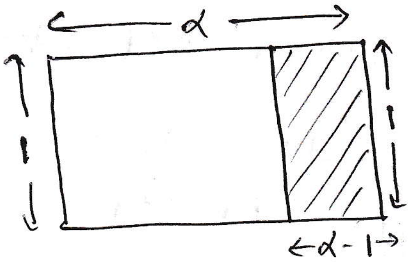
Example:

$$\alpha = \frac{1+\sqrt{5}}{2} \doteq 1.61803\ldots \quad \text{(golden ratio)}.$$

· if one starts with a $1 \times \alpha$ rectangle and removes a 1×1 rectangle from it, the remaining $\alpha - 1 \times 1$ rectangle has the same aspect ratio.



Rotate shadded sub-rectangl 90° and it has the same shape as the whole rectangle.

$$\frac{\alpha}{1} = \frac{1}{\alpha - 1} \Rightarrow \alpha^2 - \alpha - 1 = 0.$$

$$\alpha = \frac{1+\sqrt{5}}{2} \doteq 1.61803\ldots, \quad \beta = \frac{1-\sqrt{5}}{2} \doteq \cdot 61803$$

Exercise

$$\boxed{\alpha^{n-2} \leq F(n) \leq \alpha^{n-1}}$$

Exercise

Prove by induction that $\boxed{F(n) = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)},$

when $f(n)$ denotes the n-th fibonacci number.

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n \geq 2 \end{cases}$$

(verify $F(0)$, $F(1)$, use the identities $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$ to show the induction step).

Exercise $\forall n > 0$

$$\boxed{F(n) = \frac{1}{2^n\sqrt{5}}\left[\sum_{m=0}^{n}\binom{n}{m}(\sqrt{5})^m - \sum_{m=0}^{n}\binom{n}{m}(-\sqrt{5})^m\right] = \frac{1}{2^{n-1}}\sum_{m=0}^{\lfloor (n-1)/2 \rfloor}\binom{n}{2m+1}5^m.}$$

Exercise Prove ~~that~~ the matrix identity

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \quad \forall n \geq 1$$

$\to \log n$ $A^{\frac{n}{2}} A^{\frac{n}{2}}$

$T(n) = T(\frac{n}{2}) + O(1)$

$= \cdots$

$= \Theta(\log n)$

Take determinant

$$\Rightarrow \boxed{F_{n+1}F_{n-1} - F_n^2 = (-1)^n}$$

Another proof.
- Using substitution
- Using Binet formula.

$\Downarrow$

$\boxed{F_n, F_{n+1} \text{ relatively prime}}$ otherwise, a common divisor of F_n, F_{n+1} must divide $(-1)^n$ $(\to\leftarrow)$.

Another proof if $d \mid F_{n+1}, d \mid F_n$, then $d \mid F_{n-1}$
$d \mid F_n, d \mid F_{n-1} \Rightarrow d \mid F_{n-2}$
$\vdots$
$d \mid F_3, d \mid F_2 \Rightarrow d \mid F_1 = 1 \Rightarrow d = 1$

- $F_{n+1} = F_n + F_{n-1}$

$F_{n+2} = F_{n+1} + F_n$

$F_{n+3} = F_{n+2} + F_{n+1}$ $\cancel{2}F_{n+1} + F_n + F_n + F_{n-1} = $

$F_n \to 0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$= F_{n+1} + F_n + F_{n+1}$

$= 2F_{n+1} + F_n$

$F_{n+1} = F_{(n-1)+2}$
$= F_2 F_n + F_1 F_{n-1}$
~~$= F_n + F_{n-1}$~~

$F_{n+4} = \underset{F_3}{F_{n+3}} + F_{n+2}$

$= 2F_{n+1} + F_n + F_{n+1} + F_n$

$= 3F_{n+1} + 2F_n = F_4 F_{n+1} + F_3 F_n$

$F_{n+2} = F_{(n-1)+2}$
$= F_2 F_{n+1} + F_1 F_n$

by induction,

$\boxed{F_{n+m} = F_m F_{n+1} + F_{m-1} F_n}$ for any +ve integer m

- Take $m = n(k-1)$ a mul. of n $\to$ ~~$F_{nk} = F_{n(k-1)} F_{n+1} + F$~~

Generating fun. for the seq. $\{F_n\}$

→ clothesline on which we hang up a seq. of nos. for display

$$G(z) = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + \cdots$$

$$z G(z) = \quad F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + \cdots$$

$$z^2 G(z) = \quad F_0 z^2 + F_1 z^3 + F_2 z^4 + \cdots$$

$$(1 - z - z^2) G(z) = \overset{=0}{F_0} + (\overset{=1}{F_1 - F_0}) z + (\overset{=0}{F_2 - F_1 - F_0}) z^2 + (\overset{=0}{F_3 - F_2 - F_1}) z^3 + (\overset{=0}{F_4 - F_3 - F_2}) z^4 + \cdots$$

$$= z$$

$$\Rightarrow G(z) = \frac{z}{1 - z - z^2}, \text{ if exists.}$$

$$= \frac{z}{(1 - \alpha z)(1 - \beta z)}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \alpha z} - \frac{1}{1 - \beta z} \right)$$

$$= \frac{1}{\sqrt{5}} \left[(1 + \alpha z + \alpha^2 z^2 + \alpha^3 z^3 + \cdots) - (1 + \beta z + \beta^2 z^2 + \beta^3 z^3 + \cdots) \right]$$

Co-efficient of z^n in $G(z)$

$$\boxed{F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)}.$$

(Binet formula).

$1 - z - z^2 = 0$

⇕

$z^2 + z - 1 = 0$

$z = \frac{-1 \pm \sqrt{5}}{2}$

$= \frac{1}{\alpha}, \frac{1}{\beta}$

where $\alpha = \frac{1 + \sqrt{5}}{2}$ (golden ratio)

& $\beta = \frac{1 - \sqrt{5}}{2} = 1 - \alpha$

satisfying $x^2 = x + 1$

$$\frac{\cancel{1} - \beta z - \cancel{1} + \alpha z}{(1 - \alpha z)(1 - \beta z)}$$

$\alpha - \beta = \sqrt{5}$

∘ Another proof.

$$\boxed{f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}$$

- Observations.

1) $\alpha = \frac{1+\sqrt{5}}{2} = 1.61803$.

$0 < |\beta| < 1$

$|\beta^n| < |\beta|^n < 1$ for $n \geq 1$.

$\beta = 1 - \alpha = -.61803$, a ~~+~~ -ve number ~~whose ma~~ whose magnitude is less than unity.

$\Rightarrow$ β^n gets very small as n gets large

$\left|F_n - \frac{\alpha^n}{\sqrt{5}}\right| = \frac{|\beta^n|}{\sqrt{5}} < \frac{1}{\sqrt{5}} < \frac{1}{2}$

- In fact, $\frac{\beta^n}{\sqrt{5}}$ is always small enough so that we have

$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right]$, $n \geq 1$,

$F_n = \frac{1}{\sqrt{5}}\alpha^n$, rounded to the nearest integer

$[\;]$ greatest integer fun.

2) $G(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right)$.

$$G(z)^2 = \frac{1}{5}\left[\frac{1}{(1-\alpha z)^2} + \frac{1}{(1-\beta z)^2} - \frac{2}{1-z-z^2}\right]$$

co-efficient of z^n co-efficient of z^n

$$\sum_{k=0}^{n} f_k f_{n-k} \qquad \frac{1}{5}\left[(n+1)\alpha^n + (n+1)\beta^n - 2 f_{n+1}\right]$$

- $(1-\alpha z)^{-2} = 1 + 2\alpha z + 3\alpha^2 z^2 + 4\alpha^3 z^3 + \cdots + (n+1)\alpha^n z^n + \cdots$
- $(1-\beta z)^{-2} = 1 + 2\beta z + 3\beta^2 z^2 + 4\beta^3 z^3 + \cdots + (n+1)\beta^n z^n + \cdots$
- $\frac{1}{1-z-z^2} = \frac{G(z)}{z}$

$$G(z) = z(f_0 + f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots)$$
$$= z f_0 + f_1 z^2 + f_2 z^3 + f_3 z^4 + \cdots + F$$

$= \frac{1}{z}[f_0 + f_1 z + f_2 z^2 + \cdots]$

$= f_1 + f_2 z + f_3 z^2 + \cdots + f_{n+1} z^n + \cdots$

- $\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}\left[(n+1)(\alpha^n+\beta^n) - 2F_{n+1}\right]$.

