CHAPTER 1
Introduction

Notes for the Instructor

This chapter introduces the markets for futures, forward, and options contracts and explains the activities of hedgers, speculators, and arbitrageurs. Issues concerning futures contracts such as margin requirements, settlement procedures, the role of the clearinghouse, etc are covered in Chapter 2.

Some instructors prefer to avoid any mention of options until the material on linear products in Chapters 1 to 7 has been covered. I like to introduce students to options in the first class, even though they are not mentioned again for several classes. This is because most students find options to be the most interesting of the derivatives covered and I like students to be enthusiastic about the course early on.

The way in which the material in Chapter 1 is covered is likely to depend on the backgrounds of the students. If a course in investments is a prerequisite, Chapter 1 can be regarded as a review of material already familiar to the students and can be covered fairly quickly. If an investments course is not a prerequisite, more time may be required. Increasingly some aspects of derivatives markets are being covered in introductory corporate finance courses, accounting courses, strategy courses, etc. In many instances students are, therefore, likely to have had some exposure to the material in Chapter 1. I do not require an investments elective as a prerequisite for my elective on futures and options markets and find that $1\frac{1}{2}$ to 2 hours is necessary for me to introduce the course and cover the material in Chapter 1.

To motivate students at the outset of the course, I discuss the growing importance of derivatives, how much well experts in the field are paid, etc. It is not uncommon for students who join derivatives groups, and are successful, to earn (including bonus) several hundred thousand dollars a year—or even $1 million per year—three or four years after graduating.

Towards the end of the first class I usually produce a current newspaper and describe several traded futures and options. I then ask students to guess the quoted price. Sometimes votes are taken. This is an enjoyable exercise and forces students to think actively about the nature of the contracts and the determinants of price. It usually leads to a preliminary discussion of such issues as the relationship between a futures price and the corresponding spot price, the desirability of options being exercised early, why most options sell for more than their intrinsic value, etc.

While covering the Chapter 1 material, I treat futures as the same as forwards for the purposes of discussion. I try to avoid being drawn into a discussion of such issues as the mechanics of futures, margin requirements, daily settlement procedures, and so on until I am ready. These topics are covered in Chapter 2.

As will be evident from the slides that go with this chapter, I usually introduce students to a little of the Chapter 5 material during the first class. I discuss how arbitrage arguments tie the futures price of gold to its spot price and why the futures price of a consumption commodity such as oil is not tied to its spot price in the same way. Problem 1.26 can be used to initiate the discussion.

I find that Problems 1.27 and 1.31 work well as an assignment questions. (1.31 has the advantage that it introduces students to DerivaGem early in the course.) Problem 1.28 usually generates a lively discussion. I sometimes ask students to consider it between the first and second class. We then discuss it at the beginning of the second class. Problems 1.29, 1.30, and 1.32 can be used either as assignment question or for class discussion.

QUESTIONS AND PROBLEMS

Problem 1.1.

What is the difference between a long forward position and a short forward position?

When a trader enters into a long forward contract, she is agreeing to *buy* the underlying asset for a certain price at a certain time in the future. When a trader enters into a short forward contract, she is agreeing to *sell* the underlying asset for a certain price at a certain time in the future.

Problem 1.2.

Explain carefully the difference between hedging, speculation, and arbitrage.

A trader is *hedging* when she has an exposure to the price of an asset and takes a position in a derivative to offset the exposure. In a *speculation* the trader has no exposure to offset. She is betting on the future movements in the price of the asset. *Arbitrage* involves taking a position in two or more different markets to lock in a profit.

Problem 1.3.

What is the difference between entering into a long forward contract when the forward price is \$50 and taking a long position in a call option with a strike price of \$50?

In the first case the trader is obligated to buy the asset for \$50. (The trader does not have a choice.) In the second case the trader has an option to buy the asset for \$50. (The trader does not have to exercise the option.)

Problem 1.4.

Explain carefully the difference between selling a call option and buying a put option.

Selling a call option involves giving someone else the right to buy an asset from you. It gives you a payoff of

$$-\max(S_T - K,\ 0) = \min(K - S_T,\ 0)$$

Buying a put option involves buying an option from someone else. It gives a payoff of

$$\max(K - S_T,\ 0)$$

In both cases the potential payoff is $K - S_T$. When you write a call option, the payoff is negative or zero. (This is because the counterparty chooses whether to exercise.) When you buy a put option, the payoff is zero or positive. (This is because you choose whether to exercise.)

Problem 1.5.

An investor enters into a short forward contract to sell 100,000 British pounds for U.S. dollars at an exchange rate of 1.9000 U.S. dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.8900 and (b) 1.9200?

(a) The investor is obligated to sell pounds for 1.9000 when they are worth 1.8900. The gain is $(1.9000 - 1.8900) \times 100,000 = \$1,000$.

(b) The investor is obligated to sell pounds for 1.9000 when they are worth 1.9200. The loss is $(1.9200 - 1.9000) \times 100,000 = \$2,000$.

Problem 1.6.

A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?

(a) The trader sells for 50 cents per pound something that is worth 48.20 cents per pound. Gain $= (\$0.5000 - \$0.4820) \times 50,000 = \$900$.

(b) The trader sells for 50 cents per pound something that is worth 51.30 cents per pound. Loss $= (\$0.5130 - \$0.5000) \times 50,000 = \$650$.

Problem 1.7.

Suppose that you write a put contract with a strike price of \$40 and an expiration date in three months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?

You have sold a put option. You have agreed to buy 100 shares for \$40 per share if the party on the other side of the contract chooses to exercise the right to sell for this price. The option will be exercised only when the price of stock is below \$40. Suppose, for example, that the option is exercised when the price is \$30. You have to buy at \$40 shares that are worth \$30; you lose \$10 per share, or \$1,000 in total. If the option is exercised when the price is \$20, you lose \$20 per share, or \$2,000 in total. The worst that can happen is that the price of the stock declines to almost zero during the three-month period. This highly unlikely event would cost you \$4,000. In return for the possible future losses, you receive the price of the option from the purchaser.

Problem 1.8.

What is the difference between the over-the-counter market and the exchange-traded market? What are the bid and offer quotes of a market maker in the over-the-counter market?

The over-the-counter market is a telephone- and computer-linked network of financial institutions, fund managers, and corporate treasurers where two participants can enter into any mutually acceptable contract. An exchange-traded market is a market organized by an exchange where traders either meet physically or communicate electronically and the contracts that can be traded have been defined by the exchange. When a market maker quotes a bid and an offer, the bid is the price at which the market maker is prepared to buy and the offer is the price at which the market maker is prepared to sell.

Problem 1.9.

You would like to speculate on a rise in the price of a certain stock. The current stock price is \$29, and a three-month call with a strike of \$30 costs \$2.90. You have \$5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

One strategy would be to buy 200 shares. Another would be to buy 2,000 options. If the share price does well the second strategy will give rise to greater gains. For example, if the share price goes up to \$40 you gain $[2,000 \times (\$40 - \$30)] - \$5,800 = \$14,200$ from the second strategy and only $200 \times (\$40 - \$29) = \$2,200$ from the first strategy. However, if the share price does badly, the second strategy gives greater losses. For example, if the share price goes down to \$25, the first strategy leads to a loss of $200 \times (\$29 - \$25) = \$800$, whereas the second strategy leads to a loss of the whole \$5,800 investment. This example shows that options contain built in leverage.

Problem 1.10.

Suppose that you own 5,000 shares worth \$25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?

You could buy 5,000 put options (or 50 contracts) with a strike price of \$25 and an expiration date in 4 months. This provides a type of insurance. If at the end of 4 months the stock price proves to be less than \$25 you can exercise the options and sell the shares for \$25 each. The cost of this strategy is the price you pay for the put options.

Problem 1.11.

When first issued, a stock provides funds for a company. Is the same true of a stock option? Discuss.

A stock option provides no funds for the company. It is a security sold by one trader to another. The company is not involved. By contrast, a stock when it is first issued is a claim sold by the company to investors and does provide funds for the company.

Problem 1.12.

Explain why a forward contract can be used for either speculation or hedging.

If a trader has an exposure to the price of an asset, she can hedge with a forward contract. If the exposure is such that the trader will gain when the price decreases and option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than $60.00. Note that if the stock price is between $56.00 and $60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S1.2.

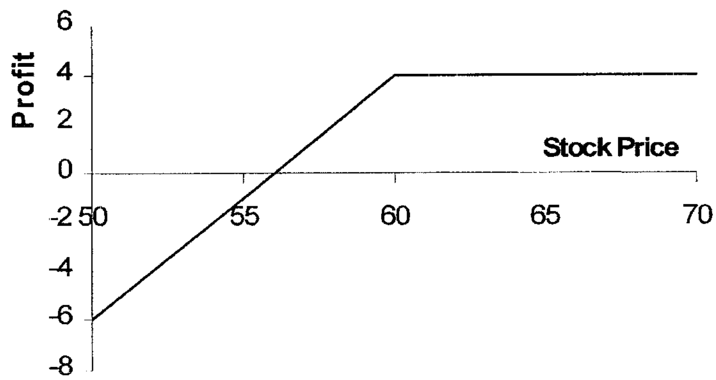

Figure S1.2 Profit from short position In Problem 1.14

Problem 1.15.

It is May and a trader writes a September call option with a strike price of $20. The stock price is $18, and the option price is $2. Describe the trader's cash flows if the option is held until September and the stock price is $25 at that time.

The trader receives an inflow of $2 in May. Since the option is exercised, the trader also has an outflow of $5 in September. The $2 is the cash received from the sale of the option. The $5 is the result of buying the stock for $25 in September and selling it to the purchaser of the option for $20. One contract consists of 100 options and so the cash flows for a contract are multiplied by 100.

Problem 1.16.

A trader writes a December put option with a strike price of $30. The price of the option is $4. Under what circumstances does the trader make a gain?

The trader makes a gain if the price of the stock is above $26 in December. (This ignores the time value of money.)

Problem 1.17.

A company knows that it is due to receive a certain amount of a foreign currency in four months. What type of option contract is appropriate for hedging?

A long position in a four-month put option can provide insurance against the exchange rate falling below the strike price. It ensures that the foreign currency can be sold for at least the strike price.

Problem 1.18.

A United States company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract; (b) an option.

The company could enter into a long forward contract to buy 1 million Canadian dollars in six months. This would have the effect of locking in an exchange rate equal to the current forward exchange rate. Alternatively the company could buy a call option giving it the right (but not the obligation) to purchase 1 million Canadian dollar at a certain exchange rate in six months. This would provide insurance against a strong Canadian dollar in six months while still allowing the company to benefit from a weak Canadian dollar at that time.

Problem 1.19.

A trader enters into a short forward contract on 100 million yen. The forward exchange rate is \$0.0080 per yen. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) \$0.0074 per yen; (b) \$0.0091 per yen?

(a) The trader sells 100 million yen for \$0.0080 per yen when the exchange rate is \$0.0074 per yen. The gain is 100×0.0006 millions of dollars or \$60,000.

(b) The trader sells 100 million yen for \$0.0080 per yen when the exchange rate is \$0.0091 per yen. The loss is 100×0.0011 millions of dollars or \$110,000.

Problem 1.20.

The Chicago Board of Trade offers a futures contract on long-term Treasury bonds. Characterize the traders likely to use this contract.

Most traders who use the contract will wish to do one of the following:

(a) Hedge their exposure to long-term interest rates

(b) Speculate on the future direction of long-term interest rates

(c) Arbitrage between cash and futures markets

This contract is discussed in Chapter 6.

Problem 1.21.

"Options and futures are zero-sum games." What do you think is meant by this statement?

The statement means that the gain (loss) to the party with a short position in an option is always equal to the loss (gain) to the party with the long position. The sum of the gains is zero.

Problem 1.22.

Describe the profit from the following portfolio: a long forward contract on an asset and a long European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up.

The terminal value of the long forward contract is:

$$S_T - F_0$$

Terminal Value of a long forward contract.

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is F_0.)

The terminal value of the put option is:

$$\max(F_0 - S_T,\, 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T,\, 0)$$

$$= \max(0,\, S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0. This result is illustrated in the Figure S1.3. The profit equals the terminal value less the amount paid for the option.

Problem 1.23.

In the 1980s, Bankers Trust developed index currency option notes (ICONs). These are bonds in which the amount received by the holder at maturity varies with a foreign exchange rate. One example was its trade with the Long Term Credit Bank of Japan. The ICON specified that if the yen–U.S. dollar exchange rate, S_T, is greater than 169 yen per dollar at maturity (in 1995), the holder of the bond receives \$1,000. If it is less than 169 yen per dollar, the amount received by the holder of the bond is

$$1,000 - \max\left[0,\, 1,000\left(\frac{169}{S_T} - 1\right)\right]$$

When the exchange rate is below 84.5, nothing is received by the holder at maturity. Show that this ICON is is a combination of a regular bond and two options.

Suppose that the yen exchange rate (yen per dollar) at maturity of the ICON is S_T. The payoff from the ICON is

$$\begin{array}{lcl} 1,000 & \text{if} & S_T > 169 \\ 1,000 - 1,000(\frac{169}{S_T} - 1) & \text{if} & 84.5 \le S_T \le 169 \\ 0 & \text{if} & S_T < 84.5 \end{array}$$

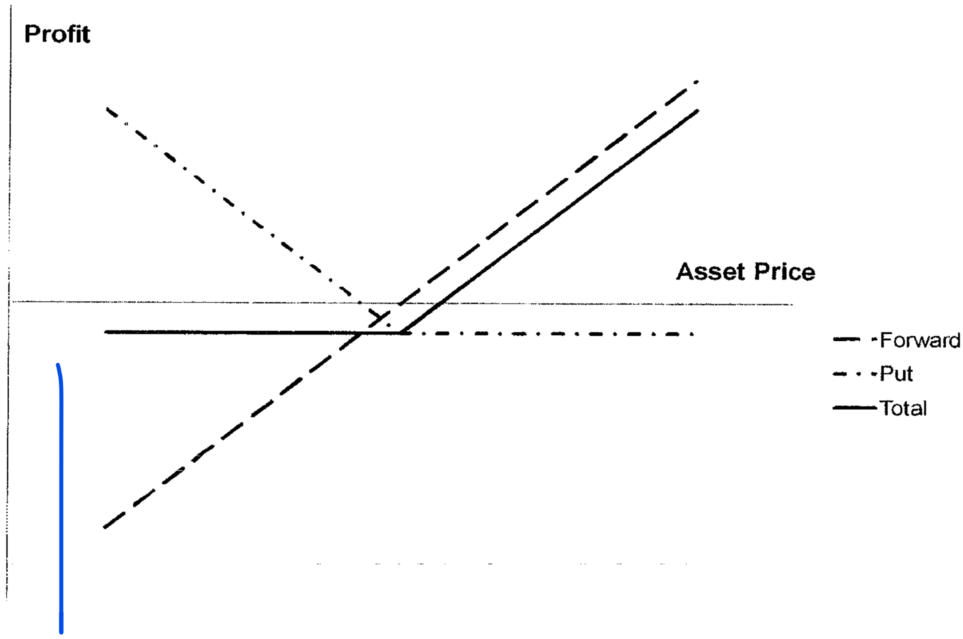

Figure S1.3 Profit from portfolio in Problem 1.22

When $84.5 \leq S_T \leq 169$ the payoff can be written

$$2,000 - \frac{169,000}{S_T}$$

The payoff from an ICON is the payoff from:
(a) A regular bond
(b) A short position in call options to buy 169,000 yen with an exercise price of 1/169
(c) A long position in call options to buy 169,000 yen with an exercise price of 1/84.5
This is demonstrated by the following table

	Terminal Value of Regular Bond	Terminal Value of Short Calls	Terminal Value of Long Calls	Terminal Value of Whole Position
$S_T > 169$	1,000	0	0	1000
$84.5 \leq S_T \leq 169$	1000	$-169,000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	0	$2000 - \frac{169,000}{S_T}$
$S_T < 84.5$	1000	$-169,000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	$169,000\left(\frac{1}{S_T} - \frac{1}{84.5}\right)$	0

Problem 1.24.

On July 1, 2008, a company enters into a forward contract to buy 10 million Japanese yen on January 1, 2009. On September 1, 2008, it enters into a forward contract to sell 10 million Japanese yen on January 1, 2009. Describe the payoff from this strategy.

Suppose that the forward price for the contract entered into on July 1, 2008 is F_1 and that the forward price for the contract entered into on September 1, 2008 is F_2 with both F_1 and F_2 being measured as dollars per yen. If the value of one Japanese yen (measured in U.S. dollars) is S_T on January 1, 2009, then the value of the first contract (in millions of dollars) at that time is

$$10(S_T - F_1)$$

while the value of the second contract (per yen sold) at that time is:

$$10(F_2 - S_T)$$

The total payoff from the two contracts is therefore

$$10(S_T - F_1) + 10(F_2 - S_T) = 10(F_2 - F_1)$$

Thus if the forward price for delivery on January 1, 2009 increases between July 1, 2008 and September 1, 2008 the company will make a profit.

Problem 1.25.

Suppose that USD-sterling spot and forward exchange rates are as follows:

Spot	*2.0080*
90-day forward	*2.0056*
180-day forward	*2.0018*

What opportunities are open to an arbitrageur in the following situations?

a. *A 180-day European call option to buy £1 for \$1.97 costs 2 cents.*

b. *A 90-day European put option to sell £1 for \$2.04 costs 2 cents.*

(a) The trader buys a 180-day call option and takes a short position in a 180-day forward contract. If S_T is the terminal spot rate, the profit from the call option is

$$\max(S_T - 1.97, 0) - 0.02$$

The profit from the short forward contract is

$$2.0018 - S_T$$

The profit from the strategy is therefore

$$\max(S_T - 1.97, 0) - 0.02 + 2.0018 - S_T$$

or

$$\max(S_T - 1.97, 0) + 1.9818 - S_T$$

This is

$$\begin{array}{lll} 1.9818 - S_T & \text{when} & S_T < 1.97 \\ 0.0118 & \text{when} & S_T > 1.97 \end{array}$$

This shows that the profit is always positive. The time value of money has been ignored in these calculations. However, when it is taken into account the strategy is still likely to be profitable in all circumstances. (We would require an extremely high interest rate for \$0.0118 interest to be required on an outlay of \$0.02 over a 180-day period.)

(b) The trader buys 90-day put options and takes a long position in a 90 day forward contract. If S_T is the terminal spot rate, the profit from the put option is

$$\max(2.04 - S_T, 0) - 0.020$$

The profit from the long forward contract is

$$S_T - 2.0056$$

The profit from this strategy is therefore

$$\max(2.04 - S_T, 0) - 0.020 + S_T - 2.0056$$

or

$$\max(2.04 - S_T, 0) + S_T - 2.0256$$

This is

$$\begin{array}{lll} S_T - 2.0256 & \text{when} & S_T > 2.04 \\ 0.0144 & \text{when} & S_T < 2.04 \end{array}$$

The profit is therefore always positive. Again, the time value of money has been ignored but is unlikely to affect the overall profitability of the strategy. (We would require interest rates to be extremely high for \$0.0144 interest to be required on an outlay of \$0.02 over a 90-day period.)

ASSIGNMENT QUESTIONS

Problem 1.26.

The price of gold is currently \$600 per ounce. The forward price for delivery in one year is \$800. An arbitrageur can borrow money at 10% per annum. What should the arbitrageur do? Assume that the cost of storing gold is zero and that gold provides no income.

The arbitrageur could borrow money to buy 100 ounces of gold today and short futures contracts on 100 ounces of gold for delivery in one year. This means that gold is purchased for \$600 per ounce and sold for \$800 per ounce. The return (33.3% per annum) is far greater than the 10% cost of the borrowed funds. This is such a profitable opportunity that the arbitrageur should buy as many ounces of gold as possible and short futures contracts on the same number of ounces. Unfortunately arbitrage opportunities as profitable as this rarely arise in practice.

Problem 1.27.

The current price of a stock is $94, and three-month European call options with a strike price of $95 currently sell for $4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying 2,000 call options (= 20 contracts). Both strategies involve an investment of $9,400. What advice would you give? How high does the stock price have to rise for the option strategy to be more profitable?

The investment in call options entails higher risks but can lead to higher returns. If the stock price stays at $94, an investor who buys call options loses $9,400 whereas an investor who buys shares neither gains nor loses anything. If the stock price rises to $120, the investor who buys call options gains

$$2000 \times (120 - 95) - 9400 = \$40,600$$

An investor who buys shares gains

$$100 \times (120 - 94) = \$2,600$$

The strategies are equally profitable if the stock price rises to a level, S, where

$$100 \times (S - 94) = 2000(S - 95) - 9400$$

or

$$S = 100$$

The option strategy is therefore more profitable if the stock price rises above $100.

Problem 1.28.

On September 12, 2006, an investor owns 100 Intel shares. As indicated in Table 1.2 the share price is \$19.56 *and a January put option with a strike price $17.50 costs* \$0.475. *The investor is comparing two alternatives to limit downside risk. The first is to buy one January put option contract with a strike price of $17.50. The second involves instructing a broker to sell the 100 shares as soon as Intel's price reaches 2$17.50. Discuss the advantages and disadvantages of the two strategies.*

The second alternative involves what is known as a stop or stop-loss order. It costs nothing and ensures that $1,750, or close to $1,750 is realized for the holding in the event the stock price ever falls to $17.50. The put option costs $47.50 and guarantees that the holding can be sold for $1,750 any time up to January. If the stock price falls marginally below $17.50 and then rises the option will not be exercised, but the stop-loss order will lead to the holding being liquidated. There are some circumstances where the put option alternative leads to a better outcome and some circumstances where the stop-loss order leads to a better outcome. If the stock price ends up below $17.50, the stop-loss order alternative leads to a better outcome because the cost of the option is avoided. If the stock price falls to \$17 in October and then rises to \$30 by January, the put option alternative leads to a better outcome. The investor is paying \$47.50 for the chance to benefit from this second type of outcome.

Problem 1.29.

A bond issued by Standard Oil some time ago worked as follows. The holder received no interest. At the bond's maturity the company promised to pay \$1,000 plus an additional amount based on the price of oil at that time. The additional amount was equal to the product of 170 and the excess (if any) of the price of a barrel of oil at maturity over \$25. The maximum additional amount paid was \$2,550 (which corresponds to a price of \$40 per barrel). Show that the bond is a combination of a regular bond, a long position in call options on oil with a strike price of \$25, and a short position in call options on oil with a strike price of \$40.

Suppose S_T is the price of oil at the bond's maturity. In addition to \$1000 the Standard Oil bond pays:

$$\begin{array}{ccc} S_T < \$25 & : & 0 \\ \$40 > S_T > \$25 & : & 170(S_T - 25) \\ S_T > \$40 & : & 2,550 \end{array}$$

This is the payoff from 170 call options on oil with a strike price of 25 less the payoff from 170 call options on oil with a strike price of 40. The bond is therefore equivalent to a regular bond plus a long position in 170 call options on oil with a strike price of \$25 plus a short position in 170 call options on oil with a strike price of \$40. The investor has what is termed a bull spread on oil. This is discussed in Chapter 10.

Problem 1.30.

Suppose that in the situation of Table 1.1 a corporate treasurer said: "I will have £1 million to sell in six months. If the exchange rate is less than 2.02 I want you to give me 2.02. If it is greater than 2.09 I will accept 2.09. If the exchange rate is between 2.02 and 2.09 I will sell the sterling for the exchange rate." How could you use options to satisfy the treasurer?

You sell the Treasurer a put option on GBP with a strike price of 2.02 and buy from the treasurer a call option on GBP with a strike price of 2.09. Both options are on one million pounds and have a maturity of six months. This is known as a range forward contract.

Problem 1.31.

Describe how foreign currency options can be used for hedging in the situation considered in Section 1.7 so that (a) ImportCo is guaranteed that its exchange rate will be less than 2.0700, and (b) ExportCo is guaranteed that its exchange rate will be at least 2.0400. Use DerivaGem to calculate the cost of setting up the hedge in each case assuming that the exchange rate volatility is 12%, interest rates in the United States are 5% and *interest rates in Britain are 5.7%. Assume that the current exchange rate is the average of the bid and offer in Table 1.1.*

ImportCo should buy three-month call options on £10 million with a strike price of 2.0700. ExportCo should buy three-month put options on £10 million with a strike price of 2.0400. In this case the foreign exchange rate is 2.0560 (the average of the bid and offer quotes in Table 1.1.), the (domestic) risk-free rate is 5%, the foreign risk-free rate is 5.7%, the volatility is 12%, and the time to exercise is 0.25. Using the Equity_FX_Index_Futures_Options worksheet in the DerivaGem Options Calculator select Currency as the underlying and Analytic European as the option type. The software shows that a call with a strike price of 2.07 is worth 0.0405 and a put with a strike of 2.04 is worth 0.0425. This means that the hedging would cost $0.0405 \times 10,000,000$ or \$405,000 for ImportCo and $\$0.0425 \times 10,000,000$ or \$425,000 for ExportCo.

Problem 1.32.

A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price and maturity. Describe the trader's position. Under what circumstances does the price of the call equal the price of the put?

The trader has a long European call option with strike price K and a short European put option with strike price K. Suppose the price of the underlying asset at the maturity of the option is S_T. If $S_T > K$, the call option is exercised by the investor and the put option expires worthless. The payoff from the portfolio is $S_T - K$. If $S_T < K$, the call option expires worthless and the put option is exercised against the investor. The cost to the investor is $K - S_T$. Alternatively we can say that the payoff to the investor is $S_T - K$ (a negative amount). In all cases, the payoff is $S_T - K$, the same as the payoff from the forward contract. The trader's position is equivalent to a forward contract with delivery price K.

Suppose that F is the forward price. If $K = F$, the forward contract that is created has zero value. Because the forward contract is equivalent to a long call and a short put, this shows that the price of a call equals the price of a put when the strike price is F.

CHAPTER 2
Mechanics of Futures Markets

Notes for the Instructor

This chapter explains the functioning of futures markets. I do not spend a great deal of time in class going over most of the details of how futures markets work. I let students read these for themselves. But I do find it worth spending some time going through Table 2.1 to explain the way in which margin accounts work. I also draw students' attention to the patterns of futures prices in Figure 2.2. After the essentials of the operations of futures markets have been explained, I ask students to consider Problem 2.22 in class because I find that this often reveals gaps in their understanding. I usually use about $1\frac{1}{2}$ hours to cover the material in the chapter.

There are many ways of making a discussion of futures markets fun. An easy-to-organize trading game that was explained to me by a Wall Street training manager works as follows. The instructor chooses two students to keep trading records on the front board and divides the rest of the students into about ten groups. Each group is given an identifier (e.g., A, B, C, etc) and a card with the identifier shown in big letters. They display the card when they want to make trades. The instructor chooses a seven-digit telephone number, but does not reveal this to students. The groups trade the sum of digits of the telephone number by entering long or short positions. For example, group B might bid (i.e. offer to buy) at 35. If this is accepted by another group (say group D), the record keepers show that B is long one contract at 35 and D is short one contract at 35. (If the actual sum of digits is 32, B is -3 on the trade and D is $+3$. The instructor controls the trading, asks for bids or offers as appropriate, and shouts trades to the record keepers. Every two minutes the instructor reveals one of the digits of the number. This game nearly always works very well for me. Trading typically starts slowly and then becomes very intense. The game gives students a sense of what futures trading is like. I insist that they use the words bid and offer rather than buy and sell.) It shows how prices are formed in markets. (After the game is over we discuss how the market price moved during the game.) The records also usually show different trading strategies. Some groups are usually speculators (all trades are long or all are short) and others are like day traders (e.g., buy at 35, sell at 36, buy at 38, sell at 39, etc). I point out to students that we need both types of traders to make the market work.

There are many stories that can be told about futures markets. Students are often interested in attempts to corner markets. I explain that the Hunt brothers' exploits in the silver market (See footnote 2 in 2.8) bankrupted them because the exchange forced them to close out their positions prior to the delivery month and as a result the price dropped. The brothers tried unsuccessfully to sue the exchange.

Business Snapshot 2.1 is an amusing story that I have often told in class. Business Snapshot 2.2 (on Long Term Capital Management) fits in well when the operation of margin accounts is being explained.

I sometimes use Problems 2.26, 2.27, and 2.28 as short hand-in assignments. Problem 2.29 is more challenging, but can be a good learning experience for students.

QUESTIONS AND PROBLEMS

Problem 2.1.

Distinguish between the terms open interest and trading volume.

The *open interest* of a futures contract at a particular time is the total number of long positions outstanding. (Equivalently, it is the total number of short positions outstanding.) The *trading volume* during a certain period of time is the number of contracts traded during this period.

Problem 2.2.

What is the difference between a local and a commission broker?

A *commission broker* trades on behalf of a client and charges a commission. A *local* trades on his or her own behalf.

Problem 2.3.

Suppose that you enter into a short futures contract to sell July silver for $10.20 per ounce on the New York Commodity Exchange. The size of the contract is 5,000 ounces. The initial margin is $4,000, and the maintenance margin is $3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

There will be a margin call when $1,000 has been lost from the margin account. This will occur when the price of silver increases by $1{,}000/5{,}000 = \$0.20$. The price of silver must therefore rise to $10.40 per ounce for there to be a margin call. If the margin call is not met, your broker closes out your position.

Problem 2.4.

Suppose that in September 2009 a company takes a long position in a contract on May 2010 crude oil futures. It closes out its position in March 2010. The futures price (per barrel) is $68.30 when it enters into the contract, $70.50 when it closes out its position, and $69.10 at the end of December 2009. One contract is for the delivery of 1,000 barrels. What is the company's total profit? When is it realized? How is it taxed if it is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year-end.

The total profit is $(\$70.50 - \$68.30) \times 1{,}000 = \$2{,}200$. Of this $(\$69.10 - \$68.30) \times 1{,}000 = \$800$ is realized on a day-by-day basis between September 2009 and December 31, 2009. A further $(\$70.50 - \$69.10) \times 1{,}000 = \$1{,}400$ is realized on a day-by-day basis between January 1, 2009, and March 2010. A hedger would be taxed on the whole profit of $2,200 in 2010. A speculator would be taxed on $800 in 2009 and $1,400 in 2010.

Problem 2.5.

What does a stop order to sell at \$2 mean? When might it be used? What does a limit order to sell at \$2 mean? When might it be used?

A *stop order* to sell at \$2 is an order to sell at the best available price once a price of \$2 or less is reached. It could be used to limit the losses from an existing long position. A *limit order* to sell at \$2 is an order to sell at a price of \$2 or more. It could be used to instruct a broker that a short position should be taken, providing it can be done at a price more favorable than \$2.

Problem 2.6.

What is the difference between the operation of the margin accounts administered by a clearinghouse and those administered by a broker?

The margin account administered by the clearinghouse is marked to market daily, and the clearinghouse member is required to bring the account back up to the prescribed level daily. The margin account administered by the broker is also marked to market daily. However, the account does not have to be brought up to the initial margin level on a daily basis. It has to be brought up to the initial margin level when the balance in the account falls below the maintenance margin level. The maintenance margin is usually about 75% of the initial margin.

Problem 2.7.

What differences exist in the way prices are quoted in the foreign exchange futures market, the foreign exchange spot market, and the foreign exchange forward market?

In futures markets, prices are quoted as the number of U.S. dollars per unit of foreign currency. Spot and forward rates are quoted in this way for the British pound, euro, Australian dollar, and New Zealand dollar. For other major currencies, spot and forward rates are quoted as the number of units of foreign currency per U.S. dollar.

Problem 2.8.

The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.

These options make the contract less attractive to the party with the long position and more attractive to the party with the short position. They therefore tend to reduce the futures price.

Problem 2.9.

What are the most important aspects of the design of a new futures contract?

The most important aspects of the design of a new futures contract are the specification of the underlying asset, the size of the contract, the delivery arrangements, and the delivery months.

Problem 2.10.

Explain how margins protect investors against the possibility of default.

A margin is a sum of money deposited by an investor with his or her broker. It acts as a guarantee that the investor can cover any losses on the futures contract. The balance in the margin account is adjusted daily to reflect gains and losses on the futures contract. If losses are above a certain level, the investor is required to deposit a further margin. This system makes it unlikely that the investor will default. A similar system of margins makes it unlikely that the investor's broker will default on the contract it has with the clearinghouse member and unlikely that the clearinghouse member will default with the clearinghouse.

Problem 2.11.

A trader buys two long July futures contracts on orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is $6,000 per contract, and the maintenance margin is $4,500 per contract. What price change would lead to a margin call? Under what circumstances could $2,000 be withdrawn from the margin account?

There is a margin call if $1,500 is lost on one contract. This happens if the futures price of orange juice falls by 10 cents to 150 cents per lb. $2,000 can be withdrawn from the margin account if there is a gain on one contract of $1,000. This will happen if the futures price rises by 6.67 cents to 166.67 cents per lb.

Problem 2.12.

Show that if the futures price of a commodity is greater than the spot price during the delivery period there is an arbitrage opportunity. Does an arbitrage opportunity exist if the futures price is less than the spot price? Explain your answer.

If the futures price is greater than the spot price during the delivery period, an arbitrageur buys the asset, shorts a futures contract, and makes delivery for an immediate profit. If the futures price is less than the spot price during the delivery period, there is no similar perfect arbitrage strategy. An arbitrageur can take a long futures position but cannot force immediate delivery of the asset. The decision on when delivery will be made is made by the party with the short position. Nevertheless companies interested in acquiring the asset will find it attractive to enter into a long futures contract and wait for delivery to be made.

Problem 2.13.

Explain the difference between a market-if-touched order and a stop order.

A market-if-touched order is executed at the best available price after a trade occurs at a specified price or at a price more favorable than the specified price. A stop order is executed at the best available price after there is a bid or offer at the specified price or at a price less favorable than the specified price.

Problem 2.14.

Explain what a stop-limit order to sell at 20.30 with a limit of 20.10 means.

A stop-limit order to sell at 20.30 with a limit of 20.10 means that as soon as there is a bid at 20.30 the contract should be sold providing this can be done at 20.10 or a higher price.

Problem 2.15.

At the end of one day a clearinghouse member is long 100 contracts, and the settlement price is \$50,000 per contract. The original margin is \$2,000 per contract. On the following day the member becomes responsible for clearing an additional 20 long contracts, entered into at a price of \$51,000 per contract. The settlement price at the end of this day is \$50,200. How much does the member have to add to its margin account with the exchange clearinghouse?

The clearinghouse member is required to provide $20 \times \$2,000 = \$40,000$ as initial margin for the new contracts. There is a gain of (50,200 − 50,000)× 100 = \$20,000 on the existing contracts. There is also a loss of $(51,000 - 50,200) \times 20 = \$16,000$ on the new contracts. The member must therefore add

$$40,000 - 20,000 + 16,000 = \$36,000$$

to the margin account.

Problem 2.16.

On July 1, 2009, a Japanese company enters into a forward contract to buy \$1 million on January 1, 2010. On September 1, 2009, it enters into a forward contract to sell \$1 million on January 1, 2010. Describe the profit or loss the company will make in yen as a function of the forward exchange rates on July 1, 2009 and September 1, 2009.

Suppose F_1 and F_2 are the forward exchange rates for the contracts entered into July 1, 2009 and September 1, 2009, and S is the spot rate on January 1, 2010. (All exchange rates are measured as yen per dollar). The payoff from the first contract is $(S - F_1)$ million yen and the payoff from the second contract is $(F_2 - S)$ million yen. The total payoff is therefore $(S - F_1) + (F_2 - S) = (F_2 - F_1)$ million yen.

Problem 2.17.

The forward price on the Swiss franc for delivery in 45 days is quoted as 1.2500. The futures price for a contract that will be delivered in 45 days is 0.7980. Explain these two quotes. Which is more favorable for an investor wanting to sell Swiss francs?

The 1.2500 forward quote is the number of Swiss francs per dollar. The 0.7980 futures quote is the number of dollars per Swiss franc. When quoted in the same way as the futures price the forward price is $1/1.2500 = 0.8000$. The Swiss franc is therefore more valuable in the forward market than in the futures market. The forward market is therefore more attractive for an investor wanting to sell Swiss francs.

Problem 2.18.

Suppose you call your broker and issue instructions to sell one July hogs contract. Describe what happens.

Hog futures are traded on the Chicago Mercantile Exchange. (See Table 2.2). The broker will request some initial margin. The order will be relayed by telephone to your broker's trading desk on the floor of the exchange (or to the trading desk of another broker).

It will be sent by messenger to a commission broker who will execute the trade according to your instructions. Confirmation of the trade eventually reaches you. If there are adverse movements in the futures price your broker may contact you to request additional margin.

Problem 2.19.

"Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

Speculators are important market participants because they add liquidity to the market. However, contracts must be useful for hedging as well as speculation. This is because regulators generally only approve contracts when they are likely to be of interest to hedgers as well as speculators.

Problem 2.20.

Identify the contracts that have the highest open interest in Table 2.2.

The table does not show contracts for all maturities. In the Metals and Petroleum category it appears that crude oil has the highest open interest. In the agricultural category it appears that corn has the highest open interest.

Problem 2.21.

What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

The contract would not be a success. Parties with short positions would hold their contracts until delivery and then deliver the cheapest form of the asset. This might well be viewed by the party with the long position as garbage! Once news of the quality problem became widely known no one would be prepared to buy the contract. This shows that futures contracts are feasible only when there are rigorous standards within an industry for defining the quality of the asset. Many futures contracts have in practice failed because of the problem of defining quality.

Problem 2.22.

"When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.

If both sides of the transaction are entering into a new contract, the open interest increases by one. If both sides of the transaction are closing out existing positions, the open interest decreases by one. If one party is entering into a new contract while the other party is closing out an existing position, the open interest stays the same.

Problem 2.23.

Suppose that on October 24, 2009, a company sells one April 2010 live-cattle futures contract. It closes out its position on January 21, 2010. The futures price (per pound) is 91.20 cents when it enters into the contract, 88.30 cents when it close out its position, and 88.80 cents at the end of December 2009. One contract is for the delivery of 40,000 pounds of cattle. What is the total profit? How is it taxed if the company is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year end.

The total profit is

$$40,000 \times (0.9120 - 0.8830) = \$1,160$$

If you are a hedger this is all taxed in 2009. If you are a speculator

$$40,000 \times (0.9120 - 0.8880) = \$960$$

is taxed in 2009 and

$$40,000 \times (0.8880 - 0.8830) = \$200$$

is taxed in 2010.

Problem 2.24.

A cattle farmer expects to have 120,000 pounds of live cattle to sell in three months. The live-cattle futures contract on the Chicago Mercantile Exchange is for the delivery of 40,000 pounds of cattle. How can the farmer use the contract for hedging? From the farmer's viewpoint, what are the pros and cons of hedging?

The farmer can short 3 contracts that have 3 months to maturity. If the price of cattle falls, the gain on the futures contract will offset the loss on the sale of the cattle. If the price of cattle rises, the gain on the sale of the cattle will be offset by the loss on the futures contract. Using futures contracts to hedge has the advantage that it can at no cost reduce risk to almost zero. Its disadvantage is that the farmer no longer gains from favorable movements in cattle prices.

Problem 2.25.

It is now July 2008. A mining company has just discovered a small deposit of gold. It will take six months to construct the mine. The gold will then be extracted on a more or less continuous basis for one year. Futures contracts on gold are available on the New York Commodity Exchange. There are delivery months every two months from August 2008 to December 2009. Each contract is for the delivery of 100 ounces. Discuss how the mining company might use futures markets for hedging.

The mining company can estimate its production on a month by month basis. It can then short futures contracts to lock in the price received for the gold. For example, if a total of 3,000 ounces are expected to be produced in January 2009 and February 2009, the price received for this production can be hedged by shorting a total of 30 February 2009 contracts.

ASSIGNMENT QUESTIONS

Problem 2.26.

A company enters into a short futures contract to sell 5,000 bushels of wheat for 450 cents per bushel. The initial margin is \$3,000 and the maintenance margin is \$2,000. What price change would lead to a margin call? Under what circumstances could \$1,500 be withdrawn from the margin account?

There is a margin call if \$1000 is lost on the contract. This will happen if the price of wheat futures rises by 20 cents from 450 cents to 470 cents per bushel. \$1500 can be withdrawn if the futures price falls by 30 cents to 420 cents per bushel.

Problem 2.27.

Suppose that there are no storage costs for crude oil and the interest rate for borrowing or lending is 5% per annum. How could you make money on January 8, 2007 by trading June 2007 and December 2007 contracts on crude oil? Use Table 2.2.

The June 2007 settlement price for oil is \$60.01 per barrel. The December 2007 settlement price for oil is \$62.94 per barrel. You could go long one June 2007 oil contract and short one December 2007 contract. In June 2007 you take delivery of the oil borrowing \$60.01 per barrel at 5% to meet cash outflows. The interest accumulated in six months is about $60.01 \times 0.05 \times 0.5$ or \$1.50. In December the oil is sold for \$62.94 and $60.01 + 1.50 =$ \$61.51 per barrel has to be repaid on the loan. The strategy therefore leads to a profit of $62.94 - 61.51$ or \$1.43 per barrel. Note that this profit is independent of the actual price of oil in June 2007 or December 2007. It will be slightly affected by the daily settlement procedures.

Problem 2.28.

What position is equivalent to a long forward contract to buy an asset at K on a certain date and a put option to sell it for K on that date.

Suppose an investor has a long European call option with strike price K and a short European put option with strike price K. Suppose the price of the underlying asset at the maturity of the option is S_T. If $S_T > K$, the call option is exercised by the investor and the put option expires worthless. The payoff from the portfolio is $S_T - K$. If $S_T < K$, the call option expires worthless and the put option is exercised against the investor. The cost to the investor is $K - S_T$. Alternatively we can say that the payoff to the investor is $S_T - K$ (a negative amount). In all cases, the payoff is $S_T - K$, the same as the payoff from the forward contract.

Suppose that F is the forward price. If $K = F$, the forward contract that is created has zero value. Because the forward contract is equivalent to a long call and a short put, this shows that the price of a call equals the price of a put when the strike price is F.

Problem 2.29.

The author's Web page (www.rotman.utoronto.ca/~hull/data) contains daily closing prices for crude oil futures and gold futures contracts. (Both contracts are traded on NYMEX.) You are required to download the data and answer the following:

a. *How high do the maintenance margin levels for oil and gold have to be set so that there is a 1% chance that an investor with a balance slightly above the maintenence margin level on a particular day has a negative balance two days later? How high do they have to be for a 0.1% chance? Assume daily price changes are normally distributed with mean zero. Explain why NYMEX might be interested in this calculation.*

b. *Imagine an investor who starts with a long position in the oil contract at the beginning of the period covered by the data and keeps the contract for the whole of the period of time covered by the data. Margin balances in excess of the initial margin are withdrawn. Use the maintenance margin you calculated in part (a) for a 1% risk level and assume that the maintenance margin is 75% of the initial margin. Calculate the number of margin calls and the number of times the investor has a negative margin balance. Assume that all margin calls are met in your calculations. Repeat the calculations for an investor who starts with a short position in the gold contract.*

(a) For gold the standard deviation of daily changes is \$2.77 per ounce or \$277 per contract. For a 1% risk this means that the maintenance margin should be set at $277 \times \sqrt{2} \times 2.33 = 912$. For a 0.1% risk the maintenance margin should be set at $277 \times \sqrt{2} \times 3.09 = 1,210$.
For crude oil the standard deviation of daily changes is \$0.31 per barrel or \$310 per contract. For a 1% risk this means that the maintenance margin should be set at $310 \times \sqrt{2} \times 2.33 = 1,021$. For a 0.1% risk the maintenance margin should be set at $310 \times \sqrt{2} \times 3.09 = 1,355$.
NYMEX is interested in these types of calculations because it wants to set the maintenance margin level so that the balance in a trader's margin account has a very low probability of becoming negative. If a trader started with a balance just above the maintenance margin level and the market moved against her, there would be a margin call at the end of the first day and the trader would have until the end of the second day to meet the margin call. It is therefore the possibility of a large futures price movement over a two-day period that is of concern to NYMEX.

(b) The initial margin is set at 1,362 for crude oil. (This is the maintenance margin divided by 0.75.) There are 151 margin calls and 7 times (out of 1201 days) where the investor is tempted to walk away. The initial margin is set at 1,215 for gold. There are 111 margin calls and 3 times (out of 826 days) when the investor is tempted to walk away. When the 0.1% risk level is used there are 3 times when the oil investor might walk away and 6 times when the gold investor might do so. These results suggest that extreme movements occur more often that the normal distribution would suggest. Here are some notes on how I handled the Excel calculations. Suppose that the initial margin is in cell Q1 and the maintenance margin is in cell Q2. Suppose further that the change in the oil futures price is in column D of the spreadsheet and the margin balance is in column E. Consider cell E7. This is updated with an instruction of the form:

$$= \text{IF}(\text{E6} < \$\text{Q}\$2, \$\text{Q}\$1, \text{IF}(\text{E6} + \text{D7} * 1000 > \$\text{Q}\$1, \$\text{Q}\$1, \text{E6} + \text{D7} * 1000))$$

Returning 1 in cell F7 if there has been a margin call and zero otherwise requires an instruction of the form:

$$= \text{IF}(\text{E7} < \$\text{Q}\$2, 1, 0)$$

Returning 1 in cell G7 if there has been an incentive to walk away and zero otherwise requires an instruction of the form:

$$= \text{IF}(\text{E6} + \text{D7} * 1000 < 0, 1, 0)$$

CHAPTER 3

Hedging Strategies Using Futures

Notes for the Instructor

This chapter discusses how long and short futures positions are used for hedging. It covers basis risk, hedge ratios, the use of stock index futures, and how to roll a hedge forward.

A number of people have pointed out a small inconsistency between the material in Chapter 3 and the CFA material in the previous edition. The issue is whether you base the number of contracts used for hedging on the futures price of the assets underlying a futures contract or the spot price of these assets. To be consistent with CFA, this edition does the former. The argument for doing so is that it is a way of adjusting for the marking to market of futures contracts. (See "tailing the hedge" material on page 58 and Problem 5.23 in Chapter 5.)

As will be evident from the slides, I cover the material in the chapter in the order in which it is presented. The section on arguments for and against hedging often generates a lively discussion. It is important to emphasize that the purpose of hedging is to reduce the standard deviation of the outcome, not to increase its expected value. I usually discuss Problem 3.17 at some stage to emphasize the point that, even in relatively simple situations, it is easy to make incorrect hedging decisions when you do not look at the big picture.

Business Snapshot 3.1 discusses hedging by gold mining companies. I use this to emphasize the importance of communicating with shareholders. I also like to discuss how investment banks hedge their risks when they enter into forward contracts with gold producers. (This is the second part of Business Snapshot 3.1.) I also like to ask students about the determinants of gold lease rates. If more gold producers choose to hedge, does the gold lease rate go up or down? (The answer is that it goes up because there is a greater demand on the part of investment banks for gold borrowing.)

Any of the Problems 3.23 to 3.26 can be used as assignment questions. My favorite is Problem 3.26.

QUESTIONS AND PROBLEMS

Problem 3.1.

Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?

A *short hedge* is appropriate when a company owns an asset and expects to sell that asset in the future. It can also be used when the company does not currently own the asset but expects to do so at some time in the future. A *long hedge* is appropriate when a company knows it will have to purchase an asset in the future. It can also be used to offset the risk from an existing short position.

Problem 3.2.

Explain what is meant by basis risk when futures contracts are used for hedging.

Basis risk arises from the hedger's uncertainty as to the difference between the spot price and futures price at the expiration of the hedge.

Problem 3.3.

Explain what is meant by a perfect hedge. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.

A *perfect hedge* is one that completely eliminates the hedger's risk. A perfect hedge does not always lead to a better outcome than an imperfect hedge. It just leads to a more certain outcome. Consider a company that hedges its exposure to the price of an asset. Suppose the asset's price movements prove to be favorable to the company. A perfect hedge totally neutralizes the company's gain from these favorable price movements. An imperfect hedge, which only partially neutralizes the gains, might well give a better outcome.

Problem 3.4.

Under what circumstances does a minimum-variance hedge portfolio lead to no hedging at all?

A minimum variance hedge leads to no hedging when the coefficient of correlation between the futures price changes and changes in the price of the asset being hedged is zero.

Problem 3.5.

Give three reasons that the treasurer of a company might not hedge the company's exposure to a particular risk.

(a) If the company's competitors are not hedging, the treasurer might feel that the company will experience less risk if it does not hedge. (See Table 3.1.) (b) The shareholders might not want the company to hedge. (c) If there is a loss on the hedge and a gain from the company's exposure to the underlying asset, the treasurer might feel that he or she will have difficulty justifying the hedging to other executives within the organization.

Problem 3.6.

Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a three-month contract? What does it mean?

The optimal hedge ratio is

$$0.8 \times \frac{0.65}{0.81} = 0.642$$

This means that the size of the futures position should be 64.2% of the size of the company's exposure in a three-month hedge.

Problem 3.7.

A company has a \$20 million portfolio with a beta of 1.2. It would like to use futures contracts on the S&P 500 to hedge its risk. The index futures is currently standing at 1080, and each contract is for delivery of \$250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6?

The formula for the number of contracts that should be shorted gives

$$1.2 \times \frac{20,000,000}{1080 \times 250} = 88.9$$

Rounding to the nearest whole number, 89 contracts should be shorted. To reduce the beta to 0.6, half of this position, or a short position in 44 contracts, is required.

Problem 3.8.

In the Chicago Board of Trade's corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in

a. *June*
b. *July*
c. *January*

A good rule of thumb is to choose a futures contract that has a delivery month as close as possible to, but later than, the month containing the expiration of the hedge. The contracts that should be used are therefore (a) July, (b) September, and (c) March.

Problem 3.9.

Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.

No. Consider, for example, the use of a forward contract to hedge a known cash inflow in a foreign currency. The forward contract locks in the forward exchange rate — which is in general different from the spot exchange rate.

