EC 813A Recitation 6

Yogeshwar Bharat

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Two Period Production Economy and Infinite Horizon Model

Question 1. (Mid Sem 2010, Q2) Consider an Infinite Horizon open, production economy studied in lectures. The representative household follows the utility function given by $u(c_s)$, where c_s are consumption in period s. The representative agent can produce productive capital from final good through 1:1 technology. The production function is given by $Y_s = F(K_s)$ and satisfies standard assumptions. The capital depreciates at rate δ. There is also a government that issues bonds and taxes private sector. The government imposes in each period a proportional tax on total output, with tax rate being τ. However, the government allows the private sector to exempt investment expenditure from the tax base. Productions are completely owned by households.

a) Write down inter-temporal budget constraint.

b) Derive the Euler condition and write down the whole system of equation that define the equilibrium generically. Discuss the impact of eliminating the tax exemption of investment expenditure.

Question 2. (Mid-Sem 2015, Q2) Consider an Infinite Horizon economy studied in lectures. The representative household follows the utility function given by "$u(c_s) - v(n_s)$", where c_s and n_s are consumption and labor supply in period s respectively. The production function is given by $Y_s = A_s F(K_s, n_s)$ and satisfies standard properties. The total factor of productivity A_s is a random variable and i.i.d (hence adding uncertainty in the model). Capital doesn't follow a linear relation with investment and grows as per: $K_{s+1} = K_s + f(I_s)$, where $f' > 0$ and $f'' > 0$. Consider capital depreciate at rate δ. Also, there is no government present.

a) Write down inter-temporal budget constraint.

b) Derive the Euler condition and write down the whole system of equation that define the equilibrium generically. Work on the equilibrium system again by putting βR = 1.

Question 3. (Mid Sem 2012, Q2) Consider an Infinite horizon "full model" small open production economy studied in lectures (allow for the presence of government). Derive the Euler condition and equilibrium system of the economy by keeping firms separated from households (this will allow you to solve explicitly for wage rate). Capital depreciates at rate δ.

Question 1.

τ tax rate on $Y_s - I_s$, $Y_s = F(k_s)$, $k_{s+1} = (1-\delta)\cdot k_s + I_s$,

HH: $\max \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$ s.t $B_{s+1} = RB_s + Y_s - I_s - C_s - \tau\cdot(Y_s - I_s)$.

LOMOC $k_{s+1} = (1-\delta) k_s + I_s$,

Govt. BC: $B^G_{s+1} = R B^G_s + G_s - \tau(Y_s - I_s)$.

State = B_s, k_s,

control = C_s, I_s

Bellmen Equation, $V(B_s, k_s) = \max_{C_s, I_s} \{ u(C_s) + \beta\cdot V(B_{s+1}, k_{s+1}) \}$.

s.t $B_{s+1} = R\cdot B_s + Y_s - I_s - C_s - \tau(Y_s - I_s)$,

$k_{s+1} = (1-\delta)\cdot k_s + I_s$,

F.O.C.

C_s: $u'(C_s) + \beta\cdot \frac{\partial V_{B_{s+1}}}{\partial B_{s+1}} \frac{\partial B_{s+1}}{\partial C_s} = u'(C_s) + \beta V_{B_{s+1}}\cdot -1 = 0 \Rightarrow u'(C_s) = \beta V_{B_{s+1}}$.

$V_{B_s} = \underbrace{\beta\cdot V_{B_{s+1}}}_{u'(C_s)}\cdot R = R\cdot u'(C_s)$. $\Rightarrow V_{B_{s+1}} = R u'(C_{s+1})$

$\Rightarrow \underline{u'(C_s) = \beta\cdot R u'(C_{s+1})}$.

I.C.C.

I_s: $\beta\cdot V_{B_{s+1}}(-1+\tau) + \beta\cdot V_{k_{s+1}}(1) \stackrel{!}{=} 0 \Rightarrow \beta\cdot V_{k_{s+1}} = (1-\tau)\cdot \beta\cdot V_{B_{s+1}} = (1-\tau)\cdot\beta R\cdot u'(C_{s+1})$.

