Predicates & Quantifiers

- **predicate**: refers to a property that a subject of the statement can have.

Example: " x is greater than 3" → denote by $P(x)$

→ $P(4)$ true
→ $P(2)$ false

the variable x is the subject of the statement

"is greater than 3" is the predicate.

- $P(x)$ is a statement, P is also called propositional fn. of x.
- $P(x)$ becomes a proposition once a value has been assigned to the variable x, and has a truth value.

~~n-ary~~ (Generalization)

- n-ary predicate → $P(x_1, x_2, \ldots, x_n)$, a statement involving n variables $x_1, x_2, \ldots, x_n$.
  ~~n-place predicate.~~
  - P → a propositional fn.
  - $P(x_1, x_2, \ldots, x_n)$ → value of the propositional fn. P at the n-tuple $(x_1, x_2, \ldots, x_n)$.

Example: Let $R(x, y, z)$ denote the statement

" $x + y = z$ ".

What are the truth values of the propositions $R(1,2,3)$ & $R(0,0,1)$?

→ $R(1,2,3)$ true as $1+2=3$ ~~holds~~

→ $R(0,0,1)$ false as $0+0 \neq 1$.

- Propositional fu^{ns}. occur in computer programs.

Example: if $x > 0$ then $x := x + 1$.

$P(x)$ is the statement "$x > 0$"

- for a particular value of x, if $P(x)$ is true, then the assignment $x := x + 1$ is executed ~~for x~~, so the value of x is incremented by 1.
- if $P(x)$ is false for this value of x, then the assignment is not executed, so the value of x is not changed

- ~~Preconditions~~
- Predicates are also used in the verification that computer program always produce the desired output when given valid input.
  - Preconditions: The statements that describe valid input
  - Post conditions: The conditions that the output should satisfy when the program has run.

Example:

temp := x
x := y
y := temp.

// interchange the values of two variables x & y

find the predicates that we can use as the precondition and the post condition to verify the correctness of this program.

predicate $P(x, y)$: "$x = a$ and $y = b$" → precondition
predicate $Q(x, y)$: "$x = b$ and $y = a$" → postcondition

<u>Explain how to use them to verify that all valid input of the program</u> does what is intended.

Let $P(x,y)$ is the Statement " $x=a$ and $y=b$ ".

Let $P(x,y)$ is true.

i.e. $x=a$, $y=b$ holds.

So

~~$temp := a$~~
~~$x := b$~~
~~$y := a$~~

$P(x,y)$ true ~~$x := a$ temp :=~~

$\Downarrow$

$temp := x$
$x := y$
$y := temp.$

$x=a,\ y=b,\ temp=a$

$y=b,\ temp=a,\ x=b$

$temp=a,\ x=b,\ y=a$

$\Downarrow$

$Q(x,y)$ is true.

- Propositional fu^n. $P(x)$

  $\downarrow$ assign a value to x

  a proposition with certain truth value.

<u>Quantifiers</u> → another means to create a proposition from a propositional fu^n.

→ a predicate is true over a range of elements.

→

<u>Example:</u> Universal quantifier : $\forall x\, P(x)$

Existential quantifier : $\exists x\, P(x)$

Uniqueness quantifier : $\exists! x\, P(x)$

" there are exactly two "

" There are no more than three "

" there are at least 100 "

So on ...

• Predicate logic & predicate calculus

- The area of logic that deals with predicates & quantifiers is called predicate calculus.
- Predicate logic is a powerful type of logic
  - used to express the meaning of a wide range of statements in mathematics & computer science
  - permits us to reason & explore relationship between objects.

Statement	When true?	When false?
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x.

Note: (i) - Domain of discourse / domain / Universe of discourse is assumed to be non-empty for quantifiers.

(ii) - If the domain is empty, then $\forall x\, P(x)$ is true for all propositional fun. $P(x)$ because there are no element x in the domain for which $P(x)$ is false.

- The domain must always be specified when a statement $\exists x\, P(x)$ is used.
- Without specifying the domain, the statement $\exists x\, P(x)$ is meaningless.
- If the domain is empty, then $\exists x\, P(x)$ is false whenever $P(x)$ is a propositional fun. because when the domain is empty, then there can be no element

Example: $P(x) = $ "$x+1 > x$".

