Multivariate Analysis.

If $\sum |x_i| f(x_i) < \infty$ or $\int |x_i| f(x_i)\, dx < \infty$ then the expectation of X is defined as **absolute** summablity / Integrability.

$$E(X) = \begin{cases} \sum_i x_i f(x_i) & \text{if } X \text{ is discrete.} \\ \int_x x f(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

If we rearrange the series / integration then we may get different / inconsistent out comes.

we don't want in $E(X)$

$$\left\{ \begin{array}{l} 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} \cdots\cdots = \log_e 2 . \\ \underbrace{\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots\right)}_{\infty} - \underbrace{\left(\tfrac{1}{2} + \tfrac{1}{4} + \cdots\cdots\right)}_{\infty} \end{array} \right.$$

$$\Sigma_x = D(\underset{\sim}{X}) = \left(\left(\text{cov}(X_i, X_j) \right)\right)_{ij} = \left(\left(\underbrace{E(X_i X_j) - E(X_i)E(X_j)} \right)\right)_{ij}$$

$i = 1, 2 \cdots n$

$j = 1, 2, \cdots n.$

$E(\underset{\sim}{X}) = (\underset{\sim}{\mu})$

$$= E\left[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T \right]$$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & \cdot & \cdot & \cdot \\ x_2 & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot \end{array}$$

HW $\quad = E(\underset{\sim}{X} X^T) - \underset{\sim}{\mu}\underset{\sim}{\mu}^T.$

$E(\underset{\sim}{x}+\underset{\sim}{b}) = E(\underset{\sim}{x}) + \underset{\sim}{b} \rightarrow$ location equivariante.

$D(\underset{\sim}{x}+\underset{\sim}{b}) = D(\underset{\sim}{x}) \rightarrow$ Location invariante.

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot (v_1\ v_2\ v_3)$ 2×3

$cov(u_i, v_j)$

$E(A\underset{\sim}{x}) = [\underset{\sim}{a}_{\cdot 1}\ \underset{\sim}{a}_{\cdot 2} \cdots \underset{\sim}{a}_{\cdot n}]\,\underset{\sim}{x}$ — colums.

$= E\begin{bmatrix} a_{1\cdot}\,\underset{\sim}{x} \\ a_{2\cdot}\,\underset{\sim}{x} \\ \vdots \\ a_{n\cdot}\,\underset{\sim}{x} \end{bmatrix}$ $a_{i\cdot}$ is the ith row.

$= \begin{bmatrix} E(a_{1\cdot}\,\underset{\sim}{x}) \\ E(a_{2\cdot}\,\underset{\sim}{x}) \\ \vdots \\ E(a_{n\cdot}\,\underset{\sim}{x}) \end{bmatrix}$

$E(a_{i\cdot}\,x)$

$= E\left(\sum_{j=1}^{n} a_{ij}\,x_j\right)$

$= \sum_{j=1}^{n} a_{ij}\,\mu_j$

$= a_{i\cdot}\,\underset{\sim}{\mu}$

$= \begin{pmatrix} \underset{\sim}{a}_{1\cdot}\,\underset{\sim}{\mu} \\ \underset{\sim}{a}_{2\cdot}\,\underset{\sim}{\mu} \\ \vdots \\ \underset{\sim}{a}_{n\cdot}\,\underset{\sim}{\mu} \end{pmatrix}$

$= A\underset{\sim}{\mu}$

If $A = \ell^T$

then $E(\underset{\sim}{\ell}^T\underset{\sim}{x}) = \underset{\sim}{\ell}^T\underset{\sim}{\mu}$

$(1\ 1\ 1)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \sum\sum$

$= a + b + c + d + e + f + g + h + i$

$E(x) = \mu, \quad D(\underset{\sim}{x})$

To show $D(x) = \Sigma$ is p.s.d.

$\Leftrightarrow \underset{\sim}{a}^T \Sigma \underset{\sim}{a} \geq 0 \quad \forall\ \underset{\sim}{a} \neq \underset{\sim}{0}$.

