Satisfiability Problem

AIFA (AI61005)
2021 Autumn

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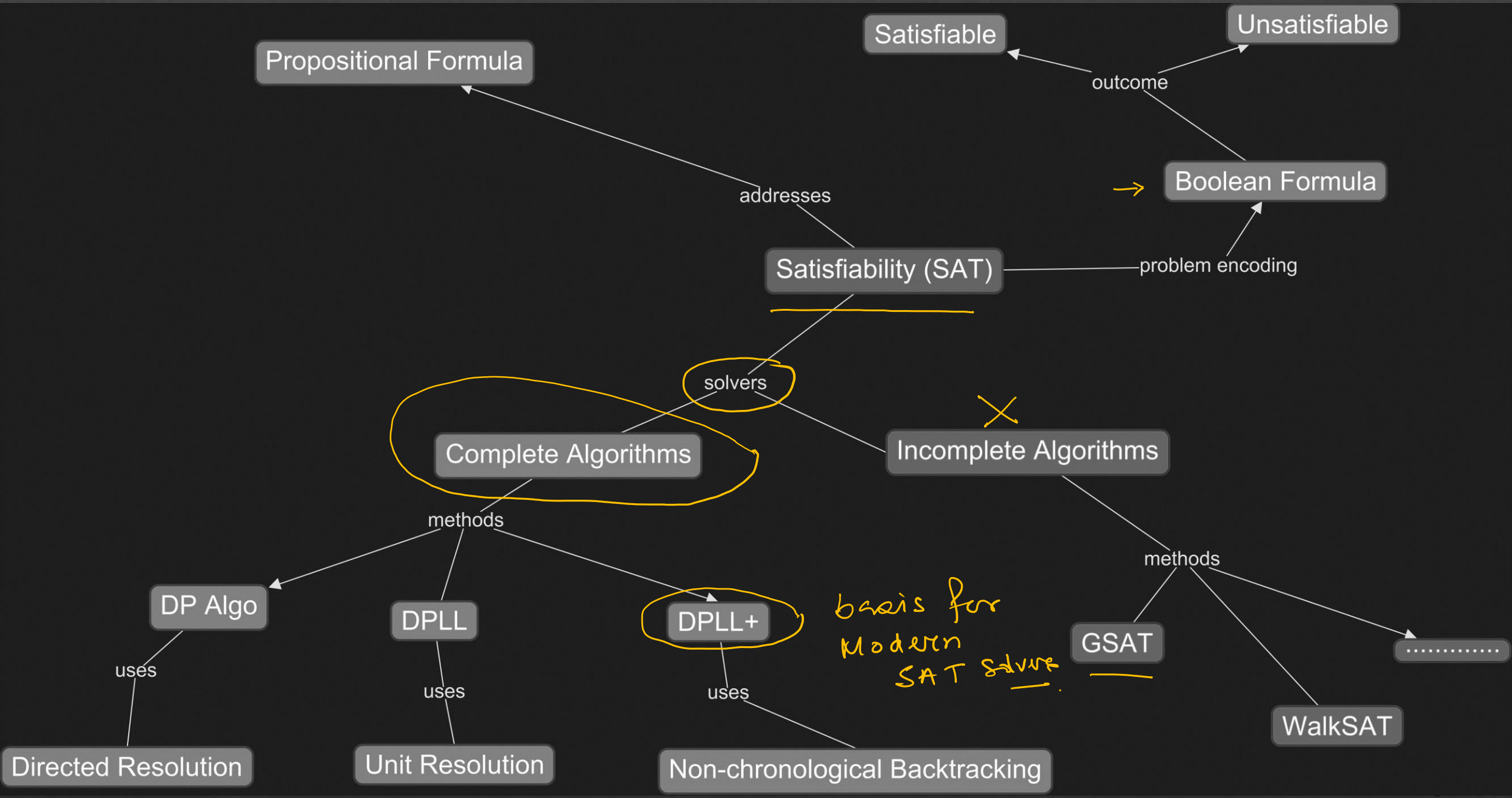

Propositional Formula
Satisfiable
Unsatisfiable
outcome
addresses
Boolean Formula
Satisfiability (SAT)
problem encoding
solvers
Complete Algorithms
Incomplete Algorithms
methods
methods
DP Algo
DPLL
DPLL+
GSAT
.....
uses
uses
uses
WalkSAT
Directed Resolution
Unit Resolution
Non-chronological Backtracking
basis for Modern SAT solvers

SAT: Why and What

Propositional Logic

- A *literal* is a Boolean variable x or its negation $\bar{x}$
- A *clause* is a disjunction ($\vee$) of literals A ∨ B
- A *CNF formula* is a conjunction ($\wedge$) of clauses

→ $F = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2) \wedge (x_2 \vee \overline{x_3})$ CNF

$Vars(F) = \{x_1, x_2, x_3\}$

$Literals(F) = \{x_1, x_2, \overline{x_2}, x_3, \overline{x_3}\}$

$Clauses(F) = \{(x_1 \vee \overline{x_2} \vee \overline{x_3}), (x_2), (x_2 \vee \overline{x_3})\}$

Satisfiability

- Truth assignment ϕ (variables)
  - $\phi(x)$ = True or $\phi(x)$ = False
- ϕ satisfies → assignment
  - x if $\phi(x)$ = True
  - $\bar{x}$ if $\phi(x)$ = False
  - A clause if it satisfies at least one of its literals
  - A CNF formula if it satisfies all of its clauses

ϕ = {A = true or B = true or C = true}

A ∨ B ∨ C

Satisfiability

- ϕ is a *satisfying assignment* of a CNF F if it satisfies F
- F is *satisfiable* if there exists a ϕ that satisfies F
- The *Satisfiability Problem* determines whether a given F is satisfiable or not.

ϕ for which F is true

Satisfiability: Example

Satisfiable Formula

(x_1)

Unsatisfiable Formula

$(x_1) \wedge (\overline{x_1})$

$(x_3) \vee (\overline{x_2}) \vee (x_7)$

$(\overline{x_1} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1})$

$(x_2) \wedge (\overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$

$(x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_2})$

Satisfiability: Scheduling Meeting

- A can only meet on Monday or Wednesday
- B cannot meet on Wednesday
- C cannot meet on Friday
- D can only meet on Thursday or Friday

→ Boolean Formula

SAT problem instance

boolean

$$F = (x_1 \vee x_3) \wedge (\overline{x_3}) \wedge (\overline{x_5}) \wedge (x_4 \vee x_5) \wedge AtMostOne(x_1, x_2, x_3, x_4, x_5)$$

Satisfiability: Scheduling Meeting

A can only meet on Monday or Wednesday

B cannot meet on Wednesday

C cannot meet on Friday

D can only meet on Thursday or Friday

SAT problem instance

$$F = (x_1 \vee x_3) \wedge (\overline{x_3}) \wedge (\overline{x_5}) \wedge (x_4 \vee x_5)$$
$$\wedge\ (\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_5})$$
$$\wedge\ (\overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_2} \vee \overline{x_5})$$
$$\wedge\ (\overline{x_3} \vee \overline{x_4}) \wedge (\overline{x_3} \vee \overline{x_5})$$
$$\wedge\ (\overline{x_4} \vee \overline{x_5})$$

SAT Solver → Unsatisfiable

Hardness of SAT

- SAT is NP-Complete
- No polynomial time algorithm for SAT yet
- Known complete algorithms have exponential runtime in worst case

Applications of SAT

- Hardware Model Checking
  - Does a circuit exhibit intended behavior?
  - Verify chip design (e.g., intel)
- Software Verification
  - SMT solvers to verify Microsoft products
  - Verification of embedded software in cars, airplanes etc.
- One of the best approaches in planning and scheduling
- Number theoretic problems
- Connection with NP-hard problems

SAT Encoding: Pythagorean Triple

Can the set of integers {1,2,3,….,} be partitioned into two sets such that no part contains a Pythagorean triple ($a, b, c \in \mathbb{N}$ st $a^2 + b^2 = c^2$)?