$= (1-\tau)\cdot u'(C_s)$

$V_{k_s} = \beta\cdot V_{B_{s+1}} \cdot \underbrace{\frac{\partial B_{s+1}}{\partial k_s}}_{(1-\tau)\cdot F'(k_s)} + \beta\cdot V_{k_{s+1}}\cdot \underbrace{\frac{\partial k_{s+1}}{\partial k_s}}_{1-\delta} = (1-\tau)\cdot F'(k_s)\cdot u'(C_s) + \underbrace{\beta\cdot V_{k_{s+1}}}_{u'(C_s)(1-\tau)}(1-\delta)$.

$V_{k_s} = u'(C_s)(1-\tau) \cdot [F'(k_s)+(1-\delta)]$

$V_{k_{s+1}} = u'(C_{s+1})(1-\tau) \cdot [F'(k_{s+1})+(1-\delta)]$.

but $\beta \cdot V_{k_{s+1}} = u'(C_s)(1-\tau)$,

Lump-sum tax,

Ricardian Equivalence holds.

$\Rightarrow \beta u'(C_{s+1})(1-\tau) \cdot [F'(k_{s+1})+(1-\delta)]$

$\Rightarrow [F'(k_{s+1})+(1-\delta)] = \beta R u'(C_{s+1})$

$\Rightarrow F'(k_{s+1}) = r+\delta$, ECI.

(ii) Equilibrium:

1. ECC.
2. ECI: $F'(k_{s+1}) = r+\delta$
3. LOMOU
4. I.B.C. $\sum_{s=t}^{\infty} \frac{C_s+(1-\tau)I_s}{R^{s-t}} = RB_t + \sum_{s=t}^{\infty} \frac{Y_s(1-\tau)}{R^{s-t}}$.
5. G.I.B.C: $\sum_{s=t}^{\infty} \frac{\tau(Y_s-I_s)}{R^{s-t}} = RB_t^G + \sum_{s=t}^{\infty} \frac{G_s}{R^{s-t}}$

(iii) No subsidy on investment:

$B_{s+1} = RB_s + Y_s - C_s - I_s - \tau \cdot Y_s$,

ECI: $\frac{r+\delta}{1-\tau} = F'(k_{s+1})$.

Proportion tax,
Ricardian Equivalence does not hold.

② (2015).

HH_s, C_s, n_s, $Y_s = A_s \cdot F(K_s, n_s)$ $A_s \overset{iid}{\sim}$ R.V.

$$HH \max_{c_s, n_s} E\left\{\sum_{s=t}^{\infty} \beta^{s-t} \, u(C_s) - v(n_s)\right\}$$

s.t $B_{s+1} = RB_s + Y_s - C_s - I_s$

CoMOC: $K_{s+1} = (1-\delta)\cdot K_s + f(I_s)$, $f'>0$, $f''>0$,

State: B_s, K_s

control: C_s, I_s, n_s,

Bellman Equation,

$$V(B_s, K_s) = \max_{C_s, I_s, n_s} \left\{ \left(u(C_s) - v(n_s)\right) + \beta \cdot E\left[V(B_{s+1}, K_{s+1}) \,\middle|\, A_s\right]\right\}.$$

s.t. $B_{s+1} = R\cdot B_s + Y_s - C_s - I_s$

$K_{s+1} = (1-\delta)\cdot K_s + f(I_s)$.

F.O.C.