- ~~α^n~~ Exercise (by induction)

$\alpha^n = F_n\alpha + F_{n-1}$

$\beta^n = F_n\beta + F_{n-1}$;

~~for~~ all integers n.

α, β roots of $x^2 - x - 1 = 0$

$\Rightarrow \alpha+\beta = 1$

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}\left[(n+1)\{F_n + 2F_{n-1}\} - 2F_{n+1}\right] \quad (\because \alpha+\beta=1)$$

$$= \frac{1}{5}\left[(n+1)F_n + 2(n+\cancel{1})F_{n-1} - 2(F_n + \cancel{F_{n-1}})\right]$$

$$= \frac{1}{5}(n-1)F_n + \frac{2n}{5}F_{n-1}.$$

- $\alpha^n = F_n\alpha + F_{n+1}$

proof.

$$F_n\alpha + F_{n-1} = \frac{\alpha^n - \beta^n}{\sqrt{5}}\alpha + \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}$$

$$= \frac{\alpha^{n+1} - \cancel{(\alpha\beta)}^{-1}\cancel{\beta^{n-1}} + \alpha^{n-1} - \cancel{\beta^{n-1}}}{\sqrt{5}}$$

$\alpha\beta = -1$

$$= \frac{\alpha^{n-1}(\alpha^2+1)}{\sqrt{5}} = \frac{\alpha^{n-1}\cdot\cancel{\sqrt{5}}\alpha}{\cancel{\sqrt{5}}}$$

$$= \alpha^n.$$

$\alpha = \frac{1+\sqrt{5}}{2}$

$\alpha^2+1 = \frac{1+5+2\sqrt{5}+4}{4}$

$= \frac{10+2\sqrt{5}}{4}$

$= \frac{5+\sqrt{5}}{2} = \sqrt{5}\cdot\frac{1+\sqrt{5}}{2} = \sqrt{5}\alpha$

Generating fun

The (ordinary) generating fun. for the seq. $\{a_k\}$ is defined to be

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$$

The sum is finite if the seq. is finite, infinite if the seq. is infinite.

↪ x is chosen so that the sum converges.

Example: (The number of labeled Graphs)

- for a fixed n, we let

$$a_k = L(n,k),\ k = 0, 1, \ldots, \binom{n}{2}$$

- n vertex
- e edges

$$L(n,e) = \binom{\binom{n}{2}}{e} = \binom{r}{e}$$

$\binom{n}{2} \to r$

$=$ # of labeled graph with n vertices & e edges.

$0 \le e \le \binom{n}{2} = r = \frac{n(n-1)}{2}$

- Consider the generating fun.

$$G_n(x) = \sum_{k=0}^{\binom{n}{2}} a_k x^k$$

$$= \sum_{k=0}^{\binom{n}{2}} \binom{r}{k} x^k$$

$$= (1+x)^{\binom{n}{2}}$$

↪ Simple way of summarize our knowledge of the nos. $L(n,e)$.

- $L(n)$ = # of labeled graphs of n vertices

$$= \binom{r}{0} + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r = 2^{\binom{n}{2}}.$$

$n = 3$

$L(3) = 2^{\frac{3 \cdot 2}{2}} = 8$

• # of labeled digraphs

n vertices $\rightarrow$ $n(n-1)$ possible arcs.

$M(n,a) \rightarrow$ # of labeled digraphs with n vertices & a arcs.

$$M(n,a) = \binom{n(n-1)}{a}, \quad 0 \leq a \leq n(n-1).$$

$\underline{n=3,\ a=4}$

$M(3,4) = \binom{6}{4} = \binom{6}{2} = \frac{6 \cdot 5}{\lfloor 4\ \lfloor 2} = 15$

$M(n) \rightarrow$ # of labeled digraphs with n vertices.

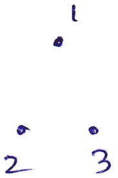
$$M(n) = \sum_{a=0}^{n(n-1)} \binom{n(n-1)}{a} \; 22^{n(n-1)}$$

$n(n-1) = 3.2 = 6$

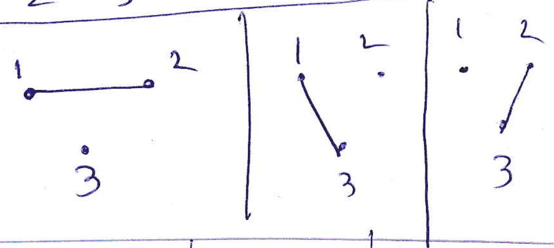
$\underline{n=3}$

$$M(3) = \sum_{a=0}^{6} \binom{6}{a} = 2^6 = 64.$$

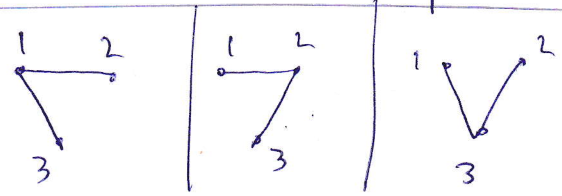
Labeled graphs with 3 vertices



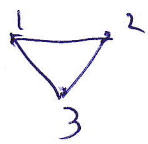
<u>0 edges</u> $L(3,0) = 1$



<u>1 edge</u> $L(3,1) = 3$



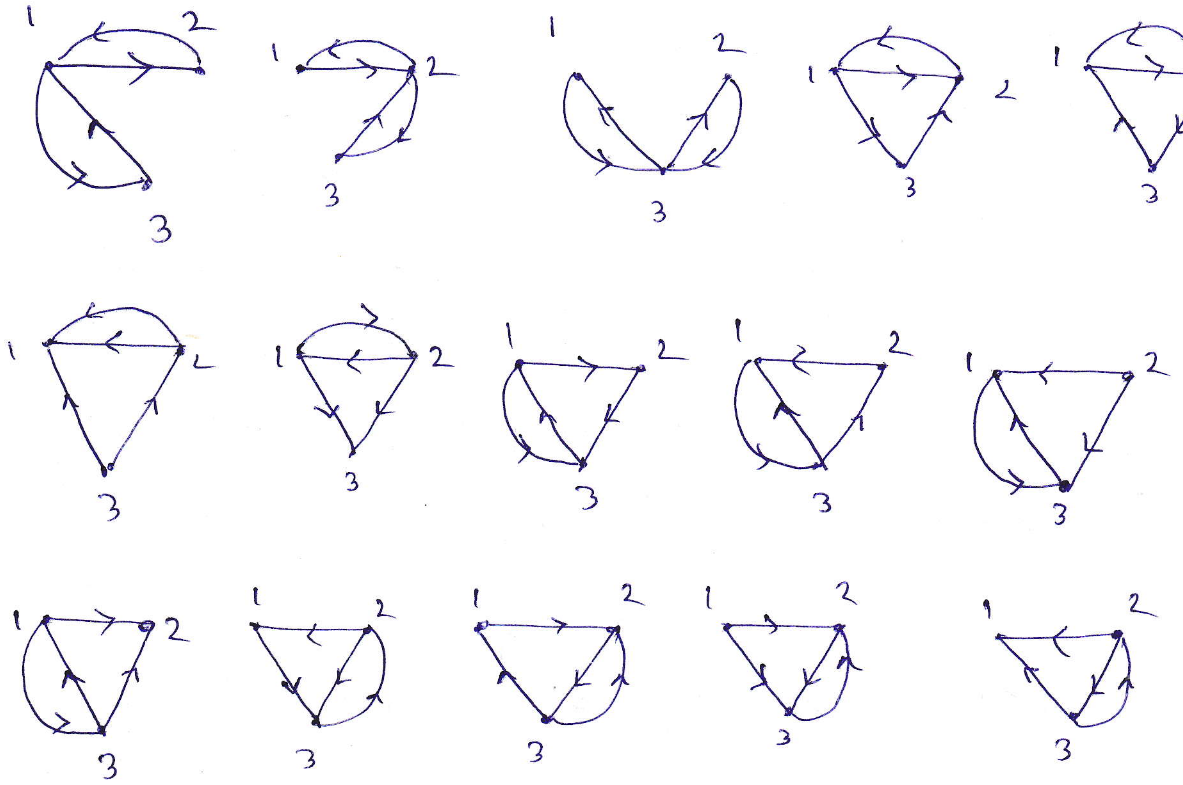
<u>2 edges</u> $L(3,2) = 3$



<u>3 edges</u> $L(3,3) = 1$

$M(3,4) = 15$

• # of distinct labeled trees of n vertices $\rightarrow n^{n-2}$ (Cayley). ③



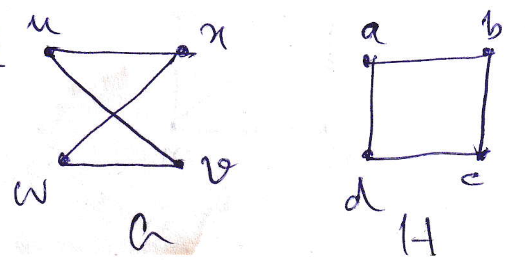
Isomorphism

• Two unlabeled graphs (digraphs) G & H, each having n vertices, are considered the same if the vertices of both can be labeled with integers $1, 2, \ldots, n$ so that the edge sets (arc sets) consist of the same unordered (ordered) pairs

i.e. if two graphs (digraphs) can be given a labeling that shows them to be the same as labeled graphs (digraphs).