Problem 3.10.

Explain why a short hedger's position improves when the basis strengthens unexpectedly and worsens when the basis weakens unexpectedly.

The basis is the amount by which the spot price exceeds the futures price. A short hedger is long the asset and short futures contracts. The value of his or her position therefore improves as the basis increases. Similarly it worsens as the basis decreases.

Problem 3.11.

Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.

The simple answer to this question is that the treasurer should (a) estimate the company's future cash flows in Japanese yen and U.S. dollars and (b) enter into forward and futures contracts to lock in the exchange rate for the U.S. dollar cash flows.

However, this is not the whole story. As the gold jewelry example in Table 3.1 shows, the company should examine whether the magnitudes of the foreign cash flows depend on the exchange rate. For example, will the company be able to raise the price of its product in U.S. dollars if the yen appreciates? If the company can do so, its foreign exchange exposure may be quite low. The key estimates required are those showing the overall effect on the company's profitability of changes in the exchange rate at various times in the future. Once these estimates have been produced the company can choose between using futures and options to hedge its risk. The results of the analysis should be presented carefully to other executives. It should be explained that a hedge does not ensure that profits will be higher. It means that profit will be more certain. When futures/forwards are used both the downside and upside are eliminated. With options a premium is paid to eliminate only the downside.

Problem 3.12.

Suppose that in Example 3.2 of Section 3.3 the company decides to use a hedge ratio of 0.8. How does the decision affect the way in which the hedge is implemented and the result?

If the hedge ratio is 0.8, the company takes a long position in 16 NYM December oil futures contracts on June 8 when the futures price is \$68.00. It closes out its position on November 10. The spot price and futures price at this time are \$70.00 and \$69.10. The gain on the futures position is

$$(69.10 - 68.00) \times 16,000 = 17,600$$

The effective cost of the oil is therefore

$$20,000 \times 70 - 17,600 = 1,382,400$$

or \$69.12 per barrel. (This compares with \$68.90 per barrel when the company is fully hedged.)

Problem 3.13.

"If the minimum-variance hedge ratio is calculated as 1.0, the hedge must be perfect." Is this statement true? Explain your answer.

The statement is not true. The minimum variance hedge ratio is

$$\rho \frac{\sigma_S}{\sigma_F}$$

It is 1.0 when $\rho = 0.5$ and $\sigma_S = 2\sigma_F$. Since $\rho < 1.0$ the hedge is clearly not perfect.

Problem 3.14.

"If there is no basis risk, the minimum variance hedge ratio is always 1.0." Is this statement true? Explain your answer.

The statement is true. Using the notation in the text, if the hedge ratio is 1.0, the hedger locks in a price of $F_1 + b_2$. Since both F_1 and b_2 are known this has a variance of zero and must be the best hedge.

Problem 3.15.

"For an asset where futures prices are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

A company that knows it will purchase a commodity in the future is able to lock in a price close to the futures price. This is likely to be particularly attractive when the futures price is less than the spot price.

Problem 3.16.

The standard deviation of monthly changes in the spot price of live cattle is (in cents per pound) 1.2. The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

The optimal hedge ratio is

$$0.7 \times \frac{1.2}{1.4} = 0.6$$

The beef producer requires a long position in $200000 \times 0.6 = 120,000$ lbs of cattle. The beef producer should therefore take a long position in 3 December contracts closing out the position on November 15.

Problem 3.17.

A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather."Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?

Suppose that the weather is bad and the farmer's production is lower than expected. Other farmers are likely to have been affected similarly. Corn production overall will be low and as a consequence the price of corn will be relatively high. The farmer is likely to be overhedged relative to actual production. The farmer's problems arising from the bad harvest will be made worse by losses on the short futures position. This problem emphasizes the importance of looking at the big picture when hedging. The farmer is correct to question whether hedging price risk while ignoring other risks is a good strategy.

Problem 3.18.

On July 1, an investor holds 50,000 shares of a certain stock. The market price is \$30 per share. The investor is interested in hedging against movements in the market over the next month and decides to use the September Mini S&P 500 futures contract. The index futures price is currently 1,500 and one contract is for delivery of \$50 times the index. The beta of the stock is 1.3. What strategy should the investor follow?

A short position in

$$1.3 \times \frac{50,000 \times 30}{50 \times 1,500} = 26$$

contracts is required.

Problem 3.19.

Suppose that in Table 3.5 the company decides to use a hedge ratio of 1.5. How does the decision affect the way the hedge is implemented and the result?

If the company uses a hedge ratio of 1.5 in Table 3.5 it would at each stage short 150 contracts. The gain from the futures contracts would be

$$1.50 \times 1.70 = \$2.55 \text{ per barrel}$$

and the company would be \$0.85 per barrel better off.

Problem 3.20.

A futures contract is used for hedging. Explain why the marking to market of the contract can give rise to cash flow problems.

Suppose that you enter into a short futures contract to hedge the sale of a asset in six months. If the price of the asset rises sharply during the six months, the futures price will also rise and you may get margin calls. The margin calls will lead to cash outflows. Eventually the cash outflows will be offset by the extra amount you get when you sell the asset, but there is a mismatch in the timing of the cash outflows and inflows. Your cash outflows occur earlier than your cash inflows. A similar situation could arise if you used a long position in a futures contract to hedge the purchase of an asset and the asset's price fell sharply. An extreme example of what we are talking about here is provided by Metallgesellschaft (see Business Snapshot 3.2).

Problem 3.21.

An airline executive has argued: "There is no point in our using oil futures. There is just as much chance that the price of oil in the future will be less than the futures price as there is that it will be greater than this price." Discuss the executive's viewpoint.

It may well be true that there is just as much chance that the price of oil in the future will be above the futures price as that it will be below the futures price. This means that the use of a futures contract for speculation would be like betting on whether a coin comes up heads or tails. But it might make sense for the airline to use futures for hedging rather than speculation. The futures contract then has the effect of reducing risks. It can be argued that an airline should not expose its shareholders to risks associated with the future price of oil when there are contracts available to hedge the risks.

Problem 3.22.

Suppose the one-year gold lease rate is 1.5% and the one-year risk-free rate is 5.0%. Both rates are compounded annually. Use the discussion in Business Snapshot 3.1 to calculate the maximum one-year forward price Goldman Sachs should quote for gold when the spot price is $600.

Goldman Sachs can borrow 1 ounce of gold and sell it for $600. It invests the $600 at 5% so that it becomes $630 at the end of the year. It must pay the lease rate of 1.5% on $600. This is $9 and leaves it with $621. It follows that if it agrees to buy the gold for less than $621 in one year it will make a profit.

ASSIGNMENT QUESTIONS

Problem 3.23.

The following table gives data on monthly changes in the spot price and the futures price for a certain commodity. Use the data to calculate a minimum variance hedge ratio.

Spot Price Change	+0.50	+0.61	−0.22	−0.35	+0.79
Futures Price Change	+0.56	+0.63	−0.12	−0.44	+0.60
Spot Price Change	+0.04	+0.15	+0.70	−0.51	−0.41
Futures price change	−0.06	+0.01	+0.80	−0.56	−0.46

Denote x_i and y_i by the i-th observation on the change in the futures price and the change in the spot price respectively.

$$\sum x_i = 0.96 \quad \sum y_i = 1.30$$

$$\sum x_i^2 = 2.4474 \quad \sum y_i^2 = 2.3594$$

$$\sum x_i y_i = 2.352$$

An estimate of σ_F is

$$\sqrt{\frac{2.4474}{9} - \frac{0.96^2}{10 \times 9}} = 0.5116$$

An estimate of σ_S is

$$\sqrt{\frac{2.3594}{9} - \frac{1.30^2}{10 \times 9}} = 0.4933$$

An estimate of ρ is

$$\frac{10 \times 2.352 - 0.96 \times 1.30}{\sqrt{(10 \times 2.4474 - 0.96^2)(10 \times 2.3594 - 1.30^2)}} = 0.981$$

The minimum variance hedge ratio is

$$\rho\frac{\sigma_S}{\sigma_F} = 0.981 \times \frac{0.4933}{0.5116} = 0.946$$

Problem 3.24.

It is July 16. A company has a portfolio of stocks worth $100 million. The beta of the portfolio is 1.2. The company would like to use the CME December futures contract on the S&P 500 to change the beta of the portfolio to 0.5 during the period July 16 to November 16. The index futures price is currently 1,000, and each contract is on $250 times the index.

a. *What position should the company take?*
b. *Suppose that the company changes its mind and decides to increase the beta of the portfolio from 1.2 to 1.5. What position in futures contracts should it take?*

(a) The company should short

$$\frac{(1.2 - 0.5) \times 100,000,000}{1000 \times 250}$$

or 280 contracts.

(b) The company should take a long position in

$$\frac{(1.5 - 1.2) \times 100,000,000}{1000 \times 250}$$

or 120 contracts.

Problem 3.25.

A fund manager has a portfolio worth $50 million with a beta of 0.87. The manager is concerned about the performance of the market over the next two months and plans to use three-month futures contracts on the S&P 500 to hedge the risk. The current level of the index is 1250, one contract is on 250 times the index, the risk-free rate is 6% per annum, and the dividend yield on the index is 3% per annum. The current 3 month futures price is 1259.

a. *What position should the fund manager take to eliminate all exposure to the market over the next two months?*
b. *Calculate the effect of your strategy on the fund manager's returns if the level of the market in two months is 1,000, 1,100, 1,200, 1,300, and 1,400. Assume that the one-month futures price is 0.25% higher than the index level at this time.*

(a) The number of contracts the fund manager should short is

$$0.87 \times \frac{50,000,000}{1259 \times 250} = 138.20$$

Rounding to the nearest whole number, 138 contracts should be shorted.

(b) The following table shows that the strategy has the effect of locking in a return of close to \$490,000. To illustrate the calculations in the table consider the first column. If the index in two months is 1,000, the futures price is $1000 \times 1.0025 = 1002.50$ The gain on the short futures position is therefore

$$(1259 - 1002.50) \times 250 \times 138 = \$8,849,250$$

The return on the index is $3 \times 2/12$=0.5% in the form of dividend and $-250/1250 =$ -20% in the form of capital gains. The total return on the index is therefore -19.5%. The risk-free rate is 1% per two months. The return is therefore -20.5% in excess of the risk-free rate. From the capital asset pricing model we expect the return on the portfolio to be $0.87 \times -20.5\% = -17.835\%$ in excess of the risk-free rate. The portfolio return is therefore -16.835%. The loss on the portfolio is $0.16835 \times 50,000,000$ or \$8,417,500. When this is combined with the gain on the futures the total gain is \$431,750.

Index in Two months	1000	1100	1200	1300	1400
Futures Price (\$)	1002.50	1102.75	1203.00	1303.25	1403.50
Gain on Futures (\$)	8,849,250	5,390,625	1,932,000	-1,526,625	-4,985,250
Index Return	-19.5%	-11.5%	-3.5%	4.5%	12.5%
Excess Ind. Return	-20.5%	-12.5%	-4.5%	3.5%	11.5%
Excess Port. Return	-17.835%	-10.875%	-3.915%	3.045%	10.005%
Port. Return	-16.835%	-9.875%	-2.915%	4.045%	11.005%
Port. Gain (\$)	-8,417,500	-4,937,500	-1,457,500	2,022,500	5,502,500
Total Gain (\$)	431,750	453,125	488,500	495,875	517,250

Problem 3.26.

It is now October 2007. A company anticipates that it will purchase 1 million pounds of copper in each of February 2008, August 2008, February 2009, and August 2009. The company has decided to use the futures contracts traded in the COMEX division of the New York Mercantile Exchange to hedge its risk. One contract is for the delivery of 25,000 pounds of copper. The initial margin is \$2,000 per contract and the maintenance margin is \$1,500 per contract. The company's policy is to hedge 80% of its exposure. Contracts with maturities up to 13 months into the future are considered to have sufficient liquidity to meet the company's needs. Devise a hedging strategy for the company. Do not make the tailing adjustments described in Section 3.4.

Assume the market prices (in cents per pound) today and at future dates are as follows. What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2004? Is the company subject to any margin calls?

Date	*Oct 2007*	*Feb 2008*	*Aug 2008*	*Feb 2009*	*Aug 2009*
Spot Price	*372.00*	*369.00*	*365.00*	*377.00*	*388.00*
Mar 2008 Futures Price	*372.30*	*369.10*			
Sep 2008 Futures Price	*372.80*	*370.20*	*364.80*		
Mar 2009 Futures Price		*370.70*	*364.30*	*376.70*	
Sep 2009 Futures Price			*364.20*	*376.50*	*388.20*

What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2007? Is the company subject to any margin calls?

To hedge the February 2008 purchase the company should take a long position in March 2008 contracts for the delivery of 800,000 pounds of copper. The total number of contracts required is $800,000/25,000 = 32$. Similarly a long position in 32 September 2008 contracts is required to hedge the August 2008 purchase. For the February 2009 purchase the company could take a long position in 32 September 2008 contracts and roll them into March 2009 contracts during August 2008. (As an alternative, the company could hedge the February 2009 purchase by taking a long position in 32 March 2008 contracts and rolling them into March 2009 contracts.) For the August 2009 purchase the company could take a long position in 32 September 2008 and roll them into September 2009 contracts during August 2008.

The strategy is therefore as follows

Oct. 2007: Enter into long position in 96 Sept. 2008 contracts
Enter into a long position in 32 Mar. 2008 contracts
Feb. 2008: Close out 32 Mar. 2008 contracts
Aug. 2008: Close out 96 Sept. 2008 contracts
Enter into long position in 32 Mar. 2009 contracts
Enter into long position in 32 Sept. 2009 contracts
Feb. 2009: Close out 32 Mar. 2009 contracts
Aug. 2009: Close out 32 Sept. 2009 contracts

With the market prices shown the company pays

$$369.00 + 0.8 \times (372.30 - 369.10) = 371.56$$

for copper in February, 2008. It pays

$$365.00 + 0.8 \times (372.80 - 364.80) = 371.40$$

for copper in August 2008. As far as the February 2009 purchase is concerned, it loses $372.80 - 364.80 = 8.00$ on the September 2008 futures and gains $376.70 - 364.30 = 12.40$

on the February 2009 futures. The net price paid is therefore

$$377.00 + 0.8 \times 8.00 - 0.8 \times 12.40 = 373.48$$

As far as the August 2009 purchase is concerned, it loses $372.80 - 364.80 = 8.00$ on the September 2008 futures and gains $388.20 - 364.20 = 24.00$ on the September 2009 futures. The net price paid is therefore

$$388.00 + 0.8 \times 8.00 - 0.8 \times 24.00 = 375.20$$

The hedging scheme succeeds in keeping the price paid in the range 371.40 to 375.20.

In October 2007 the initial margin requirement on the 128 contracts is $128 \times \$2,000$ or \$256,000. There is a margin call when the futures price drops by more than 2 cents. This happens to the March 2008 contract between October 2007 and February 2008, to the September 2008 contract between October 2007 and February 2008, and to the September 2008 contract between February 2008 and August 2008.

CHAPTER 4

Interest Rates

Notes for the Instructor

This chapter together with Chapters 6 and 7 emphasizes that, for a derivatives trader, risk-free rates are the rates derived from LIBOR markets, Eurodollar futures, and swap markets. The reasons why derivatives traders do not use Treasury rates as risk-free rates are outlined in Business Snapshot 4.1. Chapter 7 continues this discussion by explaining that swap rates have very little credit risk because a bank can earn the swap rate by making a series of short term loans to AA-rated companies.

A new feature of this chapter is the expanded treatment of liquidity preference theory in Section 4.10.

I like to spend a some time explaining compounding frequency issues. I make it clear to students that we are talking about nothing more than a unit of measurement for interest rates. Moving from quarterly compounding to continuous compounding is like changing the unit of measurement of distance from miles to kilometers. When students are introduced to continuous compounding early in a course, I find they have very little difficulty with it.

The first part of the chapter discusses zero rates, bond valuation, bond yields, par yields, and the calculation of the Treasury zero curve. The slides mirror the examples in the text. When covering the bootstrap method to calculate the Treasury zero curve, I mention that Chapter 7 explains how the same procedure can be used to calculate the LIBOR/swap zero curve. I also point out that the bootstrap method is a very popular approach, but it is not the only one that is used in practice. For example, some analysts use cubic or exponential splines.

I spend some time on the relationship between spot and forward interest rates and combine this with a discussion of FRAs and theories of the term structure. I explain that it is possible to enter into transactions that lock in the forward rate for a future time period and then discuss the Orange County story (Business Snapshot 4.2). Orange County entered into contracts (often highly levered) that paid off if the forward rate was higher than the realized future spot rate (An example of such a contract is an FRA where fixed is received and floating is paid). This worked well in 1992 and 1993, but led to a huge loss in 1994.

Sections 4.8 and 4.9 cover duration and convexity. Duration is a widely used concept in derivatives markets. The chapter explains that the $\Delta B/B = -D\Delta y$ relationship holds when rates are continuously compounded. When some other compounding frequency is used the same relationship is true provided D is defined as the modified duration. I like to illustrate the truth of the duration relationship with numerical example similar to those in the text.

Problems 4.24 to 4.28 can be used as assignment questions. My favorites are 4.27 and 4.28.

QUESTIONS AND PROBLEMS

Problem 4.1.

A bank quotes you an interest rate of 14% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

(a) The rate with continuous compounding is

$$4\ln\left(1+\frac{0.14}{4}\right)=0.1376$$

or 13.76% per annum.

(b) The rate with annual compounding is

$$\left(1+\frac{0.14}{4}\right)^4-1=0.1475$$

or 14.75% per annum.

Problem 4.2.

What is meant by LIBOR and LIBID. Which is higher?

LIBOR is the London InterBank Offered Rate. It is the rate a bank quotes for deposits it is prepared to place with other banks. LIBID is the London InterBank Bid rate. It is the rate a bank quotes for deposits from other banks. LIBOR is greater than LIBID.

Problem 4.3.

The six-month and one-year zero rates are both 10% per annum. For a bond that has a life of 18 months and pays a coupon of 8% per annum (with semiannual payments and one having just been made), the yield is 10.4% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

Suppose the bond has a face value of \$100. Its price is obtained by discounting the cash flows at 10.4%. The price is

$$\frac{4}{1.052}+\frac{4}{1.052^2}+\frac{104}{1.052^3}=96.74$$

If the 18-month zero rate is R, we must have

$$\frac{4}{1.05}+\frac{4}{1.05^2}+\frac{104}{(1+R/2)^3}=96.74$$

which gives $R=10.42\%$.

Problem 4.4.

An investor receives \$1,100 in one year in return for an investment of \$1,000 now. Calculate the percentage return per annum with a) Annual compounding, b) Semiannual compounding, c) Monthly compounding and d) Continuous compounding.

(a) With annual compounding the return is

$$\frac{1100}{1000} - 1 = 0.1$$

or 10% per annum.

(b) With semi-annual compounding the return is R where

$$1000\left(1 + \frac{R}{2}\right)^2 = 1100$$

i.e.,

$$1 + \frac{R}{2} = \sqrt{1.1} = 1.0488$$

so that $R = 0.0976$. The percentage return is therefore 9.76% per annum.

(c) With monthly compounding the return is R where

$$1000\left(1 + \frac{R}{12}\right)^{12} = 1100$$

i.e.,

$$\left(1 + \frac{R}{12}\right) = \sqrt[12]{1.1} = 1.00797$$

so that $R = 0.0957$ The percentage return is therefore 9.57% per annum.

(d) With continuous compounding the return is R where:

$$1000e^R = 1100$$

i.e.,

$$e^R = 1.1$$

so that $R = \ln 1.1 = 0.0953$. The percentage return is therefore 9.53% per annum.

Problem 4.5.

Suppose that zero interest rates with continuous compounding are as follows:

Maturity (months)	*Rate (% per annum)*
3	*8.0*
6	*8.2*
9	*8.4*
12	*8.5*
15	*8.6*
18	*8.7*

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

The forward rates with continuous compounding are as follows

Qtr 2:	8.4%
Qtr 3:	8.8%
Qtr 4:	8.8%
Qtr 5:	9.0%
Qtr 6:	9.2%

Problem 4.6.

Assuming that zero rates are as in Problem 4.5, what is the value of an FRA that enables the holder to earn 9.5% for a three-month period starting in one year on a principal of $1,000,000? The interest rate is expressed with quarterly compounding.

The forward rate is 9.0% with continuous compounding or 9.102% with quarterly compounding. From equation (4.9), the value of the FRA is therefore

$$[1,000,000 \times 0.25 \times (0.095 - 0.09102)]e^{-0.086 \times 1.25} = 893.56$$

or $893.56.

Problem 4.7.

The term structure of interest rates is upward sloping. Put the following in order of magnitude:

a. The five-year zero rate
b. The yield on a five-year coupon-bearing bond
c. The forward rate corresponding to the period between 4.75 and 5 years in the future

What is the answer to this question when the term structure of interest rates is downward sloping?

When the term structure is upward sloping, $c > a > b$. When it is downward sloping, $b > a > c$.

Problem 4.8.

What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

Duration provides information about the effect of a small parallel shift in the yield curve on the value of a bond portfolio. The percentage decrease in the value of the portfolio equals the duration of the portfolio multiplied by the amount by which interest rates are increased in the small parallel shift. The duration measure has the following limitation. It applies only to parallel shifts in the yield curve that are small.

Problem 4.9.

What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

The rate of interest is R where:

$$e^R = \left(1 + \frac{0.15}{12}\right)^{12}$$

i.e.,

$$\begin{aligned} R &= 12\ln\left(1 + \frac{0.15}{12}\right) \\ &= 0.1491 \end{aligned}$$

The rate of interest is therefore 14.91% per annum.

Problem 4.10.

A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a \$10,000 deposit?

The equivalent rate of interest with quarterly compounding is R where

$$e^{0.12} = \left(1 + \frac{R}{4}\right)^4$$

or

$$R = 4(e^{0.03} - 1) = 0.1218$$

The amount of interest paid each quarter is therefore:

$$10,000 \times \frac{0.1218}{4} = 304.55$$

or \$304.55.

Problem 4.11.

Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum with continuous compounding respectively. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months pays a coupon of 4% per annum semiannually.

The bond pays \$2 in 6, 12, 18, and 24 months, and \$102 in 30 months. The cash price is

$$2e^{-0.04\times0.5}+2e^{-0.042\times1.0}+2e^{-0.044\times1.5}+2e^{-0.046\times2}+102e^{-0.048\times2.5}=98.04$$

Problem 4.12.

A three-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

The bond pays \$4 in 6, 12, 18, 24, and 30 months, and \$104 in 36 months. The bond yield is the value of y that solves

$$4e^{-0.5y}+4e^{-1.0y}+4e^{-1.5y}+4e^{-2.0y}+4e^{-2.5y}+104e^{-3.0y}=104$$

Using the *Goal Seek* tool in Excel $y=0.06407$ or 6.407%.

Problem 4.13.

Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7% respectively. What is the two-year par yield?

Using the notation in the text, $m=2$, $d=e^{-0.07\times2}=0.8694$. Also

$$A=e^{-0.05\times0.5}+e^{-0.06\times1.0}+e^{-0.065\times1.5}+e^{-0.07\times2.0}=3.6935$$

The formula in the text gives the par yield as

$$\frac{(100-100\times0.8694)\times2}{3.6935}=7.072$$

To verify that this is correct we calculate the value of a bond that pays a coupon of 7.072% per year (that is 3.5365 every six months). The value is

$$3.536e^{-0.05\times0.5}+3.5365e^{-0.06\times1.0}+3.536e^{-0.065\times1.5}+103.536e^{-0.07\times2.0}=100$$

verifying that 7.072% is the par yield.

Problem 4.14.

Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	*Rate (% per annum)*
1	*2.0*
2	*3.0*
3	*3.7*
4	*4.2*
5	*4.5*

Calculate forward interest rates for the second, third, fourth, and fifth years.

The forward rates with continuous compounding are as follows:

Year 2:	4.0%
Year 3:	5.1%
Year 4:	5.7%
Year 5:	5.7%

Problem 4.15.

Use the rates in Problem 4.14 to value an FRA where you will pay 5% for the third year on \$1 million.

The forward rate is 5.1% with continuous compounding or $e^{0.051\times1} - 1 = 5.232\%$ with annual compounding. The 3-year interest rate is 3.7% with continuous compounding. From equation (4.10), the value of the FRA is therefore

$$[1,000,000 \times (0.05232 - 0.05) \times 1]e^{-0.037\times3} = 2,078.85$$

or \$2,078.85.

Problem 4.16.

A 10-year, 8% coupon bond currently sells for \$90. A 10-year, 4% coupon bond currently sells for \$80. What is the 10-year zero rate? (Hint: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

Taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds leads to the following cash flows

$$\text{Year } 0 : 90 - 2 \times 80 = -70$$
$$\text{Year } 10 : 200 - 100 = 100$$

because the coupons cancel out. \$100 in 10 years time is equivalent to \$70 today. The 10-year rate, R, (continuously compounded) is therefore given by

$$100 = 70e^{10R}$$

The rate is

$$\frac{1}{10}\ln\frac{100}{70} = 0.0357$$

or 3.57% per annum.

Problem 4.17.

Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward sloping more often than it is downward sloping.

If long-term rates were simply a reflection of expected future short-term rates, we would expect the term structure to be downward sloping as often as it is upward sloping. (This is based on the assumption that half of the time investors expect rates to increase and half of the time investors expect rates to decrease). Liquidity preference theory argues that long term rates are high relative to expected future short-term rates. This means that the term structure should be upward sloping more often than it is downward sloping.

Problem 4.18.

"When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain why this is so.

The par yield is the yield on a coupon-bearing bond. The zero rate is the yield on a zero-coupon bond. When the yield curve is upward sloping, the yield on an N-year coupon-bearing bond is less than the yield on an N-year zero-coupon bond. This is because the coupons are discounted at a lower rate than the N-year rate and drag the yield down below this rate. Similarly, when the yield curve is downward sloping, the yield on an N-year coupon bearing bond is higher than the yield on an N-year zero-coupon bond.

Problem 4.19.

Why are U.S. Treasury rates significantly lower that other rates that are close to risk free?

There are three reasons (see Business Snapshot 4.1).

(i) Treasury bills and Treasury bonds must be purchased by financial institutions to fulfill a variety of regulatory requirements. This increases demand for these Treasury instruments driving the price up and the yield down.

(ii)The amount of capital a bank is required to hold to support an investment in Treasury bills and bonds is substantially smaller than the capital required to support a similar investment in other very-low-risk instruments.

(iii) In the United States, Treasury instruments are given a favorable tax treatment compared with most other fixed-income investments because they are not taxed at the state level.

Problem 4.20.

Why does a loan in the repo market involve very little credit risk?

A repo is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The other company is providing a loan to the investment dealer. This loan involves very little credit risk. If the borrower does not honor the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

Problem 4.21.

Explain why an FRA is equivalent to the exchange of a floating rate of interest for a fixed rate of interest.

A FRA is an agreement that a certain specified interest rate, R_K, will apply to a certain principal, L, for a certain specified future time period. Suppose that the rate observed in the market for the future time period at the beginning of the time period proves to be R_M. If the FRA is an agreement that R_K will apply when the principal is invested, the holder of the FRA can borrow the principal at R_M and then invest it at R_K. The net cash flow at the end of the period is then an inflow of $R_K L$ and an outflow of $R_M L$. If the FRA is an agreement that R_K will apply when the principal is borrowed, the holder of the FRA can invest the borrowed principal at R_M. The net cash flow at the end of the period is then an inflow of $R_M L$ and an outflow of $R_K L$. In either case we see that the FRA involves the exchange of a fixed rate of interest, R_K, on the principal of L for the floating rate of interest observed in the market, R_M.

Problem 4.22.

A five-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.

a. *What is the bond's price?*
b. *What is the bond's duration?*
c. *Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.*
d. *Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).*

(a) The bond's price is

$$8e^{-0.11} + 8e^{-0.11\times 2} + 8e^{-0.11\times 3} + 8e^{-0.11\times 4} + 108e^{-0.11\times 5} = 86.80$$

(b) The bond's duration is

$$\frac{1}{86.80}\left[8e^{-0.11} + 2\times 8e^{-0.11\times 2} + 3\times 8e^{-0.11\times 3} + 4\times 8e^{-0.11\times 4} + 5\times 108e^{-0.11\times 5}\right]$$

$$= 4.256 \text{ years}$$

(c) Since, with the notation in the chapter

$$\Delta B = -BD\Delta y$$

the effect on the bond's price of a 0.2% decrease in its yield is

$$86.80\times 4.256\times 0.002 = 0.74$$

The bond's price should increase from 86.80 to 87.54.

(d) With a 10.8% yield the bond's price is

$$8e^{-0.108} + 8e^{-0.108\times 2} + 8e^{-0.108\times 3} + 8e^{-0.108\times 4} + 108e^{-0.108\times 5} = 87.54$$

This is consistent with the answer in (c).

Problem 4.23.

The cash prices of six-month and one-year Treasury bills are 94.0 and 89.0. A 1.5-year bond that will pay coupons of $4 every six months currently sells for $94.84. A two-year bond that will pay coupons of $5 every six months currently sells for $97.12. Calculate the six-month, one-year, 1.5-year, and two-year zero rates.

The 6-month rate (with continuous compounding) is $2\ln(1+6/94) = 12.38\%$. The 12-month rate is $\ln(1+11/89) = 11.65\%$.

For the 1.5-year bond we must have

$$4e^{-0.1238\times 0.5} + 4e^{-0.1165\times 1.0} + 104e^{-1.5R} = 94.84$$

where R is the 1.5-year spot rate. It follows that

$$3.76 + 3.56 + 104e^{-1.5R} = 94.84$$

$$e^{-1.5R} = 0.8415$$

$$R = 0.115$$

or 11.5%. For the 2-year bond we must have

$$5e^{-0.1238\times 0.5} + 5e^{-0.1165\times 1.0} + 5e^{-0.115\times 1.5} + 105e^{-2R} = 97.12$$

where R is the 2-year spot rate. It follows that

$$e^{-2R} = 0.7977$$
$$R = 0.113$$

or 11.3%.

ASSIGNMENT QUESTIONS

Problem 4.24.

An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

(a) With annual compounding the rate is $1.025^2 - 1 = 0.050625$ or 5.0625%

(b) With monthly compounding the rate is $12 \times (1.025^{1/6} - 1) = 0.04949$ or 4.949%.

(c) With continuous compounding the rate is $2 \times \ln 1.025 = 0.04939$or 4.939%.

Problem 4.25.

The 6-month, 12-month. 18-month,and 24-month zero rates are 4%, 4.5%, 4.75%, and 5% with semiannual compounding.
(a) What are the rates with continuous compounding?
(b) What is the forward rate for the six-month period beginning in 18 months
(c) What is the value of an FRA that promises to pay you 6% (compounded semiannually) on a principal of \$1 million for the six-month period starting in 18 months?

(a) With continuous compounding the 6-month rate is $2\ln 1.02 = 0.039605$ or 3.961%. The 12-month rate is $2\ln 1.0225 = 0.044501$ or 4.4501%. The 18-month rate is $2\ln 1.02375 = 0.046945$ or 4.6945%. The 24-month rate is $2\ln 1.025 = 0.049385$ or 4.9385%.

(b) The forward rate (expressed with continuous compounding) is from equation (4.5)

$$\frac{4.9385 \times 2 - 4.6945 \times 1.5}{0.5}$$

or 5.6707%. When expressed with semiannual compounding this is $2(e^{0.056707 \times 0.5} - 1) = 0.057518$ or 5.7518%.

(c) The value of an FRA that where you will receive 6% for the six month period starting in 18 months is from equation (4.9)

$$1,000,000 \times (0.06 - 0.057518) \times 0.5e^{-0.049385 \times 2} = 1,124$$

or \$1,124.

Problem 4.26.

What is the two-year par yield when the zero rates are as in Problem 4.25? What is the yield on a two-year bond that pays a coupon equal to the par yield?

The value, A of an annuity paying off \$1 every six months is

$$e^{-0.039605 \times 0.5} + e^{-0.044501 \times 1} + e^{-0.046945 \times 1.5} + e^{-0.049385 \times 2} = 3.7748$$

The present value of \$1 received in two years, d, is $e^{-0.049385 \times 2} = 0.90595$. From the formula in Section 4.4 the par yield is

$$\frac{(100 - 100 \times 0.90595) \times 2}{3.7748} = 4.983$$

or 4.983%.

Problem 4.27.

The following table gives the prices of bonds

Bond Principal ($)	*Time to Maturity (years)*	*Annual Coupon ($)**	*Bond Price ($)*
100	*0.50*	*0.0*	*98*
100	*1.00*	*0.0*	*95*
100	*1.50*	*6.2*	*101*
100	*2.00*	*8.0*	*104*

** Half the stated coupon is assumed to be paid every six months.*

a. *Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.*
b. *What are the forward rates for the periods: 6 months to 12 months, 12 months to 18 months, 18 months to 24 months?*
c. *What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?*
d. *Estimate the price and yield of a two-year bond providing a semiannual coupon of 7% per annum.*

(a) The zero rate for a maturity of six months, expressed with continuous compounding is $2\ln(1+2/98) = 4.0405\%$. The zero rate for a maturity of one year, expressed with continuous compounding is $\ln(1+5/95) = 5.1293$. The 1.5-year rate is R where

$$3.1e^{-0.040405\times 0.5} + 3.1e^{-0.051293\times 1} + 103.1e^{-R\times 1.5} = 101$$

The solution to this equation is $R = 0.054429$. The 2.0-year rate is R where

$$4e^{-0.040405\times 0.5} + 4e^{-0.051293\times 1} + 4e^{-0.054429\times 1.5} + 104e^{-R\times 2} = 104$$

The solution to this equation is $R = 0.058085$. These results are shown in the table below

Maturity (years)	Zero Rate (%)	Forward Rate (%)	Par Yield semi ann. (%)	Par Yield cont. comp.
0.5	4.0405	4.0405	4.0816	4.0405
1.0	5.1293	6.2181	5.1813	5.1154
1.5	5.4429	6.0700	5.4986	5.4244
2.0	5.8085	6.9054	5.8620	5.7778

(b) The continuously compounded forward rates calculated using equation (4.5) are shown in the third column of the table

(c) The par yield, expressed with semiannual compounding, can be calculated from the formula in Section 4.4. It is shown in the fourth column of the table. In the fifth column of the table it is converted to continuous compounding

(d) The price of the bond is

$$3.5e^{-0.040405\times 0.5} + 3.5e^{-0.051293\times 1} + 3.5e^{-0.054429\times 1.5} + 103.5e^{-0.058085\times 2} = 102.13$$

The yield on the bond, y satisfies

$$3.5e^{-y\times 0.5} + 3.5e^{-y\times 1.0} + 3.5e^{-y\times 1.5} + 103.5e^{-y\times 2.0} = 102.13$$

The solution to this equation is $y = 0.057723$. The bond yield is therefore 5.7723%.

Problem 4.28.

Portfolio A consists of a one-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum.

a. *Show that both portfolios have the same duration.*
b. *Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.*
c. *What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?*

(a) The duration of Portfolio A is

$$\frac{1 \times 2000e^{-0.1\times 1} + 10 \times 6000e^{-0.1\times 10}}{2000e^{-0.1\times 1} + 6000e^{-0.1\times 10}} = 5.95$$

Since this is also the duration of Portfolio B, the two portfolios do have the same duration.

(b) The value of Portfolio A is

$$2000e^{-0.1} + 6000e^{-0.1\times 10} = 4016.95$$

When yields increase by 10 basis points its value becomes

$$2000e^{-0.101} + 6000e^{-0.101\times 10} = 3993.18$$

The percentage decrease in value is

$$\frac{23.77 \times 100}{4016.95} = 0.59\%$$

The value of Portfolio B is

$$5000e^{-0.1\times 5.95} = 2757.81$$

When yields increase by 10 basis points its value becomes

$$5000e^{-0.101\times 5.95} = 2741.45$$

The percentage decrease in value is

$$\frac{16.36 \times 100}{2757.81} = 0.59\%$$

The percentage changes in the values of the two portfolios for a 10 basis point increase in yields are therefore the same.

(c) When yields increase by 5% the value of Portfolio A becomes

$$2000e^{-0.15} + 6000e^{-0.15 \times 10} = 3060.20$$

and the value of Portfolio B becomes

$$5000e^{-0.15 \times 5.95} = 2048.15$$

The percentage reduction in the values of the two portfolios are:

$$\text{Portfolio A:} \quad \frac{956.75}{4016.95} \times 100 = 23.82$$
$$\text{Portfolio B:} \quad \frac{709.66}{2757.81} \times 100 = 25.73$$

Since the percentage decline in value of Portfolio A is less than that of Portfolio B, Portfolio A has a greater convexity (see Figure 4.2 in text).

CHAPTER 5

Determination of Forward and Futures Prices

Notes for the Instructor

This chapter covers the relationship between forward/futures prices and spot prices. The approach used in the chapter is to produce results for forward prices first and then argue that futures prices are very close to forward prices. The early part of the chapter explains short selling and the difference between investment and consumption assets.

I usually go through the material in Section 5.4 fairly carefully to make sure that students understand the nature of the arguments that are used. (Business Snapshot 5.1, a description of Joseph Jett's trading at Kidder Peabody, helps to explain why equation 5.1 holds.) I then go through Sections 5.5 and 5.6 fairly quickly because the arguments in those sections are really just extensions of the argument in Section 5.4. However, it is necessary to explain carefully the difference between a known cash dividend and a known dividend yield.

When covering Section 5.7, I emphasize the distinction between f (the value of a long forward contract) and F_0 (the forward price). This often causes confusion. I like to go through Business Snapshot 5.2 to help students understand the issue.

If time permits I like to go through the material in the appendix. It reinforces the students' understanding of how futures contracts work and provides an interesting pure arbitrage argument.

The material in Sections 5.9 and 5.10 follows naturally from the material in Sections 5.4 to 5.6. I try to illustrate all of the formulas with numerical examples taken from current market quotes. The interpretation of a foreign currency as an investment providing a yield equal to the foreign risk-free rate needs to be explained carefully. I also like to spend some time discussing the fact that the variable underlying the CME Nikkei futures contract is not something that can be traded (see Business Snapshot 5.3).

It is important that students understand the distinction between assets that are held solely for investment by a significant number of investors and those that are not. This distinction is made right at the beginning of the chapter.

Section 5.14 ties the relationship between a futures prices and an expected future spot price to the notion of systematic risk, which will probably be familiar to students from other courses they have taken.

Problems 5.26 and 5.28 can be used for discussion in class. Problems 5.24, 5.25, and 5.27 can be used as assignment questions. My favorites are 5.25 and 5.27.

QUESTIONS AND PROBLEMS

Problem 5.1.

Explain what happens when an investor shorts a certain share.

The investor's broker borrows the shares from another client's account and sells them in the usual way. To close out the position, the investor must purchase the shares. The broker then replaces them in the account of the client from whom they were borrowed. The party with the short position must remit to the broker dividends and other income paid on the shares. The broker transfers these funds to the account of the client from whom the shares were borrowed. Occasionally the broker runs out of places from which to borrow the shares. The investor is then short squeezed and has to close out the position immediately.

Problem 5.2.

What is the difference between the forward price and the value of a forward contract?

The forward price of an asset today is the price at which you would agree to buy or sell the asset at a future time. The value of a forward contract is zero when you first enter into it. As time passes the underlying asset price changes and the value of the contract may become positive or negative.

Problem 5.3.

Suppose that you enter into a six-month forward contract on a non-dividend-paying stock when the stock price is \$30 and the risk-free interest rate (with continuous compounding) is 12% per annum. What is the forward price?

The forward price is

$$30e^{0.12\times 0.5} = \$31.86$$

Problem 5.4.

A stock index currently stands at 350. The risk-free interest rate is 8% per annum (with continuous compounding) and the dividend yield on the index is 4% per annum. What should the futures price for a four-month contract be?

The futures price is

$$350e^{(0.08-0.04)\times 0.3333} = \$354.7$$

Problem 5.5.

Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.

Gold is an investment asset. If the futures price is too high, investors will find it profitable to increase their holdings of gold and short futures contracts. If the futures price is too low, they will find it profitable to decrease their holdings of gold and go long in the futures market. Copper is a consumption asset. If the futures price is too high, a strategy of buy copper and short futures works. However, because investors do not in general hold the asset, the strategy of sell copper and buy futures is not available to them. There is therefore an upper bound, but no lower bound, to the futures price.

Problem 5.6.

Explain carefully the meaning of the terms convenience yield and cost of carry. What is the relationship between futures price, spot price, convenience yield, and cost of carry?

Convenience yield measures the extent to which there are benefits obtained from ownership of the physical asset that are not obtained by owners of long futures contracts. The *cost of carry* is the interest cost plus storage cost less the income earned. The futures price, F_0, and spot price, S_0, are related by

$$F_0 = S_0 e^{(c-y)T}$$

where c is the cost of carry, y is the convenience yield, and T is the time to maturity of the futures contract.

Problem 5.7.

Explain why a foreign currency can be treated as an asset providing a known yield.

A foreign currency provides a known interest rate, but the interest is received in the foreign currency. The value in the domestic currency of the income provided by the foreign currency is therefore known as a percentage of the value of the foreign currency. This means that the income has the properties of a known yield.

Problem 5.8.

Is the futures price of a stock index greater than or less than the expected future value of the index? Explain your answer.

The futures price of a stock index is always less than the expected future value of the index. This follows from Section 5.14 and the fact that the index has positive systematic risk. For an alternative argument, let μ be the expected return required by investors on the index so that $E(S_T) = S_0 e^{(\mu-q)T}$. Because $\mu > r$ and $F_0 = S_0 e^{(r-q)T}$, it follows that $E(S_T) > F_0$.

Problem 5.9.

A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free rate of interest is 10% per annum with continuous compounding.

a. What are the forward price and the initial value of the forward contract?

b. Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

(a) The forward price, F_0, is given by equation (5.1) as:

$$F_0 = 40e^{0.1\times 1} = 44.21$$

or $44.21. The initial value of the forward contract is zero.

(b) The delivery price K in the contract is $44.21. The value of the contract, f, after six months is given by equation (5.5) as:

$$f = 45 - 44.21e^{-0.1\times0.5}$$

$$= 2.95$$

i.e., it is $2.95. The forward price is:

$$45e^{0.1\times0.5} = 47.31$$

or $47.31.

Problem 5.10.

The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3.2% per annum. The current value of the index is 150. What is the six-month futures price?

Using equation (5.3) the six month futures price is

$$150e^{(0.07-0.032)\times0.5} = 152.88$$

or $152.88.

Problem 5.11.

Assume that the risk-free interest rate is 9% per annum with continuous compounding and that the dividend yield on a stock index varies throughout the year. In February, May, August, and November, dividends are paid at a rate of 5% per annum. In other months, dividends are paid at a rate of 2% per annum. Suppose that the value of the index on July 31 is 1,300. What is the futures price for a contract deliverable on December 31 of the same year?

The futures contract lasts for five months. The dividend yield is 2% for three of the months and 5% for two of the months. The average dividend yield is therefore

$$\frac{1}{5}(3\times2+2\times5) = 3.2\%$$

The futures price is therefore

$$1300e^{(0.09-0.032)\times0.4167} = 1,331.80$$

or $1331.80.

Problem 5.12.

Suppose that the risk-free interest rate is 10% per annum with continuous compounding and that the dividend yield on a stock index is 4% per annum. The index is standing at 400, and the futures price for a contract deliverable in four months is 405. What arbitrage opportunities does this create?

The theoretical futures price is

$$400e^{(0.10-0.04)\times 4/12} = 408.08$$

The actual futures price is only 405. This shows that the index futures price is too low relative to the index. The correct arbitrage strategy is

1. Buy futures contracts
2. Short the shares underlying the index.

Problem 5.13.

Estimate the difference between short-term interest rates in Mexico and the United States on January 8, 2007 from the information in Table 5.4.

The settlement prices for the futures contracts are

Jan	0.91250
Mar	0.91025

The March 2007 price is about 0.25% below the January 2007 price. This suggests that the short-term interest rate in the Mexico exceeded short-term interest rates in the United States by about 0.25% per two months or about 1.5% per year.

Problem 5.14.

The two-month interest rates in Switzerland and the United States are 2% and 5% per annum, respectively, with continuous compounding. The spot price of the Swiss franc is \$0.8000. The futures price for a contract deliverable in two months is \$0.8100. What arbitrage opportunities does this create?

The theoretical futures price is

$$0.8000e^{(0.05-0.02)\times 2/12} = 0.8040$$

The actual futures price is too high. This suggests that an arbitrageur should buy Swiss francs and short Swiss francs futures.

Problem 5.15.

The spot price of silver is \$9 per ounce. The storage costs are \$0.24 per ounce per year payable quarterly in advance. Assuming that interest rates are 10% per annum for all maturities, calculate the futures price of silver for delivery in nine months.

The present value of the storage costs for nine months are

$$0.06 + 0.06e^{-0.10\times 0.25} + 0.06e^{-0.10\times 0.5} = 0.176$$

or \$0.176. The futures price is from equation (5.11) given by F_0 where

$$F_0 = (9.000 + 0.176)e^{0.1\times 0.75} = 9.89$$

i.e., it is \$9.89 per ounce.

Problem 5.16.

Suppose that F_1 and F_2 are two futures contracts on the same commodity with times to maturity, t_1 and t_2, where $t_2 > t_1$. Prove that

$$F_2 \leq F_1 e^{r(t_2-t_1)}$$

where r is the interest rate (assumed constant) and there are no storage costs. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

If

$$F_2 > F_1 e^{r(t_2-t_1)}$$

an investor could make a riskless profit by

1. Taking a long position in a futures contract which matures at time t_1
2. Taking a short position in a futures contract which matures at time t_2

When the first futures contract matures, the asset is purchased for F_1 using funds borrowed at rate r. It is then held until time t_2 at which point it is exchanged for F_2 under the second contract. The costs of the funds borrowed and accumulated interest at time t_2 is $F_1 e^{r(t_2-t_1)}$ A positive profit of

$$F_2 - F_1 e^{r(t_2-t_1)}$$

is then realized at time t_2. This type of arbitrage opportunity cannot exist for long. Hence:

$$F_2 \leq F_1 e^{r(t_2-t_1)}$$

Problem 5.17.

When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the marking-to-market process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when

a. The value of the foreign currency falls rapidly during the life of the contract
b. The value of the foreign currency rises rapidly during the life of the contract
c. The value of the foreign currency first rises and then falls back to its initial value
d. The value of the foreign currency first falls and then rises back to its initial value

Assume that the forward price equals the futures price.

In total the gain or loss under a futures contract is equal to the gain or loss under the corresponding forward contract. However the timing of the cash flows is different. When the time value of money is taken into account a futures contract may prove to be more valuable or less valuable than a forward contract. Of course the company does not know in advance which will work out better. The long forward contract provides a perfect hedge. The long futures contract provides a slightly imperfect hedge.

(a) In this case the forward contract would lead to a slightly better outcome. The company will make a loss on its hedge. If the hedge is with a forward contract the whole of the loss will be realized at the end. If it is with a futures contract the loss will be realized day by day throughout the contract. On a present value basis the former is preferable.

(b) In this case the futures contract would lead to a slightly better outcome. The company will make a gain on the hedge. If the hedge is with a forward contract the gain will be realized at the end. If it is with a futures contract the gain will be realized day by day throughout the life of the contract. On a present value basis the latter is preferable.

(c) In this case the futures contract would lead to a slightly better outcome. This is because it would involve positive cash flows early and negative cash flows later.

(d) In this case the forward contract would lead to a slightly better outcome. This is because, in the case of the futures contract, the early cash flows would be negative and the later cash flow would be positive.

Problem 5.18.

It is sometimes argued that a forward exchange rate is an unbiased predictor of future exchange rates. Under what circumstances is this so?

From the discussion in Section 5.14 of the text, the forward exchange rate is an unbiased predictor of the future exchange rate when the exchange rate has no systematic risk. To have no systematic risk the exchange rate must be uncorrelated with the return on the market.

Problem 5.19.

Show that the growth rate in an index futures price equals the excess return of the index over the risk-free rate. Assume that the risk-free interest rate and the dividend yield are constant.

Suppose that F_0 is the futures price at time zero for a contract maturing at time T and F_1 is the futures price for the same contract at time t_1. It follows that

$$F_0 = S_0 e^{(r-q)T}$$

$$F_1 = S_1 e^{(r-q)(T-t_1)}$$

where S_0 and S_1 are the spot price at times zero and t_1, r is the risk-free rate, and q is the dividend yield. These equations imply that

$$\frac{F_1}{F_0} = \frac{S_1}{S_0} e^{-(r-q)t_1}$$

Define the excess return of the index over the risk-free rate as x. The total return is $r + x$ and the return realized in the form of capital gains is $r + x - q$. It follows that $S_1 = S_0 e^{(r+x-q)t_1}$ and the equation for F_1/F_0 reduces to

$$\frac{F_1}{F_0} = e^{xt_1}$$

which is the required result.

Problem 5.20.

Show that equation (5.3) is true by considering an investment in the asset combined with a short position in a futures contract. Assume that all income from the asset is reinvested in the asset. Use an argument similar to that in footnotes 2 and 4 and explain in detail what an arbitrageur would do if equation (5.3) did not hold.

Suppose we buy N units of the asset and invest the income from the asset in the asset. The income from the asset causes our holding in the asset to grow at a continuously compounded rate q. By time T our holding has grown to Ne^{qT} units of the asset. Analogously to footnotes 2 and 4 of Chapter 5, we therefore buy N units of the asset at time zero at a cost of S_0 per unit and enter into a forward contract to sell Ne^{qT} unit for F_0 per unit at time T. This generates the following cash flows:

Time 0: $-NS_0$

Time T: NF_0e^{qT}

Because there is no uncertainty about these cash flows, the present value of the time T inflow must equal the time zero outflow when we discount at the risk-free rate. This means that

$$NS_0 = (NF_0e^{qT})e^{-rT}$$

or

$$F_0 = S_0e^{(r-q)T}$$

This is equation (5.3).