C_s: $u'(C_s) + \beta \cdot E\left[V_{B_{s+1}} \cdot \underbrace{\frac{\partial B_{s+1}}{\partial C_s}}_{=-1}\right] = 0$, $\Rightarrow u'(C_s) = \beta \cdot E(V_{B_{s+1}})$.

$$V_{B_s} = \beta \cdot E\left(V_{B_{s+1}} \cdot \underbrace{\frac{\partial B_{s+1}}{\partial B_s}}_{=R}\right) = \beta \cdot E(V_{B_{s+1}}) \cdot R = u'(C_s)\cdot R$$

$$\Rightarrow u'(C_s) = \beta \cdot R \; E[u'(C_{s+1})].$$

E.C.C.

$$n_s: \quad -v_{n_s} + \beta \cdot E\left\{ V_{B_{s+1}} \cdot \frac{\partial B_{s+1}}{\partial n_s} \right\} = 0.$$

$$\frac{\partial B_{s+1}}{\partial n_s} = A_s \cdot F_{n_s}(k_s, n_s)$$

$$\Rightarrow v_{n_s} = \underbrace{\beta E\left(V_{B_{s+1}}\right)}_{= u'(C_s).} \cdot A_s \cdot F_{n_s}(k_s, n_s)$$

$$\Rightarrow \frac{v_{n_s}}{u'(C_s)} = A_s \cdot F_{n_s}(k_s, n_s). \qquad \text{EC Labor.}$$

$$I_s: \quad \beta \cdot E\left(V_{B_{s+1}} \cdot \frac{\partial B_{s+1}}{\partial I_s}\right) + \beta \cdot E\left(V_{k_{s+1}} \cdot \frac{\partial k_{s+1}}{\partial I_s}\right)$$

$$= \beta \cdot E(V_{B_{s+1}}) \cdot -1 + \beta \cdot E(V_{k_{s+1}}) \cdot f'(I_s).$$

$$\Rightarrow \underbrace{\beta E(V_{B_{s+1}})}_{} = \beta E(V_{k_{s+1}}) \cdot f'(I_s).$$

$$u'(C_s) = \beta \cdot E(V_{k_{s+1}}) \cdot f'(I_s).$$

$$V_{k_s} = \beta E\Big(V_{B_{s+1}} \underbrace{\frac{\partial B_{s+1}}{\partial k_s}}_{= A_s \cdot F_{k_s}(k_s, n_s)} + V_{k_{s+1}} \cdot \underbrace{\frac{\partial k_{s+1}}{\partial k_s}}_{1-\delta}\Big) = \underbrace{\beta \cdot E(V_{B_{s+1}})}_{u'(C_s)} \cdot A_s \cdot F_{k_s}(k_s, n_s) + (1-\delta) \cdot E(V_{k_{s+1}})$$

$$= u'(C_s) \cdot A_s \cdot F_{k_s}(k_s, n_s) + (1-\delta) \frac{u'(C_s)}{f'(I_s)}$$

$$V_{k_{s+1}} = u'(C_{s+1}) A_{s+1} F_{k_{s+1}}(k_{s+1}, n_{s+1}) + (1-\delta) \cdot \frac{u'(C_{s+1})}{f'(I_{s+1})}$$

$$\text{ECI:} \quad u'(C_s) = \beta \cdot E\left[u'(C_{s+1}) \left(A_{s+1} F_{k_{s+1}}(k_{s+1}, n_{s+1})\right) + \frac{u'(C_{s+1})}{f'(I_{s+1})} \right]$$

$$\Rightarrow \frac{u'(C_s)}{f'(I_s)} = \beta E\left[u'(C_{s+1}) A_{s+1} F_{k_{s+1}}(k_{s+1}, n_{s+1}) + \cdots \right]$$

3. HH, firm, govt., small open economy.

HHs: $\max_{C_s, L_s} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left(u(C_s) - v(L_s) \right) \right\}$

s.t $B_{s+1} = RB_s + W_s \cdot L_s^s + D_s - C_s - T_s^h$. —— ①

firms: $\max \sum_{s=t}^{\infty} \frac{D_s}{R^{s-t}}$ s.t $D_s = F(K_s, L_s) - W_s \cdot L_s^D - I_s - RB_s^f$
$+ B_{s+1}^f - T_s^f$ —— ②.

$K_{s+1} = (1-\delta) \cdot K_s + I_s$.

Govt: $B_{s+1}^G = RB_s^G + G_s - T_s^h - T_s^f$.

