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The foundations: logic & proofs.

1. Discrete Mathematics & its applications
– Kenneth H Rosen. 1

Def^n^. (Proposition) A proposition is a declarative sentence that is either true or false, but not both.

Examples

a) John dislikes mathematics.

✓ b) Napoléon is dead.

c) $4+5 > 3$.

d) The world ~~is~~ will end on 4 March 2045.

- Interogative statements, exclamatory statements, imperative statements are not used in logic.

Examples (not propositions)

✓ a) Are you Indian?

b) Report to the Commandant's office.

c) Watch out!

d) $x+y=2$

✓ e) All hignals are prognals.
→ not a proposition as we do not know ~~the meaning of. hign~~ what hignals & prognals are.
→ can become a proposition if hignals & prognals are defined.

f) He enjoys mathematics.
→ not a proposition as the pronoun 'he' acts as a variable which can take values from a domain
→ replace 'he' by a suitable name, then proposition

✓g) $x+7=10$ → is not a proposition as the domain of x is not defined.

$x+7=10$, where $x=4$ → is a proposition (albeit a false one).

• Propositional calculus
- → area of logic that deals with propositions
- → first developed systematically by Aristotle (more than 2300 yrs. ago).
- → a system of manipulating symbols instead of manipulating words is devised by George Boole & John Venn (19th. Century, 1854).

Propositional calculus
- ↓ propositional variables / statement variables (p, q, r, s, ...)
- ↓ truth value (true or false) (T) (F) ↓ two labels

• When we evaluate the truth of a proposition, we assign one of the two labels, true or false
→ these labels are called truth values of a proposition.

proposition
- prime proposition (a simple statement that expresses a single complete thought, and whose truth value is true or false, but never both)
- Compound proposition (made of two or more propositions, & whose truth value is true or false, but never both).

Examples: Identify the prime propositions & the compound propositions.

prime (a) In 2015, Barack Obama was the president of the US

Compound (b) 5 is an even no. or 17 is an even no.

× (c) What time is it?

Compound (d) It is raining right now, and I'm wet.

× (e) Come to class!

Defn. (paradox): A paradox is a statement that apparently contradicts itself and yet might be true.

Example (Liar's paradox)

i) p : This proposition is false.

→ Trying to assign this statement as true or false leads to a contradiction.

~~→ if we assign the statement p true~~

→ if p is infact true, then the ~~statement~~ proposition is false

→ if p is assigned false, then the proposition is true.

ii) (Pinocchio paradox).

- Pinocchio is a wooden puppet who dreams of one day becoming a real boy.
- Pinocchio had ~~a n~~. Such a nose when he tells a lie, his nose grows.

- A paradox occurs when Pinocchio says
  p: "My nose is growing"

→ Pinocchio's nose is growing
iff
what he is saying is false
and Pinocchio's is saying p: "My nose is growing"

→ Pinocchio's nose is growing iff it is not growing.
(Veronique Eldridge-Smith, 2010 (when she was 11 yrs. old))

Exercise: Is Plato's proposition
"I know one thing: that I know nothing"
a paradox?
(sometimes it is called Socratic paradox).

- Russell's paradox.

$S = \{x \mid x \notin x\}$ → the set that contains a set x iff x does not belong to itself.

- Barbe's paradox An ancient Sicilian legend says that the barber in a remote town who can be reached only by travelling a dangereous mountains road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

- Logical connectives → logical operators used to combine ~~operators~~. propositions using a mathematical table, called a truth table.

1. Negation : We let $\neg p$ denote "not p"

p	$\neg p$
T	F
F	T

2. Conjunction: We let $p \wedge q$ denote "p and q"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction: We let $p \vee q$ denote "p or q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

~~Example:~~

4. Exclusive Disjunction (exclusive or) : We let $p \oplus q$ denote "p or q and not both"

$$p \oplus q = (p \vee q) \wedge (\neg(p \wedge q))$$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5. **Implication:** (Conditional Statement) We let $p \to q$ denote 'if p, then q'

$$p \to q \text{ is } \begin{cases} \text{false} & \text{when } p \text{ is true, but } q \text{ is false.} \\ \text{true} & \text{otherwise.} \end{cases}$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: (falling in Poison Ivy)

Consider the propositions,

p = I fell into Poison Ivy

q = I have a rash

The implication $p \to q$ would be

"if I fell in posion ivy then I will get a rash."