• G, H are then said to be isomorphic.

Example:



G, H isomorphic as labeling

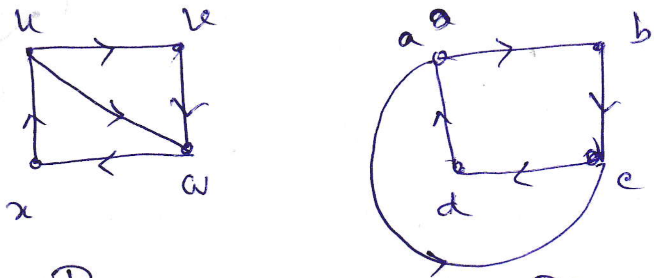
$u, a \rightarrow 1$

$v, b \rightarrow 2$

$w, c \rightarrow 3$

$x, d \rightarrow 4$

Example:



D_1 D_2

digraphs D_1, D_2 isomorphic.

labeling as
u, a → 1
v, b → 2
w, c → 3
x, d → 4

- It is easy to tell whether or not two labeled graphs or digraphs are the same.
- Determining whether or not two unlabeled graphs or digraphs are the same (isomorphic), is very difficult. (Isomorphism problem).

Naive approach

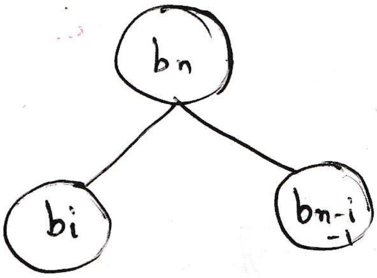
- Fix a labeling of G using integers 1, 2, ..., n (n vertices)
- Then try out all possible labelings of H using these integers.

Computational complexity → n!

- Better algms. known
- Best algm.s for solving the isomorphism problem have computational complexity exponential in the size of the problem.
- infeasibly large no. of steps to compute if the # of vertices gets moderately large.

Number of Distinct Binary Trees

$b_n \to$ # of distinct binary trees with n nodes.



$0 \le i < n$

b_n = Sum of all possible binary trees formed as above.

$$= \sum_{i=0}^{n-1} b_i \, b_{n-i-1} \quad , \quad b_0 = 1 \; ; \; n \geq 1 \quad \text{—(1)}$$

Let $B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots\ldots$

$B(x) - 1 = b_1 x + b_2 x^2 + b_3 x^3 + \ldots\ldots$

$x B(x) = b_0 x + b_1 x^2 + b_2 x^3 + b_3 x^4 + \ldots\ldots$
$= d_0 + d_1 x + d_2 x^2 + \ldots\ldots$ $\quad d_0 = 0$

$B(x) \times [x B(x)]$ $\cancel{= b_0 d_1 + b_1 d}$

$= c_0 + c_1 x + c_2 x^2 + \ldots\ldots$ $\quad c_0 = 0$.

$c_n = b_n * d_n$

$= b_n * b_{n-1}$

$$= \sum_{i=0}^{n-1} b_i \, b_{n-i-1}$$

$\therefore$ LHS of (1) $\to$ coefficient of x^n in $B(x) - 1$
RHS of (1) $\to$ coefficient of x^n in $x B^2(x)$
equal.

n	b_n	
	1	
0	1	$\to (M_1 M_2)$ 2 matrices
1	2	$\to (M_1 M_2) M_3$ or $M_1 (M_2 M_3)$ 3 matrices
2	5	$\to ((M_1 M_2) M_3) M_4$ or … 4 matrices
3	14	
4	42	
5	132	
6	429	
7	1430	
8	4862	
9	16 796. $\to$	
10		

$n \longrightarrow \frac{1}{n+1} \cdot \binom{2n}{n}$

Catalan number

given $n+1$ matrices, # of ways they can be multiplied is $\frac{1}{n+1}\binom{2n}{n}$.

exponential.

Stirling's Formula

$$\underline{n} \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

$$\therefore \frac{1}{n+1}\binom{2n}{n} = \frac{\underline{2n}}{(n+1)\,\underline{n}\,\underline{n}} = \frac{\sqrt{\cancel{4\pi n}} \left(\frac{\cancel{2n}}{\cancel{e}}\right)^{2n} (2)^{2n}}{(n+1)\ \frac{\cancel{2\pi n}}{\sqrt{\pi n}} \left(\frac{\cancel{n}}{\cancel{e}}\right)^{2n}}$$

$$\sim \frac{4^n}{\sqrt{\pi}\ n^{3/2}} = O\left(\frac{4^n}{n^{3/2}}\right).$$

③

$B(0) = b_0 = 1.$

$\therefore\ x B^2(x) = B(x) - 1.$

$x B^2(x) - B(x) + 1 = 0$

$$\therefore\ B(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$= \frac{1}{2x}\left[1 - \sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4x)^n\right]$$

$$\cancel{= \sum_{m\geq 0}\binom{\frac{1}{2}}{m+1}\frac{(-4x)^{m+1}}{2x}}$$

$$= \frac{1}{2x}\left[-\sum_{n\geq 1}\binom{\frac{1}{2}}{n}(-4x)^n\right]$$

$n-1=m$

i.e. $n=m+$

$$= \sum_{n\geq 1}\binom{\frac{1}{2}}{n}(-1)^{n-1}2^{2n-1}x^{n-1}$$

$$= \sum_{m\geq 0}\binom{\frac{1}{2}}{m+1}(-1)^{m}2^{2m+1}x^{m} = \sum_{m\geq 0}\frac{\underline{|2m}}{\underline{|m}\,\underline{|m+1}}x^m$$

$$\Rightarrow\ b_n = \frac{1}{n+1}\binom{2n}{n} \sim O\left(\frac{4^n}{n^{3/2}}\right) \quad \text{exponential order.}$$

$$\binom{\frac{1}{2}}{m+1} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-\overline{m+1-1}\right)\Big/\underline{|m+1}$$

$$= \frac{1}{2}(-1)^m\,\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2m-1}{2}\Big/\underline{|m+1}$$

$$= \frac{1}{2}\,\frac{(-1)^m}{2^m}\,1\cdot 3\cdots 2m-1\cdot\frac{2\cdot 4\cdots 2m}{2\cdot 4\cdots 2m}\Big/\underline{|m+1}$$

$$= \frac{1}{2}\,\frac{(-1)^m}{2^m}\,\frac{\underline{|2m}}{2^m\,\underline{|m}\,\underline{|m+1}} = \frac{(-1)^m}{2^{2m+1}}\,\frac{\underline{|2m}}{\underline{|m}\,\underline{|m+1}}$$

Catalan n

$b'_n \rightarrow$ the number of ways to multiply n matrices

$n=2$ $\quad M_1 \times M_2 \rightarrow$ one way

$M_1 \times M_2 \times \cdots \times M_n$.

$b'_2 = 1$ $= b_1$

$n=3$

$(M_1 \times M_2) \times M_3$

$M_1 \times (M_2 \times M_3)$

$\rightarrow$ 2 ways

$b'_3 = 2$. $= b_2$

$n=4$

$((M_1 \times M_2) \times M_3) \times M_4$

$(M_1 \times (M_2 \times M_3)) \times M_4$

$M_1 \times ((M_2 \times M_3) \times M_4)$

$M_1 \times (M_2 \times (M_3 \times M_4))$

$((M_1 \times M_2) \times (M_3 \times M_4))$

$\rightarrow$ 5 ways.