Is it possible to assign each integer one of two colors such that if $a^2 + b^2 = c^2$ then a, b, c do not all have the same color?

Not possible for 7825

Proof obtained by a SAT solver has 200 Terrabytes

Largest math proof ever

SAT Encoding: Pythagorean Triple

Is it possible to assign each integer one of two colors such that if $a^2 + b^2 = c^2$ then a, b, c do not all have the same color?

For each integer i, a Boolean variable x_i:

$x_i = 1$ if color(i)= B else $x_i = 0$

For each $\langle a, b, c \rangle$ st $a^2 + b^2 = c^2$:

R R B

$(x_a \vee x_b \vee x_c) \wedge (\overline{x_a} \vee \overline{x_b} \vee \overline{x_c})$

0 0 1 — True

1 1 0 — True

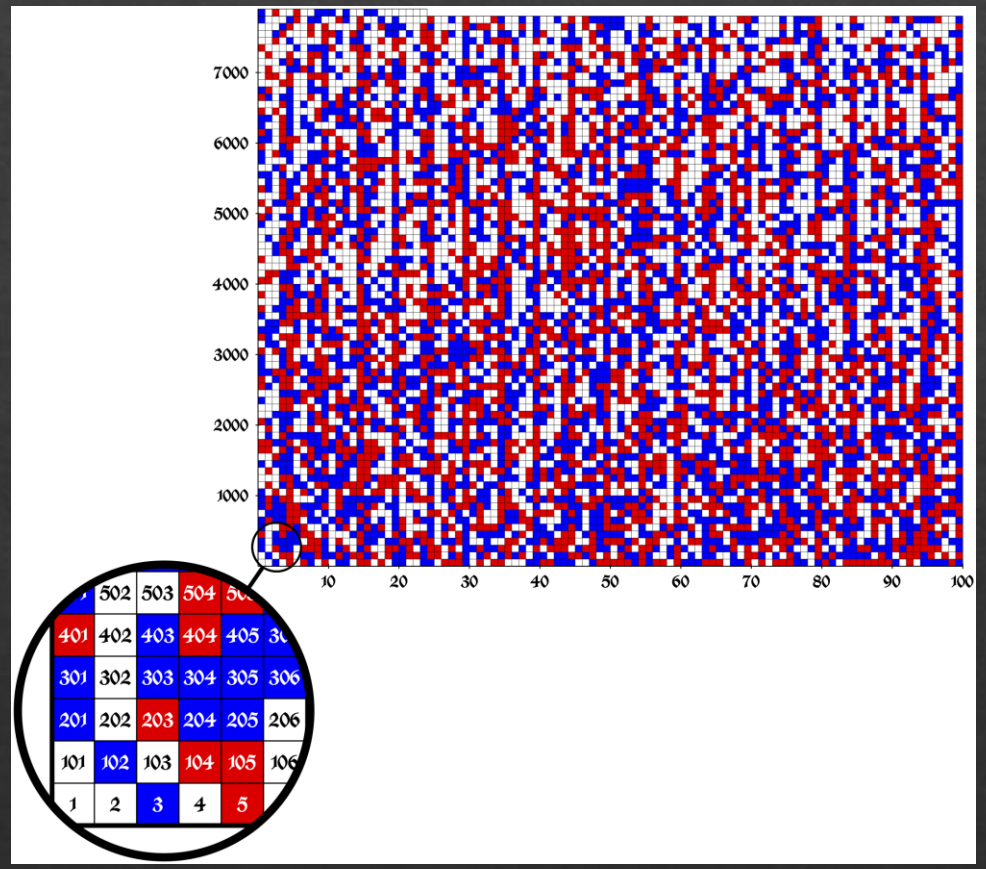
⇒ True

B, B, B

1 1 1 — True

0 0 0 — False

⇒ False



https://www.nature.com/articles/nature.2016.19990

SAT Encoding: Pythagorean Triple

Is it possible to assign each integer one of two colors such that if $a^2 + b^2 = c^2$ then a, b, c do not all have the same color?

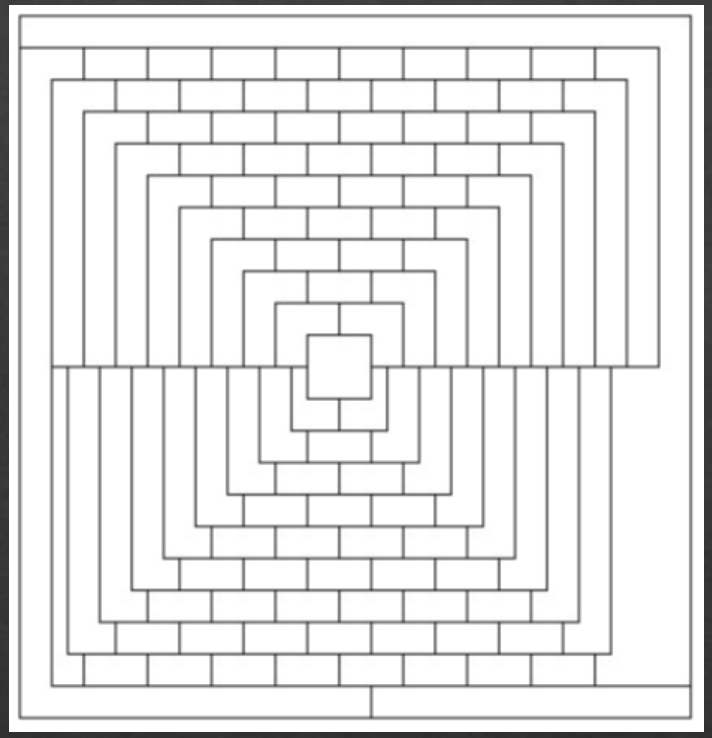
$$
\begin{aligned}
&(x_3 \vee x_4 \vee x_5) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5) \wedge (x_5 \vee x_{12} \vee x_{13}) \wedge (\bar{x}_5 \vee \bar{x}_{12} \vee \bar{x}_{13}) \wedge \\
&(x_7 \vee x_{24} \vee x_{25}) \wedge (\bar{x}_7 \vee \bar{x}_{24} \vee \bar{x}_{25}) \wedge (x_9 \vee x_{40} \vee x_{41}) \wedge (\bar{x}_9 \vee \bar{x}_{40} \vee \bar{x}_{41}) \wedge \\
&(x_6 \vee x_8 \vee x_{10}) \wedge (\bar{x}_6 \vee \bar{x}_8 \vee \bar{x}_{10}) \wedge (x_8 \vee x_{15} \vee x_{17}) \wedge (\bar{x}_8 \vee \bar{x}_{15} \vee \bar{x}_{17}) \wedge \\
&(x_{10} \vee x_{24} \vee x_{26}) \wedge (\bar{x}_{10} \vee \bar{x}_{24} \vee \bar{x}_{26}) \wedge (x_{12} \vee x_{35} \vee x_{37}) \wedge (\bar{x}_{12} \vee \bar{x}_{35} \vee \bar{x}_{37}) \wedge \\
&(x_{14} \vee x_{48} \vee x_{50}) \wedge (\bar{x}_{14} \vee \bar{x}_{48} \vee \bar{x}_{50}) \wedge (x_9 \vee x_{12} \vee x_{15}) \wedge (\bar{x}_9 \vee \bar{x}_{12} \vee \bar{x}_{15}) \wedge \\
&(x_{15} \vee x_{36} \vee x_{39}) \wedge (\bar{x}_{15} \vee \bar{x}_{36} \vee \bar{x}_{39}) \wedge (x_{12} \vee x_{16} \vee x_{20}) \wedge (\bar{x}_{12} \vee \bar{x}_{16} \vee \bar{x}_{20}) \wedge \\
&(x_{16} \vee x_{30} \vee x_{34}) \wedge (\bar{x}_{16} \vee \bar{x}_{30} \vee \bar{x}_{34}) \wedge (x_{20} \vee x_{48} \vee x_{52}) \wedge (\bar{x}_{20} \vee \bar{x}_{48} \vee \bar{x}_{52}) \wedge \\
&(x_{15} \vee x_{20} \vee x_{25}) \wedge (\bar{x}_{15} \vee \bar{x}_{20} \vee \bar{x}_{25}) \wedge (x_{18} \vee x_{24} \vee x_{30}) \wedge (\bar{x}_{18} \vee \bar{x}_{24} \vee \bar{x}_{30}) \wedge \\
&(x_{24} \vee x_{45} \vee x_{51}) \wedge (\bar{x}_{24} \vee \bar{x}_{45} \vee \bar{x}_{51}) \wedge (x_{21} \vee x_{28} \vee x_{35}) \wedge (\bar{x}_{21} \vee \bar{x}_{28} \vee \bar{x}_{35}) \wedge \\
&(x_{20} \vee x_{21} \vee x_{29}) \wedge (\bar{x}_{20} \vee \bar{x}_{21} \vee \bar{x}_{29}) \wedge (x_{24} \vee x_{32} \vee x_{40}) \wedge (\bar{x}_{24} \vee \bar{x}_{32} \vee \bar{x}_{40}) \wedge \\
&(x_{28} \vee x_{45} \vee x_{53}) \wedge (\bar{x}_{28} \vee \bar{x}_{45} \vee \bar{x}_{53}) \wedge (x_{27} \vee x_{36} \vee x_{45}) \wedge (\bar{x}_{27} \vee \bar{x}_{36} \vee \bar{x}_{45}) \wedge \\
&(x_{30} \vee x_{40} \vee x_{50}) \wedge (\bar{x}_{30} \vee \bar{x}_{40} \vee \bar{x}_{50}) \wedge (x_{33} \vee x_{44} \vee x_{55}) \wedge (\bar{x}_{33} \vee \bar{x}_{44} \vee \bar{x}_{55})
\end{aligned}
$$