$\boxed{\text{if } p \text{ then } q}$

		$p \to q$
if q then	p (Row 1)	T
	not p (Row 3)	T
if not p then	q (Row 3)	T
	not q (Row 4)	T
if p then	q (Row 1)	T
	not q (Row 2)	F
if not q then	p	T
	$\neg p$	T

- From personal knowledge, this is infact true. after coming into contact with poison ivy (p true) one will always get a rash (q true) → Row 1
- if you have a rash (q true), but we didn't step into poison ivy, ~~then~~ (p false), then it is still possible → Row 2 (you may have gotten rash from something else)
- if you did not fall in ~~posi~~ poison ivy (p false) & you didn't get a rash (q false), then it also a possibility → Row 4.
- Only impossibility is when you come into contact with

poison ivy (p true) & you didn't get a rash (q false)
→ Row 2.

- for $p \to q$, this may be read as any of the following:

  a) if p, then q
  b) p implies q
  c) q, if p
  d) p is sufficient for q
  e) p, only if q
  f) q is necessary for p.
  etc.

$p \xrightarrow{\text{suff for } q} q_{\text{necessary for } p}$

- Think of the implication as cause → effect.
- for the implication $p \to q$, we call p as the ~~called~~ the { hypothesis or antecedent or premise.

  & q as the { conclusion or consequent.

- "q unless $\neg p$" is same as $p \to q$, always have the same truth value.

↓

- ~~q is true if $\neg p$ is false~~
  ~~i.e. if p is true~~
- ~~q is false if $\neg p$ is true~~
  ~~i.e. if p is false~~

~~The statement is true means~~
~~if $\neg p$ false, then q is true~~
~~The statement is false means~~
~~if $\neg p$ false, then q is ~~false~~ false~~
~~i.e. if p true, but q false~~

- "q unless $\neg p$" ^true^ means that → think it as a loop

if $\neg p$ false, then q must be true.

i.e. the statement "q unless $\neg p$" is false when

p is true, but q is false;

Otherwise, true always.

- Consequently, "q unless $\neg p$" and $p \to q$ always have the same truth table.

Example: Decide if the proposition is true or false:

"if the moon is made of cheese, then Socrates is the Prime Minister of Canada"

- the antecedent is false: the moon is ~~never~~ not made of cheese

- the consequent is also false: Socrates has been long dead before even the creation of Canada.

- Since both the antecedent & the consequent are false, the proposition must be true.

Note:

1. Propositional language is an artificial language
2. We use conditional statements of a more general sort than we use in ~~ordinary~~ English.
3. The mathematical concept of conditional statement is independent of a cause-and-effect relationship between hypothesis & conclusion.

4. Our defn. of conditional statement specifies its truth value; it is not based on English usage.

5. if-then construction used in many programming languages is different from that used in logic.

if p then S // p proposition, S program ~~statement~~ segment

in most programming languages executes S if p is true, but does not execute S if p is false.

Converse, contrapositive, & inverse (Derived logical implications).

Let us consider $p \to q$

- Then ~~$q \to p$ is~~ the converse of $p \to q$ is $q \to p$.
- The contrapositive of $p \to q$ is $\neg q \to \neg p$
- The inverse of $p \to q$ is $\neg p \to \neg q$

Note: 1. Only the contrapositive $\neg q \to \neg p$ has the same truth value as $p \to q$.

~~2. Converse~~

2. Neither the converse $(q \to p)$ nor the inverse $(\neg p \to \neg q)$, has the same truth value as $p \to q$ for all possible truth values of p & q.

✓ 3. A conditional statement & its contrapositive are equivalent

✓ 4. The converse & inverse of a conditional statement are equivalent.