What is the truth value of $\forall x P(x)$, where the domain consists of all real numbers.

Ans. $\forall x P(x)$ is true, $\exists x P(x)$

Example: $Q(x) = $ "$x < 2$", domain consists of all real nos.

Ans. (1) $\forall x Q(x)$ false as $Q(3)$ is false
$x = 3$ is counter example for the statement $\forall x Q(x)$.

(2) $\exists x Q(x)$ true as $Q(1)$ is true

Note: When all the elements in the domain of disclosure can be listed → say, $x_1, x_2, \ldots, x_n$

then i) $\forall x P(x)$ is the same as the conjunction
$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$. [as it is true iff $P(x_1), \ldots, P(x_n)$ all true]

ii) $\exists P(x)$ is the same as the disjunction
$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$. [as it is true iff at least one of $P(x_1), \ldots, P(x_n)$ is true]

Example: $P(x) = $ "$x^2 > 10$", domain consists of the +ve integers not exceeding 4.

1) $\forall x P(x)$ is the same as the conjunction
$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ false as $P(1)$ is false ($P(2), P(3)$ also false)

2) $\exists x P(x)$ is the same as the disjunction
$P(1) \vee P(2) \vee P(3) \vee P(4)$ true as $P(4)$ is true.

Example: (Quantifiers with Restricted domains):

i) $\forall x < 0\ (x^2 > 0)$, domain = Real nos.

"The square of a -ve real no. is positive"

↓ Same as the statement

$\forall x\ (x < 0 \rightarrow x^2 > 0)$.

ii) $\forall y \neq 0\ (y^3 \neq 0)$, domain = Real nos.

"The ~~cube~~ cube of every non-zero real no. is non-zero"

↓ Same as the statement

$\forall y\ (y \neq 0 \rightarrow y^3 \neq 0)$.

iii) $\exists z > 0\ (z^2 = 2)$, domain = Real nos.

"There is a +ve ~~real nos~~ square root of 2"

↓ Same as the statement

Lewis Carroll exa

$\exists z\ (z > 0 \wedge z^2 = 2)$.

Note: 1) Restriction of a universal quantification
→ Same as the universal quantification of a conditional statement

2) Restriction of an existential quantification
→ Same as the existential quantification of a conjunction.

Precedence of Quantifiers

- $\forall, \exists$ have higher precedence than all logical operations from propositional calculus.

Example:

$$\boxed{\begin{array}{ll} \forall x\, P(x) \vee Q(x) \text{ is } & (\forall x\, P(x)) \vee Q(x). \\ \text{rather than} & \forall x\, (P(x) \vee Q(x)) \end{array}}$$

Binding variables:

- a variable x is bound if a quantifier is used on x; otherwise x is said to be free.

- all variables in a propositional fn. must be bound or set equal to a particular value to turn it into a proposition.

- a combination of universal quantifier, existential quantifier, and value assignment can be used ~~for this~~ to get a proposition from a propositional fn.

- Scope of a quantifier is the part of a logical expression to which this quantifier is applied.
- a variable is free if it is outside the scope of all quantifier in the formula that specifies this variable.

Example:

a) $\exists x\ (x+y=1)$
→ x is bound by the existential quantifier $\exists x$
→ y is free

b) $\exists x\ (P(x) \wedge Q(x)) \vee \forall x\, R(x)$
→ The scope of the first quantifier $\exists x$, is the expression $P(x) \wedge Q(x)$
→ The scope of the 2nd quantifier, $\forall x$, is the expression $R(x)$.

Logical equivalence involving Quantifiers

Defⁿ: Statements involving predicates & quantifiers are logically equivalent iff they have the same truth value no matter which predicates are substituted into these statements and which domain of disclosure is used for the variables in these propositional fuⁿs.

- $S \Leftrightarrow T$ indicates that two statements S & T involving predicates & quantifiers are logically equivalent.

Examples i) $\forall x \left(P(x) \wedge Q(x)\right) \Leftrightarrow \forall x P(x) \wedge \forall x Q(x)$

(Exercise) ii) $\exists x \left(P(x) \vee Q(x)\right) \Leftrightarrow \exists x P(x) \vee \exists x Q(x)$

i.e. we can distribute a universal quantifier over a conjunction & an existential quantifier over a disjunction.