Hh: state: B_s
control: C_s, L_s^s,

Firm: state: B_s^f, K_s
control: I_s, L_s^D.

Hh. Bellman:

$V(B_s) = \max_{C_s, L_s^s} \{ u(C_s) - v(L_s) + \beta V(B_{s+1}) \}$.

C_s: $u'(C_s) - \beta V_{B_{s+1}} \cdot R = 0 \Rightarrow u'(C_s) = \beta \cdot R \cdot u'(C_{s+1})$.

L_s^s: $-v'(L_s^s) + \beta V_{B_{s+1}} \cdot \frac{\partial B_{s+1}}{\partial L_s^s} = 0$

$v'(L_s^s) = \underbrace{\beta \cdot V_{B_{s+1}}}_{u'(C_s)} \cdot W_s = u'(C_s) \cdot W_s$. EC labor.

firm Bellman. $V(B_s^f, K_s) = \max_{I_s, L_s^D} \{ D_s + \frac{1}{R} V(B_{s+1}^f, K_{s+1}) \}$.

L_s^D: $\frac{\partial D_s}{\partial L_s^D} = F_{L_s}(K_s, L_s) - W_s = 0 \Rightarrow F_{L_s}(K_s, L_s) = W_s$. EC labor.

I_s: $\underbrace{\frac{\partial D_s}{\partial I_s}}_{=-1} + \frac{1}{R} V_{B_{s+1}} \underbrace{\frac{\partial B_{s+1}}{\partial I_s}}_{=0} + \frac{1}{R} V_{K_{s+1}} \cdot \underbrace{\frac{\partial K_{s+1}}{\partial I_s}}_{=1} = 0$

$\Rightarrow R = V_{K_{s+1}}$

$V_{K_s} = \underbrace{\frac{\partial D}{\partial K_s}}_{F_{K_s}(K_s, L_s)} + \frac{1}{R} V_{B_{s+1}} \underbrace{\frac{\partial B_{s+1}}{\partial K_s}}_{=0} + \frac{1}{R} V_{K_{s+1}} \cdot \underbrace{\frac{\partial K_{s+1}}{\partial K_s}}_{=(1-\delta)} = F_{K_s}(K_s, L_s) + \underbrace{\frac{1}{R} V_{K_{s+1}}}_{=1} \cdot (1-\delta)$

$\Rightarrow V_{K_s} = F_{K_s}(K_s, L_s) + (1-\delta)$.

$\Rightarrow V_{K_{s+1}} = F_{K_{s+1}}(K_{s+1}, L_{s+1}) + (1-\delta) = R$.

$\Rightarrow F_{K_{s+1}}(K_{s+1}, L_{s+1}) = r + \delta$. EC I.

Equilibrium. ① $L_s^S = L_s^D$;

② ECC

③ ECL (Hh) $= \frac{v'(L_s^S)}{u'(C_s)} = W_s$.

④. ECL (firm) $= F_{L_s}(K_s, L_s^D) = W_s$.

⑤ ECI: $r+\delta = F_{K_{s+1}}(K_{s+1}, L_s^D)$

⑥ $K_{s+1} = (1-\delta)\cdot K_s + I_s$,

⑦ I.B.C Hh: $\sum_{s=t}^{\infty} \frac{1}{R^{s-t}} C_s = RB_t + \sum_{s=t}^{\infty} \frac{1}{R^{s-t}} (W_s \cdot L_s + D_s - T_s^h)$,

⑧ I.B.C. firm: $\sum_{s=t}^{\infty} \frac{D_s}{R^{s-t}} = \sum_{s=t}^{\infty} \frac{1}{R^{s-t}} \left(F(K_s, L_s) - W_s L_s^D - I_s - T_s^f \right)$

⑨ G.I.B.C: $\sum_{s=t}^{\infty} \frac{T_s^h + T_s^f}{R^{s-t}} = RB_t^G + \sum_{s=t}^{\infty} \frac{G_s}{R^{s-t}}$