- When two compound ~~statement~~ propositions always has the same truth value, we call them <u>equivalent</u> (will be formally defined later).
- The converse or inverse of a conditional statement is never equivalent to this conditional statement.

<u>Example:</u> Suppose we ~~condi~~ consider the following proposition:

"If a man can march, then he is a soldier."

State the contrapositive, the converse, and the inverse of the proposition.

<u>Sol^n.</u> p = A man can march

q = He is a soldier

$p \to q$

(a) (Contrapositive) $(\neg q \to \neg p)$ — if a man is not a soldier, then he can not march

(b) (Converse) $(q \to p)$ — if a man is a soldier, then he can march.

(c) (inverse) $(\neg p \to \neg q)$ — if a man cannot march, then he ~~cannot~~ is be a soldier.

Example: Consider Fermat's Little Theorem:

$\boxed{\text{" if } n \text{ is prime and } \gcd(a,n)=1 \text{, then } a^{n-1} \equiv 1 \pmod{n} \text{ "}}$

Soln Let $P = [\, n \text{ is prime and } \gcd(a,n)=1 \,]$

$Q = [\, a^{n-1} \equiv 1 \pmod{n} \,]$

$P \to Q$

a) **Contrapositive:** if $a^{n-1} \not\equiv 1 \pmod{n}$ and $\gcd(a,n)=1$, then n is composite

$\neg Q \to \neg P$

b) **Converse** if $a^{n-1} \equiv 1 \pmod{n}$ and $\gcd(a,n)=1$,

$Q \to P$ then n is prime

c) **Inverse** if n is composite & $\gcd(a,n)=1$,

$\neg P \to \neg Q$ then $a^{n-1} \not\equiv 1 \pmod{n}$.

Used to test primality of n.
(Fermat's primality test).

- equivalence $\Leftrightarrow$

$p \to q \Leftrightarrow \neg q \to \neg p$ (contrapositive)

$p \to q \not\Leftrightarrow q \to p$ (converse)

$p \to q \not\Leftrightarrow \neg p \to \neg q$ (inverse)

Biconditional : We let $p \leftrightarrow q$ denote "p if and only if q"
or Bi-implication

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $p \leftrightarrow q$ has the same truth value as
$$(p \to q) \wedge (q \to p)$$

Example: Consider the following statement :
" I am a member at the gym if and only if I have paid my gym fee."

p = I am a member at the gym
q = I have paid my gym fee

Truth table for $p \leftrightarrow q$??

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

- The biconditional $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.

$\therefore$ $p \leftrightarrow q$ may be read as

a) if $p_{(q)}$, then $q_{(p)}$, and conversely
b) $p_{(q)}$ is necessary & sufficient condition of $q(p)$
c) p iff q.

Example: (Truth table of compound propositions)

Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Order of precedence

Sometimes sentences have the form 'p and q', but in English, what it really mean is 'p and then q'

Example:

a) I studied for the exam, and I passed the exam.
b) I passed the exam, and studied for the exam.

p = I studied for the exam
q = I passed the exam

a) $p \rightarrow q$, b) $q \rightarrow p$ (not $p \wedge q$).

- Merely knowing that p is true & that q is true, one does not automatically know the order of the two events.
- You cannot study for the exam & pass the exam concurrently.
- You would study (p) and then write and pass the exam (q)
- The following list ~~dis~~ dictates the order of precedence for logical connectives.

$$(\ ), \neg, \wedge, \vee, \rightarrow, \leftrightarrow.$$

Thus, connectives inside of the parentheses ( ) should be applied first, followed by the connective $\neg$ next, and so on.

Example: a) $p \wedge q \vee r$ means $(p \wedge q) \vee r$

rather than $p \wedge (q \vee r)$

b) $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$

Example: Put the following into symbolic notation
"You can access the Internet from campus only if you are a compute science major or you are not a freshman."

let p = you can access the Internet from campus
q = you are a computer science major
r = you are a freshman

Converse of $(q \wedge \neg r) \rightarrow p$ i.e. $p \rightarrow (q \wedge \neg r)$.