Note: We cannot distribute $\exists$ over disjunction ($\wedge$) & we cannot distribute $\forall$ over conjunction ($\vee$)

~~Proof of (i)~~: To show $S = \forall x \left(P(x) \wedge Q(x)\right)$

and $T = \forall x P(x) \wedge \forall x Q(x)$

are logically equivalent, we must show that S & T take the same truth value, no matter what the predicates P & Q are, and no matter which domain of disclosure is used.

Suppose we have particular predicates P and Q, with a common domain.

We will show the following:

a) if $\overset{S}{\cancel{\forall x\,(P(x) \wedge Q(x))}}$ is true, then $\cancel{\forall x}$ T is true, and

b) if T is true, then S is true.

(a) Let $\underset{\forall x\,(P(x) \wedge Q(x))}{S}$ be true.

Then if a is in the domain, $P(a) \wedge Q(a)$ is true.

So $P(a)$ is true & $Q(a)$ is true

As $P(a)$, $Q(a)$ true for every a in the domain, we conclude that $\forall x P(x)$ & $\forall x\, Q(x)$ true

This means T is true.

(b) Let $\underset{\forall x\, P(x) \wedge \forall x\, Q(x)}{T}$ be true

It follows that $\forall x\, P(x)$ & $\forall x\, Q(x)$ true

Hence if a is in the domain, then $P(a)$, $Q(a)$ both true

It follows that for all a, $P(a) \wedge Q(a)$ is true

Hence $\forall x\,(P(x) \wedge Q(x))$ is true, meaning S is true

Combining (a) & (b), we get $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

<u>Negating Quantified Expressions</u> (De Morgan's laws for quantifiers).

$$\boxed{\begin{array}{ll} \text{i)} & \neg \forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x) \\ \text{ii)} & \neg \exists x\, P(x) \Leftrightarrow \forall x\, \neg P(x) \end{array}}$$ (Exercise)

Example: (Illustration of De Morgan's laws for quantifiers)

"Every Student in you class has taken a course in calculus."

$P(x)$ = "x has taken a course in calculus"

domain = the students in your class

$\forall x\, P(x)$.

negation

"It is not the case that every student in your class has taken a course in calculus."

equivalent to

"There is a student in your class who has not taken a course in calculus."

$\exists x \neg P(x)$

- $\neg \forall x P(x) \iff \forall x \neg P(x)$

proof. $\neg \forall x P(x)$ true iff $\forall x P(x)$ false

iff $\exists$ an element x in the domain for which $P(x)$ false

iff $\exists$ an element x in the domain for which $\neg P(x)$ true

iff $\exists x \neg P(x)$.

- $\neg \exists x\, Q(x) \iff \forall x\, \neg Q(x)$.

proof

$\neg \exists x\, Q(x)$ true iff $\exists x\, Q(x)$ false

iff no x exists in the domain for which $Q(x)$ is true

iff for every x in the domain, $Q(x)$ is false

iff $\forall x\, \neg Q(x)$.

Negation	equivalent Statement	When Is Negation True?	When false?
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	for every x, $P(x)$ is false	there is an x for which $P(x)$ is true.
$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x.

Note: - When the domain of a predicate $P(x)$ consists of n elements, where n is a +ve integer, the rules for ~~negation~~ negating quantified statements are exactly the same as De Morgan's laws.

- When the domain has n elements, $x_1, x_2, \ldots, x_n$, then $\neg \forall x\, P(x)$ is same as $\neg [P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)]$ which is equivalent to $[\neg P(x_1)] \vee [\neg P(x_2)] \vee \cdots \vee \neg [P(x_n)]$ by De Morgan's law, which is the same as $\exists x\, \neg P(x)$.

- Similarly, $\neg \exists P(x)$ is the same as $\neg [P(x_1) \vee \cdots \vee P(x_n)]$, which is the same as $[\neg P(x_1)] \wedge \cdots \wedge [\neg P(x_n)]$ & this is the same as $\neg P(x$ [illegible]

Example: 1) $\forall x\,(x^2 > x)$, negation is $\exists x\,(x^2 \leq x)$.