If $F_0 > S_0e^{(r-q)T}$, an arbitrageur should borrow money at rate r and buy N units of the asset. At the same time the arbitrageur should enter into a forward contract to sell Ne^{qT} units of the asset at time T. As income is received, it is reinvested in the asset. At time T the loan is repaid and the arbitrageur makes a profit of $N(F_0e^{qT} - S_0e^{rT})$ at time T.

If $F_0 < S_0e^{(r-q)T}$, an arbitrageur should short N units of the asset investing the proceeds at rate r. At the same time the arbitrageur should enter into a forward contract to buy Ne^{qT} units of the asset at time T. When income is paid on the asset, the arbitrageur owes money on the short position. The investor meets this obligation from the cash proceeds of shorting further units. The result is that the number of units shorted grows at rate q to Ne^{qT}. The cumulative short position is closed out at time T and the arbitrageur makes a profit of $N(S_0e^{rT} - F_0e^{qT})$.

Problem 5.21.

Explain carefully what is meant by the expected price of a commodity on a particular future date. Suppose that the futures price of crude oil declines with the maturity of the contract at the rate of 2% per year. Assume that speculators tend to be short crude oil futures and hedgers tended to be long. What does the Keynes and Hicks argument imply about the expected future price of oil?

To understand the meaning of the expected future price of a commodity, suppose that there are N different possible prices at a particular future time: P_1, P_2, ..., P_N. Define

q_i as the (subjective) probability the price being P_i (with $q_1 + q_2 + \ldots + q_N = 1$). The expected future price is

$$\sum_{i=1}^{N} q_i P_i$$

Different people may have different expected future prices for the commodity. The expected future price in the market can be thought of as an average of the opinions of different market participants. Of course, in practice the actual price of the commodity at the future time may prove to be higher or lower than the expected price.

Keynes and Hicks argue that speculators on average make money from commodity futures trading and hedgers on average lose money from commodity futures trading. If speculators tend to have short positions in crude oil futures, the Keynes and Hicks argument implies that futures prices overstate expected future spot prices. If crude oil futures prices decline at 2% per year the Keynes and Hicks argument therefore implies an even faster decline for the expected price of crude oil.

Problem 5.22.

The Value Line Index is designed to reflect changes in the value of a portfolio of over 1,600 equally weighted stocks. Prior to March 9, 1988, the change in the index from one day to the next was calculated as the geometric average of the changes in the prices of the stocks underlying the index. In these circumstances, does equation (5.8) correctly relate the futures price of the index to its cash price? If not, does the equation overstate or understate the futures price?

When the geometric average of the price relatives is used, the changes in the value of the index do not correspond to changes in the value of a portfolio that is traded. Equation (5.8) is therefore no longer correct. The changes in the value of the portfolio is monitored by an index calculated from the arithmetic average of the prices of the stocks in the portfolio. Since the geometric average of a set of numbers is always less than the arithmetic average, equation (5.8) overstates the futures price. It is rumored that at one time (prior to 1988), equation (5.8) did hold for the Value Line Index. A major Wall Street firm was the first to recognize that this represented a trading opportunity. It made a financial killing by buying the stocks underlying the index and shorting the futures.

Problem 5.23.

A U.S. company is interested in using the futures contracts traded on the CME to hedge its Australian dollar exposure. Define r as the interest rate (all maturities) on the U.S. dollar and r_f as the interest rate (all maturities) on the Australian dollar. Assume that r and r_f are constant and that the company uses a contract expiring at time T to hedge an exposure at time t ($T > t$).

a. *Show that the optimal hedge ratio is*

$$e^{(r_f - r)(T-t)}$$

b. *Show that, when t is one day, the optimal hedge ratio is almost exactly S_0/F_0 where S_0 is the current spot price of the currency and F_0 is the current futures price of the currency for the contract maturing at time T.*

c. Show that the company can take account of the daily settlement of futures contracts for a hedge that lasts longer than one day by adjusting the hedge ratio so that it always equals the spot price of the currency divided by the futures price of the currency.

(a) The relationship between the futures price F_t and the spot price S_t at time t is

$$F_t = S_t e^{(r-r_f)(T-t)}$$

Suppose that the hedge ratio is h. The price obtained with hedging is

$$h(F_0 - F_t) + S_t$$

where F_0 is the initial futures price. This is

$$hF_0 + S_t - hS_t e^{(r-r_f)(T-t)}$$

If $h = e^{(r_f-r)(T-t)}$, this reduces to hF_0 and a zero variance hedge is obtained.

(b) When t is one day, h is approximately $e^{(r_f-r)T} = S_0/F_0$. The appropriate hedge ratio is therefore S_0/F_0.

(c) When a futures contract is used for hedging, the price movements in each day should in theory be hedged separately. This is because the daily settlement means that a futures contract is closed out and rewritten at the end of each day. From (b) the correct hedge ratio at any given time is, therefore, S/F where S is the spot price and F is the futures price. Suppose there is an exposure to N units of the foreign currency and M units of the foreign currency underlie one futures contract. With a hedge ratio of 1 we should trade N/M contracts. With a hedge ratio of S/F we should trade

$$\frac{SN}{FM}$$

contracts. In other words we should calculate the number of contracts that should be traded as the dollar value of our exposure divided by the dollar value of one futures contract (This is not the same as the dollar value of our exposure divided by the dollar value of the assets underlying one futures contract.) Since a futures contract is settled daily, we should in theory rebalance our hedge daily so that the outstanding number of futures contracts is always $(SN)/(FM)$. This is known as tailing the hedge. (See Section 3.4 of the text.)

ASSIGNMENT QUESTIONS

Problem 5.24.

A stock is expected to pay a dividend of \$1 per share in two months and in five months. The stock price is \$50, and the risk-free rate of interest is 8% per annum with *continuous compounding for all maturities. An investor has just taken a short position in a six-month forward contract on the stock.*

a. What are the forward price and the initial value of the forward contract?

b. Three months later, the price of the stock is \$48 and the risk-free rate of interest is still 8% per annum. What are the forward price and the value of the short position in the forward contract?

(a) The present value, I, of the income from the security is given by:

$$I = 1 \times e^{-0.08\times2/12} + 1 \times e^{-0.08\times5/12} = 1.9540$$

From equation (5.2) the forward price, F_0, is given by:

$$F_0 = (50 - 1.9540)e^{0.08\times0.5} = 50.01$$

or \$50.01. The initial value of the forward contract is (by design) zero. The fact that the forward price is very close to the spot price should come as no surprise. When the compounding frequency is ignored the dividend yield on the stock equals the risk-free rate of interest.

(b) In three months:

$$I = e^{-0.08\times2/12} = 0.9868$$

The delivery price, K, is 50.01. From equation (5.6) the value of the short forward contract, f, is given by

$$f = -(48 - 0.9868 - 50.01e^{-0.08\times3/12}) = 2.01$$

and the forward price is

$$(48 - 0.9868)e^{0.08\times3/12} = 47.96$$

Problem 5.25.

A bank offers a corporate client a choice between borrowing cash at 11% per annum and borrowing gold at 2% per annum. (If gold is borrowed, interest must be repaid in gold. Thus, 100 ounces borrowed today would require 102 ounces to be repaid in one year.) The risk-free interest rate is 9.25% per annum, and storage costs are 0.5% per annum. Discuss whether the rate of interest on the gold loan is too high or too low in relation to the rate of interest on the cash loan. The interest rates on the two loans are expressed with annual compounding. The risk-free interest rate and storage costs are expressed with continuous compounding.

My explanation of this problem to students usually goes as follows. Suppose that the price of gold is \$550 per ounce and the corporate client wants to borrow \$550,000. The client has a choice between borrowing \$550,000 in the usual way and borrowing 1,000 ounces of gold. If it borrows \$550,000 in the usual way, an amount equal to $550,000\times1.11 =$

$610,500 must be repaid. If it borrows 1,000 ounces of gold it must repay 1,020 ounces. In equation (5.12), $r = 0.0925$ and $u = 0.005$ so that the forward price is

$$550e^{(0.0925+0.005)\times 1} = 606.33$$

By buying 1,020 ounces of gold in the forward market the corporate client can ensure that the repayment of the gold loan costs

$$1,020 \times 606.33 = \$618,457$$

Clearly the cash loan is the better deal $(618,457 > 610,500)$.

This argument shows that the rate of interest on the gold loan is too high. What is the correct rate of interest? Suppose that R is the rate of interest on the gold loan. The client must repay $1,000(1+R)$ ounces of gold. When forward contracts are used the cost of this is

$$1,000(1+R) \times 606.33$$

This equals the $610,500 required on the cash loan when $R = 0.688\%$. The rate of interest on the gold loan is too high by about 1.31%. However, this might be simply a reflection of the higher administrative costs incurred with a gold loan.

It is interesting to note that this is not an artificial question. Many banks are prepared to make gold loans at interest rates of about 2% per annum.

Problem 5.26.

A company that is uncertain about the exact date when it will pay or receive a foreign currency may try to negotiate with its bank a forward contract that specifies a period during which delivery can be made. The company wants to reserve the right to choose the exact delivery date to fit in with its own cash flows. Put yourself in the position of the bank. How would you price the product that the company wants?

It is likely that the bank will price the product on assumption that the company chooses the delivery date least favorable to the bank. If the foreign interest rate is higher than the domestic interest rate then

1. The earliest delivery date will be assumed when the company has a long position.
2. The latest delivery date will be assumed when the company has a short position.

If the foreign interest rate is lower than the domestic interest rate then

1. The latest delivery date will be assumed when the company has a long position.
2. The earliest delivery date will be assumed when the company has a short position.

If the company chooses a delivery which, from a purely financial viewpoint, is suboptimal the bank makes a gain.

Problem 5.27.

A trader owns gold as part of a long-term investment portfolio. The trader can buy gold for \$550 per ounce and sell gold for \$549 per ounce. The trader can borrow funds at 6% per year and invest funds at 5.5% per year. (Both interest rates are expressed with *annual compounding.) For what range of one-year forward prices of gold does the trader have no arbitrage opportunities? Assume there is no bid–offer spread for forward prices.*

Suppose that F_0 is the one-year forward price of gold. If F_0 is relatively high, the trader can borrow \$550 at 6%, buy one ounce of gold and enter into a forward contract to sell gold in one year for F_0. The profit made in one year is

$$F_0 - 550 \times 1.06 = F_0 - 583$$

If F_0 is relatively low, the trader can sell one ounce of gold for \$549, invest the proceeds at 5.5%, and enter into a forward contract to buy the gold back for F_0. The profit (relative to the position the trader would be in if the gold were held in the portfolio during the year) is

$$549 \times 1.055 - F_0 = 579.195 - F_0$$

This shows that there is no arbitrage opportunity if the forward price is between \$579.195 and \$583 per ounce.

Problem 5.28.

A company enters into a forward contract with a bank to sell a foreign currency for K_1 at time T_1. The exchange rate at time T_1 proves to be S_1 ($> K_1$). The company asks the bank if it can roll the contract forward until time T_2 ($> T_1$) rather than settle at time T_1. The bank agrees to a new delivery price, K_2. Explain how K_2 should be calculated.

The value of the contract to the bank at time T_1 is $S_1 - K_1$. The bank will choose K_2 so that the new (rolled forward) contract has a value of $S_1 - K_1$. This means that

$$S_1 e^{-r_f(T_2-T_1)} - K_2 e^{-r(T_2-T_1)} = S_1 - K_1$$

where r and r_f and the domestic and foreign risk-free rate observed at time T_1 and applicable to the period between time T_1 and T_2. This means that

$$K_2 = S_1 e^{(r-r_f)(T_2-T_1)} - (S_1 - K_1)e^{r(T_2-T_1)}$$

This equation shows that there are two components to K_2. The first is the forward price at time T_1. The second is an adjustment to the forward price equal to the bank's gain on the first part of the contract compounded forward at the domestic risk-free rate.

CHAPTER 6
Interest Rate Futures

Notes for the Instructor

This chapter discusses how interest rate futures contracts are quoted, how they work, and how they are used for hedging. I start by discussing the material in Sections 6.1 and 6.2 on day counts and how prices are quoted in the spot market. (It is fun to talk about Business Snapshot 6.1 when day count conventions are discussed.) I like to spend some time making sure students are comfortable with the Treasury bond futures contracts and the Eurodollar futures contract. In the case of the Treasury bond futures contract they should understand where conversion factors come from, the cheapest-to-deliver bond calculations, and the wild card play (see Business Snapshot 6.2). In the case of Eurodollar futures they should understand the quotation system, that the contract's value changes by \$25 for each basis point change in the quote, and how the final cash settlement works. In the slides I have included a numerical example help explain this.

Students should also appreciate that a convexity adjustment is necessary to calculate a forward rate from a Eurodollar futures quote. They will not at this stage understand where equation 6.3 comes from, but they should understand that there are two reasons why forward and futures interest rates are different. The first is that futures are settled daily; forwards are not. The second is that futures (if not daily settled) would provide a payoff at the beginning of the period covered by the rate; forwards provide a payoff at the end of the period covered by the rate.

The final part of the chapter covers the use of interest rate futures for duration-based hedging. I usually illustrate this material with a numerical example.

I sometimes use Problem 6.23 in class to help explain how Eurodollar futures contracts work and the impact of day count conventions. (Without adjusting for the day count convention, the arbitrage opportunity appears to be the other way round.) Problems 6.24 and 6.26 can be used as assignment questions.

QUESTIONS AND PROBLEMS

Problem 6.1.

A U.S. Treasury bond pays a 7% coupon on January 7 and July 7. How much interest accrue per \$100 of principal to the bond holder between July 7, 2009 and August 9, 2009? How would your answer be different if it were a corporate bond?

There are 33 calendar days between July 7, 2009 and August 9, 2009. There are 184 calendar days between July 7, 2009 and January 7, 2010. The interest earned per \$100 of principal is therefore $3.5 \times 33/184 = \$0.6277$. For a corporate bond we assume 32 days between July 7 and August 9, 2009 and 180 days between July 7, 2009 and January 7, 2010. The interest earned is $3.5 \times 32/180 = \$0.6222$.

Problem 6.2.

It is January 9, 2009. The price of a Treasury bond with a 12% coupon that matures on October 12, 2020, is quoted as 102-07. What is the cash price?

There are 89 days between October 12, 2009, and January 9, 2010. There are 182 days between October 12, 2009, and April 12, 2010. The cash price of the bond is obtained by adding the accrued interest to the quoted price. The quoted price is $102\frac{7}{32}$ or 102.21875. The cash price is therefore

$$102.21875 + \frac{89}{182} \times 6 = \$105.15$$

Problem 6.3.

How is the conversion factor of a bond calculated by the Chicago Board of Trade? How is it used?

The conversion factor for a bond is equal to the quoted price the bond would have per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equals 6% per annum (with semiannual compounding). The bond maturity and the times to the coupon payment dates are rounded down to the nearest three months for the purposes of the calculation. The conversion factor defines how much an investor with a short bond futures contract receives when bonds are delivered. If the conversion factor is 1.2345 the amount investor receives is calculated by multiplying 1.2345 by the most recent futures price and adding accrued interest.

Problem 6.4.

A Eurodollar futures price changes from 96.76 to 96.82. What is the gain or loss to an investor who is long two contracts?

The Eurodollar futures price has increased by 6 basis points. The investor makes a gain per contract of $25 \times 6 = \$150$ or \$300 in total.

Problem 6.5.

What is the purpose of the convexity adjustment made to Eurodollar futures rates? Why is the convexity adjustment necessary?

Suppose that a Eurodollar futures quote is 95.00. This gives a futures rate of 5% for the three-month period covered by the contract. The convexity adjustment is the amount by which futures rate has to be reduced to give an estimate of the forward rate for the period The convexity adjustment is necessary because a) the futures contract is settled daily and b) the futures contract expires at the beginning of the three months. Both of these lead to the futures rate being greater than the forward rate.

Problem 6.6.

The 350-day LIBOR rate is 3% with continuous compounding and the forward rate calculated from a Eurodollar futures contract that matures in 350 days is 3.2% with continuous compounding. Estimate the 440-day zero rate.

From equation (6.4) the rate is

$$\frac{3.2 \times 90 + 3 \times 350}{440} = 3.0409$$

or 3.0409%.

Problem 6.7.

It is January 30. You are managing a bond portfolio worth $6 million. The duration of the portfolio in six months will be 8.2 years. The September Treasury bond futures price is currently 108-15, and the cheapest-to-deliver bond will have a duration of 7.6 years in September. How should you hedge against changes in interest rates over the next six months?

The value of a contract is $108\frac{15}{32} \times 1,000 = \$108,468.75$. The number of contracts that should be shorted is

$$\frac{6,000,000}{108,468.75} \times \frac{8.2}{7.6} = 59.7$$

Rounding to the nearest whole number, 60 contracts should be shorted. The position should be closed out at the end of July.

Problem 6.8.

The price of a 90-day Treasury bill is quoted as 10.00. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?

The cash price of the Treasury bill is

$$100 - \frac{90}{360} \times 10 = \$97.50$$

The annualized continuously compounded return is

$$\frac{365}{90} \ln\left(1 + \frac{2.5}{97.5}\right) = 10.27\%$$

Problem 6.9.

It is May 5, 2008. The quoted price of a government bond with a 12% coupon that matures on July 27, 2011, is 110-17. What is the cash price?

The number of days between January 27, 2008 and May 5, 2008 is 99. The number of days between January 27, 2008 and July 27, 2008 is 182. The accrued interest is therefore

$$6 \times \frac{99}{182} = 3.2637$$

The quoted price is 110.5312. The cash price is therefore

$$110.5312 + 3.2637 = 113.7949$$

or \$113.79.

Problem 6.10.

Suppose that the Treasury bond futures price is 101-12. Which of the following four bonds is cheapest to deliver?

Bond	*Price*	*Conversion Factor*
1	*125-05*	*1.2131*
2	*142-15*	*1.3792*
3	*115-31*	*1.1149*
4	*144-02*	*1.4026*

The cheapest-to-deliver bond is the one for which

$$\text{Quoted Price} - \text{Futures Price} \times \text{Conversion Factor}$$

is least. Calculating this factor for each of the 4 bonds we get

$$\text{Bond 1: } 125.15625 - 101.375 \times 1.2131 = 2.178$$
$$\text{Bond 2: } 142.46875 - 101.375 \times 1.3792 = 2.652$$
$$\text{Bond 3: } 115.96875 - 101.375 \times 1.1149 = 2.946$$
$$\text{Bond 4: } 144.06250 - 101.375 \times 1.4026 = 1.874$$

Bond 4 is therefore the cheapest to deliver.

Problem 6.11.

It is July 30, 2009. The cheapest-to-deliver bond in a September 2009 Treasury bond futures contract is a 13% coupon bond, and delivery is expected to be made on September 30, 2009. Coupon payments on the bond are made on February 4 and August 4 each year. The term structure is flat, and the rate of interest with semiannual compounding is 12% per annum. The conversion factor for the bond is 1.5. The current quoted bond price is \$110. Calculate the quoted futures price for the contract.

There are 176 days between February 4 and July 30 and 181 days between February 4 and August 4. The cash price of the bond is, therefore:

$$110 + \frac{176}{181} \times 6.5 = 116.32$$

The rate of interest with continuous compounding is $2\ln 1.06 = 0.1165$ or 11.65% per annum. A coupon of 6.5 will be received in 5 days (= 0.01370 years) time. The present value of the coupon is

$$6.5e^{-0.01370\times 0.1165} = 6.490$$

The futures contract lasts for 62 days (= 0.1699 years). The cash futures price if the contract were written on the 13% bond would be

$$(116.32 - 6.490)e^{0.1699\times 0.1165} = 112.03$$

At delivery there are 57 days of accrued interest. The quoted futures price if the contract were written on the 13% bond would therefore be

$$112.03 - 6.5 \times \frac{57}{184} = 110.01$$

Taking the conversion factor into account the quoted futures price should be:

$$\frac{110.01}{1.5} = 73.34$$

Problem 6.12.

An investor is looking for arbitrage opportunities in the Treasury bond futures market. What complications are created by the fact that the party with a short position can choose to deliver any bond with a maturity of over 15 years?

If the bond to be delivered and the time of delivery were known, arbitrage would be straightforward. When the futures price is too high, the arbitrageur buys bonds and shorts an equivalent number of bond futures contracts. When the futures price is too low, the arbitrageur sells bonds and goes long an equivalent number of bond futures contracts.

Uncertainty as to which bond will be delivered introduces complications. The bond that appears cheapest-to-deliver now may not in fact be cheapest-to-deliver at maturity. In the case where the futures price is too high, this is not a major problem since the party with the short position (i.e., the arbitrageur) determines which bond is to be delivered. In the case where the futures price is too low, the arbitrageur's position is far more difficult since he or she does not know which bond to buy; it is unlikely that a profit can be locked in for all possible outcomes.

Problem 6.13.

Suppose that the nine-month LIBOR interest rate is 8% per annum and the six-month LIBOR interest rate is 7.5% per annum (both with actual/365 and continuous compounding). Estimate the three-month Eurodollar futures price quote for a contract maturing in six months.

The forward interest rate for the time period between months 6 and 9 is 9% per annum with continuous compounding. This is because 9% per annum for three months when combined with $7\frac{1}{2}\%$ per annum for six months gives an average interest rate of 8% per annum for the nine-month period.

With quarterly compounding the forward interest rate is

$$4(e^{0.09/4} - 1) = 0.09102$$

or 9.102%. This assumes that the day count is actual/actual. With a day count of actual/360 the rate is $9.102 \times 360/365 = 8.977$. The three-month Eurodollar quote for a contract maturing in six months is therefore

$$100 - 8.977 = 91.02$$

This assumes no difference between futures and forward prices.

Problem 6.14.

Suppose that the 300-day LIBOR zero rate is 4% and Eurodollar quotes for contracts maturing in 300, 398 and 489 days are 95.83, 95.62, and 95.48. Calculate 398-day and 489-day LIBOR zero rates. Assume no difference between forward and futures rates for the purposes of your calculations.

The forward rates calculated form the first two Eurodollar futures are 4.17% and 4.38%. These are expressed with an actual/360 day count and quarterly compounding. With continuous compounding and an actual/365 day count they are $(365/90)\ln(1 + 0.0417/4) = 4.2060\%$ and $(365/90)\ln(1 + 0.0438/4) = 4.4167\%$. It follows from equation (6.4) that the 398 day rate is

$$\frac{4 \times 300 + 4.2060 \times 98}{398} = 4.0507$$

or 4.0507%. The 489 day rate is

$$\frac{4.0507 \times 398 + 4.4167 \times 91}{489} = 4.1188$$

or 4.1188%. We are assuming that the first futures rate applies to 98 days rather than the usual 91 days. The third futures quote is not needed.

Problem 6.15.

Suppose that a bond portfolio with a duration of 12 years is hedged using a futures contract in which the underlying asset has a duration of four years. What is likely to be the impact on the hedge of the fact that the 12-year rate is less volatile than the four-year rate?

Duration-based hedging schemes assume parallel shifts in the yield curve. Since the 12-year rate tends to move by less than the 4-year rate, the portfolio manager may find that he or she is over-hedged.

Problem 6.16.

Suppose that it is February 20 and a treasurer realizes that on July 17 the company will have to issue $5 million of commercial paper with a maturity of 180 days. If the paper were issued today, the company would realize $4,820,000. (In other words, the company would receive $4,820,000 for its paper and have to redeem it at $5,000,000 in 180 days' time.) The September Eurodollar futures price is quoted as 92.00. How should the treasurer hedge the company's exposure?

The company treasurer can hedge the company's exposure by shorting Eurodollar futures contracts. The Eurodollar futures position leads to a profit if rates rise and a loss if they fall.

The duration of the commercial paper is twice that of the Eurodollar deposit underlying the Eurodollar futures contract. The contract price of a Eurodollar futures contract is 980,000. The number of contracts that should be shorted is, therefore,

$$\frac{4,820,000}{980,000} \times 2 = 9.84$$

Rounding to the nearest whole number 10 contracts should be shorted.

Problem 6.17.

On August 1 a portfolio manager has a bond portfolio worth $10 million. The duration of the portfolio in October will be 7.1 years. The December Treasury bond futures price is currently 91-12 and the cheapest-to-deliver bond will have a duration of 8.8 years at maturity. How should the portfolio manager immunize the portfolio against changes in interest rates over the next two months?

The treasurer should short Treasury bond futures contract. If bond prices go down, this futures position will provide offsetting gains. The number of contracts that should be shorted is

$$\frac{10,000,000 \times 7.1}{91,375 \times 8.8} = 88.30$$

Rounding to the nearest whole number 88 contracts should be shorted.

Problem 6.18.

How can the portfolio manager change the duration of the portfolio to 3.0 years in Problem 6.17?

The answer in Problem 6.17 is designed to reduce the duration to zero. To reduce the duration from 7.1 to 3.0 instead of from 7.1 to 0, the treasurer should short

$$\frac{4.1}{7.1} \times 88.30 = 50.99$$

or 51 contracts.

Problem 6.19.

Between October 30, 2009, and November 1, 2009, you have a choice between owning a U.S. government bond paying a 12% coupon and a U.S. corporate bond paying a 12% coupon. Consider carefully the day count conventions discussed in this chapter and decide which of the two bonds you would prefer to own. Ignore the risk of default.

You would prefer to own the Treasury bond. Under the 30/360 day count convention there is one day between October 30, 2009 and November 1, 2009. Under the actual/actual (in period) day count convention, there are two days. Therefore you would earn approximately twice as much interest by holding the Treasury bond. This assumes that the quoted prices of the two bonds are the same.

Problem 6.20.

Suppose that a Eurodollar futures quote is 88 for a contract maturing in 60 days. What is the LIBOR forward rate for the 60- to 150-day period? Ignore the difference between futures and forwards for the purposes of this question.

The Eurodollar futures contract price of 88 means that the Eurodollar futures rate is 12% per annum. This is the forward rate for the 60- to 150-day period with quarterly compounding and an actual/360 day count convention.

Problem 6.21.

The three-month Eurodollar futures price for a contract maturing in six years is quoted as 95.20. The standard deviation of the change in the short-term interest rate in one year is 1.1%. Estimate the forward LIBOR interest rate for the period between 6.00 and 6.25 years in the future.

Using the notation of Section 6.4, $\sigma = 0.011$, $T_1 = 6$, and $T_2 = 6.25$. The convexity adjustment is

$$\frac{1}{2} \times 0.011^2 \times 6 \times 6.25 = 0.002269$$

or about 23 basis points. The futures rate is 4.8% with quarterly compounding and an actual/360 day count. $(365/90)ln(1.012) = 0.0484$ or 4.84% with continuous compounding and actual/365 day count. The forward rate is therefore $4.84 - 0.23 = 4.61\%$ with continuous compounding.

Problem 6.22.

Explain why the forward interest rate is less than the corresponding futures interest rate calculated from a Eurodollar futures contract.

Suppose that the contracts apply to the interest rate between times T_1 and T_2. There are two reasons for a difference between the forward rate and the futures rate. The first is that the futures contract is settled daily whereas the forward contract is settled once at time T_2. The second is that without daily settlement a futures contract would be settled at time T_1 not T_2. Both reasons tend to make the futures rate greater than the forward rate.

ASSIGNMENT QUESTIONS

Problem 6.23.

Assume that a bank can borrow or lend money at the same interest rate in the LIBOR market. The 90-day rate is 10% per annum, and the 180-day rate is 10.2% per annum, both expressed with continuous compounding and actual/actual day count. The Eurodollar futures price for a contract maturing in 91 days is quoted as 89.5. What arbitrage opportunities are open to the bank?

The Eurodollar futures contract price of 89.5 means that the Eurodollar futures rate is 10.5% per annum with quarterly compounding and an actual/360 day count. This becomes $10.5 \times 365/360 = 10.646\%$ with an actual/actual day count. This is

$$4 \ln(1 + 0.25 \times 0.10646) = 0.1051$$

or 10.51% with continuous compounding. The forward rate given by the 91-day rate and the 182-day rate is 10.4% with continuous compounding. This suggests the following arbitrage opportunity:

1. Buy Eurodollar futures.
2. Borrow 182-day money.
3. Invest the borrowed money for 91 days.

Problem 6.24.

A Canadian company wishes to create a Canadian LIBOR futures contract from a U.S. Eurodollar futures contract and forward contracts on foreign exchange. Using an example, explain how the company should proceed. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

The U.S. Eurodollar futures contract maturing at time T enables an investor to lock in the forward rate for the period between T and T^* where T^* is three months later than T. If $\hat{r}$ is the forward rate, the U.S. dollar cash flows that can be locked in are

$$\begin{array}{rll} -Ae^{-\hat{r}(T^*-T)} & \text{at time} & T \\ +A & \text{at time} & T^* \end{array}$$

where A is the principal amount. To convert these to Canadian dollar cash flows, the Canadian company must enter into a short forward foreign exchange contract to sell Canadian dollars at time T and a long forward foreign exchange contract to buy Canadian dollars at time T^*. Suppose F and F^* are the forward exchange rates for contracts maturing at times T and T^*. (These represent the number of Canadian dollars per U.S. dollar.) The Canadian dollars to be sold at time T are

$$Ae^{-\hat{r}(T^*-T)}F$$

and the Canadian dollars to be purchased at time T^* are

$$AF^*$$

The forward contracts convert the U.S. dollar cash flows to the following Canadian dollar cash flows:

$$\begin{array}{ll} -Ae^{-\hat{r}(T^*-T)}F & \text{at time} \quad T \\ +AF^* & \text{at time} \quad T^* \end{array}$$

This is a Canadian dollar LIBOR futures contract where the principal amount is AF^*.

Problem 6.25.

The futures price for the June 2009 CBOT bond futures contract is 118-23.

a. *Calculate the conversion factor for a bond maturing on January 1, 2025, paying a coupon of 10%.*
b. *Calculate the conversion factor for a bond maturing on October 1, 2030, paying a coupon of 7%.*
c. *Suppose that the quoted prices of the bonds in (a) and (b) are 169.00 and 136.00, respectively. Which bond is cheaper to deliver?*
d. *Assuming that the cheapest-to-deliver bond is actually delivered, what is the cash price received for the bond?*

(a) On the first day of the delivery month the bond has 15 years and 7 months to maturity. The value of the bond assuming it lasts 15.5 years and all rates are 6% per annum with semiannual compounding is

$$\sum_{i=1}^{31} \frac{5}{1.03^i} + \frac{100}{1.03^{31}} = 140.00$$

The conversion factor is therefore 1.4000.

(b) On the first day of the delivery month the bond has 21 years and 4 months to maturity. The value of the bond assuming it lasts 21.25 years and all rates are 6% per annum with semiannual compounding is

$$\frac{1}{\sqrt{1.03}}\left[3.5 + \sum_{i=1}^{42} \frac{3.5}{1.03^i} + \frac{100}{1.03^{42}}\right] = 113.66$$

Subtracting the accrued interest of 1.75, this becomes 111.91. The conversion factor is therefore 1.1191.

(c) For the first bond, the quoted futures price times the conversion factor is

$$118.71825 \times 1.4000 = 166.2056$$

This is 2.7944 less than the quoted bond price. For the second bond, the quoted futures price times the conversion factor is

$$118.71825 \times 1.1191 = 132.8576$$

This is 3.1424 less than the quoted bond price. The first bond is therefore the cheapest to deliver.

(d) The price received for the bond is 166.2056 plus accrued interest.[1] There are 176 days between January 1, 2009 and June 25, 2009. There are 181 days between January 1, 2009 and July 1, 2009. The accrued interest is therefore

$$5 \times \frac{176}{181} = 4.8619$$

The cash price received for the bond is therefore 171.0675.

Problem 6.26.

A portfolio manager plans to use a Treasury bond futures contract to hedge a bond portfolio over the next three months. The portfolio is worth $100 million and will have a duration of 4.0 years in three months. The futures price is 122, and each futures contract is on $100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 9.0 years at the maturity of the futures contract. What position in futures contracts is required?

a. What adjustments to the hedge are necessary if after one month the bond that is expected to be cheapest to deliver changes to one with a duration of seven years?

b. Suppose that all rates increase over the next three months, but long-term rates increase less than short-term and medium-term rates. What is the effect of this on the performance of the hedge?

The number of short futures contracts required is

$$\frac{100,000,000 \times 4.0}{122,000 \times 9.0} = 364.3$$

Rounding to the nearest whole number 364 contracts should be shorted.

(a) This increases the number of contracts that should be shorted to

$$\frac{100,000,000 \times 4.0}{122,000 \times 7.0} = 468.4$$

or 468 when we round to the nearest whole number.

(b) In this case the gain on the short futures position is likely to be less than the loss on the bond portfolio. This is because the gain on the short futures position depends on the size of the movement in long-term rates and the loss on the bond portfolio depends on the size of the movement in medium-term rates. Duration-based hedging assumes that the movements in the two rates are the same.

[1] Note that the delivery date was not specified in the first printing of the book. We assume it is June 25.

CHAPTER 7
Swaps

Notes for the Instructor

This chapter covers the nature of swaps and how they are valued. I believe that it makes sense to teach swaps soon after forward contracts are covered because a swap is nothing more than a convenient way of bundling forward contracts. The growth of the swaps since the early 1980s makes them one of the most important derivative instruments. This chapter covers interest rate and currency swaps and provides a brief review of nonstandard swaps. More details on nonstandard swaps are in Chapter 32.

After explaining how swaps work and the way they can be used to transform assets and liabilities, I present the traditional comparative advantage argument for plain vanilla interest rate swaps and then proceed to explain why it is flawed. This usually generates a lively discussion. The key point is that the comparative advantage argument compares apples with oranges. Suppose a BBB-rated company wants to borrow at a fixed rate for five years and can choose between a fixed rate of 8% and a floating rate of LIBOR+1%. Borrowing floating and swapping to fixed appears attractive. But this ignores a key point. A fixed-rate loan will lead to exactly the same rate of interest applying each year for five years. By contrast, the spread over LIBOR on the floating-rate loan is usually guaranteed for only 6 months. If the creditworthiness of the company declines, the rate is liable to increase when the loan is rolled over. This means that borrowing floating and swapping to fixed subjects the BBB to "rollover risk". If a financial institution offers LIBOR+1% and guarantees that the spread over LIBOR will not change, we are comparing apples with apples. However, when a table similar to 7.4 is constructed, there is then found to be no comparative advantage.

A useful exercise is to take a situation such as that shown in Figure 7.7 and ask students to identify the credit risk and rollover risk of AAACorp, BBBCorp, and the financial institution.

This is the time when the nature of the LIBOR/swap zero curve can be explained to students. I go over the arguments in Section 7.5 carefully and explain the procedure (outlined in Section 7.6) for calculating the LIBOR/swap zero curve.

In the case of currency swaps the exchange of principal needs to be explained. Valuation methods are structurally very similar to those for interest rate swaps and can usually be covered fairly quickly.

Problems 7.19, 7.20, 7.21, 7.22, and 7.23 all work well as assignment questions. I usually ask students to hand in two of them.

QUESTIONS AND PROBLEMS

Problem 7.1.

Companies A and B have been offered the following rates per annum on a \$20 million five-year loan:

	Fixed Rate	*Floating Rate*
Company A	*5.0%*	*LIBOR + 0.1%*
Company B	*6.4%*	*LIBOR + 0.6%*

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

A has an apparent comparative advantage in fixed-rate markets but wants to borrow floating. B has an apparent comparative advantage in floating-rate markets but wants to borrow fixed. This provides the basis for the swap. There is a 1.4% per annum differential between the fixed rates offered to the two companies and a 0.5% per annum differential between the floating rates offered to the two companies. The total gain to all parties from the swap is therefore $1.4 - 0.5 = 0.9\%$ per annum. Because the bank gets 0.1% per annum of this gain, the swap should make each of A and B 0.4% per annum better off. This means that it should lead to A borrowing at LIBOR − 0.3% and to B borrowing at 6.0%. The appropriate arrangement is therefore as shown in Figure S7.1.

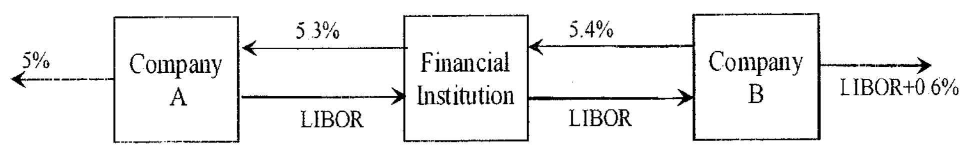

Figure S7.1 Swap for Problem 7.1

Problem 7.2.

Company X wishes to borrow U.S. dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes:

	Yen	*Dollars*
Company X	*5.0%*	*9.6%*
Company Y	*6.5%*	*10.0%*

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

X has a comparative advantage in yen markets but wants to borrow dollars. Y has a comparative advantage in dollar markets but wants to borrow yen. This provides the basis for the swap. There is a 1.5% per annum differential between the yen rates and a 0.4% per annum differential between the dollar rates. The total gain to all parties from the swap is therefore $1.5 - 0.4 = 1.1\%$ per annum. The bank requires 0.5% per annum, leaving 0.3% per annum for each of X and Y. The swap should lead to X borrowing dollars at $9.6 - 0.3 = 9.3\%$ per annum and to Y borrowing yen at $6.5 - 0.3 = 6.2\%$ per annum. The appropriate arrangement is therefore as shown in Figure S7.2. All foreign exchange risk is borne by the bank.

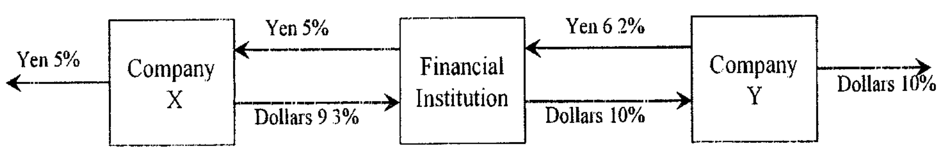

Figure S7.2 Swap for Problem 7.2

Problem 7.3.

A \$100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, six-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid–offer rate being exchanged for six-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding. The six-month LIBOR rate was 4.6% per annum two months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

In four months \$3.5 million ($= 0.5 \times 0.07 \times \$100$ million) will be received and \$2.3 million ($= 0.5 \times 0.046 \times \$100$ million) will be paid. (We ignore day count issues.) In 10 months \$3.5 million will be received, and the LIBOR rate prevailing in four months' time will be paid. The value of the fixed-rate bond underlying the swap is

$$3.5e^{-0.05\times4/12} + 103.5e^{-0.05\times10/12} = \$102.718 \text{ million}$$

The value of the floating-rate bond underlying the swap is

$$(100 + 2.3)e^{-0.05\times 4/12} = \$100.609 \text{ million}$$

The value of the swap to the party paying floating is $\$102.718 - \$100.609 = \$2.109$ million. The value of the swap to the party paying fixed is $-\$2.109$ million.

These results can also be derived by decomposing the swap into forward contracts. Consider the party paying floating. The first forward contract involves paying \$2.3 million and receiving \$3.5 million in four months. It has a value of $1.2e^{-0.05\times 4/12} = \$1.180$ million. To value the second forward contract, we note that the forward interest rate is 5% per annum with continuous compounding, or 5.063% per annum with semiannual compounding. The value of the forward contract is

$$100 \times (0.07 \times 0.5 - 0.05063 \times 0.5)e^{-0.05\times 10/12} = \$0.929 \text{ million}$$

The total value of the forward contracts is therefore $\$1.180 + \$0.929 = \$2.109$ million.

Problem 7.4.

Explain what a swap rate is. What is the relationship between swap rates and par yields?

A swap rate for a particular maturity is the average of the bid and offer fixed rates that a market maker is prepared to exchange for LIBOR in a standard plain vanilla swap with that maturity. The swap rate for a particular maturity is the LIBOR/swap par yield for that maturity.

Problem 7.5.

A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on £20 million for interest at 6% on \$30 million once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding. The current exchange rate (dollars per pound sterling) is 1.8500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

The swap involves exchanging the sterling interest of $20 \times 0.10 = 2.0$ million for the dollar interest of $30 \times 0.06 = \$1.8$ million. The principal amounts are also exchanged at the end of the life of the swap. The value of the sterling bond underlying the swap is

$$\frac{2}{(1.07)^{1/4}} + \frac{22}{(1.07)^{5/4}} = 22.182 \text{ million pounds}$$

The value of the dollar bond underlying the swap is

$$\frac{1.8}{(1.04)^{1/4}} + \frac{31.8}{(1.04)^{5/4}} = \$32.061 \text{ million}$$

The value of the swap to the party paying sterling is therefore

$$32.061 - (22.182 \times 1.85) = -\$8.976 \text{ million}$$

The value of the swap to the party paying dollars is +\$8.976 million. The results can also be obtained by viewing the swap as a portfolio of forward contracts. The continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum. The 3-month and 15-month forward exchange rates are $1.85e^{(0.03922-0.06766)\times 0.25} = 1.8369$ and $1.85e^{(0.03922-0.06766)\times 1.25} = 1.7854$. The values of the two forward contracts corresponding to the exchange of interest for the party paying sterling are therefore

$$(1.8 - 2 \times 1.8369)e^{-0.03922\times 0.25} = -\$1.855 \text{ million}$$

$$(1.8 - 2 \times 1.7854)e^{-0.03922\times 1.25} = -\$1.686 \text{ million}$$

The value of the forward contract corresponding to the exchange of principals is

$$(30 - 20 \times 1.7854)e^{-0.03922\times 1.25} = -\$5.435 \text{ million}$$

The total value of the swap is $-\$1.855 - \$1.686 - \$5.435 = -\$8.976$ million.

Problem 7.6.

Explain the difference between the credit risk and the market risk in a financial contract.

Credit risk arises from the possibility of a default by the counterparty. Market risk arises from movements in market variables such as interest rates and exchange rates. A complication is that the credit risk in a swap is contingent on the values of market variables. A company's position in a swap has credit risk only when the value of the swap to the company is positive.

Problem 7.7.

A corporate treasurer tells you that he has just negotiated a five-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at six-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

The rate is not truly fixed because, if the company's credit rating declines, it will not be able to roll over its floating rate borrowings at LIBOR plus 150 basis points. The effective fixed borrowing rate then increases. Suppose for example that the treasurer's spread over LIBOR increases from 150 basis points to 200 basis points. The borrowing rate increases from 5.2% to 5.7%.

Problem 7.8.

Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

At the start of the swap, both contracts have a value of approximately zero. As time passes, it is likely that the swap values will change, so that one swap has a positive value to the bank and the other has a negative value to the bank. If the counterparty on the other side of the positive-value swap defaults, the bank still has to honor its contract with the other counterparty. It is liable to lose an amount equal to the positive value of the swap.

Problem 7.9.

Companies X and Y have been offered the following rates per annum on a \$5 million 10-year investment:

	Fixed Rate	*Floating Rate*
Company X	*8.0%*	*LIBOR*
Company Y	*8.8%*	*LIBOR*

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

The spread between the interest rates offered to X and Y is 0.8% per annum on fixed rate investments and 0.0% per annum on floating rate investments. This means that the total apparent benefit to all parties from the swap is 0.8% per annum. Of this 0.2% per annum will go to the bank. This leaves 0.3% per annum for each of X and Y. In other words, company X should be able to get a fixed-rate return of 8.3% per annum while company Y should be able to get a floating-rate return LIBOR + 0.3% per annum. The required swap is shown in Figure S7.3. The bank earns 0.2%, company X earns 8.3%, and company Y earns LIBOR + 0.3%.

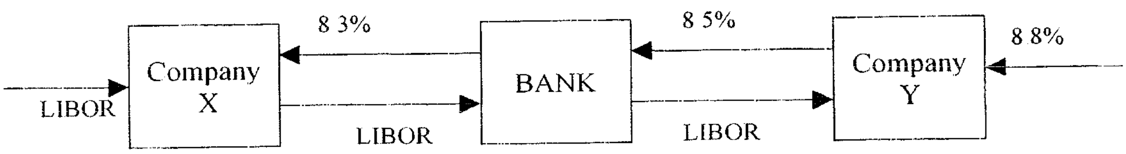

Figure S7.3 Swap for Problem 7.9

Problem 7.10.

A financial institution has entered into an interest rate swap with company X. Under the terms of the swap, it receives 10% per annum and pays six-month LIBOR on a principal of \$10 million for five years. Payments are made every six months. Suppose that company X defaults on the sixth payment date (end of year 3) when the interest rate (with semiannual compounding) is 8% per annum for all maturities. What is the loss to the financial institution? Assume that six-month LIBOR was 9% per annum halfway through year 3.

At the end of year 3 the financial institution was due to receive \$500,000 (= $0.5 \times 10\%$ of \$10 million) and pay \$450,000 (= $0.5 \times 9\%$ of \$10 million). The immediate loss is therefore \$50,000. To value the remaining swap we assume than forward rates are realized. All forward rates are 8% per annum. The remaining cash flows are therefore valued on the assumption that the floating payment is $0.5 \times 0.08 \times 10{,}000{,}000 = \$400{,}000$ and the net payment that would be received is $500{,}000 - 400{,}000 = \$100{,}000$. The total cost of default is therefore the cost of foregoing the following cash flows:

year 3:	\$50,000
year $3\frac{1}{2}$:	\$100,000
year 4:	\$100,000
year $4\frac{1}{2}$:	\$100,000
year 5:	\$100,000

Discounting these cash flows to year 3 at 4% per six months we obtain the cost of the default as \$413,000.

Problem 7.11.

Companies A and B face the following interest rates (adjusted for the differential impact of taxes):

	A	*B*
U.S. dollars (floating rate)	*LIBOR + 0.5%*	*LIBOR + 1.0%*
Canadian dollars (fixed rate)	*5.0%*	*6.5%*

Assume that A wants to borrow U.S. dollars at a floating rate of interest and B wants to borrow Canadian dollars at a fixed rate of interest. A financial institution is planning to arrange a swap and requires a 50-basis-point spread. If the swap is equally attractive to A and B, what rates of interest will A and B end up paying?

Company A has a comparative advantage in the Canadian dollar fixed-rate market. Company B has a comparative advantage in the U.S. dollar floating-rate market. (This may be because of their tax positions.) However, company A wants to borrow in the U.S. dollar floating-rate market and company B wants to borrow in the Canadian dollar fixed-rate market. This gives rise to the swap opportunity.

The differential between the U.S. dollar floating rates is 0.5% per annum, and the differential between the Canadian dollar fixed rates is 1.5% per annum. The difference between the differentials is 1% per annum. The total potential gain to all parties from the swap is therefore 1% per annum, or 100 basis points. If the financial intermediary requires 50 basis points, each of A and B can be made 25 basis points better off. Thus a swap can be designed so that it provides A with U.S. dollars at LIBOR + 0.25% per annum, and B with Canadian dollars at 6.25% per annum. The swap is shown in Figure S7.4.

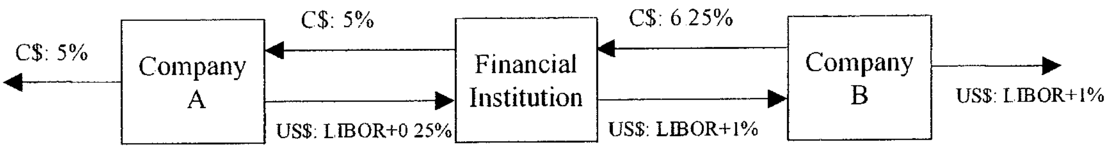

Figure S7.4 Swap for Problem 7.11

Principal payments flow in the opposite direction to the arrows at the start of the life of the swap and in the same direction as the arrows at the end of the life of the swap. The financial institution would be exposed to some foreign exchange risk which could be hedged using forward contracts.

Problem 7.12.

A financial institution has entered into a 10-year currency swap with company Y. Under the terms of the swap, the financial institution receives interest at 3% per annum in Swiss francs and pays interest at 8% per annum in U.S. dollars. Interest payments are exchanged once a year. The principal amounts are 7 million dollars and 10 million francs. Suppose that company Y declares bankruptcy at the end of year 6, when the exchange rate is \$0.80 per franc. What is the cost to the financial institution? Assume that, at the end of year 6, the interest rate is 3% per annum in Swiss francs and 8% per annum in U.S. dollars for all maturities. All interest rates are quoted with annual compounding.

When interest rates are compounded annually

$$F_0 = S_0 \left(\frac{1+r}{1+r_f}\right)^T$$

where F_0 is the T-year forward rate, S_0 is the spot rate, r is the domestic risk-free rate, and r_f is the foreign risk-free rate. As $r = 0.08$ and $r_f = 0.03$, the spot and forward exchange rates at the end of year 6 are

spot:	0.8000
1 year forward:	0.8388
2 year forward:	0.8796
3 year forward:	0.9223
4 year forward:	0.9670

The value of the swap at the time of the default can be calculated on the assumption that forward rates are realized. The cash flows lost as a result of the default are therefore as follows:

Year	Dollar Paid	Swiss Franc Received	Forward Rate	Dollar Equivalent of Swiss Franc Received	Cash Flow Lost
6	560,000	300,000	0.8000	240,000	(320,000)
7	560,000	300,000	0.8388	251,600	(308,400)
8	560,000	300,000	0.8796	263,900	(296,100)
9	560,000	300,000	0.9223	276,700	(283,300)
10	7,560,000	10,300,000	0.9670	9,960,100	2,400,100

Discounting the numbers in the final column to the end of year 6 at 8% per annum, the cost of the default is $679,800.

Note that, if this were the only contract entered into by company Y, it would make no sense for the company to default at the end of year six as the exchange of payments at that time has a positive value to company Y. In practice company Y is likely to be defaulting and declaring bankruptcy for reasons unrelated to this particular contract and payments on the contract are likely to stop when bankruptcy is declared.