Exercise: let p = I work hard

q = I pass

r = I get promoted.

Translate each of the following into symbolic form.

a) I get promoted, if I pass. $q \to r$

b) I work hard implies I pass. $p \to q$

c) Getting ~~pro~~ promoted is necessary for passing. $q \to r$

d) If I don't work hard, then I won't get promoted. $\neg p \to \neg r$

- **nand** $(\uparrow)$: another important logical connective to computer science

→ produces a truth value of true if <u>at least</u> one of the two propositions is false.

p	q	$p \uparrow q$	$p \land q$	$\neg(p \land q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T

$p \uparrow q \Leftrightarrow \neg(p \land q)$

Exercise: Put the following into ~~logical symbol~~ symbolic notation:

a) "Either the child is admitted to school immediately or he is sent home and notified of his admission."

p = the child is admitted to school immediately

q = the child is sent home & notified of his admission.

$p \oplus q$.

X **Example:** Rewrite each of the following propositions as an implication in the if-then form.

(a) Practicing her serve daily is a sufficient condition for Dercy to have a good chance of winning the tennis tournament.

p q $p \to q$

p is sufficient for q, $p \to q$, q is necessary for p

q is the necessary condition for p

(b) The computer is brand-new or I won't buy it.

Solⁿ.

a) If Dercy practices her serve daily, then she has a good chance of winning the tennis tournament.

b) If the computer is not brand-new, then I won't buy it.

X **Example:** Construct a truth table for the formula $\neg(\neg p \wedge q)$

p	q	$\neg p$	$\neg p \wedge q$	$\neg(\neg p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Propositional Equivalences

• **Tautology:** A proposition is said to be a tautology if it is true under all possible truth assignments of the variables contained.

Example: The proposition $(p \vee \neg p)$ is a tautology.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

• **Contradiction:** A proposition is said to be a contradiction if it is false under all possible truth assignments of the variables contained.

Example: The proposition $(p \wedge \neg p)$ is a contradiction

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

• **Contingency:** A proposition is said to be contingency if it is neither a tautology nor a contradiction.

Example: The proposition $(p \rightarrow q)$ is a contingency.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- <u>logical equivalence</u>: Two propositions, p and q, are logically equivalent iff $p \leftrightarrow q$ is a tautology.

We will denote logical equivalence by the meta-operator $\Leftrightarrow$ or $\equiv$.

<u>Note</u>: $\Leftrightarrow$ is not a logical connective & $p \Leftrightarrow q$ is not a compound proposition, but rather is ~~a~~ the statement that

$p \leftrightarrow q$ is a tautology.

- <u>Summary of logical connective meaning</u>

Name	Symbol	Meaning	Condition for a true truth value
Negation	$\neg p$	not p	p is false
Conjunction	$p \wedge q$	p and q	both p & q are true
Disjunction	$p \vee q$	p or q (or both)	at least one of p, q is true
Exclusive Disjunction	$p \oplus q$	p or q (not both)	either p is true or q is true, but not both.
Conditional	$p \rightarrow q$	p implies q	q is true whenever p is true
Biconditional	$p \leftrightarrow q$	p iff q	p & q have same truth value
nand	$p \uparrow q$	~~not~~ $\neg (p \wedge q)$	produces a truth value of true if at least ~~a~~ one of p & q is false.

Summary

- logic → the study of the methods & principles used in distinguishing correct from incorrect arguments.
- logical inference → the process of using the knowledge that we know, & drawing ~~a~~ conclusion.
- Truth value → either true or false
- To decide if a statement is a proposition, you must ask yourself if it is possible to assign a truth value. A proposition cannot be a question, an order, or an exclamation.
- prime proposition → a simple complete thought.
- compound proposition → made of two or more prime propositions.
- paradox → a statement that contradicts itself & yet might be true.
- Given an implication $p \to q$, then
  - the contrapositive is $\neg q \to \neg p$
  - the converse is $q \to p$
  - the inverse is $\neg p \to \neg q$.
- Necessary & Sufficient.
  a) "p is sufficient for q" means $p \to q$
  b) "p is necessary for q" means $q \to p$.
  c) If "p is necessary and sufficient for q", then $p \Leftrightarrow q$

~~NOR~~ NAND

not AND

- logical connective truth tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$	$p \uparrow q$
T	T	F	T	T	F	T	T	F
T	F	F	F	T	T	F	F	T
F	T	T	F	T	T	T	F	T
F	F	T	F	F	F	T	T	T

Determining whether two compound propositions are equivalent or not

Approach 1 : Use truth table

$p \Leftrightarrow q$ iff the column giving their truth values agree.