2) $\exists x\,(x^2 = 2)$, negation is $\forall x\, \neg(x^2 = 2)$

Example:

$$\neg \forall x\,[P(x) \rightarrow Q(x)] \Leftrightarrow \exists x\,[P(x) \wedge \neg Q(x)]$$

proof:

$$\neg \forall x\,[P(x) \rightarrow Q(x)] \Leftrightarrow \exists x\, \neg[P(x) \rightarrow Q(x)]$$

$$\Leftrightarrow \exists x\,[P(x) \wedge \neg Q(x)]$$

[using $\boxed{\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q}$ negation implication

proof:

$$\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \vee q)$$

$$\Leftrightarrow (\neg\neg p) \wedge (\neg q)$$

$$\Leftrightarrow p \wedge \neg q$$

using $\boxed{p \rightarrow q \Leftrightarrow \neg p \vee q}$

proof:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Translating from English into logical expression.

Example:

1) "Every Student in this class has Studied calculus."

$$\cancel{P(x)} = \boxed{\forall x \, S(x)}$$

When $S(x)$ = x has Studied calculus

domain = ~~in the~~ the Students in this class.

- if interested in the background of people in subjects besides calculus.

$$\boxed{\forall x \, Q(x, \text{calculus})}$$

When $Q(x,y)$ = x has Studied subject y

Change the domain

2) "for every ~~people~~ person x, if person x is a Student in this class then x has Studied calculus"

$A(x)$ = x is a student in this class

~~$Q(x)$ = x has studied calculus~~.

$$\boxed{\forall x \, (A(x) \rightarrow Q(x))}$$

or

$$\boxed{\forall x \, (A(x) \rightarrow Q(x, \text{calculus}))}$$

3) "Some Student in this class visited Maxico"

$$\boxed{\exists x \, M(x)}$$, $M(x)$ = x visited Maxico

domain = the Students in this class

4) "Every Student in this class has visited either Canada or Maxico".

$$\boxed{\forall x \, (M(x) \vee C(x))}$$, $M(x)$ = x visited Maxico

$C(x)$ = x visited Canada.

Two-variable/
Two-place predicate $W(x,y) = x$ visited country y

$$\boxed{\forall x \left(W(x, \text{Maxico}) \vee W(x, \text{canada}) \right)}.$$

x 5. "for a person x, if x is a student in this class, then x has visited Maxico or x has visited Canada."

- domain consiste of all people.

$S(x) = x$ is a student in this class.

$M(x) = x$ has visited Maxico

$C(x) = x$ has visited Canada.

$$\boxed{\forall x \left(S(x) \rightarrow M(x) \vee C(x) \right)}.$$

or

$$\boxed{\forall x \left(S(x) \rightarrow W(x, \text{Maxico}) \vee W(x, \text{Canada}) \right)}$$

X
6. "Every mail message larger than one megabyte will be compressed"

subject — mail message; predicate — larger than one megabyte; predicate — compressed

$S(m, y)$ = mail message m larger than y megabyte

$C(m)$ = mail message m is compressed.

$$\boxed{\forall m \; S(m, 1) \rightarrow C(m)}.$$

X
7. "If a user is active, at least one network link will be available."

$A(u)$ = user u is active

$S(n, x)$ = Network link n is in state x

$$\boxed{\exists u \; A(u) \rightarrow S(n, \text{available})}$$

Example: (Lewis Carroll)

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink cofee."

Domain $\rightarrow$ all creatures.

$P(x) = x$ is a lion

$Q(x) = x$ is fierce

$R(x) = x$ drinks coffee.

$\forall x \, (P(x) \rightarrow Q(x))$

$\exists x \, (P(x) \wedge \neg R(x)) \rightsquigarrow$ cannot be $\exists x \, (P(x) \rightarrow \neg R(x))$

$\exists x \, (Q(x) \wedge \neg R(x))$

cannot be written as

$\exists x \, (Q(x) \rightarrow \neg R(x))$

For a specific x

$P(x)$	$\neg R(x)$	$P(x) \rightarrow \neg R(x)$
T	T	T
T	F	F
F	T	T
F	F	T

$P(x) \rightarrow \neg R(x)$ true even when $P(x)$ false i.e. x is not a lion

So $\exists x \, (P(x) \rightarrow \neg R(x))$ is true as long as ~~x is~~ a there exists at least one creature which is not a lion even if every lion drinks coffee.

