

# A Specification Test for Dynamic Conditional Distribution Models with Function-Valued Parameters

## Appendix

### A.1. Preliminary Results

In this subsection, we provide preliminary results used in the proofs of the propositions. Let  $\mathcal{G}$  be a permissible class of functions in such a way that the following holds: (a)  $\mathcal{T}$  is a Suslin metric space (a Hausdorff topological space that is the continuous image of a Polish space) with Borel  $\sigma$ -field  $\mathcal{B}(\mathcal{T}, \Theta)$ , and (b)  $g(\cdot, \cdot, \cdot)$  is a  $\mathcal{B}(\mathcal{T}, \Theta)$ -measurable function from  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^K$  to  $\mathbb{R}$  (see Kosorok, 2007, Section 11.6). Let  $E_Q g = \int g(W_t, \theta, \tau) dQ(W_t, \theta, \tau)$ , for  $g \in \mathcal{G}$ , with  $\mathcal{G} := \{W_t \mapsto g(W_t, \theta, \tau) : \theta \in \Theta, \tau \in \mathcal{T}\}$ . We assume that the  $\mathcal{G}$  class of functions forms a so-called Vapnik-Chervonenkis subgraph (VC-subgraph) class of functions (see Dudley, 1978). The VC-subgraph class is an extension of the class of indicator functions and is useful for most statistical applications (Arcones and Yu, 1994; Radulović, 1996). If  $\mathcal{G}$  is a VC-subgraph class, then for any given  $1 \leq p < \infty$ , there are constants  $a$  and  $b$  satisfying

$$N(\varepsilon, \mathcal{G}, \|\cdot\|) \leq a \left( \frac{(E_Q |\mathbb{F}|^p)^{1/p}}{\varepsilon} \right)^b,$$

for all  $\varepsilon > 0$  and all probability measures  $Q$ , with  $E_Q|\mathbb{F}|^p < \infty$ , where  $N(\varepsilon, \mathcal{G}, \|\cdot\|)$  is the covering number of  $\mathcal{G}$  with respect to  $\|\cdot\|$ , i.e., the minimal number of  $L_2(Q)$ -balls of radius  $\varepsilon$  needed to cover  $\mathcal{G}$ , where a  $L_2(Q)$ -ball of radius  $\varepsilon$  around a function  $g \in L_2(Q)$  is the set  $\{h \in L_2(Q) : \|h - g\| < \varepsilon\}$  (see Pollard, 1984). Moreover, the class of functions  $\mathcal{G}$  has a finite and integrable envelope function  $\mathbb{F} := \sup_{g \in \mathcal{G}} |g(W_t, \theta, \tau)|$ , and it can be covered by a finite number of elements, not necessarily in  $\mathcal{G}$ .

The following result establishes a central limit theorem for strong mixing processes for the empirical distribution,  $\hat{Z}_T(y, x)$ , under the null and the alternative hypothesis.

**Lemma A.1.** If Assumption 1 holds, under  $H_0$  of (2.1) or  $H_A$  of (2.2),

$$v_T(y, x) := \sqrt{T}(\hat{Z}_T(y, x) - F_{YX}(y, x)) \Rightarrow \mathbb{H}_1(y, x), \text{ in } \ell^\infty(\mathcal{W}),$$

where  $\mathbb{H}_1$  is a tight mean zero Gaussian process in  $\ell^\infty(\mathcal{W})$  with covariance function

$$\text{Cov}(\mathbb{H}_1(y, x), \mathbb{H}_1(y', x')) = \sum_{k=-\infty}^{\infty} \text{Cov}(\mathbb{1}\{Y_0 \leq y\} \mathbb{1}\{X_0 \leq x\}, \mathbb{1}\{Y_k \leq y'\} \mathbb{1}\{X_k \leq x'\}).$$

**Proof:** Parts (a) and (b) of Assumption 1 imply the conditions (2.3) and (2.4) in Theorem 2.1 in Arcones and Yu (1994), respectively. Then the results follow from a direct application of Theorem 2.1 in Arcones and Yu (1994).  $\square$

The following result establishes a functional delta method for the empirical analog  $\hat{G}(\theta, \tau)$  of the moment conditions in (2.4) and for a consistent estimator of the function-valued parameter  $\hat{\theta}_T(\cdot)$ .