Approach 2 : Use the fundamental logical ~~identities~~ equivalences listed below & the fact that [$p \Leftrightarrow r$ and $r \Leftrightarrow q$ iff $p \Leftrightarrow q$]

Let p, q, r be any three propositions (prime or compound).

- (Idempotence) $p \wedge p \Leftrightarrow p$

  $p \vee p \Leftrightarrow p$.

- (Associativity) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

- (Commutativity) $p \wedge q \Leftrightarrow q \wedge p$

  $p \vee q \Leftrightarrow q \vee p$

  $p \leftrightarrow q \Leftrightarrow q \leftrightarrow p$.

3. (Distributivity) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

- (De Morgan) $\boxed{\neg (p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)}$

  $\neg (p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

- (Identity) $p \wedge T \Leftrightarrow p$

  $p \vee F \Leftrightarrow p$

- (Domination) $p \wedge F \Leftrightarrow F$

  $p \vee T \Leftrightarrow T$

- (Simplification) $((p \wedge q) \rightarrow p) \Leftrightarrow T$

  $((p \wedge q) \rightarrow q) \Leftrightarrow T$

- (Negation) $p \wedge (\neg p) \Leftrightarrow F$ (Contradiction)

  $p \vee (\neg p) \Leftrightarrow T$ (Excluded Middle).

- (Double Negation) $\neg(\neg p) \Leftrightarrow p$.

- (Absorption) $\boxed{p \vee (p \wedge q) \Leftrightarrow p}$

  $p \wedge (p \vee q) \Leftrightarrow p$

- (Importation) $[p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge q) \rightarrow r]$

- (Exportation) $[(p \wedge q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$

- (Simplification) $[(p \wedge q) \rightarrow p] \Leftrightarrow T$

  $[(p \wedge q) \rightarrow q] \Leftrightarrow T$.

LHS

$(p \vee p) \wedge (p \vee q)$

(Dist.)

$\Leftrightarrow p \wedge (p \vee q)$

(Idempotency

$\Leftrightarrow (p \wedge p) \vee (p \wedge q)$

(Dist)

$\Leftrightarrow$

Absorption $p \vee (p \wedge q) \Leftrightarrow p$.

$LHS = p \vee (p \wedge q)$

$= (p \wedge T) \vee (p \wedge q)$ [using Identity law]

$\cancel{= p \vee (T \wedge q)}$ [using Distributive law]

$\cancel{= p \vee}$

$= p \wedge (T \vee q)$

$= p \wedge T$ [Using Domination law].

$= p$ [Using Identity law].

De Morgan $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$.

p	q	$(p \vee q)$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The truth values of ~~the compous~~ $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ agree for all possible combination of the truth values of p and q.

So, it follows that $\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$ is a tautology & hence the compound propositions $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are logically equivalent.

Logical equivalences involving conditional statements

- $\boxed{p \to q \iff \neg p \vee q}$ (Implication) prove it using truth table
- ~~$p \to q \iff \neg q \to \neg q$~~
- $p \to q \iff [\neg q \to \neg p]$ (contrapositive)
- $\boxed{p \vee q \iff \neg p \to q}$
- $\boxed{p \wedge q \iff \neg[q \to \neg p]}$
- $\boxed{\neg(p \to q) \iff p \wedge \neg q}$ (Neg Implication)
- $(p \to q) \wedge (p \to r) \iff p \to (q \wedge r)$
- $(p \to r) \wedge (q \to r) \iff (p \vee q) \to r$
- $(p \to q) \vee (p \to r) \iff p \to (q \vee r)$
- $(p \to r) \vee (q \to r) \iff (p \wedge q) \to r$.