Example: (Argument).

premises
"All humming birds are richly colored".
"No large birds live on honey".
"Birds that do not live on honey are dull in colors".
not richly colored.

Conclusion "Hummingbirds are small"
not large

$P(x)$ = x is a hummingbird
$Q(x)$ = x is large
$R(x)$ = x lives on honey
$S(x)$ = x is richly colored.

C_1: $\forall x \; (\neg P(x) \vee S(x))$
C_2:

C_1: $\boxed{\forall x \; (P(x) \to S(x))}$
C_2: $\neg \exists x \; (Q(x) \wedge R(x))$ i.e. $\forall x \; \boxed{(\neg Q(x) \vee \neg R(x))}$
C_3: $\forall x \; (\neg R(x) \to \neg S(x))$
$\forall x \; (P(x) \to \neg Q(x))$

C_4: $\neg \forall x \; (P(x) \to \neg Q(x))$ (negation of conclusion)
C_5: $\exists x \; \neg(P(x) \to \neg Q(x))$.
C_6: $\neg(P(c) \to \neg Q(c))$, c is a particular element in the domain
C_7: $\neg(\neg P(c) \vee \neg Q(c))$
C_8: $P(c) \wedge Q(c)$
C_9: $\boxed{P(c)}$
C_{10}: $\boxed{Q(c)}$

C_1, C_9 Modus ponens
C_{11}: $S(c)$
C_{12}: $\neg R(c)$ (C_{10}, C_2 resolution)
C_3, C_{12} modus ponens
C_{13}: $\neg S(c)$
C_{13}, C_3 modus tollens
C_{14}: $\neg\neg R(c) = R$
C_{15}: $\square$ (C_{14}, C_{12} resolution)

Nested quantifiers

- Two quantifiers are nested if one is within the scope of the other.

e.g. $\boxed{\forall x\ \exists y\ (x+y=0)}$

- Everything within the scope of a quantifier can be thought of as a propositional funⁿ.

e.g. $\forall x\ \exists y\ (x+y)=0$ is same as $\forall x\ Q(x)$, where
$Q(x)$ is $\exists y\ P(x,y)$, where
$P(x,y)$ is $x+y=0$.

x **Example:** Assume that the domain for the variables x & y consists of all real nos. The statement

- $\forall x\ \forall y\ (x+y=y+x)$

~~which~~ This says that $x+y=y+x$ for all real nos. x & y.
This is commutative law for addition of real nos.

- $\forall x\ \exists y\ (x+y=0)$.

This says that for every real no. x, there is a real no. y s.t. $x+y=0$.
This states that every real no. has an additive inverse.

- $\forall x\ \forall y\ \forall z\ (x+(y+z))=((x+y)+z)$.

This is the associative law for addition of real nos.

Example: Translate into English the statement

$\boxed{\forall x\ \forall y\ ((x>0 \wedge y<0) \rightarrow (xy<0)),}$ domain of x,y all real nos

This says that "The product of a +ve real no. & a −ve real no. is −ve real no."

The order of quantifiers

Example: Let $P(x,y)$ be the statement "$x+y=y+x$". What are the truth values of the quantifications $\forall x \forall y\, P(x,y)$ and $\forall y \forall x\, P(x,y)$ when domain for all variables consists of all real nos.?

Sol^n^

$\forall x \forall y\, P(x,y)$ means "for all real no. x, for all real no. y, $x+y=y+x$"

true as addition is commutative for real nos.

same as

"for all real no. y, for all real no. x, $x+y=y+x$"

means

$\forall y \forall x\, P(x,y)$.

true

Note: Order of nested universal quantifier ~~without~~ in a statement without ~~any~~ other quantifiers can be ~~AS~~ changed without changing ~~the~~ meaning of a quantified statement.

Example: Let $Q(x,y)$ denote "$x+y=0$". What are the truth values of the quantifications $\exists y \forall x\, Q(x,y)$ and $\forall x \exists y\, Q(x,y)$, where the domain for all variables consists of all real nos.?