$b'_4 = 5$ $= b_3$

M_{ij}, $i \le j$

$= M_i \times M_{i+1} \times \cdots \times M_j$

To find $M_{1n} \rightarrow$ can be computed by computing any one of the following products

$M_{1i} \times M_{i+1,n}$, $1 \le i \le n$.

$\therefore$ # of distinct ways to obtain M_{1i} & $M_{i+1,n}$ are b'_i & b'_{n-i}, respectively.

$\therefore$ letting $b'_1 = 1$, we have $\quad b'_n = \sum_{i=1}^{n-1} b'_i b'_{n-i}$, $n > 1$.

$b_n = b'_{n+1}$ $\quad$ Then $\quad b'_{n+1} = \sum_{i=1}^{n+1-1} b'_i b'_{n+1-i}$ i.e. $b_n = \sum_{i=1}^{n} b_{i-1} b_{n-i}$

$= \sum_{j=0}^{n-1} b_j b_{n-j-1}$

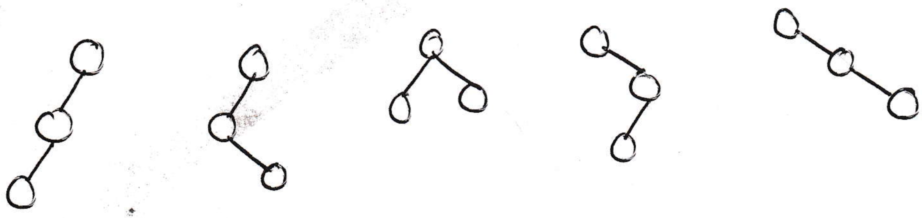
α

α^{i}

$i \quad \phi(p)$

n=2

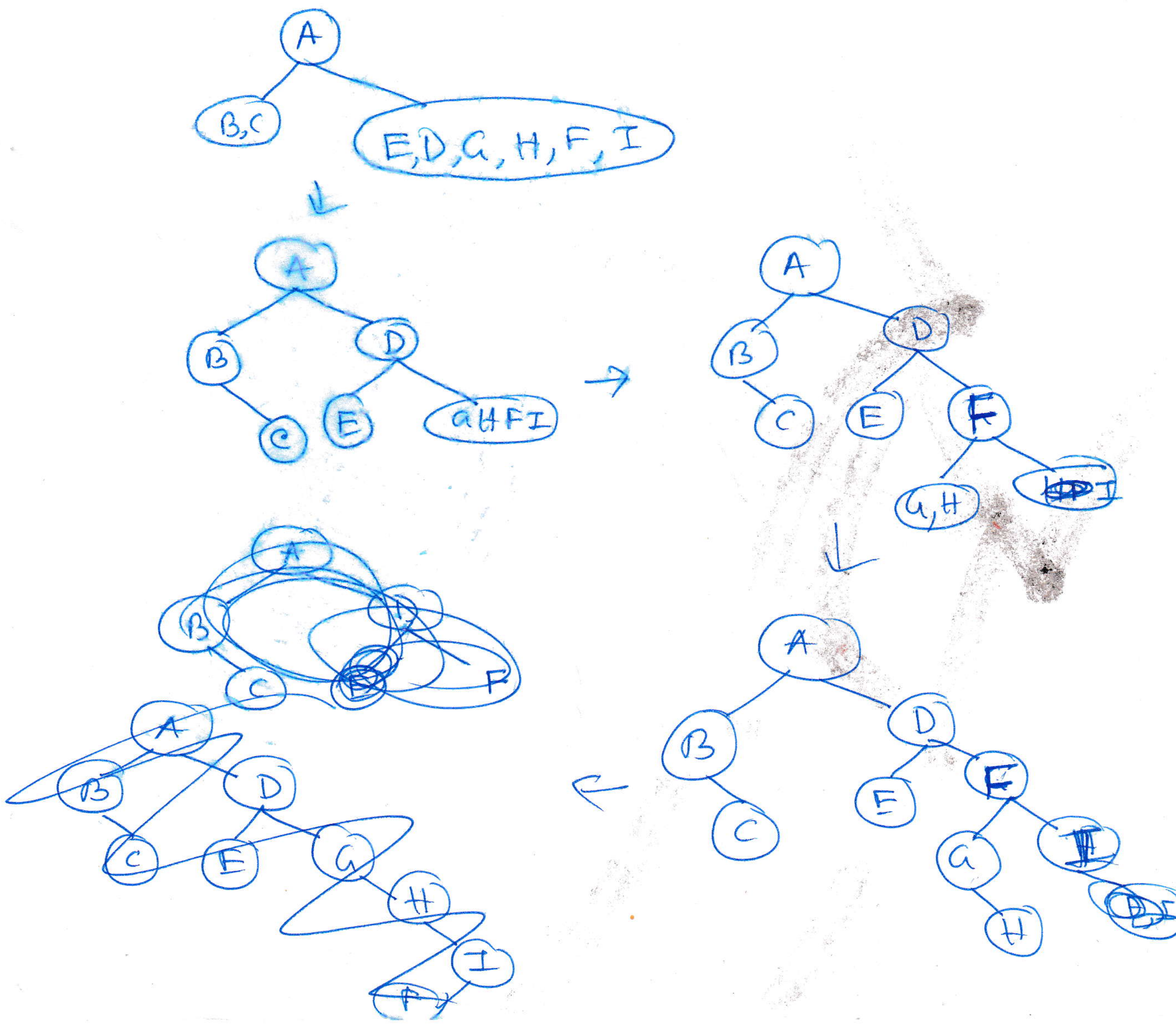
n=3



Example

preorder → A B C D E F G H I (VLR)

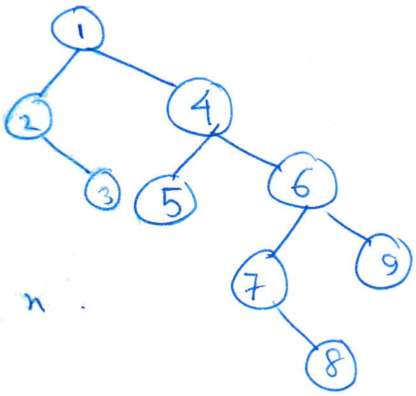
inorder → B C A E D G H F I. (LVR).



n node binary tree

↓

nodes be numbered from 1 through n.



o preorder permutation → preorder ~~traversal~~ → 1, 2, 3, 4, 5, 6, 7, 8, 9

inorder permutation → inorder traversal → 2, 3, 1, 5, 4, 7, 8, 6, 9

find. # of distinct binary trees.

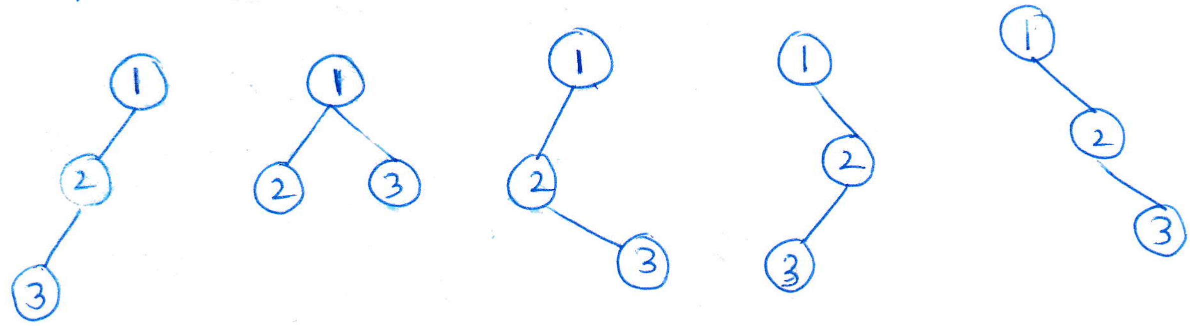
→ # of distinct inorder permutations given a fixed preorder permutation of nodes.

The pair (pre-order, inorder) seq. $\xrightarrow[\text{define}]{\text{uniquely}}$ a binary tree.

proof : Exercise.

Example 1, 2, 3 (preorder permutation) VLR.

inorder permutations LVR.



3, 2, 1   2, 1, 3   2, 3, 1   1, 3, 2   1, 2, 3.

preorder not 1,2,3

(3, 1, 2) is impossible inorder permutation

- The exponential generating fun. for the seq. $\{a_k\}$ is defined as

$$H(x) = a_0\frac{x^0}{0!} + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + \cdots + a_k\frac{x^k}{k!} + \cdots$$

(x as being chosen so that the sum converges).

