9 Calculating Efficient Portfolios

9.1 Overview

This chapter covers the theory and calculations necessary for both versions of the classical capital asset pricing model (CAPM)—both that which is based on a risk-free asset (also known as the Sharp-Lintner-Mossin model) and Black's (1972) zero-beta CAPM (which does not require the assumption of a risk-free asset). You will find that using a spreadsheet enables you to do the necessary calculations easily.

The structure of the chapter is as follows: We begin with some preliminary definitions and notation. We then state the major results (proofs are given in the appendix to the chapter). In succeeding sections we implement these results, showing you:

- How to calculate efficient portfolios.
- How to calculate the efficient frontier.

This chapter includes more theoretical material than most chapters in this book: Section 9.2 contains the propositions on portfolios which underlie the calculations of both efficient portfolios and the security market line (SML) in Chapter 11. If you find the theoretical material in section 9.2 difficult, skip it at first and try to follow the illustrative calculations in section 9.3. This chapter assumes that the variance-covariance matrix is given; we delay a discussion of various methods of computing the variance-covariance matrix until Chapter 10.

9.2 Some Preliminary Definitions and Notation

Throughout this chapter we use the following notation: There are N risky assets, each of which has expected return $E(r_i)$. The matrix $E(r)$ is the column vector of expected returns of these assets:

$$E(r) = \begin{bmatrix} E(r_1) \\ E(r_2) \\ \vdots \\ E(r_N) \end{bmatrix}$$

and S is the $N \times N$ variance-covariance matrix:

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{N1} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{N2} \\ \vdots & & & \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_{NN} \end{bmatrix}$$

A *portfolio of risky assets* (when our intention is clear, we shall just use the word *portfolio*) is a column vector x whose coordinates sum to 1:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \sum_{i=1}^{N} x_i = 1$$

Each coordinate x_i represents the proportion of the portfolio invested in risky asset i.

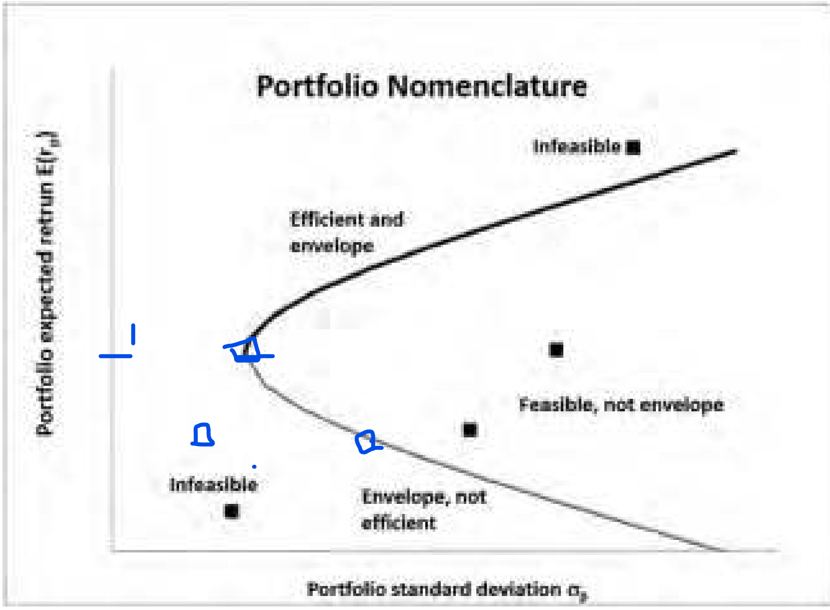
The *expected portfolio return* $E(r_x)$ of a portfolio x is given by the product of x and R:

$$E(r_x) = x^T \cdot R \equiv \sum_{i=1}^{N} x_i E(r_i).$$

The *variance of portfolio* x's *return*, $\sigma_x^2 \equiv \sigma_{xx}$, is given by the product $x^T Sx = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$.

The *covariance between the return of two portfolios x and y*, $Cov(r_x, ry)$, is defined by the product $\sigma_{xy} = x^T Sy = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j \sigma_{ij}$. Note that $\sigma_{xy} = \sigma_{yx}$.

The following graph illustrates four concepts. A *feasible* portfolio is any portfolio whose proportions sum to 1. The *feasible set* is the set of portfolio means and standard deviations generated by the feasible portfolios; this feasible set is the area inside and to the right of the curved line. A feasible portfolio is on the *envelope* of the feasible set if for a given mean return it has minimum variance. Finally, a portfolio x is *an efficient portfolio* if it maximizes the return given the portfolio variance (or standard deviation). That is: x is efficient if there is no other portfolio y such that $E(R_y) > E(R_x)$ and $\sigma_y < \sigma_x$. The set of all efficient portfolios is called the *efficient frontier*; this frontier is the heavier line in the graph.



9.3 Five Propositions on Efficient Portfolios and the CAPM

In the appendix to this chapter we prove the following results, which are basic to the calculations of the CAPM. All of these propositions are used in deriving the efficient frontier and the security market line; numerical illustrations are given in the next section and in succeeding chapters.

PROPOSITION 1 Let c be a constant. We use the notation $E(r) - c$ to denote the following column vector:

$$E(r) - c = \begin{bmatrix} E(r_1) - c \\ E(r_2) - c \\ \vdots \\ E(r_N) - c \end{bmatrix}$$

Let the vector z solve the system of simultaneous linear equations $E(r) - c = Sz$. Then this solution produces a portfolio x on the envelope of the feasible set in the following manner:

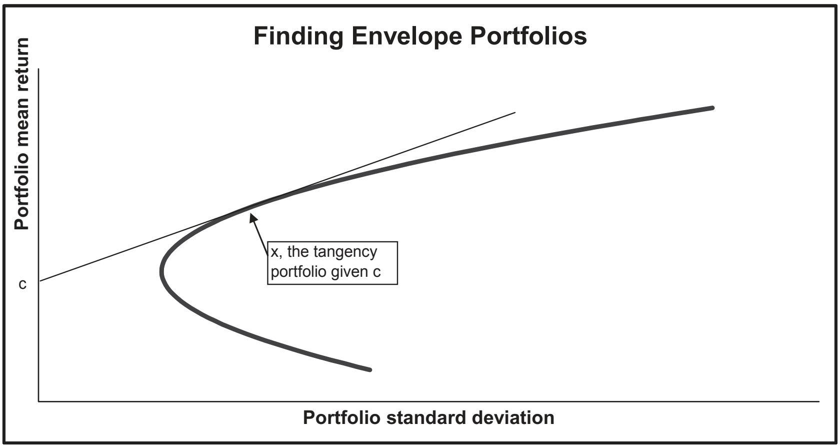
$$z = S^{-1}\{E(r) - c\}$$
$$x = \{x_1, \dots, x_N\},$$

where

$$x_i = \frac{z_i}{\sum_{j=1}^{N} z_j}$$

Furthermore, all envelope portfolios are of this form.

Intuition A formal proof of the proposition is given in the appendix to this chapter, but the intuition is simple and geometric. Suppose we pick a constant c and we try to find an efficient portfolio x for which there is a tangency between c and the feasible set:



Proposition 1 gives a procedure for finding x; furthermore, the proposition states that all envelope portfolios (in particular, all efficient portfolios) are the result of the procedure outlined in the proposition. That is, if x is any envelope portfolio, then there exists a constant c and a vector z such that $Sz = E(r) - c$ and $x = z \Big/ \sum_i z_i$.

PROPOSITION 2 By a theorem first proved by Black (1972), any two envelope portfolios are enough to establish the whole envelope. Given any two envelope portfolios $x = \{x_1, \dots, x_N\}$ and $y = \{y_1, \dots, y_N\}$, all envelope portfolios are convex combinations of x and y. This means that given any constant a, the portfolio

$$ax+(1-a)y=\begin{bmatrix} ax_1+(1-a)y_1 \\ ax_2+(1-a)y_2 \\ \vdots \\ ax_N+(1-a)y_N \end{bmatrix}$$

is an envelope portfolio.

PROPOSITION 3 If y is any envelope portfolio, then for any other portfolio (envelope or not) x, we have the relationship:

$$E(r_x)=c+\beta_x\left[E(r_y)-c\right]$$

where

$$\beta_x=\frac{Cov(x,y)}{\sigma_y^2}$$

Furthermore, c is the expected return of all portfolios z whose covariance with y is zero:

$$c=E(r_z),\text{ where } Cov(y,z)=0$$

Notes If y is on the envelope, the regression of any and all portfolios x on y gives a linear relationship. In this version of the CAPM (usually known as "Black's zero-beta CAPM," in honor of Fischer Black, whose 1972 paper proved this result) the Sharpe-Lintner-Mossin security market line (SML) is replaced with an SML in which the role of the risk-free asset is played by a portfolio with a zero beta with respect to the particular envelope portfolio y. Note that this result is true for any envelope portfolio y.

The converse of Proposition 3 is also true:

PROPOSITION 4 Suppose that there exists a portfolio y such that for any portfolio x the following relation holds:

$$E(r_x)=c+\beta_x\left[E(r_y)-c\right]$$

where

$$\beta_x=\frac{Cov(x,y)}{\sigma_y^2}$$

Then the portfolio y is an envelope portfolio.

Propositions 3 and 4 show that *an SML relation holds if and only if we regress all portfolio returns on an envelope portfolio with an* $R^2 = 100\%$. As Roll (1977, 1978) has forcefully pointed out, these propositions show that it is not enough to run a test of the CAPM by showing that the SML holds.[1] The only real test of the CAPM is *whether the true market portfolio is mean-variance efficient*. We shall return to this topic in Chapter 10.

THE MARKET PORTFOLIO The *market portfolio M* is a portfolio composed of *all the risky assets in the economy*, with each asset taken in proportion to its value. To make this more specific: Suppose that there are N risky assets and that the market value of asset i is V_i. Then the market portfolio has the following weights:

$$Proportion\ of\ asset\ i\ in\ M = \frac{V_i}{\sum_{h=1}^{N} V_h}$$

If the market portfolio M is efficient (this is a big "if" as we shall see in Chapters 11 and 13, Proposition 3 is also true for the market portfolio. That is, the SML holds with $E(r_z)$ substituted for c:

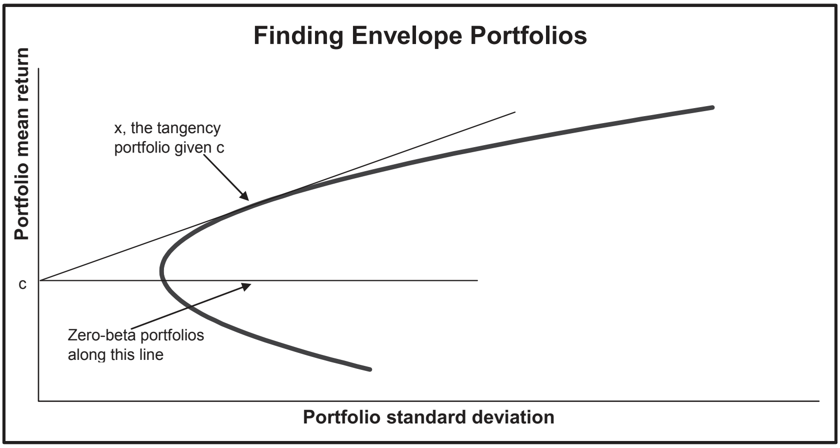
$$E(r_x) = E(r_z) + \beta_x [E(r_M) - E(r_z)]$$

where

$$\beta_x = \frac{Cov(x, M)}{\sigma_M^2} \quad \text{and} \quad Cov(z, M) = 0$$

This version of the SML has received the most empirical attention of all of the CAPM results. In Chapter 11 we show how to calculate β and how to calculate the SML; we go on to examine Roll's criticism of these empirical tests. From the following graph, it is easy to see how to locate a zero-beta portfolio on the envelope of the feasible set:

1. Roll's 1977 paper is more often cited and more comprehensive, but his 1978 paper is much easier to read and intuitive. If you're interested in this literature, start there.



When there is a risk-free asset, Proposition 3 specializes to the security market line of the classic capital asset pricing model:

PROPOSITION 5 If there exists a risk-free asset with return r_f, then there exists an envelope portfolio M such that:

$$E(r_x) = r_f + \beta_x \left[E(r_M) - r_f \right]$$

where

$$\beta_x = \frac{Cov(x, M)}{\sigma_M^2}$$

As shown in the classic papers by Sharpe (1964), Lintner (1965), and Mossin (1966), if all investors choose their portfolios only on the basis of portfolio mean and standard deviation, then the portfolio x of Proposition 5 is the market portfolio M.

In the remainder of this chapter, we explore the meaning of these propositions using numerical examples worked out on Excel.

9.4 Calculating the Efficient Frontier: An Example

In this section we calculate the efficient frontier using Excel. We consider a world with four risky assets having the following expected returns and variance-covariance matrix:

	A	B	C	D	E	F	G	H
1	**CALCULATING THE FRONTIER**							
2	**Variance-covariance, S**					**Mean returns E(r)**	**E(r) minus constant**	
3	0.10	0.01	0.03	0.05		6%	2.00%	<-- =F3-$B$8
4	0.01	0.30	0.06	-0.04		8%	4.00%	
5	0.03	0.06	0.40	0.02		10%	6.00%	
6	0.05	-0.04	0.02	0.50		15%	11.00%	
7								
8	Constant, c	4.00%						

Each cell of the column vector labeled **E(r) minus constant** contains the mean return of the given asset minus the value of the constant c (in this case c = 4%). We use this column in finding the second envelope portfolio below.

We separate our calculations into two parts: In the next subsection we calculate two portfolios on the envelope of the feasible set. In the subsequent subsection we calculate the efficient frontier.

Calculating Two Envelope Portfolios

By Proposition 2, we have to find two efficient portfolios in order to identify the whole efficient frontier. By Proposition 1 each envelope portfolio solves the system $R - c = Sz$ for z. To identify two efficient portfolios, we use two different values for c. For each value of c, we solve for z and then set $x_i = z_i \Big/ \sum_h z_h$ to find an efficient portfolio.

The c's we solve for are arbitrary (see section 9.6), but to make life easy, we first solve this system for $c = 0$. This gives the following results:

	A	B	C	D	E	F	G	H
10	**Computing an envelope portfolio with constant = 0**							
11	z					Envelope portfolio x		
12	0.3861	<-- {=MMULT(MINVERSE(A3:D6),F3:F6)}				0.3553	<-- =A12/SUM($A$12:$A$15)	
13	0.2567					0.2362		
14	0.1688					0.1553		
15	0.2752					0.2532		
16					Sum	1.0000	<-- =SUM(F12:F15)	

The formulas in the cells are:

- For z we use the array function **MMult(MInverse(A3:D6),F3:F6)**. The range A3:D6 contains the variance-covariance matrix and the cells F3:F6 contain the expected returns of the assets.
- For x: Each cell contains the associated value of z divided by the sum of all the z's. Thus, for example, cell F12 contains the formula =**A12/SUM($A$12:$A$15)**.

To find the second envelope portfolio we now solve this system for $c = 0.04$ (cell B8).

	A	B	C	D	E	F	G	H
18	**Computing an envelope portfolio with constant = 4.00%**							
19	**z**					**Envelope portfolio y**		
20	0.0404	<-- {=MMULT(MINVERSE(A3:D6),G3:G6)}				0.0782	<-- =A20/SUM($A$20:$A$23)	
21	0.1386					0.2684		
22	0.1151					0.2227		
23	0.2224					0.4307		
24					Sum	1.0000	<-- =SUM(F20:F23)	

The portfolio y in cells F20:F23 is, by the results of Proposition 1, an envelope portfolio. This vector z associated with y is calculated in a manner similar to that of the first vector, except that the array function in the cells is **MMult(MInverse(A3:D6),G3:G6)**, where G3:G6 contains the vector of expected returns minus the constant 0.04.

To complete the basic calculations, we compute the means, standard deviations, and covariance of returns for portfolios x and y:

	A	B	C	D	E	F	G	H	I	J	K
26	E(x)	9.37%			E(y)	11.30%	<-- {=MMULT(TRANSPOSE(F20:F23),F3:F6)}				
27	Var(x)	0.0862			Var(y)	0.1414	<-- {=MMULT(MMULT(TRANSPOSE(F20:F23),A3:D6),F20:F23)}				
28	Sigma(x)	29.37%			Sigma(y)	37.60%	<-- =SQRT(F27)				
29											
30	Cov(x,y)	0.1040	<-- {=MMULT(MMULT(TRANSPOSE(F12:F15),A3:D6),F20:F23)}								
31	Corr(x,y)	0.9419	<-- =B30/(B28*F28)								

The transpose vectors of x and of y are inserted using the array function **Transpose** (see Chapter 34 for a discussion of array functions). This now enables us to calculate the mean, variance, and covariance as follows:

E(x) uses the array formula **MMult(transpose_x,means)**. Note that we could have also used the function **SumProduct(x,means)**.

Var(x) uses the array formula **MMult(MMult(transpose_x, var_cov),x)**.

Sigma(x) uses the formula **Sqrt(var_x)**.

Cov(x,y) uses the array formula **MMult(MMult(transpose_x,var_cov),y)**.

Corr(x,y) uses the formula **cov(x,y)/(sigma_x*sigma_y)**.

The following spreadsheet illustrates everything that has been done in this subsection:

	A	B	C	D	E	F	G	H
1	**CALCULATING THE FRONTIER**							
2	**Variance-covariance, S**					**Mean returns E(r)**	**E(r) minus constant**	
3	0.10	0.01	0.03	0.05		6%	2.00%	<-- =F3-$B$8
4	0.01	0.30	0.06	-0.04		8%	4.00%	
5	0.03	0.06	0.40	0.02		10%	6.00%	
6	0.05	-0.04	0.02	0.50		15%	11.00%	
7								
8	Constant, c	4.00%						
9								
10	**Computing an envelope portfolio with constant = 0**							
11	**z**					**Envelope portfolio x**		
12	0.3861	<-- {=MMULT(MINVERSE(A3:D6),F3:F6)}				0.3553	<-- =A12/SUM($A$12:$A$15)	
13	0.2567					0.2362		
14	0.1688					0.1553		
15	0.2752					0.2532		
16					Sum	1.0000	<-- =SUM(F12:F15)	
17								
18	**Computing an envelope portfolio with constant = 4.00%**							
19	**z**					**Envelope portfolio y**		
20	0.0404	<-- {=MMULT(MINVERSE(A3:D6),G3:G6)}				0.0782	<-- =A20/SUM($A$20:$A$23)	
21	0.1386					0.2684		
22	0.1151					0.2227		
23	0.2224					0.4307		
24					Sum	1.0000	<-- =SUM(F20:F23)	
25								
26	E(x)	9.37%			E(y)	11.30%	<-- {=MMULT(TRANSPOSE(F2	
27	Var(x)	0.0862			Var(y)	0.1414	<-- {=MMULT(MMULT(TRANS	
28	Sigma(x)	29.37%			Sigma(y)	37.60%	<-- =SQRT(F27)	
29								
30	Cov(x,y)	0.1040	<-- {=MMULT(MMULT(TRANSPOSE(F12:F15),A3:D6),F20:F23)}					
31	Corr(x,y)	0.9419	<-- =B30/(B28*F28)					

Calculating the Envelope

By Proposition 2 of section 9.3, convex combinations of the two portfolios calculated in the previous subsection allow us to calculate the whole envelope of the feasible set (which, of course, includes the efficient frontier). Suppose we let p be a portfolio that has proportion a invested in portfolio x and proportion (1 − a) invested in y. Then—as discussed in Chapter 8—the mean and standard deviation of p's return are:

$$E(r_p) = aE(r_x) + (1-a)E(r_y)$$

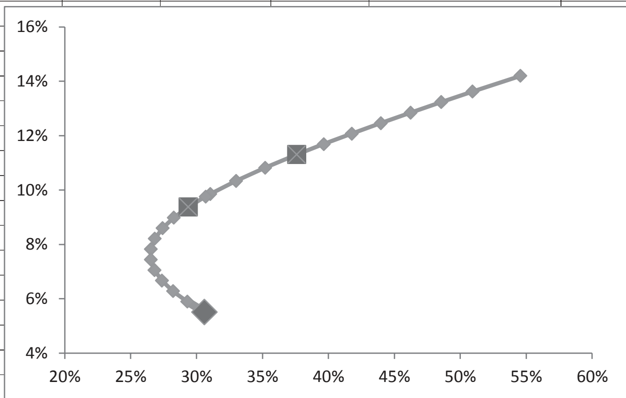
$$\sigma_p = \sqrt{a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + 2a(1-a)Cov(x, y)}$$

Here's a sample calculation for our two portfolios:

	A	B	C	D	E	F	G
34	**A single portfolio calculation**						
35	Proportion of	0.3					
36	$E(r_p)$	10.72%	<-- =B35*B26+(1-B35)*F26				
37	σ_p^2	0.1207	<-- =B35^2*B27+(1-B35)^2*F27+2*B35*(1-B35)*B30				
38	σ_p	34.75%	<-- =SQRT(B37)				

We can turn this calculation into a data table (see Chapter 31) to get the following table:

	A	B	C	D	E	F	G	H	I
34	**A single portfolio calculation**								
35	Proportion of x	0.3							
36	$E(r_p)$	10.72%	<-- =B35*B26+(1-B35)*F26						
37	σ_p^2	0.1207	<-- =B35^2*B27+(1-B35)^2*F27+2*B35*(1-B35)*B30						
38	σ_p	34.75%	<-- =SQRT(B37)						
39									
40	**Data table: We vary the proportion of x to produce a graph of the frontier**								
41	**Proportion of x**	**Sigma**	**Return**						
42		34.75%	10.72%	<-- =B36, data table header					
43	-1.5	54.56%	14.20%						
44	-1.2	50.93%	13.62%						
45	-1.0	48.56%	13.23%						
46	-0.8	46.24%	12.85%						
47	-0.6	43.97%	12.46%						
48	-0.4	41.77%	12.08%						
49	-0.2	39.64%	11.69%						
50	0.0	37.60%	11.30%						
51	0.3	35.20%	10.82%						
52	0.5	33.00%	10.34%						
53	0.8	31.04%	9.86%						
54	0.8	30.68%	9.76%						
55	1.0	29.37%	9.37%						
56	1.2	28.27%	8.99%						
57	1.4	27.42%	8.60%						
58	1.6	26.83%	8.21%						
59	1.8	26.53%	7.83%						
60	2.0	26.52%	7.44%						
61	2.2	26.80%	7.06%						
62	2.4	27.37%	6.67%						
63	2.6	28.21%	6.28%						
64	2.8	29.30%	5.90%						
65	3.0	30.60%	5.51%						



The two portfolios, *x* and *y*, whose convex combinations compose the envelope are marked. Also marked are other portfolios, some of which contain short positions of either *x* or *y*. Note that the convex combinations all lie on the envelope, but may not necessarily be efficient. An example is the last point in the data table—300% in *x* and –200% in portfolio *y*. Thus, while every efficient portfolio is a convex combination of any two efficient portfolios, it is *not true* that every convex combination of any two efficient portfolios is efficient.