take $\neg$ on both sides

proof *

another proof

$\neg(p \to q)$
$\iff \neg(\neg p \vee q)$
$\iff p \wedge \neg q$

proof using -ve implication

$p \wedge q \iff \neg(p \to \neg q)$
$\iff \neg(\neg\neg q \to \neg p)$
$\iff \neg(q \to \neg p)$

change & implication equivalent

change imp. $\neg q \to \neg p$, $\neg p \to q$ contrapositive.

$*$ $P \Leftrightarrow Q$ mean $P \leftrightarrow Q \Leftrightarrow T$

contrapositive $\neg Q \leftrightarrow \neg P$

$T \Leftrightarrow (P \to Q) \wedge (Q \to P)$
$\Leftrightarrow (\neg Q \to \neg P) \wedge (\neg P \to \neg Q)$
$\Leftrightarrow \neg P \leftrightarrow \neg Q$
$\Leftrightarrow \neg(p \to q) \wedge \neg(\neg p \ldots)$

i.e. $\neg P \Leftrightarrow \neg Q$

Logical equivalences involving biconditionals

P-⟨I⟩ $p \leftrightarrow q \iff (p \to q) \wedge (q \to p)$

P-⟨II⟩ $p \leftrightarrow q \iff (\neg p \leftrightarrow \neg q)$ * (Contrapositive)

P-⟨III⟩ $\boxed{p \leftrightarrow q \iff (p \wedge q) \vee (\neg p \wedge \neg q)}$ proof

P-⟨IV⟩ $\boxed{\neg(p \leftrightarrow q) \iff [p \leftrightarrow \neg q]}$.

LHS $= p \leftrightarrow q$
$\iff (p \to q) \wedge (q \to p)$
$\iff (\neg p \vee q) \wedge (\neg q \vee p)$
$\iff (\neg p \wedge \neg q) \vee \underbrace{(\neg p \wedge p)}_{F} \vee \underbrace{(q \wedge \neg q)}_{F} \vee (q \wedge p)$
$=$ RHS

Note. De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) = (\neg p_1) \wedge (\neg p_2) \wedge \cdots \wedge (\neg p_n)$$

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) = (\neg p_1) \vee (\neg p_2) \vee \cdots \vee (\neg p_n).$$

Example: prove that

P IV : $\neg(p \leftrightarrow q) \Leftrightarrow [p \leftrightarrow \neg q]$

Sol^n. Using truth table :

p	q	$p \Leftrightarrow q$	$\neg(p \Leftrightarrow q)$	$\neg q$	$[p \Leftrightarrow \neg q]$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

Using logical equivalences :

$LHS = \neg(p \Leftrightarrow q) \Leftrightarrow \neg[(p \rightarrow q) \wedge (q \rightarrow p)]$

$\Leftrightarrow [\neg(p \rightarrow q)] \wedge [\neg(q \rightarrow p)]$ (Dist. law)

$\Leftrightarrow (p \wedge \neg q) \wedge (q \wedge \neg p)$

$RHS = (p \leftrightarrow \neg q) \Leftrightarrow (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$

$\Leftrightarrow (\neg p \vee \neg q) \wedge (\neg\neg q \wedge \neg p)$

$q \wedge p$.

$LHS = \neg(p \Leftrightarrow q) \Leftrightarrow \neg[(p \wedge q) \vee (\neg p \wedge \neg q)]$ [P-III]

$\Leftrightarrow [\neg(p \wedge q)] \wedge [\neg(\neg p \wedge \neg q)]$ [Dist]

[De Morgan]

$\Leftrightarrow (\neg p \vee \neg q) \wedge (p \wedge q)$.

$RHS = (p \Leftrightarrow \neg q) \Leftrightarrow (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$ (P-I)

$\Leftrightarrow (\neg p \wedge \neg q) \wedge (\neg(\neg q) \wedge p) = RHS$.