**Lemma A.2.** Let  $v_T(y, x) := \sqrt{T}(\hat{Z}_T(y, x) - F_{YX}(y, x))$  be the empirical process of Lemma A.1 and define the empirical process  $r_T(\theta, \tau) := \sqrt{T}(\hat{G}(\theta, \tau) - G(\theta, \tau))$ . If Assumption 1 is

satisfied, under  $H_0$  of (2.1) or  $H_A$  of (2.2),

$$\begin{aligned}(v_T(y, x), r_T(\theta, \tau)) &\Rightarrow (\mathbb{H}_1(y, x), \tilde{\mathbb{H}}_2(\theta, \tau)), \text{ in } \ell^\infty(\mathcal{W} \times \Theta \times \mathcal{T}), \\ \sqrt{T}(\hat{\theta}_T(\cdot) - \theta_0(\cdot)) &\Rightarrow -\dot{G}_{\theta_0, \cdot}^{-1}(\tilde{\mathbb{H}}_2(\theta_0(\cdot), \cdot)) \text{ in } \ell^\infty(\mathcal{T}),\end{aligned}$$

where  $\tilde{\mathbb{H}}_2$  is a tight mean zero Gaussian process with covariance function

$$\text{Cov}(\tilde{\mathbb{H}}_2(\theta, \tau), \tilde{\mathbb{H}}_2(\theta', \tau')) = \sum_{k=-\infty}^{\infty} \text{Cov}(g(W_0, \theta, \tau), g(W_k, \theta', \tau')).$$

**Proof:** By Lemma E.1 in Chernozhukov et al. (2013), Assumption 1 implies that (a) the inverse of  $G(\cdot, \tau)$  defined as  $G^{-1}(x, \tau) := \{\theta \in \Theta : G(\theta, \tau) = x\}$  is continuous at  $x = 0$  uniformly in  $\tau \in \mathcal{T}$  with respect to the Hausdorff distance, (b) there exists  $\dot{G}_{\theta_0, \tau}$  such that

$$\lim_{t \rightarrow 0} \sup_{\tau \in \mathcal{T}, \|h\|=1} |t^{-1}(G(\theta_0(\tau) + th, \tau) - G(\theta_0(\tau), \tau)) - \dot{G}_{\theta_0, \tau}h| = 0,$$

where  $\inf_{\tau \in \mathcal{T}} \inf_{\|h\|=1} \|\dot{G}_{\theta_0, \tau}h\| > 0$ , (c) the maps  $\tau \mapsto \theta_0(\tau)$  and  $\tau \mapsto \dot{G}_{\theta_0, \tau}$  are continuous, and (d) the mapping  $\tau \mapsto \theta_0(\tau)$  is continuously differentiable. Under the previous conditions, Lemma E.2 in Chernozhukov et al. (2013) holds, and the process  $r_T(\theta, \tau)$  weakly converges to  $\tilde{\mathbb{H}}_2(\theta, \tau)$  in  $\ell^\infty(\Theta \times \mathcal{T})$  and the map  $\theta \mapsto G(\theta, \cdot)$  is Hadamard differentiable at  $\theta_0$  with continuously invertible derivative  $\dot{G}_{\theta_0, \cdot}$ . By Hadamard differentiability of the map  $\theta \mapsto G(\theta, \cdot)$ , it follows the weak convergence of the process  $\sqrt{T}(\hat{\theta}_T(\cdot) - \theta_0(\cdot))$  in  $\ell^\infty(\mathcal{T})$ .

□

**Lemma A.3.** Let  $v_T(y, x) := \sqrt{T}(\hat{Z}_T(y, x) - F_{YX}(y, x))$  be the empirical process of Lemma A.1 and define the empirical process  $v_T^{\theta_0}(y, x) := \sqrt{T}(\hat{F}_T(y, x, \hat{\theta}_T) - F(y, x, \theta_0))$ . If Assumption 1 holds, under  $H_0$  of (2.1) or  $H_A$  of (2.2),

$$(v_T(y, x), v_T^{\theta_0}(y, x)) \Rightarrow (\mathbb{H}_1(y, x), \mathbb{H}_2(y, x)) \text{ in } \ell^\infty(\mathcal{W} \times \mathcal{W}),$$

where  $\mathbb{H}_2$  is a tight mean zero Gaussian process in  $\ell^\infty(\mathcal{W})$ .