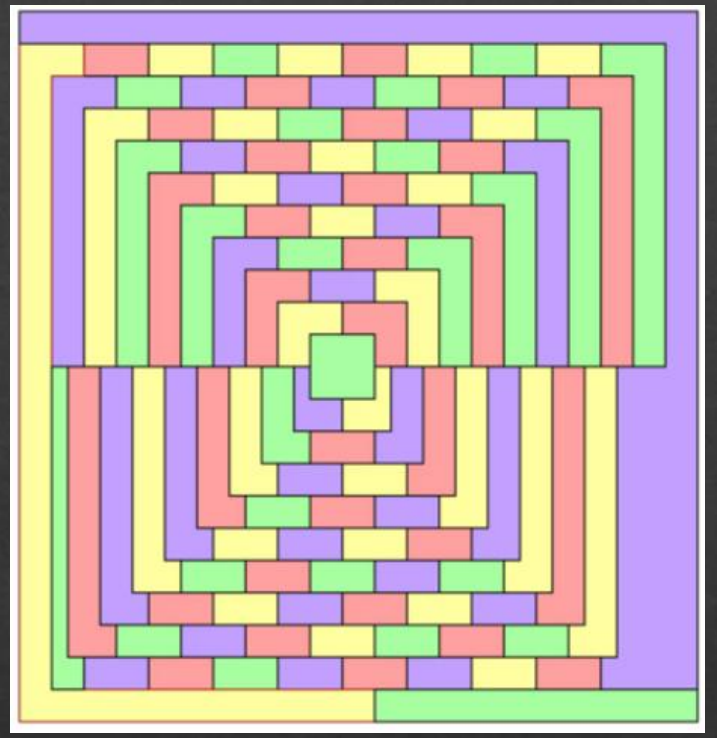
$i = 55$

N=55

7825

SAT Encoding: Graph Coloring





McGregor graph (of order 10): planar, 110 nodes

Cannot be colored with less than 5 colors

K- Coloring

SAT Encoding: Graph Coloring

A graph coloring using at most k colors is called k-coloring. The Graph Coloring Problem seeks whether k-coloring for a graph $G = \langle V, E \rangle$ exists.

5 v_1 v_2 v_3 v_4 v_5

Variables:

$k.|V|$ Boolean variables v_j for $v \in V, 1 \leq j \leq k, v_j = \text{True}$ if node v gets color j

j = Color

Clauses:

→ Every node gets a color: $(v_1 \vee v_2 \vee \cdots \vee v_k)$ for $v \in V$

→ Adjacent nodes have diff color: $(\overline{u_j} \vee \overline{v_j})$ for $(u, v) \in E, 1 \leq j \leq k$

$\overline{u_j}$ = False

j th Color u u_j = True

v_j = False

SAT Encoding: Graph Coloring

$V = \{u, v, w, x, y\}$ $k = 1,2,3$

$Variables = \{u_1, u_2, u_3, \dots, y_1, y_2, y_3\}$

Clauses:

Every node gets a color:

$(u_1 \vee u_2 \vee u_3) \wedge$

$\dots \dots \dots \dots \wedge$

$(y_1 \vee y_2 \vee y_3)$

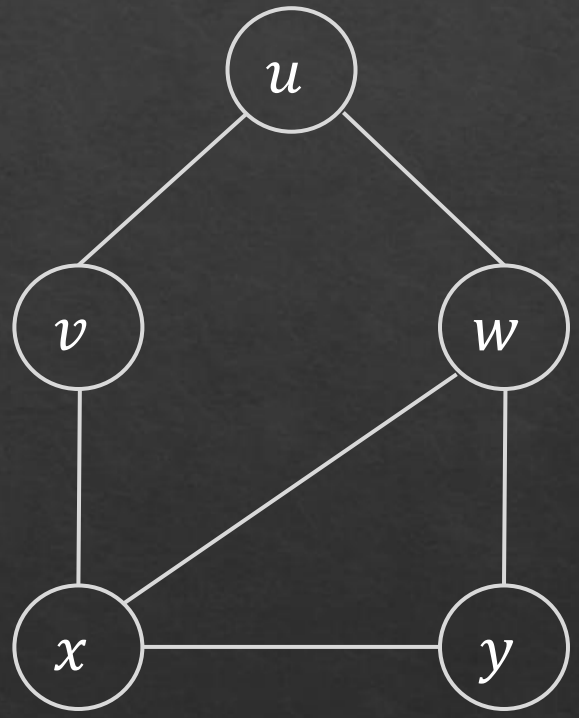
Adjacent nodes have diff color:

$(\overline{u_1} \vee \overline{v_1}) \wedge \cdots \wedge (\overline{u_3} \vee \overline{v_3})$

$\dots \dots \dots \dots$

$(\overline{x_1} \vee \overline{y_1}) \wedge \cdots \wedge (\overline{x_3} \vee \overline{y_3})$

For every edge



SAT Solvers

SAT Solver with inference

Resolution: $P = x$ $\quad \neg P = \neg x$

$(a_1 \vee a_2 \vee x \vee a_3 \vee a_4) \quad (b_1 \vee b_2 \vee \neg x \vee b_3 \vee b_4)$

$\Rightarrow (a_1 \vee a_2 \vee a_3 \vee a_4 \vee b_1 \vee b_2 \vee b_3 \vee b_4)$

$P \rightarrow$ Boolean variable

$\Delta \rightarrow$ CNF with clauses C_i and C_j

$P \in C_i \quad \neg P \in C_j$

Derived clause: $(C_i - \{P\}) \vee (C_j - \{\neg P\}) \rightarrow$ P_resolvant

$\Delta = \{(A, B, \neg C), (\neg B, D)\} \Rightarrow (A, \neg C, D) \rightarrow$ B-resolvant

SATISFIABILITY We derive empty clause

$\Delta = \{ \underset{\wedge}{\_\_}, \underset{\wedge}{\_\_}, \underset{\wedge}{\_\_}, \underset{\text{False}}{\Phi} \} = \{ \underset{\text{False}}{\Phi} \} \rightarrow$ UNSATISFIABILITY

→ Resolution is sound but Incomplete

not guaranteed to derive every clause that is implied by Δ.