Example: (Eulerian graphs) → has Euler's circuit. ↓ traverses every edge exactly once & come back to the start vertex

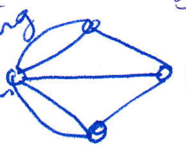
- u_n = # of labeled, Eulerian graphs of n vertices.

The exponential generating fun. $U(x)$ for the seq. $\{u_n\}$ is

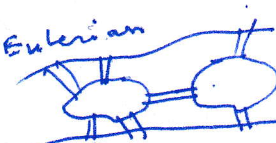
$$U(x) = x + \frac{x^3}{3!} + \frac{3x^4}{4!} + \frac{38x^5}{5!} + \cdots$$

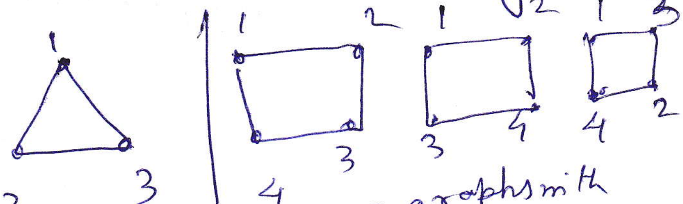
A graph without isolated vertices (vertices of degree 0) will be called Eulerian if it is connected & every vertex has even degree.

Königsberg bridge problem



Not Eulerian





1 graph with 3 vertices

3 graphs with 4 vertices.

Example: (Counting Permutations).

Suppose that we have p types of objects, with n_i indistinguishable objects of type i, $i = 1, 2, \ldots, p$.
The no. of distinguishable permutations of length k with upto n_i objects of type i is the coeffficient of $\frac{x^k}{k!}$ in the exponential generating fun.

$$\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right)\cdots\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_p}}{n_p!}\right)$$

Example:

- A code can use three different letters, a, b or c.
- A sequence of five or fewer letters gives a codewords.
- The codeword can use at most one b, at most one c, and up to three a's.
- How many possible codewords are there of length k, with $k \le 5$?

Ordinary generating f^n

$$(1+ax+a^2x^2+a^3x^3)(1+bx)(1+cx)$$

$$= 1+(a+b+c)x+(bc+a^2+ab+ac)x^2$$
$$+(a^3+abc+a^2b+a^2c)\cancel{x^2}\,x^3$$
$$+(a^2bc+a^3b+a^3c)x^4+a^3bc\,x^5$$

co-efficient of x^k
$\hookrightarrow$ ways of obtaining k letters.

e.g. 3 letters can be obtained as

$\frac{3!}{3!} \longleftrightarrow$ 3 a's $\longleftrightarrow$ 1 permutation aaa

$\frac{3!}{1!\,1!\,1!} \longleftrightarrow$ 1 a, 1 b, 1 c $\longrightarrow$ 3! permutations abc, acb, bac, bca, cab, cba.

$\frac{3!}{2!\,1!} \longleftrightarrow$ 2 a's, 1 b $\longrightarrow$ aab, aba, baa

$\frac{3!}{2!\,1!} \longleftrightarrow$ 2 a's, 1 c $\longrightarrow$ aac, aca, caa

- The proper information for the way to obtain codewords if three letters are chosen is given by

$$\boxed{\frac{3!}{3!}a^3 + \frac{3!}{1!1!1!}abc + \frac{3!}{2!1!}a^2b + \frac{3!}{2!1!}a^2c} \to u_3.$$

- Setting $a=b=c=1$ would yield the proper count of # of such codewords of 3 letters.

- So we consider $\frac{(ax)^p}{p!}$ instead of $a^p x^p$ to desire the generating fun^n.

- In the above example, we have

$$① — \left(1+\frac{a}{1!}x+\frac{a^2}{2!}x^2+\frac{a^3}{3!}x^3\right)\left(1+\frac{b}{1!}x\right)\left(1+\frac{c}{1!}x\right)$$

$$= 1 + 1!\left(\frac{a}{1!}+\frac{b}{1!}+\frac{c}{1!}\right)\frac{x}{1!} + 2!\left(\frac{bc}{1!1!}+\frac{a^2}{2!}+\frac{ab}{1!1!}+\frac{ac}{1!1!}\right)\frac{x^2}{2!}$$
$$+ \boxed{3!\left(\frac{a^3}{3!}+\frac{abc}{1!1!1!}+\frac{a^2b}{2!1!}+\frac{a^2c}{2!1!}\right)}\frac{x^3}{3!} \to u_3.$$
$$+ 4!\left(\frac{a^2bc}{2!1!1!}+\frac{a^3b}{3!1!}+\frac{a^3c}{3!1!}\right)\frac{x^4}{4!} + 5!\frac{a^3bc}{3!1!1!}\frac{x^5}{5!}$$

- Set $a=b=c=1$, take the coefficient of $\frac{x^k}{k!}$, we get the appropriate no. of codewords (permutations).

e.g. the # of length 3 codewords

coeff. of $\frac{x^k}{k!}$ in ①

$$u_3 = 3!\left(\frac{1}{3!}+\frac{1}{1!1!1!}+\frac{1}{2!1!}+\frac{1}{2!1!}\right)$$
$$= 6.\left(\frac{1}{6}+1+\frac{1}{2}+\frac{1}{2}\right) = 1+12 = 13.$$

In general

n_i objects of type i $1 \le i \le p$

of ~~disting~~ distinguishable permutations of length k with up to n_i objects of type i

= ~~the~~ coefficient of $\frac{x^k}{k!}$ in the exponential generating fn.

$$\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right) \cdots \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_p}}{n_p!}\right)$$

~~Distinguishas~~

Probability Generating functions

- Suppose that after an experiment is performed, it is known that one and only one of a (finite or countably infinite) set of possible events will occur.

- Let p_k be the probability that the k-th event occurs, $k = 0, 1, 2, \ldots$ (notation does not work if there is a continuum of possible events) ↳ uncountable

- The ordinary generating fun.

$$G(x) = \sum_{k=0}^{\infty} p_k x^k$$

is called the probability generating funⁿ.

(Series converges at least for $|x| \leq 1$ since $p_0 + p_1 + \cdots + p_k + \cdots = 1$)

- Extremely useful in evaluating experiments, in particular in analyzing roughly what we "expect" the outcomes to be.

Example: (Coin tossing)

$$G(x) = \frac{1}{2} + \frac{1}{2}x.$$

Tossing a fair coin.

events → H, T

$p_0 = P(H) = \frac{1}{2}$

$p_1 = P(T) = \frac{1}{2}$

Example: (Bernoulli Trials)

- n independent repeated trials of an experiment with each trial leading to a success with probability p and a failure with probability $q = 1-p$.

e.g. a testing whether a product is defective or nondefective.

or a testing for ~~present~~ presence or absence of a deases.

or a decision about whether to accept or reject a candidate for a job.

- the probability that in n trials there will be k success

$$b(k,n,p) = \binom{n}{k} p^k q^{n-k}.$$

The probability generating fun. for the no. of successes in n trials is given by

$$G(x) = \sum_{k=0}^{n} b(k,n,p)\, x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} x^k$$

$$= (px+q)^n$$

system

$$X = \begin{pmatrix} 1 & 2 & \cdots & k & \cdots & n \\ p_1 & p_2 & & p_k & \cdots & p_n \end{pmatrix}$$

$$G(x) = \sum p_k x^k$$

$$G'(x) = \sum k p_k x^{k-1}$$

$$G'(1) = \sum k p_k$$

kth event gives value k = expectation (provided series converges)

- $G(1) = 1$
- $G'(1) = \text{Exp}$

$x_k \to k$ th event

$p_k \to$ prob. of event x_k

- $\text{Var} = G''(1) + G'(1) - [G'(1)]^2$

$$\text{Exp} = \sum_k k p_k$$

$$\text{Var} = \sum_k k^2 p_k - \left(\sum_k k p_k\right)^2$$

$$= G''(1) + G'(1) - [G'(1)]^2$$

$$G(x) = \sum_k p_k x^k$$

$$G'(x) = \sum k p_k x^{k-1}$$

$$G''(x) = \sum k(k-1) p_k x^k$$

$$= \sum_k k^2 p_k x^{k-2} - \sum_k k p_k x^k$$

$$G''(1) = \sum_k k^2 p_k - \sum_k k p_k$$

$$G''(1) + G'(1) = \sum_k k^2 p_k$$

Example: Bernoulli trials

$$G(x) = (px+q)^n$$

$$G'(1) = \left[np(px+q)^{n-1}\right]_{x=1}$$

$$= np \longrightarrow$$ expected value ~~Expectation of the~~ $E = np$

$$G''(1) = \left[n(n-1)p^2(px+q)^{n-2}\right]_{x=1}$$

$$= n(n-1)p^2$$

V = Variance

$$= G''(1) + G'(1) - [G'(1)]^2 = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq.$$

Solving recurrences using Generating functions

Example: (The Grain of Wheat)

Story of King Shirham of India, who wanted to reward his Grand Vizier, Sissa Ben Dahir, for inventing the game of chess.