**Proof:** By Lemma A.2,  $\sqrt{T}(\hat{\theta}_T(\cdot) - \theta_0(\cdot)) \Rightarrow -\dot{G}_{\theta_0, \cdot}^{-1}(\tilde{\mathbb{H}}_2(\theta_0(\cdot), \cdot))$  in  $\ell^\infty(\mathcal{T})$ , where  $\tilde{\mathbb{H}}_2$  is a tight mean zero Gaussian process. Similarly to Lemma A.1, under  $H_0$  of (2.1) or  $H_A$  of (2.2), if parts (a)-(b) of Assumption 1 hold, then  $\sqrt{T}(\hat{F}_X(x^*) - F_X(x^*))$  weakly converges to a tight mean zero Gaussian process. Now, let the measurable functions  $\Gamma : \mathcal{W} \mapsto [0, 1]$  be defined by  $(y, x) \mapsto \Gamma(y, x)$ , and the bounded maps  $\Pi : \mathcal{H} \mapsto \mathbb{R}$  be defined by  $f \mapsto \int f d\Pi$ . Then it follows from Lemma D.1 in Chernozhukov et al. (2013) that the mapping  $(\Gamma, \Pi) \mapsto \int \Gamma(\cdot, x) d\Pi(x)$ , with  $\Gamma(\cdot, x) = \mathbb{1}\{x^* \leq x\}F(\cdot|x)$  and  $\Pi = F_X(\cdot)$ , is well defined and Hadamard differentiable at  $(\Gamma, \Pi)$ . Under  $H_0$  of (2.1) or  $H_A$  of (2.2), we can write  $\hat{F}_T(y, x, \hat{\theta}_T) = \int F(y|x^*, \hat{\theta}_T) \mathbb{1}\{x^* \leq x\} d\hat{F}_X(x^*)$  and  $F(y, x, \theta_0) = \int F(y|x^*) \mathbb{1}\{x^* \leq x\} dF_X(x^*)$ . Then, by the functional delta method from Lemma B.1 of Chernozhukov et al. (2013), it follows that

$$\begin{aligned} \sqrt{T}(\hat{F}_T(y, x, \hat{\theta}_T) - F(y, x, \theta_0)) &= \int \sqrt{T} \left[ F(y|x^*, \hat{\theta}_T) - F(y|x^*) \right] \mathbb{1}\{x^* \leq x\} dF_X(x^*) \\ &+ \int F(y|x^*) \mathbb{1}\{x^* \leq x\} \sqrt{T} d \left( \hat{F}_X(x^*) - F_X(x^*) \right) + o_p(1). \end{aligned}$$

Using the same arguments of the Proof of Lemma A.2, we can show that the map  $\theta \mapsto F(\cdot|\cdot, \theta(\cdot))$  is Hadamard differentiable. Thus, we apply the functional delta method, for fixed  $y$  and  $x$ , as follows:

$$\begin{aligned} \sqrt{T} \left( F(y|x, \hat{\theta}_T) - F(y|x) \right) &\Rightarrow -\dot{F}(y|x, \theta_0)(-\dot{G}_{\theta_0, \cdot}^{-1}(\tilde{\mathbb{H}}_2(\theta_0(\cdot), \cdot))) \\ &:= \mathbb{H}_2^*(y, x) \text{ in } \ell^\infty(\mathcal{W}). \end{aligned}$$

Given the Hadamard differentiability of the mapping  $(\Gamma, \Pi) \mapsto \int \Gamma(\cdot, x) d\Pi(x)$ , the result follows from an application of the functional delta method, where the Gaussian process  $\mathbb{H}_2$

is given by

$$\mathbb{H}_2(y, x) := \int \mathbb{H}_2^*(y, x^*) \mathbb{1}\{x^* \leq x\} dF_X(x^*) + \int F(y|x^*) \mathbb{1}\{x^* \leq x\} d\mathbb{H}_1(\infty, x^*),$$

where  $\mathbb{H}_1$  is the tight mean zero Gaussian process defined in Lemma A.1.  $\square$