$\Leftrightarrow D(\underset{\sim}{a}^T\underset{\sim}{x}) \geq 0$

$\Leftrightarrow Var(\underset{\sim}{a}^T\underset{\sim}{x}) \geq 0$

$\Sigma = D(x)$ is p.s.d and symmetric.

(15)

$E(X) = \mu$, $D(\underset{\sim}{X}) = \Sigma$,

$$E(X^T A X) = tr(\Sigma A) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}$$

$E(X_i) = \mu_i$, $V(X_i) = \sigma_i^2$

Case 1. $A = I_n$.

$$\Rightarrow E(\underset{\sim}{x}^T I_n \underset{\sim}{x}) = E(\underset{\sim}{x}^T \underset{\sim}{x}) = E\left(\sum_{i=1}^{n} x_i^2\right)$$

$$= \sum_{i=1}^{n} E(x_i^2) = \sum_{i=1}^{n} (\sigma_i^2 + \mu_i^2).$$

$$= \sum_{i=1}^{n} \sigma_i^2 + \sum_{i=1}^{n} \mu_i^2$$

$$= tr(\Sigma) + \underset{\sim}{\mu}^T \underset{\sim}{\mu}$$

$$= tr(\Sigma I) + \underset{\sim}{\mu}^T I \underset{\sim}{\mu}$$

$$E(\underset{\sim}{X}^T A \underset{\sim}{X}).$$

$$= E[tr(X^T A X.)]$$

$$= E[tr(A X X^T)]$$

$$= tr[A\, E(X X^T)]$$

$$= tr[A(\Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T)]$$

$$= tr(A\Sigma) + tr(A \underset{\sim}{\mu}\underset{\sim}{\mu}^T)$$

$$= tr(\Sigma A) + tr(\underset{\sim}{\mu}^T A \underset{\sim}{\mu})$$

$$= tr(\Sigma A) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}.$$

$D(\underset{\sim}{X}) = \Sigma$

$$\Rightarrow E(X X^T) - \underset{\sim}{\mu}\underset{\sim}{\mu}^T = \Sigma$$

$$\Rightarrow E(X X^T) = \Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T$$

$\sigma_{ij} = cov(X_i X_j)$

$$= E(X_i X_j) - \mu_i \mu_j$$

If A is p.s.d.

$\underset{\sim}{\mu}^T A \underset{\sim}{\mu} \geq 0$ NOTE.

Case 2. If $x_1 x_2 \cdots x_n \overset{iid}{\sim} N(0,1)$.

$$E(\underset{\sim}{X}) = \underset{\sim}{0} \quad D(\underset{\sim}{X}) = I_n.$$

$$E(X^T X) = E\left(\sum x_i^2\right) = n$$

$$\sum_{i=1}^{n} X_i^2 \sim \chi^2_n \equiv G\left(\tfrac{n}{2}, \tfrac{1}{2}\right)$$

$$\qquad\qquad \alpha \qquad \lambda$$

$$E(\chi^2_n) = \alpha/\lambda = \frac{n/2}{1/2} = n.$$

$$E\,tr(X^T A X)$$

$$= E\left(tr \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j\right)$$

$$= \sum_i \sum_j a_{ij} E(x_i x_j)$$

$$= \sum_i \sum_j a_{ij} \left(cov(x_i x_j) + \mu_i \mu_j \right).$$

Th. $\underset{\sim}{X} \in \mathbb{R}^n$, $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $D(\underset{\sim}{X}) = \Sigma$, To show $P(\underset{\sim}{X} - \underset{\sim}{\mu} \in \mathcal{C}(\Sigma)) = 1$ (13)