- The vizier made a modest request: Give me one grain of wheat for the 1st. square on a chessboard, two grains for the second square, four grains for the 3rd square, eight grain for the fourth sq., and so on untill all the squares are covered.
- The King was delighted at ~~to~~ the modesty of Vizier's request, & grated it immediately.
- Did the king do a very wise thing?

$t_k \rightarrow$ # of grains for the k-th square.

$$\boxed{t_{k+1} = 2t_k, \quad t_1 = 1.}$$

$t_2 = 2t_1$
$t_3 = 2^2 t_1$
$\vdots$
$t_k = 2t_{k-1} = \dots = 2^{k-1} t_1$

- 64 squares
- # of grains of wheat the Vizier asked for is $2^{64} - 1$, which is

18,446,744,073,709,551,615.

- $t_{k+1} = 2t_k$, $t_1 = 1$, $t_0 = \frac{1}{2}t_1 = \frac{1}{2}$
  (t_0 not defined, we define t_0 as $\frac{1}{2}t_1 = \frac{1}{2}$).

- The generating fun. for $\{t_n\}$ is

$$G(x) = \sum_{k=0}^{\infty} t_k x^k = t_0 + t_1 x + t_2 x^2 + \cdots$$

$$\sum_{k=0}^{\infty} t_{k+1} x^k = \sum_{k=0}^{\infty} 2t_k x^k \Rightarrow \frac{1}{x}\left[G(x) - t_0\right] = 2G(x)$$

$$t_1 + t_2 x + t_3 x^2 + \cdots = \frac{1}{x}\left[G(x) - t_0\right]$$

$$\Rightarrow G(x) - t_0 = 2x\,G(x)$$

$$\Rightarrow G(x) = \frac{t_0}{1-2x} = \frac{\frac{1}{2}}{1-2x} = \frac{1}{2}\left[1 + (2x) + (2x)^2 + (2x)^3 + \cdots\right]$$

$$\therefore t_k = \frac{1}{2}\cdot 2^k = 2^{k-1}.$$

<u>Example:</u> (legitimate Codewords).

- Codewords from the alphabet $\{0,1,2,3\}$ are to be recognized as legitimate iff they have an even no. of 0's.
- How many legitimate codewords of length k are there?
- Let a_k be the answer.
- Consider $(k+1)$-length legitimate Codewords.
  - Starts with 1 or 2 or 3, ↳ last k-digits legitimate codewords
  - Starts with 0. ↳ last k digit form illegitimate codeword

k-length legitimate codewords $= a_k$

k-length illegitimate codewords $= 4^k - a_k$

$$\therefore\ a_{k+1} = (a_k + a_k + a_k) + (4^k - a_k)$$

$$\boxed{a_{k+1} = 2a_k + 4^k, \quad a_1 = 3}$$

We use the generating fun. for $\{a_n\}$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

• need to define a_0

$$a_1 = 2a_0 + 4^0$$

$$3 = 2a_0 + 1 \Rightarrow a_0 = 1$$

$$\sum_{k=0}^{\infty} a_{k+1} x^k = \sum_{k=0}^{\infty} 2a_k x^k + \sum_{k=0}^{\infty} 4^k x^k$$

$$\frac{1}{x}\sum_{k=0}^{\infty} a_{k+1} x^{k+1} = [2G(x)] + \frac{1}{1-4x}$$

$$\frac{1}{x}[a_1 x + a_2 x^2 + \cdots]$$

$$= \frac{1}{x}[G(x) - a_0].$$

$$\therefore\ \frac{1}{x}[G(x) - 1] = 2G(x) + \frac{1}{1-4x}$$

$$\text{or}\quad G(x)[1-2x] = \frac{x}{1-4x} + 1$$

$$G(x) = \frac{x}{(1-2x)(1-4x)} + \frac{1}{(1-2x)}$$

$$= -\frac{1}{2(1-2x)} + \frac{1}{2(1-4x)} + \frac{1}{(1-2x)}$$

$$= \frac{1/2}{(1-4x)} + \frac{1/2}{(1-2x)}$$

$$\boxed{\therefore a_k = \frac{1}{2}4^k + \frac{1}{2}\cdot 2^k}$$

$$\frac{x}{(1-2x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-4x}$$

$$x \equiv A(1-4x) + B(1-2x)$$

$$A + B = 0 \Rightarrow A = -B$$

$$-4A - 2B = 1$$

$$4B - 2B = 1$$

$$\Rightarrow B = \frac{1}{2}$$

$$A = -\frac{1}{2}$$

The Hatcheck problem

- Imagine that n gentlemen attend a party and check their hats. The checker has a little too much to drink, and returns the hats at random.
  What is the probability that no gentleman receives his own hat? → (very rapidly becomes independent of the # of gentlemen).
- Can be answered by studying the notion of Derangement. ($= D_n/n!$ which converges rapidly to e^{-1})

Derangement → a permutation or arrangement in which object i is not placed in the i-th place for any i (n objects labeled $1, 2, \ldots, n$)

e.g. $n=3$ → 231 is a derangement, but 213 is not

D_n → # of derangements of n objects.

- $D_1 = 0$ — 1 element, no arrangement in which the element does not appear in its proper place
- $D_2 = 1$ — 21 is the only derangement

$$\boxed{\to D_{n+1} = n(D_{n-1} + D_n),\quad n \geq 2}$$

Deriving recurrence relation for D_n

- Consider $n+1$ elements $1, 2, \ldots, n+1$.

K ... 1 ...

1st ... kth ....

n choices

K 2 3 ... $k-1$ 1 $k+1$ .. $n+1$

1st. 2nd. 3rd ... $(k-1)$st kth $(k+1)$th ... $(n+1)$th

$(k \neq 1)$

Total $n-1$ elements → D_{n-1} # of derangements

- put k at 1st position
  - kth position has 1 → D_{n-1}
  - k-th position does not have 1 → ~~Total~~ D_n
    place 1 at kth spot & find derangements from 2nd to $(n+1)$th spot

Derangements

$$\boxed{\overline{\text{Ans}} \quad D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]}$$

(16)

- D_n = # of derangements of n elements.

Then $D_{n+1} = n(D_{n-1} + D_n)$, $n \geq 2$.

$D_2 = 1$, $D_1 = 0$.

$\left[D_2 = 1(D_0 + D_1) \Rightarrow D_0 = 1 - 0 = 1. \right]$

$$\therefore \boxed{D_{n+1} = n(D_{n-1} + D_n), \ n \geq 1} \quad \text{---} \ (1).$$

Algebraic Manipulation

$$\boxed{D_{n+1} = (n+1)D_n + (-1)^{n+1}, \ n \geq 0}$$

$D_{n+1} - (n+1)D_n$

$= D_{n+1} - nD_n - D_n$

$= nD_{n-1} - D_n$ (By (1))

$= (-1)[D_n - nD_{n-1}]$

$= (-1)^2[D_{n-1} - (n-1)D_{n-2}]$

$= (-1)^{n-j+1}[D_j - jD_{j-1}]$

$= \cdots$

$= (-1)^{n-k+1}[D_k - kD_{k-1}]$

$(-1)^j[D_j - jD_{j-1}]$

$= (-1)^k[D_k - kD_{k-1}]$

$(-1)^2[D_2 - 2D_1]$

$= [1 - 2\cdot 0] = 1$

$(-1)^{n-1}[D_2 - 2D_1]$ $\quad n-(n-1)$

$= (-1)^{n-1}$

$= (-1)^{n+1}$

Try to calculate the ordinary generating fu^n.

$$G(x) = \sum_{n=0}^{\infty} D_n x^n$$

$$\sum_{n=0}^{\infty} D_{n+1}x^n = \sum_{n=0}^{\infty}(n+1)D_n x^n + \sum_{n=0}^{\infty}(-1)^{n+1}x^n$$

$D_1 + D_2 x + D_3 x^2 + \cdots$

$= \frac{1}{x}[G(x) - D_0]$

$\sum_{n=0}^{\infty} nD_n x^n + \sum_{n=0}^{\infty} D_n x^n$

$= x\sum_{n=0}^{\infty} nD_n x^{n-1} + \sum_{n=0}^{\infty} D_n x^n$

$= xG'(x) + G(x)$.