**Lemma A.4.** Under the local alternatives  $H_{A,T}$  in (2.7) and Assumptions 1-2, let  $F_T^A(y, x) = \int F_T(y|x^*) \mathbb{1}\{x^* \leq x\} dF_X(x^*)$  and  $G_{F_T}(\theta, \tau) = E_{F_T}[g(W_t, \theta, \tau)]$ , then:

$$\begin{pmatrix} \sqrt{T} \left( \hat{Z}_T(y, x) - F_T^A(y, x) \right) \\ \sqrt{T} \left( \hat{G}(\theta, \tau) - G_{F_T}(\theta, \tau) \right) \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbb{H}_1(y, x) \\ \tilde{\mathbb{H}}_2(\theta, \tau) \end{pmatrix} \text{ in } \ell^\infty(\mathcal{W} \times \Theta \times \mathcal{T}),$$

where  $(\mathbb{H}_1, \tilde{\mathbb{H}}_2)$  are the tight mean zero Gaussian processes defined in Lemmas A.1-A.2.

**Proof:** Under Assumption 2,  $F_T^A(y, x)$  is contiguous to  $F(y, x, \theta_0)$ , then under the sequence of local alternatives  $H_{A,T}$  in (2.7) and Assumptions 1-2,  $F_T(y|x)$  of (2.7) is a linear combination of two measures that are VC-subgraph class with a  $p$ -integrable envelope, for some  $2 < p < \infty$ . From an application of Lemma 2.8.7 in Van der Vaart and Wellner (2000), we have that  $(\sqrt{T}(\hat{Z}_T(y, x) - F_T^A(y, x)), \sqrt{T}(\hat{G}(\theta, \tau) - G_{F_T}(\theta, \tau))) \Rightarrow (\mathbb{H}_1(y, x), \tilde{\mathbb{H}}_2(\theta, \tau))$  in  $\ell^\infty(\mathcal{W} \times \Theta \times \mathcal{T})$ .

$\square$

## A.2. Proofs of the Propositions

**Proof of Proposition 1:** To prove part (a), we consider the empirical processes  $v_T(y, x) = \sqrt{T}(\hat{Z}_T(y, x) - F_{YX}(y, x))$  and  $v_T^{\theta_0}(y, x) = \sqrt{T}(\hat{F}_T(y, x, \hat{\theta}_T) - F(y, x, \theta_0))$  defined in Lemma A.1 and in Lemma A.3, respectively. Under  $H_0$  in (2.1), we have that  $F_{YX}(y, x) \equiv F(y, x, \theta_0)$ .

Then:

$$\begin{aligned}
S_T^{CM} &= \int T \left( \hat{Z}_T(y, x) - \hat{F}_T(y, x, \hat{\theta}_T) \right)^2 d\hat{Z}_T(y, x) \\
&= \int T \left( \hat{Z}_T(y, x) - \hat{F}_T(y, x, \hat{\theta}_T) \pm F_{YX}(y, x) \right)^2 d\hat{Z}_T(y, x) \\
&= \int \left( v_T(y, x) - v_T^{\theta_0}(y, x) \right)^2 d\hat{Z}_T(y, x) \\
&= \int \left( v_T(y, x) - v_T^{\theta_0}(y, x) \right)^2 dF_{YX}(y, x) \\
&+ \int \left( v_T(y, x) - v_T^{\theta_0}(y, x) \right)^2 d(\hat{Z}_T(y, x) - F_{YX}(y, x)).
\end{aligned}$$

By Lemma A.1,  $\sqrt{T}(\hat{Z}_T(y, x) - F_{YX}(y, x)) \Rightarrow \mathbb{H}_1(y, x)$ , where  $\mathbb{H}_1(y, x)$  is a tight mean zero Gaussian process in  $\ell^\infty(\mathcal{W})$ . Then, it follows that

$$S_T^{CM} = \int \left( v_T(y, x) - v_T^{\theta_0}(y, x) \right)^2 dF_{YX}(y, x) + o_p(1).$$

By Lemma A.3,  $(v_T(y, x), v_T^{\theta_0}(y, x)) \Rightarrow (\mathbb{H}_1(y, x), \mathbb{H}_2(y, x))$  in  $\ell^\infty(\mathcal{W} \times \mathcal{W})$ , where  $\mathbb{H}_2(y, x)$  is a tight mean zero Gaussian process in  $\ell^\infty(\mathcal{W})$ . Then, the result follows by an application of the continuous mapping theorem.