It is equivalent to show that. $\underset{\sim}{l} \in (\mathcal{C}(\Sigma))^\perp$ then $\underset{\sim}{l} \perp (\underset{\sim}{X} - \underset{\sim}{\mu})$

$$\underset{\sim}{l} \in (\mathcal{C}(\Sigma))^\perp.$$

$$\Leftrightarrow \quad \underset{\sim}{l}^T \Sigma = \underset{\sim}{0}^T$$

$$\Leftrightarrow \quad \underset{\sim}{l}^T \Sigma \underset{\sim}{l} = \underset{\sim}{0}^T \underset{\sim}{l}.$$

$$\Leftrightarrow \quad \mathrm{Var}(\underset{\sim}{l}^T \underset{\sim}{X}) = 0.$$

$$\Leftrightarrow \quad \mathrm{Var}(\underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu})) = 0.$$

$$\begin{cases} E(\underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu})) = 0 \\ \mathrm{Var}(\underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu})) = 0. \end{cases}$$

$$Y = \underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu})$$

$$E(Y) = 0 \quad \mathrm{Var}(Y) = 0.$$

$$P(Y = 0) = 1.$$

$$\Rightarrow P(\underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu}) = 0) = 1$$

$$\Rightarrow P(\underset{\sim}{l}^T \perp (\underset{\sim}{X} - \underset{\sim}{\mu})) = 1.$$

$$E(\underset{\sim}{l}^T (\underset{\sim}{X} - \underset{\sim}{\mu}))$$

$$= E(\underset{\sim}{l}^T \underset{\sim}{X}) - E(\underset{\sim}{l}^T \underset{\sim}{\mu})$$

$$= \underset{\sim}{l}^T \underset{\sim}{\mu} - \underset{\sim}{l}^T \underset{\sim}{\mu} = \underline{0}$$

$$\boxed{\begin{array}{c} X - \mu = \underline{k\sigma}. \\ \text{in } \underline{\text{1-D}}. \end{array}}$$

Normal distribute.

If $x_1, x_2 \ldots x_n$ are normally distributed then.

$\sum_{i=1}^{n} c_i x_i$ is also normally distributed for any $c_i \in \mathbb{R}$. $i=1,2,\ldots n$.

If $\sum c_i x_i$ is normally distributed for any $c_i \in \mathbb{R}$ $i=1,2\ldots n$

then x_i's are ~~no~~ normally distribed.

$$f(x) = \frac{\exp\left\{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})\right\}}{(\sqrt{2\pi})^n\sqrt{|\Sigma|}}. \qquad |\Sigma| \neq 0.$$

$$\rho = \frac{\mathrm{cov}(x_1 x_2)}{\sqrt{v(x_1)v(x_2)}} = \frac{\mathrm{cov}(x_1,x_2)}{\sigma_1\sigma_2}$$

Find the expression of Bivariate normal

$$(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$$f(x_1,x_2) = \frac{\exp\left\{\frac{-1}{2}\frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}}{2\pi\ \sigma_1\sigma_2\sqrt{1-\rho^2}}$$

H.W

Marginal distn. of X_1

$X_1 \sim N(\mu_1, \sigma_1^2)$ Marginal distn of X_2.

$X_2 \sim N(\mu_2, \sigma_2^2)$

$$\underset{\sim}{\mu} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$$

$$\Sigma = \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}$$

$$X_1 = \underset{\sim}{l}^T \underset{\sim}{X} = \begin{pmatrix}1\\0\end{pmatrix}^T\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$E(x_1) = \underset{\sim}{l}^T\underset{\sim}{\mu} = \mu_1$

$V(x_1) = \underset{\sim}{l}^T\Sigma\underset{\sim}{l} = \sigma_1^2$

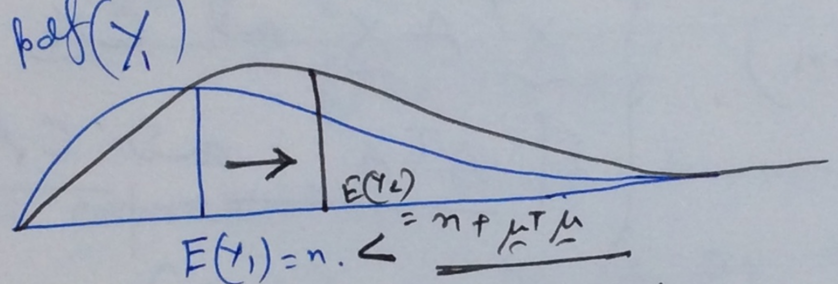
15

Case 1: $x_i \overset{iid}{\sim} N(0,1) \iff \underset{\sim}{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N(\underset{\sim}{0}, I_n)$

then $Y_1 = x^T x = \sum_{i=1}^{n} x_i^2 \sim \chi^2_{n,\ ncp=0}$

Case 2: $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ then $Y_2 = X^T X = \sum_{i=1}^{n} x_i^2 \sim \chi^2_{df=n,\ ncp=\mu^T\mu}$