COMPLETE SAT

→ Resolution is refutation Complete

guaranteed to derive an empty clause if Δ is unsatisfiable

⇓

Resolution based complete Algo for SAT testing

1. $\{\neg P, R\}$
2. $\{\neg Q, R\}$
3. $\{\neg R\}$ ⇒
4. $\{P, Q\}$

---

5. $\{\neg P\}$ (1,3) → Keep applying resolution until we derive empty clause ⇒ UNSAT
6. $\{\neg Q\}$ (2,3)
7. $\{Q\}$ (4,5)
8. $\{\phi\}$ (6,7) → until we have no option to apply resolution rule ⇒ SAT

↓ UNSAT

Unit Resolution: At least one of resolved clauses has only one literal

$$\frac{C_1 \quad C_2}{}$$

CONDITIONING

$\Delta|L$ ⇒ Replacing every occurrence of L by 'True' and $\neg L$ by 'False'

$$\Delta|L = \{\alpha - \{\neg L\} \mid \alpha \in \Delta, L \notin \alpha\}$$

$C = \alpha \wedge \beta \wedge \gamma$

↳ true

clauses in Δ can be partition into:

$\alpha = (A \vee L \vee \neg B)$

$\alpha = (A \vee \text{True} \vee \neg B)$

$\alpha = (A \vee \neg L \vee \neg B)$

↓ False

$(A \vee \neg B)$

1. $L \in \alpha$ ⇒ α does not appear in $\Delta|L$
2. $\neg L \in \alpha$ ⇒ occurrence of $\neg L$ in α does not have any effect. ⇒ $\{\alpha - \{\neg L\}\}$
3. L or $\neg L \notin \alpha$ ⇒ α will appear in $\Delta|L$ as it is $\alpha = (A \vee B \vee C)$

Example

$\Delta = \{\{A, B, \neg C\}, \{\neg A, D\}, \{B, C, D\}\}$

$\neg C \in \alpha \rightarrow \Delta|_C = \{\{A, B\}, \{\neg A, D\}\}$

$\alpha = \{\neg C\}$ → $\{A, B\}$

$\Delta|_{\neg C} = \{\{\neg A, D\}, \{B, D\}\}$

$\Delta|_{C,A} = \{\{D\}\}$

$\Delta|_{C,A,\neg D} = \{\square\} \rightarrow$ Contradiction.

$\{\square\}$: Contains an empty clause = False

$\Delta|_{C,A,D} = \{\} = \top \rightarrow$ Satisfaction

$\{\}$

$\Delta|_{\neg C,D} = \{\} \rightarrow$ Consistent Satisfiable.

DP Algo (Davis-Putnam)

Directed Resolution → Repeated application of resolution
↳ P-resolvant
Bucket Elimination

→ A bucket for each var.
→ Choose a variable ordering (π)
→ Construct and fill the buckets.
→ $\alpha \in \Delta$ - add α to the first bucket P from the top such that P appears in α

$(P, \neg P)$

Bucket Elimination Example

$\Delta = \{\{\neg A, B\}, \{\neg A, C\}, \{\neg B, D\}, \{\neg C, \neg D\}, \{A, \neg C, E\}\}$

① $\pi = C, B, A, D, E$

C-resolution → $\{\neg A, \neg D\}$ (True) → True

C: $\{\neg A, C\}, \{\neg C, \neg D\}, \{A, \neg C, E\}$

B: $\{\neg A, B\}, \{\neg B, D\}$

A:

D:

E:

SATISFACTION

$\{\neg A, \neg D\}, \{\neg A, D\}$ →

→ No new resolution

② $\pi = E, A, B, C, D$

E: $\{A, \neg C, E\}$

A: $\{\neg A, B\}, \{\neg A, C\}$

B: $\{\neg B, D\}$

C: $\{\neg C, \neg D\}$

D:

SATISFACTION

DP(Δ, π) returns UNSAT or SAT

For each variable V:
    create empty bucket B_V

For each clause $C \in \Delta$:
    V ← first var. of C as per π
    $B_V \leftarrow B_V \cup \{C\}$

For each var V as per π:
    If B_V is not empty:
        for each V-resolvant of $C \in B_V$:
            if C is empty ($\{\}$):
                return UNSAT
        → U ← first var of C as per π
        $B_U \leftarrow B_U \{C\}$

return SAT

Forgetting (Existential Quantification)

$B \Rightarrow C,\ C \Rightarrow D \Rightarrow B \Rightarrow D$

$\exists C\ \Delta \models B \Rightarrow D$

$\Delta = \begin{matrix} A \Rightarrow B \\ B \Rightarrow C \\ C \Rightarrow D \\ D \Rightarrow E \end{matrix} \models A \Rightarrow E$

$A \Rightarrow D$
$D \Rightarrow E$
$A \Rightarrow E$

$\exists C, B, D\ \Delta \models A \Rightarrow E$

$\exists x \Delta = (\Delta|x) \vee (\Delta|\neg x)$

forgetting x

Δ is satisfiable if $\exists P \Delta$ is satisfiable.

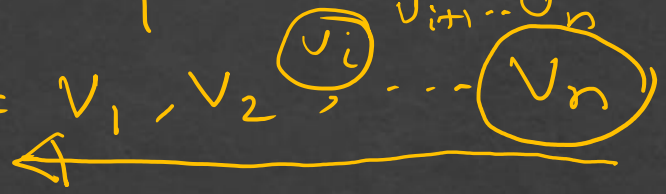
↳ fewer variables

$\{ A \Rightarrow B,\ B \Rightarrow C \mid A \Rightarrow C \}$

Extract Satisfying Assignment (ϕ)

→ Process vars. from bottom to top.

$\pi = v_1, v_2, \ldots, v_i, v_{i+1} \ldots v_n$



→ B_{v_i} is empty ⇒ Assign any value.

✓ else ⇒ Assign v_i such a way

$i < j < n$ v_j

C : $\{\neg A, C\}$, $\{\overline{\neg C}, \neg D\}$, $\{A, \neg C, E\}$

B : $\{\neg A, B\}$, $\{\neg B, D\}$

True True

$\{\neg A, \neg D\}$ $\{\neg A, D\}$

$E = T$, $D = F$ $A = F$

$B = F$, $C = T$

→ A :

