1 Properties of Relations

- Reflexivity: $\forall x \in S : (x, x) \in \mathcal{R}$
- Completeness: $\forall x, y \in S : x \neq y \Rightarrow (x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$
- Transitivity: $\forall x, y, z \in S : ((x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}) \Rightarrow (x, z) \in \mathcal{R}$
- Symmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$
- Anti-symmetry: $\forall x, y \in S : ((x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}) \Rightarrow x = y$
- Asymmetry: $\forall x, y \in S : (x, y) \in \mathcal{R} \Rightarrow (y, x) \notin \mathcal{R}$
- Negative transitivity: $\forall x, y, z \in S : ((x, y) \notin \mathcal{R}$ and $(y, z) \notin \mathcal{R}) \Rightarrow (x, z) \notin \mathcal{R}$
- Equivalence: Relation which is symmetric, reflexive and transitive.

1.1 The indifference Relation

We know that the prefernce relation $\succsim$ on A is Reflexive, Transitive and Complete. Define the indiffernce relation on A as follows;

$$\sim \; = \{(a, b) \in A \times A \mid (a, b) \in \; \succsim \wedge \; (b, a) \in \; \succsim\}$$

Now we can check for the reflexivity, transitivity, completeness and symmetry of the indiffernce relation $\sim$;

- Reflexivity: By the definiton of the indifference relation $\sim$ on A we have,

$$\begin{aligned}
&\sim \text{ on } A = \{(a, a) \in A \times A \mid (a, a) \in \; \succsim \wedge \; (a, a) \in \; \succsim\} \\
&\implies (a, a) \in \; \sim
\end{aligned}$$

- Construct an example by yourself to show that $\sim$ is not complete.
- Transitivity: We want to show, $(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete) ($\sim$ relation derived in the above manner is transitive)

  or,

$$(\forall a, b, c \in A)\,((a, b) \in \; \sim \wedge \, (b, c) \in \; \sim) \implies ((a, c) \in \; \sim)$$

Proof; Consider any arbitrary A, and any arbitrary $\succsim$ on A that is transitive, reflexive and complete.
Now consider the indifference relation

$$\sim = \{(a,b) \in A \times A |\, (a,b) \in \succsim \wedge (b,a) \in \succsim\}$$

and any arbitrary $a, b, c \in A$ such that $(a,b) \in \sim$ and $(b,c) \in \sim$
Now since $(a,b) \in \sim$ and $(b,c) \in \sim$ then by the definiton of the indiffernce relation,

$$(a,b) \in \succsim \wedge (b,a) \in \succsim \tag{1}$$
$$(b,c) \in \succsim \wedge (c,b) \in \succsim \tag{2}$$

Then from the transitivty of $\succsim$ and from $(1), (2)$ we get,

$$\begin{gathered} (a,c) \in \succsim \text{ and } (c,a) \in \succsim \\ \text{so } (a,a) \in \succsim \text{ or by the definiton of } \sim, (a,c) \in \sim \end{gathered}$$

Hence proved that the indifference relation $\sim$ is transitive.

- Symmetry: We want to show, $(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete) ($\sim$ relation derived in the above manner is symmetric)

  Proof; Consider any arbitrary A, and any arbitrary $\succsim$ on A that is transitive, reflexive and complete.
  Now consider the indifference relation

  $$\sim = \{(a, b) \in A \times A |\, (a, b) \in \succsim \wedge (b, a) \in \succsim\}$$

  and the symmetric relation $\mathcal{R}$ on A defined as follows,

  $$(\forall a, b \in A)\,((a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R})$$

  Now consider arbitrary a, b such that $(a, b) \in \sim$,
  Then by the definiton of the indifference relation we get,

  $$(a, b) \in \succsim \wedge (b, a) \in \succsim$$

  which can be rewritten as

  $$(b, a) \in \succsim \wedge (a, b) \in \succsim$$

  and therefore $(b, a) \in \sim$

So we have shown that if the prefernce relation $\succsim$ is reflexive, transitive and complete, then the indifference relation $\sim$ derived from it is reflexive, transitive and symmetric.