To prove part (b), under the alternative hypothesis  $H_A$  of (2.2),  $F_{YX}(y, x) \neq F(y, x, \theta)$  for some  $(y, x) \in \mathcal{W}$  and for all  $\theta \in \mathcal{B}(\mathcal{T}, \Theta)$ , and  $v_T^{\theta_0}(y, x)$  becomes  $v_T^\theta(y, x) = \sqrt{T}(\hat{F}_T(y, x, \hat{\theta}_T) - F_T(y, x, \theta))$ . [Then](#),

$$\begin{aligned}
S_T^{CM} &= \int T \left( \hat{Z}_T(y, x) - \hat{F}_T(y, x, \hat{\theta}_T) \pm F_{YX}(y, x) \pm F(y, x, \theta) \right)^2 d\hat{Z}_T(y, x) \\
&= \int \left( v_T(y, x) - v_T^\theta(y, x) + \sqrt{T}(F_{YX}(y, x) - F(y, x, \theta)) \right)^2 dF_{YX}(y, x) + o_P(1).
\end{aligned}$$

As a corollary of Lemma A.3,  $(v_T(y, x), v_T^\theta(y, x)) \Rightarrow (\mathbb{H}_1(y, x), \mathbb{H}_2(y, x))$  in  $\ell^\infty(\mathcal{W} \times \mathcal{W})$ .

Therefore, for all fixed constants  $C > 0$ , we have  $\lim_{T \rightarrow \infty} P(S_T^{CM} > C) = 1$ , and the result follows.  $\square$

**Proof of Proposition 2:** Under the local alternative  $H_{A,T}$  in (2.7), consider the empirical processes

$$\begin{aligned} v_T^1(y, x) &= \sqrt{T} \left( \hat{Z}_T(y, x) - \int F(y|x^*, \theta_0) \mathbb{1}\{x^* \leq x\} dF_X(x^*) \right), \text{ and} \\ r_T^1(\theta, \tau) &= \sqrt{T} (\hat{G}(\theta, \tau) - E_F[g(W_t, \theta, \tau)]), \end{aligned}$$

where  $G_F(\theta, \tau) := E_F[g(W_t, \theta, \tau)]$ , with  $E_F[\cdot]$  defined as the expectation w.r.t.  $F = F(y|x, \theta_0)$  in (2.7). Then

$$\begin{aligned} v_T^1(y, x) &= \sqrt{T} \left( \hat{Z}_T(y, x) - \int F(y|x^*, \theta_0) \mathbb{1}\{x^* \leq x\} dF_X(x^*) \right) \\ &= \sqrt{T} \hat{Z}_T(y, x) \\ &\quad - \sqrt{T} \int \left( F_T(y|x^*) + \frac{\delta}{\sqrt{T}} [F(y|x^*, \theta_0) - J(y|x^*)] \right) \mathbb{1}\{x^* \leq x\} dF_X(x^*) \\ &= \sqrt{T} \left( \hat{Z}_T(y, x) - F_T^A(y, x) \right) + \delta \int [J(y|x^*) - F(y|x^*, \theta_0)] \mathbb{1}\{x^* \leq x\} dF_X(x^*). \end{aligned}$$