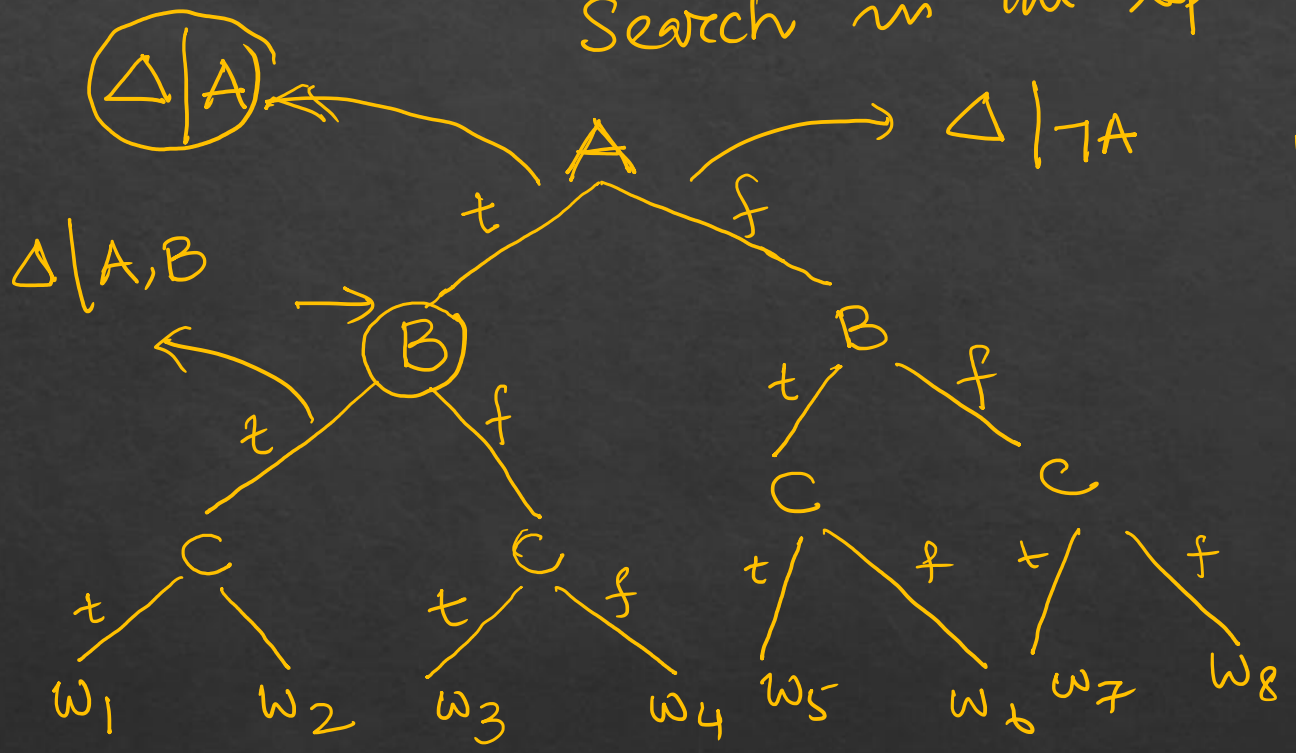
→ D :

E :

SAT by Search

DPLL (Davis-Putnam, Logemann-Loveland)

Search in the space of truth assignment



$w_1 \ldots w_8$

$w_3 = \{A=t, B=f, C=t\}$

leaf → a complete truth assignment

↳ Satisfiability

↳ finding a leaf node that satisfies the CNF

A tree that has depth n.

→ DFS to explore the search space

DPLL- (Δ, d) return SAT + variable instantiations or UNSAT

If $\Delta = \{\}$ then [// No Contradiction]
return $\{\}$

DPLL- $(\Delta, 0)$ else if $\{\} \in \Delta$ then [// Contradiction]
return UNSAT

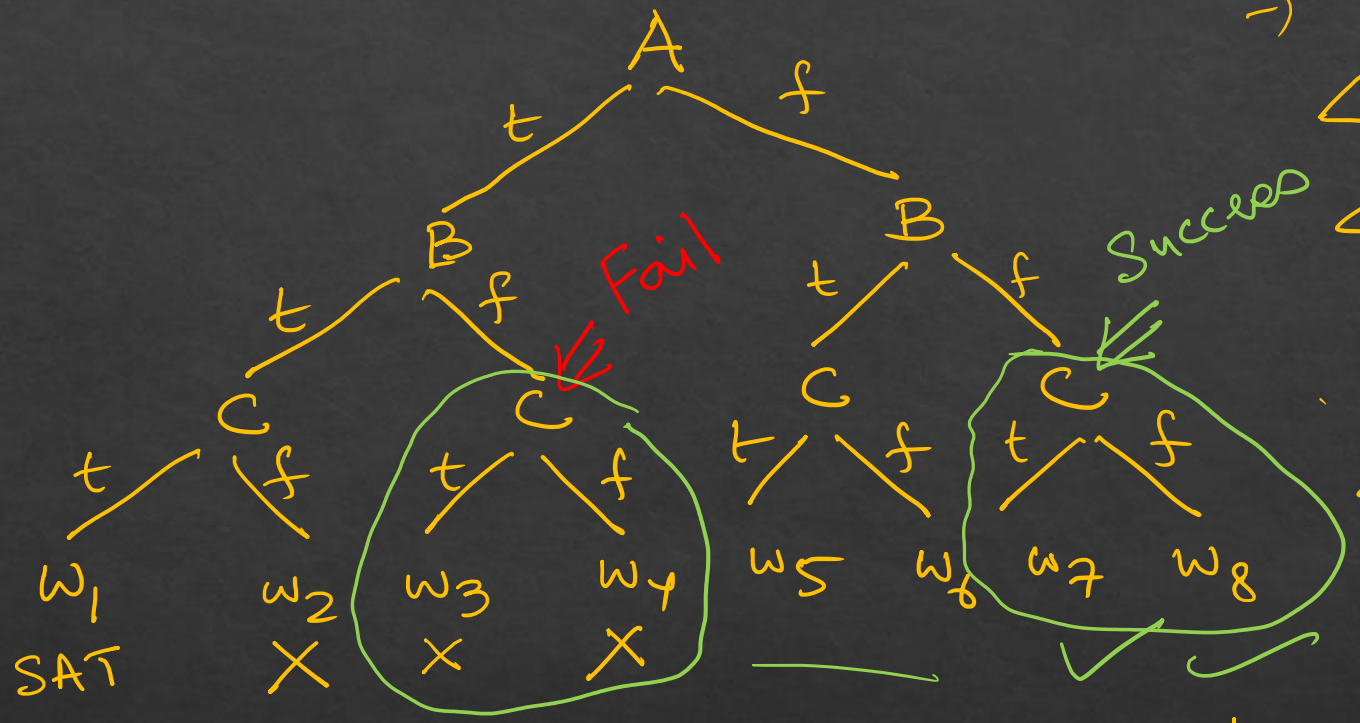
else if $L = \text{DPLL-}(\Delta | P_{d+1}, d+1) \neq \text{UNSAT}$ then
return $L \cup \{P_{d+1}\}$

else if $L = \text{DPLL-}(\Delta | \neg P_{d+1}, d+1) \neq \text{UNSAT}$ then
return $L \cup \{\neg P_{d+1}\}$