$E(\underset{\sim}{x}) = \underset{\sim}{\mu}$. $\quad E(\underset{\sim}{X}^T \underset{\sim}{X}) = n + \mu^T\mu$. $\quad$ If $\underset{\sim}{\mu} \neq \underset{\sim}{0}$. then $\mu^T\mu > 0$.



Large values of test statistic $X^T X$ can lead us to reject Ho in favour of H_1.

Test: $H_0: \underset{\sim}{\mu} = \underset{\sim}{0}$ (null) vs $H_1: \underset{\sim}{\mu} \neq \underset{\sim}{0}$ (alternative).

Key construction of ANOVA. (Analysis of variance).

$$\underset{\sim}{A} = I_3 = A_1 + A_2 \equiv \underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{A_1} + \underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{A_2}.$$

$X^T A X$

$X^T A_1 \underset{\sim}{x}$ $\quad$ $x^T A_2 \underset{\sim}{x}$

$X \sim N(\mu, I_3)$ $\quad$ $\underset{\sim}{x}^T A_1 \underset{\sim}{x}$ and $\underset{\sim}{X}^T A_2 \underset{\sim}{x}$ are independent.

p.s.d. then $\underset{\sim}{x}^T A_2 \underset{\sim}{x} \sim \chi^2$

$$A = \sum_{i=1}^{k} A_i$$

$$\underline{x}^T A \underline{x} = \sum_{i=1}^{k} \underline{x}^T A_i \underline{x}$$

$$\left.\right\} \quad \underline{x}^T A_i \underline{x} \sim \chi^2_{df = \text{rank}(A_i),\ ncp = \underline{\mu}^T A_i \underline{\mu}}$$

$\underline{A^T = A}$ and $\underline{A_i^2 = A}$.

$X \sim N(\mu, I_n)$ $\left\{ \begin{array}{l} \underline{x}^T A \underline{x} \text{ and } C\underline{x} \text{ are indepedent.} \\ \text{iff. } \underline{A^T = A}, \text{ and } \underline{CA = \underset{\sim}{0}} \end{array} \right.$

(matrix)

$$Z \sim N(0,1), \quad Y \sim \chi^2_K \quad \text{independent.}$$

$$T = \frac{Z}{\sqrt{Y/K}} \sim t_K$$

$$Y_1 \sim \chi^2_{n_1}, \quad Y_2 \sim \chi^2_{n_2} \quad \text{independent.}$$

$$F = \frac{Y_1/n_1}{Y_2/n_2} \sim F_{n_1, n_2}.$$

$$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \sigma^2 I_n)$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \underset{\sim}{\mu} = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \mu \underset{\sim}{1}, \quad \Sigma = \sigma^2 I_n.$$

<u>standardization.</u>

As $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\Rightarrow \bar{X} - \mu \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1). \qquad (1)$$

<u>distn. of sample mean</u>

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$= \left(\frac{1}{n}\right) \underset{\sim}{1}^T \underset{\sim}{X}. \qquad \underset{\sim}{l} = (1/n)\underset{\sim}{1}$$

$$\equiv \underset{\sim}{l}^T \underset{\sim}{X}.$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$E(\underset{\sim}{l}^T \underset{\sim}{X}) = \underset{\sim}{l}^T \underset{\sim}{\mu}$$

$$= 1/n \, \underset{\sim}{1}^T \mu \cdot \underset{\sim}{1}$$

$$= \frac{\mu}{n} \underset{\sim}{1}^T \underset{\sim}{1}$$

$$= \mu.$$

$$V(\underset{\sim}{l}^T x) = \underset{\sim}{l}^T \Sigma \underset{\sim}{l}$$

$$= \sigma^2 \left(\frac{1}{n}\right)^2 \left(\underset{\sim}{1}^T I_n \underset{\sim}{1}\right)$$

$$= \sigma^2 \frac{1}{n^2} \cdot n$$

$$= \frac{\sigma^2}{n}.$$

(18)