9.5 Finding Efficient Portfolios in One Step

The examples in section 9.4 find efficient portfolios by writing out most of the components of the portfolio separately on the spreadsheet. However, for some uses we will want to calculate the efficient portfolio in one step. Here's an example:

	A	B	C	D	E	F	G
1	**FINDING ENVELOPE PORTFOLIOS IN ONE STEP**						
2	**Variance-covariance, S**					**Mean returns E(r)**	
3	0.10	0.01	0.03	0.05		6%	
4	0.01	0.30	0.06	-0.04		8%	
5	0.03	0.06	0.40	0.02		10%	
6	0.05	-0.04	0.02	0.50		15%	
7							
8	Constant	4%					
9							
10	**Envelope portfolio**						
11	0.0782	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)/SUM(MMULT(MINVERSE(A3:D6),F3:F6-B8))}					
12	0.2684						
13	0.2227						
14	0.4307						
15							
16	Portfolio mean	11.30%	<-- =SUMPRODUCT(A11:A14,F3:F6)				
17	Portfolio sigma	37.60%	<-- {=SQRT(MMULT(MMULT(TRANSPOSE(A11:A14),A3:D6),A11:A14))}				

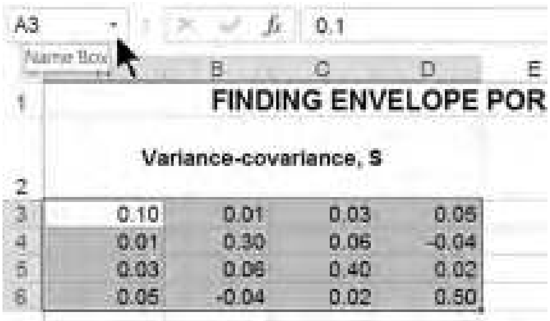
This approach requires a number of Excel tricks, most of which relate to the correct use of array functions. The end result is that we can write the Proposition 1 expression for an envelope portfolio, $x = \frac{S^{-1}\{E(r)-c\}}{Sum\left[S^{-1}\{E(r)-c\}\right]}$, in one cell:

- In cells A11:A14 we have used the array formula F3:F6-B8 to indicate the expected returns minus the constant in cell B8.
- In these same cells we have used **SUM(MMult(MInverse(A3:D6),F3:F6-B8))** to give the denominator of the expression

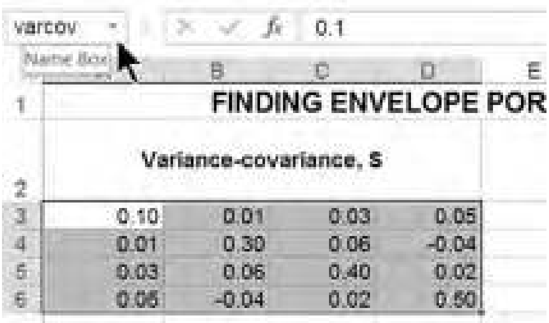
$$x = \frac{S^{-1}\{E(r)-c\}}{Sum\left[S^{-1}\{E(r)-c\}\right]}.$$

Using Cell Names for Clarity

We can make the whole process even clearer by using cell names. To define a cell name, simply mark the cell or a range of cells and go to the **Name Box**, as shown below:



You can now go into the box and type a name:



The name can now be used, as illustrated below:

	A	B	C	D	E	F	G
1	**FINDING ENVELOPE PORTFOLIOS IN ONE STEP**						
2	**Variance-covariance, S**					**Mean returns E(r)**	
3	0.10	0.01	0.03	0.05		6%	
4	0.01	0.30	0.06	-0.04		8%	
5	0.03	0.06	0.40	0.02		10%	
6	0.05	-0.04	0.02	0.50		15%	
7							
8	Constant	4%					
9							
10	**Envelope portfolio**						
11	0.0782						
12	0.2684	<-- {=MMULT(MINVERSE(varcov),means-					
13	0.2227	B8)/SUM(MMULT(MINVERSE(varcov),means-B8))}					
14	0.4307						
15							
16	Portfolio mean	11.30%	<-- =SUMPRODUCT(portx,means)				
17	Portfolio sigma	37.60%	<-- {=SQRT(MMULT(MMULT(TRANSPOSE(portx),varcov),portx))}				

9.6 Three Notes on the Optimization Procedure

In this section we note three additional facts about the optimization procedure of Proposition 1, which leads to the computation of envelope portfolios.

Note 1: All Roads Lead to Rome: Envelope Is Determined by Any Two *c*'s

By Proposition 2, the envelope is determined by any two of its portfolios. This means that for the determination of the envelope it is irrelevant which two portfolios we use. To drive home this point, the spreadsheet below computes three envelope portfolios:

- The envelope portfolio x is computed with a constant $c = 0\%$.
- The envelope portfolio y is computed with a constant $c = 4\%$.
- A third envelope portfolio z is computed with a constant $c = 6\%$ (cells D11:D14). As shown in rows 20–26, portfolio z is composed of a convex combination of x and y. This is true for any x, y, and z.

This little exercise shows that the constants c which determine the envelope are completely arbitrary. Any two constants will determine the same envelope.

	A	B	C	D	E	F	G
1	**CALCULATING THE ENVELOPE All constants c lead to the same envelope**						
2	**Variance-covariance, S**					**Mean returns E(r)**	
3	0.10	0.01	0.03	0.05		6%	
4	0.01	0.30	0.06	-0.04		8%	
5	0.03	0.06	0.40	0.02		10%	
6	0.05	-0.04	0.02	0.50		15%	
7							
8	Constant	0%	4%	6%			
9							
10		**Portfolio x**	**Portfolio y**	**Portfolio z**			
11		0.3553	0.0782	-0.5724	<-- {=MMULT(MINVERSE(varcov),means-D8)/SUM(MMULT(MINVERSE(varcov),means-D8))}		
12		0.2362	0.2684	0.3439			
13		0.1553	0.2227	0.3811			
14		0.2532	0.4307	0.8474			
15							
16	Portfolio mean	9.37%	11.30%	15.84%	<-- =SUMPRODUCT(portfolioz,means)		
17	Portfolio sigma	29.37%	37.60%	65.20%	<-- {=SQRT(MMULT(MMULT(TRANSPOSE(portfolioz),varcov),portfolioz))}		
18							
19							
20	**Show: portfolio z is a linear proportion of portfolio x and portfolio y**						
21	Proportion	-2.34822	<-- =(D11-C11)/(B11-C11)				
22	Check						
23	z1	-0.5724	<-- =$B$21*B11+(1-$B$21)*C11				
24	z2	0.3439	<-- =$B$21*B12+(1-$B$21)*C12				
25	z3	0.3811	<-- =$B$21*B13+(1-$B$21)*C13				
26	z4	0.8474	<-- =$B$21*B14+(1-$B$21)*C14				

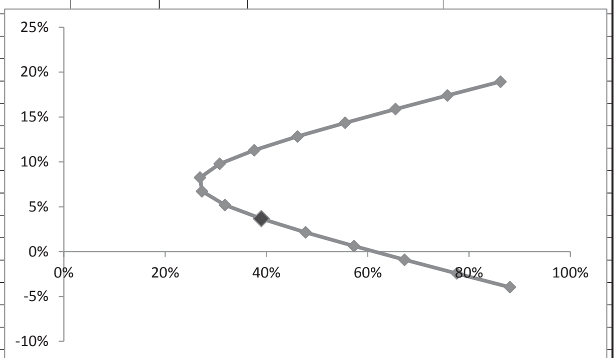
Note 2: Some Values of *c* Locate Non-Efficient Envelope Portfolios

The optimization procedure of Proposition 1 locates a portfolio *x,* which has proportions:

$$x = \frac{S^{-1}\{E(r)-c\}}{Sum\left[S^{-1}\{E(r)-c\}\right]}$$

Although always on the envelope, this portfolio is not necessarily efficient, as is shown in the example below, where a constant $c = 0.11$ leads to an inefficient portfolio.

	A	B	C	D	E	F	G	H
8	Constant	11%	4%					
9								
10		**Portfolio x**	**Portfolio y**					
11		1.1728	0.0782					
12		0.1413	0.2684					
13		-0.0437	0.2227					
14		-0.2704	0.4307					
15								
16	Portfolio mean	3.67%	11.30%					
17	Portfolio sigma	39.01%	37.60%					
18	Cov(x,y)	-0.00631						
19								
20	Single portfolio calculation							
21	Proportion of x	0.6						
22	Mean	6.73%	<-- =B21*B16+(1-B21)*C16					
23	Sigma	27.27%	<-- =SQRT(B21^2*B17^2+(1-B21)^2*C17^2+2*B21*(1-B21)*B18)					
24								
25								
26	**Data table to determine the frontier**							
27	**Proportion of x**	**Sigma**	**Mean**					
28		27.27%	6.73%					
29	-1.0	86.20%	18.93%					
30	-0.8	75.74%	17.41%					
31	-0.6	65.49%	15.88%					
32	-0.4	55.55%	14.36%					
33	-0.2	46.12%	12.83%					
34	0.0	37.60%	11.30%					
35	0.2	30.75%	9.78%					
36	0.4	26.87%	8.25%					
37	0.6	27.27%	6.73%					
38	0.8	31.78%	5.20%					
39	1.0	39.01%	3.67%					
40	1.2	47.73%	2.15%					
41	1.4	57.27%	0.62%					
42	1.6	67.27%	-0.90%					
43	1.8	77.57%	-2.43%					
44	2.0	88.05%	-3.96%					



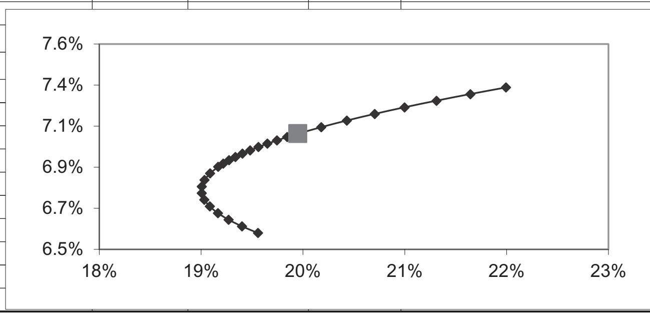
Note 3: The Portfolio Associated with $c = r_f$ Is Optimal

We've said all this before in our discussion of Proposition 1, but it's worth repeating.[2] If we set c to be equal to the risk-free rate of interest, and if the resulting optimizing portfolio $x = \dfrac{S^{-1}\{E(r)-c\}}{Sum\left[S^{-1}\{E(r)-c\}\right]}$ is efficient, then this portfolio is the optimal investment portfolio for an investor whose preferences are defined solely in terms of the mean and standard deviation of portfolio

2. And it forms the basis of our discussion of the Black-Litterman model in Chapter 13.

returns. In the example below, we assume that r_f = 4%. Locating the optimizing portfolio $x = \dfrac{S^{-1}\{E(r)-c\}}{Sum\left[S^{-1}\{E(r)-c\}\right]}$ on the envelope shows that it is efficient. Therefore, the *optimal investment portfolio* for this case is given by x.

	A	B	C	D	E	F	G
1	**IF c = r_f AND THE OPTIMIZING PORTFOLIO IS EFFICIENT, THEN THE ENVELOPE PORTFOLIO IS OPTIMAL** **The portfolio *x* determined by the constant c = 4% is optimal**						
2	**Variance-covariance matrix**					**Expected returns**	
3	0.40	0.03	0.02	0.00		0.06	
4	0.03	0.20	0.00	-0.06		0.05	
5	0.02	0.00	0.30	0.03		0.07	
6	0.00	-0.06	0.03	0.10		0.08	
7							
8	Constant	0.04					
9							
10	**Computing an envelope portfolio with constant = 0.04**						
11	**z**					**Envelope portfolio x**	
12	0.0330	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)}				0.0423	<-- {=A12:A15/SUM(A12:A15)}
13	0.1959					0.2514	
14	0.0468					0.0601	
15	0.5035					0.6462	
16					Sum	1.0000	<-- =SUM(F12:F15)
17							
18							
19	$E(r_x)$	0.0710					
20	σ_x	0.1995					
21							
22							
23							
24							
25							
26							
27							
28							
29							
30							



9.7 Finding the Market Portfolio: The Capital Market Line (CML)

Suppose a risk-free asset exists, and suppose that this asset has expected return r_f. Let M be the efficient portfolio which is the solution to the system of equations:

$$E(r) - r_f = Sz$$

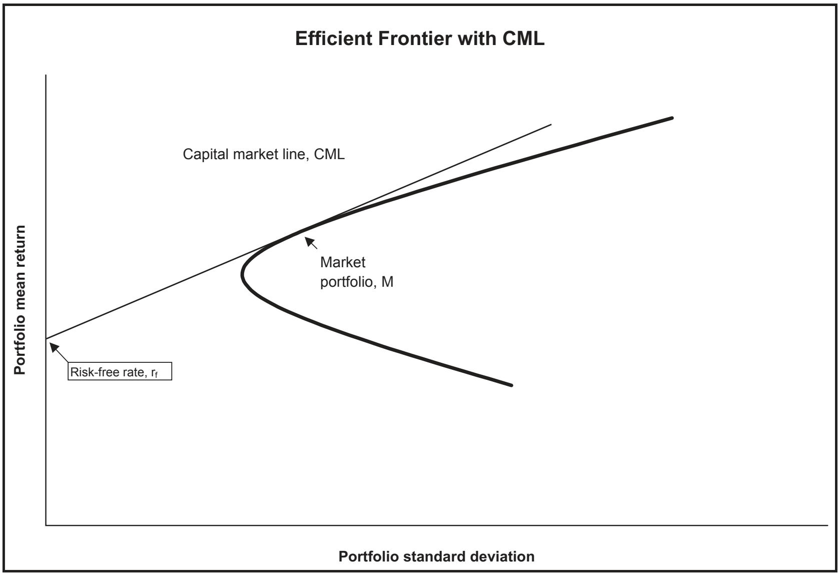
$$M_i = \frac{z_i}{\sum_{i=1}^{N} z_i}$$

Now consider a convex combination of the portfolio M and the risk-free asset r_f; for example, suppose that the weight of the risk-free asset in such a portfolio is a. It follows from the standard equations for portfolio return and σ that:

$$E(r_p) = ar_f + (1-a)E(r_M)$$

$$\sigma_p = \sqrt{a^2\sigma_{r_f}^2 + (1-a)^2\sigma_M^2 + 2a(1-a)Cov(r_f, r_y)} = (1-a)\sigma_M$$

The locus of all such combinations for $a \geq 0$ is known as the *capital market line* (CML). It is graphed below along with the efficient frontier:



The portfolio M is called the *market portfolio* for several reasons:

• Suppose investors agree about the statistical portfolio information (i.e., the vector of expected returns $E(r)$ and the variance-covariance matrix S). Suppose furthermore that investors are interested only in maximizing expected portfolio return given portfolio standard deviation σ. Then it follows that *all optimal portfolios will lie on the CML.*

- In the case above, it further follows that *the portfolio M is the only portfolio of risky assets included in any optimal portfolio*. *M* must therefore include *all the risky assets, with each asset weighted in proportion to its market value*. That is:

$$\textit{weight of risky asset i in portfolio M} = \frac{V_i}{\sum_{i=1}^{N} V_i}$$

where V_i *is the market value of asset i*

It is not difficult to find *M* when we know r_f: We merely have to solve for the efficient portfolio given that the constant $c = r_f$. When r_f changes, we get a different "market" portfolio—this is just the efficient portfolio given a constant of r_f. For example, in our numerical example, suppose that the risk-free rate is $r_f = 5\%$. Then solving the system $E(r) - r_f = Sz$ gives:

	A	B	C	D	E	F	G	H
1	**WHEN c = r$_f$, THE ENVELOPE PORTFOLIO IS THE MARKET PORTFOLIO M**							
2	**Variance-covariance matrix**					**Expected returns E(r)**		
3	0.40	0.03	0.02	0.00		0.06		
4	0.03	0.20	0.00	-0.06		0.05		
5	0.02	0.00	0.30	0.03		0.07		
6	0.00	-0.06	0.03	0.10		0.08		
7								
8	Constant	0.05						
9								
10	**Envelope portfolio is market portfolio M**							
11	0.0314	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)/SUM(MMULT(MINVERSE(A3:D6),F3:F6-B8))}						
12	0.2059							
13	0.0597							
14	0.7031							
15								
16	Portfolio expected return, E(r$_M$)	7.26%	<-- =SUMPRODUCT(A11:A14,F3:F6)					
17	Portfolio standard devation, σ_M	21.21%	<-- {=SQRT(MMULT(MMULT(TRANSPOSE(A11:A14),A3:D6),A11:A14))}					

9.8 Testing the SML—Implementing Propositions 3–5

To illustrate Propositions 3–5 consider the following data for four risky assets:

	A	B	C	D	E	F
1	**ILLUSTRATING PROPOSITIONS 3-5**					
2	**Dates**	**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**	
3	1	-6.63%	-2.49%	-4.27%	11.72%	
4	2	8.53%	2.44%	-3.15%	-8.33%	
5	3	1.79%	4.46%	1.92%	19.18%	
6	4	7.25%	17.90%	-6.53%	-7.41%	
7	5	0.75%	-8.22%	-1.76%	-1.44%	
8	6	-1.57%	0.83%	12.88%	-5.92%	
9	7	-2.10%	5.14%	13.41%	-0.46%	
10						
11	Mean	1.15%	2.87%	1.79%	1.05%	<-- =AVERAGE(E3:E9)

The asset returns on seven dates are given in rows 3–7, and the average return is given in row 11.

We use some sophisticated array functions to compute the variance-covariance matrix:

	A	B	C	D	E	F	G
13	**Variance-covariance matrix**						
14		Asset 1	Asset 2	Asset 3	Asset 4		
15	Asset 1	0.0024	0.0019	-0.0015	-0.0024		
16	Asset 2	0.0019	0.0056	-0.0007	-0.0016	cells B15:E18 contain the formula	
17	Asset 3	-0.0015	-0.0007	0.0057	-0.0005	{=MMULT(TRANSPOSE(B3:E9-B11:E11),B3:E9-	
18	Asset 4	-0.0024	-0.0016	-0.0005	0.0094	B11:E11)/7}	
19							
20	**Finding an efficient portfolio w**						
21	Constant	0.50%					
22							
23	Asset 1	0.3129					
24	Asset 2	0.2464		Cells B23:B26 contain the formula			
25	Asset 3	0.2690		{=MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-			
26	Asset 4	0.1717		B21)/SUM(MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-B21))}			

The efficient portfolio given the constant $c = 0.5\%$ is given in cells B23:B26; we compute this portfolio using the method of Proposition 1.[3] We call this portfolio w. The returns of portfolio w on dates 1–7 are given in column G below:

3. Following the discussion in section 9.6, a careful reader will recall that Proposition 1 only guarantees that this portfolio is on the envelope. But it is, in fact, efficient.

	A	B	C	D	E	F	G	H
2	**Dates**	**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**		**Efficient portfolio w**	
3	1	-6.63%	-2.49%	-4.27%	11.72%		-1.82%	<-- {=MMULT(B3:E9,B23:B26)}
4	2	8.53%	2.44%	-3.15%	-8.33%		0.99%	
5	3	1.79%	4.46%	1.92%	19.18%		5.47%	
6	4	7.25%	17.90%	-6.53%	-7.41%		3.65%	
7	5	0.75%	-8.22%	-1.76%	-1.44%		-2.51%	
8	6	-1.57%	0.83%	12.88%	-5.92%		2.16%	
9	7	-2.10%	5.14%	13.41%	-0.46%		4.14%	
10								
11	Mean	1.15%	2.87%	1.79%	1.05%	<-- =AVERAGE(E3:E9)	1.73%	

We illustrate Propositions 3–5 in two steps:

- Step 1: We regress the returns of each asset on the returns of the efficient portfolio: For $i = 1, \ldots, 4$ we run the regression $r_{it} = \alpha_i + \beta_i r_{wt} + \varepsilon_{it}$. This regression is often called the *first pass regression*. The results are given below.

	A	B	C	D	E	F	G
29	**Implementing propositions 3-5—finding the SML**						
30	Step 1: Regress each asset's returns on those of the efficient portfolio w						
31		**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**		
32	Alpha	0.0024	-0.0047	-0.0002	0.0028	<-- =INTERCEPT(E3:E9,$G$3:$G$9)	
33	Beta	0.5284	1.9301	1.0490	0.4478	<-- =SLOPE(E3:E9,$G$3:$G$9)	
34	R-squared	0.0897	0.5241	0.1505	0.0167	<-- =RSQ(E3:E9,$G$3:$G$9)	

- Step 2: We now regress the betas of the assets on their mean returns. Running this regression, $\overline{r}_i = \gamma_0 + \gamma_1\beta_i + \varepsilon_i$, gives:

	A	B	C	D	E
36	Step 2: Regress the asset mean returns on their betas				
37	Intercept	0.005	<-- =INTERCEPT(B11:E11,B33:E33)		
38	Slope	0.0123	<-- =SLOPE(B11:E11,B33:E33)		
39	R-squared	1.0000	<-- =RSQ(B11:E11,B33:E33)		

To check the results of Propositions 3–5, we run a test:

	A	B	C	D	E
41	**Check Propositions 3 & 4: Step 2 coefficients should be: Intercept = c, Slope = E(r_w) - c**				
42	Intercept = c ?	yes	<-- =IF(B36=B20,"yes","no")		
43	Slope = E(r_w) - c ?	yes	<-- =IF(B38=G11-B21,"yes","no")		

The "perfect" regression results (note the $R^2 = 1$ in cell B39) are the results promised us by Propositions 3–5:

- The second-pass regression intercept is equal to c and the slope is equal to $E(r_w) - c$.
- If there is a riskless asset with return $c = r_f$, then Proposition 5 promises that in the second-pass regression $\overline{r_i} = \gamma_0 + \gamma_1 \beta_i + \varepsilon_i$, $\gamma_0 = r_f$ and $\gamma_1 = E(r_w) - r_f$.
- If there is no riskless asset, then Proposition 3 states that in the second-pass regression $\gamma_0 = E(r_z)$ and $\gamma_1 = E(r_w) - E(r_z)$, where z is a portfolio whose covariance with w is zero.
- Finally, if we run a two-stage regression of the type described on *any portfolio w* and get a "perfect regression," then Proposition 4 guarantees that w is in fact efficient.

To drive home the point that this technique always works, we show you all the calculations using a different value for c (cell B21, highlighted below). As proved in Propositions 3–5, the result is still a perfect regression of the means on the betas:

	A	B	C	D	E	F	G	H
1	**ILLUSTRATING PROPOSITIONS 3-5** **This time the constant is 2% (cell B21)**							
2	**Dates**	**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**		Efficient portfolio w	
3	1	-6.63%	-2.49%	-4.27%	11.72%		-2.95%	<-- {=MMULT(B3:E9,B23:B26)}
4	2	8.53%	2.44%	-3.15%	-8.33%		3.64%	
5	3	1.79%	4.46%	1.92%	19.18%		5.16%	
6	4	7.25%	17.90%	-6.53%	-7.41%		-2.40%	
7	5	0.75%	-8.22%	-1.76%	-1.44%		2.24%	
8	6	-1.57%	0.83%	12.88%	-5.92%		0.01%	
9	7	-2.10%	5.14%	13.41%	-0.46%		-0.26%	
10								
11	Mean	1.15%	2.87%	1.79%	1.05%	<-- =AVERAGE(E3:E9)	0.78%	
12								
13	**Variance-covariance matrix**							
14		Asset 1	Asset 2	Asset 3	Asset 4			
15	Asset 1	0.0024	0.0019	-0.0015	-0.0024	<-- {=MMULT(TRANSPOSE(B3:E9-B11:E11),B3:E9-B11:E11)/7}		
16	Asset 2	0.0019	0.0056	-0.0007	-0.0016			
17	Asset 3	-0.0015	-0.0007	0.0057	-0.0005			
18	Asset 4	-0.0024	-0.0016	-0.0005	0.0094			
19								
20	**Finding an efficient portfolio w**							
21	Constant	2.00%						
22								
23	Asset 1	0.8234	<-- {=MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-B21)/SUM(MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-B21))}					
24	Asset 2	-0.2869						
25	Asset 3	0.2278						
26	Asset 4	0.2357						
27								
28								
29	**Implementing propositions 3-5--finding the SML**							
30	Step 1: Regress each asset's returns on those of the efficient portfolio w							
31		**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**			
32	Alpha	0.0061	0.0342	0.0165	0.0044	<-- =INTERCEPT(E3:E9,$G$3:$G$9)		
33	Beta	0.6968	-0.7075	0.1752	0.7776	<-- =SLOPE(E3:E9,$G$3:$G$9)		
34	R-squared	0.1570	0.0709	0.0042	0.0506	<-- =RSQ(E3:E9,$G$3:$G$9)		
35								
36	Step 2: Regress the asset mean returns on their betas							
37	Intercept	0.02	<-- =INTERCEPT(B11:E11,B33:E33)					
38	Slope	-0.0122	<-- =SLOPE(B11:E11,B33:E33)					
39	R-squared	1.0000	<-- =RSQ(B11:E11,B33:E33)					
40								
41	**Check Propositions 3 & 4: Step 2 coefficients should be:** **Intercept = c, Slope = $E(r_w)$ - c**							
42	Intercept = c ?	yes	<-- =IF(B36=B20,"yes","no")					
43	Slope = $E(r_w)$ - c ?	yes	<-- =IF(B38=G11-B21,"yes","no")					

9.9 Summary

In this chapter we have presented theorems relating to efficient portfolios and then showed how to implement these theorems to find the efficient frontier. Two basic propositions allow us to derive portfolios on the envelope of the feasible set of portfolios and the envelope itself. Three further propositions relate the expected returns of any asset or portfolio to the expected returns on any efficient portfolio. Under certain circumstances, this allows us to derive the security market line (SML) and the capital market line (CML) of the classic capital asset pricing model (CAPM).