Problem 7.13.

After it hedges its foreign exchange risk using forward contracts, is the financial institution's average spread in Figure 7.10 likely to be greater than or less than 20 basis points? Explain your answer.

The financial institution will have to buy 1.1% of the AUD principal in the forward market for each year of the life of the swap. Since AUD interest rates are higher than dollar interest rates, AUD is at a discount in forward markets. This means that the AUD purchased for year 2 is less expensive than that purchased for year 1; the AUD purchased for year 3 is less expensive than that purchased for year 2; and so on. This works in favor of the financial institution and means that its spread increases with time. The spread is always above 20 basis points.

Problem 7.14.

"Companies with high credit risks are the ones that cannot access fixed-rate markets directly. They are the companies that are most likely to be paying fixed and receiving floating in an interest rate swap." Assume that this statement is true. Do you think it increases or decreases the risk of a financial institution's swap portfolio? Assume that companies are most likely to default when interest rates are high.

Consider a plain-vanilla interest rate swap involving two companies X and Y. We suppose that X is paying fixed and receiving floating while Y is paying floating and receiving fixed.

The quote suggests that company X will usually be less creditworthy than company Y. (Company X might be a BBB-rated company that has difficulty in accessing fixed-rate markets directly; company Y might be a AAA-rated company that has no difficulty accessing fixed or floating rate markets.) Presumably company X wants fixed-rate funds and company Y wants floating-rate funds.

The financial institution will realize a loss if company Y defaults when rates are high or if company X defaults when rates are low. These events are relatively unlikely since (a)

Y is unlikely to default in any circumstances and (b) defaults are less likely to happen when rates are low. For the purposes of illustration, suppose that the probabilities of various events are as follows:

Default by Y:	0.001
Default by X:	0.010
Rates high when default occurs:	0.7
Rates low when default occurs:	0.3

The probability of a loss is

$$0.001 \times 0.7 + 0.010 \times 0.3 = 0.0037$$

If the roles of X and Y in the swap had been reversed the probability of a loss would be

$$0.001 \times 0.3 + 0.010 \times 0.7 = 0.0073$$

Assuming companies are more likely to default when interest rates are high, the above argument shows that the observation in quotes has the effect of decreasing the risk of a financial institution's swap portfolio. It is worth noting that the assumption that defaults are more likely when interest rates are high is open to question. The assumption is motivated by the thought that high interest rates often lead to financial difficulties for corporations. However, there is often a time lag between interest rates being high and the resultant default. When the default actually happens interest rates may be relatively low.

Problem 7.15.

Why is the expected loss from a default on a swap less than the expected loss from the default on a loan with the same principal?

In an interest-rate swap a financial institution's exposure depends on the difference between a fixed-rate of interest and a floating-rate of interest. It has no exposure to the notional principal. In a loan the whole principal can be lost.

Problem 7.16.

A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?

The bank is paying a floating-rate on the deposits and receiving a fixed-rate on the loans. It can offset its risk by entering into interest rate swaps (with other financial institutions or corporations) in which it contracts to pay fixed and receive floating.

Problem 7.17.

Explain how you would value a swap that is the exchange of a floating rate in one currency for a fixed rate in another currency.

The floating payments can be valued in currency A by (i) assuming that the forward rates are realized, and (ii) discounting the resulting cash flows at appropriate currency A discount rates. Suppose that the value is V_A. The fixed payments can be valued in currency B by discounting them at the appropriate currency B discount rates. Suppose that the value is V_B. If Q is the current exchange rate (number of units of currency A per unit of currency B), the value of the swap in currency A is $V_A - QV_B$. Alternatively, it is $V_A/Q - V_B$ in currency B.

Problem 7.18.

The LIBOR zero curve is flat at 5% (continuously compounded) out to 1.5 years. Swap rates for 2- and 3-year semiannual pay swaps are 5.4% and 5.6%, respectively. Estimate the LIBOR zero rates for maturities of 2.0, 2.5, and 3.0 years. (Assume that the 2.5-year swap rate is the average of the 2- and 3-year swap rates.)

The two-year swap rate is 5.4%. This means that a two-year LIBOR bond paying a semiannual coupon at the rate of 5.4% per annum sells for par. If R_2 is the two-year LIBOR zero rate

$$2.7e^{-0.05\times0.5} + 2.7e^{-0.05\times1.0} + 2.7e^{-0.05\times1.5} + 102.7e^{-R_2\times2.0} = 100$$

Solving this gives $R_2 = 0.05342$. The 2.5-year swap rate is assumed to be 5.5%. This means that a 2.5-year LIBOR bond paying a semiannual coupon at the rate of 5.5% per annum sells for par. If $R_{2.5}$ is the 2.5-year LIBOR zero rate

$$2.75e^{-0.05\times0.5}+2.75e^{-0.05\times1.0}+2.75e^{-0.05\times1.5}+2.75e^{-0.05342\times2.0}+102.75e^{-R_{2.5}\times2.5} = 100$$

Solving this gives $R_{2.5} = 0.05442$. The 3-year swap rate is 5.6%. This means that a 3-year LIBOR bond paying a semiannual coupon at the rate of 5.6% per annum sells for par. If R_3 is the three-year LIBOR zero rate

$$2.8e^{-0.05\times0.5} + 2.8e^{-0.05\times1.0} + 2.8e^{-0.05\times1.5} + 2.8e^{-0.05342\times2.0} + 2.8e^{-0.05442\times2.5}$$
$$+102.8e^{-R_3\times3.0} = 100$$

Solving this gives $R_3 = 0.05544$. The zero rates for maturities 2.0, 2.5, and 3.0 years are therefore 5.342%, 5.442%, and 5.544%, respectively.

ASSIGNMENT QUESTIONS

Problem 7.19.

The one-year LIBOR rate is 10%. A bank trades swaps where a fixed rate of interest is exchanged for 12-month LIBOR with payments being exchanged annually. Two- and three-year swap rates (expressed with annual compounding) are 11% and 12% per annum. Estimate the two- and three-year LIBOR zero rates.

The two-year swap rate implies that a two-year LIBOR bond with a coupon of 11% sells for par. If R_2 is the two-year zero rate

$$11e^{-0.10\times1.0} + 111e^{-R_2\times2.0} = 100$$

so that $R_2 = 0.1046$ The three-year swap rate implies that a three-year LIBOR bond with a coupon of 12% sells for par. If R_3 is the three-year zero rate

$$12e^{-0.10\times 1.0} + 12e^{-0.1046\times 2.0} + 112e^{-R_3\times 3.0} = 100$$

so that $R_3 = 0.1146$ The two- and three-year rates are therefore 10.46% and 11.46% with continuous compounding.

Problem 7.20.

Company A, a British manufacturer, wishes to borrow U.S. dollars at a fixed rate of interest. Company B, a U.S. multinational, wishes to borrow sterling at a fixed rate of interest. They have been quoted the following rates per annum (adjusted for differential tax effects):

	Sterling	*U.S. Dollars*
Company A	*11.0%*	*7.0%*
Company B	*10.6%*	*6.2%*

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 15 basis points per annum for each of the two companies.

The spread between the interest rates offered to A and B is 0.4% (or 40 basis points) on sterling loans and 0.8% (or 80 basis points) on U.S. dollar loans. The total benefit to all parties from the swap is therefore

$$80 - 40 = 40 \text{ basis points}$$

It is therefore possible to design a swap which will earn 10 basis points for the bank while making each of A and B 15 basis points better off than they would be by going directly to financial markets. One possible swap is shown in Figure M7.1. Company A borrows at an effective rate of 6.85% per annum in U.S. dollars.

Company B borrows at an effective rate of 10.45% per annum in sterling. The bank earns a 10-basis-point spread. The way in which currency swaps such as this operate is as follows. Principal amounts in dollars and sterling that are roughly equivalent are chosen. These principal amounts flow in the opposite direction to the arrows at the time the swap is initiated. Interest payments then flow in the same direction as the arrows during the life of the swap and the principal amounts flow in the same direction as the arrows at the end of the life of the swap.

Note that the bank is exposed to some exchange rate risk in the swap. It earns 65 basis points in U.S. dollars and pays 55 basis points in sterling. This exchange rate risk could be hedged using forward contracts.

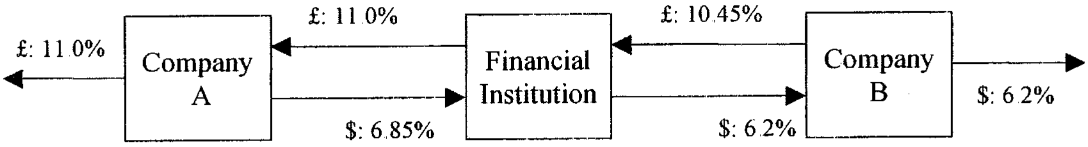

Figure M7.1 One Possible Swap for Problem 7.20

Problem 7.21.

Under the terms of an interest rate swap, a financial institution has agreed to pay 10% per annum and to receive three-month LIBOR in return on a notional principal of $100 million with payments being exchanged every three months. The swap has a remaining life of 14 months. The average of the bid and offer fixed rates currently being swapped for three-month LIBOR is 12% per annum for all maturities. The three-month LIBOR rate one month ago was 11.8% per annum. All rates are compounded quarterly. What is the value of the swap?

The swap can be regarded as a long position in a floating-rate bond combined with a short position in a fixed-rate bond. The correct discount rate is 12% per annum with quarterly compounding or 11.82% per annum with continuous compounding.

Immediately after the next payment the floating-rate bond will be worth $100 million. The next floating payment ($ million) is

$$0.118 \times 100 \times 0.25 = 2.95$$

The value of the floating-rate bond is therefore

$$102.95e^{-0.1182 \times 2/12} = 100.941$$

The value of the fixed-rate bond is

$$2.5e^{-0.1182 \times 2/12} + 2.5e^{-0.1182 \times 5/12} + 2.5e^{-0.1182 \times 8/12}$$

$$+2.5e^{-0.1182 \times 11/12} + 102.5e^{-0.1182 \times 14/12} = 98.678$$

The value of the swap is therefore

$$100.941 - 98.678 = \$2.263 \text{ million}$$

As an alternative approach we can value the swap as a series of forward rate agreements. The calculated value is

$$(2.95 - 2.5)e^{-0.1182 \times 2/12} + (3.0 - 2.5)e^{-0.1182 \times 5/12}$$

$$+(3.0 - 2.5)e^{0.1182 \times 8/12} + (3.0 - 2.5)e^{-0.1182 \times 11/12}$$

$$+(3.0 - 2.5)e^{-0.1182 \times 14/12} = \$2.263 \text{ million}$$

which is in agreement with the answer obtained using the first approach.

Problem 7.22.

Suppose that the term structure of interest rates is flat in the United States and Australia. The USD interest rate is 7% per annum and the AUD rate is 9% per annum. The current value of the AUD is 0.62 USD. Under the terms of a swap agreement, a financial institution pays 8% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $12 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last two more years. What is the value of the swap to the financial institution? Assume all interest rates are continuously compounded.

The financial institution is long a dollar bond and short a USD bond. The value of the dollar bond (in millions of dollars) is

$$0.48e^{-0.07\times 1} + 12.48e^{-0.07\times 2} = 11.297$$

The value of the AUD bond (in millions of AUD) is

$$1.6e^{-0.09\times 1} + 21.6e^{-0.09\times 2} = 19.504$$

The value of the swap (in millions of dollars) is therefore

$$11.297 - 19.504 \times 0.62 = -0.795$$

or −$795,000.

As an alternative we can value the swap as a series of forward foreign exchange contracts. The one-year forward exchange rate is $0.62e^{-0.02} = 0.6077$. The two-year forward exchange rate is $0.62e^{-0.02\times 2} = 0.5957$. The value of the swap in millions of dollars is therefore

$$(0.48 - 1.6 \times 0.6077)e^{-0.07\times 1} + (12.48 - 21.6 \times 0.5957)e^{-0.07\times 2} = -0.795$$

which is in agreement with the first calculation.

Problem 7.23.

Company X is based in the United Kingdom and would like to borrow $50 million at a fixed rate of interest for five years in U.S. funds. Because the company is not well known in the United States, this has proved to be impossible. However, the company has been quoted 12% per annum on fixed-rate five-year sterling funds. Company Y is based in the United States and would like to borrow the equivalent of $50 million in sterling funds for five years at a fixed rate of interest. It has been unable to get a quote but has been offered U.S. dollar funds at 10.5% per annum. Five-year government bonds currently yield 9.5% per annum in the United States and 10.5% in the United Kingdom. Suggest an appropriate currency swap that will net the financial intermediary 0.5% per annum.

There is a 1% differential between the yield on sterling and dollar 5-year bonds. The financial intermediary could use this differential when designing a swap. For example, it could (a) allow company X to borrow dollars at 1% per annum less than the rate offered on sterling funds, that is, at 11% per annum and (b) allow company Y to borrow sterling at 1% per annum more than the rate offered on dollar funds, that is, at $11\frac{1}{2}\%$ per annum. However, as shown in Figure M7.2, the financial intermediary would not then earn a positive spread.

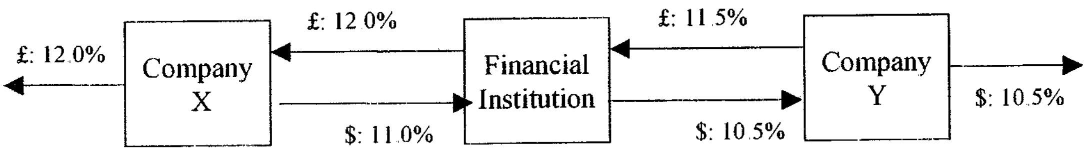

Figure M7.2 First attempt at designing swap for Problem 7.23

To make 0.5% per annum, the financial intermediary could add 0.25% per annum, to the rates paid by each of X and Y. This means that X pays 11.25% per annum, for dollars and Y pays 11.75% per annum, for sterling and leads to the swap shown in Figure M7.3. The financial intermediary would be exposed to some foreign exchange risk in this swap. This could be hedged using forward contracts.

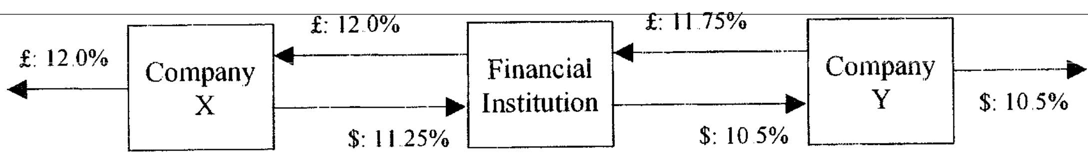

Figure M7.3 Final Swap for Problem 7.23

CHAPTER 8
Mechanics of Options Markets

Notes for the Instructor

This chapter provides information on how options markets work. I usually go through the chapter fairly quickly leaving students to read the details for themselves. Points I spend time on are the payoffs from the four option positions and how the terms of options change when there are dividends and stock splits. There is less material on employee stock options in this chapter than before. This is because there is now a whole chapter (Chapter 14) devoted to this topic.

Problems 8.24 and 8.26 work well for class discussion. Problems 8.23 and 8.25 can be used as assignment questions.

QUESTIONS AND PROBLEMS

Problem 8.1.

An investor buys a European put on a share for $3. The stock price is $42 and the strike price is $40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock on the expiration date is less than $37. In these circumstances the gain from exercising the option is greater than $3. The option will be exercised if the stock price is less than $40 at the maturity of the option. The variation of the investor's profit with the stock price in Figure S8.1.

Problem 8.2.

An investor sells a European call on a share for $4. The stock price is $47 and the strike price is $50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock is below $54 on the expiration date. If the stock price is below $50, the option will not be exercised, and the investor makes a profit of $4. If the stock price is between $50 and $54, the option is exercised and the investor makes a profit between $0 and $4. The variation of the investor's profit with the stock price is as shown in Figure S8.2.

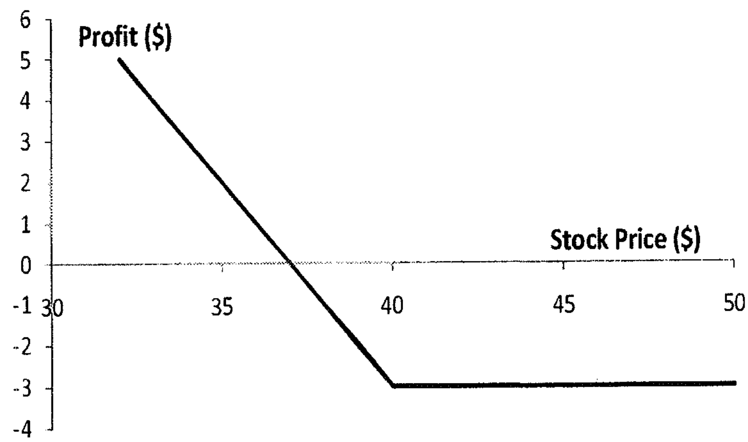

Figure S8.1 Investor's profit in Problem 8.1

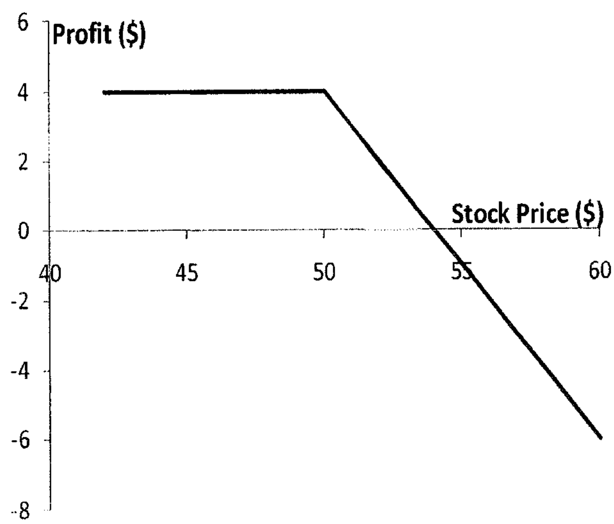

Figure S8.2 Investor's profit in Problem 8.2

Problem 8.3.

An investor sells a European call option with strike price of K and maturity T and buys a put with the same strike price and maturity. Describe the investor's position.

The payoff to the investor is

$$-\max(S_T - K, 0) + \max(K - S_T, 0)$$

This is $K - S_T$ in all circumstances. The investor's position is the same as a short position in a forward contract with delivery price K.

Problem 8.4.

Explain why brokers require margins when clients write options but not when they buy options.

When an investor buys an option, cash must be paid up front. There is no possibility of future liabilities and therefore no need for a margin account. When an investor sells an option, there are potential future liabilities. To protect against the risk of a default, margins are required.

Problem 8.5.

A stock option is on a February, May, August, and November cycle. What options trade on (a) April 1 and (b) May 30?

On April 1 options trade with expiration months of April, May, August, and November. On May 30 options trade with expiration months of June, July, August, and November.

Problem 8.6.

A company declares a 2-for-1 stock split. Explain how the terms change for a call option with a strike price of \$60.

The strike price is reduced to \$30, and the option gives the holder the right to purchase twice as many shares.

Problem 8.7.

"Employee stock options issued by a company are different from regular exchange-traded call options on the company's stock because they can affect the capital structure of the company." Explain this statement.

The exercise of employee stock options usually leads to new shares being issued by the company and sold to the employee. This changes the amount of equity in the capital structure. When a regular exchange-traded option is exercised no new shares are issued and the company's capital structure is not affected.

Problem 8.8.

A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the Philadelphia Stock Exchange and (b) the over-the-counter market for trading?

The Philadelphia Exchange offers European and American options with standard strike prices and times to maturity. Options in the over-the-counter market have the advantage that they can be tailored to meet the precise needs of the treasurer. Their disadvantage is that they expose the treasurer to some credit risk. Exchanges organize their trading so that there is virtually no credit risk.

Problem 8.9.

Suppose that a European call option to buy a share for $100.00 costs $5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the holder of the option will make a profit if the stock price at maturity of the option is greater than $105. This is because the payoff to the holder of the option is, in these circumstances, greater than the $5 paid for the option. The option will be exercised if the stock price at maturity is greater than $100. Note that if the stock price is between $100 and $105 the option is exercised, but the holder of the option takes a loss overall. The profit from a long position is as shown in Figure S8.3.

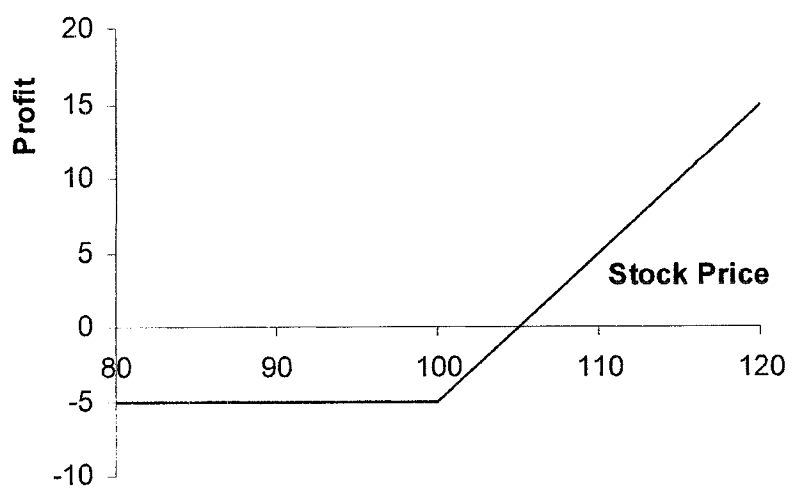

Figure S8.3 Profit from long position in Problem 8.9

Problem 8.10.

Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the seller of the option will make a profit if the stock price at maturity is greater than \$52.00. This is because the cost to the seller of the option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than \$60.00. Note that if the stock price is between \$52.00 and \$60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S8.4.

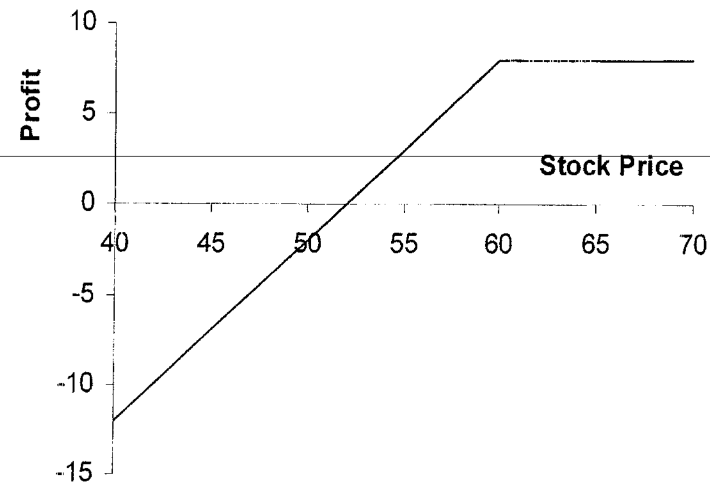

Figure S8.4 Profit from short position in Problem 8.10

Problem 8.11.

Describe the terminal value of the following portfolio: a newly entered-into long forward contract on an asset and a long position in a European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up. Show that the European put option has the same value as a European call option with the same strike price and maturity.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is also F_0.)

The terminal value of the put option is:

$$\max(F_0 - S_T,\ 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T,\ 0)$$

$$= \max(0,\ S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0.

We have shown that the forward contract plus the put is worth the same as a call with the same strike price and time to maturity as the put. The forward contract is worth zero at the time the portfolio is set up. It follows that the put is worth the same as the call at the time the portfolio is set up.

Problem 8.12.

A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price.

Figure S8.5 shows the variation of the trader's position with the asset price. We can divide the alternative asset prices into three ranges:

(a) When the asset price less than \$40, the put option provides a payoff of $40 - S_T$ and the call option provides no payoff. The options cost \$7 and so the total profit is $33 - S_T$.

(b) When the asset price is between \$40 and \$45, neither option provides a payoff. There is a net loss of \$7.

(c) When the asset price greater than \$45, the call option provides a payoff of $S_T - 45$ and the put option provides no payoff. Taking into account the \$7 cost of the options, the total profit is $S_T - 52$.

The trader makes a profit (ignoring the time value of money) if the stock price is less than \$33 or greater than \$52. This type of trading strategy is known as a strangle and is discussed in Chapter 10.

Problem 8.13.

Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

The holder of an American option has all the same rights as the holder of a European option and more. It must therefore be worth at least as much. If it were not, an arbitrageur could short the European option and take a long position in the American option.

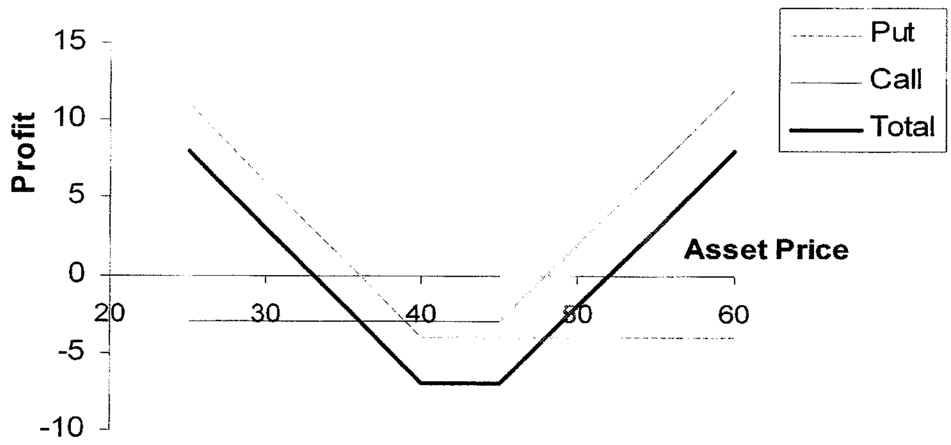

Figure S8.5 Profit from trading strategy in Problem 8.12

Problem 8.14.

Explain why an American option is always worth at least as much as its intrinsic value.

The holder of an American option has the right to exercise it immediately. The American option must therefore be worth at least as much as its intrinsic value. If it were not an arbitrageur could lock in a sure profit by buying the option and exercising it immediately.

Problem 8.15.

Explain carefully the difference between writing a put option and buying a call option.

Writing a put gives a payoff of $\min(S_T - K, 0)$. Buying a call gives a payoff of $\max(S_T - K, 0)$. In both cases the potential payoff is $S_T - K$. The difference is that for a written put the counterparty chooses whether you get the payoff (and will allow you to get it only when it is negative to you). For a long call you decide whether you get the payoff (and you choose to get it when it is positive to you.)

Problem 8.16.

The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

Forward contracts lock in the exchange rate that will apply to a particular transaction in the future. Options provide insurance that the exchange rate will not be worse than some level. The advantage of a forward contract is that uncertainty is eliminated as far as possible. The disadvantage is that the outcome with hedging can be significantly worse than the outcome with no hedging. This disadvantage is not as marked with options. However, unlike forward contracts, options involve an up-front cost.

Problem 8.17.

Consider an exchange-traded call option contract to buy 500 shares with a strike price of \$40 and maturity in four months. Explain how the terms of the option contract change when there is

a. A 10% stock dividend
b. A 10% cash dividend
c. A 4-for-1 stock split

(a) The option contract becomes one to buy $500 \times 1.1 = 550$ shares with an exercise price $40/1.1 = 36.36$.
(b) There is no effect. The terms of an options contract are not normally adjusted for cash dividends.
(c) The option contract becomes one to buy $500 \times 4 = 2,000$ shares with an exercise price of $40/4 = \$10$.

Problem 8.18.

"If most of the call options on a stock are in the money, it is likely that the stock price has risen rapidly in the last few months." Discuss this statement.

The exchange has certain rules governing when trading in a new option is initiated. These mean that the option is close-to-the-money when it is first traded. If all call options are in the money it is therefore likely that the stock price has risen since trading in the option began.

Problem 8.19.

What is the effect of an unexpected cash dividend on (a) a call option price and (b) a put option price?

An unexpected cash dividend would reduce the stock price on the ex-dividend date. This stock price reduction would not be anticipated by option holders. As a result there would be a reduction in the value of a call option and an increase the value of a put option. (Note that the terms of an option are adjusted for cash dividends only in exceptional circumstances.)

Problem 8.20.

Options on General Motors stock are on a March, June, September, and December cycle. What options trade on (a) March 1, (b) June 30, and (c) August 5?

(a) March, April, June and September
(b) July, August, September, December
(c) August, September, December, March.
Longer dated options may also trade.

Problem 8.21.

Explain why the market maker's bid-offer spread represents a real cost to options investors.

A "fair" price for the option can reasonably be assumed to be half way between the bid and the offer price quoted by a market maker. An investor typically buys at the market maker's offer and sells at the market maker's bid. Each time he or she does this there is a hidden cost equal to half the bid-offer spread.

Problem 8.22.

A United States investor writes five naked call option contracts. The option price is \$3.50, the strike price is \$60.00, and the stock price is \$57.00. What is the initial margin requirement?

The two calculations are necessary to determine the initial margin. The first gives

$$500 \times (3.5 + 0.2 \times 57 - 3) = 5,950$$

The second gives

$$500 \times (3.5 + 0.1 \times 57) = 4,600$$

The initial margin is the greater of these, or \$5,950. Part of this can be provided by the initial amount of $500 \times 3.5 = \$1,750$ received for the options.

ASSIGNMENT QUESTIONS

Problem 8.23.

The price of a stock is \$40. The price of a one-year European put option on the stock with a strike price of \$30 is quoted as \$7 and the price of a one-year European call option on the stock with a strike price of \$50 is quoted as \$5. Suppose that an investor buys 100 shares, shorts 100 call options, and buys 100 put options. Draw a diagram illustrating how the investor's profit or loss varies with the stock price over the next year. How does your answer change if the investor buys 100 shares, shorts 200 call options, and buys 200 put options?

Figure M8.1 shows the way in which the investor's profit varies with the stock price in the first case. For stock prices less than \$30 there is a loss of \$1,200. As the stock price increases from \$30 to \$50 the profit increases from –\$1,200 to \$800. Above \$50 the profit is \$800. Students may express surprise that a call which is \$10 out of the money is less expensive than a put which is \$10 out of the money. This could be because of dividends or the crashophobia phenomenon discussed in Chapter 18.

Figure M8.2 shows the way in which the profit varies with stock price in the second case. In this case the profit pattern has a zigzag shape. The problem illustrates how many

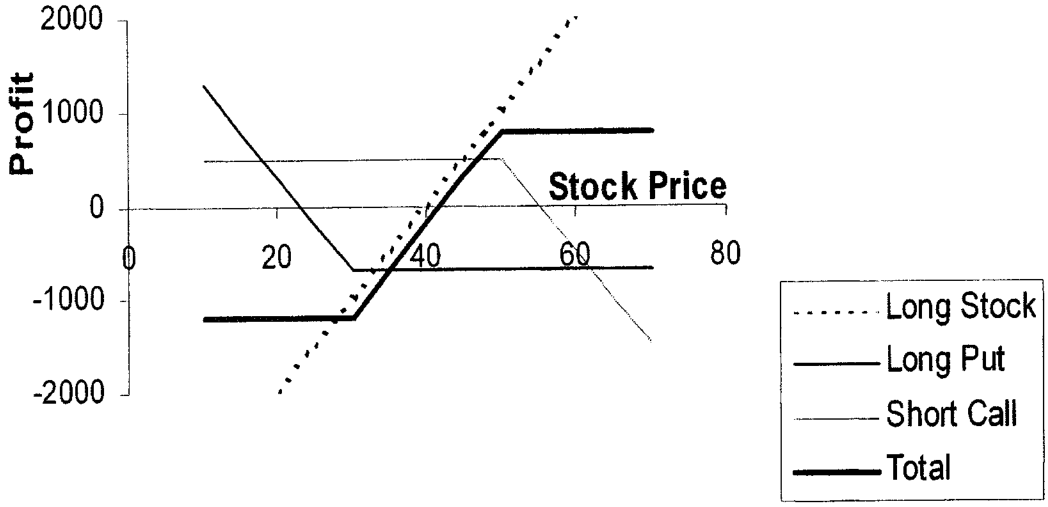

Figure M8.1 Profit in first case considered Problem 8.25

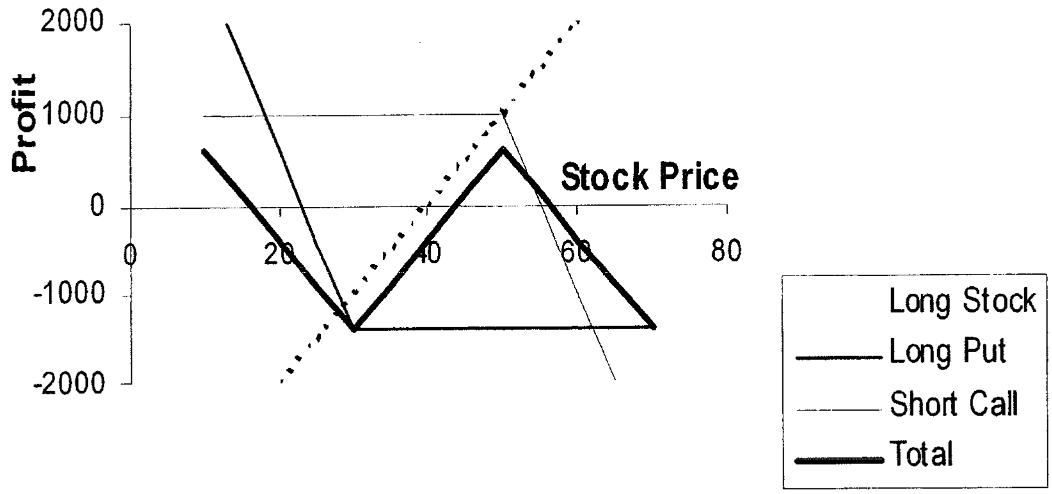

Figure M8.2 Profit for the second case considered Problem 8.25

different patterns can be obtained by including calls, puts, and the underlying asset in a portfolio.

Problem 8.24.

"If a company does not do better than its competitors but the stock market goes

up, executives do very well from their stock options. This makes no sense" Discuss this viewpoint. Can you think of alternatives to the usual executive stock option plan that take the viewpoint into account.

Executive stock option plans account for a high percentage of the total remuneration received by executives. When the market is rising fast (as it was for much of the 1990s) many corporate executives do very well out of their stock option plans — even when their company does worse than its competitors. Large institutional investors have argued that executive stock options should be structured so that the payoff depends how the company has performed relative to an appropriate industry index. In a regular executive stock option the strike price is the stock price at the time the option is issued. In the type of relative-performance stock option favored by institutional investors, the strike price at time t is $S_0 I_t / I_0$ where S_0 is the company's stock price at the time the option is issued, I_0 is the value of an equity index for the industry in which the company operates at the time the option is issued, and I_t is the value of the index at time t. If the company's performance equals the performance of the industry, the options are always at-the-money. If the company outperforms the industry, the options become in the money. If the company underperforms the industry, the options become out of the money. Note that a relative performance stock option can provide a payoff when both the market and the company's stock price decline.

Relative performance stock options clearly provide a better way of rewarding senior management for superior performance. Some companies have argued that, if they introduce relative performance options when their competitors do not, they will lose some of their top management talent.

Problem 8.25.

Use DerivaGem to calculate the value of an American put option on a nondividend paying stock when the stock price is \$30, the strike price is \$32, the risk-free rate is 5%, the volatility is 30%, and the time to maturity is 1.5 years. (Choose "Binomial American" for the "option type" and 50 time steps.)

a. *What is the option's intrinsic value?*
b. *What is the option's time value?*
c. *What would a time value of zero indicate? What is the value of an option with zero time value?*
d. *Using a trial and error approach calculate how low the stock price would have to be for the time value of the option to be zero.*

DerivaGem shows that the value of the option is 4.57. The option's intrinsic value is $32 - 30 = 2.00$. The option's time value is therefore $4.57 - 2.00 = 2.57$. A time value of zero would indicate that it is optimal to exercise the option immediately. In this case the value of the option would equal its intrinsic value. When the stock price is 20, DerivaGem gives the value of the option as 12, which is its intrinsic value. When the stock price is 25, DerivaGem gives the value of the options as 7.54, indicating that the time value is still positive ($= 0.54$). Keeping the number of time steps equal to 50, trial and error indicates the time value disappears when the stock price is reduced to 21.69 or lower. (With 500 time steps this estimate of how low the stock price must become is reduced to 21.35.)

Problem 8.26.

On July 20, 2004 Microsoft surprised the market by announcing a \$3 dividend. The ex-dividend date was November 17, 2004 and the payment date was December 2, 2004. Its stock price at the time was about \$28. It also changed the terms of its employee stock options so that each exercise price was adjusted downward to

$$\textit{Pre-dividend Exercise Price} \times \frac{\text{Closing Price}-\$3.00}{\text{Closing Price}}$$

The number of shares covered by each stock option outstanding was adjusted upward to

$$\textit{Number of Shares Pre-dividend} \times \frac{\text{Closing Price}}{\text{Closing Price}-\$3.00}$$

"Closing Price" means the official NASDAQ closing price of a share of Microsoft common stock on the last trading day before the ex-dividend date.

Evaluate this adjustment. Compare it with the system used by exchanges to adjust for extraordinary dividends (see Business Snapshot 8.1).

Suppose that the closing stock price is \$28 and an employee has 1000 options with a strike price of \$24. Microsoft's adjustment involves changing the strike price to $24\times 25/28 = 21.4286$ and changing the number of options to $1000 \times 28/25 = 1,120$. The system used by exchanges would involve keeping the number of options the same and reducing the strike price by \$3 to \$21.

The Microsoft adjustment is more complicated than that used by the exchange because it requires a knowledge of the Microsoft's stock price immediately before the stock goes ex-dividend. However, arguably it is a better adjustment than the one used by the exchange. Before the adjustment the employee has the right to pay \$24,000 for Microsoft stock that is worth \$28,000. After the adjustment the employee also has the option to pay \$24,000 for Microsoft stock worth \$28,000. Under the adjustment rule used by exchanges the employee would have the right to buy stock worth \$25,000 for \$21,000. If the volatility of Microsoft remains the same this is a less valuable option.

One complication here is that Microsoft's volatility does not remain the same. It can be expected to go up because some cash (a zero risk asset) has been transferred to shareholders. The employees therefore have the same basic option as before but the volatility of Microsoft can be expected to increase. The employees are slightly better off because the value of an option increases with volatility.

CHAPTER 9

Properties of Stock Options

Notes for the Instructor

This chapter outlines a number of relationships between a stock option price and the underlying stock price that do not involve any assumptions about the volatility of the stock's price. As will be evident from the slides, I like to present students with sets of numerical data for c, C, p, P, S_0, K, r, T, and D that violate the relationships in the chapter and ask them what trades they would do. This usually results in good classroom interaction.

I devote most time in class to

1. The $c \geq S_0 - Ke^{-rT}$ result
2. The early exercise arguments for American calls
3. The put–call parity result for European options

When discussing the early exercise of American calls on non-dividend-paying stocks I present students with a situation where the option is deep-in-the-money ($S_0 = 100$; $T = 0.25$; $K = 60$) and ask whether they would exercise early when (a) they want to keep the stock as part of their portfolio and (b) when they think the stock is a "dog". In the first case, they should delay exercise and thereby delay paying the strike price. In the second case they should sell the option to an investor who does want to keep the stock as part of his or her portfolio. (Such an investor must exist as otherwise the stock price would not be \$100). This investor will pay at least 100 minus the present value of 60 for the option. So, if the possibility of the stock price falling below \$60 is ignored, the option should not be exercised early. When the possibility of the stock price falling below \$60 is recognized, we become even less inclined to exercise early.

Two follow-up questions are "Why does this argument not work for put options?" and "Why are employee stock options frequently exercised early?" The answer to the first question is that the strike price is paid not received in the case of put options and so the time value of money argument does not work. The answer to the second question is that employee stock options cannot be traded.

This can be a good time to introduce the Excel-based software DerivaGem that goes with the book (if the instructor has not already done so). The software can be used to plot the relationship between call/put prices and the variables: S_0, K, r, σ, and T. It can also be used to check put–call parity and investigate the difference between American and European option prices.

As already mentioned I often use Problem 9.20 in class. Problem 9.22 is a short assignment question. Problems 9.23, 9.24, and 9.25 are more challenging and can be used for assignments or class discussion. Problem 9.26 gets students started with the DerivaGem software.

QUESTIONS AND PROBLEMS

Problem 9.1.

List the six factors affecting stock option prices.

The six factors affecting stock option prices are the stock price, strike price, risk-free interest rate, volatility, time to maturity, and dividends.

Problem 9.2.

What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?

The lower bound is

$$28 - 25e^{-0.08\times 0.3333} = \$3.66$$

Problem 9.3.

What is a lower bound for the price of a one-month European put option on a non-dividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

The lower bound is

$$15e^{-0.06\times 0.08333} - 12 = \$2.93$$

Problem 9.4.

Give two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second reason should apply even if interest rates are zero.

Delaying exercise delays the payment of the strike price. This means that the option holder is able to earn interest on the strike price for a longer period of time. Delaying exercise also provides insurance against the stock price falling below the strike price by the expiration date. Assume that the option holder has an amount of cash K and that interest rates are zero. Exercising early means that the option holder's position will be worth S_T at expiration. Delaying exercise means that it will be worth $\max(K,\, S_T)$ at expiration.

Problem 9.5.

"The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.

An American put when held in conjunction with the underlying stock provides insurance. It guarantees that the stock can be sold for the strike price, K. If the put is exercised early, the insurance ceases. However, the option holder receives the strike price immediately and is able to earn interest on it between the time of the early exercise and the expiration date.

Problem 9.6.

Explain why an American call option on a dividend-paying stock is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.

An American call option can be exercised at any time. If it is exercised its holder gets the intrinsic value. It follows that an American call option must be worth at least its intrinsic value. A European call option can be worth less than its intrinsic value. Consider, for example, the situation where a stock is expected to provide a very high dividend during the life of an option. The price of the stock will decline as a result of the dividend. Because the European option can be exercised only after the dividend has been paid, its value may be less than the intrinsic value today.

Problem 9.7.

The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of \$20?

In this case $c = 1$, $T = 0.25$, $S_0 = 19$, $K = 20$, and $r = 0.04$. From put–call parity

$$p = c + Ke^{-rT} - S_0$$

or

$$p = 1 + 20e^{-0.04\times0.25} - 19 = 1.80$$

so that the European put price is \$1.80.

Problem 9.8.

Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

When early exercise is not possible, we can argue that two portfolios that are worth the same at time T must be worth the same at earlier times. When early exercise is possible, the argument falls down. Suppose that $P + S > C + Ke^{-rT}$. This situation does not lead to an arbitrage opportunity. If we buy the call, short the put, and short the stock, we cannot be sure of the result because we do not know when the put will be exercised.

Problem 9.9.

What is a lower bound for the price of a six-month call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum?

The lower bound is

$$80 - 75e^{-0.1\times0.5} = \$8.66$$

Problem 9.10

What is a lower bound for the price of a two-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?

The lower bound is

$$65e^{-0.05\times 2/12} - 58 = \$6.46$$

Problem 9.11.

A four-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in one month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?

The present value of the strike price is $60e^{-0.12\times 4/12} = \$57.65$. The present value of the dividend is $0.80e^{-0.12\times 1/12} = 0.79$. Because

$$5 < 64 - 57.65 - 0.79$$

the condition in equation (9.5) is violated. An arbitrageur should buy the option and short the stock. This generates $64 - 5 = \$59$. The arbitrageur invests \$0.79 of this at 12% for one month to pay the dividend of \$0.80 in one month. The remaining \$58.21 is invested for four months at 12%. Regardless of what happens a profit will materialize.

If the stock price declines below \$60 in four months, the arbitrageur loses the \$5 spent on the option but gains on the short position. The arbitrageur shorts when the stock price is \$64, has to pay dividends with a present value of \$0.79, and closes out the short position when the stock price is \$60 or less. Because \$57.65 is the present value of \$60, the short position generates at least $64 - 57.65 - 0.79 = \$5.56$ in present value terms. The present value of the arbitrageur's gain is therefore at least $5.56 - 5.00 = \$0.56$.

If the stock price is above \$60 at the expiration of the option, the option is exercised. The arbitrageur buys the stock for \$60 in four months and closes out the short position. The present value of the \$60 paid for the stock is \$57.65 and as before the dividend has a present value of \$0.79. The gain from the short position and the exercise of the option is therefore exactly $64 - 57.65 - 0.79 = \$5.56$. The arbitrageur's gain in present value terms is exactly $5.56 - 5.00 = \$0.56$.

Problem 9.12.

A one-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

In this case the present value of the strike price is $50e^{-0.06\times 1/12} = 49.75$. Because

$$2.5 < 49.75 - 47.00$$

the condition in equation (9.2) is violated. An arbitrageur should borrow \$49.50 at 6% for one month, buy the stock, and buy the put option. This generates a profit in all circumstances.

If the stock price is above \$50 in one month, the option expires worthless, but the stock can be sold for at least \$50. A sum of \$50 received in one month has a present value of \$49.75 today. The strategy therefore generates profit with a present value of at least \$0.25.

If the stock price is below \$50 in one month the put option is exercised and the stock owned is sold for exactly \$50 (or \$49.75 in present value terms). The trading strategy therefore generates a profit of exactly \$0.25 in present value terms.

Problem 9.13.

Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

The early exercise of an American put is attractive when the interest earned on the strike price is greater than the insurance element lost. When interest rates increase, the value of the interest earned on the strike price increases making early exercise more attractive. When volatility decreases, the insurance element is less valuable. Again this makes early exercise more attractive.

Problem 9.14.

The price of a European call that expires in six months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in six months and has a strike price of \$30?

Using the notation in the chapter, put-call parity [equation (9.7)] gives

$$c + Ke^{-rT} + D = p + S_0$$

or

$$p = c + Ke^{-rT} + D - S_0$$

In this case

$$p = 2 + 30e^{-0.1\times 6/12} + (0.5e^{-0.1\times 2/12} + 0.5e^{-0.1\times 5/12}) - 29 = 2.51$$

In other words the put price is \$2.51.

Problem 9.15.

Explain carefully the arbitrage opportunities in Problem 9.14 if the European put price is \$3.

If the put price is \$3.00, it is too high relative to the call price. An arbitrageur should buy the call, short the put and short the stock. This generates $-2 + 3 + 29 =$ \$30 in cash which is invested at 10%. Regardless of what happens a profit with a present value of $3.00 - 2.51 =$ \$0.49 is locked in.

If the stock price is above \$30 in six months, the call option is exercised, and the put option expires worthless. The call option enables the stock to be bought for \$30, or $30e^{-0.10\times6/12} =$ \$28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1\times2/12} + 0.5e^{-0.1\times5/12} =$ \$0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 =$ \$0.49.

If the stock price is below \$30 in six months, the put option is exercised and the call option expires worthless. The short put option leads to the stock being bought for \$30, or $30e^{-0.10\times6/12} =$ \$28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1\times2/12} + 0.5e^{-0.1\times5/12} =$ \$0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 =$ \$0.49.

Problem 9.16.

The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in three months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

From equation (9.4)

$$S_0 - K \le C - P \le S_0 - Ke^{-rT}$$

In this case

$$31 - 30 \le 4 - P \le 31 - 30e^{-0.08\times0.25}$$

or

$$1.00 \le 4.00 - P \le 1.59$$

or

$$2.41 \le P \le 3.00$$

Upper and lower bounds for the price of an American put are therefore \$2.41 and \$3.00.

Problem 9.17.

Explain carefully the arbitrage opportunities in Problem 9.16 if the American put price is greater than the calculated upper bound.

If the American put price is greater than \$3.00 an arbitrageur can sell the American put, short the stock, and buy the American call. This realizes at least $3 + 31 - 4 =$ \$30 which can be invested at the risk-free interest rate. At some stage during the 3-month period either the American put or the American call will be exercised. The arbitrageur then pays \$30, receives the stock and closes out the short position. The cash flows to the arbitrageur are +\$30 at time zero and −\$30 at some future time. These cash flows have a positive present value.

Problem 9.18.

Prove the result in equation (9.4). (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to K and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. Because $P \geq p$, it follows from put–call parity that

$$P \geq c + Ke^{-rT} - S_0$$

and since $c = C$,

$$P \geq C + Ke^{-rT} - S_0$$

or

$$C - P \leq S_0 - Ke^{-rT}$$

For a further relationship between C and P, consider

Portfolio I: One European call option plus an amount of cash equal to K.

Portfolio J: One American put option plus one share.

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early portfolio J is worth

$$\max(S_T, K)$$

at time T. Portfolio I is worth

$$\max(S_T - K, 0) + Ke^{rT} = \max(S_T, K) - K + Ke^{rT}$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ. This means that portfolio J is worth K at time τ. However, even if the call option were worthless, portfolio I would be worth $Ke^{r\tau}$ at time τ. It follows that portfolio I is worth at least as much as portfolio J in all circumstances. Hence

$$c + K \geq P + S_0$$

Since $c = C$,

$$C + K \geq P + S_0$$

or

$$C - P \geq S_0 - K$$

Combining this with the other inequality derived above for $C - P$, we obtain

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

Problem 9.19.

Prove the result in equation (9.8). (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to $D + K$ and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. The present value of the dividends will be denoted by D. As shown in the answer to Problem 9.18, when there are no dividends

$$C - P \leq S_0 - Ke^{-rT}$$

Dividends reduce C and increase P. Hence this relationship must also be true when there are dividends.

