exists.

Proof

- Since X is compact, f(X) is compact (by proposition 22) and hence bounded (by proposition 19).
- There exists an increasing sequence (y_n) in f(X) such that $y_n \to \sup f(X)$ (by proposition 16).
- f(X) is also closed (by proposition 19), so we have $\sup f(X) \in f(X)$ (by proposition 15).
- Thus, there exists $x^* \in X$ such that

$$f(x^*) \ge f(x) \quad \forall x \in X$$

This result is applicable with Euclidean Metric.

So in other words if $X \neq \phi$ is a compact set of \mathbb{R}^m and $f: X \to \mathbb{R}$ is a continuous function on X, then the solution to the problem

$$\max_{x \in X} f(x)$$

exists.

notes that from the last result $f(X) \subset \mathbb{R}$ is compact, and since it is compact it is also bounded and we have a finite supremum, $\sup f(X) < \infty$.

Now we need to show that $\exists x^* \in X \text{ s.t. } f(x^*) = \sup f(X)$

Notice that by a previous result we proved earlier, $\exists (y_n)$ in f(X) s.t. $y_n \to \sup f(X) \in f(X)$ because f(X) is compact and hence closed the limit of the sequence y_n that is $\sup f(X)$ also lies in f(X).

In a similar way the the solution to a minimization problem exists if the conditions of this proposition holds, since mimizing f(X) is the same thing as maximizing -f(x).

3 Convexity

Our primary focus is to deal with the following type of problems,

$$\max_{x \in X} f(x)$$

or,

$$\min_{x \in X} f(x)$$

and we will primarily be focusing on the functions with convex domains, such that $f: X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$

Theorems and Proofs on concavity and convexity of funcitons.

let's focus on the following optimization problems;

What can you say about the set of solutions to the problem;

$$\max_{x \in X} f(x)$$

where f is quasi-concave and X is a convex set.

Yes, the set of solutions to the above problem will be a convex set.

Proof

Suppose M denotes the set of solutions, i.e.,

$$M = \{x^* \in X \mid f(x^*) \ge f(x) \quad \forall \ x \in X\}$$

Suppose $x', x'' \in M$ then because f is quasiconcave,

$$f(\lambda x' + (1 - \lambda)x'') \ge \min(f(x'), f(x''))$$

= $f(x') \ge f(x) \quad \forall \ x \in X$

Therefore $\lambda x' + (1 - \lambda)x'' \in M$

Note that if f is strictly quasi concave then it can not have more than one solution that is M is either empty or singleton set, because when f is strictly quasi concave then the following holds,

$$f(\lambda x' + (1 - \lambda)x'') > \min(f(x'), f(x''))$$

which contadicts the fact that x', x'' are maximizers.

3.1 The cost minimization probelm

$$C(w,q) = \max_{x \in \mathbb{R}_+^m} w \cdot x$$

$$s.t. \quad f(x) \ge q$$

where x(w, q) is the solution to this problem and f is the production function and q is some target level of output and C is a concave function of w, then we want to prove that holding q fixed C is concave in input prices;

$$C(\lambda w' + (1 - \lambda)w'', q) \ge \lambda C(w', q) + (1 - \lambda)C(w'', q) \quad \forall q$$

Now
$$C(w', q) = w' \cdot x(w', q)$$

 $C(w'', q) = w'' \cdot x(w'', q)$
 $C(\lambda w' + (1 - \lambda)w'', q) = (\lambda w' + (1 - \lambda)w'') \cdot x(\lambda w' + (1 - \lambda)w'', q)$
 $= \lambda w' \cdot x(\lambda w' + (1 - \lambda)w'', q) + (1 - \lambda)w'' \cdot x(\lambda w' + (1 - \lambda)w'', q)$
 $\geq \lambda w' \cdot x(w', q) + (1 - \lambda)w'' \cdot x(w'', q)$
 $= \lambda C(w', q) + (1 - \lambda)C(w'', q)$

A few examples;

$$f(l,k) = \min(l,k)$$

$$C(w,r,q) = (\omega + r)q$$

$$f(l,k) = l + k$$

$$C(\omega,r,q) = (\min(w,r))q$$

$$f(l,k) = lk$$

$$C(w,r,q) = (2\sqrt{wr})\sqrt{q}$$

Holding q fixed all the cost functions are concave.