In subsequent chapters we discuss the implementation of the CAPM. We show how to compute the variance-covariance matrix (Chapter 10), how to test the SML (Chapter 11), how to optimize in the presence of short-sale constraints (Chapter 12), and how to derive useful portfolio optimization routines from our knowledge of efficient set mathematics (Chapter 13, which discusses the Black-Litterman model).

Exercises

1. Consider the data below for six furniture companies.

	A	B	C	D	E	F	G	H	I
2	**Variance-covariance matrix**	**La-Z-Boy**	**Kimball**	**Flexsteel**	**Leggett**	**Miller**	**Shaw**		**Means**
3	**La-Z-Boy**	0.1152	0.0398	0.1792	0.0492	0.0568	0.0989		29.24%
4	**Kimball**	0.0398	0.0649	0.0447	0.0062	0.0349	0.0269		20.68%
5	**Flexsteel**	0.1792	0.0447	0.3334	0.0775	0.0886	0.1487		25.02%
6	**Leggett**	0.0492	0.0062	0.0775	0.1033	0.0191	0.0597		31.64%
7	**Miller**	0.0568	0.0349	0.0886	0.0191	0.0594	0.0243		15.34%
8	**Shaw**	0.0989	0.0269	0.1487	0.0597	0.0243	0.1653		43.87%

a. Given this matrix, and assuming that the risk-free rate is 0%, calculate the efficient portfolio of these six firms.

b. Repeat, assuming that the risk-free rate is 10%.

c. Use these two portfolios to generate an efficient frontier for the six furniture companies. Plot this frontier.

d. Is there an efficient portfolio with only positive proportions of all the assets?

2. A sufficient condition to produce positively weighted efficient portfolios is that the variance-covariance matrix be diagonal: That is, that $\sigma_{ij} = 0$, for $i \neq j$. By continuity, positively weighted portfolios will result if the off-diagonal elements of the variance-covariance matrix are sufficiently small compared to the diagonal. Consider a transformation of the above matrix in which:

$$\sigma_{ij} = \begin{cases} \varepsilon\sigma_{ij}^{original} & \text{if } i \neq j \\ \sigma_{ii}^{original} \end{cases}$$

When $\varepsilon = 1$, this transformation will give the original variance-covariance matrix and when $\varepsilon = 0$, the transformation will give a fully diagonal matrix.

For $r = 10\%$ find the maximum ε for which all portfolio weights are positive.

3. In the example below, use Excel to find an envelope portfolio whose β with respect to the efficient portfolio y is zero. *Hint*: Notice that because the covariance is linear, so is β: Suppose that $z = \lambda x + (1 - \lambda)y$ is a convex combination of x and y, and that we are trying to find the β_z. Then

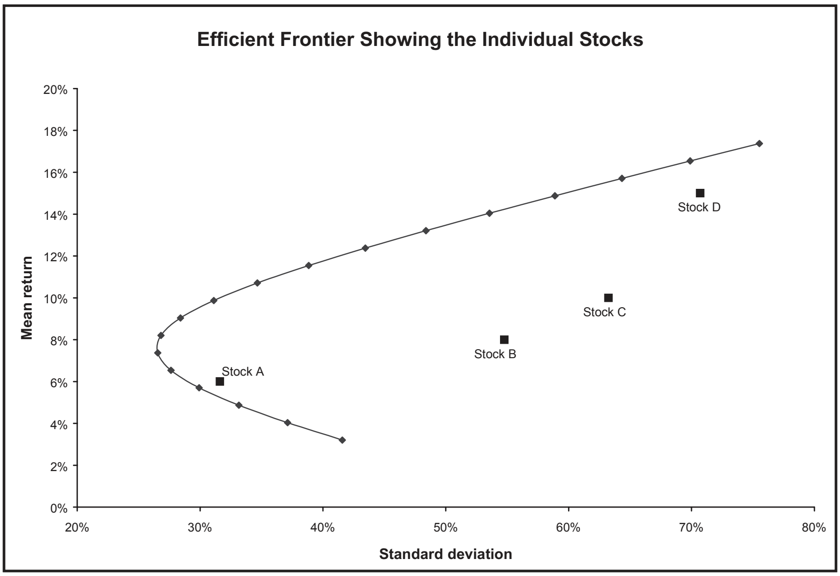
$$\beta_z = \frac{Cov(z, y)}{\sigma_y^2} = \frac{Cov(\lambda x + (1-\lambda)y, y)}{\sigma_y^2}$$
$$= \frac{\lambda Cov(x, y)}{\sigma_y^2} + \frac{(1-\lambda)Cov(y, y)}{\sigma_y^2} = \lambda\beta_x + (1-\lambda)$$

	A	B	C	D	E	F
1	Variance-covariance matrix					Mean returns
2	0.400	0.030	0.020	0.000		0.06
3	0.030	0.200	0.001	-0.060		0.05
4	0.020	0.001	0.300	0.030		0.07
5	0.000	-0.060	0.030	0.100		0.08

4. Calculate the envelope set for the four assets below and show that the individual assets all lie within this envelope set.

	A	B	C	D	E	F
1	A FOUR-ASSET PORTFOLIO PROBLEM					
2	Variance-covariance					Mean returns
3	0.10	0.01	0.03	0.05		6%
4	0.01	0.30	0.06	-0.04		8%
5	0.03	0.06	0.40	0.02		10%
6	0.05	-0.04	0.02	0.50		15%

You should get a graph which looks something like the following:



Mathematical Appendix

In this appendix we collect the various proofs of statements made in the chapter. As in the chapter, we assume that we are examining data for N risky assets. It is important to note that all the definitions of "feasibility" and "optimality" are made relative to this set. Thus the phrase "efficient" really means "efficient relative to the set of the N assets being examined."

PROPOSITION 0 The set of all feasible portfolios of risky assets is convex.

Proof A portfolio x is feasible if and only if the proportions of the portfolio add up to 1; i.e., $\sum_{i=1}^{N} x_i = 1$, where N is the number of risky assets. Suppose that x and y are feasible portfolios and suppose that λ is some number between 0 and 1. Then it is clear that $z = \lambda x + (1 - \lambda)y$ is also feasible.

PROPOSITION 1 Let c be a constant and denote by R the vector of mean returns. A portfolio x is on the envelope relative to the sample set of N assets if and only if it is the normalized solution of the system:

$$R - c = Sz$$

$$x_i = \frac{z_i}{\sum_h z_h}$$

Proof A portfolio x is on the envelope of the feasible set of portfolios if and only if it lies on the tangency of a line connecting some point c on the y-axis to the feasible set. Such a portfolio must either maximize or minimize the ratio $\frac{x(R-c)}{\sigma^2(x)}$, where $x(R - c)$ is the vector product which gives the portfolio's expected excess return over c, and $\sigma^2(x)$ is the portfolio's variance. Let this ratio's value, when maximized (or minimized), be λ. Then our portfolio must satisfy

$$\frac{x(R-c)}{\sigma^2(x)} = \lambda$$

$$\Rightarrow x(R-c) = \sigma^2(x)\lambda = xSx^T\lambda$$

Let h be a particular asset and differentiate this last expression with respect to x_h. This gives: $\bar{R}_h - c = Sx^T\lambda$. Writing $z_h = \lambda x_h$, we see that a portfolio is efficient if and only if it solves the system $R - c = Sz$. Normalizing z so that its coordinates add to 1 gives the desired result.

PROPOSITION 2 The convex combination of any two envelope portfolios is on the envelope of the feasible set.

Proof Let x and y be portfolios on the envelope. By the above theorem, it follows that there exist two vectors, z_x and z_y, and two constants c_x and c_y, such that:

- x is the normalized-to-unity vector of z_x; i.e., $x_i = \frac{z_{xi}}{\sum_h z_{xh}}$, and y is the normalized-to-unity vector of z_y.
- $R - c_x = Sz_x$ and $R - c_y = Sz_y$

Furthermore, since z maximizes the ratio $\frac{z(R-c)}{\sigma^2(z)}$, it follows that any normalization of z also maximizes this ratio. With no loss in generality, therefore, we can assume that z sums to 1.

It follows that for any real number a the portfolio $az_x + (1-a)z_y$ solves the system $R - (ac_x + (1-a)c_y) = Sz$. This proves our claim.

PROPOSITION 3 Let y be any envelope portfolio of the set of N assets. Then for any other portfolio x (including, possibly, a portfolio composed of a single asset) there exists a constant c such that the following relation holds between the expected return on x and the expected return on portfolio y:

$$E(r_x) = c + \beta_x\left[E(r_y) - c\right]$$

where

$$\beta_x = \frac{Cov(x,y)}{\sigma_y^2}$$

Furthermore, $c = E(r_z)$, where z is any portfolio for which $Cov(z,y) = 0$.

Proof Let y be a particular envelope portfolio and let x be any other portfolio. We assume that both portfolios x and y are column vectors. Note that

$$\beta_x = \frac{Cov(x,y)}{\sigma_y^2} = \frac{x^T S y}{y^T S y}$$

Now since y is on the envelope, we know that there exist a vector w and a constant c which solves the system $Sw = R - c$ and that $y = w \Big/ \sum_i w_i = w/a$. Substituting this in the expression for β_x, we get:

$$\beta_x = \frac{Cov(x,y)}{\sigma_y^2} = \frac{x^T S y}{y^T S y} = \frac{x^T(R-c)/a}{y^T(R-c)/a} = \frac{x^T(R-c)}{y^T(R-c)}$$

Next note that since $\sum_i x_i = 1$, it follows that $x^T I(R - c) = E(r_x) - c$ and that $y^T I(R - c) = E(r_y) - c$. This shows that

$$\beta_x = \frac{E(r_x) - c}{E(r_y) - c}$$

which can be rewritten as:

$$E(r_x) = c + \beta_x\left[E(r_y) - c\right]$$

To finish the proof, let z be a portfolio which has zero covariance with y. Then the above logic shows that $c = E(r_z)$. This proves the claim.

PROPOSITION 4 If in addition to the N risky assets, there exists a risk-free asset with return r_f, then the standard security market line holds:

$$E(r_x) = r_f + \beta_x\left[E(r_M) - r_f\right],$$

where

$$\beta_x = \frac{Cov(x,M)}{\sigma_M^2}$$

Proof If there exists a risk-free security, then the tangent line from this security to the efficient frontier dominates all other feasible portfolios. Call the point of tangency on the efficient frontier M; then the result follows.

Note: It is important to repeat again that the terminology "Market portfolio" refers in this case to the "Market portfolio relative to the sample set of N assets."

PROPOSITION 5 Suppose that there exists a portfolio y such that for any portfolio x the following relation holds:

$$E(r_x) = c + \beta_x [E(r_y) - c]$$

where

$$\beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Then the portfolio y is on the envelope.

Proof Substituting in for the definition of β_x it follows that for any portfolio x the following relation holds:

$$\frac{x^T S y}{\sigma_y^2} = \frac{x^T R - c}{y^T R - c}$$

Let x be the vector composed solely of the first risky asset: $x = \{1, 0, \dots, 0\}$. Then the above equation becomes:

$$S_1 y \frac{y^T R - c}{\sigma_y^2} = E(r_1) - c$$

which we write:

$$S_1 a y = E(r_1) - c$$

where S_1 is the first row of the variance-covariance matrix S. Note that $a = \dfrac{y^T R - c}{\sigma_y^2}$ is a constant whose value is independent of the vector x. If we let x be a vector composed solely of the ith risky asset, we get:

$$S_i a y = E(r_i) - c$$

This proves that the vector $z = ay$ solves the system $Sz = R - c$; by Proposition 1 this means that the normalization of z is on the envelope. But this normalization is simply the vector y.

10 Calculating the Variance-Covariance Matrix

10.1 Overview

In order to calculate efficient portfolios, we must be able to compute the variance-covariance matrix from return data for stocks. In this chapter we discuss this computation, showing how to do the calculations in Excel. The most obvious calculation is the *sample variance-covariance matrix*: This is the matrix computed directly from the historic returns. We illustrate several methods for calculating the sample variance-covariance matrix, including a direct calculation in the spreadsheet using the excess return matrix and an implementation of this method with VBA.

While the sample variance-covariance matrix may appear to be an obvious choice, a large literature recognizes that it may not be the best estimate of variances and covariances. Disappointment with the sample variance-covariance matrix stems both from its often unrealistic parameters and from its inability to predict. These issues are discussed briefly in sections 10.5 and 10.6. As an alternative to the sample matrix, sections 10.7–10.10 discuss so-called "shrinkage" methods for improving the estimate of the variance-covariance matrix.[1]

Before starting this chapter, you may want to peruse Chapter 34 which discusses *array functions*. These are Excel functions whose arguments are vectors and matrices; their implementation is slightly different from standard Excel functions. This chapter makes heavy use of the array functions **Transpose( )** and **MMult( )** as well as some other "home-grown" array functions.

10.2 Computing the Sample Variance-Covariance Matrix

Suppose we have return data for N assets over M periods. Writing the return of asset i in period t as r_{it}, we write the *mean return* of asset i as:

$$\overline{r_i} = \frac{1}{M}\sum_{t=1}^{M} r_{it}, \ i = 1, \ldots, N$$

Then the covariance of the return of asset i and asset j is calculated as:

$$\sigma_{ij} = Cov(i, j) = \frac{1}{M-1}\sum_{t=1}^{M}(r_{it} - \overline{r_i})\cdot(r_{jt} - \overline{r_j}), \ i, j = 1, \ldots, N$$

1. We return to the issue of prediction in Chapter 13, which discusses the Black-Litterman model of portfolio optimization.

The matrix of these covariances (which includes, of course, the variances when $i = j$) is the *sample variance-covariance matrix*. Our problem is to calculate these covariances efficiently. Define the *excess return matrix* to be:

$$A = matrix\ of\ excess\ returns = \begin{bmatrix} r_{11}-\overline{r}_1 & \cdots & r_{N1}-\overline{r}_N \\ r_{12}-\overline{r}_1 & \cdots & r_{N2}-\overline{r}_N \\ \vdots & & \vdots \\ r_{1M}-\overline{r}_1 & \cdots & r_{NM}-\overline{r}_N \end{bmatrix}$$

Columns of matrix A subtract the mean asset return from the individual asset returns. The transpose of this matrix is:

$$A^T = \begin{bmatrix} r_{11}-\overline{r}_1 & r_{12}-\overline{r}_1 & \cdots & \cdots & r_{1M}-\overline{r}_1 \\ \vdots & \vdots & & & \vdots \\ r_{N1}-\overline{r}_N & r_{N2}-r_N & \cdots & \cdots & r_{NM}-\overline{r}_N \end{bmatrix}$$

Multiplying A^T times A and dividing through by $M - 1$ gives the sample variance-covariance matrix:

$$S = [\sigma_{ij}] = \frac{A^T \cdot A}{M-1}$$

To consider the computational aspects, we use $M = 60$ months of return data for $N = 10$ stocks. The spreadsheet below shows the price data (adjusted for dividends) and the computed returns:

	A	B	C	D	E	F	G	H	I	J	K	L
1	**FIVE YEARS OF PRICES FOR 10 STOCKS AND THE SP500**											
2		**McDonalds**	**US Steel**	**Arcelor-Mittal**	**Microsoft**	**Apple**	**Kellogg**	**General Electric**	**Bank of America**	**Pfizer**	**Exxon**	**S&P500**
3	**Date**	**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	**^GSPC**
4	1-Feb-07	37.57	84.74	44.84	25.53	84.61	43.45	28.97	44.68	19.57	64.20	1406.82
5	1-Mar-07	38.74	94.76	46.63	25.26	92.91	44.83	29.34	44.85	19.80	67.58	1420.86
6	2-Apr-07	41.52	97.03	47.10	27.14	99.80	46.12	30.59	44.74	20.74	71.10	1482.37
57	1-Jul-11	85.26	39.80	30.54	27.02	390.48	54.85	17.57	9.68	18.65	78.37	1292.28
58	1-Aug-11	89.74	30.01	21.74	26.40	384.83	53.84	16.00	8.15	18.60	73.18	1218.89
59	1-Sep-11	87.16	21.94	15.74	24.70	381.32	52.72	15.07	6.11	17.32	71.81	1131.42
60	3-Oct-11	92.16	25.27	20.51	26.43	404.78	53.73	16.55	6.82	18.87	77.20	1253.30
61	1-Nov-11	95.52	27.26	18.89	25.58	382.20	49.16	15.76	5.44	19.86	80.00	1246.96
62	1-Dec-11	100.33	26.42	18.19	25.96	405.00	50.57	17.91	5.56	21.42	84.30	1257.60
63	3-Jan-12	99.05	30.14	20.52	29.53	456.48	49.52	18.71	7.13	21.18	83.28	1312.41
64	1-Feb-12	99.99	31.01	23.30	30.77	493.17	50.21	19.13	8.18	21.14	84.88	1351.95

Using the Excel function **Ln(P$_t$/P$_{t-1}$)**, we compute the monthly returns:

	A	B	C	D	E	F	G	H	I	J	K	L
1	FIVE YEARS OF MONTHLY RETURNS FOR 10 STOCKS AND THE SP500											
2	**Date**	**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	**^GSPC**
3	1-Mar-07	3.07%	11.18%	3.91%	-1.06%	9.36%	3.13%	1.27%	0.38%	1.17%	5.13%	0.99%
4	2-Apr-07	6.93%	2.37%	1.00%	7.18%	7.15%	2.84%	4.17%	-0.25%	4.64%	5.08%	4.24%
5	1-May-07	4.59%	11.02%	12.16%	2.80%	19.42%	2.55%	1.91%	0.73%	4.89%	5.09%	3.20%
6	1-Jun-07	0.41%	-3.98%	3.93%	-4.06%	0.70%	-4.14%	2.60%	-3.66%	-7.23%	0.85%	-1.80%
7	2-Jul-07	-5.85%	-10.11%	-2.23%	-1.66%	7.66%	0.04%	1.24%	-3.06%	-8.39%	1.49%	-3.25%
8	1-Aug-07	2.83%	-3.72%	8.65%	-0.53%	4.97%	6.42%	0.28%	6.66%	6.70%	1.09%	1.28%
54	1-Jun-11	3.35%	-0.15%	3.86%	3.86%	-3.56%	-2.97%	-3.24%	-6.90%	-4.07%	-2.53%	-1.84%
55	1-Jul-11	2.53%	-14.09%	-10.97%	5.24%	15.12%	0.82%	-5.16%	-12.05%	-6.79%	-1.97%	-2.17%
56	1-Aug-11	5.12%	-28.23%	-33.99%	-2.32%	-1.46%	-1.86%	-9.36%	-17.20%	-0.27%	-6.85%	-5.85%
57	1-Sep-11	-2.92%	-31.32%	-32.29%	-6.66%	-0.92%	-2.10%	-5.99%	-28.81%	-7.13%	-1.89%	-7.45%
58	3-Oct-11	5.58%	14.13%	26.47%	6.77%	5.97%	1.90%	9.37%	10.99%	8.57%	7.24%	10.23%
59	1-Nov-11	3.58%	7.58%	-8.23%	-3.27%	-5.74%	-8.89%	-4.89%	-22.61%	5.11%	3.56%	-0.51%
60	1-Dec-11	4.91%	-3.13%	-3.78%	1.47%	5.79%	2.83%	12.79%	2.18%	7.56%	5.24%	0.85%
61	3-Jan-12	-1.28%	13.17%	12.05%	12.88%	11.97%	-2.10%	4.37%	24.87%	-1.13%	-1.22%	4.27%
62	1-Feb-12	0.94%	2.85%	12.71%	4.11%	7.73%	1.38%	2.22%	13.74%	-0.19%	1.90%	2.97%
63												
64	Mean	1.63%	-1.68%	-1.09%	0.31%	2.94%	0.24%	-0.69%	-2.83%	0.13%	0.47%	-0.07%

Below we compute the excess returns and the variance-covariance matrix:

	A	B	C	D	E	F	G	H	I	J	K	L
66	Variance-Covariance Matrix											
67		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	
68	**MCD**	0.0020	0.0037	0.0028	0.0015	0.0017	0.0007	0.0020	0.0031	0.0015	0.0011	
69	**X**	0.0037	0.0380	0.0284	0.0076	0.0111	0.0031	0.0127	0.0176	0.0043	0.0043	
70	**MT**	0.0028	0.0284	0.0267	0.0065	0.0097	0.0031	0.0102	0.0133	0.0038	0.0039	
71	**MSFT**	0.0015	0.0076	0.0065	0.0063	0.0049	0.0010	0.0046	0.0079	0.0018	0.0014	
72	**AAPL**	0.0017	0.0111	0.0097	0.0049	0.0126	0.0016	0.0049	0.0049	0.0007	0.0020	
73	**K**	0.0007	0.0031	0.0031	0.0010	0.0016	0.0026	0.0028	0.0046	0.0011	0.0003	
74	**GE**	0.0020	0.0127	0.0102	0.0046	0.0049	0.0028	0.0122	0.0163	0.0041	0.0022	
75	**BAC**	0.0031	0.0176	0.0133	0.0079	0.0049	0.0046	0.0163	0.0393	0.0080	0.0017	
76	**PFE**	0.0015	0.0043	0.0038	0.0018	0.0007	0.0011	0.0041	0.0080	0.0041	0.0011	
77	**XOM**	0.0011	0.0043	0.0039	0.0014	0.0020	0.0003	0.0022	0.0017	0.0011	0.0026	
78			<-- {=MMULT(TRANSPOSE(B83:K142),B83:K142)/59}									
79												
80												
81	Excess returns: r_{ij}-r_i											
82		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	
83	1-Mar-07	0.0144	0.1285	0.0501	-0.0137	0.0642	0.0289	0.0196	0.0321	0.0104	0.0467	<-- =K3-K$64
84	2-Apr-07	0.0530	0.0404	0.0209	0.0687	0.0422	0.0260	0.0486	0.0258	0.0451	0.0461	<-- =K4-K$64
85	1-May-07	0.0296	0.1269	0.1325	0.0249	0.1648	0.0231	0.0260	0.0356	0.0476	0.0462	<-- =K5-K$64
86	1-Jun-07	-0.0122	-0.0231	0.0502	-0.0437	-0.0224	-0.0438	0.0329	-0.0083	-0.0736	0.0039	<-- =K6-K$64
87	2-Jul-07	-0.0748	-0.0843	-0.0114	-0.0197	0.0473	-0.0020	0.0193	-0.0023	-0.0852	0.0102	
88	1-Aug-07	0.0119	-0.0205	0.0974	-0.0084	0.0204	0.0618	0.0097	0.0949	0.0657	0.0063	
89	4-Sep-07	0.0845	0.1313	0.1795	0.0217	0.0734	0.0168	0.0768	0.0327	-0.0180	0.0720	
90	1-Oct-07	0.0762	0.0351	0.0310	0.2197	0.1839	-0.0615	0.0009	-0.0121	0.0058	-0.0108	
91	1-Nov-07	-0.0120	-0.0806	-0.0644	-0.0911	-0.0709	0.0268	-0.0652	-0.0171	-0.0238	-0.0321	
92	3-Dec-07	-0.0088	0.2299	0.0576	0.0547	0.0541	-0.0327	-0.0170	-0.0690	-0.0458	0.0449	
93	2-Jan-08	-0.1111	-0.1537	-0.1419	-0.0913	-0.4101	-0.0948	-0.0403	0.0960	0.0260	-0.0937	
94	1-Feb-08	0.0003	0.0810	0.1464	-0.1801	-0.1088	0.0629	-0.0489	-0.0770	-0.0345	0.0148	
95	3-Mar-08	0.0140	0.1737	0.0892	0.0392	0.1085	0.0332	0.1172	-0.0022	-0.0637	-0.0331	
96	1-Apr-08	0.0497	0.2102	0.0963	0.0019	0.1631	-0.0292	-0.1169	0.0186	-0.0416	0.0910	
97	1-May-08	-0.0207	0.1332	0.1197	-0.0066	0.0523	0.0163	-0.0553	-0.0705	-0.0233	-0.0476	
98	2-Jun-08	-0.0635	0.0843	0.0120	-0.0319	-0.1492	-0.0786	-0.1228	-0.3064	-0.1040	-0.0118	

A VBA Function to Compute the Variance-Covariance Matrix

To automate this procedure, we write a VBA function that computes the variance-covariance matrix using the Excel function **Covariance.S**. When Excel functions with periods, such as **Covariance.S**, are used in VBA, the period becomes an underscore: **Covariance_S**:

```
'My thanks to Amir Kirsh
'Revised 2012 by Benjamin Czaczkes and _
Simon Benninga
Function VarCovar(rng As Range) As Variant
  Dim i As Integer
  Dim j As Integer
  Dim numcols As Integer
  numcols = rng.Columns.Count
  numrows = rng.Rows.Count
  Dim matrix() As Double
  ReDim matrix(numcols - 1, numcols - 1)
  For i = 1 To numcols
    For j = 1 To numcols
        matrix(i - 1, j - 1) = _
      Application.WorksheetFunction.
Covariance_S(rng.Columns(i), _
    rng.Columns(j))
    Next j
  Next i
  VarCovar = matrix
End Function
```

The VBA computes **Covariance_S** for every entry of the variance-covariance matrix.[2] Here's the result:

	A	B	C	D	E	F	G	H	I	J	K	L
1	**PORTFOLIO ANALYSIS FOR MONTHLY DATA**											
2		Variance-Covariance Matrix										
3		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	
4	**MCD**	0.0020	0.0037	0.0028	0.0015	0.0017	0.0007	0.0020	0.0031	0.0015	0.0011	<-- {=varcovar('Page 253'!B3:K62)}
5	**X**	0.0037	0.0380	0.0284	0.0076	0.0111	0.0031	0.0127	0.0176	0.0043	0.0043	
6	**MT**	0.0028	0.0284	0.0267	0.0065	0.0097	0.0031	0.0102	0.0133	0.0038	0.0039	
7	**MSFT**	0.0015	0.0076	0.0065	0.0063	0.0049	0.0010	0.0046	0.0079	0.0018	0.0014	
8	**AAPL**	0.0017	0.0111	0.0097	0.0049	0.0126	0.0016	0.0049	0.0049	0.0007	0.0020	
9	**K**	0.0007	0.0031	0.0031	0.0010	0.0016	0.0026	0.0028	0.0046	0.0011	0.0003	
10	**GE**	0.0020	0.0127	0.0102	0.0046	0.0049	0.0028	0.0122	0.0163	0.0041	0.0022	
11	**BAC**	0.0031	0.0176	0.0133	0.0079	0.0049	0.0046	0.0163	0.0393	0.0080	0.0017	
12	**PFE**	0.0015	0.0043	0.0038	0.0018	0.0007	0.0011	0.0041	0.0080	0.0041	0.0011	
13	**XOM**	0.0011	0.0043	0.0039	0.0014	0.0020	0.0003	0.0022	0.0017	0.0011	0.0026	

Should We Divide by *M* – 1 or by *M*

In the above calculations, we use the sample covariance (in Excel **Covariance.S** and in VBA **Covariance_S**) to divide by *M* – 1 instead of *M* in order to get the unbiased estimate of the variances and covariances. We don't think this matters very much, but for reference to a higher authority, we suggest our discussion of *M* versus *M* – 1 in section 8.2.