Sol^n^

- $\exists y \forall x\, Q(x,y)$

means

"There exists a real no. y such that for all real no. x, $Q(x,y)$ holds"

No matter what value of y is chosen, there is only one x for which $Q(x,y)$ is true i.e. $x+y=0$.

So $\exists y \forall x\, Q(x,y)$ is false.

- $\forall x \exists y\, Q(x,y)$ is true as "for all real x, $\exists$ a real y s.t. $x+y=0$" is true

Example: $P(x,y,z)$: "$x+y=z$", domain of x,y,z consists of all real nos.

- $\forall x \forall y \exists z\, P(x,y)$ true
- $\exists z \forall x \forall y\, P(x,y)$ false.

There exists real z such that for all real x & for all real y $x+y=z$ holds.

Which is not true as there exists no real z ~~for~~ ~~each~~ satisfying the eqn. $x+y=z$ for all real x & y.

Translating Mathematical Statements into Statements involving nested quantifiers

Example: "The sum of two +ve integers is always +ve"

$\forall x \forall y\, (x>0 \wedge y>0 \Rightarrow x+y>0)$, where domain for both the variables consists of all integers.

Example: "Every real no. except zero has a multiplicative inverse"

~~$\forall x \exists y\, (x \neq 0 \rightarrow xy =$~~

$\forall x\, (x \neq 0 \longrightarrow \exists y\, (xy=1))$, domain of x, y consists of all real nos.

Example: $\lim_{x \to a} f(x) = L$, express the ε-δ defn. of this limit using quantifiers.

Discuss limit does not exist Example here

$\forall \varepsilon \exists \delta \forall x\, (0 < |x-a| < \delta \longrightarrow |f(x)-L| < \varepsilon)$

domain of ε, δ consists of all +ve real nos. & x consists of all real nos.

Quantification of two variables

Statement	When true	When false
• $\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x,y for which $P(x,y)$ is false.
• $\forall x \exists y\, P(x,y)$	for every x, there is a y for which $P(x,y)$ is true	There is an x such that for all y, $P(x,y)$ is false.
• $\exists x \forall y\, P(x,y)$	There is an x such that $P(x,y)$ is true for all y	For every x, there is an y such that $P(x,y)$ is false
• $\exists x \exists y\, P(x,y)$ $\exists y \exists x\, P(x,y)$	There is a pair x,y for which $P(x,y)$ is true	for every pair x,y, $P(x,y)$ is false.

Translating from Nested Quantifiers into English.

Example: Translate the statement

$$\forall x\, (C(x) \vee \exists y\, (C(y) \wedge F(x,y)))$$

into English, when $C(x)$ is "x has a computer"
$F(x,y)$ is "x & y are friends",

and the domain of both x & y consists of all students in your school.

→ "Every student in your class has a computer or has a friend who has a computer."

Example: Translate the statement

$$\exists x\ \forall y\ \forall z\ \big(\big(F(x,y) \land F(x,z) \land (y \neq z)\big) \rightarrow \neg F(y,z)\big)$$

into English, where $F(a,b)$ means "a & b are friends" and the domain for x, y and z consists of all students in your school.

→ "There is a student in your school none of whose friends are also friends with each other."

Translating English Sentences into Logical Expressions

Example: Express the statement

"If a person is female and is a parent, then this person is someone's mother"

~~into~~ as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

- $F(x)$ = "x is female"
- $P(x)$ = "x is a parent"
- $M(x,y)$ = "x is a mother of y"

$$\forall x\big((F(x) \land P(x)) \rightarrow \exists y\ M(x,y)\big)$$

same as

$$\forall x\ \exists y\ \big((F(x) \land P(x)) \rightarrow M(x,y)\big).$$

<u>Example</u>: Express the statement

"Everyone has <u>exactly</u> one best friend"

as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Let $B(x,y)$ be the statement "y is the best friend of x"

$$\forall x \left(\exists y\, B(x,y) \wedge \left(\forall z\, (z \neq y) \rightarrow \neg B(x,z) \right) \right)$$

<u>Example</u>: Use quantifiers to express the statement

"<u>There is</u> a woman who has taken <u>a flight</u> on <u>every</u> airline in the world"

- $R(w,f,a)$ = "The woman w has taken a flight f on airline a".

$$\exists w \forall a \exists f \left(R(w,f,a) \right)$$

<u>or</u>

- Let $P(w,f)$ = "The woman w has taken a flight f"

  & $Q(f,a)$ = "f is a flight on airline a"

$$\exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right).$$

<u>Negating nested quantifiers</u>

$$\forall w \exists a \forall f \left(\neg P(w,f) \vee \neg Q(f,a) \right).$$

"For every woman there is an airline such that for all flights, that woman has not taken that flight or that flight is not on that airline"

Example: Use quantifiers & predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist.