else
return UNSAT

Example DPLL-

$\Delta = \{\{\neg A, B\}, \{\neg B, C\}\}$



$\Delta|_A = \{\{\overline{B}\}, \{\neg B, C\}\}$

$\Delta|_{A,B} = \{\{C\}\}$

$\Delta|_{A,B,C} = \{\} \rightarrow$ SAT

$\Delta|_{A,B,\neg C} = \{\{\}\} \rightarrow$ UNSAT

$\Delta|_{A,\neg B} = \{\{\}, \{C\}\}$

$\{\} \in \Delta|_{A,\neg B}$

↳ UNSAT

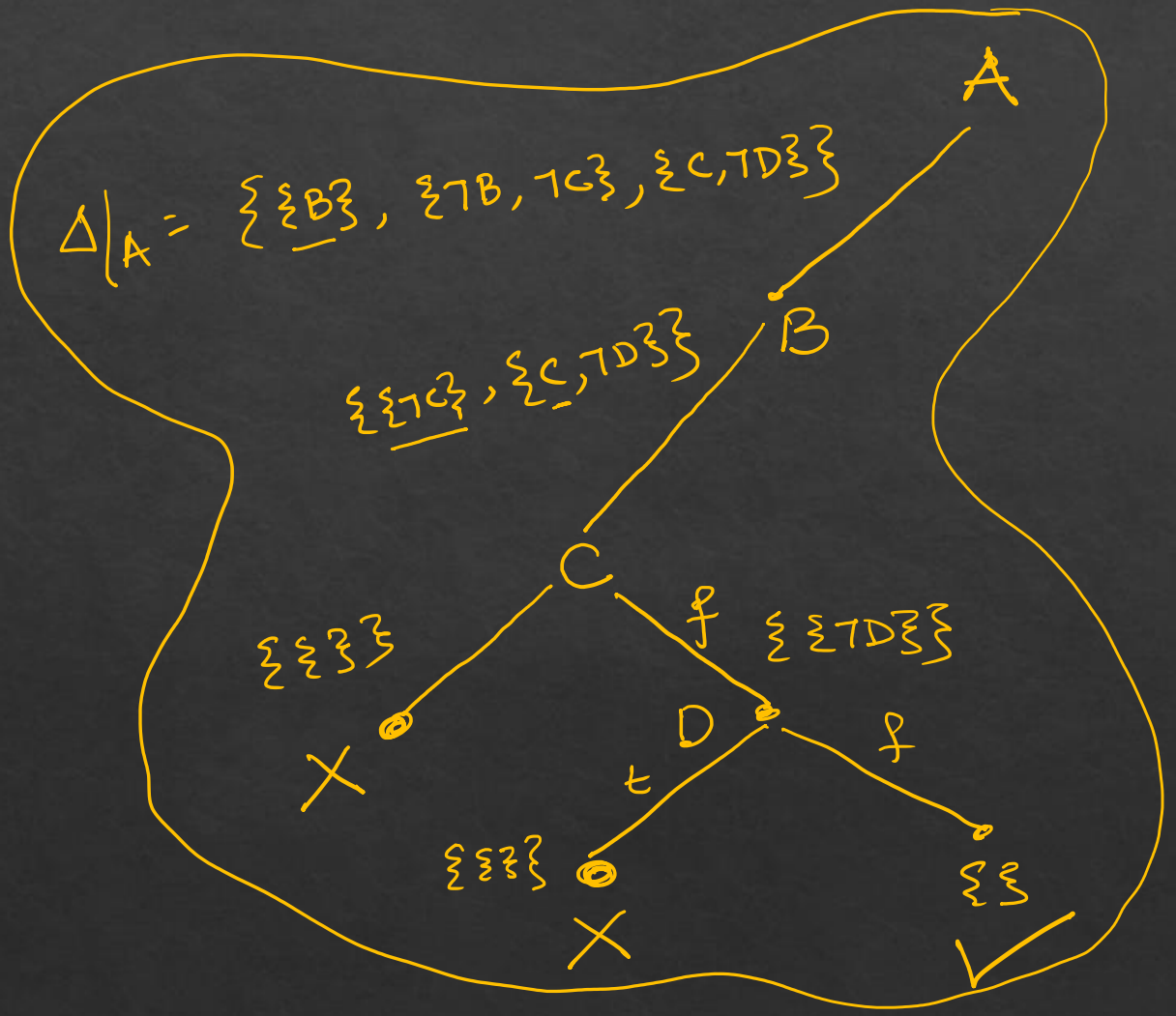
$\Delta|_{\neg A} = \{\{\neg B, C\}\}$

$\Delta|_{\neg A,\neg B} = \{\}$ → SAT

DPLL- may detect success before reaching the leaf.

Amount of work by DPLL-

$\rightarrow \Delta = \{\{\neg A, B\}, \{\neg B, \neg C\}, \{C, \neg D\}\}$



$\Delta|_A = \{\{B\}, \{\neg B, \neg C\}, \{C, \neg D\}\}$

$\Delta|_A = \{\}$ SAT

$\{\} \in \Delta|_A$ UNSAT

DPLL- cannot decide failure or success at this pt.

$\Delta|_{A,B} = \{\{\neg C\}, \{C, \neg D\}\}$

$\Delta|_{A,B,\neg C} = \{\{\neg D\}\}$

$\Delta|_{A,B,C,\neg D} = \{\} \rightarrow$ sat.

$\Delta|_A \rightarrow \{A\}$ $\{\neg A, B\}$

$\{B\}$

First: Closing Δ under unit resolution collecting all the unit clauses.

Second: Set the variables to satisfy unit clauses.

$\{P\} \Rightarrow$ Set P = True

$\{\neg P\} \rightarrow$ Set P = False

Third: Checking for success or fail.

Unit Propagation → Applying unit resolution in chain

I: A set of literals either present in unit clauses or were derived from Δ using unit resolution

Γ : $\Delta \mid I$

$\Delta = \{\{\neg A, \neg B\}, \{B, C\}, \{\neg C, D\}, \{A\}\}$ $I = \{A, \neg B, C, D\}$

$\Delta \mid I = \Gamma = \{\}$ $\Delta = \{\{\neg A, \neg B\}, \{B, C\}, \{\neg C, D\}, \{C\}\}$

$I = \{C, D\}$ $\Delta \mid I = \Gamma = \{\{\neg A, \neg B\}\}$

```
DPLL(Δ)
    (I, Γ) = UNIT-PROPAGATION(Δ)
    if Γ = {} then
        return I
    else if {} ∈ Γ then
        return UNSAT
    else
        Choose a literal X in Γ
        if L = DPLL(Γ|X) ≠ UNSAT then
            return L ∪ I ∪ {X}
        else L = DPLL(Γ|¬X) ≠ UNSAT then
            return L ∪ I ∪ {¬X}
    else
        return UNSAT
```

SAT by Search + Inference

GRASP, MiniSAT, zChaff

Limitation of DPLL

Chronological Backtracking:

→ try both values of V at level l.

→ trace contradiction?

↳ move to l-1, undo intermediate assignments.

→ try a new value at l-1

→ for l-2 ... So on

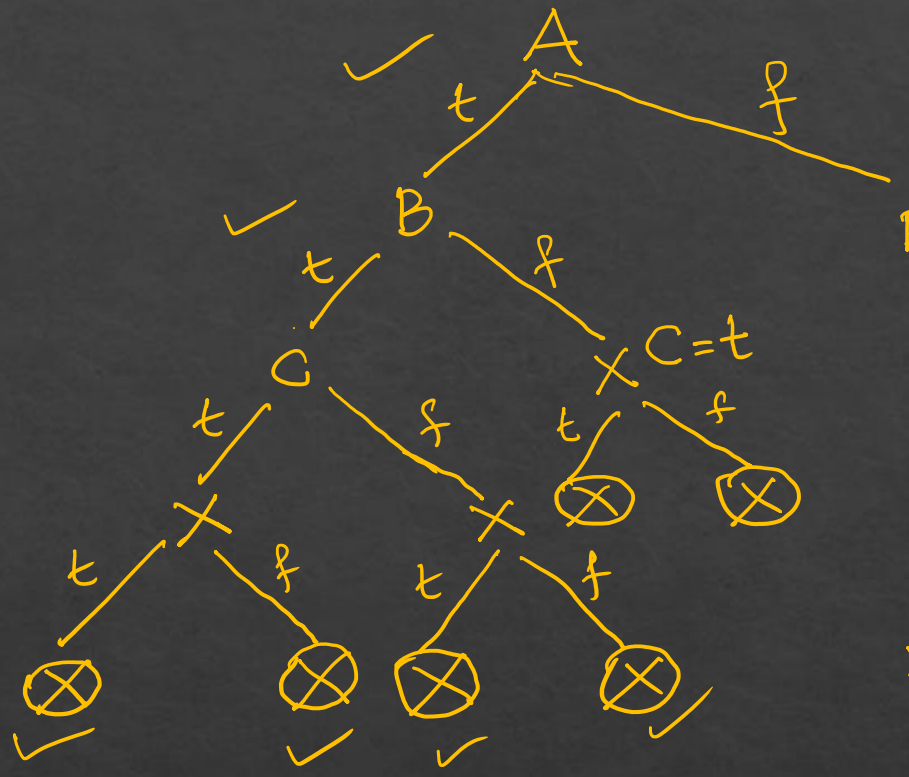
Final: move to level 0, always a contradiction

→ inconsistent

Not keeping track of the reason behind contradiction

Backtracking at level l is done after trying out all the possibilities at l+1

$$\Delta = \begin{cases} 1.\ \{A, B\} \\ 2.\ \{B, C\} \\ 3.\ \{\neg A, \neg X, Y\} \\ 4.\ \{\neg A, X, Z\} \\ 5.\ \{\neg A, \neg Y, Z\} \\ 6.\ \{\neg A, X, \neg Z\} \\ 7.\ \{\neg A, \neg Y, \neg Z\} \end{cases}$$