For a further relationship between C and P, consider

Portfolio I: one European call option plus an amount of cash equal to $D + K$

Portfolio J: one American put option plus one share

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early, portfolio J is worth

$$\max(S_T, K) + De^{rT}$$

at time T. Portfolio I is worth

$$\max(S_T - K, 0) + (D + K)e^{rT} = \max(S_T, K) + De^{rT} + Ke^{rT} - K$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ. This means that portfolio J is worth at most $K + De^{r\tau}$ at time τ. However, even if the call option were worthless, portfolio I would be worth $(D + K)e^{r\tau}$ at time τ. It follows that portfolio I is worth more than portfolio J in all circumstances. Hence

$$c + D + K \geq P + S_0$$

Because $C \geq c$

$$C - P \geq S_0 - D - K$$

Problem 9.20.

Consider a 5-year employee stock option on a non-dividend-paying stock. The option can be exercised at any time after the end of the first year. Unlike a regular exchange-traded call option, the employee stock option cannot be sold. What is the likely impact of this restriction on the early exercise decision?

Executive stock options may be exercised early because the executive needs the cash or because he or she is uncertain about the company's future prospects. Regular call options can be sold in the market in either of these two situations, but executive stock options cannot be sold. In theory an executive can short the company's stock as an alternative to exercising. In practice this is not usually encouraged and may even be illegal.

Problem 9.21.

Use the software DerivaGem to verify that Figures 9.1 and 9.2 are correct.

The graphs can be produced from the first worksheet in DerivaGem. Select Equity as the Underlying Type. Select Analytic European as the Option Type. Input stock price as 50, volatility as 30%, risk-free rate as 5%, time to exercise as 1 year, and exercise price as 50. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 7.15562248. Move to the Graph Results on the right hand side of the worksheet. Enter Option Price for the vertical axis and Asset price for the horizontal axis. Choose the minimum strike price value as 10 (software will not accept 0) and the maximum strike price value as 100. Hit *Enter* and click on *Draw Graph.* This will produce Figure 9.1a. Figures 9.1c, 9.1e, 9.2a, and 9.2c can be produced similarly by changing the horizontal axis. By selecting put instead of call and recalculating the rest of the figures can be produced. You are encouraged to experiment with this worksheet. Try different parameter values and different types of options.

ASSIGNMENT QUESTIONS

Problem 9.22.

A European call option and put option on a stock both have a strike price of $20 and an expiration date in three months. Both sell for $3. The risk-free interest rate is 10% per annum, the current stock price is $19, and a $1 dividend is expected in one month. Identify the arbitrage opportunity open to a trader.

If the call is worth $3, put-call parity shows that the put should be worth

$$3 + 20e^{-0.10\times 3/12} + e^{-0.1\times 1/12} - 19 = 4.50$$

This is greater than $3. The put is therefore undervalued relative to the call. The correct arbitrage strategy is to buy the put, buy the stock, and short the call. This costs $19. If the stock price in three months is greater than $20, the call is exercised. If it is less than $20, the put is exercised. In either case the arbitrageur sells the stock for $20 and collects the $1 dividend in one month. The present value of the gain to the arbitrageur is

$$-3 - 19 + 3 + 20e^{-0.10\times 3/12} + e^{-0.1\times 1/12} = 1.50$$

Problem 9.23.

Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

(Hint: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2. The value of the portfolio can be worked out in four different situations

$$
\begin{aligned}
S_T \le K_1 :&\ \text{Portfolio Value} &&= 0 \\
K_1 < S_T \le K_2 :&\ \text{Portfolio Value} &&= S_T - K_1 \\
K_2 < S_T \le K_3 :&\ \text{Portfolio Value} &&= S_T - K_1 - 2(S_T - K_2) \\
& &&= K_2 - K_1 - (S_T - K_2) \ge 0 \\
S_T > K_3 :&\ \text{Portfolio Value} &&= S_T - K_1 - 2(S_T - K_2) + S_T - K_3 \\
& &&= K_2 - K_1 - (K_3 - K_2) \\
& &&= 0
\end{aligned}
$$

The value is always either positive or zero at the expiration of the option. In the absence of arbitrage possibilities it must be positive or zero today. This means that

$$c_1 + c_3 - 2c_2 \ge 0$$

or

$$c_2 \le 0.5(c_1 + c_3)$$

Note that students often think they have proved this by writing down

$$c_1 \le S_0 - K_1 e^{-rT}$$

$$2c_2 \le 2(S_0 - K_2 e^{-rT})$$

$$c_3 \le S_0 - K_3 e^{-rT}$$

and subtracting the middle inequality from the sum of the other two. But they are deceiving themselves. Inequality relationships cannot be subtracted. For example, $9 > 8$ and $5 > 2$, but it is not true that $9 - 5 > 8 - 2$

Problem 9.24.

What is the result corresponding to that in Problem 9.23 for European put options?

The corresponding result is

$$p_2 \le 0.5(p_1 + p_3)$$

where p_1, p_2 and p_3 are the prices of European put option with the same maturities and strike prices K_1, K_2 and K_3 respectively. This can be proved from the result in Problem 9.23 using put-call parity. Alternatively we can consider a portfolio consisting of a long position in a put option with strike price K_1, a long position in a put option with strike price K_3, and a short position in two put options with strike price K_2. The value of this portfolio in different situations is given as follows

$$S_T \le K_1 : \ \text{Portfolio Value} = K_1 - S_T - 2(K_2 - S_T) + K_3 - S_T$$

$$
\begin{aligned}
& & &= K_3 - K_2 - (K_2 - K_1) \\
& & &= 0 \\
K_1 < S_T \leq K_2 : &\text{ Portfolio Value} & &= K_3 - S_T - 2(K_2 - S_T) \\
& & &= K_3 - K_2 - (K_2 - S_T) \\
& & &\geq 0 \\
K_2 < S_T \leq K_3 : &\text{ Portfolio Value} & &= K_3 - S_T \\
S_T > K_3 : &\text{ Portfolio Value} & &= 0
\end{aligned}
$$

Because the portfolio value is always zero or positive at some future time the same must be true today. Hence

$$p_1 + p_3 - 2p_2 \geq 0$$

or

$$p_2 \leq 0.5(p_1 + p_3)$$

Problem 9.25.

Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in one year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.

a. *Express your position as an option on the value of the company.*
b. *Express the position of the debt holders in terms of options on the value of the company.*
c. *What can you do to increase the value of your position?*

(a) Suppose V is the value of the company and D is the face value of the debt. The value of the manager's position in one year is

$$\max(V - D,\ 0)$$

This is the payoff from a call option on V with strike price D.

(b) The debt holders get

$$\min(V,\ D)$$

$$= D - \max(D - V,\ 0)$$

Since $\max(D - V,\ 0)$ is the payoff from a put option on V with strike price D, the debt holders have in effect made a risk-free loan (worth D at maturity with certainty) and written a put option on the value of the company with strike price D. The position of the debt holders in one year can also be characterized as

$$V - \max(V - D,\ 0)$$

This is a long position in the assets of the company combined with a short position in a call option on the assets with a strike price of D. The equivalence of the two characterizations can be presented as an application of put–call parity. (See Business Snapshot 9.1.)

(c) The manager can increase the value of his or her position by increasing the value of the call option in (a). It follows that the manager should attempt to increase both V and the volatility of V. To see why increasing the volatility of V is beneficial, imagine what happens when there are large changes in V. If V increases, the manager benefits to the full extent of the change. If V decreases, much of the downside is absorbed by the company's lenders.

Problem 9.26.

Consider an option on a stock when the stock price is \$41, the strike price is \$40, the risk-free rate is 6%, the volatility is 35%, and the time to maturity is 1 year. Assume that a dividend of \$0.50 is expected after six months.

a. *Use DerivaGem to value the option assuming it is a European call.*
b. *Use DerivaGem to value the option assuming it is a European put.*
c. *Verify that put–call parity holds.*
d. *Explore using DerivaGem what happens to the price of the options as the time to maturity becomes very large. For this purpose assume there are no dividends. Explain the results you get.*

DerivaGem shows that the price of the call option is 6.9686 and the price of the put option is 4.1244. In this case

$$c + D + Ke^{-rT} = 6.9686 + 0.5e^{-0.06\times 0.5} + 40e^{-0.06\times 1} = 45.1244$$

Also

$$p + S = 4.1244 + 41 = 45.1244$$

As the time to maturity becomes very large and there are no dividends, the price of the call option approaches the stock price of 41. (For example when $T = 100$ it is 40.94.) This is because the call option can be regarded as a position in the stock where the price does not have to be paid for a very long time. The present value of what has to be paid is close to zero. As the time to maturity becomes very large the price of the European put option becomes close to zero. (For example when $T = 100$ it is 0.04.) This is because the present value of what might be received from the put option becomes close to zero.

CHAPTER 10

Trading Strategies Involving Options

Notes for the Instructor

This chapter covers various ways in which traders can form portfolios of calls and puts to get interesting payoff patterns. For ease of exposition, the time value of money is ignored in payoff diagrams and payoff tables.

Students usually enjoy the chapter. As each spread strategy is covered, I like to use put–call parity to relate the cost of the spread created using calls to the cost of a spread created using puts. (See Problems 10.8 and 10.11) This reinforces the Chapter 9 material on put–call parity. It can be useful to cover Business Snapshot 10.1 in class.

Problem 10.19 can be used for class discussion. Problems 10.20, 10.21, 10.22, and 10.23 can be used as hand-in assignments.

QUESTIONS AND PROBLEMS

Problem 10.1.

What is meant by a protective put? What position in call options is equivalent to a protective put?

A protective put consists of a long position in a put option combined with a long position in the underlying shares. It is equivalent to a long position in a call option plus a certain amount of cash. This follows from put–call parity:

$$p + S_0 = c + Ke^{-rT} + D$$

Problem 10.2.

Explain two ways in which a bear spread can be created.

A bear spread can be created using two call options with the same maturity and different strike prices. The investor shorts the call option with the lower strike price and buys the call option with the higher strike price. A bear spread can also be created using two put options with the same maturity and different strike prices. In this case, the investor shorts the put option with the lower strike price and buys the put option with the higher strike price.

Problem 10.3.

When is it appropriate for an investor to purchase a butterfly spread?

A butterfly spread involves a position in options with three different strike prices (K_1, K_2, and K_3). A butterfly spread should be purchased when the investor considers that the price of the underlying stock is likely to stay close to the central strike price, K_2.

Problem 10.4.

Call options on a stock are available with strike prices of \$15, $\$17\frac{1}{2}$, and \$20 and expiration dates in three months. Their prices are \$4, \$2, and $\$\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.

An investor can create a butterfly spread by buying call options with strike prices of \$15 and \$20 and selling two call options with strike prices of $\$17\frac{1}{2}$. The initial investment is $4 + \frac{1}{2} - 2 \times 2 = \$\frac{1}{2}$. The following table shows the variation of profit with the final stock price:

Stock Price S_T	Profit
$S_T < 15$	$-\frac{1}{2}$
$15 < S_T < 17\frac{1}{2}$	$(S_T - 15) - \frac{1}{2}$
$17\frac{1}{2} < S_T < 20$	$(20 - S_T) - \frac{1}{2}$
$S_T > 20$	$-\frac{1}{2}$

Problem 10.5.

What trading strategy creates a reverse calendar spread?

A reverse calendar spread is created by buying a short-maturity option and selling a long-maturity option, both with the same strike price.

Problem 10.6.

What is the difference between a strangle and a straddle?

Both a straddle and a strangle are created by combining a long position in a call with a long position in a put. In a straddle the two have the same strike price and expiration date. In a strangle they have different strike prices and the same expiration date.

Problem 10.7.

A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?

A strangle is created by buying both options. The pattern of profits is as follows:

Stock Price S_T	Profit
$S_T < 45$	$(45 - S_T) - 5$
$45 < S_T < 50$	-5
$S_T > 50$	$(S_T - 50) - 5$

Problem 10.8.

Use put–call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.

A bull spread using calls provides a profit pattern with the same general shape as a bull spread using puts (see Figures 10.2 and 10.3 in the text). Define p_1 and c_1 as the prices of put and call with strike price K_1 and p_2 and c_2 as the prices of a put and call with strike price K_2. From put-call parity

$$p_1 + S = c_1 + K_1 e^{-rT}$$

$$p_2 + S = c_2 + K_2 e^{-rT}$$

Hence:

$$p_1 - p_2 = c_1 - c_2 - (K_2 - K_1)e^{-rT}$$

This shows that the initial investment when the spread is created from puts is less than the initial investment when it is created from calls by an amount $(K_2 - K_1)e^{-rT}$. In fact as mentioned in the text the initial investment when the bull spread is created from puts is negative, while the initial investment when it is created from calls is positive.

The profit when calls are used to create the bull spread is higher than when puts are used by $(K_2 - K_1)(1 - e^{-rT})$. This reflects the fact that the call strategy involves an additional risk-free investment of $(K_2 - K_1)e^{-rT}$ over the put strategy. This earns interest of $(K_2 - K_1)e^{-rT}(e^{rT} - 1) = (K_2 - K_1)(1 - e^{-rT})$.

Problem 10.9.

Explain how an aggressive bear spread can be created using put options.

An aggressive bull spread using call options is discussed in the text. Both of the options used have relatively high strike prices. Similarly, an aggressive bear spread can be created using put options. Both of the options should be out of the money (that is, they should have relatively low strike prices). The spread then costs very little to set up because both of the puts are worth close to zero. In most circumstances the spread will provide zero payoff. However, there is a small chance that the stock price will fall fast so that on expiration both options will be in the money. The spread then provides a payoff equal to the difference between the two strike prices, $K_2 - K_1$.

Problem 10.10.

Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Construct a table that shows the profit and payoff for both spreads.

A bull spread is created by buying the \$30 put and selling the \$35 put. This strategy gives rise to an initial cash inflow of \$3. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	3
$30 \leq S_T < 35$	$S_T - 35$	$S_T - 32$
$S_T < 30$	-5	-2

A bear spread is created by selling the \$30 put and buying the \$35 put. This strategy costs \$3 initially. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	-3
$30 \leq S_T < 35$	$35 - S_T$	$32 - S_T$
$S_T < 30$	5	2

Problem 10.11.

Use put–call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

Define c_1, c_2, and c_3 as the prices of calls with strike prices K_1, K_2 and K_3. Define p_1, p_2 and p_3 as the prices of puts with strike prices K_1, K_2 and K_3. With the usual notation

$$c_1 + K_1 e^{-rT} = p_1 + S$$

$$c_2 + K_2 e^{-rT} = p_2 + S$$

$$c_3 + K_3 e^{-rT} = p_3 + S$$

Hence

$$c_1 + c_3 - 2c_2 + (K_1 + K_3 - 2K_2)e^{-rT} = p_1 + p_3 - 2p_2$$

Because $K_2 - K_1 = K_3 - K_2$, it follows that $K_1 + K_3 - 2K_2 = 0$ and

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2$$

The cost of a butterfly spread created using European calls is therefore exactly the same as the cost of a butterfly spread created using European puts.

Problem 10.12.

A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

A straddle is created by buying both the call and the put. This strategy costs \$10. The profit/loss is shown in the following table:

Stock Price	Payoff	Profit
$S_T > 60$	$S_T - 60$	$S_T - 70$
$S_T \leq 60$	$60 - S_T$	$50 - S_T$

This shows that the straddle will lead to a loss if the final stock price is between \$50 and \$70.

Problem 10.13.

Construct a table showing the payoff from a bull spread when puts with strike prices K_1 and K_2 are used ($K_2 > K_1$).

The bull spread is created by buying a put with strike price K_1 and selling a put with strike price K_2. The payoff is calculated as follows:

Stock Price Range	Payoff from Long Put Option	Payoff from Short Put Option	Total Payoff
$S_T \geq K_2$	0	0	0
$K_1 < S_T < K_2$	0	$S_T - K_2$	$-(K_2 - S_T)$
$S_T \leq K_1$	$K_1 - S_T$	$S_T - K_2$	$-(K_2 - K_1)$

Problem 10.14.

An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

Possible strategies are:

Strangle
Straddle
Strip
Strap
Reverse calendar spread
Reverse butterfly spread

The strategies all provide positive profits when there are large stock price moves. A strangle is less expensive than a straddle, but requires a bigger move in the stock price in order to provide a positive profit. Strips and straps are more expensive than straddles but provide bigger profits in certain circumstances. A strip will provide a bigger profit when there is a large downward stock price move. A strap will provide a bigger profit when there is a large upward stock price move. In the case of strangles, straddles, strips and straps, the profit increases as the size of the stock price movement increases. By contrast in a reverse calendar spread and a reverse butterfly spread there is a maximum potential profit regardless of the size of the stock price movement.

Problem 10.15.

How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

Suppose that the delivery price is K and the delivery date is T. The forward contract is created by buying a European call and selling a European put when both options have strike price K and exercise date T. This portfolio provides a payoff of $S_T - K$ under all circumstances where S_T is the stock price at time T. Suppose that F_0 is the forward price. If $K = F_0$, the forward contract that is created has zero value. This shows that the price of a call equals the price of a put when the strike price is F_0.

Problem 10.16.

"A box spread comprises four options. Two can be combined to create a long forward position and two can be combined to create a short forward position." Explain this statement.

A box spread is a bull spread created using calls and a bear spread created using puts. With the notation in the text it consists of a) a long call with strike K_1, b) a short call with strike K_2, c) a long put with strike K_2, and d) a short put with strike K_1. a) and d) give a long forward contract with delivery price K_1; b) and c) give a short forward contract with delivery price K_2. The two forward contracts taken together give the payoff of $K_2 - K_1$.

Problem 10.17.

What is the result if the strike price of the put is higher than the strike price of the call in a strangle?

The result is shown in Figure S10.1. The profit pattern from a long position in a call and a put when the put has a higher strike price than a call is much the same as when the call has a higher strike price than the put. Both the initial investment and the final payoff are much higher in the first case.

Problem 10.18.

One Australian dollar is currently worth \$0.64. A one-year butterfly spread is set up using European call options with strike prices of \$0.60, \$0.65, and \$0.70. The risk-free

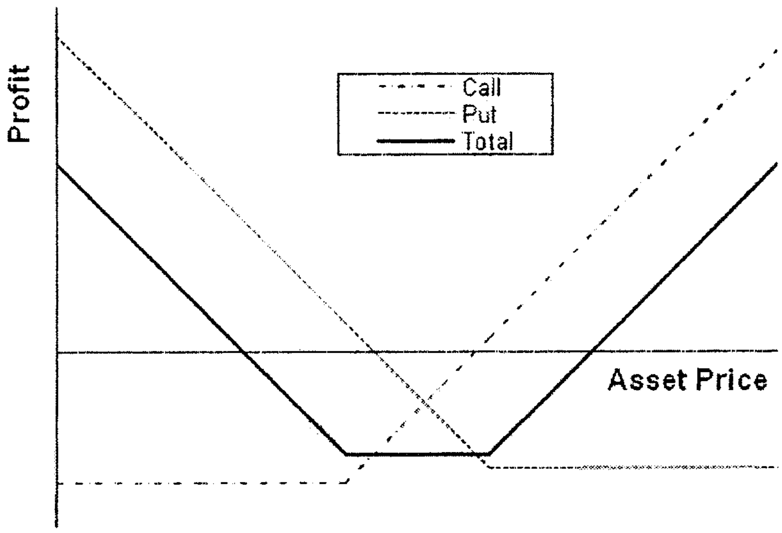

Figure S10.1 Profit Pattern in Problem 10.17

interest rates in the United States and Australia are 5% and 4% respectively, and the volatility of the exchange rate is 15%. Use the DerivaGem software to calculate the cost of setting up the butterfly spread position. Show that the cost is the same if European put options are used instead of European call options.

To use DerivaGem select the first worksheet and choose Currency as the Underlying Type. Select Analytic European as the Option Type. Input exchange rate as 0.64, volatility as 15%, risk-free rate as 5%, foreign risk-free interest rate as 4%, time to exercise as 1 year, and exercise price as 0.60. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.0618. Change the exercise price to 0.65, hit *Enter*, and click on calculate again. DerivaGem will show the value of the option as 0.0352. Change the exercise price to 0.70, hit *Enter*, and click on calculate. DerivaGem will show the value of the option as 0.0181.

Now select the button corresponding to put and repeat the procedure. DerivaGem shows the values of puts with strike prices 0.60, 0.65, and 0.70 to be 0.0176, 0.0386, and 0.0690, respectively.

The cost of setting up the butterfly spread when calls are used is therefore

$$0.0618 + 0.0181 - 2 \times 0.0352 = 0.0095$$

The cost of setting up the butterfly spread when puts are used is

$$0.0176 + 0.0690 - 2 \times 0.0386 = 0.0094$$

Allowing for rounding errors these two are the same.

Problem 10.19.

Three put options on a stock have the same expiration date and strike prices of \$55, \$60, and \$65. The market prices are \$3, \$5, and \$8, respectively. Explain how a butterfly spread can be created. Construct a table showing the profit from the strategy. For what range of stock prices would the butterfly spread lead to a loss?

A butterfly spread is created by buying the \$55 put, buying the \$65 put and selling two of the \$60 puts. This costs $3 + 8 - 2 \times 5 = \$1$ initially. The following table shows the profit/loss from the strategy.

Stock Price	Payoff	Profit
$S_T \geq 65$	0	−1
$60 \leq S_T < 65$	$65 - S_T$	$64 - S_T$
$55 \leq S_T < 60$	$S_T - 55$	$S_T - 56$
$S_T < 55$	0	−1

The butterfly spread leads to a loss when the final stock price is greater then \$64 or less than \$56.

Problem 10.20.

A diagonal spread is created by buying a call with strike price K_2 and exercise date T_2 and selling a call with strike price K_1 and exercise date T_1 $(T_2 > T_1)$. Draw a diagram showing the profit when (a) $K_2 > K_1$ and (b) $K_2 < K_1$.

There are two alternative profit patterns for part (a). These are shown in Figures M10.1 and M10.2. In Figure M10.1 the long maturity (high strike price) option is worth more than the short maturity (low strike price) option. In Figure M10.2 the reverse is true. There is no ambiguity about the profit pattern for part (b). This is shown in Figure M10.3.

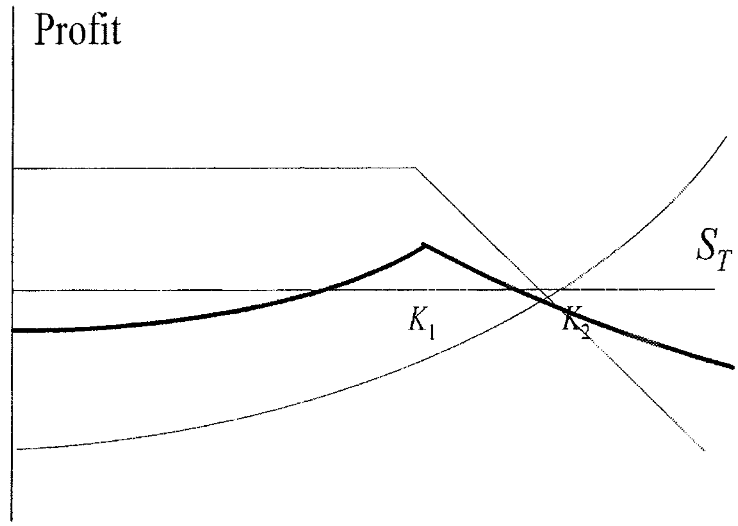

Figure M10.1 Investor's Profit/Loss in Problem 10.20a when long maturity call is worth more than short maturity call

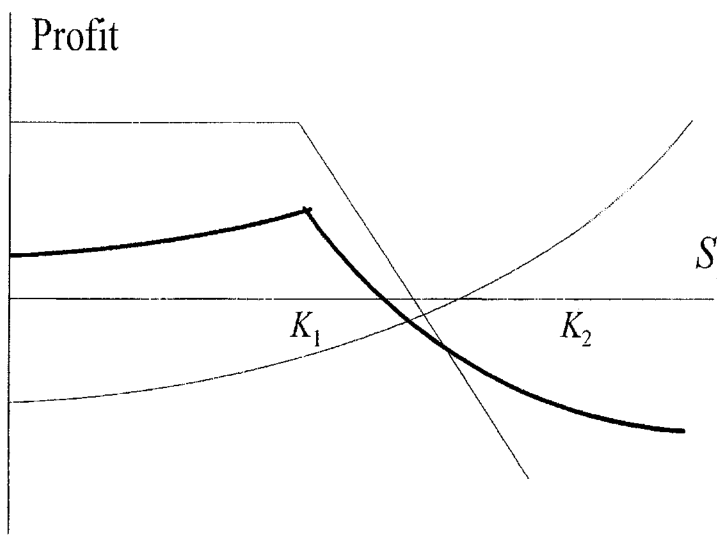

Figure M10.2 Investor's Profit/Loss in Problem 10.20a when short maturity call is worth more than long maturity call

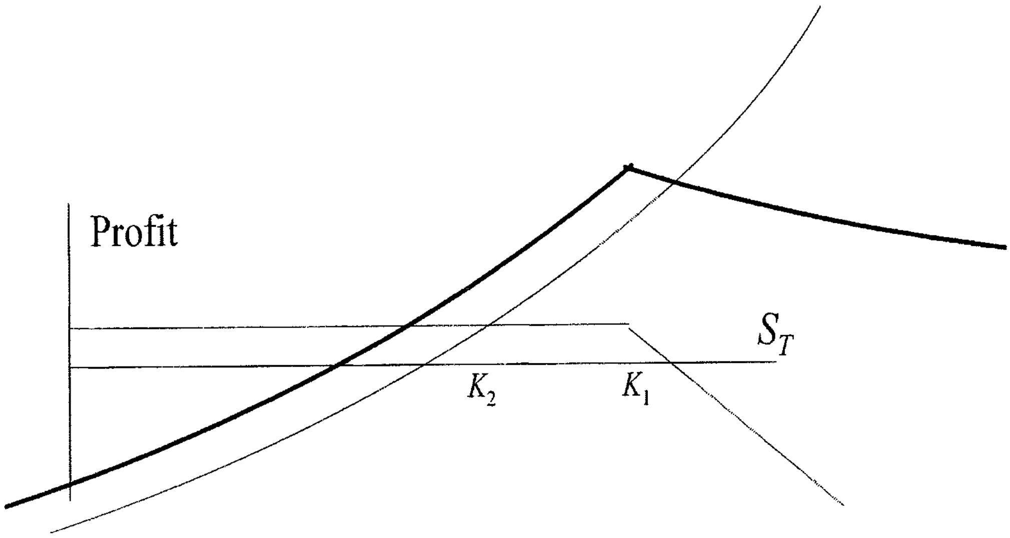

Figure M10.3 Investor's Profit/Loss in Problem 10.20b

Problem 10.21.

Draw a diagram showing the variation of an investor's profit and loss with the terminal stock price for a portfolio consisting of

a. *One share and a short position in one call option*
b. *Two shares and a short position in one call option*
c. *One share and a short position in two call options*
d. *One share and a short position in four call options*

In each case, assume that the call option has an exercise price equal to the current stock price.

The variation of an investor's profit/loss with the terminal stock price for each of the four strategies is shown in Figure M10.4. In each case the dotted line shows the profits from the components of the investor's position and the solid line shows the total net profit.

Problem 10.22.

Suppose that the price of a non-dividend-paying stock is \$32, its volatility is 30%, and the risk-free rate for all maturities is 5% per annum. Use DerivaGem to calculate the cost of setting up the following positions. In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

a. *A bull spread using European call options with strike prices of \$25 and \$30 and a maturity of six months.*
b. *A bear spread using European put options with strike prices of \$25 and \$30 and a maturity of six months*
c. *A butterfly spread using European call options with strike prices of \$25, \$30, and \$35 and a maturity of one year.*
d. *A butterfly spread using European put options with strike prices of \$25, \$30, and \$35 and a maturity of one year.*
e. *A straddle using options with a strike price of \$30 and a six-month maturity.*

f. A strangle using options with strike prices of \$25 and \$35 and a six-month maturity.

(a) A call option with a strike price of 25 costs 7.90 and a call option with a strike price of 30 costs 4.18. The cost of the bull spread is therefore $7.90 - 4.18 = 3.72$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-3.72
$25 < S_T < 30$	$S_T - 28.72$
$S_T \geq 30$	1.28

(b) A put option with a strike price of 25 costs 0.28 and a put option with a strike price of 30 costs 1.44. The cost of the bear spread is therefore $1.44 - 0.28 = 1.16$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	$+3.84$
$25 < S_T < 30$	$28.84 - S_T$
$S_T \geq 30$	-1.16

(c) Call options with maturities of one year and strike prices of 25, 30, and 35 cost 8.92, 5.60, and 3.28, respectively. The cost of the butterfly spread is therefore $8.92 + 3.28 - 2 \times 5.60 = 1.00$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-1.00
$25 < S_T < 30$	$S_T - 26.00$
$30 \leq S_T < 35$	$34.00 - S_T$
$S_T \geq 35$	-1.00

(d) Put options with maturities of one year and strike prices of 25, 30, and 35 cost 0.70, 2.14, 4.57, respectively. The cost of the butterfly spread is therefore $0.70 + 4.57 - 2 \times 2.14 = 0.99$. Allowing for rounding errors, this is the same as in (c). The profits are the same as in (c).

(e) A call option with a strike price of 30 costs 4.18. A put option with a strike price of 30 costs 1.44. The cost of the straddle is therefore $4.18 + 1.44 = 5.62$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 30$	$24.58 - S_T$
$S_T > 30$	$S_T - 35.62$

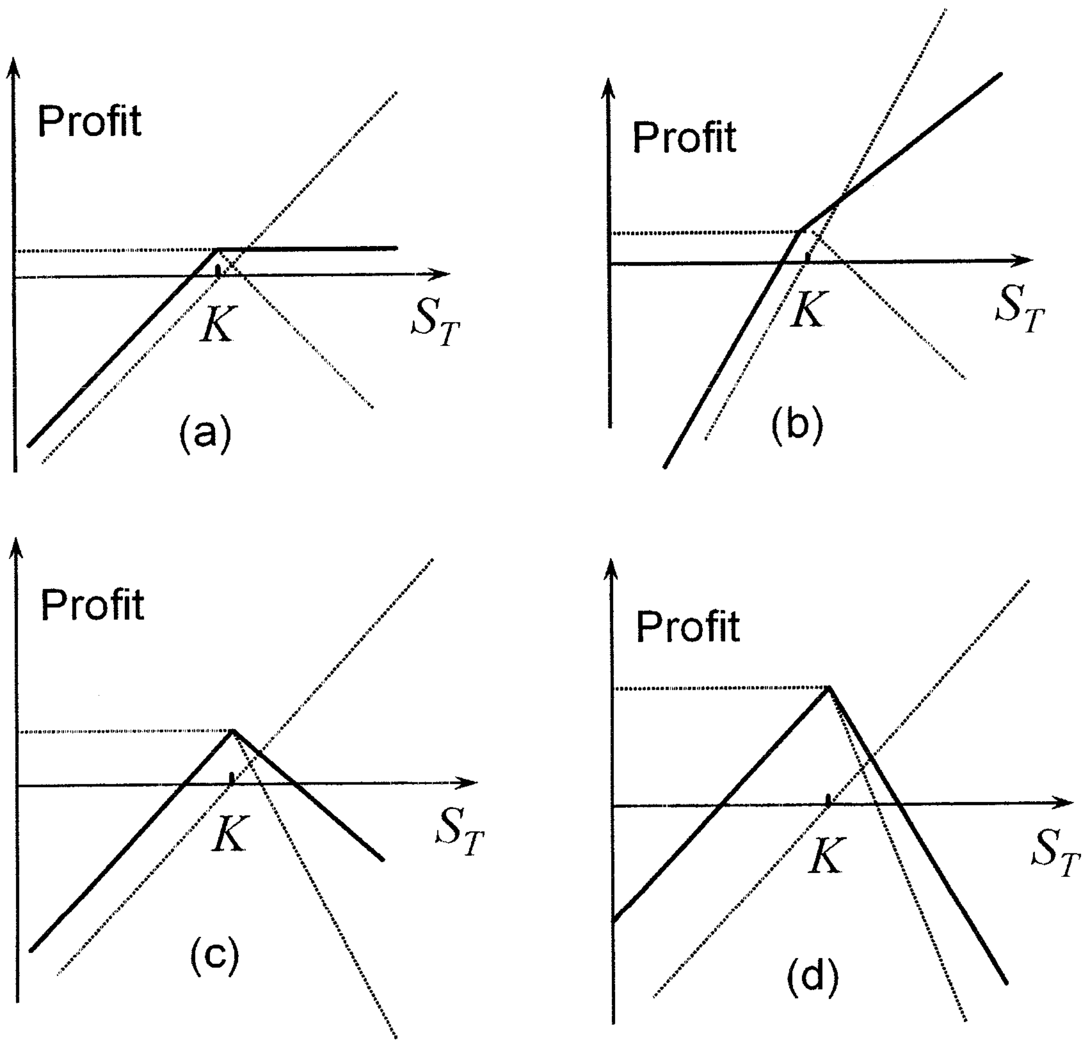

Figure M10.4 Investor's Profit/Loss in Problem 10.21

(f) A six-month call option with a strike price of 35 costs 1.85. A six-month put option with a strike price of 25 costs 0.28. The cost of the strangle is therefore $1.85 + 0.28 = 2.13$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	$22.87 - S_T$
$25 < S_T < 35$	-2.13
$S_T \geq 35$	$S_T - 37.13$

Problem 10.23.

What trading position is created from a long strangle and a short straddle when both have the same time to maturity? Assume that the strike price in the straddle is half way between the two strike prices in the strangle.

A butterfly spread is created.

CHAPTER 11
Binomial Trees

Notes for the Instructor

This chapter discusses binomial trees. It enables some of the key concepts in option valuation to be introduced at a relatively early stage in a course. It includes material on the use of binomial trees for index options, currency options, and futures options (see Section 11.9).

The one-step binomial model can be used to demonstrate both no-arbitrage and risk-neutral valuation arguments. I like to first go through the arguments using the numerical example in the text and then generalize them by introducing some notation. After two- and three-step trees have been covered students should have a good appreciation of the way in which multistep trees are used to value options. DerivaGem provides a convenient way of displaying trees in class. The material on delta serves as an introduction to the hedging material in Chapter 17.

Any of Problems 11.16 to 11.22 can be used as assignments. I usually discuss 11.22 in class.

QUESTIONS AND PROBLEMS

Problem 11.1.

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

Consider a portfolio consisting of

-1 :	Call option
$+\Delta$:	Shares

If the stock price rises to \$42, the portfolio is worth $42\Delta - 3$. If the stock price falls to \$38, it is worth 38Δ. These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is 28.5 for both stock prices. Its value today must be the present value of 28.5, or $28.5e^{-0.08\times 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where f is the call price. Because $\Delta = 0.75$, the call price is $40 \times 0.75 - 28.31 = \$1.69$. As an alternative approach, we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1-p) = 40e^{0.08\times 0.08333}$$

so that

$$4p = 40e^{0.08\times 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08\times 0.08333} = 1.69$$

or \$1.69. This agrees with the previous calculation.

Problem 11.2.

Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 11.3.

What is meant by the delta of a stock option?

The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 11.4.

A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a six-month European put option with a strike price of \$50?

Consider a portfolio consisting of

-1 :	Put option
$+\Delta$:	Shares

If the stock price rises to \$55, this is worth 55Δ. If the stock price falls to \$45, the portfolio is worth $45\Delta - 5$. These are the same when

$$45\Delta - 5 = 55\Delta$$

or $\Delta = -0.50$. The value of the portfolio in six months is -27.5 for both stock prices. Its value today must be the present value of -27.5, or $-27.5e^{-0.1\times 0.5} = -26.16$. This means that

$$-f + 50\Delta = -26.16$$

where f is the put price. Because $\Delta = -0.50$, the put price is \$1.16. As an alternative approach we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$55p + 45(1-p) = 50e^{0.1\times 0.5}$$

so that

$$10p = 50e^{0.1\times 0.5} - 45$$

or $p = 0.7564$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[0 \times 0.7564 + 5 \times 0.2436]e^{-0.1\times 0.5} = 1.16$$

or \$1.16. This agrees with the previous calculation.

Problem 11.5.

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08\times 0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown in Figure S11.1. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly from equation (11.10):

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2\times 0.08\times 0.5} = 9.61$$

or \$9.61.

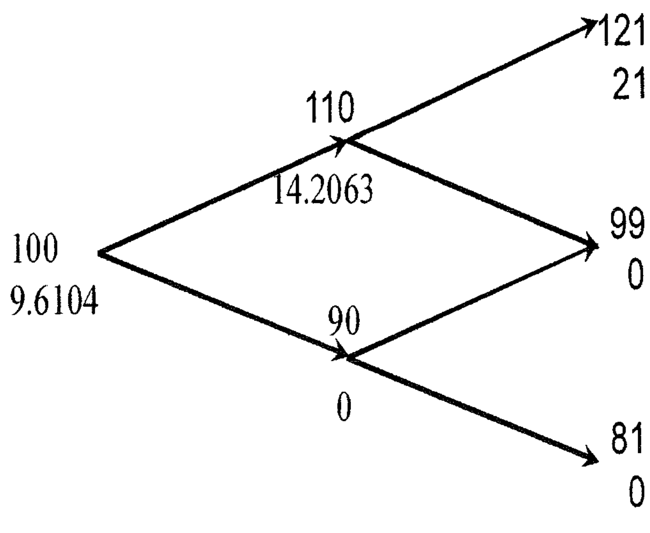

Figure S11.1 Tree for Problem 11.5

Problem 11.6.

For the situation considered in Problem 11.5, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put–call parity.

Figure S11.2 shows how we can value the put option using the same tree as in Problem 11.5. The value of the option is \$1.92. The option value can also be calculated directly from equation (11.10):

$$e^{-2\times0.08\times0.5}[0.7041^2\times0+2\times0.7041\times0.2959\times1+0.2959^2\times19]=1.92$$

or \$1.92. The stock price plus the put price is $100 + 1.92 = \$101.92$. The present value of the strike price plus the call price is $100e^{-0.08\times1} + 9.61 = \$101.92$. These are the same, verifying that put–call parity holds.

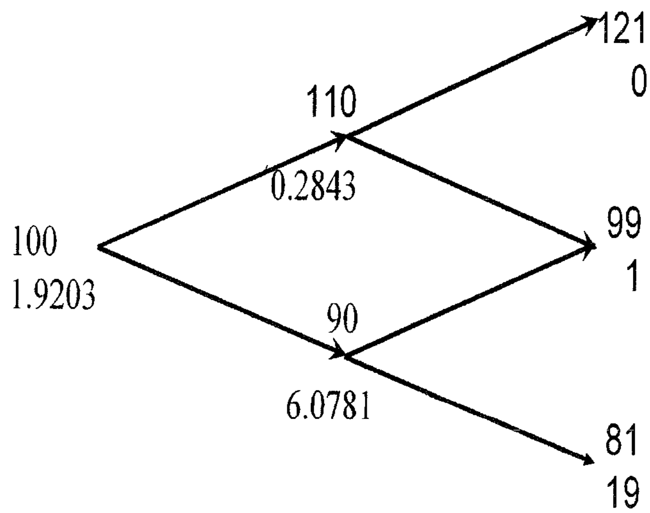

Figure S11.2 Tree for Problem 11.6

Problem 11.7.

What are the formulas for u and d in terms of volatility?

$u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$

Problem 11.8.

Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

Problem 11.9.

A stock price is currently $50. It is known that at the end of two months it will be either $53 or $48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of $49? Use no-arbitrage arguments.

At the end of two months the value of the option will be either $4 (if the stock price is $53) or $0 (if the stock price is $48). Consider a portfolio consisting of:

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{option} \end{array}$$

The value of the portfolio is either 48Δ or $53\Delta - 4$ in two months. If

$$48\Delta = 53\Delta - 4$$

i.e.,

$$\Delta = 0.8$$

the value of the portfolio is certain to be 38.4. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.8 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.8 \times 50 - f)e^{0.10\times 2/12} = 38.4$$

i.e.,

$$f = 2.23$$

The value of the option is therefore $2.23.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.06$, $d = 0.96$ so that

$$p = \frac{e^{0.10\times 2/12} - 0.96}{1.06 - 0.96} = 0.5681$$

and

$$f = e^{-0.10\times 2/12} \times 0.5681 \times 4 = 2.23$$

Problem 11.10.

A stock price is currently \$80. It is known that at the end of four months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of \$80? Use no-arbitrage arguments.

At the end of four months the value of the option will be either \$5 (if the stock price is \$75) or \$0 (if the stock price is \$85). Consider a portfolio consisting of:

$$\begin{array}{rcl} -\Delta & : & \text{shares} \\ +1 & : & \text{option} \end{array}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta + 5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5 \times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5 \times 80 + f)e^{0.05 \times 4/12} = 42.5$$

i.e.,

$$f = 1.80$$

The value of the option is therefore \$1.80.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.0625$, $d = 0.9375$ so that

$$p = \frac{e^{0.05 \times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05 \times 4/12} \times 0.3655 \times 5 = 1.80$$

Problem 11.11.

A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with *an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.*

At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$\begin{array}{rcl} -\Delta & : & \text{shares} \\ +1 & : & \text{option} \end{array}$$

(Note: The delta, Δ, of a put option is negative. We have constructed the portfolio so that it is $+1$ option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta + 5$ or -45Δ. If:

$$-35\Delta + 5 = -45\Delta$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta + f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40 \times 0.5 + f) \times 1.02 = 22.5$$

Hence

$$f = 2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p + 35(1 - p) = 40 \times 1.02$$

i.e.,

$$10p = 5.8$$

or:

$$p = 0.58$$

The expected value of the option in a risk-neutral world is:

$$0 \times 0.58 + 5 \times 0.42 = 2.10$$

This has a present value of

$$\frac{2.10}{1.02} = 2.06$$

This is consistent with the no-arbitrage answer.

Problem 11.12.

A stock price is currently \$50. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of \$51?

A tree describing the behavior of the stock price is shown in Figure S11.3. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05\times 3/12} - 0.95}{1.06 - 0.95} = 0.5689$$

There is a payoff from the option of $56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves) zero in all other cases. The value of the option is therefore

$$5.18 \times 0.5689^2 \times e^{-0.05\times 6/12} = 1.635$$

This can also be calculated by working back through the tree as indicated in Figure S11.3. The value of the call option is the lower number at each node in the figure.

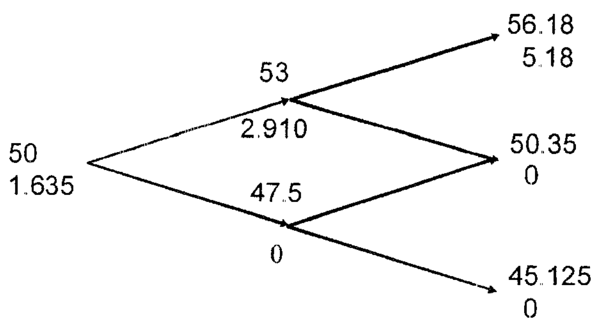

Figure S11.3 Tree for Problem 11.12

Problem 11.13.

For the situation considered in Problem 11.12, what is the value of a six-month European put option with a strike price of \$51? Verify that the European call and European put prices satisfy put–call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

The tree for valuing the put option is shown in Figure S11.4. We get a payoff of $51 - 50.35 = 0.65$ if the middle final node is reached and a payoff of $51 - 45.125 = 5.875$ if the lowest final node is reached. The value of the option is therefore

$$(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2)e^{-0.05\times6/12} = 1.376$$

This can also be calculated by working back through the tree as indicated in Figure S11.4.

The value of the put plus the stock price is from Problem 11.12

$$1.376 + 50 = 51.376$$

The value of the call plus the present value of the strike price is

$$1.635 + 51e^{-0.05\times6/12} = 51.376$$

This verifies that put–call parity holds

To test whether it worth exercising the option early we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is $51 - 47.5 = 3.5$. Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

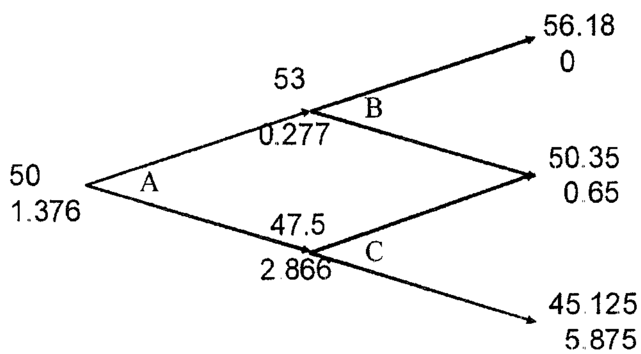

Figure S11.4 Tree for Problem 11.13

Problem 11.14.

A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$\begin{array}{rcc} +\Delta & : & \text{shares} \\ -1 & : & \text{derivative} \end{array}$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e.,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10\times 2/12} = 621$$

i.e.,

$$f = 639.3$$

The value of the option is therefore \$639.3.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.08$, $d = 0.92$ so that

$$p = \frac{e^{0.10\times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10\times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

Problem 11.15.

Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

In this case

$$a = e^{(0.05-0.08)\times 1/12} = 0.9975$$

$$u = e^{0.12\sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$

ASSIGNMENT QUESTIONS

Problem 11.16.

A stock price is currently \$50. It is known that at the end of six months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a six-month European call option on the stock with an exercise price of \$48. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

At the end of six months the value of the option will be either \$12 (if the stock price is \$60) or \$0 (if the stock price is \$42). Consider a portfolio consisting of:

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{option} \end{array}$$

The value of the portfolio is either 42Δ or $60\Delta - 12$ in six months. If

$$42\Delta = 60\Delta - 12$$

i.e.,

$$\Delta = 0.6667$$

the value of the portfolio is certain to be 28. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.6667 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.6667 \times 50 - f)e^{0.12 \times 0.5} = 28$$

i.e.,

$$f = 6.96$$

The value of the option is therefore \$6.96.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$60p + 42(1 - p) = 50 \times e^{0.06}$$

i.e.,

$$18p = 11.09$$

or:

$$p = 0.6161$$

The expected value of the option in a risk-neutral world is:

$$12 \times 0.6161 + 0 \times 0.3839 = 7.3932$$

This has a present value of

$$7.3932e^{-0.06} = 6.96$$

Hence the above answer is consistent with risk-neutral valuation.

Problem 11.17.

A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

a. *What is the value of a six-month European put option with a strike price of \$42?*
b. *What is the value of a six-month American put option with a strike price of \$42?*

(a) A tree describing the behavior of the stock price is shown in Figure M11.1. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.12\times 3/12} - 0.90}{1.1 - 0.9} = 0.6523$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[2.4 \times 2 \times 0.6523 \times 0.3477 + 9.6 \times 0.3477^2]e^{-0.12\times 6/12} = 2.118$$

The value of the European option is 2.118. This can also be calculated by working back through the tree as shown in Figure M11.1. The second number at each node is the value of the European option.

(b) The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.

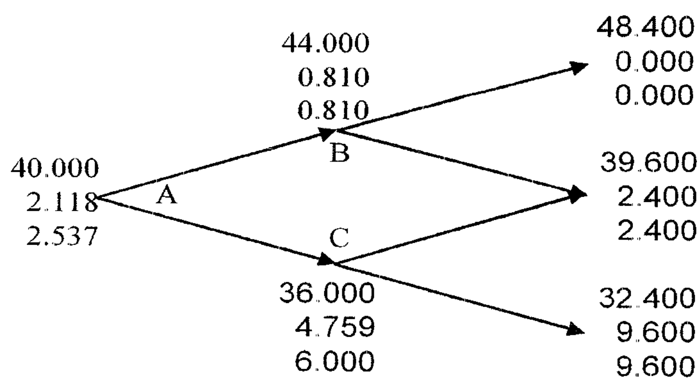

Figure M11.1 Tree to evaluate European and American put options in Problem 11.17. At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 11.18.

Using a "trial-and-error" approach, estimate how high the strike price has to be in Problem 11.17 for it to be optimal to exercise the option immediately.

Trial and error shows that immediate early exercise is optimal when the strike price is above 43.2.

This can be also shown to be true algebraically. Suppose the strike price increases by a relatively small amount q. This increases the value of being at node C by q and the value of being at node B by $0.3477e^{-0.03}q = 0.3374q$. It therefore increases the value of being at node A by

$$(0.6523 \times 0.3374q + 0.3477q)e^{-0.03} = 0.551q$$

For early exercise at node A we require $2.537+0.551q < 2+q$ or $q > 1.196$. This corresponds to the strike price being greater than 43.196.

Problem 11.19.

A stock price is currently \$30. During each two-month period for the next four months it is expected to increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $[\max(30 - S_T, 0]^2$ where S_T is the stock price in four months? If the derivative is American-style, should it be exercised early?

This type of option is known as a power option. A tree describing the behavior of the stock price is shown in Figure M11.2. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05 \times 2/12} - 0.9}{1.08 - 0.9} = 0.6020$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[0.7056 \times 2 \times 0.6020 \times 0.3980 + 32.49 \times 0.3980^2]e^{-0.05 \times 4/12} = 5.394$$

The value of the European option is 5.394. This can also be calculated by working back through the tree as shown in Figure M11.2. The second number at each node is the value of the European option.

Early exercise at node C would give 9.0 which is less than 13.2449. The option should therefore not be exercised early if it is American.

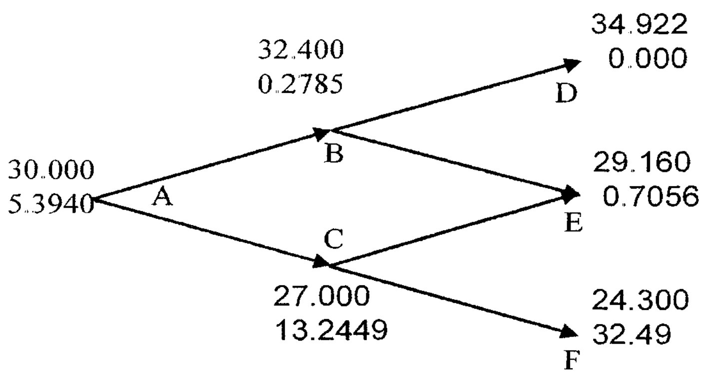

Figure M11.2 Tree to evaluate European power option in Problem 11.19. At each node, upper number is the stock price; next number is the option price

Problem 11.20.

Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is six months.

a. *Calculate u, d, and p for a two step tree*
b. *Value the option using a two step tree.*
c. *Verify that DerivaGem gives the same answer*
d. *Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.*

(a) This problem is based on the material in Section 11.8. In this case $\Delta t = 0.25$ so that $u = e^{0.30\times\sqrt{0.25}} = 1.1618$, $d = 1/u = 0.8607$, and

$$p = \frac{e^{0.04\times 0.25} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

(b) and (c) The value of the option using a two-step tree as given by DerivaGem is shown in Figure M11.3 to be 3.3739. To use DerivaGem choose the first worksheet, select Equity as the underlying type, and select Binomial European as the Option Type. After carrying out the calculations select Display Tree.

(d) With 5, 50, 100, and 500 time steps the value of the option is 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Values in red are a result of early exercise.

Strike price = 40
Discount factor per step = 0.9900
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.0101
Probability of up move, p = 0.4959
Up step size, u = 1.1618
Down step size, d = 0.8607

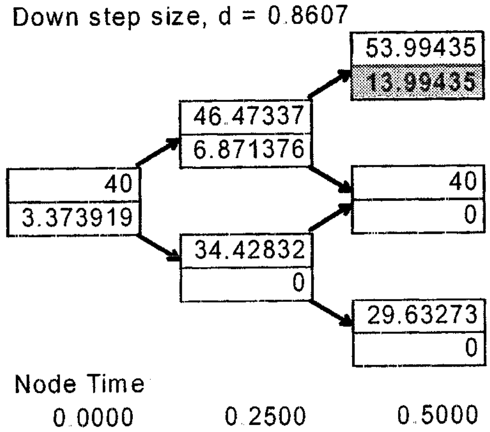

Figure M11.3 Tree produced by DerivaGem to evaluate European option in Problem 11.20.

Problem 11.21.

Repeat Problem 11.20 for an American put option on a futures contract. The strike price and the futures price are \$50, the risk-free rate is 10%, the time to maturity is six months, and the volatility is 40% per annum.

(a) In this case $\Delta t = 0.25$ and $u = e^{0.40\times\sqrt{0.25}} = 1.2214$, $d = 1/u = 0.8187$, and

$$p = \frac{e^{0.1\times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.4502$$

(b) and (c) The value of the option using a two-step tree is 4.8604.

(d) With 5, 50, 100, and 500 time steps the value of the option is 5.6858, 5.3869, 5.3981, and 5.4072, respectively.

Problem 11.22.

Footnote 1 shows that the correct discount rate to use for the real world expected payoff in the case of the call option considered in Figure 11.1 is 42.6%. Show that if the option is a put rather than a call the discount rate is –52.5%. Explain why the two real-world discount rates are so different.

The value of the put option is

$$(0.6523 \times 0 + 0.3477 \times 3)e^{-0.12 \times 3/12} = 1.0123$$

The expected payoff in the real world is

$$(0.7041 \times 0 + 0.2959 \times 3) = 0.8877$$

The discount rate R that should be used in the real world is therefore given by solving

$$1.0123 = 0.8877e^{-0.25R}$$

The solution to this is -0.525 or 52.5%.

The call option has a high positive discount rate because it has high positive systematic risk. The put option has a high negative discount rate because it has high negative systematic risk.

CHAPTER 12
Wiener processes and Itô's Lemma

Notes for the Instructor

The chapter provides some basic knowledge about Wiener processes, develops the geometric Brownian motion model of stock price behavior, and covers Itô's lemma. The book has been designed so that this chapter can be skipped if the instructor considers it too technical. For example, a popular way of using the book for a first MBA elective in derivatives is to assign the first 18 chapters, excluding Chapter 12 and Section 13.6 of Chapter 13.

I find that most students have surprisingly little difficulty with the material in Chapter 12. I usually start by discussing the distinction between continuous time and discrete time stochastic processes and the distinction between continuous variable and discrete variable stochastic processes. I do this with simple examples of models of stock price movements. A discrete time, discrete variable model would be one where every day a stock price has a probability p_1 of moving up by \$1, a probability p_2 of remaining the same, and a probability p_3 of moving down by \$1. A continuous time, discrete variable model would be one where price changes of \$1 are generated by a Poisson process. A discrete time, continuous variable model would be one where in each small time interval, stock price movements are sampled from a continuous distribution. The main purpose of the chapter is to develop a continuous time, continuous variable model as a limiting case of this last model.

The nature of Markov processes and the fact that they are consistent with market efficiency needs to be explained carefully. I find it useful to discuss Problems 12.1 and 12.2 in class.

I explain Wiener processes by starting with a discrete time, continuous variable model where values of the variable are observed every year. I assume that the change in the value of the variable during the year is a random sample from the distribution $\phi(0, 1)$. (Note that the second argument of ϕ is the variance not the standard deviation.) I then suppose that values of the variable are observed every 6 months and ask what distribution for 6-month changes is consistent with the distribution of 1-year changes. The answer is $\phi(0, \frac{1}{2})$. When 3-month changes are considered, the distribution is $\phi(0, 0.25)$. When time intervals of length Δt are considered, the distribution is $\phi(0, \Delta t)$. A random sample from $\phi(0, \Delta t)$ is $\epsilon\sqrt{\Delta t}$. This explains where the definition of a Wiener process comes from. In particular it explains why we take the square root of time. Once Wiener processes are understood, generalized Wiener processes and Itô processes should not cause too much difficulty.

In understanding a stochastic process, I find that an explanation of how it can be simulated is useful and usually go through an example such as that in Section 12.3.

The amount of time spent on Itô's lemma will depend on the mathematical backgrounds of students. Mathematically inclined students generally feel quite a sense of achievement when they have managed to work through the material in the appendix to Chapter 12.

All of Problems 12.12 to 12.16 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 12.1.

What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in the last week would be irrelevant.

To answer the second part of the question you might like to consider the following scenario for the first week in May:

(i) Monday to Thursday are warm days; today, Friday, is a very cold day.

(ii) Monday to Friday are all very cold days.

What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

Problem 12.2.

Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear.

As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Papers were published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

Problem 12.3.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of one year?

Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\phi(x + 4 \times 0.5, 4 \times 4) = \phi(x + 2.0,\ 16)$$

where $\phi(m, v)$ is a normal probability distribution with mean m and variance v. The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x+2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From normal distribution tables

$$N\left(-\frac{x+2.0}{4}\right) = 0.05$$

when:

$$-\frac{x+2.0}{4} = -1.6449$$

i.e., when $x = 4.5796$. The initial cash position must therefore be \$4.58 million.

Problem 12.4.

Variables X_1 and X_2 follow generalized Wiener processes with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2. What process does $X_1 + X_2$ follow if:

(a) The changes in X_1 and X_2 in any short interval of time are uncorrelated?

(b) There is a correlation ρ between the changes in X_1 and X_2 in any short interval of time?

(a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution

$$\phi(a_1 + \mu_1 T,\ \sigma_1^2 T)$$

and X_2 has a probability distribution

$$\phi(a_2 + \mu_2 T,\ \sigma_2^2 T)$$

From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution

$$\phi\left(a_1 + \mu_1 T + a_2 + \mu_2 T,\ \sigma_1^2 T + \sigma_2^2 T\right)$$

i.e.,

$$\phi\left[a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2)T\right]$$

This shows that $X_1 + X_2$ follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2$.

(b) In this case the change in the value of $X_1 + X_2$ in a short interval of time Δt has the probability distribution:

$$\phi\left[(\mu_1 + \mu_2)\Delta t, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)\Delta t\right]$$

If μ_1, μ_2, σ_1, σ_2 and ρ are all constant, arguments similar to those in Section 12.2 show that the change in a longer period of time T is

$$\phi\left[(\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)T\right]$$

The variable, $X_1 + X_2$, therefore follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$.

Problem 12.5.

Consider a variable, S, that follows the process

$$dS = \mu\, dt + \sigma\, dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year six?

The change in S during the first three years has the probability distribution

$$\phi(2 \times 3,\ 9 \times 3) = \phi(6,\ 27)$$

The change during the next three years has the probability distribution

$$\phi(3 \times 3,\ 16 \times 3) = \phi(9,\ 48)$$

The change during the six years is the sum of a variable with probability distribution $\phi(6,\ 27)$ and a variable with probability distribution $\phi(9,\ 48)$. The probability distribution of the change is therefore

$$\phi(6 + 9, 27 + 48)$$
$$= \phi(15,\ 75)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\phi(20,\ 75)$$

Problem 12.6.

Suppose that G is a function of a stock price, S and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

From Itô's lemma

$$\sigma_G G = \frac{\partial G}{\partial S}\sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S}\lambda\sigma_S S$$

or

$$\lambda\sigma_G G$$

The growth rate of G, therefore, increases by $\lambda\sigma_G$.

Problem 12.7.

Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

Define S_A, μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B, μ_B and σ_B as the stock price, expected return and volatility for stock B. Define ΔS_A and ΔS_B as the change in S_A and S_B in time Δt. Since each of the two stocks follows geometric Brownian motion,

$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \epsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \epsilon_B \sqrt{\Delta t}$$

where ϵ_A and ϵ_B are independent random samples from a normal distribution.

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B)\Delta t + (\sigma_A S_A \epsilon_A + \sigma_B S_B \epsilon_B)\sqrt{\Delta t}$$

This *cannot* be written as

$$\Delta S_A + \Delta S_B = \mu(S_A + S_B)\Delta t + \sigma(S_A + S_B)\epsilon\sqrt{\Delta t}$$

for any constants μ and σ. (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

Problem 12.8.

The process for the stock price in equation (12.8) is

$$\Delta S = \mu S\,\Delta t + \sigma S \epsilon\,\sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu\,\Delta t + \sigma\epsilon\,\sqrt{\Delta t}$$
$$\Delta S = \mu S\,\Delta t + \sigma\epsilon\,\sqrt{\Delta t}$$
$$\Delta S = \mu\,\Delta t + \sigma S\epsilon\,\sqrt{\Delta t}$$

Why is the model in equation (12.8) a more appropriate model of stock price behavior than any of these three alternatives?

In:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price

In:

$$\Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is \$5 per annum when the stock price is \$25, it is also \$5 per annum when it is \$100. If the standard deviation of weekly stock price movements is \$1 when the price is \$25, it is also \$1 when the price is \$100.

In:

$$\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.

In:

$$\Delta S = \mu \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.

The model:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval are constant.

Problem 12.9.

It has been suggested that the short-term interest rate, r, follows the stochastic process

$$dr = a(b - r)\,dt + rc\,dz$$

where a, b, and c are positive constants and dz is a Wiener process. Describe the nature of this process.

The drift rate is $a(b-r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b, the drift rate is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility equal to c is superimposed upon the "pull" or the drift.

Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact it is slightly greater than this, because as the interest rate decreases, the "pull" decreases.) Superimposed upon the drift is a volatility of 15% per annum.

Problem 12.10.

Suppose that a stock price, S, follows geometric Brownian motion with expected return μ and volatility σ:

$$dS = \mu S\,dt + \sigma S\,dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion.

If $G(S,t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Itô's lemma:

$$dG = [\mu nG + \frac{1}{2}n(n-1)\sigma^2 G]\,dt + \sigma nG\,dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

Problem 12.11.

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

Since:

$$B = e^{-x(T-t)}$$

the required partial derivatives are

$$\begin{aligned}
\frac{\partial B}{\partial t} &= xe^{-x(T-t)} = xB \\
\frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\
\frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B
\end{aligned}$$

Hence:

$$dB = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right] Bdt - sx(T-t)Bdz$$

ASSIGNMENT QUESTIONS

Problem 12.12.

Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:

(a) The expected stock price at the end of the next day.
(b) The standard deviation of the stock price at the end of the next day.
(c) The 95% confidence limits for the stock price at the end of the next day.

With the notation in the text

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t,\, \sigma^2 \Delta t)$$

In this case $S = 50$, $\mu = 0.16$, $\sigma = 0.30$ and $\Delta t = 1/365 = 0.00274$. Hence

$$\begin{aligned}\frac{\Delta S}{50} &\sim \phi(0.16 \times 0.00274,\, 0.09 \times 0.00274)\\ &= \phi(0.00044,\, 0.000247)\end{aligned}$$

and

$$\Delta S \sim \phi(50 \times 0.00044,\, 50^2 \times 0.000247)$$

that is,

$$\Delta S \sim \phi(0.022,\, 0.6164)$$

(a) The expected stock price at the end of the next day is therefore 50.022
(b) The standard deviation of the stock price at the end of the next day is $\sqrt{0.6154} = 0.785$
(c) 95% confidence limits for the stock price at the end of the next day are

$$50.022 - 1.96 \times 0.785 \quad \text{and} \quad 50.022 + 1.96 \times 0.785$$

i.e.,

$$48.48 \quad \text{and} \quad 51.56$$

Note that some students may consider one trading day rather than one calendar day. Then $\Delta t = 1/252 = 0.00397$. The answer to (a) is then 50.032. The answer to (b) is 0.945. The answers to part (c) are 48.18 and 51.88.

Problem 12.13.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

(a) What are the probability distributions of the cash position after one month, six months, and one year?

(b) What are the probabilities of a negative cash position at the end of six months and one year?

(c) At what time in the future is the probability of a negative cash position greatest?

(a) The probability distributions are:

$$\phi(2.0 + 0.1,\ 0.16) = \phi(2.1, 0.16)$$

$$\phi(2.0 + 0.6,\ 0.16 \times 6) = \phi(2.6,\ 0.96)$$

$$\phi(2.0 + 1.2,\ 0.16 \times 12) = \phi(3.2,\ 1.96)$$

(b) The chance of a random sample from $\phi(2.6,\ 0.96)$ being negative is

$$N\left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65)$$

where $N(x)$ is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution $\phi(0,\ 1)$] is less than x. From normal distribution tables $N(-2.65) = 0.0040$. Hence the probability of a negative cash position at the end of six months is 0.40%.

Similarly the probability of a negative cash position at the end of one year is

$$N\left(-\frac{3.2}{\sqrt{1.96}}\right) = N(-2.30) = 0.0107$$

or 1.07%.

(c) In general the probability distribution of the cash position at the end of x months is

$$\phi(2.0 + 0.1x,\ 0.16x)$$

The probability of the cash position being negative is maximized when:

$$\frac{2.0 + 0.1x}{\sqrt{0.16x}}$$

is minimized. Define

$$y = \frac{2.0 + 0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}}$$

$$= x^{-\frac{3}{2}}(-2.5 + 0.125x)$$

This is zero when $x = 20$ and it is easy to verify that $d^2y/dx^2 > 0$ for this value of x. It therefore gives a minimum value for y. Hence the probability of a negative cash position is greatest after 20 months.

Problem 12.14.

Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

In this case

$$B = \frac{1}{x}$$

so that:

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence

$$\begin{aligned} dB &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{1}{2}s^2x^2\frac{2}{x^3}\right]dt - \frac{1}{x^2}sxdz \\ &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right]dt - \frac{s}{x}dz \end{aligned}$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore:

$$1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is:

$$\left(1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right) / \left(\frac{1}{x}\right)$$

$$= x - \frac{a}{x}(x_0 - x) + s^2$$

Problem 12.15.

If S follows the geometric Brownian motion process in equation (12.6), what is the process followed by

a. $y = 2S$

b. $y = S^2$

c. $y = e^S$

d. $y = \frac{e^{r(T-t)}}{S}$

In each case express the coefficients of dt and dz in terms of y rather than S.

(a) In this case $\partial y/\partial S = 2$, $\partial^2 y/\partial S^2 = 0$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = 2\mu S\,dt + 2\sigma S\,dz$$

or

$$dy = \mu y\,dt + \sigma y\,dz$$

(b) In this case $\partial y/\partial S = 2S$, $\partial^2 y/\partial S^2 = 2$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (2\mu S^2 + \sigma^2 S^2)\,dt + 2\sigma S^2\,dz$$

or

$$dy = (2\mu + \sigma^2)y\,dt + 2\sigma y\,dz$$

(c) In this case $\partial y/\partial S = e^S$, $\partial^2 y/\partial S^2 = e^S$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (\mu S e^S + \sigma^2 S^2 e^S/2)\,dt + \sigma S e^S\,dz$$

or

$$dy = [\mu y \ln y + \sigma^2 y(\ln y)^2/2]\,dt + \sigma y \ln y\,dz$$

(d) In this case $\partial y/\partial S = -e^{r(T-t)}/S^2 = -y/S$, $\partial^2 y/\partial S^2 = 2e^{r(T-t)}/S^3 = 2y/S^2$, and $\partial y/\partial t = -re^{r(T-t)}/S = -ry$ so that Itô's lemma gives

$$dy = (-ry - \mu y + \sigma^2 y)\,dt - \sigma y\,dz$$

or

$$dy = -(r + \mu - \sigma^2)y\,dt - \sigma y\,dz$$

Problem 12.16.

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.3^2/2)2 = 4.062$ and $0.3\sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$1 - N\left(\frac{4.382 - 4.062}{0.424}\right) = 1 - N(0.754)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x. From the tables at the back of the book $N(0.754) = 0.775$ so that the required probability is 0.225.

CHAPTER 13
The Black–Scholes–Merton Model

Notes for the Instructor

This chapter covers important material: the lognormality of stock prices, the calculation of volatility from historical data, the Black–Scholes–Merton differential equation, risk-neutral valuation, the Black–Scholes–Merton option pricing formulas, implied volatilities, and the impact of dividends. Section 13.6 should be skipped if Chapter 12 has not already been covered.

The distinction between

μ: the expected rate of return in a short period of time, and

$\mu - \sigma^2/2$: the expected continuously compounded rate of return over any period of time

usually causes some problems. I have tried a few different approaches and think that the one that is now in the text works reasonably well.

Business Snapshot 13.2 on the causes of volatility generally leads to a lively discussion. I find that students have an easier time than academics in accepting that trading itself causes volatility!

When presenting Black–Scholes–Merton arguments I point out that in any small interval of time Δt, the stock price and the option price are perfectly correlated. This is the same as saying that the ratio $\Delta c/\Delta S$ is constant where Δc and ΔS are the change in c and S in time Δt respectively. It is possible to set up a portfolio consisting of a position in the derivative and a position in the stock which is, for the next small interval of time Δt, riskless. (For example, if

$$\frac{\Delta c}{\Delta S} = 0.4$$

a short position in 100 of the derivative security when combined with a long position in 40 of the stock is riskless for time Δt.) This is essentially what Black, Scholes, and Merton did to derive their differential equation. After presenting the Black–Scholes–Merton differential equation I like to got through Example 13.5 on forward contracts in class. Later the same example can be used to illustrate risk-neutral valuation.

The risk-neutral valuation argument must be covered carefully. It cannot be emphasized often enough that we are not assuming risk neutrality. It just happens that the value of a derivative security is independent of risk preferences.

When the Black–Scholes–Merton equation for pricing a call option is presented, students sometimes ask for the intuition behind it and are frustrated that they cannot easily derive it. I point out that a European call option holder gets $S_T - K$ whenever $S_T > K$. This means that the option holder is long a security that pays off S_T when $S_T > K$ and short a security that pays off K when $S_T > K$. The first security is known as an asset-or-nothing call. The second security is known as a cash-or-nothing call. The probability that $S_T > K$ in a risk-neutral world is $N(d_2)$. (See Problem 13.22). The expected payoff from the second security in a risk-neutral world is therefore $KN(d_2)$. From risk-neutral valuation, the value of the security is $KN(d_2)e^{-rT}$. The value of the first security can also be calculated using risk-neutral valuation. It turns out to be $S_0N(d_1)$. (Students have to take this on faith). Putting the two results together we get the Black–Scholes–Merton formula for a European call option.

When calculating the cumulative normal distribution function, most students will choose to use the table at the end of the book or the Excel function NORMSDIST. The polynomial approximation may be useful if they choose to write their own software. I encourage students to develop their own Excel worksheets for option pricing as well as using DerivaGem.

All the assignment questions work well. My favorite is 13.28.

QUESTIONS AND PROBLEMS

Problem 13.1.

What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 13.2.

The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 13.3.

Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 13.4.

Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times 0.25} - 50N(-0.2417)$$
$$= 50 \times 0.4634e^{-0.1\times 0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

Problem 13.5.

What difference does it make to your calculations in Problem 13.4 if a dividend of \$1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$
$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

Problem 13.6.

What is implied volatility? How can it be calculated?

The implied volatility is the volatility that makes the Black–Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

Problem 13.7.

A stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (13.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

i.e.,

$$\phi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is $\sqrt{0.03125}$ or 17.68% per annum.

Problem 13.8.

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.

(a) What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in six months will be exercised?

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

(a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5,\ 0.35^2 \times 0.5)$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.06125)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{\sqrt{0.06125}}\right) = 1 - N(0.008)$$

From normal distribution tables $N(0.008) = 0.5032$ so that the required probability is 0.4968. In general the required probability is $N(d_2)$. (See Problem 13.22).

(b) In this case the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

Problem 13.9.

Prove that with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

From equation (13.3):

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0+(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0+(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

i.e.

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$$

and

$$S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Problem 13.10.

A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

The statement is misleading in that a certain sum of money, say $1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 13.11.

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security *that pays off a dollar amount equal to $\ln S_T$ at time T where S_T denotes the value of the stock price at time T.*

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t.

(b) Confirm that your price satisfies the differential equation (13.16).

(a) At time t, the expected value of $\ln S_T$ is, from equation (13.3)

$$\ln S + (\mu - \frac{\sigma^2}{2})(T-t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore:

$$\ln S + (r - \frac{\sigma^2}{2})(T-t)$$

Using risk-neutral valuation the value of the security at time t is:

$$e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

(b) If:

$$f = e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right] - e^{-r(T-t)}(r - \frac{\sigma^2}{2})$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black–Scholes–Merton differential equation is

$$e^{-r(T-t)}\left[r\ln S + r(r - \frac{\sigma^2}{2})(T-t) - (r - \frac{\sigma^2}{2}) + r - \frac{\sigma^2}{2}\right]$$

$$= re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$= rf$$

Hence equation (13.16) is satisfied.

Problem 13.12.

Consider a derivative that pays off S_T^n at time T where S_T is the stock price at that time. When the stock price follows geometric Brownian motion, it can be shown that its price at time t ($t \leq T$) has the form

$$h(t,T)S^n$$

where S is the stock price at time t and h is a function only of t and T.

(a) By substituting into the Black–Scholes–Merton partial differential equation derive an ordinary differential equation satisfied by $h(t,T)$.

(b) What is the boundary condition for the differential equation for $h(t,T)$?

(c) Show that

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

This problem is related to Problem 12.10.

(a) If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes–Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

(c) The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

so that

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

Problem 13.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$
$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)$$
$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$
$$= 5.06$$

or \$5.06.

Problem 13.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$
$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times 0.5}N(0.0809) - 69N(-0.1666)$$
$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$
$$= 6.40$$

or \$6.40.

Problem 13.15.

Consider an American call option on a stock. The stock price is \$70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after three months and again after six months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Using the notation of Section 13.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2 - t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

Problem 13.16.

A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

Problem 13.17.

With the notation used in this chapter

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.

(f) Show that the c satisfies the Black–Scholes–Merton differential equation.

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \longrightarrow T$

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$
\begin{aligned}
N'(d_1) &= N'(d_2 + \sigma\sqrt{T-t}) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] \\
&= N'(d_2) \exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]
\end{aligned}
$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$
\begin{aligned}
c &= SN(d_1) - Ke^{-r(T-t)}N(d_2) \\
\frac{\partial c}{\partial t} &= SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}
\end{aligned}
$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$\begin{aligned} d_1 - d_2 &= \sigma\sqrt{T-t} \\ \frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} &= \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\ &= -\frac{\sigma}{2\sqrt{T-t}} \end{aligned}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes–Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\begin{aligned} \frac{\partial^2 c}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} \\ &= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \end{aligned}$$

From the results in d) and e)

$$\begin{aligned} \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} &= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \\ &+ rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \\ &= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)] \\ &= rc \end{aligned}$$

This shows that the Black–Scholes formula for a call option does indeed satisfy the Black–Scholes–Merton differential equation

(g) Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S - K,\, 0)$.

Problem 13.18.

Show that the Black–Scholes formulas for call and put options satisfy put–call parity.

From the Black–Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$ this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black–Scholes equations are therefore consistent with put–call parity.

Problem 13.19.

A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes?

	Maturity (months)		
Strike Price (\$)	*3*	*6*	*12*
45	*7.0*	*8.3*	*10.5*
50	*3.7*	*5.2*	*7.5*
55	*1.6*	*2.9*	*5.1*

This problem naturally leads on to the material in Chapter 18 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

	Maturity (months)		
Strike Price ($)	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black–Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

Problem 13.20.

Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.

It appears from this argument that Black's approach understates the true option value. However, the way in which volatility is applied can lead to Black's approach overstating the option value. Black applies the volatility to the option price. The binomial model, as we will see in Chapter 19, applies the volatility to the stock price less the present value of the dividend. This issue is also discussed in Example 13.10.

Problem 13.21.

Consider an American call option on a stock. The stock price is $50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is $55, and the volatility is 25%. Dividends of $1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1-e^{-r(t_2-t_1)}\right] = 55(1-e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1-e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or \$4.17.

Problem 13.22.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \phi[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

Problem 13.23.

Show that S^{-2r/σ^2} could be the price of a traded security.

If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2}S^{-2r/\sigma^2-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right)S^{-2r/\sigma^2-2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black–Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

Problem 13.24.

A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain you answer.

The answer is no. If markets are efficient they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot 13.3.

Problem 13.25.

A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees three million at-the-money five-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the five-year risk-free rate is 5% and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

The Black-Scholes price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 13.10 the cost to the company of the options is

$$\frac{10}{10+3} \times 16.252 = 12.5$$

or about \$12.5 per option. The total cost is therefore 3 million times this or \$37.5 million. If the market perceives no benefits from the options the stock price will fall by \$3.75.

ASSIGNMENT QUESTIONS

Problem 13.26.

A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (13.3), given by:

$$\ln S_T \sim \phi[\ln 50 + (0.18 - \frac{0.09}{2})2,\ 0.3^2 \times 2]$$

i.e.,

$$\ln S_T \sim \phi(4.18,\ 0.18)$$

The mean stock price is from equation (13.4)

$$50e^{2\times 0.18} = 50e^{0.36} = 71.67$$

and the standard deviation is, from equation (13.5),

$$50e^{2\times 0.18}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 13.27.

Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

$$30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2$$

Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Problem 13.27
Computation of Volatility

Week	Closing Stock Price ($)	Price Relative $= S_i/S_{i-1}$	Daily Return $u_i = \ln(S_i/S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	-0.02853
4	30.1	0.96785	-0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	-0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	-0.00595
15	33.2	0.99104	-0.00900

Problem 13.28.

A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

(a) *Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (Hint: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in section 13.1.)*

(b) *Confirm that your price satisfies the differential equation (13.16).*

(a) The expected value of the security is $E[(S_T)^2]$ From equations (13.4) and (13.5), at time t :

$$E(S_T) = Se^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1]$$

Since $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$E[(S_T)^2] = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1] + S^2 e^{2\mu(T-t)}$$
$$= S^2 e^{(2\mu+\sigma^2)(T-t)}$$

In a risk-neutral world $\mu = r$ so that

$$\hat{E}[(S_T)^2] = S^2 e^{(2r+\sigma^2)(T-t)}$$

Using risk-neutral valuation, the value of the derivative security at time t is

$$e^{-r(T-t)}\hat{E}[(S_T)^2]$$

$$= S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)}$$
$$= S^2 e^{(r+\sigma^2)(T-t)}$$

(b) If:

$$f = S^2 e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial f}{\partial t} = -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial f}{\partial S} = 2Se^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial^2 f}{\partial S^2} = 2e^{(r+\sigma^2)(T-t)}$$

The left-hand side of the Black-Scholes–Merton differential equation is:

$$-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2e^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 e^{(r+\sigma^2)(T-t)}$$
$$= rS^2 e^{(r+\sigma^2)(T-t)}$$
$$= rf$$

Hence the Black-Scholes equation is satisfied.

Problem 13.29.

Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.

(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is an American call?
(c) What is the price of the option if it is a European put?
(d) Verify that put–call parity holds.

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 0.3333$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

(a) The European call price is

$$30 \times 0.6637 - 29e^{-0.05\times0.3333} \times 0.6096 = 2.52$$

or \$2.52.

(b) The American call price is the same as the European call price. It is \$2.52.

(c) The European put price is

$$29e^{-0.05\times0.3333} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or \$1.05.

(d) Put-call parity states that:

$$p + S_0 = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied.

Problem 13.30.

Assume that the stock in Problem 13.29 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is a European put?

(c) If the option is an American call, are there any circumstances under which it will be exercised early?

(a) The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125\times0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.3068$$

$$d_2 = \frac{\ln(29.5031/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.1625$$

$$N(d_1) = 0.6205; \quad N(d_2) = 0.5645$$

The price of the option is therefore

$$29.5031 \times 0.6205 - 29e^{-0.3333\times0.05} \times 0.5645 = 2.21$$

or \$2.21.

(b) Since

$$N(-d_1) = 0.3795, \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.3333\times0.05} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or \$1.22.

(c) If t_1 denotes the time when the dividend is paid:

$$K[1 - e^{-r(T-t_1)}] = 29(1 - e^{-0.05\times0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 13.31.

Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. DerivaGem also shows that the correct American option price calculated with 100 time steps is 0.8243.

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \leq K[1 - e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \leq 20[1 - e^{-(0.1\times0.25)}]$$

that is,

$$D_1 \leq 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \leq K(1 - e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \leq 20(1 - e^{-0.1\times0.0833})$$

that is,

$$D_2 \leq 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.

CHAPTER 14
Employee Stock Options

Notes for the Instructor

This chapter is new to the seventh edition. Employee stock options have been much in the news in recent years and I find students enjoy talking about them. Many students hope to become rich one day by exercising such options!

The chapter covers how the options typically work, whether they align the interests of senior executives and shareholders, their accounting treatment, alternative valuation approaches, and backdating scandals. Many instructors will want to spend time on the academic research of Yermack, Lie, and Heron which was largely responsible for exposing the backdating scandals (see Section 14.5). Others may want to focus on how employee stock options can be designed to better align the interests of shareholders and senior managers. As described in Section 14.1 the traditional stock option plan is one where at-the-money options are issued periodically. For many years, companies were reluctant to move away from this type of plan because they would then be required to expense the options. Now the accounting treatment of employee stock options has now changed (with expensing being mandatory) and so there is no reason for companies not to consider nontraditional plans such as those mentioned in Section 14.3.

I recommend spending some time talking about the fact that employee stock options (unlike regular call options) cannot be sold. This leads to the situation where they tend to be exercised much earlier than regular call options (see Section 14.1). A discussion of this should reinforce a student's understanding of the arguments concerning early exercise of calls in Chapter 9.

The three assignment questions test whether students can use some of the approaches for valuing employee stock options. If 14.14 is assigned it is a good idea to suggest to students that they calculated the expected life using a tree.

QUESTIONS AND PROBLEMS

Problem 14.1.

Why was it attractive for companies to grant at-the-money stock options prior to 2005? What changed in 2005?

Prior to 2005 companies did not have to expense at-the-money options on the income statement. They merely had to report the value of the options in notes to the accounts. FAS 123 and IAS 2 required the fair value of the options to be reported as a cost on the income statement starting in 2005.

Problem 14.2.

What are the main differences between a typical employee stock option and a call option traded on an exchange or in the over-the-counter market?

The main differences are a) employee stock options last much longer than the typical exchange-traded or over-the-counter option, b) there is usually a vesting period during which they cannot be exercised, c) the options cannot be sold by the employee, d) if the employee leaves the company the options usually either expire worthless or have to be exercised immediately, and e) exercise of the options usually leads to the company issuing more shares.

Problem 14.3.

Explain why employee stock options on a non-dividend-paying stock are frequently exercised before the end of their lives whereas an exchange-traded call option on such a stock is never exercised early.

It is always better for the option holder to sell a call option on a non-dividend -paying stock rather than exercise it. Employee stock options cannot be sold and so the only way an employee can monetize the option is to exercise the option and sell the stock.

Problem 14.4.

"Stock option grants are good because they motivate executives to act in the best interests of shareholders." Discuss this viewpoint.

This is questionable. Executives benefit from share price increases but do not bear the costs of share price decreases. Employee stock options are liable to encourage executives to take decisions that boost the value of the stock in the short term at the expense of the long term health of the company. It may even be the case that executives are encouraged to take high risks so as to maximize the value of their options.

Problem 14.5.

"Granting stock options to executives is like allowing a professional footballer to bet on the outcome of games." Discuss this viewpoint.

Professional footballers are not allowed to bet on the outcomes of games because they themselves influence the outcomes. Arguably, an executive should not be allowed to bet on the future stock price of her company because her actions influence that price. However, it could be argued that there is nothing wrong with a professional footballer betting that his team will win (but everything wrong with betting that it will lose). Similarly there is nothing wrong with an executive betting that her company will do well.

Problem 14.6.

Why did some companies backdate stock option grants in the US prior to 2002? What changed in 2002?

Backdating allowed the company to issue employee stock options with a strike price equal to the price at some previous date and claim that they were at the money. At the money options did not lead to an expense on the income statement until 2005. The amount recorded for the value of the options in the notes to the income was less than the actual cost on the true grant date. In 2002 the SEC required companies to report stock option grants within two business days of the grant date. This eliminated the possibility of backdating for companies that complied with this rule.

Problem 14.7.

In what way would the benefits of backdating be reduced if a stock option grant had to be revalued at the end of each quarter?

If a stock option grant had to be revalued each quarter the value of the option of the grant date (true or fabricated) would become less important. Stock price movements following the reported grant date would be incorporated in the next revaluation. The total cost of the options would be independent of the stock price on the grant date.

Problem 14.8.

Explain how you would do the analysis to produce a chart such as the one in Figure 15.2.

It would be necessary to look at returns on each stock in the sample (possibly adjusted for the returns on the market and the beta of the stock) around the reported employee stock option grant date. One could designate Day 0 as the grant date and look at returns on each stock each day from Day –30 to Day +30. The returns would then be averaged across the stocks.

Problem 14.9.

On May 31 a company's stock price is $70. One million shares are outstanding. An executive exercises 100,000 stock options with a strike price of $50. What is the impact of this on the stock price?

There should be no impact on the stock price because the stock price will already reflect the dilution expected from the executive's exercise decision.

Problem 14.10.

The notes accompanying a company's financial statements say: "Our executive stock options last 10 years and vest after four years. We valued the options granted this year using the Black–Scholes model with an expected life of 5 years and a volatility of 20%. "What does this mean? Discuss the modeling approach used by the company.

The notes in indicate that the Black–Scholes model was used to produce the valuation with T the option life being set equal to5 years and the stock price volatility being set equal to 20%.

Problem 14.11.

In a Dutch auction of 10,000 options, bids are as follows
A bids $30 for 3,000
B bids $33 for 2,500

C bids \$29 for 5,000
D bids \$40 for 1,000
E bids \$22 for 8,000
F bids \$35 for 6,000
What is the result of the auction? Who buys how many at what price?

The price at which 10,000 options can be sold is \$30. B, D, and F get their order completely filled at this price. A buys 500 options (out of its total bid for 3,000 options) at this price.

Problem 14.12.

A company has granted 500,000 options to its executives. The stock price and strike price are both \$40. The options last for 12 years and vest after four years. The company decides to value the options using an expected life of five years and a volatility of 30% per annum. The company pays no dividends and the risk-free rate is 4%. What will the company report as an expense for the options on its income statement?

The options are valued using Black–Scholes with $S_0 = 40$, $K = 40$, $T = 5$, $\sigma = 0.3$ and $r = 0.04$. The value of each option is \$4.488. The total expense reported is $500,000 \times \$4.488$ or \$2.244 million.

Problem 14.13.

A company's CFO says: "The accounting treatment of stock options is crazy. We granted 10,000,000 at-the-money stock options to our employees last year when the stock price was \$30. We estimated the value of each option on the grant date to be \$5. At our year end the stock price had fallen to \$4, but we were still stuck with a \$50 million charge to the P&L." Discuss.

The problem is that under the current rules the options are valued only once—on the grant date. Arguably it would make sense to treat the options in the same way as other derivatives entered into by the company and revalue them on each reporting date. However, this does not happen under the current rules in the United States unless the options are settled in cash.

ASSIGNMENT QUESTIONS

Problem 14.14.

What is the (risk-neutral) expected life for the employee stock option in Example 14.2? What is the value of the option obtained by using this expected life in Black–Scholes?

The expected life at time zero can be calculated by rolling back through the tree asking the question at each node: "What is the expected life if the node is reached." This is what has been done in Figure M14.1. For example at node G (time 6 years) there is a 81% chance that the option will be exercised and a 19% chance that it will last an extra two years. The expected life if node G is reached is therefore $0.81 \times 6 + 0.19 \times 8 = 6.38$

years. Similarly, the expected life if node H is reached is $0.335 \times 6 + 0.665 \times 8 = 7.33$ years. The expected life if node I or J is reached is $0.05 \times 6 + 0.95 \times 8 = 7.90$ years. The expected life if node D is reached is

$$0.43 \times 4 + 0.57 \times (0.5158 \times 6.38 + 0.4842 \times 7.33) = 5.62$$

Continuing in this way the expected life at time zero is 6.86 years. (As in Example 14.2 we assume that no employees leave at time zero.)

The value of the option assuming an expected life of 6.86 years is given by Black–Scholes with $S_0 = 40$, $K = 40$, $r = 0.05$, $\sigma = 0.3$ and $T = 6.86$. It is 17.17. Using a four-step tree it is 16.51.

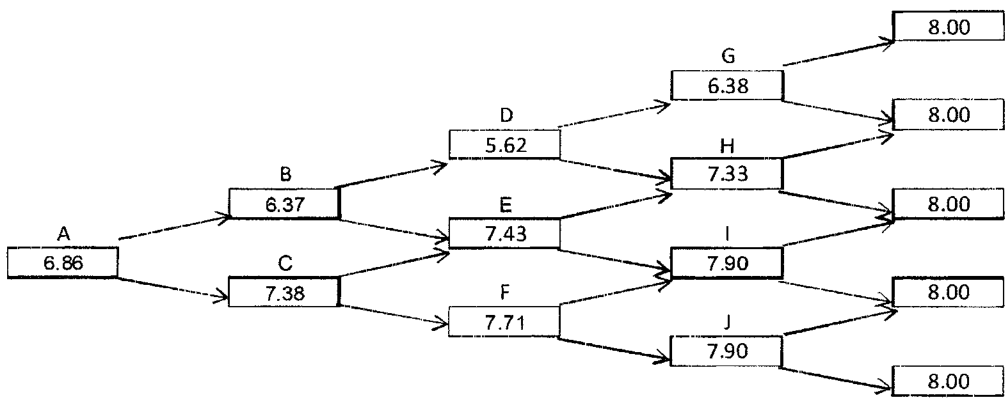

Figure M14.1 Tree for calculating expected life in Problem 14.14

Problem 14.15.

A company has granted 2,000,000 options to its employees. The stock price and strike price are both \$60. The options last for 8 years and vest after two years. The company decides to value the options using an expected life of six years and a volatility of 22% per annum. The dividend on the stock is \$1, payable half way through each year, and the risk-free rate is 5%. What will the company report as an expense for the options on its income statement.

The options are valued using Black–Scholes with $K = 60$, $T = 6$, $\sigma = 0.22$, $r = 0.05$. The present value of the dividends during the six years assumed life are

$$1 \times e^{-0.05\times0.5} + 1 \times e^{-0.05\times1.5} + 1 \times e^{-0.05\times2.5} + 1 \times e^{-0.05\times3.5} + 1 \times e^{-0.05\times4.5} + 1 \times e^{-0.05\times5.5}$$

$$= 5.183$$

The stock price, S_0, adjusted for dividend is therefore $60 - 5.183 = 54.817$. The Black–Scholes model gives the price of one option as \$16.492. The company will therefore report as an expense $2,000,000 \times \$16.492$ or \$32.984 million.

Problem 14.16.

A company has granted 1,000,000 options to its employees. The stock price and strike price are both \$20. The options last 10 years and vest after three years. The stock price volatility is 30%, the risk-free rate is 5%, and the company pays no dividends. Use a four-step tree to value the options. Assume that there is a probability of 4% that an employee leaves the company at the beginning of each the time steps on your tree. Assume also that the probability of voluntary early exercise at a node, conditional on no prior exercise, when a) the option has vested and b) the option is in the money, is

$$1 - \exp[-a(S/K - 1)/T]$$

where S is the stock price, K is the strike price, T is the time to maturity and $a = 2$.

The valuation is shown in Figure M14.2. The tree is similar to Figure 14.1 in the text. The upper number at each node is the stock price and the lower number is the value of the option. In this case $u = 1.6070$ and $p = 0.5188$. The probability of voluntary exercise at nodes A, B, and C are 0.4690, 0.9195, and 0.3846, respectively. The total probability of exercise at these nodes (including the impact of employees leaving the company) is 0.4902, 0.9227, and 0.4093. The value of each option is \$8.54 and the value of the option grant is \$8.54 million. This problem and Example 14.2 in the text specify that employees are assumed to leave at the beginning of each time period. It is questionable whether this includes time zero. Both my answer to this question and the answer to Example 14.1 assume that it does not include time zero. (On reflection, it would have been better for both questions to say that employees leave at the end of each time period.) If in this question it is assumed that 4% of employees leave the company at the initial node the answer is reduced by 4% to \$8.20 million.

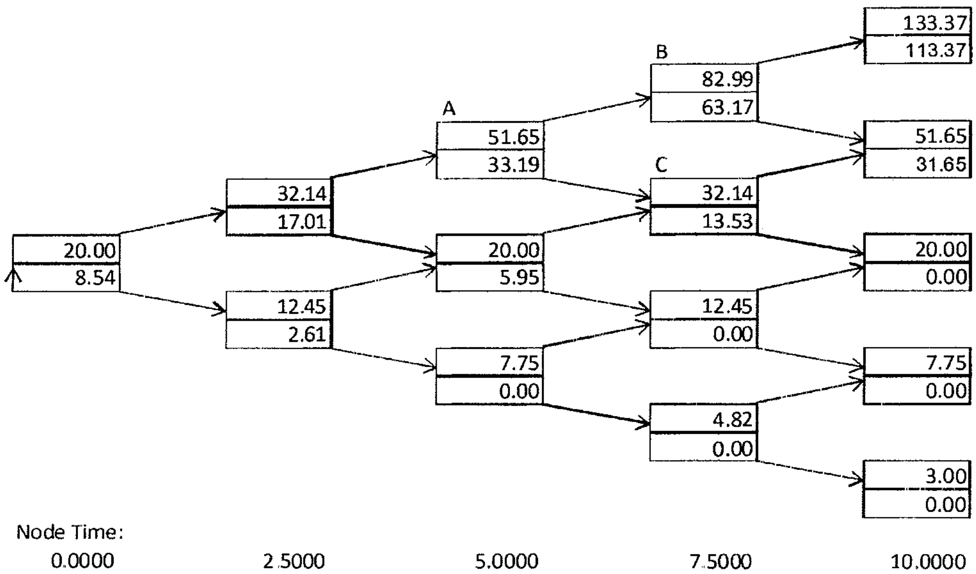

Figure M14.2 Valuation of employee stock option in Problem 14.16

CHAPTER 15
Options on Stock Indices and Currencies

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on options on stock indices and currencies has also been restructured for the seventh edition to make it more interesting. Instead of starting with valuation, it now starts with examples of how options on stock indices and options on foreign currencies are used. Range forwards are discussed here rather than later in the book.

For students who have a good knowledge of Chapter 13, the valuation material in this chapter should present few problems. The key argument is in Section 15.3 and shows how the Black-Scholes formulas can be modified to provide valuations of European call and put options on a stock paying a known dividend yield. Stock indices and currencies are analogous to stocks paying known dividend yields.

Any of Problems 15.23 to 13.28 make good assignment questions. Problem 15.22 requires Ito's lemma to have been covered.

QUESTIONS AND PROBLEMS

Problem 15.1.

A portfolio is currently worth \$10 million and has a beta of 1.0. An index is currently standing at 800. Explain how a put option on the index with a strike of 700 can be used to provide portfolio insurance.

When the index goes down to 700, the value of the portfolio can be expected to be $10 \times (700/800) =$ \$8.75 million. (This assumes that the dividend yield on the portfolio equals the dividend yield on the index.) Buying put options on 10,000,000/800 = 12,500 times the index with a strike of 700 therefore provides protection against a drop in the value of the portfolio below \$8.75 million. If each contract is on 100 times the index a total of 125 contracts would be required.

Problem 15.2.

"Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices, currencies, and futures." Explain this statement.

A stock index is analogous to a stock paying a continuous dividend yield, the dividend yield being the dividend yield on the index. A currency is analogous to a stock paying a continuous dividend yield, the dividend yield being the foreign risk-free interest rate.

Problem 15.3.

A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

The lower bound is given by equation 15.1 as

$$300e^{-0.03\times 0.5} - 290e^{-0.08\times 0.5} = 16.90$$

Problem 15.4.

A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value a) a European four-month call option with a strike price of \$0.79 and b) an American four-month call option with the same strike price

In this case $u = 1.0502$ and $p = 0.4538$. The tree is shown in Figure S15.1. The value of the option if it is European is \$0.0235. the value of the option if it is American is \$0.0250.

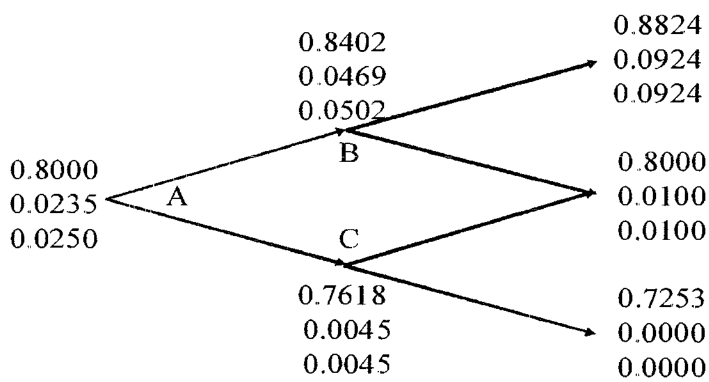

Figure S15.1 Tree to evaluate European and American put options in Problem 15.4. At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 15.5.

Explain how corporations use range forward contracts to hedge their foreign exchange risk.

A range forward contract allows a corporation to ensure that the exchange rate applicable to a transaction will not be worse that one exchange rate and will not be better than another exchange rate. Depending on the exposure being hedged a range forward contract is created by either a) buying a put with the lower exchange rate and selling a call with the higher exchange rate or b) selling a put with the lower exchange rate and buying a call with the higher exchange rate.

Problem 15.6.

Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

In this case, $S_0 = 250$, $K = 250$, $r = 0.10$, $\sigma = 0.18$, $T = 0.25$, $q = 0.03$ and

$$d_1 = \frac{\ln(250/250) + (0.10 - 0.03 + 0.18^2/2)0.25}{0.18\sqrt{0.25}} = 0.2394$$

$$d_2 = d_1 - 0.18\sqrt{0.25} = 0.1494$$

and the call price is

$$250N(0.2394)e^{-0.03\times0.25} - 250N(0.1494)e^{-0.10\times0.25}$$

$$= 250 \times 0.5946e^{-0.03\times0.25} - 250 \times 0.5594e^{-0.10\times0.25}$$

or 11.15.

Problem 15.7.

Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

In this case $S_0 = 0.52$, $K = 0.50$, $r = 0.04$, $r_f = 0.08$, $\sigma = 0.12$, $T = 0.6667$, and

$$d_1 = \frac{\ln(0.52/0.50) + (0.04 - 0.08 + 0.12^2/2)0.6667}{0.12\sqrt{0.6667}} = 0.1771$$

$$d_2 = d_1 - 0.12\sqrt{0.6667} = 0.0791$$

and the put price is

$$0.50N(-0.0791)e^{-0.04\times0.6667} - 0.52N(-0.1771)e^{-0.08\times0.6667}$$

$$= 0.50 \times 0.4685e^{-0.04\times0.6667} - 0.52 \times 0.4297e^{-0.08\times0.6667}$$

$$= 0.0162$$

Problem 15.8.

Show that the formula in equation (15.12) for a put option to sell one unit of currency A for currency B at strike price K gives the same value as equation (15.11) for a call option to buy K units of currency B for currency A at a strike price of 1/K.

A put option to sell one unit of currency A for K units of currency B is worth

$$Ke^{-r_B T}N(-d_2) - S_0 e^{-r_A T}N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

and r_A and r_B are the risk-free rates in currencies A and B, respectively. The value of the option is measured in units of currency B. Defining $S_0^* = 1/S_0$ and $K^* = 1/K$

$$d_1 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

The put price is therefore

$$S_0 K[S_0^* e^{-r_B T}N(d_1^*) - K^* e^{-r_A T}N(d_1^*)$$

where

$$d_1^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A + \sigma^2/2)T}{\sigma\sqrt{T}}$$

This shows that put option is equivalent to KS_0 call options to buy 1 unit of currency A for $1/K$ units of currency B. In this case the value of the option is measured in units of currency A. To obtain the call option value in units of currency B (the same units as the value of the put option was measured in) we must divide by S_0. This proves the result.

Problem 15.9.

A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

Lower bound for European option is

$$S_0 e^{-r_f T} - Ke^{-rT} = 1.5e^{-0.09\times 0.5} - 1.4e^{-0.05\times 0.5} = 0.069$$

Lower bound for American option is

$$S_0 - K = 0.10$$

Problem 15.10.

Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

In this case $S_0 = 250$, $q = 0.04$, $r = 0.06$, $T = 0.25$, $K = 245$, and $c = 10$. Using put–call parity

$$c + Ke^{-rT} = p + S_0e^{-qT}$$

or

$$p = c + Ke^{-rT} - S_0e^{-qT}$$

Substituting:

$$p = 10 + 245e^{-0.25\times0.06} - 250e^{-0.25\times0.04} = 3.84$$

The put price is 3.84.

Problem 15.11.

An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

In this case $S_0 = 696$, $K = 700$, $r = 0.07$, $\sigma = 0.3$, $T = 0.25$ and $q = 0.04$. The option can be valued using equation (15.5).

$$d_1 = \frac{\ln(696/700) + (0.07 - 0.04 + 0.09/2)\times 0.25}{0.3\sqrt{0.25}} = 0.0868$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.0632$$

and

$$N(-d_1) = 0.4654, \quad N(-d_2) = 0.5252$$

The value of the put, p, is given by:

$$p = 700e^{-0.07\times0.25}\times 0.5252 - 696e^{-0.04\times0.25}\times 0.4654 = 40.6$$

i.e., it is \$40.6.

Problem 15.12.

Show that if C is the price of an American call with exercise price K and maturity T on a stock paying a dividend yield of q, and P is the price of an American put on the same stock with the same strike price and exercise date,

$$S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT}$$

where S_0 is the stock price, r is the risk-free rate, and $r > 0$. (Hint: To obtain the first half of the inequality, consider possible values of:

Portfolio A: a European call option plus an amount K invested at the risk-free rate
Portfolio B: an American put option plus e^{-qT} of stock with dividends being reinvested in the stock
To obtain the second half of the inequality, consider possible values of:
Portfolio C: an American call option plus an amount Ke^{-rT} invested at the risk-free rate
Portfolio D: a European put option plus one stock with dividends being reinvested in the stock)

Following the hint, we first consider

Portfolio A: A European call option plus an amount K invested at the risk-free rate

Portfolio B: An American put option plus e^{-qT} of stock with dividends being reinvested in the stock.

Portfolio A is worth $c + K$ while portfolio B is worth $P + S_0e^{-qT}$. If the put option is exercised at time τ $(0 \le \tau < T)$, portfolio B becomes:

$$K - S_\tau + S_\tau e^{-q(T-\tau)} \le K$$

where S_τ is the stock price at time τ. Portfolio A is worth

$$c + Ke^{r\tau} \ge K$$

Hence portfolio A is worth at least as much as portfolio B. If both portfolios are held to maturity (time T), portfolio A is worth

$$\begin{aligned} &\max(S_T - K, 0) + Ke^{rT} \\ =&\max(S_T, K) + K(e^{rT} - 1) \end{aligned}$$

Portfolio B is worth $\max(S_T, K)$. Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth at least as much as portfolio B in all circumstances

$$P + S_0e^{-qT} \le c + K$$

Because $c \le C$:

$$P + S_0e^{-qT} \le C + K$$

or

$$S_0e^{-qT} - K \le C - P$$

This proves the first part of the inequality.

For the second part consider:

Portfolio C: An American call option plus an amount Ke^{-rT} invested at the risk-free rate

Portfolio D: A European put option plus one stock with dividends being reinvested in the stock.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + S_0$. If the call option is exercised at time τ $(0 \leq \tau < T)$ portfolio C becomes:

$$S_\tau - K + Ke^{-r(T-\tau)} < S_\tau$$

while portfolio D is worth

$$p + S_\tau e^{q(\tau - t)} \geq S_\tau$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(S_T, K)$ while portfolio D is worth

$$\begin{aligned} &\max(K - S_T, 0) + S_T e^{qT} \\ =&\max(S_T, K) + S_T(e^{qT} - 1) \end{aligned}$$

Hence portfolio D is worth at least as much as portfolio C.

Since portfolio D is worth at least as much as portfolio C in all circumstances:

$$C + Ke^{-rT} \leq p + S_0$$

Since $p \leq P$:

$$C + Ke^{-rT} \leq P + S_0$$

or

$$C - P \leq S_0 - Ke^{-rT}$$

This proves the second part of the inequality. Hence:

$$S_0 e^{-qT} - K \leq C - P \leq S_0 - Ke^{-rT}$$

Problem 15.13.

Show that a European call option on a currency has the same price as the corresponding European put option on the currency when the forward price equals the strike price.

This follows from put–call parity and the relationship between the forward price, F_0, and the spot price, S_0.

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

and

$$F_0 = S_0 e^{(r - r_f)T}$$

so that

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

If $K = F_0$ this reduces to $c = p$. The result that $c = p$ when $K = F_0$ is true for options on all underlying assets, not just options on currencies. An at-the-money option is frequently defined as one where $K = F_0$ (or $c = p$) rather than one where $K = S_0$.

Problem 15.14.

Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

The volatility of a stock index can be expected to be less than the volatility of a typical stock. This is because some risk (i.e., return uncertainty) is diversified away when a portfolio of stocks is created. In capital asset pricing model terminology, there exists systematic and unsystematic risk in the returns from an individual stock. However, in a stock index, unsystematic risk has been diversified away and only the systematic risk contributes to volatility.

Problem 15.15.

Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

The cost of portfolio insurance increases as the beta of the portfolio increases. This is because portfolio insurance involves the purchase of a put option on the portfolio. As beta increases, the volatility of the portfolio increases causing the cost of the put option to increase. When index options are used to provide portfolio insurance, both the number of options required and the strike price increase as beta increases.

Problem 15.16.

Suppose that a portfolio is worth \$60 million and the S&P 500 is at 1200. If the value of the portfolio mirrors the value of the index, what options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

If the value of the portfolio mirrors the value of the index, the index can be expected to have dropped by 10% when the value of the portfolio drops by 10%. Hence when the value of the portfolio drops to \$54 million the value of the index can be expected to be 1080. This indicates that put options with an exercise price of 1080 should be purchased. The options should be on:

$$\frac{60,000,000}{1200} = \$50,000$$

times the index. Each option contract is for \$100 times the index. Hence 500 contracts should be purchased.

Problem 15.17.

Consider again the situation in Problem 15.16. Suppose that the portfolio has a beta of 2.0, the risk-free interest rate is 5% per annum, and the dividend yield on both the portfolio and the index is 3% per annum. What options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

When the value of the portfolio falls to \$54 million the holder of the portfolio makes a capital loss of 10%. After dividends are taken into account the loss is 7% during the year. This is 12% below the risk-free interest rate. According to the capital asset pricing model:

$$\begin{array}{c}\text{Excess expected return of portfolio}\\ \text{above riskless interest rate}\end{array} = \beta \times \begin{array}{c}\text{Excess expected return of market}\\ \text{above riskless interest rate}\end{array}$$

Therefore, when the portfolio provides a return 12% below the risk-free interest rate, the market's expected return is 6% below the risk-free interest rate. As the index can be assumed to have a beta of 1.0, this is also the excess expected return (including dividends) from the index. The expected return from the index is therefore -1% per annum. Since the index provides a 3% per annum dividend yield, the expected movement in the index is -4%. Thus when the portfolio's value is \$54 million the expected value of the index $0.96 \times 1200 = 1152$. Hence European put options should be purchased with an exercise price of 1152. Their maturity date should be in one year.

The number of options required is twice the number required in Problem 15.16. This is because we wish to protect a portfolio which is twice as sensitive to changes in market conditions as the portfolio in Problem 15.16. Hence options on \$100,000 (or 1,000 contracts) should be purchased. To check that the answer is correct consider what happens when the value of the portfolio declines by 20% to \$48 million. The return including dividends is -17%. This is 22% less than the risk-free interest rate. The index can be expected to provide a return (including dividends) which is 11% less than the risk-free interest rate, i.e. a return of -6%. The index can therefore be expected to drop by 9% to 1092. The payoff from the put options is $(1152 - 1092) \times 100{,}000 = \$6$ million. This is exactly what is required to restore the value of the portfolio to \$54 million.

Problem 15.18.

An index currently stands at 1,500. European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six-month risk-free rate is 5%. What is the implied dividend yield?

The implied dividend yield is the value of q that satisfies the put–call parity equation. It is the value of q that solves

$$154 + 1400e^{-0.05\times0.5} = 34.25 + 1500e^{-0.5q}$$

This is 1.99%.

Problem 15.19.

A total return index tracks the return, including dividends, on a certain portfolio. Explain how you would value (a) forward contracts and (b) European options on the index.

A total return index behaves like a stock paying no dividends. In a risk-neutral world it can be expected to grow on average at the risk-free rate. Forward contracts and options on total return indices should be valued in the same way as forward contracts and options on non-dividend-paying stocks.

Problem 15.20.

What is the put–call parity relationship for European currency options

The put–call parity relationship for European currency options is

$$c + Ke^{-rT} = p + Se^{-r_fT}$$

To prove this result, the two portfolios to consider are:

Portfolio A: one call option plus one discount bond which will be worth K at time T

Portfolio B: one put option plus $e^{-r_f T}$ of foreign currency invested at the foreign risk-free interest rate.

Both portfolios are worth $\max(S_T, K)$ at time T. They must therefore be worth the same today. The result follows.

Problem 15.21.

Can an option on the yen-euro exchange rate be created from two options, one on the dollar-euro exchange rate, and the other on the dollar-yen exchange rate? Explain your answer.

There is no way of doing this. A natural idea is to create an option to exchange K euros for one yen from an option to exchange Y dollars for 1 yen and an option to exchange K euros for Y dollars. The problem with this is that it assumes that either both options are exercised or that neither option is exercised. There are always some circumstances where the first option is in-the-money at expiration while the second is not and vice versa.

Problem 15.22.

Prove the results in equation (15.1), (15.2), and (15.3) using the portfolios indicated.

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to K at time T. If $S_T > K$, the call option is exercised at time T and portfolio A is worth S_T. If $S_T < K$, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth

$$\max(S_T, K)$$

Because of the reinvestment of dividends, portfolio B becomes one share at time T. It is, therefore, worth S_T at this time. It follows that portfolio A is always worth as much as, and is sometimes worth more than, portfolio B at time T. In the absence of arbitrage opportunities, this must also be true today. Hence,

$$c + Ke^{-rT} \geq S_0 e^{-qT}$$

or

$$c \geq S_0 e^{-qT} - Ke^{-rT}$$

This proves equation (15.1)

In portfolio C, the reinvestment of dividends means that the portfolio is one put option plus one share at time T. If $S_T < K$, the put option is exercised at time T and portfolio C is worth K. If $S_T > K$, the put option expires worthless and the portfolio is worth S_T. Hence, at time T, portfolio C is worth

$$\max(S_T, K)$$

Portfolio D is worth K at time T. It follows that portfolio C is always worth as much as, and is sometimes worth more than, portfolio D at time T. In the absence of arbitrage opportunities, this must also be true today. Hence,

$$p + S_0e^{-qT} \geq Ke^{-rT}$$

or

$$p \geq Ke^{-rT} - S_0e^{-qT}$$

This proves equation (15.2)

Portfolios A and C are both worth $\max(S_T, K)$ at time T. They must, therefore, be worth the same today, and the put–call parity result in equation (15.3) follows.

ASSIGNMENT QUESTIONS

Problem 15.23.

The Dow Jones Industrial Average on January 12, 2007 was 12,556 and the price of the March 126 call was \$2.25. Use the DerivaGem software to calculate the implied volatility of this option. Assume that the risk-free rate was 5.3% and the dividend yield was 3%. The option expires on March 20, 2007. Estimate the price of a March 126 put. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.

Options on the DJIA are European. There are 47 trading days between January 12, 2007 and March 20, 2007. Setting the time to maturity equal to $47/252 = 0.1865$, DerivaGem gives the implied volatility as 10.23%. (If instead we use calendar days the time to maturity is 67/365=0.1836 and the implied volatility is 10.33%.)

From put call parity (equation 13.3) the price of the put, p, (using trading time) is given by

$$2.25 + 126e^{-0.053\times0.1865} = p + 125.56e^{-0.03\times0.1865}$$

so that $p = 2.1512$. DerivaGem shows that the implied volatility is 10.23% (as for the call). (If calendar time is used the price of the put is 2.1597 and the implied volatility is 10.33% as for the call.)

A European call has the same implied volatility as a European put when both have the same strike price and time to maturity. This is formally proved in the appendix to Chapter 17.

Problem 15.24.

A stock index currently stands at 300 and has a volatility of 20%. The risk-free interest rate is 8% and the dividend yield on the index is 3%. Use a three-step binomial tree to value a six-month put option on the index with a strike price of 300 if it is (a) European and (b) American?

A shown by DerivaGem the value of the European option is 14.39 and the value of the American option is 14.97.

Problem 15.25.

Suppose that the spot price of the Canadian dollar is U.S. \$0.85 and that the Canadian dollar/U.S. dollar exchange rate has a volatility of 4% per annum. The risk-free rates of interest in Canada and the United States are 4% and 5% per annum, respectively. Calculate the value of a European call option to buy one Canadian dollar for U.S. \$0.85 in nine months. Use put-call parity to calculate the price of a European put option to sell one Canadian dollar for U.S. \$0.85 in nine months. What is the price of a call option to buy U.S. \$0.85 with one Canadian dollar in nine months?

In this case $S_0 = 0.85$, $K = 0.85$, $r = 0.05$, $r_f = 0.04$, $\sigma = 0.04$ and $T = 0.75$. The option can be valued using equation (15.11)

$$d_1 = \frac{\ln(0.85/0.85) + (0.05 - 0.04 + 0.0016/2) \times 0.75}{0.04\sqrt{0.75}} = 0.2338$$
$$d_2 = d_1 - 0.04\sqrt{0.75} = 0.1992$$

and

$$N(d_1) = 0.5924, \quad N(d_2) = 0.5789$$

The value of the call, c, is given by

$$c = 0.85e^{-0.04\times0.75} \times 0.0.5924 - 0.85e^{-0.05\times0.75} \times 0.5789 = 0.0147$$

i.e., it is 1.47 cents. From put–call parity

$$p + S_0e^{-r_fT} = c + Ke^{-rT}$$

so that

$$p = 0.0147 + 0.85e^{-0.05\times9/12} - 0.85e^{-0.04\times9/12} = 0.00854$$

The option to buy US\$0.85 with C\$1.00 is the same as the same as an option to sell one Canadian dollar for US\$0.85. This means that it is a put option on the Canadian dollar and its price is US\$0.00854.

Problem 15.26.

A mutual fund announces that the salaries of its fund managers will depend on the performance of the fund. If the fund loses money, the salaries will be zero. If the fund makes a profit, the salaries will be proportional to the profit. Describe the salary of a fund manager as an option. How is a fund manager motivated to behave with this type of remuneration package?

Suppose that K is the value of the fund at the beginning of the year and S_T is the value of the fund at the end of the year.

The salary of a fund manager is

$$\alpha \max(S_T - K, 0)$$

where α is a constant.

This shows that a fund manager has a call option on the value of the fund at the end of the year. All of the parameters determining the value of this call option are outside the control of the fund manager except the volatility of the fund. The fund manager has an incentive to make the fund as volatile as possible! This may not correspond with the desires of the investors. One way of making the fund highly volatile would be by investing only in high-beta stocks. Another would be by using the whole fund to buy call options on a market index.

It might be argued that a fund manager would not do this because of the risk which the manager faces. If the fund earns a negative return the manager's salary is zero. However, a fund manager could hedge the risk of a negative return by, on his or her own account, taking a short position in call options on a stock market index.

The position could be chosen so that if the market goes up, the gain on salary more than offsets the losses on the call options.

If the market goes down the fund manager ends up with the price received for the call options. It is easy to see that the strategy becomes more attractive as the riskiness of the fund's portfolio increases.

To summarize, the (superficially attractive) remuneration package is open to abuse and does not necessarily motivate the fund managers to act in the best interests of the fund's investors.

Problem 15.27.

Assume that the price of currency A expressed in terms of the price of currency B follows the process

$$dS = (r_{\mathrm{B}} - r_{\mathrm{A}})S\,dt + \sigma S\,dz$$

where r_{A} is the risk-free interest rate in currency A and r_{B} is the risk-free interest rate in currency B. What is the process followed by the price of currency B expressed in terms of currency A?

The price of currency B expressed in terms of currency A is $1/S$. From Ito's lemma the process followed by $X = 1/S$ is

$$dX = [(r_{\mathrm{B}} - r_{\mathrm{A}})S \times (-1/S^2) + 0.5\sigma^2 S^2 \times (2/S^3)]dt + \sigma S \times (-1/S^2)dz$$

or

$$dX = [r_{\mathrm{A}} - r_{\mathrm{B}} + \sigma^2]X dt - \sigma X dz$$

This is Siegel's paradox and is discussed further in Business Snapshot 29.1.

Problem 15.28.

The three-month forward USD/euro exchange rate is 1.3000. The exchange rate volatility is 15%. A US company will have to pay 1 million euros in three months. The *euro and USD risk-free rates are 5% and 4%, respectively. The company decides to use a range forward contract with the lower strike price equal to 1.2500.*

(a) What should the higher strike price be to create a zero-cost contract.

(b) What position in calls and puts should the company take.

(c) Does your answer depend on the euro risk-free rate? Explain.

(d) Does your answer depend on the USD risk-free rate? Explain.

(a) A put with a strike price of 1.25 is worth \$0.019. By trial and error DerivaGem can be used to show that the strike price of a call that leads to a call having a price of \$0.019 is 1.3477. This is the higher strike price to create a zero cost contract.

(b) The company should sell a put with strike price 1.25 and buy a call with strike price 1.3477. This ensures that the exchange rate it pays for the euros is between 1.2500 and 1.3477.

(c) The answer does depend on the euro risk-free rate because the forward exchange rate depends on this rate

(d) The answer does depend on the dollar risk-free rate because the forward exchange rate depends on this rate. However, if the interest rates change so that the spread between the dollar and euro interest rates remains the same, the upper strike price is unchanged at 1.3477. This can be seen from equations (15.13) and (15.14). The forward exchange rate, F_0 is unchanged and changing r has the same percentage effect on both the call and the put.

CHAPTER 16
Futures Options

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on futures options has been restructured for the seventh edition. The chapter now spends more time discussing how Black's model can be used to price European options in terms of forward or futures prices. This is important material because in practice it is usually the case that practitioners use Black's model rather than Black-Scholes model for European options. By doing this they avoid the need to estimate the income on the underlying asset explicitly. (The Black's model material in this chapter is extended to the stochastic interest rate case in Section 27.6.) This chapter also discusses futures style options which are becoming popular at some exchanges. A futures style option is a futures contract on the payoff from an option.

The way I approach the material in the chapter is indicated by the slides. I like to use Problem 16.23 as a hand-in assignment because it provides practice using DerivaGem and links in with the material on volatility smiles in Chapter 18 and the material on American options in Chapter 19.

QUESTIONS AND PROBLEMS

Problem 16.1

Explain the difference between a call option on yen and a call option on yen futures.

A call option on yen gives the holder the right to buy yen in the spot market at an exchange rate equal to the strike price. A call option on yen futures gives the holder the right to receive the amount by which the futures price exceeds the strike price. If the yen futures option is exercised, the holder also obtains a long position in the yen futures contract.

Problem 16.2.

Why are options on bond futures more actively traded than options on bonds?

The main reason is that a bond futures contract is a more liquid instrument than a bond. The price of a Treasury bond futures contract is known immediately from trading on CBOT. The price of a bond can be obtained only by contacting dealers.

Problem 16.3.

"A futures price is like a stock paying a dividend yield." What is the dividend yield?

A futures price behaves like a stock paying a dividend yield at the risk-free interest rate.

Problem 16.4.

A futures price is currently 50. At the end of six months it will be either 56 or 46. The risk-free interest rate is 6% per annum. What is the value of a six-month European call option with a strike price of 50?

In this case $u = 1.12$ and $d = 0.92$. The probability of an up movement in a risk-neutral world is

$$\frac{1 - 0.92}{1.12 - 0.92} = 0.4$$

From risk-neutral valuation, the value of the call is

$$e^{-0.06\times0.5}(0.4 \times 6 + 0.6 \times 0) = 2.33$$

Problem 16.5.

How does the put–call parity formula for a futures option differ from put–call parity for an option on a non-dividend-paying stock?

The put–call parity formula for futures options is the same as the put–call parity formula for stock options except that the stock price is replaced by F_0e^{-rT}, where F_0 is the current futures price, r is the risk-free interest rate, and T is the life of the option.

Problem 16.6.

Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

The American futures call option is worth more than the corresponding American option on the underlying asset when the futures price is greater than the spot price prior to the maturity of the futures contract. This is the case when the risk-free rate is greater than the income on the asset plus the convenience yield.

Problem 16.7.

Calculate the value of a five-month European put futures option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

In this case $F_0 = 19$, $K = 20$, $r = 0.12$, $\sigma = 0.20$, and $T = 0.4167$. The value of the European put futures option is

$$20N(-d_2)e^{-0.12\times0.4167} - 19N(-d_1)e^{-0.12\times0.4167}$$

where

$$d_1 = \frac{\ln(19/20) + (0.04/2)0.4167}{0.2\sqrt{0.4167}} = -0.3327$$

$$d_2 = d_1 - 0.2\sqrt{0.4167} = -0.4618$$

This is

$$e^{-0.12\times0.4167}[20N(0.4618) - 19N(0.3327)]$$

$$= e^{-0.12\times0.4167}(20 \times 0.6778 - 19 \times 0.6303)$$

$$= 1.50$$

or \$1.50.

Problem 16.8.

Suppose you buy a put option contract on October gold futures with a strike price of \$700 per ounce. Each contract is for the delivery of 100 ounces. What happens if you exercise when the October futures price is \$680?

An amount $(700 - 680) \times 100 = \$2,000$ is added to your margin account and you acquire a short futures position giving you the right to sell 100 ounces of gold in October. This position is marked to market in the usual way until you choose to close it out.

Problem 16.9.

Suppose you sell a call option contract on April live cattle futures with a strike price of 90 cents per pound. Each contract is for the delivery of 40,000 pounds. What happens if the contract is exercised when the futures price is 95 cents?

In this case an amount $(0.95 - 0.90) \times 40,000 = \$2,000$ is subtracted from your margin account and you acquire a short position in a live cattle futures contract to sell 40,000 pounds of cattle in April. This position is marked to marked in the usual way until you choose to close it out.

Problem 16.10.

Consider a two-month call futures option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(F_0 - K)e^{-rT} = (47 - 40)e^{-0.1\times2/12} = 6.88$$

Lower bound if option is American is

$$F_0 - K = 7$$

Problem 16.11.

Consider a four-month put futures option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(K - F_0)e^{-rT} = (50 - 47)e^{-0.1 \times 4/12} = 2.90$$

Lower bound if option is American is

$$K - F_0 = 3$$

Problem 16.12.

A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month call option on the futures with a strike price of 60 If the call were American, would it ever be worth exercising it early?

In this case $u = e^{0.3 \times \sqrt{0.25}} = 1.1618$ and $d = 1/u = 0.8607$ the risk-neutral probability of an up move is

$$p = \frac{1 - 0.8607}{1.1618 - 0.8607} = 0.4626$$

In the tree shown in Figure S16.1 the middle number at each node is the price of the European option and the lower number is the price of the American option. The tree shows that the price of the European option is 4.3155 and the price of the American option is 4.4026. The American option should sometimes be exercised early.

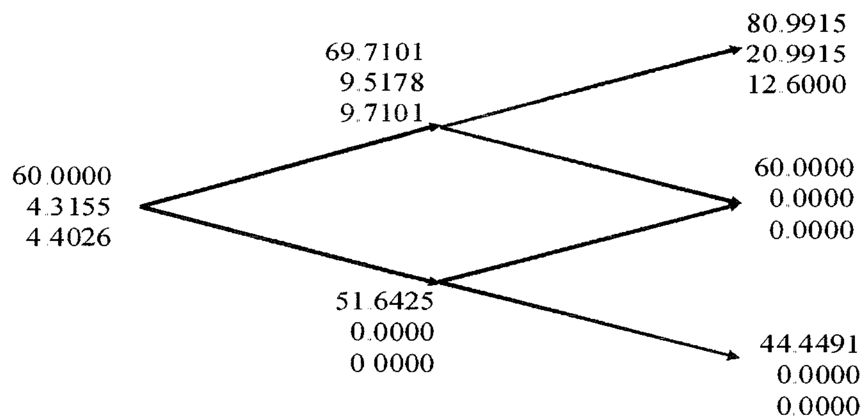

Figure S16.1 Tree to evaluate European and American call options in Problem 16.12.

Problem 16.13.

In Problem 16.12 what is the value of a six-month European put option on futures with a strike price of 60? If the put were American, would it ever be worth exercising it early? Verify that the call prices calculated in Problem 16.12 and the put prices calculated here satisfy put–call parity relationships.

The parameters u, d, and p are the same as in Problem 16.12. The tree in Figure S16.2 shows that the price of the European option is 3.0265 while the price of the American option is 3.0847.

Because $c = p$ and $F_0 = K$ the put–call parity realtionship in equation (16.1) clearly holds. For the American option prices we have:

$$C - P = 0; \qquad F_0e^{-rT} - K = -2.353; \qquad F_0 - Ke^{-rT} = 2.353$$

The put–call inequalities for American options in equation (16.2) are therefore satisfied

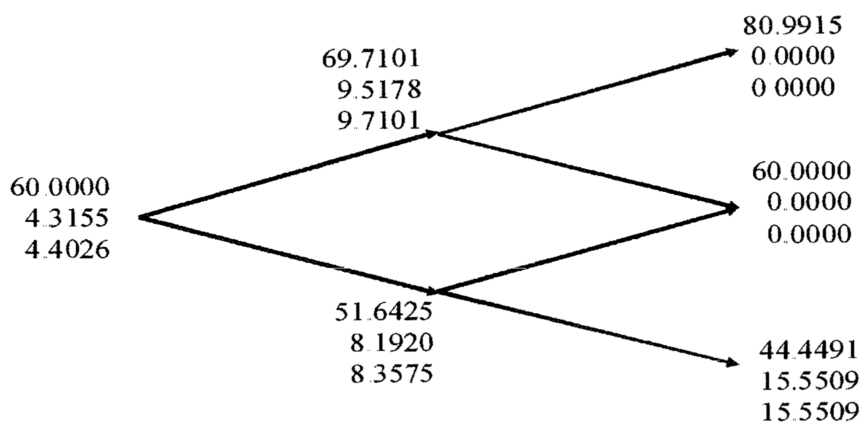

Figure S16.2 Tree to evaluate European and American put options in Problem 16.13.

Problem 16.14.

A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

In this case $F_0 = 25$, $K = 26$, $\sigma = 0.3$, $r = 0.1$, $T = 0.75$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}} = -0.0211$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2T/2}{\sigma\sqrt{T}} = -0.2809$$

$$c = e^{-0.075}[25N(-0.0211) - 26N(-0.2809)]$$

$$= e^{-0.075}[25 \times 0.4916 - 26 \times 0.3894] = 2.01$$

Problem 16.15.

A futures price is currently 70, its volatility is 20% per annum, and the risk-free interest rate is 6% per annum. What is the value of a five-month European put on the futures with a strike price of 65?

In this case $F_0 = 70$, $K = 65$, $\sigma = 0.2$, $r = 0.06$, $T = 0.4167$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = 0.6386$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = 0.5095$$

$$p = e^{-0.025}[65N(-0.5095) - 70N(-0.6386)]$$

$$= e^{-0.025}[65 \times 0.3052 - 70 \times 0.2615] = 1.495$$

Problem 16.16.

Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

In this case

$$c + Ke^{-rT} = 2 + 34e^{-0.1\times1} = 32.76$$

$$p + F_0e^{-rT} = 2 + 35e^{-0.1\times1} = 33.67$$

Put-call parity shows that we should buy one call, short one put and short a futures contract. This costs nothing up front. In one year, either we exercise the call or the put is exercised against us. In either case, we buy the asset for 34 and close out the futures position. The gain on the short futures position is $35 - 34 = 1$.

Problem 16.17.

"The price of an at-the-money European call futures option always equals the price of a similar at-the-money European put futures option." Explain why this statement is true.

The put price is

$$e^{-rT}[KN(-d_2) - F_0N(-d_1)]$$

Because $N(-x) = 1 - N(x)$ for all x the put price can also be written

$$e^{-rT}[K - KN(d_2) - F_0 + F_0N(d_1)]$$

Because $F_0 = K$ this is the same as the call price:

$$e^{-rT}[F_0N(d_1) - KN(d_2)]$$

This result can also be proved from put–call parity showing that it is not model dependent.

Problem 16.18.

Suppose that a futures price is currently 30. The risk-free interest rate is 5% per annum. A three-month American call futures option with a strike price of 28 is worth 4. Calculate bounds for the price of a three-month American put futures option with a strike price of 28.

From equation (16.2), $C - P$ must lie between

$$30e^{-0.05\times3/12} - 28 = 1.63$$

and

$$30 - 28e^{-0.05\times3/12} = 2.35$$

Because $C = 4$ we must have $1.63 < 4 - P < 2.35$ or

$$1.65 < P < 2.37$$

Problem 16.19.

Show that if C is the price of an American call option on a futures contract when the strike price is K and the maturity is T, and P is the price of an American put on the same futures contract with the same strike price and exercise date,

$$F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

where F_0 is the futures price and r is the risk-free rate. Assume that $r > 0$ and that there is no difference between forward and futures contracts. (Hint: Use an analogous approach to that indicated for Problem 15.12.)

In this case we consider

Portfolio A: A European call option on futures plus an amount K invested at the risk-free interest rate

Portfolio B: An American put option on futures plus an amount F_0e^{-rT} invested at the risk-free interest rate plus a long futures contract maturing at time T.

Following the arguments in Chapter 5 we will treat all futures contracts as forward contracts. Portfolio A is worth $c + K$ while portfolio B is worth $P + F_0e^{-rT}$. If the put option is exercised at time τ $(0 \le \tau < T)$, portfolio B is worth

$$K - F_\tau + F_0e^{-r(T-\tau)} + F_\tau - F_0$$
$$=K + F_0e^{-r(T-\tau)} - F_0 < K$$

at time τ where F_τ is the futures price at time τ. Portfolio A is worth

$$c + Ke^{r\tau} \ge K$$

Hence Portfolio A more than Portfolio B. If both portfolios are held to maturity (time T), Portfolio A is worth

$$\begin{aligned}&\max(F_T - K, 0) + Ke^{rT}\\&= \max(F_T, K) + K(e^{rT} - 1)\end{aligned}$$

Portfolio B is worth

$$\max(K - F_T, 0) + F_0 + F_T - F_0 = \max(F_T, K)$$

Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth more than portfolio B in all circumstances:

$$P + F_0 e^{-r(T-t)} < c + K$$

Because $c \leq C$ it follows that

$$P + F_0 e^{-rT} < C + K$$

or

$$F_0 e^{-rT} - K < C - P$$

This proves the first part of the inequality.

For the second part of the inequality consider:

Portfolio C: An American call futures option plus an amount Ke^{-rT} invested at the risk-free interest rate

Portfolio D: A European put futures option plus an amount F_0 invested at the risk-free interest rate plus a long futures contract.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + F_0$. If the call option is exercised at time τ $(0 \leq \tau < T)$ portfolio C becomes:

$$F_\tau - K + Ke^{-r(T-\tau)} < F_\tau$$

while portfolio D is worth

$$\begin{aligned}&p + F_0 e^{r\tau} + F_\tau - F_0\\&= p + F_0(e^{r\tau} - 1) + F_\tau \geq F_\tau\end{aligned}$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(F_T, K)$ while portfolio D is worth

$$\begin{aligned}&\max(K - F_T, 0) + F_0 e^{rT} + F_T - F_0\\&= \max(K, F_T) + F_0(e^{rT} - 1)\\&> \max(K, F_T)\end{aligned}$$

Hence portfolio D is worth more than portfolio C.

Because portfolio D is worth more than portfolio C in all circumstances

$$C + Ke^{-rT} < p + F_0$$

Because $p \leq P$ it follows that

$$C + Ke^{-rT} < P + F_0$$

or

$$C - P < F_0 - Ke^{-rT}$$

This proves the second part of the inequality. The result:

$$F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

has therefore been proved.

Problem 16.20.

Calculate the price of a three-month European call option on the spot price of silver. The three-month futures price is \$12, the strike price is \$13, the risk-free rate is 4%, and the volatility of the price of silver is 25%.

This has the same value as a three-month call option on silver futures where the futures contract expires in three months. It can therefore be valued using equation (16.9) with $F_0 = 12$, $K = 13$, $r = 0.04$, $\sigma = 0.25$ and $T = 0.25$. The value is 0.244.

Problem 16.21.

A corporation knows that in three months it will have \$5 million to invest for 90 days at LIBOR minus 50 basis points and wishes to ensure that the rate obtained will be at least 6.5%. What position in exchange-traded interest-rate options should the corporation take?

The rate received will be less than 6.5% when LIBOR is less than 7%. The corporation requires a three-month call option on a Eurodollar futures option with a strike price of 93. If three-month LIBOR is greater than 7% at the option maturity, the Eurodollar futures quote at option maturity will be less than 93 and there will be no payoff from the option. If the three-month LIBOR is less than 7%, one Eurodollar futures options provide a payoff of \$25 per 0.01%. Each 0.01% of interest costs the corporation \$500 ($= 5,000,000 \times 0.0001$). A total of $500/25 = 20$ contracts are therefore required.

ASSIGNMENT QUESTIONS

Problem 16.22.

A futures price is currently 40. It is known that at the end of three months the price will be either 35 or 45. What is the value of a three-month European call option on the futures with a strike price of 42 if the risk-free interest rate is 7% per annum?

In this case $u = 1.125$ and $d = 0.875$. The risk-neutral probability of an up move is

$$(1 - .875)/(1.125 - 0.875) = 0.5$$

The value of the option is

$$e^{-0.07\times 0.25}[0.5\times 3+0.5\times 0]=1.474$$

Problem 16.23.

It is February 4. July call options on corn futures with strike prices of 260, 270, 280, 290, and 300 cost 26.75, 21.25, 17.25, 14.00, and 11.375, respectively. July put options with these strike prices cost 8.50, 13.50, 19.00, 25.625, and 32.625, respectively. The options mature on June 19, the current July corn futures price is 278.25, and the risk-free interest rate is 1.1%. Calculate implied volatilities for the options using DerivaGem. Comment on the results you get.

There are 135 days to maturity (assuming this is not a leap year). Using DerivaGem with $F_0 = 278.25$, $r = 1.1\%$, $T = 135/365$, and 500 time steps gives the implied volatilities shown in the table below.

Strike Price	Call Price	Put Price	Call Imp Vol	Put Imp Vol
260	26.75	8.50	24.69	24.59
270	21.25	13.50	25.40	26.14
280	17.25	19.00	26.85	26.86
290	14.00	25.625	28.11	27.98
300	11.375	32.625	29.24	28.57
310	9.25		34.32	

We do not expect put–call parity to hold exactly for American options and so there is no reason why the implied volatility of a call should be exactly the same as the implied volatility of a put. Nevertheless it is reassuring that they are close.

There is a tendency for high strike price options to have a higher implied volatility. As explained in Chapter 18, this is an indication that the probability distribution for corn futures prices in the future has a heavier right tail and less heavy left tail than the lognormal distribution.

Problem 16.24.

Calculate the implied volatility of soybean futures prices from the following information concerning a European put on soybean futures:

Current futures price	*525*
Exercise price	*525*
Risk-free rate	*6% per annum*
Time to maturity	*5 months*
Put price	*20*

In this case $F_0 = 525$, $K = 525$, $r = 0.06$, $T = 0.4167$. We wish to find the value of σ for which $p = 20$ where:

$$p = Ke^{-rT}N(-d_2) - F_0e^{-rT}N(-d_1)$$

This must be done by trial and error. When $\sigma = 0.2$, $p = 26.36$. When $\sigma = 0.15$, $p = 19.78$. When $\sigma = 0.155$, $p = 20.44$. When $\sigma = 0.152$, $p = 20.04$. These calculations show that the implied volatility is approximately 15.2% per annum.

Problem 16.25.

Calculate the price of a six-month European put option on the spot value of the S&P 500. The six-month forward price of the index is 1,400, the strike price is 1,450, the risk-free rate is 5%, and the volatility of the index is 15%.

The price of the option is the same as the price of a European put option on the forward price of the index where the forward contract has a maturity of six months. It is given by equation (16.10) with $F_0 = 1400$, $K = 1450$, $r = 0.05$, $\sigma = 0.15$, and $T = 0.5$. It is 86.35.

CHAPTER 17
The Greek Letters

Notes for the Instructor

This chapter covers the way in which traders working for financial institutions and market makers on the floor of an exchange hedge portfolio of derivatives. Students generally enjoy the chapter. The software, DerivaGem for Excel, can be used to demonstrate the relationships between any of the Greek letters and variables such as S_0, K, r, σ, and T.

The chapter has been restructured for the seventh edition. Up to Section 17.12 the presentation now focuses on the calculation of Greek letters for stocks. Section 17.12 then extends the results to other underlying assets (stock indices, currencies and futures). Section 17.12 also covers the difference between the delta of futures and forward contracts. This restructuring, suggested by an instructor who adopted the book, creates a significant improvement in the way the material is presented.

It is important to make sure that students understand what is meant by hedging and in particular what constitutes a good hedge. A financial institution is well hedged with respect to an underlying variable if its wealth position is largely unaffected by changes in the value of the variable. The naked positions and covered positions described in Section 17.2 are clearly not perfect hedges. The deceptively simple stop-loss rule in Section 17.3 is also far from perfect. Delta hedging works better. In fact, it works perfectly if volatility is constant and the position in the underlying asset is changed continuously. In practice of course positions cannot be changed continuously and volatility is not constant so that delta hedging is than less than perfect (See Tables 17.2, 17.3 and 17.4 for the impact of discrete rebalancing.)

I spend some time on Figure 17.7. It shows that the error in delta hedging depends on the curvature of the relationship between the derivative's price and the price of the underlying asset. This observation provides a lead in to gamma, which measures curvature. I find it worth going through a numerical example to show how a portfolio that is both gamma-neutral and delta-neutral can be constructed.

Theta is not the same type of hedge statistic as delta and gamma because there is no uncertainty about the rate at which time will pass. It is an interesting description of one aspect of a portfolio of derivatives. When delta is zero, equation (17.4) shows that theta is a proxy for gamma. When gamma is large and negative theta is large and positive, and vice versa.

Whereas gamma hedging protects the hedger against the fact that the hedge can only be adjusted discretely (e.g. every day or every week), vega hedging protects against volatility changes. Generally gamma is more important for short-life options while vega is more important for long-life options. This is clear from the equations for them. Gamma is proportional to $1/\sqrt{T}$; vega is proportional to $\sqrt{T}$.

Portfolio insurance usually generates a lively discussion—particularly if students are familiar with the details of the October 19, 1987 crash. It is important to explain that portfolio insurance involves creating a long position in an option synthetically. By contrast, hedging a long option position involves creating a short position in the option synthetically.

Problems 17.24, 17.25, 17.26,17.27 (more difficult), and 17.30 (more difficult) all make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 17.1.

Explain how a stop-loss hedging scheme can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

Suppose the strike price is 10.00. The option writer aims to be fully covered whenever the option is in the money and naked whenever it is out of the money. The option writer attempts to achieve this by buying the assets underlying the option as soon as the asset price reaches 10.00 from below and selling as soon as the asset price reaches 10.00 from above. The trouble with this scheme is that it assumes that when the asset price moves from 9.99 to 10.00, the next move will be to a price above 10.00. (In practice the next move might back to 9.99.) Similarly it assumes that when the asset price moves from 10.01 to 10.00, the next move will be to a price below 10.00. (In practice the next move might be back to 10.01.) The scheme can be implemented by buying at 10.01 and selling at 9.99. However, it is not a good hedge. The cost of the trading strategy is zero if the asset price never reaches 10.00 and can be quite high if it reaches 10.00 many times. A good hedge has the property that its cost is always very close the value of the option.

Problem 17.2.

What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

Problem 17.3.

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

In this case $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0/K) + (0.1 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.3712$$

The delta of the option is $N(d_1)$ or 0.64.

Problem 17.4.

What does it mean to assert that the theta of an option position is –0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

A theta of -0.1 means that if Δt units of time pass with no change in either the stock price or its volatility, the value of the option declines by $0.1\Delta t$. A trader who feels that neither the stock price nor its implied volatility will change should write an option with as high a negative theta as possible. Relatively short-life at-the-money options have the most negative thetas.

Problem 17.5.

What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

The gamma of an option position is the rate of change of the delta of the position with respect to the asset price. For example, a gamma of 0.1 would indicate that when the asset price increases by a certain small amount delta increases by 0.1 of this amount. When the gamma of an option writer's position is large and negative and the delta is zero, the option writer will lose significant amounts of money if there is a large movement (either an increase or a decrease) in the asset price.

Problem 17.6.

"The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

To hedge an option position it is necessary to create the opposite option position synthetically. For example, to hedge a long position in a put it is necessary to create a short position in a put synthetically. It follows that the procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position.

Problem 17.7.

Why did portfolio insurance not work well on October 19, 1987?

Portfolio insurance involves creating a put option synthetically. It assumes that as soon as a portfolio's value declines by a small amount the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On October 19, 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

Problem 17.8.

The Black–Scholes price of an out-of-the-money call option with an exercise price of \$40 is \$4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at \$40.10 and to sell at \$39.90. Estimate the expected number of times the stock will be bought or sold.

The strategy costs the trader 0.10 each time the stock is bought or sold. The total expected cost of the strategy, in present value terms, must be \$4. This means that the expected number of times the stock will be bought or sold is approximately 40. The expected number of times it will be bought is approximately 20 and the expected number of times it will be sold is also approximately 20. The buy and sell transactions can take place at any time during the life of the option. The above numbers are therefore only approximately correct because of the effects of discounting. Also the estimate is of the number of times the stock is bought or sold in the risk-neutral world, not the real world.

Problem 17.9.

Suppose that a stock price is currently $20 and that a call option with an exercise price of $25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:

a. *Stock price increases steadily from $20 to $35 during the life of the option.*
b. *Stock price oscillates wildly, ending up at $35.*

Which scenario would make the synthetically created option more expensive? Explain your answer.

The holding of the stock at any given time must be $N(d_1)$. Hence the stock is bought just after the price has risen and sold just after the price has fallen. (This is the buy high sell low strategy referred to in the text.) In the first scenario the stock is continually bought. In second scenario the stock is bought, sold, bought again, sold again, etc. The final holding is the same in both scenarios. The buy, sell, buy, sell... situation clearly leads to higher costs than the buy, buy, buy... situation. This problem emphasizes one disadvantage of creating options synthetically. Whereas the cost of an option that is purchased is known up front and depends on the forecasted volatility, the cost of an option that is created synthetically is not known up front and depends on the volatility actually encountered.

Problem 17.10.

What is the delta of a short position in 1,000 European call options on silver futures? The options mature in eight months, and the futures contract underlying the option matures in nine months. The current nine-month futures price is $8 per ounce, the exercise price of the options is $8, the risk-free interest rate is 12% per annum, and the volatility of silver is 18% per annum.

The delta of a European futures call option is usually defined as the rate of change of the option price with respect to the futures price (not the spot price). It is

$$e^{-rT}N(d_1)$$

In this case $F_0 = 8$, $K = 8$, $r = 0.12$, $\sigma = 0.18$, $T = 0.6667$

$$d_1 = \frac{\ln(8/8) + (0.18^2/2) \times 0.6667}{0.18\sqrt{0.6667}} = 0.0735$$

$N(d_1) = 0.5293$ and the delta of the option is

$$e^{-0.12 \times 0.6667} \times 0.5293 = 0.4886$$

The delta of a short position in 1,000 futures options is therefore -488.6.

Problem 17.11.

In Problem 17.10, what initial position in nine-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If one-year silver futures are used, what is the initial position? Assume no storage costs for silver.

In order to answer this problem it is important to distinguish between the rate of change of the option with respect to the futures price and the rate of change of its price with respect to the spot price.

The former will be referred to as the futures delta; the latter will be referred to as the spot delta. The futures delta of a nine-month futures contract to buy one ounce of silver is by definition 1.0. Hence, from the answer to Problem 17.10, a long position in nine-month futures on 488.6 ounces is necessary to hedge the option position.

The spot delta of a nine-month futures contract is $e^{0.12\times0.75} = 1.094$ assuming no storage costs. (This is because silver can be treated in the same way as a non-dividend-paying stock when there are no storage costs. $F_0 = S_0e^{rT}$ so that the spot delta is the futures delta times e^{rT}) Hence the spot delta of the option position is $-488.6 \times 1.094 = -534.6$. Thus a long position in 534.6 ounces of silver is necessary to hedge the option position.

The spot delta of a one-year silver futures contract to buy one ounce of silver is $e^{0.12} = 1.1275$. Hence a long position in $e^{-0.12} \times 534.6 = 474.1$ ounces of one-year silver futures is necessary to hedge the option position.

Problem 17.12.

A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?

a. A virtually constant spot rate

b. Wild movements in the spot rate

Explain your answer.

A long position in either a put or a call option has a positive gamma. From Figure 17.8, when gamma is positive the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence the hedger will fare better in case (b).

Problem 17.13.

Repeat Problem 17.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

A short position in either a put or a call option has a negative gamma. From Figure 17.8, when gamma is negative the hedger gains from a small change in the stock price and loses from a large change in the stock price. Hence the hedger will fare better in case (a).

Problem 17.14.

A financial institution has just sold 1,000 seven-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price *is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.*

In this case $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.15$, $T = 0.5833$

$$d_1 = \frac{\ln(0.80/0.81) + \left(0.08 - 0.05 + 0.15^2/2\right) \times 0.5833}{0.15\sqrt{0.5833}} = 0.1016$$

$$d_2 = d_1 - 0.15\sqrt{0.5833} = -0.0130$$

$$N(d_1) = 0.5405; \quad N(d_2) = 0.4998$$

The delta of one call option is $e^{-r_f T}N(d_1) = e^{-0.05 \times 0.5833} \times 0.5405 = 0.5250$.

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2} = \frac{1}{\sqrt{2\pi}}e^{-0.00516} = 0.3969$$

so that the gamma of one call option is

$$\frac{N'(d_1)e^{-r_f T}}{S_0\sigma\sqrt{T}} = \frac{0.3969 \times 0.9713}{0.80 \times 0.15 \times \sqrt{0.5833}} = 4.206$$

The vega of one call option is

$$S_0\sqrt{T}N'(d_1)e^{-r_f T} = 0.80\sqrt{0.5833} \times 0.3969 \times 0.9713 = 0.2355$$

The theta of one call option is

$$\begin{aligned}
&-\frac{S_0 N'(d_1)\sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0 N(d_1)e^{-r_f T} - rKe^{-rT}N(d_2) \\
=&-\frac{0.8 \times 0.3969 \times 0.15 \times 0.9713}{2\sqrt{0.5833}} \\
&+ 0.05 \times 0.8 \times 0.5405 \times 0.9713 - 0.08 \times 0.81 \times 0.9544 \times 0.4948 \\
=&-0.0399
\end{aligned}$$

The rho of one call option is

$$\begin{aligned}
&KTe^{-rT}N(d_2) \\
=&0.81 \times 0.5833 \times 0.9544 \times 0.4948 \\
=&0.2231
\end{aligned}$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the delta increases by 4.206 times that amount. Vega can be interpreted as meaning that, when the volatility (measured in decimal form) increases by a small amount, the option's value increases by 0.2355 times that amount. When volatility increases by 1% (= 0.01) the option price increases by 0.002355. Theta can be interpreted as meaning that, when a small amount of time (measured in years) passes, the option's value decreases by 0.0399 times that amount. In particular when one calendar day passes it decreases by 0.0399/365 = 0.000109. Finally, rho can be interpreted as meaning that, when the interest rate (measured in decimal form) increases by a small amount the option's value increases by 0.2231 times that amount. When the interest rate increases by 1% (= 0.01), the options value increases by 0.002231.

Problem 17.15.

Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

Assume that S_0, K, r, σ, T, q are the parameters for the option held and S_0, K^*, r, σ, T^*, q are the parameters for another option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1^* is the value of d_1 calculated on the basis of the second set of parameters. Suppose further that w of the second option are held for each of the first option held. The gamma of the portfolio is:

$$\alpha\left[\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}+w\frac{N'(d_1^*)e^{-qT^*}}{S_0\sigma\sqrt{T^*}}\right]$$

where α is the number of the first option held.

Since we require gamma to be zero:

$$w=-\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}\sqrt{\frac{T^*}{T}}$$

The vega of the portfolio is:

$$\alpha\left[S_0\sqrt{T}N'(d_1)e^{-q(T)}+wS_0\sqrt{T^*}N'(d_1^*)e^{-q(T^*)}\right]$$

Since we require vega to be zero:

$$w=-\sqrt{\frac{T}{T^*}}\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}$$

Equating the two expressions for w

$$T^*=T$$

Hence the maturity of the option held must equal the maturity of the option used for hedging.

Problem 17.16.