Starting with Excel 2010, Microsoft has cleared up considerable confusion that once existed in Excel about whether to divide by *M* or *M* – 1. The new versions of Excel have standardized the nomenclature and computations for these functions:

Excel 2010 and later	Other (older) versions of this function (still work)	Comments	When used in VBA
Covariance.S		Sample covariance, divides by *M–1*	**Application. WorksheetFunction. Covariance_S**
Covariance.P	**Covar**	Population covariance, divides by *M*	**Application. WorksheetFunction. Covariance_P**

2. Since the covariance matrix is symmetric, we've actually done too many computations. But given the speed of our computers, who cares?

Excel 2010 and later	Other (older) versions of this function (still work)	Comments	When used in VBA
Var.S	**VarS**	Sample variance	**Application.WorksheetFunction.Var_S** **Application.WorksheetFunction.VarS**
Var.P	**VarP**	Population variance	**Application.WorksheetFunction.Var_P** **Application.WorksheetFunction.VarP**

Confused? Don't worry! As the discussion in Chapter 8 indicates, perhaps it doesn't matter very much.

10.3 The Correlation Matrix

Using the Excel function **Correl** we can compute the correlation matrix of the returns:

```
Function CorrMatrix(rng As Range) As Variant
  Dim i As Integer
  Dim j As Integer
  Dim numcols As Integer
  numcols = rng.Columns.Count
  numrows = rng.Rows.Count
  Dim matrix() As Double
  ReDim matrix(numcols - 1, numcols - 1)
```

```
  For i = 1 To numcols
    For j = 1 To numcols
    matrix(i - 1, j - 1) = _
    Application.WorksheetFunction.Correl(rng. _
    Columns(i), rng.Columns(j))
    Next j
  Next i
  CorrMatrix = matrix
End Function
```

	A	B	C	D	E	F	G	H	I	J	K
1	**CORRELATION MATRIX**										
2		**McDonalds**	**US Steel**	**Arcelor-Mittal**	**Microsoft**	**Apple**	**Kellogg**	**General Electric**	**Bank of America**	**Pfizer**	**Exxon**
3		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**
4	**MCD**	1.0000	0.4199	0.3859	0.4238	0.3379	0.2920	0.4064	0.3506	0.5411	0.4741
5	**X**	0.4199	1.0000	0.8898	0.4898	0.5062	0.3078	0.5904	0.4556	0.3491	0.4361
6	**MT**	0.3859	0.8898	1.0000	0.5044	0.5277	0.3692	0.5659	0.4103	0.3602	0.4620
7	**MSFT**	0.4238	0.4898	0.5044	1.0000	0.5497	0.2416	0.5312	0.5050	0.3542	0.3581
8	**AAPL**	0.3379	0.5062	0.5277	0.5497	1.0000	0.2827	0.3964	0.2205	0.0945	0.3425
9	**K**	0.2920	0.3078	0.3692	0.2416	0.2827	1.0000	0.4846	0.4559	0.3487	0.1234
10	**GE**	0.4064	0.5904	0.5659	0.5312	0.3964	0.4846	1.0000	0.7461	0.5842	0.3926
11	**BAC**	0.3506	0.4556	0.4103	0.5050	0.2205	0.4559	0.7461	1.0000	0.6328	0.1723
12	**PFE**	0.5411	0.3491	0.3602	0.3542	0.0945	0.3487	0.5842	0.6328	1.0000	0.3435
13	**XOM**	0.4741	0.4361	0.4620	0.3581	0.3425	0.1234	0.3926	0.1723	0.3435	1.0000
14				<-- {=CorrMatrix('Page 253'!B3:K62)}							

Here's another version of the correlation matrix, this time only the upper half:

```
'Triangular correlation matrix
Function CorrMatrixTriangular(rng As Range) _
As Variant
  Dim i As Integer
  Dim j As Integer
  Dim numcols As Integer
  numcols = rng.Columns.Count
  numrows = rng.Rows.Count
  Dim matrix() As Variant
  ReDim matrix(numcols - 1, numcols - 1)
```

```
  For i = 1 To numcols
    For j = 1 To numcols
    If i <= j Then
      matrix(i - 1, j - 1) = _
    Application.WorksheetFunction.Correl(rng. _
    Columns(i), rng.Columns(j))
    Else
    matrix(i - 1, j - 1) = ""
    End If
    Next j
    Next i
  CorrMatrixTriangular = matrix
End Function
```

	A	B	C	D	E	F	G	H	I	J	K
16		McDonalds	US Steel	Arcelor-Mittal	Microsoft	Apple	Kellogg	General Electric	Bank of America	Pfizer	Exxon
17		MCD	X	MT	MSFT	AAPL	K	GE	BAC	PFE	XOM
18	MCD	1.0000	0.4199	0.3859	0.4238	0.3379	0.2920	0.4064	0.3506	0.5411	0.4741
19	X		1.0000	0.8898	0.4898	0.5062	0.3078	0.5904	0.4556	0.3491	0.4361
20	MT			1.0000	0.5044	0.5277	0.3692	0.5659	0.4103	0.3602	0.4620
21	MSFT				1.0000	0.5497	0.2416	0.5312	0.5050	0.3542	0.3581
22	AAPL					1.0000	0.2827	0.3964	0.2205	0.0945	0.3425
23	K						1.0000	0.4846	0.4559	0.3487	0.1234
24	GE							1.0000	0.7461	0.5842	0.3926
25	BAC								1.0000	0.6328	0.1723
26	PFE									1.0000	0.3435
27	XOM										1.0000
28				<-- {=CorrMatrixTriangular('Page 253'!B3:K62)}							

Here are some statistics for the correlations. The average correlation for our sample (0.4226) is a bit high (usually a sample of stocks will give average correlation of 0.2–0.3). The largest correlations ($\rho_{\text{Arcelor,US Steel}} = 0.8898$, $\rho_{\text{GE,BankAmerica}} = 0.7461$) look quite high, though perhaps there are economic explanations.[3]

3. Arcelor and U.S. Steel are, of course, both steel companies. GE has one of the largest financing operations in the world; perhaps this explains the high correlation between the returns of Bank of America and GE? Or perhaps it's just a fluke of the data?

	A	B	C	D	E	F	G	H	I	J
30	Some correlation statistics									
31	Average	0.4226	<-- =AVERAGEIF(B18:K27,"<1")							
32	Largest	0.8898	<-- =LARGE(B18:K27,11)			Smallest	0.0945	<-- =SMALL($B$18:$K$27,1)		
33	Next largest	0.7461	<-- =LARGE(B18:K27,12)			Next small	0.1234	<-- =SMALL($B$18:$K$27,2)		
34	etc.	0.6328	<-- =LARGE(B18:K27,13)			etc.	0.1723	<-- =SMALL($B$18:$K$27,3)		
35	etc.	0.5904	<-- =LARGE(B18:K27,14)			etc.	0.2416	<-- =SMALL($B$18:$K$27,5)		
36	etc.	0.5842	<-- =LARGE(B18:K27,15)			etc.	0.2416	<-- =SMALL($B$18:$K$27,5)		

10.4 Computing the Global Minimum Variance Portfolio (GMVP)

The two most prominent uses of the variance-covariance matrix are to find the global minimum variance portfolio (GMVP) and to find efficient portfolios. Both uses illustrate the problematics of working with sample data and provide us with the introduction needed for sections 10.7–10.10, which discuss alternatives to the sample variance-covariance matrix. In this section we discuss the GMPV.

Suppose there are N assets having a variance-covariance matrix S. The GMVP is the portfolio $x = \{x_1, x_2, \ldots, x_N\}$ which has the lowest variance from among all feasible portfolios. The minimum variance portfolio is defined by

$$x_{GMVP} = \{x_{GMVP,1}, x_{GMVP,2}, \ldots, x_{GMVP,N}\} = \frac{1_{row} \cdot S^{-1}}{1_{row} \cdot S^{-1} \cdot 1_{row}^T},$$

where $1_{row} = \underbrace{\{1, 1, \ldots, 1\}}_{\substack{\uparrow \\ \text{N-dimensional} \\ \text{row vector of 1s}}} = \dfrac{1_{row} \cdot S^{-1}}{Sum(numerator)}$

$$x_{GMVP} = \begin{Bmatrix} x_{GMVP,1} \\ x_{GMVP,2} \\ \vdots \\ x_{GMVP,N} \end{Bmatrix} = \frac{S^{-1} 1_{column}}{1_{column}^T \cdot S^{-1} \cdot 1_{column}}, \text{ where } 1_{column} = \underbrace{\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}}_{\substack{\uparrow \\ \text{N-dimensional} \\ \text{column vector of 1s}}}$$

$$= \frac{S^{-1} 1_{column}}{Sum(numerator)}$$

This formula is due to Merton.[4]

The particular fascination of the minimum variance portfolio is that it is the only portfolio on the efficient frontier whose computation does not require the asset expected returns. The mean μ_{GMVP} and the variance σ^2_{GMVP} of the minimum variance portfolio are given by:

$$\mu_{GMVP} = x_{GMVP} \cdot E(r), \sigma^2_{GMVP} = x_{GMVP} \cdot S \cdot x^T_{GMVP}$$

Here's an implementation of these formulas for our particular example. We use two VBA functions for the unit column and row vectors:

```
'I thank Priyush Singh and Ayal Itzkovitz
Function UnitrowVector(numcols As Integer) _
As Variant
  Dim i As Integer
  Dim vector() As Integer
  ReDim vector(0, numcols - 1)
  For i = 1 To numcols
    vector(0, i - 1) = 1
  Next i
  UnitrowVector = vector
End Function

Function UnitColVector(numrows As Integer) _
As Variant
  Dim i As Integer
  Dim vector() As Integer
  ReDim vector(numrows - 1, 0)
  For i = 1 To numrows
    vector(i - 1, 0) = 1
  Next i
  UnitColVector = vector
End Function
```

4. Robert C. Merton, "An Analytical Derivation of the Efficient Portfolio Frontier," *Journal of Financial and Quantitative Analysis* (1973).

Applying this to our 10-asset example:

	A	B	C	D	E	F	G	H	I	J	K
1		**COMPUTING THE GLOBAL MINIMUM VARIANCE PORTFOLIO**									
2		Variance-Covariance Matrix									
3		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**
4	**MCD**	0.0020	0.0037	0.0028	0.0015	0.0017	0.0007	0.0020	0.0031	0.0015	0.0011
5	**X**	0.0037	0.0380	0.0284	0.0076	0.0111	0.0031	0.0127	0.0176	0.0043	0.0043
6	**MT**	0.0028	0.0284	0.0267	0.0065	0.0097	0.0031	0.0102	0.0133	0.0038	0.0039
7	**MSFT**	0.0015	0.0076	0.0065	0.0063	0.0049	0.0010	0.0046	0.0079	0.0018	0.0014
8	**AAPL**	0.0017	0.0111	0.0097	0.0049	0.0126	0.0016	0.0049	0.0049	0.0007	0.0020
9	**K**	0.0007	0.0031	0.0031	0.0010	0.0016	0.0026	0.0028	0.0046	0.0011	0.0003
10	**GE**	0.0020	0.0127	0.0102	0.0046	0.0049	0.0028	0.0122	0.0163	0.0041	0.0022
11	**BAC**	0.0031	0.0176	0.0133	0.0079	0.0049	0.0046	0.0163	0.0393	0.0080	0.0017
12	**PFE**	0.0015	0.0043	0.0038	0.0018	0.0007	0.0011	0.0041	0.0080	0.0041	0.0011
13	**XOM**	0.0011	0.0043	0.0039	0.0014	0.0020	0.0003	0.0022	0.0017	0.0011	0.0026
14											
15	GMVP as row	0.0326	0.2117	0.1754	0.0705	0.0873	0.0340	0.1166	0.1891	0.0493	0.0335
16			<-- {=MMULT(unitrowvector(10),B4:K13)/SUM(MMULT(unitrowvector(10),B4:K13))}								
17											
18	GMVP as column	0.0326									
19		0.2117	<-- {=MMULT(B4:K13,unitcolvector(10))/SUM(MMULT(B4:K13,unitcolvector(10)))}								
20		0.1754									
21		0.0705									
22		0.0873									
23		0.0340									
24		0.1166									
25		0.1891									
26		0.0493									
27		0.0335									
28											
29	**GMVP statistics**										
30	Mean	-0.80%	T(B15:K15,'Page 253'!B64:K64)								
31	Variance	0.0130	<-- {=MMULT(MMULT(B15:K15,B4:K13),B18:B27)}								
32	Sigma	11.40%	<-- =SQRT(B31)								

10.5 Four Alternatives to the Sample Variance-Covariance Matrix

In succeeding sections we illustrate four alternatives to the sample variance-covariance matrix:

- The **single-index model** assumes that the only sources of variance risk are the market variance and the betas of the assets.
- The **constant correlation model** assumes the correlation between all asset returns is constant, so that $\sigma_{ij} = \rho\ \sigma_i\ \sigma_j$.

- **Shrinkage methods** assume that the variance-covariance matrix is a convex combination of the sample variance-covariance and a matrix with variances on the diagonal and zeros elsewhere.
- **Option methods** use options to derive the standard deviations of returns for the assets. We combine this in section 10.9 with the constant correlation method to compute a variance-covariance matrix.

The first three models arise out of a distrust of return data for generating the covariances of the data in the future. The fourth method—using option data—goes further and assumes that even the sample variances are an inaccurate prediction of future variance.

10.6 Alternatives to the Sample Variance-Covariance: The Single-Index Model (SIM)

The single-index model (SIM) began as an attempt to simplify some of the computational complexities of calculating the variance-covariance matrix.[5] The basic assumption of the SIM is that the returns of each asset can be linearly regressed on a market index x:

$$\tilde{r}_i = \alpha_i + \beta_i \tilde{r}_x + \tilde{\varepsilon}_i$$

where the correlation between ε_i and ε_j is zero. Given this assumption, it is easy to establish the following two facts:

- $E(\tilde{r}_i) = \alpha_i + \beta_i E(\tilde{r}_x)$
- $\sigma_{ij} = \begin{cases} \beta_i \beta_j \sigma_x^2 & \text{when } i \neq j \\ \sigma_i^2 & \text{when } i = j \end{cases}$

5. W. M. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science* (1963).

Essentially the SIM involves changes in the estimates of the covariances, but not the sample variance. We can automate the procedure for computing the SIM by writing some VBA code:

```
Function sim(assetdata As Range, marketdata As Range) _
As Variant
    Dim i As Integer
    Dim j As Integer
    Dim numcols As Integer
    numcols = assetdata.Columns.Count
    Dim matrix() As Double
    ReDim matrix(numcols - 1, numcols - 1)

    For i = 1 To numcols
    For j = 1 To numcols
        If i = j Then
        matrix(i - 1, j - 1) = Application. _
        WorksheetFunction.Var_S(assetdata.Columns(i))
        Else
        matrix(i - 1, j - 1) = _
      Application.WorksheetFunction.Slope(assetdata. _
      Columns(i), marketdata) * _
      Application.WorksheetFunction.Slope(assetdata. _
      Columns(j), marketdata) * _
      Application.WorksheetFunction.Var_S(marketdata)
    End If
    Next j
    Next i
    sim = matrix
End Function
```

The two arguments of this function are the asset returns and the market returns. Applying this code in our example:

	A	B	C	D	E	F	G	H	I	J	K	L
1	**COMPUTING THE SINGLE-INDEX VARIANCE-COVARIANCE MATRIX**											
2		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	
3	**MCD**	0.0020	0.0036	0.0031	0.0013	0.0016	0.0006	0.0021	0.0032	0.0009	0.0006	
4	**X**	0.0036	0.0380	0.0198	0.0085	0.0105	0.0038	0.0137	0.0204	0.0059	0.0042	
5	**MT**	0.0031	0.0198	0.0267	0.0073	0.0090	0.0033	0.0117	0.0175	0.0051	0.0036	
6	**MSFT**	0.0013	0.0085	0.0073	0.0063	0.0038	0.0014	0.0050	0.0075	0.0022	0.0015	
7	**AAPL**	0.0016	0.0105	0.0090	0.0038	0.0126	0.0017	0.0062	0.0092	0.0027	0.0019	
8	**K**	0.0006	0.0038	0.0033	0.0014	0.0017	0.0026	0.0022	0.0034	0.0010	0.0007	
9	**GE**	0.0021	0.0137	0.0117	0.0050	0.0062	0.0022	0.0122	0.0121	0.0035	0.0025	
10	**BAC**	0.0032	0.0204	0.0175	0.0075	0.0092	0.0034	0.0121	0.0393	0.0052	0.0037	
11	**PFE**	0.0009	0.0059	0.0051	0.0022	0.0027	0.0010	0.0035	0.0052	0.0041	0.0011	
12	**XOM**	0.0006	0.0042	0.0036	0.0015	0.0019	0.0007	0.0025	0.0037	0.0011	0.0026	
13				<-- {=sim(B16:K75,L16:L75)}								
14												
15	**Date**	**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**	**^GSPC**
16	1-Mar-07	3.07%	11.18%	3.91%	-1.06%	9.36%	3.13%	1.27%	0.38%	1.17%	5.13%	0.99%
17	2-Apr-07	6.93%	2.37%	1.00%	7.18%	7.15%	2.84%	4.17%	-0.25%	4.64%	5.08%	4.24%
18	1-May-07	4.59%	11.02%	12.16%	2.80%	19.42%	2.55%	1.91%	0.73%	4.89%	5.09%	3.20%
19	1-Jun-07	0.41%	-3.98%	3.93%	-4.06%	0.70%	-4.14%	2.60%	-3.66%	-7.23%	0.85%	-1.80%
20	2-Jul-07	-5.85%	-10.11%	-2.23%	-1.66%	7.66%	0.04%	1.24%	-3.06%	-8.39%	1.49%	-3.25%
21	1-Aug-07	2.83%	-3.72%	8.65%	-0.53%	4.97%	6.42%	0.28%	6.66%	6.70%	1.09%	1.28%
67	1-Jun-11	3.35%	-0.15%	3.86%	3.86%	-3.56%	-2.97%	-3.24%	-6.90%	-4.07%	-2.53%	-1.84%
68	1-Jul-11	2.53%	-14.09%	-10.97%	5.24%	15.12%	0.82%	-5.16%	-12.05%	-6.79%	-1.97%	-2.17%
69	1-Aug-11	5.12%	-28.23%	-33.99%	-2.32%	-1.46%	-1.86%	-9.36%	-17.20%	-0.27%	-6.85%	-5.85%
70	1-Sep-11	-2.92%	-31.32%	-32.29%	-6.66%	-0.92%	-2.10%	-5.99%	-28.81%	-7.13%	-1.89%	-7.45%
71	3-Oct-11	5.58%	14.13%	26.47%	6.77%	5.97%	1.90%	9.37%	10.99%	8.57%	7.24%	10.23%
72	1-Nov-11	3.58%	7.58%	-8.23%	-3.27%	-5.74%	-8.89%	-4.89%	-22.61%	5.11%	3.56%	-0.51%
73	1-Dec-11	4.91%	-3.13%	-3.78%	1.47%	5.79%	2.83%	12.79%	2.18%	7.56%	5.24%	0.85%
74	3-Jan-12	-1.28%	13.17%	12.05%	12.88%	11.97%	-2.10%	4.37%	24.87%	-1.13%	-1.22%	4.27%
75	1-Feb-12	0.94%	2.85%	12.71%	4.11%	7.73%	1.38%	2.22%	13.74%	-0.19%	1.90%	2.97%

10.7 Alternatives to the Sample Variance-Covariance: Constant Correlation

The constant correlation model of Elton and Gruber (1973) computes the variance-covariance matrix by assuming that the variances of the asset returns are the sample returns, but that the covariances are all related by the same correlation coefficient, which is generally taken to be the average correlation coefficient of the assets in question. Since $Cov(r_i,r_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, this means that in the constant correlation model:

$$\sigma_{ij} = \begin{cases} \sigma_{ii} = \sigma_i^2 & \text{when } i = j \\ \sigma_{ij} = \rho\sigma_i\sigma_j & \text{when } i \neq j \end{cases}$$

Using our data for the 10 stocks, we can implement the constant correlation model. We first compute the correlations of all the stocks:

	A	B	C	D	E	F	G	H	I	J	K
1	ESTIMATING THE CONSTANT CORRELATION VARIANCE-COVARIANCE MATRIX										
2	Correlation	0.20									
3											
4		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**
5	**MCD**	0.0020	0.0018	0.0015	0.0007	0.0010	0.0005	0.0010	0.0018	0.0006	0.0005
6	**X**	0.0018	0.0380	0.0064	0.0031	0.0044	0.0020	0.0043	0.0077	0.0025	0.0020
7	**MT**	0.0015	0.0064	0.0267	0.0026	0.0037	0.0017	0.0036	0.0065	0.0021	0.0017
8	**MSFT**	0.0007	0.0031	0.0026	0.0063	0.0018	0.0008	0.0017	0.0031	0.0010	0.0008
9	**AAPL**	0.0010	0.0044	0.0037	0.0018	0.0126	0.0012	0.0025	0.0044	0.0014	0.0011
10	**K**	0.0005	0.0020	0.0017	0.0008	0.0012	0.0026	0.0011	0.0020	0.0007	0.0005
11	**GE**	0.0010	0.0043	0.0036	0.0017	0.0025	0.0011	0.0122	0.0044	0.0014	0.0011
12	**BAC**	0.0018	0.0077	0.0065	0.0031	0.0044	0.0020	0.0044	0.0393	0.0025	0.0020
13	**PFE**	0.0006	0.0025	0.0021	0.0010	0.0014	0.0007	0.0014	0.0025	0.0041	0.0007
14	**XOM**	0.0005	0.0020	0.0017	0.0008	0.0011	0.0005	0.0011	0.0020	0.0007	0.0026
15		<-- {=constantcorr('Page 253'!B3:K62,'Page 265'!B2)}									

We've written a VBA function to compute this matrix from the return data:

```
Function constantcorr(data As Range, corr As Double) _
As Variant
    Dim i As Integer
    Dim j As Integer
    Dim numcols As Integer
    numcols = data.Columns.Count
    numrows = data.Rows.Count
    Dim matrix() As Double
    ReDim matrix(numcols - 1, numcols - 1)
    If Abs(corr) >= 1 Then GoTo Out
    For i = 1 To numcols
    For j = 1 To numcols
        If i = j Then
        matrix(i - 1, j - 1) = Application. _
        WorksheetFunction.Var_S(data.Columns(i))
        Else
```

```
        matrix(i - 1, j - 1) = corr * jjunk(data, i) * _
        jjunk(data, j)
       End If
    Next j
    Next i
Out:
    If Abs(corr) >= 1 Then constantcorr = VarCovar(data) _
    Else constantcorr = matrix
End Function
```

10.8 Alternatives to the Sample Variance-Covariance: Shrinkage Methods

A third class of methods of estimating the variance-covariance matrix has recently achieved popularity. So-called *shrinkage methods* assume that the variance-covariance matrix is a convex combination of the sample covariance matrix and some other matrix:

$$\begin{aligned}&\textit{Shrinkage variance-covariance matrix} = \\ &\quad \lambda^* \textit{Sample var-cov} + (1-\lambda)^* \textit{Other matrix}\end{aligned}$$

In the example below, the "other" matrix is a diagonal matrix of only variances, with zeros elsewhere. The shrinkage estimator $\lambda = 0.3$ (cell B20).