Thus we get

$$\frac{1}{x}G(x) - \frac{1}{x} = x\,G'(x) + G(x) - \frac{1}{1+x}$$

or $$G'(x) + \left(\frac{1}{x} - \frac{1}{x^2}\right)G(x) = \frac{1}{x+x^2} - \frac{1}{x^2}$$

↳ a linear 1st. order DE, Unfortunately not easy to solve.

<u>Use exponential generating fn.</u>

- $H(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$

- $D_{n+1} = (n+1)D_n + (-1)^{n+1}$, $D_0 = 1$, $n \geq 1$

multiply by $\frac{x^{n+1}}{(n+1)!}$

$$\sum_{n=0}^{\infty} D_{n+1}\frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(n+1)D_n\frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{(n+1)!}$$

$D_1\frac{x^1}{1!} + D_2\frac{x^2}{2!} + \cdots$ ↙ $H(x) - D_0$

$D_0\frac{x}{1!} + D_1\frac{x^2}{2!} + \cdots$ ↙ $x\,H(x)$

$-\frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$ ↙ $e^{-x} - 1$

$$H(x) - \cancel{1} = xH(x) + e^{-x} - \cancel{1}$$

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$x - \frac{x^2}{2} + \frac{x^3}{2!} + \cdots$$

$$\Rightarrow H(x) = \frac{e^{-x}}{1-x}$$

coefficient of x^n in $H(x)$ $= \sum_{k=0}^{n}\frac{(-1)^k}{k!}$

$\therefore D_n$ = Coefficient of $\frac{x^n}{n!}$ in $H(x)$ $= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

$$= \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]\left[1 + x + x^2 + x^3 + \cdots\right]$$

$$= \sum_{n=0}^{\infty}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]x^n$$

Rook Polynomials

- $B_{n \times m}$ board such that certain squares forbidden & others acceptable (darkened).

- $r_k(B)$ = # of ways to choose k acceptable (darkened) squares, no two of which lie in the same row & no two of which lie in the same column

- B → think B as part of a chessboard.

- rook → a piece that can travel either horizontally or vertically on the ~~bead~~ board.

- one rook is said to be able to take another rook if the two are in the same row or in the same column.

- We wish to place k rooks on B in acceptable squares in such a way that no rook can take another.

- Then $r_k(B)$ counts # of ways k ~~rooks~~ nontaking rooks can be placed in acceptable squares of B.

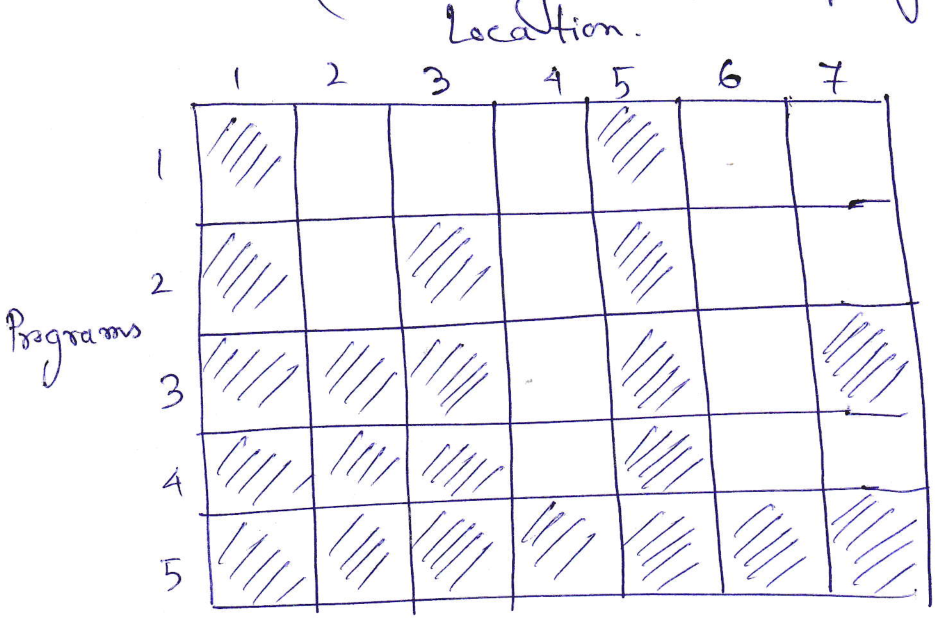
Example:

Job assignment problem

Determine # of ways in which each worker can be assigned one job, no more than one ~~job~~ worker per job so that a worker gets a job to which he or she is suited. ⟶ i.e. To find $r_5(B)$

(i,j) position darkened → worker i is suitable for job j

Example: (Storing computer programs)



(i,j)-th position darkened

$\to$ storage location has sufficient storage capacity for program i

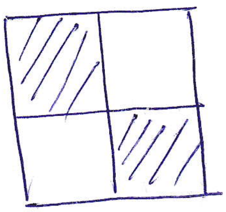
To determine # of ways to assign each program to a storage location with sufficient storage capacity, at most one program per location.

i.e. $r_5(B)$.

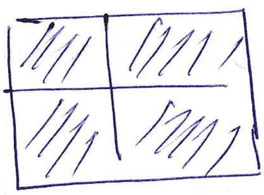
Rook polynomial for the board B.

$$R(x,B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

e.g.



B_1



B_2

$R(x,B_2) = 1 + 4x + 2x^2$

no rook, 1 rook, 2 rook

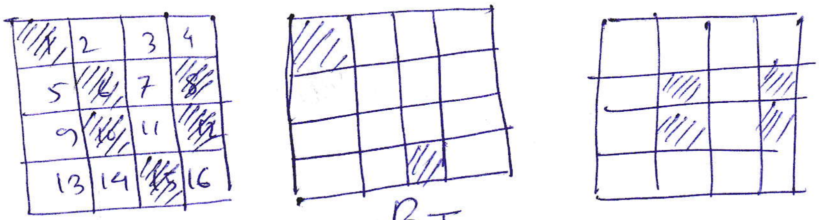
no way to place more than 2 rooks

$R(x,B_1) = 1 + 2x + x^2$

0 rook, 1 rook, 2 rook

~~A~~ Reduction for Rook Polynomials

(reduce computation of Rook Poly. of a board B to comp. of the rook poly^s. of smaller boards).



B $\qquad$ B_I $\qquad$ B_J

$I = \{1, 15\}$ $\qquad$ $J = \{6, 10, 8, 12\}$

Obtained by lightening the darkened squares in B not in I

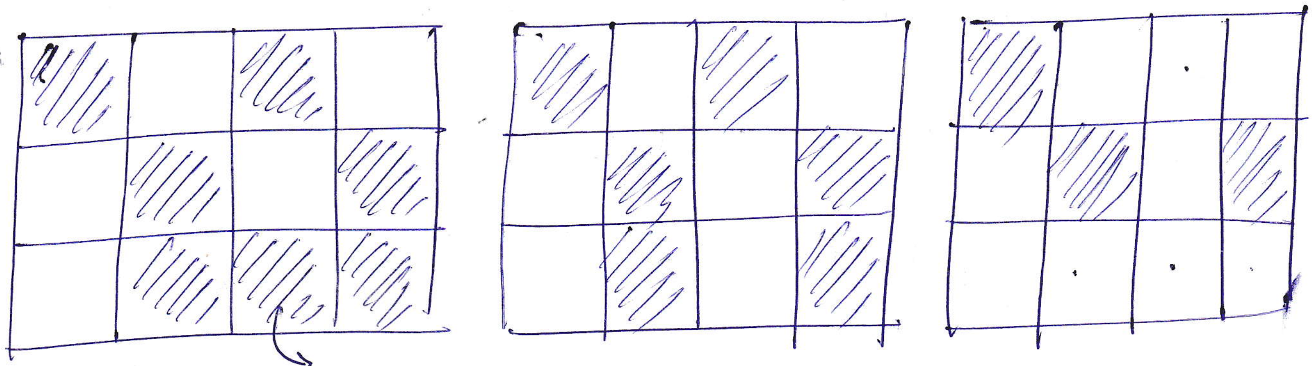
- I, J disjoint $\to$ partition of the darkened square of B. so that no square in I lies in the same row or column as any square of J.
- B_I, B_J decompose B.
- As all acceptable squares fall in B_I or B_J, and no rook of I can take a rook of J, or vice versa, to place k non-taking rooks on B, we place p nontaking rooks on B_I and then $(k-p)$ nontaking rooks in B_J, for some p.