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$$ Can we express S^2 as a quadratic from.

$$= \underset{\sim}{x}^T A \underset{\sim}{x}.$$

$$S^2 = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2$$

$$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2$$ as $\sum_1^n x_i = n\bar{x}$

$$= \sum_1^n x_i^2 - n\bar{x}^2$$ as $\underset{\sim}{1}^T \underset{\sim}{x} = n\bar{x}$

$$= \underset{\sim}{x}^T \left(I_n - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\underset{\sim}{x}.$$ as. $\underset{\sim}{x}^T \underset{\sim}{1} = n\bar{x}$

$$A = \left(I_n - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$$

$\left\{ \frac{1}{n} 1 1^T \right.$ is also square symmetric idempotent matrix.

$$A = A^T \quad \text{(symmetric).}$$

$$A^2 = \left(I_n - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\left(I_n - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$$

$$= I_n - 2\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T + \frac{1}{n^2} 1(\underset{\sim}{1}^T\underset{\sim}{1})1^T.$$

$$= I_n - 2\underset{\sim}{1}\underset{\sim}{1}^T + \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T$$

$$= A \quad \text{(idempotent).}$$

$$\text{rank}(A) = (n-1).$$

$$I_n = \left(I_n - \frac{1}{n} 1 1^T\right) + \left(\frac{1}{n} 1 1^T\right)$$

$\downarrow$ rank(n) $\quad\downarrow$ rank(n-1). $\quad\downarrow$ rank(1)

$$\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \sigma^2 I_n).$$

$$\underset{\sim}{w} = \left(\frac{\underset{\sim}{X}}{\sigma}\right) \sim N\left(\left(\frac{\underset{\sim}{\mu}}{\sigma}\right), I_n\right).$$

$$S^2 = \underset{\sim}{x}^T A \underset{\sim}{x}$$

$$= \sigma^2 \left(\frac{\underset{\sim}{x}^T}{\sigma}\right) A \left(\frac{\underset{\sim}{x}}{\sigma}\right).$$

$$= \sigma^2 (\underset{\sim}{w}^T A \underset{\sim}{w}.)$$

$$= \sigma^2 \chi^2_{df = n-1}$$

$$ncp = \frac{\mu^T}{\sigma} A \frac{\mu^T}{\sigma} = 0$$

$$S^2 \sim \sigma^2 \chi^2_{n-1,\, ncp = 0.}$$

$\underset{\sim}{l}^T A$

$\underset{\sim}{l} = \frac{1}{n} \underset{\sim}{1}$

$A = \left(I_n - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}^T\right)$

$\frac{1}{n} \underset{\sim}{1}^T \left(I_n - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}^T\right)$

$1/n\, \underset{\sim}{1}^T - \frac{1}{n^2} (\underset{\sim}{1}^T \underset{\sim}{1}) \underset{\sim}{1}$

$\frac{1}{n} \underset{\sim}{1}^T - \frac{1}{n} \underset{\sim}{1}^T = \underset{\sim}{0}^T.$

$\underset{\sim}{l}^T \underset{\sim}{X} = \bar{X}.$

$\Rightarrow X^T A X = S^2.$

are independently distributed.

$$T = \frac{\sqrt{n}(\bar{X} - \mu)/\sigma}{\sqrt{\left(S^2/\sigma^2\right) \cdot \left(\frac{1}{n-1}\right)}} = \frac{\sqrt{n}\,(\bar{X} - \mu)}{\sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}} \sim t_{n-1,\ ncp=0}$$

$$T^2 = \frac{\left(\frac{\sqrt{n}\,(\bar{x} - \mu)}{\sigma}\right)^2 / 1}{\frac{1}{n-1}\ \frac{S^2}{\sigma^2}} = \frac{n\ (\bar{X} - \mu)^2 / 1}{\frac{1}{n-1}\ S^2} \sim F_{1, n-1}$$