A=t

B=t { }

6

$\Delta|_{A,B,\neg C,X}$ $\Delta|_{A,B,\neg X}$

$\Delta|_{A,B}$

x=f : $\neg X$ –

4′ → {z} → 4″ →

6′4″ → {x} –

{ { } }

A=t : 3 → {¬x, y} → 3′

4 → {x, z} → 4′

5 → {¬y, z} → 5′ –

6 → {x, ¬z} – 6′ →

7 → {¬y, ¬z} – 7′

B=t, C=t

x=t : 3′ → {x} → 3″

5′3″ → {z} → 53″

7′3″ → {¬z} → 73″

53″ 73″ → { { } }

$\Delta|_{A,B,C,X}$ and $\Delta|_{A,B,C,\neg X}$ $\Delta|_{A,B,C}$

DPLL cannot detect

→ All the contradictions are at level 0 A = true

→ Assignment of B, C does not any effect → irrelevant

→ Chronological BT does not see it

→ Realization that A = T is a bad decision is achieved after 6 contradiction

Non-chronological BT

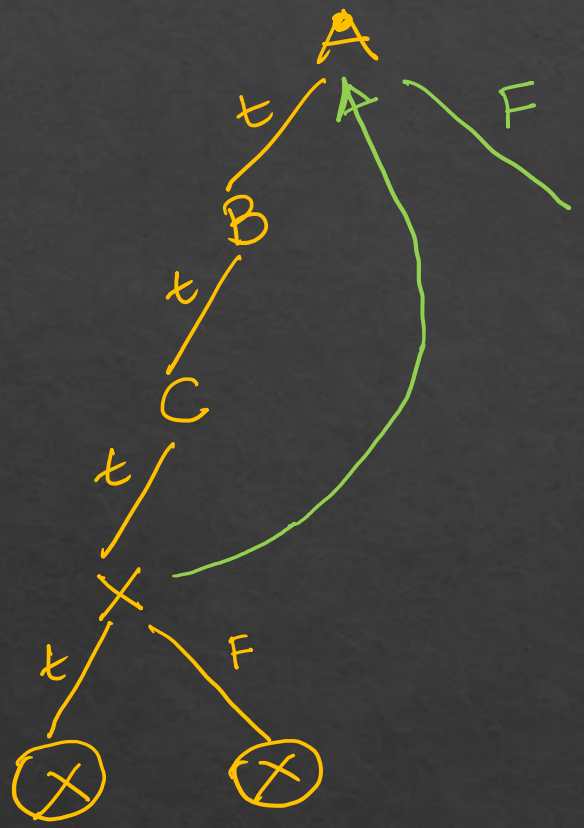
→ Probe the set of vars that actually are involved in contradiction (A)

→ technique for CSP (Backjumping)

Philosophy: Backtracking to a lower level l, without trying out every possibility between the current level and l

Intuitions: → identify every assignment that lead to contradiction
↳ set of assignments (conflict set)

→ backtrack to the most recent decision var in the conflict set



SEARCH

$A=T, B=T, C=T, X=T \Rightarrow Y=T, Z=T$

$CS1 = \{A=T, X=T, Y=T, Z=T\}$

$A=T, B=T, C=T, X=F \Rightarrow Z=T$

$CS2 = \{A=T, X=F, Z=T\}$

$A=F$

$A=F$

After 2 contradictions $\rightarrow$ $A=T$ is a bad decision

Empowering Clause: → unit Resolution DPLL

— $\alpha : \Delta \models \alpha$

— If we add α to Δ explicitly, unit resolution will have additional info

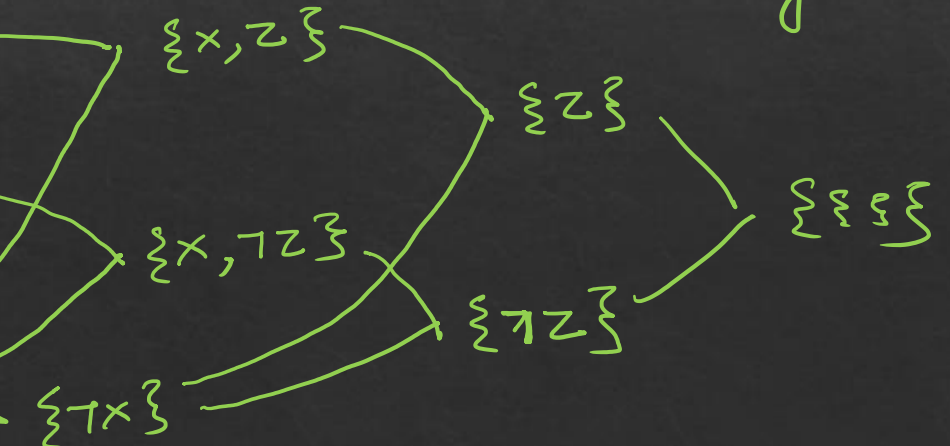
— $\Delta \models \{\neg A, \neg x\} \rightarrow$.

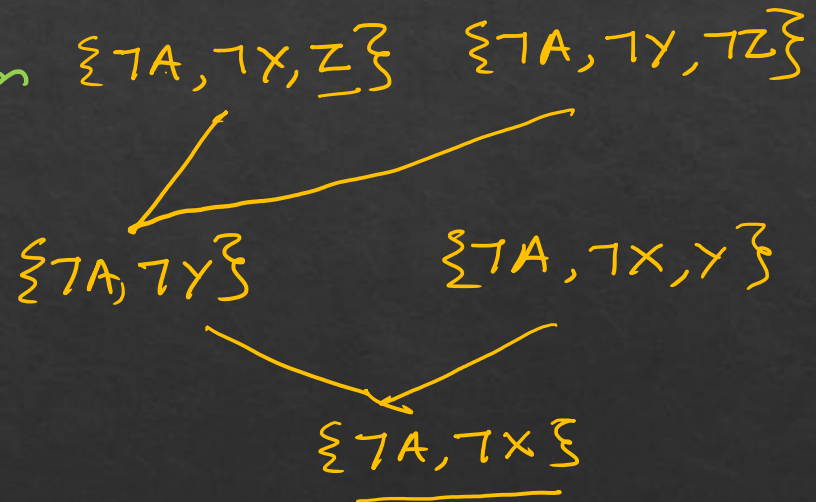
Empowering ⇒ unit resolution to detect contradiction early

1. $\{A, B\}$
2. $\{B, C\}$
3. $\{\neg A, \neg x, y\}$
4. $\{\neg A, x, z\}$
5. $\{\neg A, \neg x, z\}$
6. $\{\neg A, x, \neg z\}$
7. $\{\neg A, \neg y, \neg z\}$

---

8. $\{\neg A, \neg x\}$
9. $\{A\}$





How to identify empowering clauses?

→ Whenever UR finds a contradiction
  ↳ There is an opportunity to discover an empowering clause (new)
    ↳ allow new implications.

→ Conflict driven clauses or Conflict clause

Add these clauses → allow UR to detect contradictions earlier.

NCB and Conflic driven clauses (CDC)

→ Derive CDCs

→ Use these clauses in performing NCB.