	A	B	C	D	E	F	G	H
1	**ESTIMATING THE VARIANCE-COVARIANCE MATRIX USING THE SHRINKAGE APPROACH Gives weight 0.30 (the shrinkage factor) to sample var-cov and weight 0.70 to a diagonal matrix of only variances**							
2	**Return data**							
3	**Date**	**GE**	**MSFT**	**JNJ**	**K**	**BA**	**IBM**	
4	03-Jan-94	56.44%	-1.50%	6.01%	-9.79%	58.73%	21.51%	
5	03-Jan-95	18.23%	33.21%	41.56%	7.46%	-0.24%	6.04%	
6	02-Jan-96	56.93%	44.28%	57.71%	37.76%	65.55%	27.33%	
7	02-Jan-97	42.87%	79.12%	22.94%	-5.09%	54.34%	41.08%	
8	02-Jan-98	47.11%	38.04%	17.62%	32.04%	37.11%	2.63%	
9	04-Jan-99	34.55%	85.25%	26.62%	-10.74%	15.05%	-2.11%	
10	03-Jan-00	28.15%	11.20%	3.41%	-48.93%	43.53%	23.76%	
11	02-Jan-01	4.61%	-47.19%	10.69%	11.67%	28.29%	21.76%	
12	02-Jan-02	-19.74%	4.27%	23.11%	19.90%	-15.09%	4.55%	
13	02-Jan-03	-44.78%	-29.47%	-5.67%	10.88%	-23.23%	15.54%	
14	02-Jan-04	35.90%	18.01%	-1.27%	15.49%	39.82%	31.80%	
15								
16	**Average**	23.66%	21.38%	18.43%	5.51%	27.63%	17.63%	<-- =AVERAGE(G4:G14)
17	**Standard deviation**	32.17%	40.71%	18.97%	23.86%	29.93%	13.56%	<-- =STDEV(G4:G14)
18	**Variance**	0.1035	0.1657	0.0360	0.0570	0.0896	0.0184	<-- =VAR(G4:G14)
19								
20	**Shrinkage factor λ**	0.3	<-- This is the weight put on the sample var-cov					
21								
22	**Shrinkage matrix Uses the array formula {=B20*B34:G39+(1-B20)*B44:G49} to compute the shrinkage covariance matrix**							
23		**GE**	**MSFT**	**JNJ**	**K**	**BA**	**IBM**	
24	**GE**	0.1035	0.0228	0.0066	-0.0013	0.0257	0.0037	
25	**MSFT**	0.0228	0.1657	0.0124	-0.0016	0.0114	-0.0007	
26	**JNJ**	0.0066	0.0124	0.0360	0.0054	0.0030	-0.0012	
27	**K**	-0.0013	-0.0016	0.0054	0.0570	-0.0023	-0.0014	
28	**BA**	0.0257	0.0114	0.0030	-0.0023	0.0896	0.0074	
29	**IBM**	0.0037	-0.0007	-0.0012	-0.0014	0.0074	0.0184	
30								
31								
32	**Uses the array formula {=MMULT(TRANSPOSE(B4:G14-B16:G16),B4:G14-B16:G16)/10} to compute the constant sample covariance matrix. In the shrinkage var-cov, this matrix is given weight lambda.**							
33		**GE**	**MSFT**	**JNJ**	**K**	**BA**	**IBM**	
34	**GE**	0.1035	0.0758	0.0222	-0.0043	0.0857	0.0123	
35	**MSFT**	0.0758	0.1657	0.0412	-0.0052	0.0379	-0.0022	
36	**JNJ**	0.0222	0.0412	0.0360	0.0181	0.0101	-0.0039	
37	**K**	-0.0043	-0.0052	0.0181	0.0570	-0.0076	-0.0046	
38	**BA**	0.0857	0.0379	0.0101	-0.0076	0.0896	0.0248	
39	**IBM**	0.0123	-0.0022	-0.0039	-0.0046	0.0248	0.0184	
40								
41								
42	**Uses the array formula {=MMULT(TRANSPOSE(B4:G14-B16:G16),B4:G14-B16:G16)/10*IF(A44:A49=B43:G43,1,0)} to compute a matrix with only variances on diagonal and zeros elsewhere. In the shrinkage var-cov this matrix is given weight 1-lambda.**							
43		**GE**	**MSFT**	**JNJ**	**K**	**BA**	**IBM**	
44	**GE**	0.1035	0.0000	0.0000	0.0000	0.0000	0.0000	
45	**MSFT**	0.0000	0.1657	0.0000	0.0000	0.0000	0.0000	
46	**JNJ**	0.0000	0.0000	0.0360	0.0000	0.0000	0.0000	
47	**K**	0.0000	0.0000	0.0000	0.0570	0.0000	0.0000	
48	**BA**	0.0000	0.0000	0.0000	0.0000	0.0896	0.0000	
49	**IBM**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0184	

There is little theory about choosing the proper shrinkage estimator.[6] Our suggestion is to choose a shrinkage operator λ so that the GMVP is wholly positive (see next section for details).

10.9 Using Option Information to Compute the Variance Matrix[7]

Another way to compute the variance matrix is to use the information from the options market. We use the implied volatility for each of the stocks from their at-the-money call options and then compute the variance matrix using constant correlation:

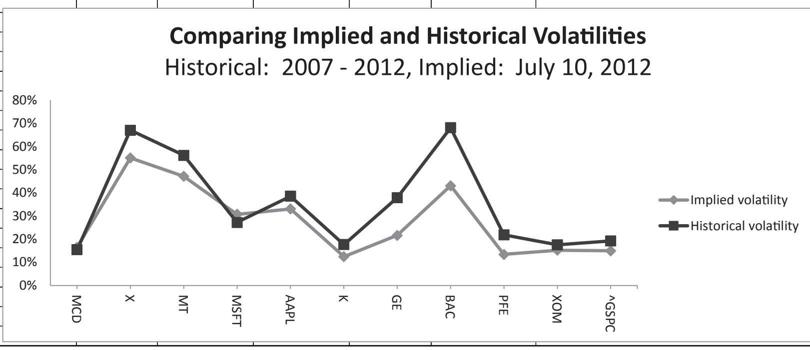
$$\sigma_{ij} = \begin{cases} \sigma^2_{i,implied} & \text{if } i = j \\ \rho\sigma_{i,implied}\sigma_{j,implied} & \text{if } i \neq j \end{cases}$$

Here's an example for our 10-stock case. We use data from the options markets and the function **CallVolatility** discussed in Chapter 17 to compute the implied volatility for each of the 10 stocks and the S&P 500. We use our data set of five years of returns to compute the historical volatility:

6. Three papers by Olivier Ledoit and Michael Wolf may offer some guidance: "Improved Estimation of the Covariance Matrix of Stock Returns with an Application to Portfolio Selection," *Journal of Empirical Finance*, 2003. "A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices," *Journal of Multivariate Analysis*, 2004. "Honey, I Shrunk the Sample Covariance Matrix," *Journal of Portfolio Management*, 2004.

7. This section uses some information from the chapters on options.

	A	B	C	D	E	F	G
1	**COMPARING IMPLIED AND HISTORICAL VOLATILITY**						
2	Current date	10-Jul-12					
3	Expire date	17-Aug-12			=callvolatility(B7,C7,($B$3-$B$2)/365,$B$4,D7)		
4	Interest	1.00%					
5							
6		**Stock price**	**Strike**	**Price**	**Implied volatility**	**Historical volatility**	
7	**MCD**	90.25	90.00	2.09	16.50%	15.42%	<-- {=TRANSPOSE('Betas etc'!B11:L11)}
8	**X**	20.28	20.00	1.58	55.05%	67.00%	
9	**MT**	14.67	15.00	0.75	47.12%	56.18%	
10	**MSFT**	29.74	30.00	1.07	30.74%	27.21%	
11	**AAPL**	608.21	610.00	25.30	33.04%	38.53%	
12	**K**	48.98	50.00	0.40	12.44%	17.67%	
13	**GE**	19.62	20.00	0.39	21.63%	37.91%	
14	**BAC**	7.48	8.00	0.22	43.04%	68.10%	
15	**PFE**	22.44	23.00	0.18	13.38%	21.87%	
16	**XOM**	83.11	85.00	0.90	15.23%	17.55%	
17	**^GSPC**	1,341.47	1,350.00	22.50	14.96%	19.21%	



We can now use the implied volatilities as the basis for a constant correlation variance-covariance matrix:

	A	B	C	D	E	F	G	H	I	J	K
1	**CONSTANT CORRELATION MATRIX WITH IMPLIED VOLATILITIES**										
2	Correlation	0.20									
3											
4		**MCD**	**X**	**MT**	**MSFT**	**AAPL**	**K**	**GE**	**BAC**	**PFE**	**XOM**
5	**MCD**	0.0272	0.0182	0.0156	0.0101	0.0109	0.0041	0.0071	0.0142	0.0044	0.0050
6	**X**	0.0182	0.3031	0.0519	0.0338	0.0364	0.0137	0.0238	0.0474	0.0147	0.0168
7	**MT**	0.0156	0.0519	0.2221	0.0290	0.0311	0.0117	0.0204	0.0406	0.0126	0.0143
8	**MSFT**	0.0101	0.0338	0.0290	0.0945	0.0203	0.0076	0.0133	0.0265	0.0082	0.0094
9	**AAPL**	0.0109	0.0364	0.0311	0.0203	0.1091	0.0082	0.0143	0.0284	0.0088	0.0101
10	**K**	0.0041	0.0137	0.0117	0.0076	0.0082	0.0155	0.0054	0.0107	0.0033	0.0038
11	**GE**	0.0071	0.0238	0.0204	0.0133	0.0143	0.0054	0.0468	0.0186	0.0058	0.0066
12	**BAC**	0.0142	0.0474	0.0406	0.0265	0.0284	0.0107	0.0186	0.1852	0.0115	0.0131
13	**PFE**	0.0044	0.0147	0.0126	0.0082	0.0088	0.0033	0.0058	0.0115	0.0179	0.0041
14	**XOM**	0.0050	0.0168	0.0143	0.0094	0.0101	0.0038	0.0066	0.0131	0.0041	0.0232
15		<-- {=ImpliedVol(B5:K14,E22:E31,B2)}									
16											
17	Current date	10-Jul-12									
18	Expire date	17-Aug-12									
19	Interest	1.00%									
20											
21		**Stock price**	**Strike**	**Price**	**Implied volatility**						
22	**MCD**	90.25	90.00	2.09	16.50%	<-- =callvolatility(B22,C22,($B$18-$B$17)/365,$B$19,D22)					
23	**X**	20.28	20.00	1.58	55.05%						
24	**MT**	14.67	15.00	0.75	47.12%						
25	**MSFT**	29.74	30.00	1.07	30.74%						
26	**AAPL**	608.21	610.00	25.30	33.04%						
27	**K**	48.98	50.00	0.40	12.44%						
28	**GE**	19.62	20.00	0.39	21.63%						
29	**BAC**	7.48	8.00	0.22	43.04%						
30	**PFE**	22.44	23.00	0.18	13.38%						
31	**XOM**	83.11	85.00	0.90	15.23%						

The programming of the **ImpliedVolVarCov** is similar to previous VBAs in this chapter:

```
Function ImpliedVolVarCov(varcovarmatrix As _
  Range, volatilities As Range, corr As Double)
  As Variant
  Dim i As Integer
  Dim j As Integer
  Dim numcols As Integer
  numcols = varcovarmatrix.Columns.Count
  numrows = numcols
  Dim matrix() As Double
  ReDim matrix(numcols - 1, numcols - 1)
  If Abs(corr) >= 1 Then GoTo Out
```

```
  For i = 1 To numcols
  For j = 1 To numcols
    If i = j Then
    matrix(i - 1, j - 1) = volatilities(i) ^ 2
    Else
    matrix(i - 1, j - 1) = corr * _
      volatilities(i) * volatilities(j)
    End If
  Next j
  Next i
Out:
  If Abs(corr) >= 1 Then ImpliedVolVarCov = _
    "ERR" Else ImpliedVolVarCov = matrix
End Function
```

10.10 Which Method to Compute the Variance-Covariance Matrix?

This chapter gives five alternatives to computing the variance-covariance matrix:

- The sample variance-covariance
- The single-index model
- The constant correlation approach
- Shrinkage methods
- Implied volatility-based variance-covariance matrices

How do we compare these alternatives? Which one should we choose? We could compare the technical outcomes of using each of these methods—for example, show the alternative values of the GMVP using different methods—but this largely misses the point.

The choice of how to compute the variance-covariance matrix is largely a question of how you view capital markets. If you strongly believe that the past predicts the future, then perhaps your choice should be to use the sample varcov matrix. This author prefers to get away from history … our preference is to use an option-based volatility model with a changing correlation: In "normal" times we would use a "normal" correlation of between 0.2 and 0.3; in times of crisis, we would use a much higher correlation, say, $\rho = 0.5 - 0.6$.

10.11 Summary

In this chapter we considered how to compute the variance-covariance matrix which is central to all portfolio optimization. Starting with the standard sample variance-covariance matrix, we also showed how to compute several alternatives that have appeared in the literature as perhaps improving portfolio computations.

Exercises

1. Below you will find annual return data for six furniture companies for the years 1982–1992. Use these data to calculate the variance-covariance matrix of the returns.

	A	B	C	D	E	F	G	H
1	**DATA FOR 6 FURNITURE COMPANIES**							
2		**La-Z-Boy**	**Kimball**	**Flexsteel**	**Leggett & Platt**	**Herman Miller**	**Shaw Industries**	
3	**1982**	36.67%	0.20%	41.54%	21.92%	26.13%	22.50%	
4	**1983**	122.82%	61.43%	195.09%	62.27%	73.38%	117.89%	
5	**1984**	14.44%	63.51%	-38.38%	-1.27%	45.15%	7.80%	
6	**1985**	21.39%	28.42%	1.30%	81.17%	24.27%	38.14%	
7	**1986**	45.36%	-7.44%	21.89%	19.83%	10.73%	54.48%	
8	**1987**	20.19%	48.27%	9.11%	-10.21%	-11.92%	26.82%	
9	**1988**	-8.94%	-11.28%	12.65%	13.77%	7.06%	-6.24%	
10	**1989**	27.02%	12.85%	12.08%	32.55%	-7.55%	123.03%	
11	**1990**	-11.64%	2.42%	-17.13%	-6.48%	1.31%	15.48%	
12	**1991**	20.29%	6.90%	3.62%	50.12%	-5.54%	19.92%	
13	**1992**	34.08%	22.21%	33.46%	84.40%	5.71%	62.76%	
14								
15	**Beta**	0.80	0.95	0.65	0.85	0.85	1.40	
16	**Mean returns**	29.24%	20.68%	25.02%	31.64%	15.34%	43.87%	<-- =AVERAGE(G3:G13)

The remaining exercises refer to the data in the tab **Price data** on the exercise spreadsheet that accompanies this book. This tab gives three years of price data for six stocks and the S&P 500 as a surrogate for the market.

2. Compute the returns of the data and the statistics for each of the assets (mean return, variance and standard deviation of return, beta).

3. Compute the sample variance-covariance matrix and the correlation matrix for the six stocks.

4. Use the function **SIM** defined in the chapter to compute the single-index variance-covariance matrix.

5. Compute the global minimum variance portfolio (GMVP) using the sample variance-covariance matrix.

6. Compute the GMVP using the constant correlation covariance matrix.

11 Estimating Betas and the Security Market Line

11.1 Overview

The capital asset pricing model (CAPM) is one of the two most influential innovations in financial theory in the latter half of the twentieth century.[1] By integrating the portfolio decision with utility theory and the statistical behavior of asset prices, the formulators of the CAPM defined the paradigm which is now generally used for the analysis of stock prices.

What does the CAPM actually say? What are its empirical implications? Roughly speaking, we can differentiate between two kinds of implications of the CAPM. First, the capital market line (CML) defines the *individual optimal portfolios* for an investor interested in the mean and variance of her optimal portfolio. Second, given agreement between investors on the statistical properties of asset returns and on the importance of mean-variance optimization, the security market line (SML) defines the *risk-return* relation for *each individual asset*.

It is useful to differentiate between the case wherein a risk-free asset exists and the case wherein there is no risk-free asset.[2]

Case 1: A Risk-Free Asset Exists

Suppose a risk-free asset exists and has return r_f. We can differentiate between the individual optimization of investors and the general equilibrium implications of the CAPM:

• *Individual optimization:* Assuming that investors optimize based on the expected return and standard deviation of their portfolio returns (in the jargon of finance—they have "mean-variance" preferences), the CAPM states that each individual investor's optimal portfolio falls on the line $E(r_p) = r_f + \sigma_p [E(r_x) - r_f]$, where portfolio x is a portfolio which maximizes $\frac{E(r_y) - r_f}{\sigma_y}$ for all feasible portfolios y. Proposition 1 of Chapter 9 shows that x can be computed by $x = \{x_1, x_2, \ldots, x_N\} = \frac{S^{-1}[E(r) - r_f]}{\sum S^{-1}[E(r) - r_f]}$, where

1. The other remarkable innovation is option pricing theory, which is discussed in Chapters 15–19. These two innovations together have accounted for a number of Nobel Prizes in Economics: Harry Markowitz (1990), William Sharpe (1990), Myron Scholes (1997), and Robert Merton (1997). But for their untimely demise, others associated with these theories—Jan Mossin (1936–1987), and Fischer Black (1938–1995)—would doubtless also have received the Nobel.

2. The existence (or non-existence) of a risk-free asset is closely related to the investment horizon. Assets which are risk-free over a short term may not be riskless over a longer term.

S is the variance-covariance matrix of risky asset returns and $E(r)=\{E(r_1), E(r_2), \ldots, E(r_N)\}$ is the vector of expected asset returns.

- *General equilibrium*: If all investors agree about the statistical assumptions of the model—the variance-covariance matrix S and the vector of expected asset returns $E(r)$—and if a risk-free asset exists, then individual asset returns are defined by the security market line (SML):

$$E(r_i)=r_f+\frac{Cov(r_i, r_M)}{\sigma_M^2}[E(r_M)-r_f]$$

where M denotes the market portfolio—the value-weighted portfolio of all risky assets. The expression $\frac{Cov(r_i, r_M)}{\sigma_M^2}$ is generally termed the asset's *beta*:

$$\beta_i=\frac{Cov(r_i, r_M)}{\sigma_M^2}.$$

Case 2: No Risk-Free Asset Exists

If there is no risk-free asset, then the implications of the CAPM both for individual optimization and for general equilibrium are defined by Black's (1972) zero-beta model (Proposition 3 of Chapter 9):

- *Individual optimization*: In the absence of a risk-free asset, individual optimal portfolios will fall along the *efficient frontier*. As shown in Proposition 2 of Chapter 9, this frontier is the upward-sloping portion of the mean-sigma combinations created by the convex combination of any two optimizing portfolios $x=\frac{S^{-1}[E(r)-c_1]}{\sum S^{-1}[E(r)-c_1]}$ and $y=\frac{S^{-1}[E(r)-c_2]}{\sum S^{-1}[E(r)-c_2]}$, where c_1 and c_2 are two arbitrary constants.
- *General equilibrium*: In the absence of a risk-free asset, if all investors agree about the statistical assumptions of the model—the variance-covariance matrix S and the vector of expected asset returns $E(r)$—then individual asset returns are defined by the security market line (SML):

$$E(r_i)=E(r_z)+\frac{Cov(r_i, r_y)}{\sigma_y^2}[E(r_y)-E(r_z)]$$

where y is any efficient portfolio and z is a portfolio which has zero covariance with y (the so-called "zero beta portfolio").

The case of no risk-free asset is obviously weaker than the case of a risk-free asset. If there is a risk-free asset, the general equilibrium version of the CAPM says that all portfolios are situated on a single, agreed-upon line. If there is no risk-free asset, then all optimal portfolios are on the same frontier; but in this case asset betas can differ, since there are many portfolios y which fulfill the equation $E(r_i) = E(r_z) + \frac{Cov(r_i, r_y)}{\sigma_y^2}\left[E(r_y) - E(r_z)\right]$.

The CAPM as a Prescriptive and a Descriptive Tool

As you can see from the discussion above, the CAPM is both *prescriptive* and *descriptive*.

As a prescriptive tool, the CAPM tells a mean-variance investor how to choose his optimal portfolio. By finding a portfolio of the form $\frac{S^{-1}[E(r) - c_1]}{\sum S^{-1}[E(r) - c_1]}$, the investor can identify an optimal portfolio from the data set.

As a descriptive tool, the CAPM gives conditions under which we can generalize about the structure of expected returns in the market. Whether or not a risk-free asset exists, these conditions assume that investors agree on the statistical structure of asset returns—the variance-covariance matrix and the expected returns. In this case all the returns are expected to lie on a security market line (SML) of the form $E(r_i) = r_f + \frac{Cov(r_i, r_M)}{\sigma_M^2}\left[E(r_M) - r_f\right]$ (if there is a risk-free asset) or of the form $E(r_i) = E(r_z) + \frac{Cov(r_i, r_y)}{\sigma_y^2}\left[E(r_y) - E(r_z)\right]$ if there is no risk-free asset.

This Chapter

In this chapter we look at some typical capital market data and replicate a simple test of the descriptive part of the CAPM. This means that we have to calculate the betas for a set of assets, and we then have to determine the equation of the security market line (SML). The test in this chapter is the simplest possible test of the CAPM. There is an enormous literature in which the possible statistical and methodological pitfalls of CAPM tests are discussed. Good places to begin are textbooks by Elton, Gruber, Brown, and Goetzmann (2009), and Bodie, Kane, and Marcus (2010).[3]

11.2 Testing the SML

Typical tests of the security market line (SML) start with return data on a set of risky assets. The steps in the test are as follows:

- Determine a candidate for the market portfolio M. In our example we will use the Standard & Poor's 500 Index (S&P 500) as a candidate for M. This is a critical step: In principle, the "true" market portfolio should—as pointed out in Chapter 9—contain all the market's risky assets in proportion to their value. It is clearly impossible to calculate this theoretical market portfolio, and we must therefore make do with a surrogate. As you will see in the next two sections, the propositions of Chapter 9 can shed much light on how the choice of the market surrogate affects the r-squared of our regression test of the CAPM.