Question

Suppose $A = \{a, b, c\}$ and $\mathcal{R} = \{(a, b), (b, a), (a, a)\}$.

Is $\mathcal{R}$ a valid indifference relation?

Answer

Note that $\mathcal{R}$ is not reflexive, not complete and neither transitive and an indifference realtion is always reflexive, transitive and symmetric so therefore, $\mathcal{R}$ is not a valid indifference realtion.

1.2 The strict prefernce relation

We know that the preference realtion $\succsim$ on A is reflexive, transitive and complete. From this we can define the strict prefernce relation as follows;

$$\begin{aligned} \succ &= \{(a,b) \in A \times A | (a,b) \in \succsim \wedge (b,a) \notin \succsim\} \\ \text{So,} \quad &(a,b) \in \succsim \wedge \neg(b,a) \in \succsim \end{aligned}$$

Now we can check for the reflexivity, transitivity, completeness and symmetry of the strict prefernce relation $\succ$ as follows;

- Reflexivity: We want to show, $(\forall A)$ ($\forall \succsim$ on A that are transitive, reflexive and complete) ($\succ$ relation derived in the above manner is reflexive)

  for the proof, try to negate the above proposition and you can see that the strict preference relation is not reflexive.

- Completeness: try to find an example to show that the strict preference relation is not complete.

  Consider $A = \{a, b, c\}$ and the prefernce relation $\succsim = A \times A$.

  Now because of the given prefernce the relation the strict prefernce relation will be an empty set or $\succ = \phi$ which is neither reflexive and nor complete.

- Transitivity: We want to show, $(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete$)$ $(\succ$ relation derived in the above manner is transitive$)$

> Proof; Consider any arbitrary A and an arbitraty $\succsim$ satisfying reflexiveness, transitivity and completeness and also consider any aribitray a, b, c in A such that $(a, b) \in \succ$ and $(b, c) \in \succ$.
> Now because $(a, b) \in \succ$ and $(b, c) \in \succ$ we get,
>
> $$(a, b) \in \succsim \wedge\ (b, a) \notin \succsim \tag{3}$$
> $$(b, c) \in \succsim \wedge\ (c, b) \notin \succsim \tag{4}$$
>
> and because the preference relation $\succsim$ is transitive and from $(3), (4)$ we get that $(a, c) \in \succsim$ and now we want to show that $(c, a) \notin \succsim$
> We will do so by the way of contradiction:-
> Suppose $(c, a) \in \succsim$, given that $(a, b) \in \succsim$ from, the previous assumption, we get that $(c, b) \in \succsim$ because the prefernce relation is transitive.
> But this is a contradiction because we know that $(c, b) \notin \succsim$ from (4) and therfore, $(c, a) \notin \succsim$
> So we have $(a, c) \in \succsim \wedge\ (c, a) \notin \succsim$, which clearly implies that the strict prefrence realtion is transitive.

So we have shown that if the prefernce relation $\succsim$ is reflexive, transitive and complete, then the strict preference relation $\succ$ derived from it is not reflexive, nor complete but transitive.

Examples

$A = \{a, b, c\}$

$\mathcal{R} = \{(a, a), (a, b), (b, b)\}$

Note that $\mathcal{R}$ is Anti-symmetric but not Asymmetric and it is also not negative transitive because $(a, c) \notin \mathcal{R}$ and $(c, b) \notin \mathcal{R}$ but $(a, b) \in \mathcal{R}$

How to negate $\neg\left((\forall x), (\forall y), (P(x) \implies Q(y))\right)$

$$\begin{aligned}\neg\left((\forall x), (\forall y), (P(x) \implies Q(y))\right) &\equiv \left((\exists x)\,(\exists y)\,(\neg(P(x) \implies Q(y)))\right)\\ &\equiv \left((\exists x)\,(\exists y)\,(P(x) \wedge \neg Q(y))\right)\end{aligned}$$

Propositon 1

$(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete$)$ $(\succ$ relation derived from $\succsim$ is asymmetric$)$

Proof; Consider any arbitrary $a, b \in A$ and $(a, b) \in \succ$
Now because $(a, b) \in \succ$ we get

$$(a, b) \in \succsim \wedge (b, a) \notin \succsim$$

but this implies that $(b, a) \notin \succ$
and therefore the strict prefernce relation $\succ$ is assymetric.