Sol^n.

$\lim_{x \to a} f(x)$ does not exist

means

$\lim_{x \to a} f(x) \neq L$ for all real nos L

$\forall L \; \exists \varepsilon > 0 \; \forall \delta > 0$

$\exists x \; (0 < |x-a| < \delta \wedge |f(x) - L| \geq \varepsilon)$.

$\lim_{x \to a} f(x) = L$

$\Downarrow$

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x$

$(0 < |x-a| < \delta \rightarrow |f(x) - L| <$

negation

$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x$

$(0 < |x-a| < \delta \wedge |f(x) - L| \geq$

which is ~~the~~ ~~range~~

$\neg \, (0 < |x-a| < \delta \rightarrow |f(x) -$

(Using $\neg (p \to q) \Leftrightarrow p \wedge \neg q$)

Def^n (Argument) : An argument consists of a set of propositions $p_1, p_2, \ldots, p_n$, called premises, and a proposition q, called conclusion. ↳ (preceding statement)

An argument is valid iff the conclusion is true whenever the premises are all true.

* An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

Example:

All males are humans.	Major premise
All boys are males	Minor Premise
Therefore, all boys are humans.	Conclusion.

- The major premise → a general proposition, usually an implication.
- The minor premise → a more specific proposition

Determine, whether the above argument is valid or not.

Sol^n. Argument is valid iff the conclusion is true whenever the premises are all true.

Convert the argument into symbolic form

m denotes males
h denotes humans
b denotes boys

$m \to h$ major premise
$b \to m$ minor premise

$\therefore b \to h$ Conclusion

Truth table

m	h	b	$m \to h$	$b \to m$	$b \to h$
T	T	T	T	T	T ✓
T	T	F	T	T	T ✓
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T ✓
F	F	T	T	F	F
F	F	F	T	T	T

When we look at the rows where the premises $m \to h, b \to m$ are both true, we see that the conclusion $b \to h$ is also true.

Therefore, the argument is valid.

Def^n. (Fallacy) An argument is not valid is called a fallacy or an invalid argument. (i.e. all premises are true, conclusion false)

arg. that has an inherent fl… in the st. of the argument it… which renders the arg. invalid

Theorem: Suppose than an argument consists of the premises $p_1, p_2, \ldots, p_n$ and conclusion q. Then the argument is valid iff the proposition $[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$ is a tautology.

<u>proof.</u>

a) if an argument is valid, then we will show that the given proposition is a tautology; and

b) if the given ~~taut~~ proposition is a tautology, then the argument is valid.

<u>proof of (a).</u> Let the argument is valid, but $[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$ is <u>not</u> a tautology.

Then we must have the following truth table:

p_1	p_2	$\cdots$	p_n	q	$p_1 \wedge p_2 \wedge \cdots \wedge p_n$	$[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$
		$\cdots$				$\vdots$
? T	? T		? T	? F	? T	F
		$\cdots$				$\vdots$

all the premises $p_1, \cdots, p_n$ true, but the conclusion q is false contradicting the fact that the argument is valid.

Hence $[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$ must be a tautology.

<u>proof of (b)</u> Let ~~the~~ $[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$ is be a tautology, but the argument is not valid.

As argument is invalid, That means, we must have the following Truth table;

p_1	p_2	$\cdots$	p_n	q	$p_1 \wedge p_2 \wedge \cdots \wedge p_n$	$[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$
T	T	$\cdots$	T	F	? T	? F contradicts $[p_1 \wedge p_2 \wedge \cdots \wedge p_n] \to q$ is a tautology.

Hence the argument must be valid.