Thus, it follows from Lemma A.4 that

$$v_T^1(y, x) \Rightarrow \mathbb{H}_1(y, x) + \delta \int [J(y|x^*) - F(y|x^*, \theta_0)] \mathbb{1}\{x^* \leq x\} dF_X(x^*),$$

where  $\mathbb{H}_1$  is a tight mean zero Gaussian process in  $\ell^\infty(\mathcal{W})$  defined in Lemma A.1. Then

$$\begin{aligned}
r_T^1(\theta, \tau) &= \sqrt{T} \left( \hat{G}(\theta, \tau) - E_F[g(W_t, \theta, \tau)] \right) \\
&= \sqrt{T} \left( \hat{G}(\theta, \tau) - (E_{F_T}[g(W_t, \theta, \tau)] + \delta E_F[g(W_t, \theta, \tau)] - \delta E_J[g(W_t, \theta, \tau)]) \right) \\
&= \sqrt{T} \left( \hat{G}(\theta, \tau) - G_{F_T}(\theta, \tau) + \delta (E_J[g(W_t, \theta, \tau)] - E_F[g(W_t, \theta, \tau)]) \right),
\end{aligned}$$

where  $G_J(\theta, \tau) := E_J[g(W_t, \theta, \tau)]$ , with  $E_J[\cdot]$  defined as the expectation w.r.t.  $J = \textcolor{blue}{J}(y|x)$  in (2.7). We define the empirical process  $v_T^{1\theta_0}(y, x)$  as follows:

$$v_T^{1\theta_0}(y, x) = \sqrt{T} \left( \int \textcolor{blue}{F}(y|x^*, \hat{\theta}_T) \mathbb{1}\{x^* \leq x\} d\hat{F}_X(x^*) - \int \textcolor{blue}{F}(y|x^*, \theta_0) \mathbb{1}\{x^* \leq x\} dF_X(x^*) \right).$$

By Lemmas A.3-A.4,

$$\begin{pmatrix} v_T^{1\theta_0}(y, x) \\ r_T^1(\theta, \tau) \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbb{H}_2(y, x) + \delta \int \dot{\textcolor{blue}{F}}(y|x^*)[h] \mathbb{1}\{x^* \leq x\} dF_X(x^*) \\ \tilde{\mathbb{H}}_2(\theta, \tau) + \delta (E_J[g(W_t, \theta, \tau)] - E_F[g(W_t, \theta, \tau)]) \end{pmatrix},$$

with  $h(\tau) = (\partial G_F(\theta_0, \tau)/\partial \theta)^{-1} G_J(\theta_0, \tau)$ , and where  $(\mathbb{H}_2, \tilde{\mathbb{H}}_2)$  are the tight mean zero Gaussian processes described in Lemmas A.2-A.3. Therefore, under  $H_{A,T}$  in (2.7),

$$\begin{aligned}
S_T^{CM} &= \int T \left( \hat{Z}_T(y, x) - \hat{F}_T(y, x, \hat{\theta}_T) \pm \int \textcolor{blue}{F}(y|x^*, \theta_0) \mathbb{1}\{x^* \leq x\} dF_X(x^*) \right)^2 d\hat{Z}_T(y, x) \\
&= \int (v_T^1(y, x) - v_T^{1\theta_0}(y, x))^2 d\hat{Z}_T(y, x) \\
&= \int (v_T^1(y, x) - v_T^{1\theta_0}(y, x))^2 dF_{YX}(y, x) + o_p(1),
\end{aligned}$$



Then,

$$S_T^{CM} \xrightarrow{d} \int (\mathbb{H}_1(y, x) - \mathbb{H}_2(y, x) + \Delta(y, x))^2 dF_{YX}(y, x),$$

with  $\Delta(y, x) = \delta \int (J(y|x^*) - F(y|x^*, \theta_0) + \dot{F}(y|x^*, \theta_0)[h]) \mathbb{1}\{x^* \leq x\} dF_X(x^*)$ , and  $h$  is the function  $h(\tau) = (\partial G_F(\theta_0, \tau) / \partial \theta)^{-1} G_J(\theta_0, \tau)$ .  $\square$

**Proof of Proposition 3:** Assumption 1 implies Assumptions 1-2 of Whang (2006). Then, parts (a) and (b) follow from an application of Theorems 2 and 3 of Whang (2006) using the convergence results of our Proposition 1. Further, Assumption 2 implies Assumption 2\* of Whang (2006). Therefore, part (c) follows the same steps of Theorem 5 of Whang (2006) using the convergence results of our Proposition 1.  $\square$

## References

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