Propositon 2

$(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete$)$ $(\succ$ relation derived from $\succsim$ is anti-symmetric$)$

this is a corollary of the above proof because

$(\forall\ \mathcal{R}$ on $A)\,(\mathcal{R}$ is asymmetric $\implies \mathcal{R}$ is anti-symmetric $)$

Propositon 3

$(\forall A)\,(\forall \succsim$ on A that are transitive, reflexive and complete) ($\succ$ relation derived from $\succsim$ is Negative transitive)

Proof. Consider any arbitrary A, and any weak preference relation $\succsim$ on set A that is reflexive, complete and transitive. Now consider arbitrary $a, b, c \in A$ such that $(a, b) \notin \succ$ and $(b, c) \notin \succ$. We will try and show that $(a, c) \notin \succ$.
Suppose, by way of contradiction, $(a, c) \in \succ$.
Step 1. By completeness of $\succsim$, we also have $(c, b) \in \succsim$. By transitivity of $\succsim$. This implies that $(a, b) \in \succsim$ and we also have $(b, a) \in \succsim$ by completeness of $\succsim$. So, we get $(a, b) \in \sim$.
Step 2. In the similar fashion as above, by completeness of $\succsim$, we also have $(b, a) \in \succsim$. By transitivity of $\succsim$. This implies that $(b, c) \in \succsim$ and we also have $(c, b) \in \succsim$ by completeness of $\succsim$. So, we get $(b, c) \in \sim$.
So, by step 1 and 2, we get that $(a, c) \in \sim$ (by transitivity of $\succsim$). This is a contradiction.
Therefore, $(a, c) \notin \succ$.
Alternative Proof (Direct Proof). Consider any arbitrary A, and any weak preference relation $\succsim$ on set A that is reflexive, complete and transitive. Now consider arbitrary $a, b, c \in A$ such that $(a, b) \notin \succ$ and $(b, c) \notin \succ$. By completeness of $\succsim$ $(b, a) \in \succsim$ and $(c, b) \in \succsim$. By transitivity of $\succsim$ $(c, a) \in \succsim$. Therefore, $(a, c) \notin \succ$.

Question

Consider a binary relation $\succsim$ on a set A. Suppose $\succsim$ is transitive. Define relations $\succ$ and $\sim$ on A by: for $x, y \in A$,

$x \succ y$ if and only if $x \succsim y$ and not $y \succsim x$

$x \sim y$ if and only if $x \succsim y$ and $y \succsim x$

Prove the following. (i) If $x \succ y$ and $y \succ z$ then $x \succ z$

(ii) If $x \sim y$ and $y \sim z$ then $x \sim z$

(iii) If $x \succ y$ and $y \succsim z$ then $x \succ z$

Proof. Let $\succsim$ be transitive, then we want to prove that
$(x \succ y \wedge y \succsim z) \implies x \succ z$
Consider any arbitrary $x, y, z \in A$ such that;
$x \succ y \wedge y \succsim z$ holds
Suppose by the way of contradiction $x \succ z$ does not hold
then by definition of strict preference,

$$\begin{aligned} &\neg(x \succsim z \wedge \neg(z \succsim x)) \\ \implies &\neg(x \succsim z) \vee z \succsim x \end{aligned}$$

But then by the transitivity of $\succsim$

$$z \succsim x \wedge y \succsim z \implies y \succsim x$$

But this contradicts the fact that $x \succ y$ holds. Hence proved!