A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth \$360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.

a. *If the fund manager buys traded European put options, how much would the insurance cost?*
b. *Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.*
c. *If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?*
d. *If the fund manager decides to provide insurance by using nine-month index futures, what should the initial position be?*

The fund is worth \$300,000 times the value of the index. When the value of the portfolio falls by 5% (to \$342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

(a) $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.30$, $T = 0.50$ and $q = 0.03$. Hence:

$$d_1 = \frac{\ln(1200/1140) + \left(0.06 - 0.03 + 0.3^2/2\right) \times 0.5}{0.3\sqrt{0.5}} = 0.4186$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.2064$$

$$N(d_1) = 0.6622; \quad N(d_2) = 0.5818$$

$$N(-d_1) = 0.3378; \quad N(-d_2) = 0.4182$$

The value of one put option is

$$\begin{aligned} &1140e^{-rT}N(-d_2) - 1200e^{-qT}N(-d_1) \\ =&1140e^{-0.06\times0.5} \times 0.4182 - 1200e^{-0.03\times0.5} \times 0.3378 \\ =&63.40 \end{aligned}$$

The total cost of the insurance is therefore

$$300,000 \times 63.40 = \$19,020,000$$

(b) From put–call parity

$$S_0e^{-qT} + p = c + Ke^{-rT}$$

or:

$$p = c - S_0e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by selling (or shorting) e^{-qT} of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

1) Sell $360e^{-0.03\times 0.5} = \$354.64$ million of stock
2) Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months.
3) Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

(c) The delta of one put option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ =& e^{-0.03\times 0.5}(0.6622 - 1) \\ -& 0.3327 \end{aligned}$$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

(d) The delta of a nine-month index futures contract is

$$e^{(r-q)T} = e^{0.03\times 0.75} = 1.023$$

The spot short position required is

$$\frac{119,770,000}{1200} = 99,808$$

times the index. Hence a short position in

$$\frac{99,808}{1.023 \times 250} = 390$$

futures contracts is required.

Problem 17.17.

Repeat Problem 17.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

When the value of the portfolio goes down 5% in six months, the total return from the portfolio, including dividends, in the six months is

$$-5 + 2 = -3\%$$

i.e., -6% per annum. This is 12% per annum less than the risk-free interest rate. Since the portfolio has a beta of 1.5 we would expect the market to provide a return of 8% per annum less than the risk-free interest rate, i.e., we would expect the market to provide a return of -2% per annum. Since dividends on the market index are 3% per annum, we would expect the market index to have dropped at the rate of 5% per annum or 2.5% per six months; i.e., we would expect the market to have dropped to 1170. A total of $450,000 = (1.5 \times 300,000)$ put options on the S&P 500 with exercise price 1170 and exercise date in six months are therefore required.

(a) $S_0 = 1200$, $K = 1170$, $r = 0.06$, $\sigma = 0.3$, $T = 0.5$ and $q = 0.03$. Hence

$$d_1 = \frac{\ln(1200/1170) + (0.06 - 0.03 + 0.09/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2961$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.0840$$

$$N(d_1) = 0.6164; \quad N(d_2) = 0.5335$$

$$N(-d_1) = 0.3836; \quad N(-d_2) = 0.4665$$

The value of one put option is

$$\begin{aligned} & Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \\ =& 1170e^{-0.06\times0.5} \times 0.4665 - 1200e^{-0.03\times0.5} \times 0.3836 \\ =& 76.28 \end{aligned}$$

The total cost of the insurance is therefore

$$450,000 \times 76.28 = \$34,326,000$$

Note that this is significantly greater than the cost of the insurance in Problem 17.16.

(b) As in Problem 17.16 the fund manager can 1) sell \$354.64 million of stock, 2) buy call options on 450,000 times the S&P 500 with exercise price 1170 and exercise date in six months and 3) invest the remaining cash at the risk-free interest rate.

(c) The portfolio is 50% more volatile than the S&P 500. When the insurance is considered as an option on the portfolio the parameters are as follows: $S_0 = 360$, $K = 342$, $r = 0.06$, $\sigma = 0.45$, $T = 0.5$ and $q = 0.04$

$$d_1 = \frac{\ln(360/342) + (0.06 - 0.04 + 0.45^2/2) \times 0.5}{0.45\sqrt{0.5}} = 0.3517$$

$$N(d_1) = 0.6374$$

The delta of the option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ =& e^{-0.04\times0.5}(0.6374 - 1) \\ =& -0.355 \end{aligned}$$

This indicates that 35.5% of the portfolio (i.e., \$127.8 million) should be sold and invested in riskless securities.

(d) We now return to the situation considered in (a) where put options on the index are required. The delta of each put option is

$$\begin{aligned} & e^{-qT}(N(d_1) - 1) \\ =& e^{-0.03\times 0.5}(0.6164 - 1) \\ =& -0.3779 \end{aligned}$$

The delta of the total position required in put options is $-450,000 \times 0.3779 = -170,000$. The delta of a nine month index futures is (see Problem 17.16) 1.023. Hence a short position in

$$\frac{170,000}{1.023 \times 250} = 665$$

index futures contracts.

Problem 17.18.

Show by substituting for the various terms in equation (17.4) that the equation is true for:

a. *A single European call option on a non-dividend-paying stock*
b. *A single European put option on a non-dividend-paying stock*
c. *Any portfolio of European put and call options on a non-dividend-paying stock*

(a) For a call option on a non-dividend-paying stock

$$\begin{aligned} \Delta &= N(d_1) \\ \Gamma &= \frac{N'(d_1)}{S_0\sigma\sqrt{T}} \\ \Theta &= -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$\begin{aligned} =& -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) + rS_0N(d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}} \\ =& r[S_0N(d_1) - Ke^{-rT}N(d_2)] \\ =& r\Pi \end{aligned}$$

(b) For a put option on a non-dividend-paying stock

$$\begin{aligned} \Delta &= N(d_1) - 1 = -N(-d_1) \\ \Gamma &= \frac{N'(d_1)}{S_0\sigma\sqrt{T}} \\ \Theta &= -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) \end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$
\begin{aligned}
&-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) - rS_0N(-d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}} \\
=&r[Ke^{-rT}N(-d_2) - S_0N(-d_1)] \\
=&r\Pi
\end{aligned}
$$

(c) For a portfolio of options, Π, Δ, Θ and Γ are the sums of their values for the individual options in the portfolio. It follows that equation (17.4) is true for any portfolio of European put and call options.

Problem 17.19

What is the equation corresponding to equation (17.4) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures contract?

A currency is analogous to a stock paying a continuous dividend yield at rate r_f. The differential equation for a portfolio of derivatives dependent on a currency is (see equation 15.6)

$$\frac{\partial \Pi}{\partial t} + (r - r_f)S\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Hence

$$\Theta + (r - r_f)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Similarly, for a portfolio of derivatives dependent on a futures price (see equation 16.8)

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Problem 17.20.

Suppose that \$70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within one year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

We can regard the position of all portfolio insurers taken together as a single put option. The three known parameters of the option, before the 23% decline, are $S_0 = 70$, $K = 66.5$, $T = 1$. Other parameters can be estimated as $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$. Then:

$$d_1 = \frac{\ln(70/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = 0.4502$$

$$N(d_1) = 0.6737$$

The delta of the option is

$$\begin{aligned}&e^{-qT}[N(d_1)-1]\\ =&e^{-0.03}(0.6737-1)\\ =&-0.3167\end{aligned}$$

This shows that 31.67% or \$22.17 billion of assets should have been sold before the decline. These numbers can also be produced from DerivaGem by selecting Underlying Type and Index and Option Type as Analytic European.

After the decline, $S_0 = 53.9$, $K = 66.5$, $T = 1$, $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$.

$$d_1 = \frac{\ln(53.9/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = -0.5953$$
$$N(d_1) = 0.2758$$

The delta of the option has dropped to

$$\begin{aligned}&e^{-0.03\times 0.5}(0.2758-1)\\ =&-0.7028\end{aligned}$$

This shows that cumulatively 70.28% of the assets originally held should be sold. An additional 38.61% of the original portfolio should be sold. The sales measured at pre-crash prices are about \$27.0 billion. At post crash prices they are about 20.8 billion.

Problem 17.21.

Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

With our usual notation the value of a forward contract on the asset is $S_0e^{-qT} - Ke^{-rT}$. When there is a small change, ΔS, in S_0 the value of the forward contract changes by $e^{-qT}\Delta S$. The delta of the forward contract is therefore e^{-qT}. The futures price is $S_0e^{(r-q)T}$. When there is a small change, ΔS, in S_0 the futures price changes by $\Delta Se^{(r-q)T}$. Given the daily settlement procedures in futures contracts, this is also the immediate change in the wealth of the holder of the futures contract. The delta of the futures contract is therefore $e^{(r-q)T}$. We conclude that the deltas of a futures and forward contract are not the same. The delta of the futures is greater than the delta of the corresponding forward by a factor of e^{rT}.

Problem 17.22.

A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of $-80,000$. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01 \times 30,000 = \$300$. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by $0.01 \times 80,000 = 800$. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $(0.93 - 0.90) \times 80,000 = 2,400$ so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 2,400 euros so that a net 27,600 have been shorted. As shown in the text (see Figure 17.8), when a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.

Problem 17.23.

Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

(a) *The delta of a European call and the delta of a European put.*
(b) *The gamma of a European call and the gamma of a European put.*
(c) *The vega of a European call and the vega of a European put.*
(d) *The theta of a European call and the theta of a European put.*

For a non-dividend paying stock, put-call parity gives at a general time t:

$$p + S = c + Ke^{-r(T-t)}$$

(a) Differentiating with respect to S:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

(b) Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence the gamma of a European put equals the gamma of a European call.

(c) Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.

(d) Differentiating the put-call parity relationship with respect to T

$$\frac{\partial p}{\partial t} = rKe^{-r(T-t)} + \frac{\partial c}{\partial t}$$

This is in agreement with the thetas of European calls and puts given in Section 17.5 since $N(d_2) = 1 - N(-d_2)$.

ASSIGNMENT QUESTIONS

Problem 17.24.

Consider a one-year European call option on a stock when the stock price is $30, the strike price is $30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to $30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is $30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem software to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

The price, delta, gamma, vega, theta, and rho of the option are 3.7008, 0.6274, 0.050, 0.1135, −0.00596, and 0.1512. When the stock price increases to 30.1, the option price increases to 3.7638. The change in the option price is $3.7638 - 3.7008 = 0.0630$. Delta predicts a change in the option price of $0.6274 \times 0.1 = 0.0627$ which is very close. When the stock price increases to 30.1, delta increases to 0.6324. The size of the increase in delta is $0.6324 - 0.6274 = 0.005$. Gamma predicts an increase of $0.050 \times 0.1 = 0.005$ which is the same. When the volatility increases from 25% to 26%, the option price increases by 0.1136 from 3.7008 to 3.8144. This is consistent with the vega value of 0.1135. When the time to maturity is changed from 1 to 1-1/365 the option price reduces by 0.006 from 3.7008 to 3.6948. This is consistent with a theta of −0.00596. Finally when the interest rate increases from 5% to 6% the value of the option increases by 0.1527 from 3.7008 to 3.8535. This is consistent with a rho of 0.1512.

Problem 17.25.

A financial institution has the following portfolio of over-the-counter options on sterling:

Type	*Position*	*Delta of Option*	*Gamma of Option*	*Vega of Option*
Call	*−1,000*	*0.50*	*2.2*	*1.8*
Call	*−500*	*0.80*	*0.6*	*0.2*
Put	*−2,000*	*−0.40*	*1.3*	*0.7*
Call	*−500*	*0.70*	*1.8*	*1.4*

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

a. *What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?*

b. What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

The delta of the portfolio is

$$-1,000 \times 0.50 - 500 \times 0.80 - 2,000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1,000 \times 2.2 - 500 \times 0.6 - 2,000 \times 1.3 - 500 \times 1.8 = -6,000$$

The vega of the portfolio is

$$-1,000 \times 1.8 - 500 \times 0.2 - 2,000 \times 0.7 - 500 \times 1.4 = -4,000$$

(a) A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000 \times 1.5 = +6,000$. The delta of the whole portfolio (including traded options) is then:

$$4,000 \times 0.6 - 450 = 1,950$$

Hence, in addition to the 4,000 traded options, a short position in £1,950 is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000 \times 0.8 = +4,000$. The delta of the whole portfolio (including traded options) is then

$$5,000 \times 0.6 - 450 = 2,550$$

Hence, in addition to the 5,000 traded options, a short position in £2,550 is necessary so that the portfolio is both vega and delta neutral.

Problem 17.26.

Consider again the situation in Problem 17.25. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option. We require:

$$6,000 = 1.5w_1 + 0.5w_2$$

$$4,000 = 0.8w_1 + 0.6w_2$$

The solution to these equations can easily be seen to be $w_1 = 3,200$, $w_2 = 2,400$. The whole portfolio then has a delta of

$$-450 + 3,200 \times 0.6 + 2,400 \times 0.1 = 1,710$$

Therefore the portfolio can be made delta, gamma and vega neutral by taking a long position in 3,200 of the first traded option, a long position in 2,400 of the second traded option and a short position in £1,710.

Problem 17.27.

A deposit instrument offered by a bank guarantees that investors will receive a return during a six-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put $100,000 in the instrument. Describe the payoff as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?

The product provides a six-month return equal to

$$\max(0,\ 0.4R)$$

where R is the return on the index. Suppose that S_0 is the current value of the index and S_T is the value in six months.

When an amount A is invested, the return received at the end of six months is:

$$\begin{aligned}&A\max\left(0,\ 0.4\frac{S_T-S_0}{S_0}\right)\\&=\frac{0.4A}{S_0}\max(0,\ S_T-S_0)\end{aligned}$$

This is $0.4A/S_0$ of at-the-money European call options on the index. With the usual notation, they have value:

$$\begin{aligned}&\frac{0.4A}{S_0}[S_0e^{-qT}N(d_1)-S_0e^{-rT}N(d_2)]\\&=0.4A[e^{-qT}N(d_1)-e^{-rT}N(d_2)]\end{aligned}$$

In this case $r=0.08$, $\sigma=0.25$, $T=0.50$ and $q=0.03$

$$\begin{aligned}d_1&=\frac{(0.08-0.03+0.25^2/2)\,0.50}{0.25\sqrt{0.50}}=0.2298\\d_2&=d_1-0.25\sqrt{0.50}=0.0530\end{aligned}$$

$$N(d_1)=0.5909;\quad N(d_2)=0.5212$$

The value of the European call options being offered is

$$\begin{aligned}&0.4A(e^{-0.03\times0.5}\times0.5909-e^{-0.08\times0.5}\times0.5212)\\&=0.0325A\end{aligned}$$

This is the present value of the payoff from the product. If an investor buys the product he or she avoids having to pay $0.0325A$ at time zero for the underlying option. The cash flows to the investor are therefore

Time 0: $-A+0.0325A = 0.9675A$

After six months: $+A$

The return with continuous compounding is $2\ln(1/0.9675) = 0.066$ or 6.6% per annum. The product is therefore slightly less attractive than a risk-free investment.

Problem 17.28.

The formula for the price of a European call futures option in terms of the futures price, F_0, is given in Chapter 16 as

$$c = e^{-rT}[F_0N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and K, r, T, and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0N'(d_1) = KN'(d_2)$

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$.

(c) Prove that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$

(d) Prove the formula for the rho of a call futures option given in Section 17.12. The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q with q replaced by r and S_0 replaced by F_0. Explain why the same is not true of the rho of a call futures option.

(a)

$$FN'(d_1) = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

$$KN'(d_2) = KN'(d_1 - \sigma\sqrt{T}) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+d_1\sigma\sqrt{T}-\sigma^2T/2}$$

Because $d_1\sigma\sqrt{T} = \ln(F/K) + \sigma^2 T/2$ the second equation reduces to

$$KN'(d_2) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+ln(F/K)} = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

The result follows.

(b)

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1) + e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial F} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial F}$$

Because

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$$

it follows from the result in (a) that

$$\frac{\partial c}{\partial F} = e^{-rT} N(d_1)$$

(c)

$$\frac{\partial c}{\partial \sigma} = e^{-rT} F N'(d_1) \frac{\partial d_1}{\partial \sigma} - e^{-rT} K N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

Because $d_1 = d_2 + \sigma\sqrt{T}$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

From the result in (a) it follows that

$$\frac{\partial c}{\partial \sigma} = e^{-rT} F N'(d_1) \sqrt{T}$$

(d) Rho is given by

$$\frac{\partial c}{\partial r} = -Te^{-rT}[FN(d_1) - KN(d_2)]]$$

or $-cT$. Because $q = r$ in the case of a futures option there are two components to rho. One arises from differentiation with respect to r, the other from differentiation with respect to q.

Problem 17.29.

Use DerivaGem to check that equation (17.4) is satisfied for the option considered in Section 17.1. (Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (17.7) is "per year.")

For the option considered in Section 17.1, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.20$, and $T = 20/52$. DerivaGem shows that $\Theta = -0.011795 \times 365 = -4.305$, $\Delta = 0.5216$, $\Gamma = 0.065544$, $\Pi = 2.4005$. The left hand side of equation (17.7)

$$-4.305 + 0.05 \times 49 \times 0.5216 + \frac{1}{2} \times 0.2^2 \times 49^2 \times 0.065544 = 0.120$$

The right hand side is

$$0.05 \times 2.4005 = 0.120$$

This shows that the result in equation (17.4) is satisfied.

Problem 17.30.

Use the DerivaGem Application Builder functions to reproduce Table 17.2. (Note that in Table 17.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (17.3) is approximately satisfied. (Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (17.3) is "per year.")

Consider the first week. The portfolio consists of a short position in 100,000 options and a long position in 52,200 shares. The value of the option changes from \$240,053 at the beginning of the week to \$188,760 at the end of the week for a gain of \$51,293. The value of the shares change from $52,200 \times 49 = \$2,557,800$ to $52,200 \times 48.12 = \$2,511,864$ for a loss of \$45,936. The net gain is $51,293 - 45,936 = \$5,357$. The gamma and theta (per year) of the portfolio are -6554.4 and 430,533 so that equation (17.3) predicts the gain as

$$430533 \times \frac{1}{52} - \frac{1}{2} \times 6554.4 \times (48.12 - 49)^2 = 5742$$

The results for all 20 weeks are shown in the following table.

Week	Actual Gain	Predicted Gain
1	5,357	5,742
2	5,689	6,093
3	-19,742	-21,084
4	1,941	1,572
5	3,706	3,652
6	9,320	9,191
7	6,249	5,936
8	9,491	9,259
9	961	870
10	-23,380	-18,992
11	1,643	2,497
12	2,645	1,356
13	11,365	10,923
14	-2,876	-3,342
15	12,936	12,302
16	7,566	8,815
17	-3,880	-2,763
18	6,764	6,899
19	4,295	5,205
20	4,804	4,805

CHAPTER 18
Volatility Smiles

Notes for the Instructor

This chapter covers volatility smiles and how they are used in practice. The approach is to start with the volatility smiles that are observed in the equity and foreign currency markets and then show what the implied distributions look like. A number of improvements have been made to the chapter. Section 18.1 reads more easily. Different ways used by practitioners to quantify the volatility smile are covered.

I find that many students are interested in the details of how one goes from a volatility smile to an implied distribution. The appendix to the chapter now has more information on this and includes a numerical example.

I focus on foreign exchange and equity markets when covering this chapter, but futures markets can also be mentioned. (Problem 16.23 from Chapter 16 derives a volatility for corn futures.)

It is not difficult to construct interesting assignments based on the material. For example, students can be asked to calculate a volatility smile for options on the S&P 500 using data obtained from a newspaper or a live data feed. Problems 18.19 and 18.26 work well for class discussion. The others make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 18.1.

What volatility smile is likely to be observed when

a. *Both tails of the stock price distribution are less heavy than those of the lognormal distribution?*

b. *The right tail is heavier, and the left tail is less heavy, than that of a lognormal distribution?*

A downward sloping volatility smile is usually observed for equities.

Problem 18.2.

What volatility smile is observed for equities?

A downward sloping volatility smile is usually observed for equities.

Problem 18.3.

What volatility smile is likely to be caused by jumps in the underlying asset price? Is the pattern likely to be more pronounced for a two-year option than for a three-month option?

Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile similar to that in Figure 18.1. The volatility smile is likely to be more pronounced for the three-month option.

Problem 18.4.

A European call and put option have the same strike price and time to maturity. The call has an implied volatility of 30% and the put has an implied volatility of 25%. What trades would you do?

The put has a price that is too low relative to the call's price. The correct trading strategy is to buy the put, buy the stock, and sell the call.

Problem 18.5.

Explain carefully why a distribution with a heavier left tail and less heavy right tail than the lognormal distribution gives rise to a downward sloping volatility smile.

The heavier left tail should lead to high prices, and therefore high implied volatilities, for out-of-the-money (low-strike-price) puts. Similarly the less heavy right tail should lead to low prices, and therefore low volatilities for out-of-the-money (high-strike-price) calls. A volatility smile where volatility is a decreasing function of strike price results.

Problem 18.6.

The market price of a European call is \$3.00 and its price given by Black–Scholes model with a volatility of 30% is \$3.50. The price given by this Black–Scholes model for a European put option with the same strike price and time to maturity is \$1.00. What should the market price of the put option be? Explain the reasons for your answer.

With the notation in the text

$$c_{\text{bs}} + Ke^{-rT} = p_{\text{bs}} + Se^{-qT}$$

$$c_{\text{mkt}} + Ke^{-rT} = p_{\text{mkt}} + Se^{-qT}$$

It follows that

$$c_{\text{bs}} - c_{\text{mkt}} = p_{\text{bs}} - p_{\text{mkt}}$$

In this case $c_{\text{mkt}} = 3.00$; $c_{\text{bs}} = 3.50$; and $p_{\text{bs}} = 1.00$. It follows that p_{mkt} should be 0.50.

Problem 18.7.

Explain what is meant by crashophobia.

The crashophobia argument is an attempt to explain the pronounced volatility skew in equity markets since 1987. (This was the year equity markets shocked everyone by crashing more than 20% in one day). The argument is that traders are concerned about another crash and as a result increase the price of out-of-the-money puts. This creates the volatility skew.

Problem 18.8.

A stock price is currently \$20. Tomorrow, news is expected to be announced that will either increase the price by \$5 or decrease the price by \$5. What are the problems in using Black–Scholes to value one-month options on the stock?

The probability distribution of the stock price in one month is not lognormal. Possibly it consists of two lognormal distributions superimposed upon each other and is bimodal. Black–Scholes is clearly inappropriate, because it assumes that the stock price at any future time is lognormal.

Problem 18.9.

What volatility smile is likely to be observed for six-month options when the volatility is uncertain and positively correlated to the stock price?

When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing less heavy left tails and heavier right tails. Implied volatility then increases with the strike price.

Problem 18.10.

What problems do you think would be encountered in testing a stock option pricing model empirically?

There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

Problem 18.11.

Suppose that a central bank's policy is to allow an exchange rate to fluctuate between 0.97 and 1.03. What pattern of implied volatilities for options on the exchange rate would you expect to see?

In this case the probability distribution of the exchange rate has a thin left tail and a thin right tail relative to the lognormal distribution. We are in the opposite situation to that described for foreign currencies in Section 18.1. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. The pattern of implied volatilities is likely to be similar to Figure 18.7.

Problem 18.12.

Option traders sometimes refer to deep-out-of-the-money options as being options on volatility. Why do you think they do this?

A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

Problem 18.13.

A European call option on a certain stock has a strike price of $30, a time to maturity of one year, and an implied volatility of 30%. A European put option on the same stock has a strike price of $30, a time to maturity of one year, and an implied volatility of 33%. What is the arbitrage opportunity open to a trader? Does the arbitrage work only when the lognormal assumption underlying Black–Scholes holds? Explain the reasons for your answer carefully.

As explained in the appendix to the chapter, put–call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put and short the stock. This does not depend on the lognormal assumption underlying Black–Scholes. Put–call parity is true for any set of assumptions.

Problem 18.14.

Suppose that the result of a major lawsuit affecting a company is due to be announced tomorrow. The company's stock price is currently $60. If the ruling is favorable to the company, the stock price is expected to jump to $75. If it is unfavorable, the stock is expected to jump to $50. What is the risk-neutral probability of a favorable ruling? Assume that the volatility of the company's stock will be 25% for six months after the ruling if the ruling is favorable and 40% if it is unfavorable. Use DerivaGem to calculate the relationship between implied volatility and strike price for six-month European options on the company today. The company does not pay dividends. Assume that the six-month risk-free rate is 6%. Consider call options with strike prices of $30, $40, $50, $60, $70, and $80.

Suppose that p is the probability of a favorable ruling. The expected price of the company's stock tomorrow is

$$75p + 50(1-p) = 50 + 25p$$

This must be the price of the stock today. (We ignore the expected return to an investor over one day.) Hence

$$50 + 25p = 60$$

or $p = 0.4$.

If the ruling is favorable, the volatility, σ, will be 25%. Other option parameters are $S_0 = 75$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 26.502.

If the ruling is unfavorable, the volatility, σ will be 40% Other option parameters are $S_0 = 50$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 6.310.

The value today of a European call option with a strike price today is the weighted average of 26.502 and 6.310 or:

$$0.4 \times 26.502 + 0.6 \times 6.310 = 14.387$$

DerivaGem can be used to calculate the implied volatility when the option has this price. The parameter values are $S_0 = 60$, $K = 50$, $T = 0.5$, $r = 0.06$ and $c = 14.387$. The implied volatility is 47.76%.

These calculations can be repeated for other strike prices. The results are shown in the table below. The pattern of implied volatilities is shown in Figure S18.1.

Strike Price	Call Option Price Favorable Outcome	Call Option Price Unfavorable Outcome	Weighted Price	Implied Volatility (%)
30	45.887	21.001	30.955	46.67
40	36.182	12.437	21.935	47.78
50	26.502	6.310	14.387	47.76
60	17.171	2.826	8.564	46.05
70	9.334	1.161	4.430	43.22
80	4.159	0.451	1.934	40.36

Figure S18.1 Implied Volatilities in Problem 18.14

Problem 18.15.

An exchange rate is currently 0.8000. The volatility of the exchange rate is quoted as 12% and interest rates in the two countries are the same. Using the lognormal assumption, estimate the probability that the exchange rate in three months will be (a) less than 0.7000, (b) between 0.7000 and 0.7500, (c) between 0.7500 and 0.8000, (d) between 0.8000 and 0.8500, (e) between 0.8500 and 0.9000, and (f) greater than 0.9000. Based on the volatility smile usually observed in the market for exchange rates, which of these estimates would you expect to be too low and which would you expect to be too high?

As pointed out in Chapters 5 and 13 an exchange rate behaves like a stock that provides a dividend yield equal to the foreign risk-free rate. Whereas the growth rate in a non-dividend-paying stock in a risk-neutral world is r, the growth rate in the exchange rate in a risk-neutral world is $r - r_f$. Exchange rates have low systematic risks and so we can reasonably assume that this is also the growth rate in the real world. In this case the foreign risk-free rate equals the domestic risk-free rate ($r = r_f$). The expected growth rate in the exchange rate is therefore zero. If S_T is the exchange rate at time T its probability distribution is given by equation (12.2) with $\mu = 0$:

$$\ln S_T \sim \phi(\ln S_0 - \sigma^2 T/2, \sigma^2 T)$$

where S_0 is the exchange rate at time zero and σ is the volatility of the exchange rate. In this case $S_0 = 0.8000$ and $\sigma = 0.12$, and $T = 0.25$ so that

$$\ln S_T \sim \phi(\ln 0.8 - 0.12^2 \times 0.25/2, 0.12^2 \times 0.25)$$

or

$$\ln S_T \sim \phi(-0.2249, 0.0036)$$

(a) $\ln 0.70 = -0.3567$. The probability that $S_T < 0.70$ is the same as the probability that $\ln S_T < -0.3567$. It is

$$N\left(\frac{-0.3567 + 0.2249}{0.06}\right) = N(-2.1955)$$

This is 1.41%.

(b) $\ln 0.75 = -0.2877$. The probability that $S_T < 0.75$ is the same as the probability that $\ln S_T < -0.2877$. It is

$$N\left(\frac{-0.2877 + 0.2249}{0.06}\right) = N(-1.0456)$$

This is 14.79%. The probability that the exchange rate is between 0.70 and 0.75 is therefore $14.79 - 1.41 = 13.38\%$.

(c) $\ln 0.80 = -0.2231$. The probability that $S_T < 0.80$ is the same as the probability that $\ln S_T < -0.2231$. It is

$$N\left(\frac{-0.2231 + 0.2249}{0.06}\right) = N(0.0300)$$

This is 51.20%. The probability that the exchange rate is between 0.75 and 0.80 is therefore $51.20 - 14.79 = 36.41\%$.

(d) $\ln 0.85 = -0.1625$. The probability that $S_T < 0.85$ is the same as the probability that $\ln S_T < -0.1625$. It is

$$N\left(\frac{-0.1625 + 0.2249}{0.06}\right) = N(1.0404)$$

This is 85.09%. The probability that the exchange rate is between 0.80 and 0.85 is therefore $85.09 - 51.20 = 33.89\%$.

(e) $\ln 0.90 = -0.1054$. The probability that $S_T < 0.90$ is the same as the probability that $\ln S_T < -0.1054$. It is

$$N\left(\frac{-0.1054 + 0.2249}{0.06}\right) = N(1.9931)$$

This is 97.69%. The probability that the exchange rate is between 0.85 and 0.90 is therefore $97.69 - 85.09 = 12.60\%$.

(f) The probability that the exchange rate is greater than 0.90 is $100 - 97.69 = 2.31\%$

The volatility smile encountered for foreign exchange options is shown in Figure 18.1 of the text and implies the probability distribution in Figure 18.2. Figure 18.2 suggests that we would expect the probabilities in (a), (c), (d), and (f) to be too low and the probabilities in (b) and (e) to be too high.

Problem 18.16.

The price of a stock is \$40. A six-month European call option on the stock with a strike price of \$30 has an implied volatility of 35%. A six month European call option on the stock with a strike price of \$50 has an implied volatility of 28%. The six-month risk-free rate is 5% and no dividends are expected. Explain why the two implied volatilities are different. Use DerivaGem to calculate the prices of the two options. Use put–call parity to calculate the prices of six-month European put options with strike prices of \$30 and \$50. Use DerivaGem to calculate the implied volatilities of these two put options.

The difference between the two implied volatilities is consistent with Figure 18.3 in the text. For equities the volatility smile is downward sloping. A high strike price option has a lower implied volatility than a low strike price option. The reason is that traders consider that the probability of a large downward movement in the stock price is higher than that predicted by the lognormal probability distribution. The implied distribution assumed by traders is shown in Figure 18.4.

To use DerivaGem to calculate the price of the first option, proceed as follows. Select Equity as the Underlying Type in the first worksheet. Select Analytic European as the Option Type. Input the stock price as 40, volatility as 35%, risk-free rate as 5%, time to exercise as 0.5 year, and exercise price as 30. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 11.155. Change the volatility to 28% and the strike price to 50. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.725.

Put–call parity is

$$c + Ke^{-rT} = p + S_0$$

so that

$$p = c + Ke^{-rT} - S_0$$

For the first option, $c = 11.155$, $S_0 = 40$, $r = 0.054$, $K = 30$, and $T = 0.5$ so that

$$p = 11.155 + 30e^{-0.05 \times 0.5} - 40 = 0.414$$

For the second option, $c = 0.725$, $S_0 = 40$, $r = 0.06$, $K = 50$, and $T = 0.5$ so that

$$p = 0.725 + 50e^{-0.05 \times 0.5} - 40 = 9.490$$

To use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 30. Input the price as 0.414 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 34.99%.

Similarly, to use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 50. Input the price as 9.490 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 27.99%.

These results are what we would expect. DerivaGem gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30. Similarly, it gives the implied volatility of a put with strike price 50 to be almost exactly the same as the implied volatility of a call with a strike price of 50.

Problem 18.17.

"The Black–Scholes model is used by traders as an interpolation tool." Discuss this view.

When plain vanilla call and put options are being priced, traders do use the Black–Scholes model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolating between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black–Scholes to calculate prices for these options. In practice much of the work in producing a table such as Table 18.2 in the over-the-counter market is done by brokers. Brokers often act as intermediaries between participants in the over-the-counter market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table such as Table 18.2 to their clients as a service.

Problem 18.18

Using Table 18.2 calculate the implied volatility a trader would use for an 8-month option with $K/S_0 = 1.04$.

13.45%. We get the same answer by (a) interpolating between strike prices of 1.00 and 1.05 and then between maturities six months and one year and (b) interpolating between maturities of six months and one year and then between strike prices of 1.00 and 1.05.

ASSIGNMENT QUESTIONS

Problem 18.19.

A company's stock is selling for $4. The company has no outstanding debt. Analysts consider the liquidation value of the company to be at least $300,000 and there are 100,000 shares outstanding. What volatility smile would you expect to see?

In liquidation the company's stock price must be at least 300,000/100,000 = $3. The company's stock price should therefore always be at least $3. This means that the stock price distribution that has a thinner left tail and fatter right tail than the lognormal distribution. An upward sloping volatility smile can be expected.

Problem 18.20.

A company is currently awaiting the outcome of a major lawsuit. This is expected to be known within one month. The stock price is currently $20. If the outcome is positive, the stock price is expected to be $24 at the end of one month. If the outcome is negative, it is expected to be $18 at this time. The one-month risk-free interest rate is 8% per annum.

a. *What is the risk-neutral probability of a positive outcome?*
b. *What are the values of one-month call options with strike prices of $19, $20, $21, $22, and $23?*
c. *Use DerivaGem to calculate a volatility smile for one-month call options.*
d. *Verify that the same volatility smile is obtained for one-month put options.*

(a) If p is the risk-neutral probability of a positive outcome (stock price rises to $24), we must have

$$24p + 18(1 - p) = 20e^{0.08 \times 0.0833}$$

so that $p = 0.356$

(b) The price of a call option with strike price K is $(24 - K)pe^{-0.08 \times 0.08333}$ when $K < 24$. Call options with strike prices of 19, 20, 21, 22, and 23 therefore have prices 1.766, 1.413, 1.060, 0.707, and 0.353, respectively.

(c) From DerivaGem the implied volatilities of the options with strike prices of 19, 20, 21, 22, and 23 are 49.8%, 58.7%, 61.7%, 60.2%, and 53.4%, respectively. The volatility smile is therefore a "frown" with the volatilities for deep-out-of-the-money and deep-in-the-money options being lower than those for close-to-the-money options.

(d) The price of a put option with strike price K is $(K - 18)(1 - p)e^{-0.08 \times 0.08333}$. Put options with strike prices of 19, 20, 21, 22, and 23 therefore have prices of 0.640, 1.280, 1.920, 2.560, and 3.200. DerivaGem gives the implied volatilities as 49.81%, 58.68%, 61.69%, 60.21%, and 53.38%. Allowing for rounding errors these are the same as the implied volatilities for put options.

Problem 18.21.

A futures price is currently \$40. The risk-free interest rate is 5%. Some news is expected tomorrow that will cause the volatility over the next three months to be either 10% or 30%. There is a 60% chance of the first outcome and a 40% chance of the second outcome. Use DerivaGem to calculate a volatility smile for three-month options.

The calculations are shown in the following table. For example, when the strike price is 34, the price of a call option with a volatility of 10% is 5.926, and the price of a call option when the volatility is 30% is 6.312. When there is a 60% chance of the first volatility and 40% of the second, the price is $0.6 \times 5.926 + 0.4 \times 6.312 = 6.080$. The implied volatility given by this price is 23.21. The table shows that the uncertainty about volatility leads to a classic volatility smile similar to that in Figure 18.1 of the text. In general when volatility is stochastic with the stock price and volatility uncorrelated we get a pattern of implied volatilities similar to that observed for currency options.

Strike Price	Call Option Price 10% Volatility	Call Option Price 30% Volatility	Weighted Price	Implied Volatility (%)
34	5.926	6.312	6.080	23.21
36	3.962	4.749	4.277	21.03
38	2.128	3.423	2.646	18.88
40	0.788	2.362	1.418	18.00
42	0.177	1.560	0.730	18.80
44	0.023	0.988	0.409	20.61
46	0.002	0.601	0.242	22.43

Problem 18.22.

Data for a number of foreign currencies are provided on the author's Web site:
http://www.rotman.utoronto.ca/~hull/data
Choose a currency and use the data to produce a table similar to Table 18.1.

The following table shows the percentage of daily returns greater than 1, 2, 3, 4, 5, and 6 standard deviations for each currency. The pattern is similar to that in Table 18.1.

	> 1sd	> 2sd	> 3sd	> 4sd	> 5sd	> 6sd
AUD	24.8	5.3	1.3	0.2	0.1	0.0
BEF	24.3	5.7	1.3	0.6	0.2	0.0
CHF	26.1	4.2	1.3	0.6	0.1	0.0
DEM	23.9	5.0	1.4	0.6	0.1	0.0
DKK	26.7	5.8	1.3	0.3	0.0	0.0
ESP	28.2	5.1	0.9	0.3	0.1	0.0
FRF	26.0	5.4	1.4	0.2	0.0	0.0
GBP	23.9	6.4	1.1	0.4	0.1	0.0
ITL	25.4	6.6	1.1	0.2	0.0	0.0
NLG	25.6	5.7	1.7	0.2	0.0	0.0
SEK	28.2	5.2	1.0	0.0	0.0	0.0
Normal	31.7	4.6	0.3	0.0	0.0	0.0

Problem 18.23.

Data for a number of stock indices are provided on the author's Web site:
http://www.rotman.utoronto.ca/~hull/data
Choose an index and test whether a three standard deviation down movement happens more often than a three standard deviation up movement.

The results are shown in the table below.

	> 3sd down	> 3sd up
TSE	0.88	0.22
S&P	0.55	0.44
FTSE	0.55	0.66
CAC	0.33	0.33
Nikkei	0.55	0.66
Total	0.57	0.46

Problem 18.24.

Consider a European call and a European put with the same strike price and time to maturity. Show that they change in value by the same amount when the volatility increases from a level, σ_1, to a new level, σ_2 within a short period of time. (Hint Use put–call parity.)

Define c_1 and p_1 as the values of the call and the put when the volatility is σ_1. Define c_2 and p_2 as the values of the call and the put when the volatility is σ_2. From put–call parity

$$p_1 + S_0e^{-qT} = c_1 + Ke^{-rT}$$
$$p_2 + S_0e^{-qT} = c_2 + Ke^{-rT}$$

If follows that

$$p_1 - p_2 = c_1 - c_2$$

Problem 18.25.

An exchange rate is currently 1.0 and the implied volatilities of six-month European options with strike prices 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, and 1.3 are 13%, 12%, 11%, 10%, 11%, 12%, and 13%. The domestic and foreign risk free rates are both 2.5%. Calculate the implied probability distribution using an approach similar to that used in the appendix for Example 18.2. Compare it with the implied distribution where all the implied volatilities are 11.5%.

Define:

$$\begin{aligned}
g(S_T) &= g_1 \quad \text{for } 0.7 \le S_T < 0.8 \\
g(S_T) &= g_2 \quad \text{for } 0.8 \le S_T < 0.9 \\
g(S_T) &= g_3 \quad \text{for } 0.9 \le S_T < 1.0 \\
g(S_T) &= g_4 \quad \text{for } 1.0 \le S_T < 1.1 \\
g(S_T) &= g_5 \quad \text{for } 1.1 \le S_T < 1.2 \\
g(S_T) &= g_6 \quad \text{for } 1.2 \le S_T < 1.3
\end{aligned}$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a six-month option with a strike price of 0.75 as 12.5%. This means that options with strike prices of 0.7, 0.75, and 0.8 have implied volatilities of 13%, 12.5% and 12%, respectively. From DerivaGem their prices are \$0.2963, \$0.2469, and \$0.1976, respectively. Using equation (18A.1) with $K = 0.75$ and $\delta = 0.05$ we get

$$g_1 = \frac{e^{0.025\times0.5}(0.2963 + 0.1976 - 2 \times 0.2469)}{0.05^2} = 0.0315$$

Similar calculations show that $g_2 = 0.7241$, $g_3 = 4.0788$, $g_4 = 3.6766$, $g_5 = 0.0.7285$, and $g_6 = 0.0898$. The total probability between 0.7 and 1.3 is the sum of these numbers multiplied by 0.1 or 0.9329. If the volatility had been flat at 11.5% the values of g_1, g_2, g_3, g_4, g_5, and g_6 would have been 0.0239, 0.9328, 4.2248, 3.7590, 0.9613, and 0.0938. The total probability between 0.7 and 1.3 is in this case 0.9996. This shows that the volatility smile gives rise to heavy tails for the distribution.

Problem 18.26.

Use Table 18.2 to calculate the implied volatility a trader would use for an 11-month option with $K/S_0 = 0.98$

Interpolation gives the volatility for a six-month option with a strike price of 98 as 12.82%. Interpolation also gives the volatility for a 12-month option with a strike price of 98 as 13.7%. A final interpolation gives the volatility of an 11-month option with a strike price of 98 as 13.55%. The same answer is obtained if the sequence in which the interpolations is done is reversed.

CHAPTER 19
Basic Numerical Procedures

Notes for the Instructor

Chapter 19 presents the standard numerical procedures used to value derivatives when analytic results are not available. These involve binomial/trinomial trees, Monte Carlo simulation, and finite difference methods.

Binomial trees are introduced in Chapter 11, and Section 19.1 and 19.2 can be regarded as a review and more in-depth treatment of that material. When covering Section 19.1, I usually go through in some detail the calculations for a number of nodes in an example such as the one in Figure 19.3. Once the basic tree building and roll back procedure has been covered it is fairly easy to explain how it can be extended to currencies, indices, futures, and stocks that pay dividends. Also the calculation of hedge statistics such as delta, gamma, and vega can be explained. The software DerivaGem is a convenient way of displaying trees in class as well as an important calculation tool for students.

The binomial tree and Monte Carlo simulation approaches use risk-neutral valuation arguments. By contrast, the finite difference method solves the underlying differential equation directly. However, as explained in the book the explicit finite difference method is essentially the same as the trinomial tree method and the implicit finite difference method is essentially the same as a multinomial tree approach where the are $M+1$ branches emanating from each node. Binomial trees and finite difference methods are most appropriate for American options; Monte Carlo simulation is most appropriate for path-dependent options.

Any of Problems 19.25 to 19.30 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 19.1.

Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

Delta, gamma, and theta can be determined from a single binomial tree. Vega is determined by making a small change to the volatility and recomputing the option price using a new tree. Rho is calculated by making a small change to the interest rate and recomputing the option price using a new tree.

Problem 19.2.

Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the risk-free interest rate is 10%

per annum, and the volatility is 45% per annum. Use a binomial tree with a time interval of one month.

In this case, $S_0 = 60$, $K = 60$, $r = 0.1$, $\sigma = 0.45$, $T = 0.25$, and $\Delta t = 0.0833$. Also

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.45\sqrt{0.0833}} = 1.1387$$

$$d = \frac{1}{u} = 0.8782$$

$$a = e^{r\Delta t} = e^{0.1\times 0.0833} = 1.0084$$

$$p = \frac{a-d}{u-d} = 0.4998$$

$$1 - p = 0.5002$$

The output from DerivaGem for this example is shown in the Figure S19.1. The calculated price of the option is \$5.16.

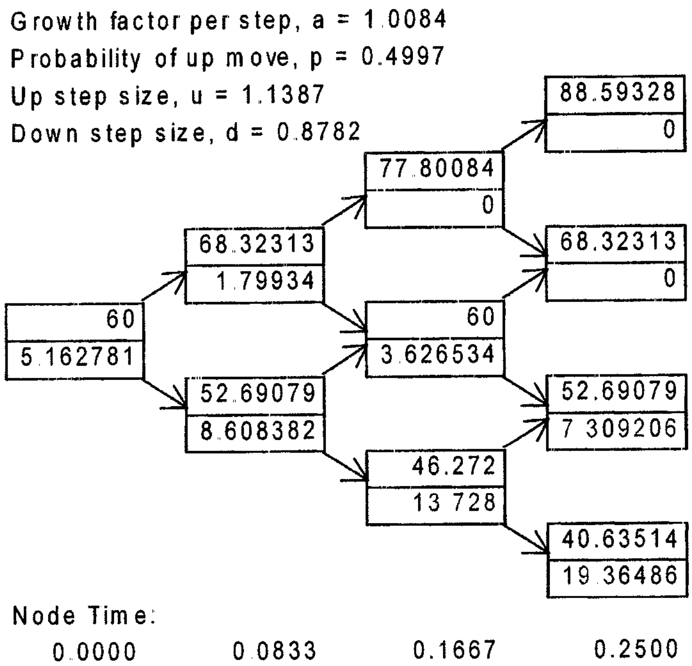

Figure S19.1 Tree for Problem 19.2

Problem 19.3.

Explain how the control variate technique is implemented when a tree is used to value American options.

The control variate technique is implemented by

(a) valuing an American option using a binomial tree in the usual way ($= f_A$).

(b) valuing the European option with the same parameters as the American option using the same tree ($= f_E$).

(c) valuing the European option using Black–Scholes ($= f_{BS}$). The price of the American option is estimated as $f_A + f_{BS} - f_E$.

Problem 19.4.

Calculate the price of a nine-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time interval of three months.

In this case $F_0 = 198$, $K = 200$, $r = 0.08$, $\sigma = 0.3$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$\begin{aligned} u &= e^{0.3\sqrt{0.25}} = 1.1618 \\ d &= \frac{1}{u} = 0.8607 \\ a &= 1 \\ p &= \frac{a-d}{u-d} = 0.4626 \\ 1-p &= 0.5373 \end{aligned}$$

The output from DerivaGem for this example is shown in the Figure S19.2. The calculated price of the option is 20.34 cents.

Problem 19.5.

Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued using the binomial tree approach? Explain your answer.

A binomial tree cannot be used in the way described in this chapter. This is an example of what is known as a history-dependent option. The payoff depends on the path followed by the stock price as well as its final value. The option cannot be valued by starting at the end of the tree and working backward since the payoff at the final branches is not known unambiguously. Chapter 26 describes an extension of the binomial tree approach that can be used to handle options where the payoff depends on the average value of the stock price.

Problem 19.6.

"For a dividend-paying stock, the tree for the stock price does not recombine; but the tree for the stock price less the present value of future dividends does recombine." Explain this statement.

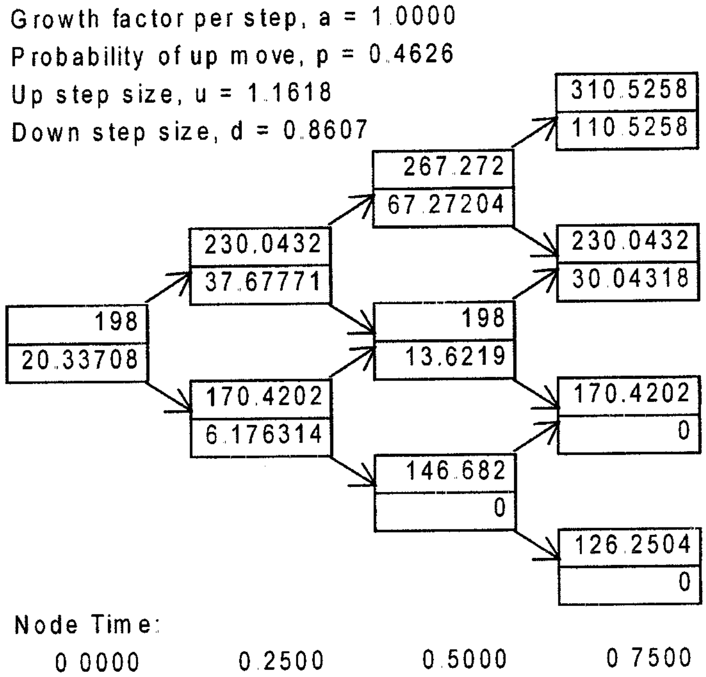

Figure S19.2 Tree for Problem 19.4

Suppose a dividend equal to D is paid during a certain time interval. If S is the stock price at the beginning of the time interval, it will be either $Su - D$ or $Sd - D$ at the end of the time interval. At the end of the next time interval, it will be one of $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$ and $(Sd - D)d$. Since $(Su - D)d$ does not equal $(Sd - D)u$ the tree does not recombine. If S is equal to the stock price less the present value of future dividends, this problem is avoided.

Problem 19.7.

Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 9 holds.

With the usual notation

$$p = \frac{a-d}{u-d}$$

$$1-p = \frac{u-a}{u-d}$$

If $a < d$ or $a > u$, one of the two probabilities is negative. This happens when

$$e^{(r-q)\Delta t} < e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t} > e^{\sigma\sqrt{\Delta t}}$$

This in turn happens when $(q-r)\sqrt{\Delta t} > \sigma$ or $(r-q)\sqrt{\Delta t} > \sigma$ Hence negative probabilities occur when

$$\sigma < |(r-q)\sqrt{\Delta t}|$$

This is the condition in footnote 9.

Problem 19.8.

Use stratified sampling with 100 trials to improve the estimate of π in Business Snapshot 19.1 and Table 19.1.

In Table 19.1 cells A1, A2, A3,..., A100 are random numbers between 0 and 1 defining how far to the right in the square the dart lands. Cells B1, B2, B3,...,B100 are random numbers between 0 and 1 defining how high up in the square the dart lands. For stratified sampling we could choose equally spaced values for the A's and the B's and consider every possible combination. To generate 100 samples we need ten equally spaced values for the A's and the B's so that there are $10 \times 10 = 100$ combinations. The equally spaced values should be 0.05, 0.15, 0.25,..., 0.95. We could therefore set the A's and B's as follows:

$$\text{A1} = \text{A2} = \text{A3} = \ldots = \text{A10} = 0.05$$

$$\text{A11} = \text{A12} = \text{A13} = \ldots = \text{A20} = 0.15$$

$$\ldots$$

$$\ldots$$

$$\text{A91} = \text{A92} = \text{A93} = \ldots = \text{A100} = 0.95$$

and

$$\text{B1} = \text{B11} = \text{B21} = \ldots. = \text{B91} = 0.05$$

$$\text{B2} = \text{B12} = \text{B22} = \ldots = \text{B92} = 0.15$$

$$\ldots$$

$$\ldots$$

$$\text{B10} = \text{B20} = \text{B30} = \ldots = \text{B100} = 0.95$$

We get a value for π equal to 3.2, which is closer to the true value than the value of 3.04 obtained with random sampling in Table 19.1. Because samples are not random we cannot easily calculate a standard error of the estimate.

Problem 19.9.

Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.