Thus
$$r_k(B) = r_0(B_I)\,r_k(B_J) + r_1(B_I)\,r_{k-1}(B_J) + \cdots + r_p(B_I)\,r_{k-p}(B_J) + \cdots + r_k(B_I)\,r_0(B_J)$$

i.e. $\{r_k(B)\}$ is simply the convolution of the two sequences $\{r_k(B_I)\}$ and $\{r_k(B_J)\}$.

Hence $R(x, B) = R(x, B_I)\, R(x, B_J)$.

Recurrence relation



B s (a darkened square) B_s B'_s

B_s: Obtained by forbidding s in B

B'_s: obtained from B by forbidding all sqs. in the same row or column as s.

- To place $k \geq 1$ rooks on B, either we use s or we do not use s.
  - if s is not used, then we have to place k rooks on the sq. B_s
  - if s is used, then we still have to place $k-1$ rooks on the sq. B'_s

  on the sq. B except those in the same row or column as s i.e.

$$\therefore\ r_k(B) = r_k(B_s) + r_{k-1}(B'_s),\ k \geq 1.$$

$$\sum_{k=1}^{\infty} r_k(B)x^k = \sum_{k=1}^{\infty} r_k(B_s)x^k + \sum_{k=1}^{\infty} r_{k-1}(B'_s)x^k$$

$$R(x,B) - \underset{1}{r_0(B)} = \left[R(x,B_s) - \underset{1}{r_0(B_s)}\right] + x\,R(x,B'_s)$$

or
$$R(x,B) = R(x,B_s) + x\,R(x,B'_s).$$

Distributions of Distinguishable Balls into Indistinguishable Cells (Occupancy Problems)

The Stirling's no. of the second kind,

$S(n,k)$ = # of distributions of n distinguishable balls into k indistinguishable cells with no cell empty.

$$= \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

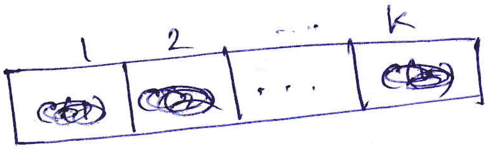
[= # of surjections from $\{1,2,..,n\}$ to $\{1,2,..,k\}$

= # of partitioning of $\{1,2..n\}$ into k sets when the order of the set matters]

Let us first find

$T(n,k)$ = # of ways to put n distinguishable balls ~~with no cells empty~~ by into k distinguishable cells, labeled $1,2,..,k$ with no cell empty.

Then $T(n,k) = k!\, S(n,k)$ [find a dist. of n distinguishable balls into k indistinguishable cells with no cell empty & then labeling (ordering) the cells.]



i-th ball goes to cell $c(i)$

$c(1)\,c(2)\cdots c(n) \to n$-permutation of k-set $\{1,2,..,k\}$

$1 \le c(i) \le k$

(with ~~at least one~~ each label j in k-set $\{1,2,..,k\}$ used at least once.

# of such permutation $c(1)\,c(2)\cdots c(n)$ is $T(n,k)$.

for a fixed k, the exponential generating fun. for $T(n,k)$ is given by

$$H(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^k = (e^x - 1)^k$$

$\therefore T(n,k) =$ coefficient of $\frac{x^n}{n!}$ in the expansion of $H(x)$

$$H(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (e^x)^{k-i}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{n=0}^{\infty} \frac{1}{n!} (k-i)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n$$

$$\therefore T(n,k) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n$$

partition 3 distinguishable balls a, b, c in 1 cell

1 cell

$a \to$ 1st ball $\to$ cell 1
$b \to$ 2nd ball $\to$ cell 1
$c \to$ 3rd ball $\to$ cell 1

$x + \frac{x^2}{2!} + \frac{x^3}{3!}$

coeff. of $\frac{x^3}{3!} = 1$

$k = 1$ cell
$n = 3$ balls

$T(3,1) = \binom{1}{0}(-1)^0(1-0)^3 + \binom{1}{1}(-1)^1(1-1)^3 = 1$

${}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_9 = 2^{10} - {}^{10}C_0 - {}^{10}C_{10} = 2^{10} - 2$

partition 10 dist. balls in 2 cells

cell 1 | cell 2
1 | 9
2 | 8
... | ...
9 | 1

$T(10,2) = \binom{2}{0}(-1)^0(2-0)^{10} + \binom{2}{1}(-1)^1(2-1)^{10} + \binom{2}{2}(-1)^2(2-2)^{10} = 2^{10} - 2$

Example: (Elevators)

- An elevator with 8 passengers stops at 6 different floors. The passengers corresponds to the balls, the floors to the cells.
- If we are interested only in the passengers who get off together, we can consider the balls distinguishable, and the cells indistinguishable.
- The # of possible distribution $= S(8,1) + S(8,2) + S(8,3) + S(8,4) + S(8,5) + S(8,6)$

(placings balls into cells)

Classification of Occupancy problems.

	Distinguishable balls ?	Distinguishable Cells ?	Can cells be empty	# of ways to place n balls into k cells
Case 1	YES	YES	YES	k^n
	YES	YES	NO	$k!\, S(n,k) = T(n,k)$
Case 2	NO	YES	YES	$C(k+n-1, n)$
	NO	YES	NO	$C(n-1, k-1)$ ← coeff of x^n in $(x+x^2+\dots)^k = x^k(1-x)^{-k}$
Case 3	YES	No	YES	$S(n,1)+S(n,2)+\dots+S(n,k)$
	YES	NO	NO	$S(n,k)$
Case 4	No	No	YES	# of partitions of n into k or fewer parts.
	NO	NO	NO	# of partitions of n into exactly k parts

e.g. $n=5$
$\{1,1,1,1,1\}, \{1,1,1,2\}, \{1,1,3\}, \{1,2,2\}, \{2,3\}, \{1,4\}, \{5\}$. $\{1,4\},\{4,1\}$ same

$n=5, k=3 \rightarrow$ # partitions with no cell empty $= 2$ (~~all but 1st two~~)

→ # of partitions with empty cells allowed $= 5$ (all but 1st. two)

Applications

1> In classifying types of accidents ~~of the cells~~ according to the day of the week in which they occur.

balls → types of accidents

~~days →~~

cells → days of the week.

Case 2

k types of cells

$(1+x+x^2+\dots)^k = (1-x)^{-k}$

empty cells allowed.

# of distinguishable ways to choose n cells with unlimited repeatations allowed

$= $ coeff. of x^n in $(1-x)^{-k}$

$= \binom{-k}{n}(-1)^n$

$= \dfrac{(-k)(-k-1)\dots(-k-n+1)\times(-1)^n}{n!} = \dfrac{(k+n-1)!}{n!\,(k-1)!}$

2> Coupon collection

balls → particular coupons

cells → types of coupons.

- For non-negative integers m and n, let $s(m,n)$ be the coefficient of x^m in $x(x-1)(x-2)\cdots(x-n+1)$ (with the convention that $s(0,0)=1$).

$s(m,n) \rightarrow$ Stirling's numbers of first kind

Show that they are defined by the recurrence relation:

$$s(0,0)=1, \quad s(m,0)=0 \ (m>0), \quad s(0,n)=0 \ (n>0).$$

and $$s(m,n) = s(m-1,n-1) - (m-1)\,s(m-1,n) \quad (m,n>0)$$

- for non-negative integers m & n, let $S(m,n)$ be the number of different partitions of the set $\{1,2,\ldots,m\}$ into n non-empty sets (with the convention that $S(0,0)=1$)

$S(m,n) \rightarrow$ Stirlings numbers of second kind.

Show that they are determined by the recurrence relation

$$S(0,0)=1, \quad S(m,0)=0 \ (m>0), \quad S(0,n)=0 \ (n>0)$$

and $$S(m,n) = S(m-1,n-1) + n\,S(m-1,n) \quad (m,n>0).$$

- $M = \big(s(m,n)\big)_{4\times 4}$, $N = \big(S(m,n)\big)_{4\times 4}$, $1 \le m,n \le 4$

Then $M^{-1} = N$

Note works for any size of matrices & is one of the key links between the two kinds of Stirling numbers.