Problem Set

Growth&Development
Juan Carlos Muñoz-Mora

Slow-Swan Model
Date: September 2, 2019
Due to: *August 14, 2019*

Instructions: .

- Please, read carefully each point before answering. Make sure you understand!
- One PS per team
- Intuition, intuition, intuition! (be concise, yet do not forget the intuition)
- Why not in LaTeX? ☺

Question1 (25 points)

Consider the production function $Y = AK + BL$ where A and B are positive constants:

(a) Is this production function neoclassical? Which of the neoclassical conditions does it satisfy and which ones does it not?

(b) Write the output per person as a function of capital per person. What is the marginal product of $k = \frac{K}{L}$? What is the average product of k?

Now let's assume that population grows at the constant rate n and that capital depreciates at the constant rate δ.

(c) Write down the fundamental equation of the Solow-Swam model.

(d) Under which conditions does this model have a steady state with no growth of per capita capital?

(e) If $s = 0.4$, $A = 1$, $B = 2$, $\delta = 0.08$ and $n = 0.02$, what is the long-run growth rate of this economy? what if B=5? Explain the diferences

Solution:

(a) We know that there a neoclassical production function fulfill the following assumptions:

- *Constant returns to scale*

$$F(\lambda K, \lambda L, T) = \lambda F(K, L, T) \forall \lambda > 0$$

- *Positive and diminishing retrns to private inputs*

$$\frac{\partial F}{\partial x} > 0, \frac{\partial^2 F}{\partial x^2} < 0 \forall x \in [K, L]$$

- *Inada Conditions*

$$\lim_{x \to 0} \frac{\partial F}{\partial x} = \infty, \lim_{x \to \infty} \frac{\partial F}{\partial x} = 0, \forall \lambda > 0$$

- *Essentiality*

$$F(x = 0) = 0 \forall x \in [K, L]$$

So we need to show whether this production function fulfill the assumption above (with one is enough).

- *Constant returns to scale*

$$F(\lambda K, \lambda L) = A\lambda K + B\lambda L = \lambda(AK + BL) = \lambda F(K, L)$$

- *Positive and diminishing returns to private inputs*

$$\frac{\partial F}{\partial K} = A, \text{ where } A > 0, \rightarrow \frac{\partial^2 F}{\partial K^2} = 0$$

Therefore, there is not a diminishing returns to private inputs. It is not a neoclassical function.

- *Inada Conditions*

$$\lim_{x \to 0} \frac{\partial F}{\partial K} = \lim_{x \to 0} A = A \neq \infty$$

- *Essentiality*

$$F(K=0)=BL$$

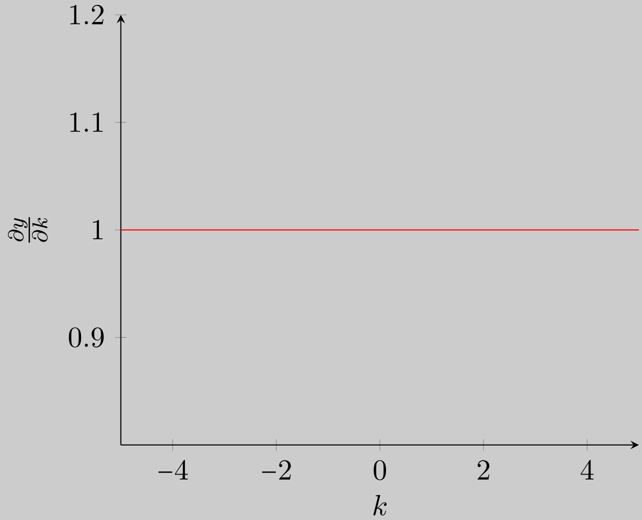
(b) The first step we need to make in this point is obtaining output per person as function of capital person, simply by dividing by L is both sides, in this way:

$$\begin{aligned} Y &= AK+BL \\ &\text{Divided by } (L) \\ \frac{Y}{L} &= A\frac{K}{L}+B\frac{L}{L} \\ y &= Ak+B \end{aligned}$$

where $k=\frac{K}{L}$ and $y=\frac{Y}{L}$. That is, $y=f(k)$. Giving this functional form we know that the marginal product of k is given by

$$\begin{aligned} \frac{\partial y}{\partial k} &= \frac{\partial f(k)}{\partial k} \\ &= A \end{aligned}$$