- For each of the assets in question, determine the asset beta, $\beta_i = \dfrac{Cov(r_i, r_M)}{\sigma_M^2}$.

This is often called the *first-pass regression*.

- Regress the mean returns of the assets on their respective betas (the *second-pass regression*):

$$\bar{r}_i = \gamma_0 + \gamma_1 \beta_i$$

If the CAPM in its descriptive format holds, then the second-pass regression should be the security market line.[4]

We illustrate the tests of the CAPM with a simple numerical example that uses data for the 30 stocks in the Dow-Jones Industrials. We start with the prices of the S&P 500 (symbol ^GSPC) and the stocks in the DJ30 (some of the rows and columns are not shown):

3. For further references, see the Selected References at the end of this book. Our personal expositional favorite is a paper by Roll, "Ambiguity When Performance Is Measured by the Securities Market Line," *Journal of Finance* (1978).

4. This is a direct consequence of Propositions 3 and 4 of Chapter 9.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**PRICE DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **July 2001 - July 2006**													
2	**Date**	**S&P500 Index ^GSPC**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3	03-Jul-01	1211.23	35.37	81.32	33.73	53.64	40.97	24.73	36.29	25.02	38.33	49.72	48.26	32.75
4	01-Aug-01	1133.58	34.50	76.43	30.46	47.06	37.49	22.45	35.01	24.14	36.04	43.14	44.06	33.27
5	04-Sep-01	1040.94	28.06	76.24	24.30	30.79	33.15	20.11	32.07	17.68	32.93	33.81	36.79	23.57
6	01-Oct-01	1059.78	29.34	76.82	24.68	29.97	37.26	20.23	34.18	17.65	32.23	32.56	36.66	26.38
7	01-Nov-01	1139.45	35.09	80.54	27.60	32.43	39.34	21.45	38.21	19.43	34.08	39.63	44.78	29.77
8	03-Dec-01	1148.08	32.32	77.64	29.93	35.83	41.46	23.63	36.63	19.88	35.63	38.75	48.97	30.38
9	02-Jan-02	1130.20	32.59	72.51	30.13	37.83	39.06	22.91	38.06	20.21	33.03	40.77	48.09	30.19
10	01-Feb-02	1106.73	34.31	72.38	30.64	42.64	37.29	25.29	40.68	22.07	34.39	42.67	48.00	34.43
51	01-Jul-05	1234.18	27.48	59.76	47.68	65.02	42.62	53.28	41.22	25.37	33.53	34.89	42.97	38.43
52	01-Aug-05	1220.33	26.38	58.92	47.89	66.26	42.89	54.85	38.55	24.93	32.66	32.86	39.92	37.66
53	01-Sep-05	1228.81	24.05	61.67	49.79	67.18	44.60	58.07	38.16	23.88	32.94	29.42	37.76	36.89
54	03-Oct-05	1207.01	23.92	64.49	49.41	63.91	44.86	52.22	40.62	24.11	33.17	26.33	40.63	33.65
55	01-Nov-05	1249.48	27.16	66.82	51.04	67.67	48.04	57.37	42.02	24.67	34.94	21.44	41.46	36.15
56	01-Dec-05	1248.29	29.30	67.91	51.08	69.71	48.02	57.36	41.77	23.97	34.53	19.01	40.17	36.85
57	03-Jan-06	1280.08	31.21	65.15	52.19	67.79	46.09	67.69	38.48	25.31	32.27	23.56	40.24	38.01

We first transform these price data to returns:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**RETURN DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **July 2001 - July 2006**													
2	**Date**	**S&P 500 Index ^GSPC**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3				=AVERAGE(C9:C68)										
4	Average return	0.07%	-0.09%	[illegible]	%	0.67%	0.30%	1.79%	0.18%	0.29%	-0.23%	-0.87%	-0.52%	0.29%
5	Beta	1.00	1.90		=SLOPE(C9:C68,$B$9:$B$68)		1.30	1.39	1.00	1.28	0.84	1.41	1.55	1.66
6	Alpha	0	-0.23%	-0.61%	0.62%	0.58%	0.21%	1.69%	0.11%	0.20%	-0.30%	-0.97%	-0.63%	0.17%
7	R-squared	1	0.6085	=INTERCEPT(C9:C68,$B$9:$B$68)			972	0.5158	0.4362	0.3845	0.3221	0.2607	0.5288	0.5473
8														
9	01-Aug-01	-6.63%	-2.49%	-6.20%	-10.20%	-13.09%	-8.88%	-9.67%	-3.59%	-3.58%	-6.16%	-14.20%	-9.11%	1.58%
10	04-Sep-01	-8.53%	-20.66%	=RSQ(C9:C68,$B$9:$B$68)			-12.30%	-11.01%	-8.77%	-31.14%	-9.02%	-24.37%	-18.03%	-34.47%
11	01-Oct-01	1.79%	4.46%	[illegible]	[illegible]	[illegible]	11.69%	0.59%	6.37%	-0.17%	-2.15%	-3.77%	-0.35%	11.26%
12	01-Nov-01	7.25%	17.90%	4.73%	11.18%	7.89%	5.43%	5.86%	11.15%	9.61%	5.58%	19.65%	20.01%	12.09%
13	03-Dec-01	0.75%	-8.22%	-3.67%	8.10%	9.97%	5.25%	9.68%	-4.22%	2.29%	4.45%	-2.25%	8.94%	2.03%
14	02-Jan-02	-1.57%	0.83%	-6.84%	0.67%	5.43%	-5.96%	-3.09%	3.83%	1.65%	-7.58%	5.08%	-1.81%	-0.63%
15	01-Feb-02	-2.10%	5.14%	-0.18%	1.68%	11.97%	-4.64%	9.88%	6.66%	8.80%	4.03%	4.55%	-0.19%	13.14%
16	01-Mar-02	3.61%	0.47%	-2.52%	11.66%	4.85%	9.02%	2.38%	0.66%	0.36%	-2.89%	13.20%	-2.72%	0.41%

The First-Pass Regression

Row 4 gives each asset's average monthly return over the 60-month period (to annualize these returns, we would multiply by 12). Rows 5–7 report the results of the *first-pass regression*. For each asset i we report the regression $r_{it} = \alpha_i + \beta_i r_{SP,t}$. We use the Excel function **Slope** to compute the β of each asset, and the functions **Intercept** and **Rsq** to compute the α and R^2 for each regression.

As a check, we also compute the α, β, and R^2 for the S&P 500 index (column B). Not surprisingly, $\alpha_{SP} = 0$, $\beta_{SP} = 1$, $R^2 = 1$.

The Second-Pass Regression

The SML postulates that the mean return of each security should be linearly related to its beta. Assuming that the historic data provide an accurate description of the distribution of future returns, we postulate that $E(R_i) = \alpha + \beta_i \Pi + \varepsilon_i$, where the definitions of α and Π depend on whether we are in Case 1 or Case 2 of section 11.1:

$$\alpha = \begin{cases} r_f & \text{Case 1: there exists a risk-free asset} \\ E(r_z) & \text{Case 2: no risk-free asset. } z \text{ has zero correlation with efficient portfolio } y \end{cases}$$

$$\Pi = \begin{cases} E(r_M) - r_f & \text{Case 1} \\ E(r_y) - E(r_z) & \text{Case 2} \end{cases}$$

In the second step of our test of the CAPM, we examine this hypothesis by regressing the mean returns on the β's.

	A	B	C	D	E	F	G	H
1	THE SECOND-PASS REGRESSION							
2	Stock	Average monthly return	Beta	Alpha				
3	Alcoa AA	-0.09%	1.9028	-0.0023		Second-pass regression, regressing monthly returns on Beta		
4	American International Group AIG	-0.54%	0.9936	-0.0061		Intercept	0.0036	<-- =INTERCEPT(B3:B32,C3:C32)
5	American Express AXP	0.72%	1.3784	0.0062		Slope	-0.0020	<-- =SLOPE(B3:B32,C3:C32)
6	Boeing BA	0.67%	1.1515	0.0058		R-squared	0.0238	<-- =RSQ(B3:B32,C3:C32)
7	Citigroup C	0.30%	1.2952	0.0021				
8	Caterpillar CAT	1.79%	1.3903	0.0169		t-statistic, intercept	1.2381	<-- =tintercept(B3:B32,C3:C32)
9	DuPont DD	0.18%	1.0009	0.0011		t-statistic, slope	-0.8254	<-- =tslope(B3:B32,C3:C32)
10	Disney DIS	0.29%	1.2805	0.0020				
11	General Electric GE	-0.23%	0.8420	-0.0030				
12	General Motors GM	-0.87%	1.4060	-0.0097				
13	Home Depot HD	-0.52%	1.5528	-0.0063				
14	Honeywell HON	0.29%	1.6640	0.0017				
15	Hewlett Packard HPQ	0.61%	1.9594	0.0046				
16	IBM	-0.47%	1.5764	-0.0058				
17	Intel INTC	-0.73%	2.2648	-0.0089				
18	Johnson & Johnson JNJ	0.34%	0.2471	0.0032				
19	JP Morgan JPM	0.18%	1.7917	0.0005				
20	Coca Cola KO	0.12%	0.3590	0.0009				
21	McDonalds MCD	0.35%	1.2646	0.0025				
22	3M MMM	0.64%	0.6504	0.0059				
23	AltriaMO	1.30%	0.6633	0.0125				
24	Merck MRK	-0.63%	0.6099	-0.0068				
25	Microsoft MSFT	-0.35%	1.1219	-0.0043				
26	Pfizer PFE	-0.74%	0.5572	-0.0078				
27	Proctor Gamble PG	0.94%	0.1687	0.0093				
28	AT&T T	-0.41%	1.1275	-0.0050				
29	United Technologies UTX	1.03%	1.0659	0.0095				
30	Verizon VZ	-0.49%	1.0231	-0.0057				
31	Walmart WMT	-0.25%	0.6000	-0.0030				
32	Exxon Mobil XOM	0.88%	0.6455	0.0083				

The results (cells F4:G6) are very disappointing. Our test yields the following SML:

$$E(r_i) = \underbrace{0.0036}_{\substack{\uparrow \\ \gamma_0}} - \underbrace{0.0020}_{\substack{\uparrow \\ \gamma_1}} \beta_i,\ R^2 = 0.0238$$

There is nothing about these numbers which inspires confidence:

- γ_0 should correspond to the risk-free rate over the period. In section 11.9 we discuss this rate, which changed wildly over the 60 months surveyed. At this point it is enough to point out that the average monthly risk-free interest rate was 0.18% (or 0.0018, exactly half of γ_0).
- γ_1 should correspond to $E(r_M) - r_f$. The average monthly return of the S&P 500 over the period was –0.10% and the average monthly risk-free interest rate was 0.18%, so that γ_1 should be approximated by –0.28% (or 0.0028).
- Both the *t*-statistics for the *i* (cell G8) and the slope (cell G9) indicate that they are not statistically different from zero.[5]

Our test of the SML has failed. The CAPM may have prescriptive validity, but it does not describe our data.

Why Are the Results So Bad?

The experiment we did—checking the CAPM by plotting the security market line—does not appear to have worked out very well. There does not appear to be much evidence in favor of the SML: Neither the R^2 of the regression nor the *t*-statistics give much evidence that there is a relation between expected return and portfolio β.

There are a number of reasons why these disappointing results may hold:

- One reason is that perhaps the CAPM itself does not hold. This could be true for a variety of reasons:
  - ◊ Perhaps in the market short sales of assets are restricted. Our derivation of the CAPM (see Chapter 9 on efficient portfolios) assumes that there are no short-sale restrictions. Clearly this is an unrealistic assumption. The computation of efficient portfolios when short sales are restricted is

5. The functions **TIntercept** and **TSlope** were created by the author. They are attached to the spreadsheet for this chapter and are discussed in Chapter 3.

considered in Chapter 12. In this case, however, there are no simple relations (such as those proved in Chapter 9) between the returns of assets and their betas. In particular, if short sales are restricted, there is no reason to expect the SML to hold.

◊ Perhaps individuals do not have homogeneous probability assessments, or perhaps they do not have the same expectations of asset returns, variances, and covariances.

• Perhaps the CAPM holds only for portfolios and not for single assets.

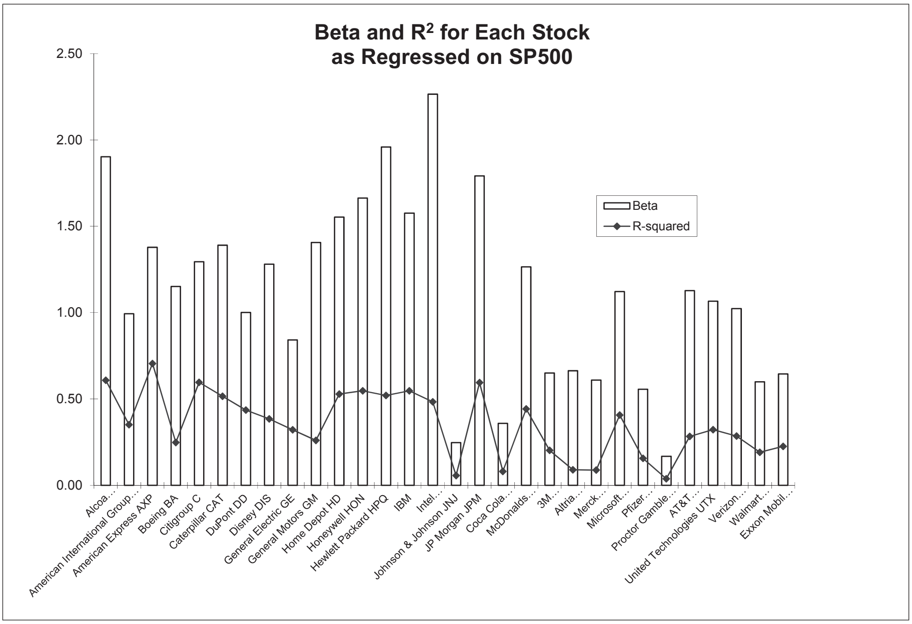
• Perhaps our set of assets isn't large enough: After all, the CAPM talks about *all risky assets*, whereas we have chosen—for illustrative purposes—to do our test on a very small subset of these assets. The literature on CAPM testing records tests in which the set of risky assets has been expanded to include bonds, real estate, and even non-diversifiable assets such as human capital.

• Perhaps the "market portfolio" isn't efficient. This possibility is suggested by the mathematics of Chapter 9 on efficient portfolios, and it is this suggestion which we further explore in the next section.

• Perhaps the CAPM holds only if the market returns are positive (in the period surveyed they were, on average, negative).

11.3 Did We Learn Something?

The results of our exercise in section 11.1 are quite disappointing. Did we learn anything positive from this exercise? Absolutely. For example, the regression model does a pretty good job of describing individual asset returns in relation to the S&P 500:



On average the S&P 500 describes about 35% of the variability of the DJ30 stocks, which have an average beta of 1.12. If we exclude the seven stocks with the lowest R^2, the S&P describes almost 43% of the variation in the stocks' returns:

	A	B	C	D	E	F	G	H	I
1	**OUR SML EXERCISE: WHAT DID WE LEARN?**								
2	Average alpha	0.06%	<-- =AVERAGE('Page 277 bottom'!C6:AF6)						
3	Average beta	1.12	<-- =AVERAGE('Page 277 bottom'!C5:AF5)						
4	Average r-squared	0.3510	<-- =AVERAGE('Page 277 bottom'!C7:AF7)						
5									
6	**Average R^2 for best regressions**								
7	Cutoff for R^2	0.2	<-- Below we count all R^2 which are greater than this number						
8		9.8258	<-- =SUMIF('Page 277 bottom'!C7:AF7,">"&TEXT(B7,"0.00"))						
9		23	<-- =COUNTIF('Page 277 bottom'!C7:AF7,">"&TEXT(B7,"0.00"))						
10	Average R^2	0.4272	<-- =B8/B9						
11									
12	**T-statistics for intercept and slope**								
13		Alcoa AA	American International Group AIG	American Express AXP	Boeing BA	Citigroup C	Caterpillar CAT	DuPont DD	Disney DIS
14	t-stat for intercept	0.3144	0.6324	-1.0525	-0.1584	-0.2013	-1.6120	-0.0192	-0.0371
15	t-stat for slope	9.4942	5.6112	11.7783	4.3815	9.2729	7.8607	6.6993	6.0199
16									
17	Average absolute t-stat for intercept	0.3998	<-- {=AVERAGE(ABS(B14:AE14))}						
18	Average t-stat for slope	5.7866	<-- =AVERAGE(B15:AE15)						

Cell B10 above computes the average R^2 for those regressions which had an $R^2 > 0.2$. This is 23 of the Dow-Jones 30. So—on average the first-pass regressions are very significant. The average R^2 of 35% that we got for our first-pass regressions of the basic SML is actually a respectable number in finance. Students—influenced by over-enthusiastic statistics instructors and an overly linear view of the world—often feel that the R^2 of any convincing regression should be at least 90%. Finance does not appear to be a highly linear profession: A good rule of thumb is that any financial regression that gives an R^2 greater than 80% is possibly misspecified and misleading.[6]

Another way to look at the significance of our results is to compute the *t*-statistics for the intercept and slope of the first-pass regressions (rows 14–15 above). While the intercepts are not significantly different from zero (since their *t*-statistic is less than 2), the slopes are very significant.

An Excel Note: Computing the Absolute Value of an Array of Numbers

In the computations above we use a neat Excel trick related to array functions (see Chapter 34). By using **Abs** as an array function (that is, by entering the function using [Ctrl]+[Shift]+[Enter]), we can compute the average of the absolute values of a vector of numbers. A simple example is shown below:

	A	B	C	D	E	F
1	**USING ABS FUNCTION IN ARRAY** **The Excel "Abs" function computes the absolute value** **If we use it as an array function, it can be applied to a range of numbers**					
2						
3	Numbers	1	-2	-3	-6	8
4						
5	Average number	-0.4000	<-- =AVERAGE(B3:F3)			
6	Average absolute number	4.0000	<-- {=AVERAGE(ABS(B3:F3))}			
7	The above, but not as array function	1.0000	<-- =AVERAGE(ABS(B3:F3))			

Notice cell B7: Using the same function as a regular function does not produce the correct answer.

6. An exception to this useful rule relates to diversified portfolios—here the R^2 increases dramatically.

11.4 The Non-Efficiency of the "Market Portfolio"

When we calculated the SML in section 11.1, we regressed the mean return of each asset on the returns of the market portfolio. The propositions of Chapter 9 on efficient portfolios suggest that our failure to find adequate results may stem from the fact that the S&P 500 portfolio is not efficient relative to the set of the six assets which we have chosen. Proposition 3 of Chapter 9 states that if we had chosen to regress our asset returns on a portfolio that is efficient with respect to the asset set itself, we would get an *r*-squared of 100%. Proposition 4 of Chapter 9 shows that if we get an *r*-squared of 100% then the portfolio on which we regress the asset returns is necessarily efficient with respect to the set of assets. In this section we give a numerical illustration of these propositions.

In the spreadsheet below we create a "mysterious portfolio" in column B. This portfolio (its construction is described in the next subsection) is efficient with respect to the Dow-Jones 30. As you can see in cells A10:B12, when we perform the second-pass regression—regressing the individual average returns of the assets on their betas computed with respect to the mysterious portfolio—the results are perfect. The resulting regression has an intercept of 0.0030 and a slope of 0.0425. Most important—it has an R^2 of 100%.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**RETURN DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **July 2001 - July 2006**													
2	**Date**	**Mysterious portfolio**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3														
4	Average return	4.55%	-0.09%	-0.54%	0.72%	0.67%	0.30%	1.79%	0.18%	0.29%	-0.23%	-0.87%	-0.52%	0.29%
5	Beta		-0.09	-0.20	0.10	0.09	0.00	0.35	-0.03	0.00	-0.13	-0.28	-0.19	0.00
6	Alpha		0.33%	0.36%	0.27%	0.27%	0.30%	0.19%	0.31%	0.30%	0.34%	0.38%	0.36%	0.30%
7	R-squared		0.0025	0.0242	0.0064	0.0024	0.0000	0.0579	0.0006	0.0000	0.0126	0.0176	0.0143	0.0000
8														
9	**SML--regressing the average returns on the betas**													
10	Intercept	0.0030	<-- =INTERCEPT(C4:AF4,C5:AF5)											
11	Slope	0.0425	<-- =SLOPE(C4:AF4,C5:AF5)											
12	R-squared	1.0000	<-- =RSQ(C4:AF4,C5:AF5)											
13														
14														
15	01-Aug-01	-1.01%	-2.49%	-6.20%	-10.20%	-13.09%	-8.88%	-9.67%	-3.59%	-3.58%	-6.16%	-14.20%	-9.11%	1.58%
16	04-Sep-01	0.40%	-20.66%	-0.25%	-22.59%	-42.42%	-12.30%	-11.01%	-8.77%	-31.14%	-9.02%	-24.37%	-18.03%	-34.47%
17	01-Oct-01	4.71%	4.46%	0.76%	1.55%	-2.70%	11.69%	0.59%	6.37%	-0.17%	-2.15%	-3.77%	-0.35%	11.26%
18	01-Nov-01	-1.33%	17.90%	4.73%	11.18%	7.89%	5.43%	5.86%	11.15%	9.61%	5.58%	19.65%	20.01%	12.09%
19	03-Dec-01	8.11%	-8.22%	-3.67%	8.10%	9.97%	5.25%	9.68%	-4.22%	2.29%	4.45%	-2.25%	8.94%	2.03%

The Mysterious Portfolio Is Efficient

The propositions of Chapter 9 leave us with only one conclusion: The "mysterious portfolio" must be efficient with respect to the DJ30. And so it is. In the spreadsheet below we show the construction of this efficient portfolio, which follows the propositions of Chapter 9.