Conflict Analysis

Δ

1. {A, B}
2. {C, D}
3. {¬A, ¬X, Y}
4. {¬A, X, Z}
5. {¬A, ¬Y, Z}
6. {¬A, X, ¬Z}
7. {¬A, ¬Y, ¬Z}

8. {¬A, ¬X}

Trace of unit resolution over Δ (Implication Graph)

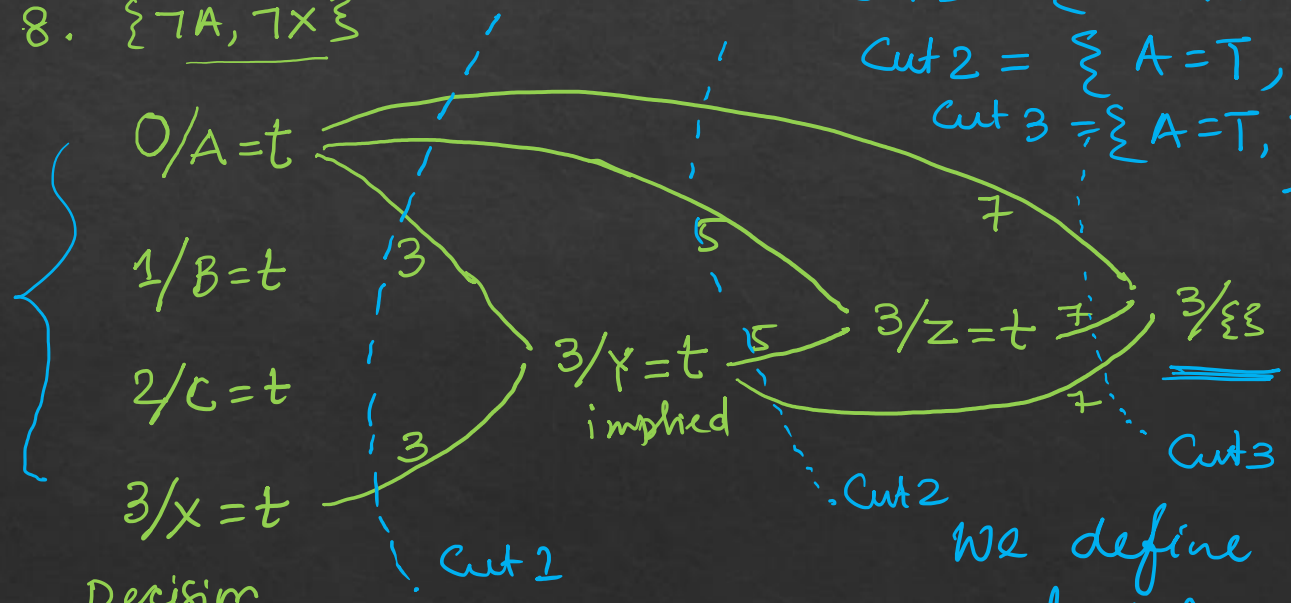
[Δ | A, B, C, X]

Decision node | implied node

Cut 1 = {A = T, X = T}

Cut 2 = {A = T, Y = T}

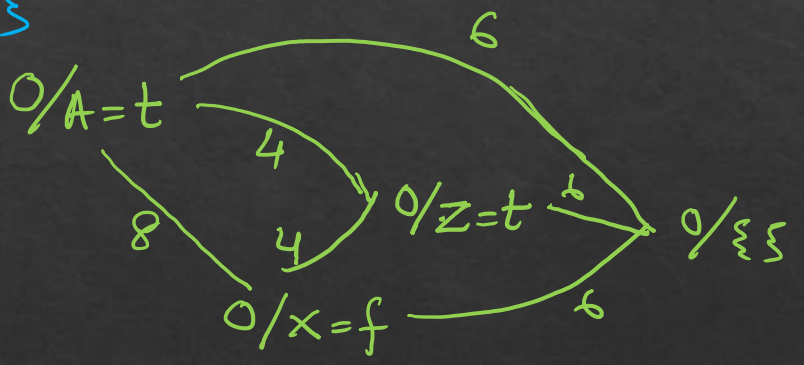
Cut 3 = {A = T, Y = T, Z = T}



We define cuts that seperates decision vars. from the conflict node

Power of the empowerment clauses

[Δ | A]



$$\left\{\begin{array}{l} Cut_1 = \{A=T,\ X=T\} \\ Cut_2 = \{A=T,\ Y=T\} \\ Cut_3 = \{A=T,\ Y=T,\ Z=T\} \end{array}\right\} \quad \text{Contradiction} \rightarrow \text{Conflict sets}$$

$\underline{A=T \wedge X=T}$ → Conflict set

↳ Adding this to Δ will lead to a contradiction.

$\alpha = \neg(A=T \wedge X=T)$ $\quad \Delta \models \alpha$

↳ $(\neg A \vee \neg X)$ → Add this to Δ

↳ learned Clauses

↳ conflict driven Clauses

Multiplicity of CDC

How to choose?

Asserting clause: Includes only one variable assignment at the last decision level

✓ $\{A^0 = T, X^3 = T\}$ ✓ $\{A^0 = T, Y^3 = T\}$

$\{A^0 = T, Y^3 = T, Z^3 = T\}$ ✗

Assertion level: second highest level in the clause

$\{A^0 = T, X^3 = T\}$ $\{A^0 = T, Y^3 = T\}$

⇓ ⇓

0 0

Backtracking to the assertion level

Modern SAT Solver

1. Identify Conflict driven clauses (asserting)
2. Backtrack to the assertion level
3. Add the CDC to Δ
4. Apply unit Resolution.

DPLL+(CNF Δ) returns SAT or UNSAT

$D \leftarrow ()$ // empty decision seq.

$\Gamma \leftarrow \{\}$ // learned clauses

While true do

If UR detects a contra in $(\Delta + \Gamma|_D)$:

Contradiction in original Δ ⟵ if $D = ()$: // contradiction w/o decision

return UNSAT

else // backtracking to AL

$\alpha \leftarrow$ asserting clause

$m \leftarrow$ AL of α

$D \leftarrow$ first m decision in D // reusing decisions l_{m+1} onward.

adding α to Γ

else // unit resolution does not lead to contra

if L is literal L or $\neg L$ ~~⊢~~ (Δ, Γ, D):
unit resolution

$D \leftarrow D; L$ ⟶ Decision.

else

return SAT

decide on variables that are still unassigned

1: {A, B}

2. {B, C} ✓

3. {¬A, ¬X, Y} ✓

→ 4. {¬A, X, Z} ✓

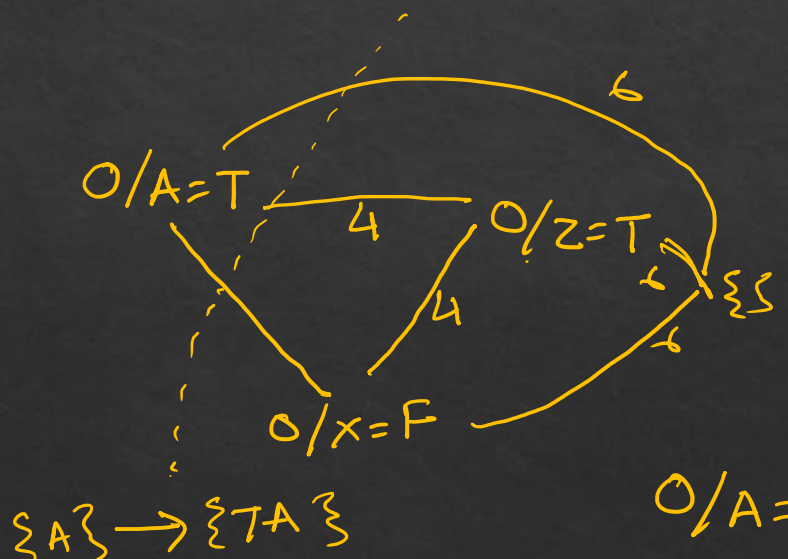
5. {¬A, ¬Y, Z} ✓

6. {¬A, X, ¬Z} ✓

7. {¬A, ¬Y, ¬Z} ✓

---

8. {¬A, ¬X}



Iteration 1:

$D = ( )$, $\Gamma = \{\ \}$

$D = (A=T, B=T, C=T, X=T)$

$\alpha = \{\neg A, \neg X\}$

$m = 0$

$D = (A=T)$, $\Gamma = \{\{\neg A, \neg X\}\}$ $\Delta' = \Delta + \Gamma$

Iteration 2

Δ, $D = (A=F)$ $\Gamma = \{\{\neg A, \neg X$

$\alpha = \{\neg A\}$ (0)

$m = -1$

$D = ( )$

$\Gamma = \{\{\neg A, \neg X\}, \{\neg A\}\}$ ↓

$A = F$

Iteration

$O/A=F$

1

$O/B=T$

SAT

$\boxed{A=F,\ B=T}$