It implies that there any marginal increase on k yields a constant and positive return of A. Just to make an



example, we have that if A=1

Now to obtain the average product of k we need to get k to the Lefthand side.

$$k=\frac{y-B}{A}$$

(c) Let's get that fundamental Solow-Model equation using the above $F(.)$. Basic inputs for the Solow model:

- $$Y(t)=C(t)+I(t)$$
- $$S(t)=Y(t)-C(t)$$
- $$S(t)\equiv I(t)$$
- $$S(t)$$

  is assumed to be constant s. Therefore,

  $$C(t)=(1-s)Y(t)$$

  and

  $$S(t)=sY(t)=I(t)$$

- $$\dot{K} = I(t) - \delta K(t) = sF[K(t), L(t), T(t)] - \delta K(t)$$
- We know that, we can divide this by L obtaining $\rightarrow$

$$\frac{\dot{K}}{L} = sf(k) - \delta k$$

where $k = \frac{K}{L}$.

- By solving the last, we obtained that the fundamental differential equation of the Solow-Swan model is

$$\dot{k} = sf(k) - (n + \delta)k$$

So, we can either go step by step or going directly to this equation. Let's write the equation directly. So, as $f(k) = Ak + B$, so the fundamental differential equation of the Solow-Swan model is

$$\dot{k} = s(Ak + B) - (n + \delta)k = sB - (n + \delta - sA)k$$

(d) We know that at steady stage $\dot{k} = 0$, therefore,

$$\begin{array}{ccc} 0 & = & sB - (n + \delta - sA)k \\ k* & = & \frac{sB}{n+\delta-sA} \end{array}$$

This implies that the steady-state level of output per capita is give by:

$$\begin{array}{ccc} y* & = & A\frac{sB}{n+\delta-sA} + B \\ & = & B \end{array}$$

(e) Le's assume that $s = 0.4$, $A = 1$, $B = 2$, $\delta = 0.08$ and $n = 0.02$. Therefore:

- $f(k) = k + 2$
- $\dot{k} = 0.8 + 0.3k$
- $k* = -\frac{0.8}{0.3} = \frac{8}{3}$ (steady-state)

So, the long-run growth rate of this economy is given by:

$$\gamma_k \equiv \frac{\dot{k}}{k} = \frac{0.8 + 0.3k}{k} = 0.3 + \frac{0.8}{k}$$

The growth rate of output per capita is given by

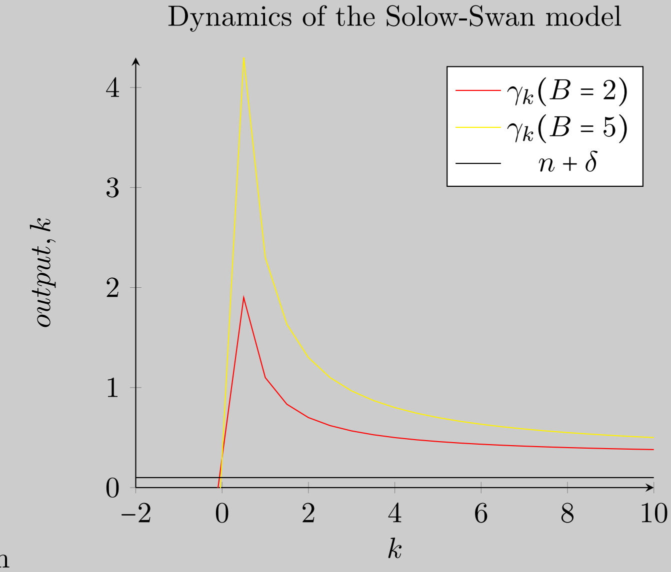
$$\gamma_y \equiv \frac{\dot{y}}{y} = \frac{f'(k)\dot{k}}{f(k)} = \frac{0.8 + 0.3k}{k + 2}$$

Now if we assume that $B = 5$, we have that $\dot{k} = 2 + 0.3k$:

- $$\gamma_k \equiv \frac{\dot{k}}{k} = \frac{2 + 0.3k}{k} = 0.3 + \frac{2}{k}$$

  The growth rate of output per capita is given by

  $$\gamma_y \equiv \frac{\dot{y}}{y} = \frac{f'(k)\dot{k}}{f(k)} = \frac{(0.8 - 0.6k)}{k + 2}$$