- We first construct the variance-covariance matrix S using the function **Varcovar** defined in Chapter 10.
- We then compute the efficient portfolio by solving $\dfrac{S^{-1}[E(r)-c]}{\sum S^{-1}[E(r)-c]}$. In the spreadsheet below we use $c = 0.0030$, which then turns out to be the intercept of the second-pass regression.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**RETURN DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **July 2001 - July 2006**													
2	**Date**	**SP 500 Index ^SPX**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3														
4	Average return	0.07%	-0.09%	=AVERAGE(C9:C68)	.72%	0.67%	0.30%	1.79%	0.18%	0.29%	-0.23%	-0.87%	-0.52%	0.29%
5	Beta	1.00	1.90	[illegible]	=SLOPE(C9:C68,$B$9:$B$68)	[illegible]	1.30	1.39	1.00	1.28	0.84	1.41	1.55	1.66
6	Alpha	0	-0.23%	[illegible]	[illegible]	[illegible]	0.21%	1.69%	0.11%	0.20%	-0.30%	-0.97%	-0.63%	0.17%
7	R-squared	1	0.6085	=INTERCEPT(C9:C68,$B$9:$B$68)	[illegible]	[illegible]	0.5972	0.5158	0.4362	0.3845	0.3221	0.2607	0.5288	0.5473
8														
9	01-Aug-01	-6.63%	-2.49%	-6.20%	-10.20%	-13.09%	-8.88%	-9.67%	-3.59%	-3.58%	-6.16%	-14.20%	-9.11%	1.58%
10	04-Sep-01	-8.53%	-20.66%	=RSQ(C9:C68,$B$9:$B$68)	[illegible]	42%	-12.30%	-11.01%	-8.77%	-31.14%	-9.02%	-24.37%	-18.03%	-34.47%
11	01-Oct-01	1.79%	4.46%	[illegible]	[illegible]	2.70%	11.69%	0.59%	6.37%	-0.17%	-2.15%	-3.77%	-0.35%	11.26%
12	01-Nov-01	7.25%	17.90%	4.73%	11.18%	7.89%	5.43%	5.86%	11.15%	9.61%	5.58%	19.65%	20.01%	12.09%
13	03-Dec-01	0.75%	-8.22%	-3.67%	8.10%	9.97%	5.25%	9.68%	-4.22%	2.29%	4.45%	-2.25%	8.94%	2.03%
14	02-Jan-02	-1.57%	0.83%	-6.84%	0.67%	5.43%	-5.96%	-3.09%	3.83%	1.65%	-7.58%	5.08%	-1.81%	-0.63%
15	01-Feb-02	-2.10%	5.14%	-0.18%	1.68%	11.97%	-4.64%	9.88%	6.66%	8.80%	4.03%	4.55%	-0.19%	13.14%
67	01-Jun-06	0.01%	2.00%	-2.92%	-2.13%	-1.62%	-2.15%	2.08%	-2.21%	-1.65%	-3.14%	10.09%	-5.91%	-2.16%
68	03-Jul-06	-0.37%	3.61%	-0.14%	-2.08%	-2.37%	1.71%	-2.58%	-2.58%	-0.57%	1.03%	-1.05%	-1.18%	-3.38%
69														
70														
71	**The cells below compute the variance-covariance matrix for the DJ30 by using the formula {=varcovar(C9:AF68)}**													
72		AA	AIG	AXP	BA	C	CAT	DD	DIS	GE	GM	HD	HON	HPQ
73	AA	0.0093	0.0032	0.0038	0.0045	0.0033	0.0046	0.0040	0.0042	0.0024	0.0046	0.0046	0.0061	0.0060
74	AIG	0.0032	0.0044	0.0020	0.0010	0.0023	0.0022	0.0022	0.0015	0.0015	0.0019	0.0021	0.0027	0.0024
75	AXP	0.0038	0.0020	0.0042	0.0030	0.0031	0.0029	0.0020	0.0037	0.0019	0.0033	0.0033	0.0042	0.0050
76	BA	0.0045	0.0010	0.0030	0.0083	0.0014	0.0029	0.0021	0.0037	0.0020	0.0043	0.0026	0.0053	0.0042
99	UTX	0.0038	0.0012	0.0031	0.0050	0.0022	0.0026	0.0021	0.0038	0.0018	0.0030	0.0023	0.0049	0.0048
100	VZ	0.0028	0.0018	0.0019	0.0007	0.0023	0.0022	0.0018	0.0014	0.0013	0.0006	0.0016	0.0024	0.0030
101	WMT	0.0018	0.0014	0.0011	0.0003	0.0015	0.0011	0.0014	0.0009	0.0009	0.0012	0.0024	0.0008	0.0008
102	XOM	0.0024	0.0010	0.0010	0.0017	0.0009	0.0021	0.0012	0.0004	0.0007	0.0020	0.0007	0.0020	0.0012
103														
104	**Finding an efficient portfolio**													
105	Constant	0.30%												
106														
107	AA	5.45%	<-- {=MMULT(MINVERSE(varcov),TRANSPOSE(C4:AF4)-B105)/SUM(MMULT(MINVERSE(varcov),TRANSPOSE(C4:AF4)-B105))}											
108	AIG	-11.77%												
109	AXP	-5.78%												
110	BA	-13.94%												
111	C	-36.60%												
112	CAT	76.26%												
113	DD	-22.55%												
114	DIS	-17.05%												
115	GE	-8.80%												
116	GM	-37.73%												
117	HD	-37.21%												
118	HON	-17.40%												
119	HPQ	39.79%												
120	IBM	-26.38%												
121	INTC	-18.62%												
122	JNJ	65.08%												
123	JPM	53.61%												
124	KO	-12.95%												
125	MCD	-12.18%												
126	MMM	-2.07%												
127	MP	42.15%												
128	MRK	8.29%												
129	MSFT	3.61%												
130	PFE	-61.19%												
131	PG	54.67%												
132	T	-8.38%												
133	UTX	44.05%												
134	VZ	-36.55%												
135	WMT	64.78%												
136	XOM	29.40%												
137	Sum	100.00%												

The “mysterious portfolio” is not unique. Following we show results using another constant *c,* which gives another version of the SML.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**RETURN DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **July 2001 - July 2006**													
2	**Date**	**Mysterious portfolio**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3														
4	Average return	8.88%	-0.09%	-0.54%	0.72%	0.67%	0.30%	1.79%	0.18%	0.29%	-0.23%	-0.87%	-0.52%	0.29%
5	Beta		-0.07	-0.12	0.03	0.02	-0.02	0.15	-0.04	-0.02	-0.09	-0.16	-0.12	-0.03
6	Alpha		0.54%	0.56%	0.49%	0.49%	0.51%	0.42%	0.52%	0.51%	0.54%	0.58%	0.56%	0.51%
7	R-squared		0.0061	0.0399	0.0019	0.0005	0.0015	0.0467	0.0045	0.0010	0.0256	0.0259	0.0237	0.0009
8														
9	**SML--regressing the average returns on the betas**													
10	Intercept	0.0050	<-- =INTERCEPT(C4:AF4,C5:AF5)											
11	Slope	0.0838	<-- =SLOPE(C4:AF4,C5:AF5)											
12	R-squared	1.0000	<-- =RSQ(C4:AF4,C5:AF5)											

Note also that even though the R^2 of the second pass regression is 100% (since the “mysterious portfolio” is efficient), the R^2’s of the individual first-pass regressions are far from notable.

11.5 So What’s the Real Market Portfolio? How Can We Test the CAPM?

A little reflection will reveal that although the “mysterious” portfolio of the previous section may be efficient with respect to the 30 stocks of the Dow-Jones, it could not be the *true market portfolio*, even if the DJ30 stocks represented the whole universe of risky securities. This is because many of the stocks appear in the “mysterious portfolio” with negative weights. Surely a minimal characteristic of the market portfolio must be that all shares appear in it with *positive proportions*.

Roll (1977, 1978) suggests that the only test of the CAPM is to answer the question: *Is the true market portfolio mean-variance efficient?* If the answer to this question is “yes,” then it follows from Proposition 3 of Chapter 9 that a linear relation holds between the mean of each portfolio and its β. In our example, we can shed some light on this question by building a table of the asset proportions of portfolios on the efficient frontier.

In the table below we give some evidence that all efficient portfolios for the DJ30 contain significant short positions. Using the wonders of Excel’s **Data Table**, we compute the largest short and long positions for a series of efficient portfolios, each defined by its own constant *c*. All of these portfolios contain large short positions (and, as you can see, also large long positions):

	A	B	C	D	E	F	G	H	I
105	**An efficient portfolio**			**Data table: computing the largest short and long position for a given constant c**					
106	Constant	0.30%			**Largest short**	**Largest long**			
107				**Constant c**			<-- Data table hidden: =B141		
108	AA	5.5%		0.00%	-32.64%	52.33%			
109	AIG	-11.8%		0.05%	-35.51%	53.58%			
110	AXP	-5.8%		0.10%	-38.87%	55.05%			
111	BA	-13.9%		0.15%	-42.86%	56.79%			
112	C	-36.6%		0.20%	-47.69%	59.70%			
113	CAT	76.3%		0.25%	-53.65%	67.01%			
114	DD	-22.6%		0.30%	-61.19%	76.26%			
115	DIS	-17.0%		0.35%	-71.01%	88.32%			
116	GE	-8.8%		0.40%	-84.36%	104.71%			
117	GM	-37.7%		0.45%	-103.56%	128.28%			
118	HD	-37.2%		0.50%	-133.51%	165.05%			
119	HON	-17.4%		0.55%	-186.77%	230.42%			
120	HPQ	39.8%		0.60%	-307.86%	379.08%			
121	IBM	-26.4%		0.65%	-853.66%	1049.09%			
122	INTC	-18.6%		0.70%	-1398.93%	1140.50%			
123	JNJ	65.1%		0.75%	-422.59%	345.18%			
124	JPM	53.6%		0.80%	-249.90%	204.50%			
125	KO	-13.0%							
126	MCD	-12.2%							
127	MMM	-2.1%							
128	MP	42.1%							
129	MRK	8.3%							
130	MSFT	3.6%							
131	PFE	-61.2%							
132	PG	54.7%							
133	T	-8.4%							
134	UTX	44.1%							
135	VZ	-36.6%							
136	WMT	64.8%							
137	XOM	29.4%							
138	Sum	100.0%							
139									
140	Largest short	-61.2%	<-- =MIN(B108:B137)						
141	Largest long	76.3%	<-- =MAX(B108:B137)						

Our depressing conclusion: If the data for the DJ30 and the S&P 500 are representative, the CAPM as a descriptive theory of capital markets appears not to work.[7]

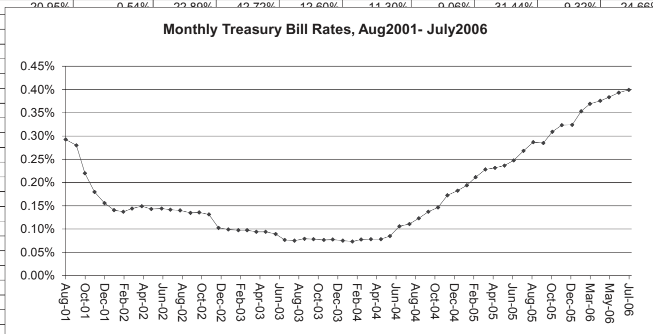
11.6 Using Excess Returns

Perhaps we should have conducted our experiment on the CAPM in terms of excess returns—the difference between the stocks' monthly returns and the risk-free rates? In this section we perform this variation on the experiment and show that it does little to improve our analysis.

7. All is not lost! In Chapter 13 we examine the Black-Litterman model, which is a more positivist approach to portfolio choice.

Below we show the same Dow-Jones data, with an additional column appended for Treasury bill returns; these varied wildly over the period:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	**EXCESS RETURN DATA FOR THE DOW-JONES INDUSTRIAL STOCKS AND THE STANDARD AND POORS 500** **Monthly returns minus monthly Treasury bill return** **July 2001 - July 2006**														
2	**Date**	**Treasury bill return risk-free rate**	**S&P 500 Index ^GSPC**	**Alcoa AA**	**American International Group AIG**	**American Express AXP**	**Boeing BA**	**Citigroup C**	**Caterpillar CAT**	**DuPont DD**	**Disney DIS**	**General Electric GE**	**General Motors GM**	**Home Depot HD**	**Honeywell HON**
3															
4	Average return		-0.22%	-0.38%	-0.83%	0.43%	0.37%	0.01%	1.50%	-0.11%	0.00%	-0.53%	-1.16%	-0.81%	0.00%
5	Beta		1.00	1.90	0.99	1.38	1.15	1.30	1.39	1.00	1.28	0.84	1.41	1.55	1.66
6	Alpha		0	0.04%	-0.61%	0.73%	0.63%	0.29%	1.81%	0.11%	0.28%	-0.34%	-0.86%	-0.47%	0.36%
7	R-squared		1	0.6085	0.3518	0.7052	0.2487	0.5972	0.5158	0.4362	0.3845	0.3221	0.2607	0.5288	0.5473
8															
9	01-Aug-01	0.29%	-6.92%	-2.78%	-6.49%	-10.49%	-13.38%	-9.17%	-9.97%	-3.88%	-3.87%	-6.45%	-14.49%	-9.40%	1.28%
10	04-Sep-01	0.28%	-8.82%											-18.33%	-34.76%
11	01-Oct-01	0.22%	1.50%											-0.65%	10.97%
12	01-Nov-01	0.18%	6.96%											19.72%	11.80%
13	03-Dec-01	0.16%	0.46%											8.65%	1.74%
14	02-Jan-02	0.14%	-1.86%											-2.11%	-0.92%
15	01-Feb-02	0.14%	-2.39%											-0.48%	12.85%
16	01-Mar-02	0.14%	3.32%											-3.02%	0.11%
17	01-Apr-02	0.15%	-6.63%											-5.00%	-4.55%
18	01-May-02	0.14%	-1.20%											-10.94%	6.84%
19	03-Jun-02	0.14%	-7.81%											-12.83%	-10.96%
20	01-Jul-02	0.14%	-8.52%											-17.65%	-8.81%
21	01-Aug-02	0.14%	0.19%											6.16%	-7.45%
22	03-Sep-02	0.14%	-11.95%											-23.41%	-32.66%
23	01-Oct-02	0.14%	8.00%											9.83%	9.71%
24	01-Nov-02	0.13%	5.26%											-9.27%	8.80%
25	02-Dec-02	0.10%	-6.52%											-9.51%	-8.34%
26	02-Jan-03	0.10%	-3.07%											-14.19%	1.50%
27	03-Feb-03	0.10%	-2.01%											11.20%	-6.01%
28	03-Mar-03	0.10%	0.54%											3.77%	-7.21%
29	01-Apr-03	0.09%	7.50%											14.09%	9.69%
30	01-May-03	0.09%	4.67%											14.14%	10.89%
31	02-Jun-03	0.09%	0.83%											1.81%	2.17%
32	01-Jul-03	0.08%	1.32%											-6.29%	4.91%
33	01-Aug-03	0.08%	1.48%	3.09%	-7.78%	1.68%	12.36%	-3.60%	5.95%	2.31%	-6.94%	3.62%	10.41%	2.73%	2.84%
34	02-Sep-03	0.08%	-1.49%	-9.09%	-3.36%	-0.29%	-8.84%	4.56%	-4.54%	-11.47%	-1.96%	1.12%	-0.71%	-1.03%	-9.82%



Running the second-pass regression shows only minor changes from the results of section 11.2:

	A	B	C	D	E	F	G	H
1	**THE SECOND-PASS REGRESSION FOR EXCESS RETURNS**							
2	**Stock**	**Average monthly excess return**	**Beta**	**Alpha**				
3	Alcoa AA	-0.38%	1.9028	0.0004		**Second-pass regression, regressing monthly returns on Beta**		
4	American International Group AIG	-0.83%	0.9936	-0.0061		Intercept	0.0007	<-- =INTERCEPT(B3:B32,C3:C32)
5	American Express AXP	0.43%	1.3784	0.0073		Slope	-0.0020	<-- =SLOPE(B3:B32,C3:C32)
6	Boeing BA	0.37%	1.1515	0.0063		R-squared	0.0238	<-- =RSQ(B3:B32,C3:C32)
7	Citigroup C	0.01%	1.2952	0.0029				
8	Caterpillar CAT	1.50%	1.3903	0.0181		t-statistic, intercept	0.2439	<-- =tintercept(B3:B32,C3:C32)
9	DuPont DD	-0.11%	1.0009	0.0011		t-statistic, slope	-0.8254	<-- =tslope(B3:B32,C3:C32)
10	Disney DIS	0.00%	1.2805	0.0028				
11	General Electric GE	-0.53%	0.8420	-0.0034				
12	General Motors GM	-1.16%	1.4060	-0.0086				

11.7 Summary: Does the CAPM Have Any Uses?

Is the game lost? Do we have to give up on the CAPM? Not totally:

• First of all, it could be that the mean returns are approximately described by their regression on a market portfolio. In this alternative description of the CAPM, we claim (with some justification) that the β of an asset (which measures the dependence of the asset's returns on the market returns) is an important measure of the asset's risk.

• Second, the CAPM might be a good normative description of how to choose portfolios. As we showed in the appendix of Chapter 3, larger diversified portfolios are quite well described by their betas, so that the average beta of a well-diversified portfolio may be a reasonable description of the portfolio's risk.

Exercises

1. In a well-known paper, Roll (1978) discusses tests of the SML in a four-asset context:

Variance-covariance matrix					Returns
0.10	0.02	0.04	0.05		0.06
0.02	0.20	0.04	0.01		0.07
0.04	0.04	0.40	0.10		0.08
0.05	0.01	0.10	0.60		0.09

a. Derive two efficient portfolios in this 4-asset model and draw a graph of the efficient frontier.

b. Show that the following four portfolios are efficient by proving that each is a convex combination of the two portfolios you derived in part a above:

Security 1	0.59600	0.40700	-0.04400	-0.49600
Security 2	0.27621	0.31909	0.42140	0.52395
Security 3	0.07695	0.13992	0.29017	0.44076
Security 4	0.05083	0.13399	0.33242	0.53129

c. Suppose that the market portfolio is composed of equal proportions of each asset (i.e., the market portfolio has proportions (0.25,0.25,0.25,0.25)). Calculate the resulting SML. Is the portfolio (0.25,0.25,0.25,0.25) efficient?

d. Repeat this exercise, but substitute one of the four portfolios of part b above as the candidate for the market portfolio.

The remaining questions relate to a data set for 10 stocks. The data are given on the exercise file with this chapter.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**PRICE DATA: 10 STOCKS AND S&P 500, Jul2007 - Jul2012** S&P 500 represented by Vanguard's Index 500 fund (includes dividends)											
2		1	2	3	4	5	6	7	8	9	10	11
3		Apple	Google	Whole Foods	Seagate	Comcast	Merck	Johnson-Johnson	General Electric	Hewlett Packard	Goldman Sachs	S&P 500
4												
5	Date	AAPL	GOOG	WFM	STX	CMCSA	MRK	JNJ	GE	HPQ	GS	S&P 500
6	2-Jul-07	131.76	510.00	35.75	21.18	24.17	39.78	51.39	31.84	43.69	178.93	125.27
7	1-Aug-07	138.48	515.25	42.72	23.37	24.00	40.20	52.84	31.93	46.85	167.21	121.40
8	4-Sep-07	153.47	567.27	47.26	23.15	22.25	41.73	56.18	34.24	47.34	205.91	123.22
9	1-Oct-07	189.95	707.00	47.98	25.29	19.37	47.04	55.73	34.04	49.14	235.90	127.81
10	1-Nov-07	182.22	693.00	41.66	23.43	18.90	47.93	58.29	31.67	48.64	215.65	129.83
11	3-Dec-07	198.08	691.48	39.52	23.16	16.80	47.23	57.39	30.92	48.07	204.62	124.39
12	2-Jan-08	135.36	564.30	38.41	18.50	16.71	37.47	54.33	29.49	41.63	190.21	123.53
13	1-Feb-08	125.02	471.18	34.23	19.69	17.98	36.00	53.67	27.89	45.49	161.69	116.10
14	3-Mar-08	143.50	440.47	32.11	19.11	17.85	31.11	56.19	31.15	43.56	157.65	112.33

2. Fill in the template file below.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**RETURN DATA: 10 STOCKS AND SP500**											
2		**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
3		**Apple**	**Google**	**Whole Foods**	**Seagate**	**Comcast**	**Merck**	**Johnson-Johnson**	**General Electric**	**Hewlett Packard**	**Goldman Sachs**	**S&P 500**
4	**Monthly statistics**											
5	Mean											
6	Variance											
7	Sigma											
8												
9	**Annual statistics**											
10	Mean											
11	Variance											
12	Sigma											
13												
14	**Regressing individual assets on the S&P 500**											
15	Alpha											
16	Beta											
17	Rsq											
18	T-test, intercept											
19	T-test, slope											
20												
21												
22	**RETURN DATA**											
23	**Date**	**AAPL**	**GOOG**	**WFM**	**STX**	**CMCSA**	**MRK**	**JNJ**	**GE**	**HPQ**	**GS**	**SP500**
24	1-Aug-07											
25	4-Sep-07											
26	1-Oct-07											
27	1-Nov-07											
28	3-Dec-07											
29	2-Jan-08											
30	1-Feb-08											
31	3-Mar-08											
32	1-Apr-08											
33	1-May-08											
34	2-Jun-08											
35	1-Jul-08											

3. Perform the second-pass regression: Regress the monthly average returns on the betas of the assets. Does this confirm that the S&P 500 is efficient?

4. Compute the variance-covariance matrix for the 10 stocks. Using the monthly average returns and a monthly risk-free interest rate of 0.20%, compute an efficient portfolio. Here's the template:

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**COMPUTING AN EFFICIENT PORTFOLIO OF THE 10 STOCKS**												
2													
3	**Variance-covariance matrix**												
4		**AAPL**	**GOOG**	**WFM**	**STX**	**CMCSA**	**MRK**	**JNJ**	**GE**	**HPQ**	**GS**		**Average returns**
5	**AAPL**												
6	**GOOG**												
7	**WFM**												
8	**STX**												
9	**CMCSA**												
10	**MRK**												
11	**JNJ**												
12	**GE**												
13	**HPQ**												
14	**GS**												
15													
16													
17	**Risk-free**	0.20%											
18													
19		**Efficient portfolio**											
20	**AAPL**												
21	**GOOG**												
22	**WFM**												
23	**STX**												
24	**CMCSA**												
25	**MRK**												
26	**JNJ**												
27	**GE**												
28	**HPQ**												
29	**GS**												

5. Using the efficient portfolio instead of the S&P 500:

   a. Compute the monthly returns on the efficient portfolio.

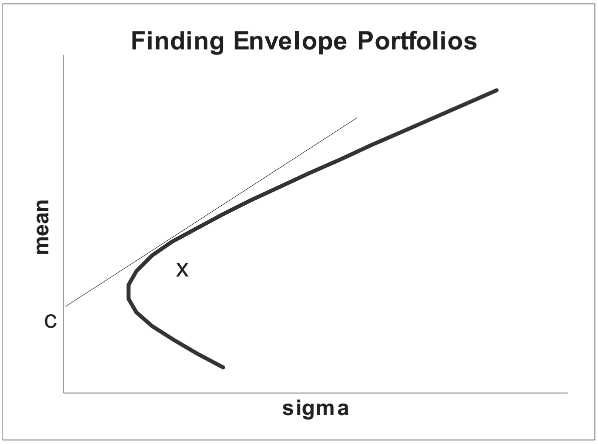
   b. Regress the average monthly returns of the stocks on their betas with respect to the efficient portfolio.

   c. Explain your results in light of Propositions 3 and 4 from Chapter 9.

12 Efficient Portfolios Without Short Sales

12.1 Overview

In Chapter 9 we discussed the problem of finding an efficient portfolio. As shown there, this problem can be written as finding a tangent portfolio on the envelope of the feasible set of portfolios:



The proof in the Appendix to Chapter 9 for solving for such an efficient portfolio involved finding the solution to the following problem:

$$\max \Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$\sum_{i=1}^{N} x_i = 1$$

where

$$E(r_x) = x^T \cdot R = \sum_{i=1}^{N} x_i E(r_i)$$

$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

Proposition 1 of Chapter 9 gives a methodology for solving this problem. Solutions to the maximization problem allow *negative* portfolio proportions;

when $x_i < 0$, this assumes that the *i*th security is sold short by the investor and that the proceeds from this short sale become immediately available to the investor.

Reality is, of course, considerably more complicated than this academic model of short sales. In particular, it is rare for all of the short-sale proceeds to become available to the investor at the time of investment, since brokerage houses typically escrow some or even all of the proceeds. It may also be that the investor is completely prohibited from making any short sales (indeed, most small investors seem to proceed on the assumption that short sales are impossible).[1]

In this chapter we investigate these problems. We show how to use Excel's **Solver** to find efficient portfolios of assets when we restrict short sales.[2]

12.2 A Numerical Example

We start with the problem of finding an optimal portfolio when no short sales are allowed. The problem we solve is similar to the maximization problem stated above, with the addition of the short-sales constraint $x_i > 0$ for the asset proportions:

$$\max \Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$\sum_{i=1}^{N} x_i = 1$$

$$x_i \geq 0, i = 1, \ldots, N$$

where

1. The actual procedures for implementing a short sale are not simple. A well-written academic survey is a recent paper by Gene D'Avolio, "The Market for Borrowing Stock," *Journal of Financial Economics* (2003). There's also a wonderful article, "Get Shorty," in the 1 December 2003 issue of *The New Yorker* magazine by James Surowiecki.

2. We do not go into the efficient set mathematics when short sales of assets are restricted. This involves the Kuhn-Tucker conditions, a discussion of which can be found in Edwin Elton, Martin Gruber, Stephen Brown, and W. N. Goetzmann, *Modern Portfolio Theory and Investment Analysis* (Wiley, 8th edition, 2009).

$$E(r_x) = x^T \cdot R = \sum_{i=1}^{N} x_i E(r_i)$$

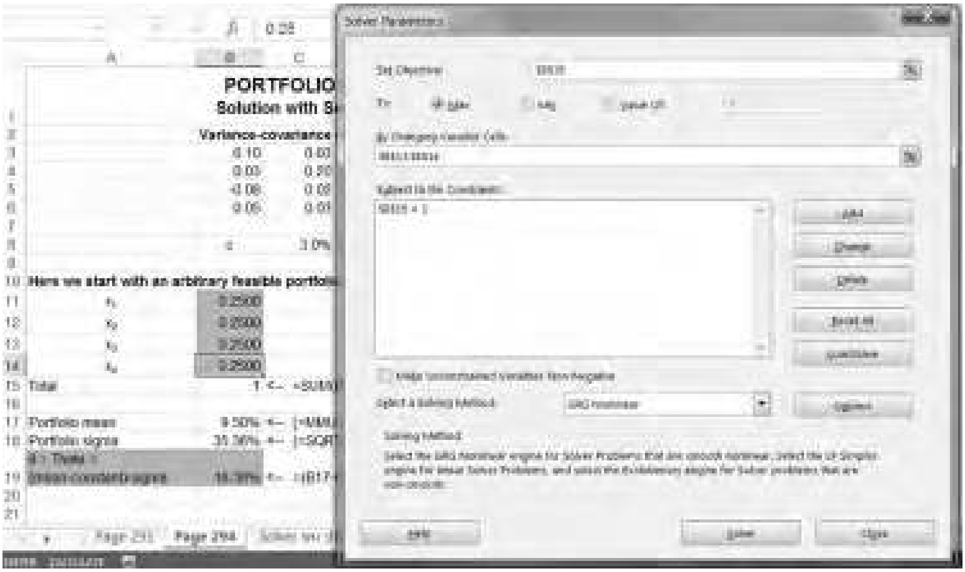
$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

Solving an Unconstrained Portfolio Problem

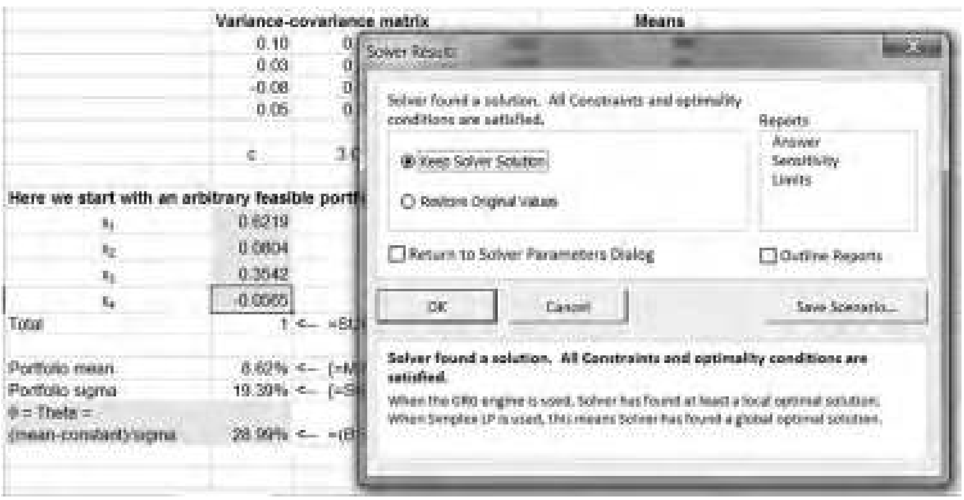
To set the scene, we consider the optimization problem below, which we solve without any short-sale constraints. The spreadsheet shows a four-asset variance-covariance matrix and associated expected returns. Given a constant $c = 8\%$, the optimal portfolio is given in cells B11:B14. Notice θ in cell B19: This is the *Sharpe ratio* of the portfolio, the ratio of its excess return over the constant c to its standard deviation: $\theta = \frac{E(r_x) - c}{\sigma_x}$. The optimal portfolio maximizes the Sharpe ratio θ.

	A	B	C	D	E	F	G	H
1	**PORTFOLIO OPTIMIZATION ALLOWING SHORT SALES** **Follows Proposition 1, Chapter 9**							
2		**Variance-covariance matrix**					**Means**	
3		0.10	0.03	-0.08	0.05		8%	
4		0.03	0.20	0.02	0.03		9%	
5		-0.08	0.02	0.30	0.20		10%	
6		0.05	0.03	0.20	0.90		11%	
7								
8		c	3.0%	<-- This is the constant				
9								
10	**Optimal portfolio without short sale restrictions (Chapter 9, Proposition 1)**							
11	x_1	0.6219	<-- {=MMULT(MINVERSE(B3:E6),G3:G6-C8)/SUM(MMULT(MINVERSE(B3:E6),G3:G6-C8))}					
12	x_2	0.0804						
13	x_3	0.3542						
14	x_4	-0.0565						
15	Total	1	<-- =SUM(B11:B14)					
16								
17	Portfolio mean	8.62%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}					
18	Portfolio sigma	19.39%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}					
19	θ = Theta = (mean-constant)/sigma	28.99%	<-- =(B17-C8)/B18					

There is another way to solve this unconstrained problem. Starting from an arbitrary portfolio (the spreadsheet below uses $x_1 = x_2 = x_3 = x_4 = 0.25$), we use **Solver** to find a solution:



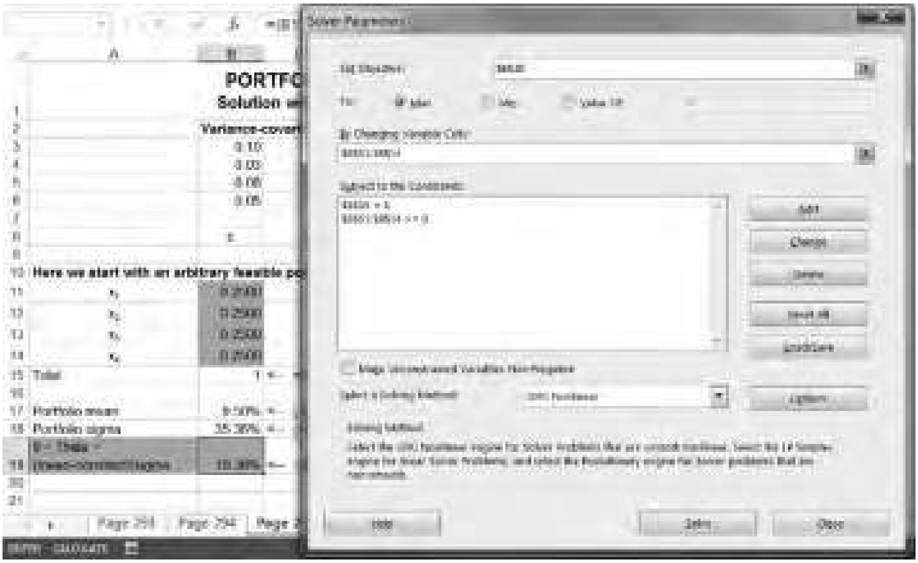
The **Solver** solution maximizes θ (cell B19) subject to the constraint that cell B15, which contains the sum of the portfolio positions, equals 1.[3] When we press **Solve** we get the solution we achieved before:



3. If **Tools|Solver** doesn't work, you may not have loaded the Solver add-in. To do so, go to **Tools|Add-ins** and click next to the **Solver Add-in**.

Solving a Constrained Portfolio Problem

The optimal solution above contains a short position in asset 4. To restrict the short selling, we add a no-short-sale constraint to **Solver**. Starting from an arbitrary solution, we bring up **Solver** as shown below:



Pressing **Solve** yields the following solution:

	A	B	C	D	E	F	G	H
1	**PORTFOLIO OPTIMIZATION WITHOUT SHORT SALES** **Solution with Solver, starting from an arbitrary feasible portfolio**							
2		**Variance-covariance matrix**					**Means**	
3		0.10	0.03	-0.08	0.05		8%	
4		0.03	0.20	0.02	0.03		9%	
5		-0.08	0.02	0.30	0.20		10%	
6		0.05	0.03	0.20	0.90		11%	
7								
8		c	3.0%	<-- This is the constant				
9								
10	**Here we start with an arbitrary feasible portfolio and use Solver**							
11	x_1	0.5856						
12	x_2	0.0965						
13	x_3	0.3179						
14	x_4	0.0000						
15	Total	1	<-- =SUM(B11:B14)					
16								
17	Portfolio mean	8.73%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}					
18	Portfolio sigma	20.32%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}					
19	θ = Theta = (mean-constant)/sigma	28.21%	<-- =(B17-C8)/B18					

The non-negativity constraints is added by clicking on the **Add** button in the **Solver** dialogue box. This brings up the following window (shown here filled in):

The second constraint (which constrains the portfolio proportions to sum to 1) is added in a similar fashion.

An Alternative Method

There's another way of doing this. **Solver** has an option: "Make Unconstrained Variables Non-Negative." Clicking this option gives the same result:

By changing the value of c in the spreadsheet, we can compute other portfolios; in the following example, we have set the constant $c = 8.5\%$:

	A	B	C	D	E	F	G	H
1	**PORTFOLIO OPTIMIZATION WITHOUT SHORT SALES** **Solution with Solver, starting from an arbitrary feasible portfolio**							
2		**Variance-covariance matrix**					**Means**	
3		0.10	0.03	-0.08	0.05		8%	
4		0.03	0.20	0.02	0.03		9%	
5		-0.08	0.02	0.30	0.20		10%	
6		0.05	0.03	0.20	0.90		11%	
7								
8		c	8.5%	<-- This is the constant				
9								
10	**Here we start with an arbitrary feasible portfolio and use Solver**							
11	x_1	0.0000						
12	x_2	0.2515						
13	x_3	0.4885						
14	x_4	0.2601						
15	Total	1	<-- =SUM(B11:B14)					
16								
17	Portfolio mean	10.01%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}					
18	Portfolio sigma	45.25%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}					
19	θ = Theta = (mean-constant)/sigma	3.33%	<-- =(B17-C8)/B18					

In both examples, the short-sale restriction is effective, with zero positions in some asset. However, not all values of c give portfolios in which the short-sale constraints are effective. For example, if the constant is 8%, we get:

	A	B	C	D	E	F	G	H
1	**PORTFOLIO OPTIMIZATION WITHOUT SHORT SALES** **Solution with Solver, starting from an arbitrary feasible portfolio**							
2		**Variance-covariance matrix**					**Means**	
3		0.10	0.03	-0.08	0.05		8%	
4		0.03	0.20	0.02	0.03		9%	
5		-0.08	0.02	0.30	0.20		10%	
6		0.05	0.03	0.20	0.90		11%	
7								
8		c	8.0%	<-- This is the constant				
9								
10	**Here we start with an arbitrary feasible portfolio and use Solver**							
11	x_1	0.2004						
12	x_2	0.2587						
13	x_3	0.4219						
14	x_4	0.1190						
15	Total	1	<-- =SUM(B11:B14)					
16								
17	Portfolio mean	9.46%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}					
18	Portfolio sigma	31.91%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}					
19	θ = Theta = (mean-constant)/sigma	4.57%	<-- =(B17-C8)/B18					

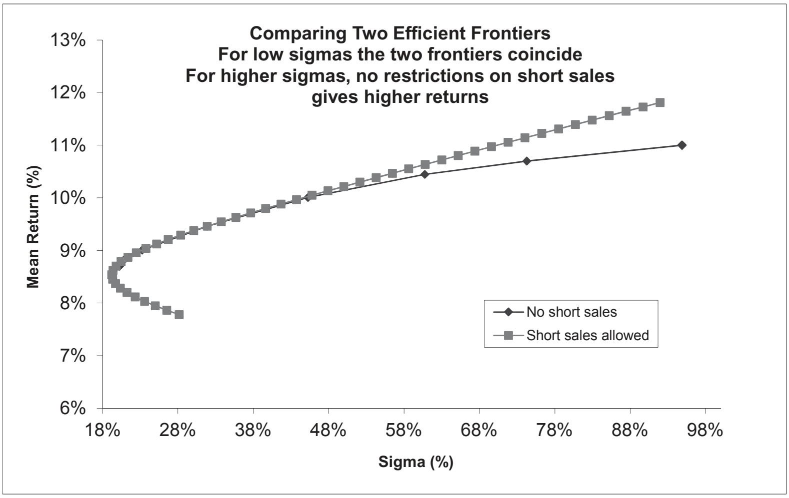
As we saw for the example where $c = 3\%$, as c gets lower, the short-sale constraint begins to be effective with respect to asset 4. For very high c's (the case below illustrates $c = 11\%$), only asset 4 is included in the maximizing portfolio:

	A	B	C	D	E	F	G	H
8		c	11.0%	<-- This is the constant				
9								
10	**Here we start with an arbitrary feasible portfolio and use Solver**							
11	x_1	0.0000						
12	x_2	0.0000						
13	x_3	0.0000						
14	x_4	1.0000						
15	Total	1	<-- =SUM(B11:B14)					
16								
17	Portfolio mean	11.00%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}					
18	Portfolio sigma	94.87%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}					
19	θ = Theta = (mean-constant)/sigma	0.00%	<-- =(B17-C8)/B18					

12.3 The Efficient Frontier with Short-Sale Restrictions

We want to graph the efficient frontier with short-sale restrictions. Recall that in the case of no short-sale restrictions discussed in Chapter 9, it was enough to find two efficient portfolios in order to determine the whole efficient frontier (this was proved in Proposition 2 of Chapter 9). When we impose short-sale restrictions, this statement is no longer true. In this case the determination of the efficient frontier requires the plotting of a large number of points. The only efficient (pardon the pun!) way of doing this is with a VBA program which repeatedly applies the **Solver** and puts the solutions in a table.

In section 12.3 we describe such a program. Once we have the program and the graph of the efficient frontier without short sales, we can compare this efficient frontier to the efficient frontier *with* short sales allowed:



The relation between these two graphs is not all that surprising:

- In general, the efficient frontier with short sales dominates the efficient frontier without short sales. This must clearly be so, since the short-sales restriction imposes an extra constraint on the maximization problem.
- For some cases, the two efficient frontiers coincide. One such point occurs, as we saw above, when $c = 8\%$.

Putting these two graphs on one set of axes shows that the effect of the short-sale restrictions is mainly for portfolios with higher returns and sigmas.

12.4 A VBA Program for the Efficient Frontier Without Short Sales

The output for the restricted short-sale case shown in section 12.3 was produced with the following VBA program:

```
Sub Solve()
       SolverOk SetCell:="$B$19", MaxMinVal:=1,
       ValueOf:="0", ByChange:="$B$11:$B$14"
       SolverSolve UserFinish:=True
End Sub
Sub Doit()
    Range("Results").ClearContents
    For counter = 1 To 40
        Range("constant") = -0.04 + counter * 0.005
        Solve
        Application.SendKeys ("{Enter}")
        Range("Results").Cells(counter, 1) _
        ActiveSheet.Range("constant")
        Range("Results").Cells(counter, 2) _
        ActiveSheet.Range("portfolio_sigma")
        Range("Results").Cells(counter, 3) _
        ActiveSheet.Range("portfolio_mean")
        Range("Results").Cells(counter, 4) _
        ActiveSheet.Range("x_1")
        Range("Results").Cells(counter, 5) _
        ActiveSheet.Range("x_2")
        Range("Results").Cells(counter, 6) _
        ActiveSheet.Range("x_3")
        Range("Results").Cells(counter, 7) _
        ActiveSheet.Range("x_4")
    Next counter
End Sub             ActiveSheet.Range("x_3")
        Range("Results").Cells(counter, 7) = _
            ActiveSheet.Range("x_4")
    Next counter
End Sub
```

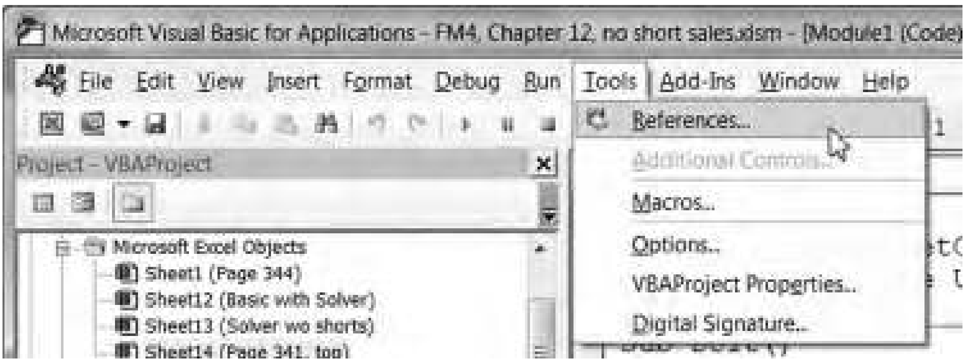
The program includes two subroutines: `Solve` calls the Excel Solver; and the subroutine `Doit` repeatedly calls the solver for different values of the range named `Constant` (this is cell C8 in the spreadsheet), putting the output in a range called "Results."

The final output looks like this:

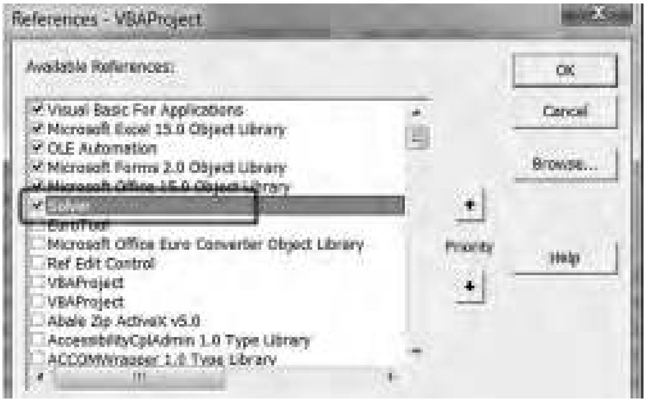
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	**PORTFOLIO OPTIMIZATION WITHOUT SHORT SALES**									**RESULTS**					
2		**Variance-covariance matrix**					**Means**			**c**	**Sigma**	**Mean**	**x_1**	**x_2**	**x_3**
3		0.10	0.03	-0.08	0.05		8%		Ctrl+A works the VBA program	-0.035	20.24%	8.70%	0.6049	0.0885	0.3066
4		0.03	0.20	0.02	0.03		9%		which calculates efficient	-0.03	20.25%	8.70%	0.6042	0.0887	0.3070
5		-0.08	0.02	0.30	0.20		10%		portfolios for no-short sales.	-0.025	20.25%	8.70%	0.6035	0.0890	0.3075
6		0.05	0.03	0.20	0.90		11%		This program iteratively	-0.02	20.25%	8.71%	0.6027	0.0893	0.3080
7									substitutes a constant ranging	-0.015	20.25%	8.71%	0.6017	0.0897	0.3086
8		c	16.0%	<-- This is the constant					from -3.5% 'till 16% (1/2%	-0.01	20.26%	8.71%	0.6007	0.0901	0.3092
9									jumps) and calculates the	-0.005	20.26%	8.71%	0.5994	0.0908	0.3098
10									optimal portfolio.	0	20.27%	8.71%	0.5982	0.0912	0.3106
11	x_1	0.0000	0							0.005	20.27%	8.71%	0.5968	0.0917	0.3115
12	x_2	0.0000	0							0.01	20.28%	8.72%	0.5950	0.0926	0.3123
13	x_3	0.0000	0							0.015	20.29%	8.72%	0.5932	0.0934	0.3134
14	x_4	1.0000	0							0.02	20.30%	8.72%	0.5910	0.0943	0.3147
15	Total	1.0000	<-- =SUM(B11:B14)							0.025	20.31%	8.73%	0.5885	0.0953	0.3161
16										0.03	20.32%	8.73%	0.5856	0.0965	0.3179
17	Portfolio mean	11.00%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}							0.035	20.34%	8.74%	0.5821	0.0980	0.3199
18	Portfolio sigma	94.87%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B3:E6,B11:B14)))}							0.04	20.37%	8.74%	0.5779	0.0998	0.3224
19	Theta	-5.27%	<-- =(B17-C8)/B18							0.045	20.41%	8.75%	0.5726	0.1019	0.3255
20										0.05	20.46%	8.76%	0.5659	0.1047	0.3294
21										0.055	20.54%	8.78%	0.5572	0.1083	0.3345
22										0.06	20.67%	8.80%	0.5452	0.1133	0.3415
23										0.065	20.90%	8.82%	0.5277	0.1205	0.3518
24										0.07	21.36%	8.87%	0.4992	0.1324	0.3684
25										0.075	23.27%	9.01%	0.4267	0.1630	0.3856

Adding a Reference to Solver in VBA

If the above routine does not work, you may need to add a reference to **Solver** in the VBA editor. Press [Alt] + F11 to get to the editor and then go to **Tools|References**:

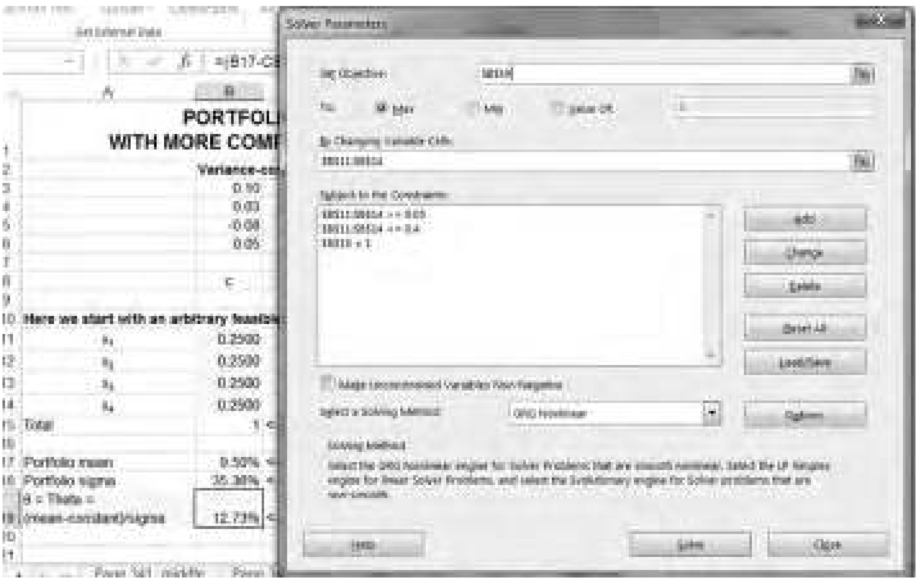


If this reference is missing, go to **Tools|References** on the VBA menu and make sure that **Solver** is checked:



12.5 Other Position Restrictions

It goes without saying that Excel and **Solver** can accommodate other position limits. Suppose, for example, that the investor wants at least 5% of her portfolio invested in any asset and no more than 40% of the portfolio invested in any single asset. This is easily set up in **Solver**:



This solves to give:

	A	B	C	D	E	F	G
1	**PORTFOLIO OPTIMIZATION WITH MORE COMPLICATED CONSTRAINTS**						
2		**Variance-covariance matrix**					**Means**
3		0.10	0.03	-0.08	0.05		8%
4		0.03	0.20	0.02	0.03		9%
5		-0.08	0.02	0.30	0.20		10%
6		0.05	0.03	0.20	0.90		11%
7							
8		c	5.0%	<-- This is the constant			
9							
10	**Here we start with an arbitrary feasible portfolio and use Solver**						
11	x_1	0.4000					
12	x_2	0.2270					
13	x_3	0.3230					
14	x_4	0.0500					
15	Total	1	<-- =SUM(B11:B14)				
16							
17	Portfolio mean	9.02%	<-- {=MMULT(TRANSPOSE(B11:B14),G3:G6)}				
18	Portfolio sigma	23.81%	<-- {=SQRT(MMULT(TRANSPOSE(B11:B14),MMULT(B				
19	θ = Theta = (mean-constant)/sigma	16.89%	<-- =(B17-C8)/B18				

12.6 Summary

No one would claim that Excel offers a quick way to solve for portfolio maximization, with or without short-sale constraints. However, it can be used to illustrate the principles involved, and the Excel **Solver** provides an easy-to-use and intuitive interface for setting up these problems.

Exercise

Given the data below:

a. Calculate the efficient frontier assuming no short sales are allowed.
b. Calculate the efficient frontier assuming that short sales are allowed.
c. Graph both frontiers on the same set of axes.

	A	B	C	D	E	F	G	H	I
3		A	B	C	D	E	F		Mean returns
4	A	0.0100	0.0000	0.0000	0.0000	0.0000	0.0000		0.0100
5	B	0.0000	0.0400	0.0000	0.0000	0.0000	0.0000		0.0200
6	C	0.0000	0.0000	0.0900	0.0000	0.0000	0.0000		0.0300
7	D	0.0000	0.0000	0.0000	0.1500	0.0000	0.0000		0.0400
8	E	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000		0.0500
9	F	0.0000	0.0000	0.0000	0.0000	0.0000	0.3000		0.0550