So, we have the two potential equilibrium

So, we can conclude that

- There is always a positive growth rate ¿0 under this model
- In the long-term there is always growth

Question2 (25 points)

Let us introduce government spending in the basic Solow-Swan Model. Consider the basic model without technological change and suppose that:

$$Y(t) = C(t) + I(t) + G(t)$$

with $G(t)$ denoting government spending at time t. Image that government spending is given by $G(t) = \sigma Y(t)$

(a) Discuss how the relationship between income and consumption should be changed. Is it reasonable to assume that $C(t) = sY(t)$

(b) Suppose that government spending partly comes out of private consumption, so that $C(t) = (s - \lambda\sigma)Y(t)$, where $\lambda \in [0,1]$. What is the effect of higher government spending (in the form of higher σ) on the equilibrium of the Solow model?

(c) Now suppose that a fraction ϕ of $G(t)$ is invested in the capital, so that total investment at time t is given by

$$I(t) = (1 - s - (1-\lambda)\sigma + \phi\sigma)Y(t)$$

Show that if ϕ is is sufficiently high, the steady-state level of capital-labor ratio will increase as a result of higher government spending (corresponding to higher σ) Is this reasonable? How would you alternatively introduce public investments in this model?

Solution:

(a) We know that $Y(t) = C(t) + I(t) + G(t)$ where $G(t) = \sigma Y(t)$. So, $Y(t) = C(t) + I(t) + \sigma Y(t)$, then, $Y(t) = \frac{C(t)+I(t)}{1-\sigma}$. There is now a new element into the traditional model, $\frac{1}{1-\sigma}$, that implies that

$$Y(t) = \begin{cases} \sigma = 0 & \rightarrow & C(t) + I(t) \\ \sigma = 1 & \rightarrow & ND \\ \sigma > 1, \sigma < -1 & \rightarrow & Y(t) < 0 \\ \sigma \in (-1,1) & \rightarrow & Y(t) > 0 \end{cases}$$

There is a part of the income that it has to go the goverment, so there is not sense to assume that $C(t) = sY(t)$.

(b) If $C(t) = (s - \lambda\sigma)Y(t)$, then

$$I(t) = (1 - s + (1 - \lambda)\sigma)Y(t)$$

So, the equation the transition of capital given by:

$$\dot{K} = (1 - s - (1 - \lambda)\sigma)Y(t) - \delta K(t)$$

So we can divide by L and assuming a neo-classical function $Y(t) = F(K(t), L(t), T(t))$, then we can write the fundamental differential equation of the Solow-Swan model:

$$\dot{k} = (1 - s + (1 - \lambda)\sigma)f(k) - (n + \delta)k$$

At the steady-state we have that:

$$k^* = \frac{(1 - s - (1 - \lambda)\sigma)f(k^*)}{(n + \delta)}$$

Any increase in σ, *cetirus paribus*, will decrease k

(c) We can directly write down the

$$\dot{K} = (1 - s - (1 - \lambda + \phi)\sigma)Y(t) - \delta K(t)$$

then,

$$\dot{k} = (1 - s - (1 - \lambda + \phi)\sigma)f(k) - (n + \delta)k$$

Therefore, if $\phi \to \infty$

$$\lim_{\phi\to\infty} \frac{\dot{k}}{k} = \lim_{\phi\to\infty} (1 - s - (1 - \lambda + \phi)\sigma)\frac{f(k)}{k} - (n + \delta)$$

Question3 (20 points)
Let us consider that a economy follows this production function:

$$Y = AK^{\lambda}H^{\eta}[T(t)L]^{1-\alpha-\eta}$$

(a) Is this production function neoclassical?
(b) What the growth rate of the physical capita?
(c) what is the main steady-state condition?
(d) what is the the convergence coefficient in the steady state?

Solution:

(a) If only if, we have a Cobb-Douglas conditions $\lambda + \eta + (1 - \alpha - \eta) = 1$, that is, $\lambda + 1 - \alpha = 1$. So it is neoclassic iff $\lambda = \alpha$. Therefore, the function is

$$Y = AK^{\alpha}H^{\eta}[T(t)L]^{1-\alpha-\eta}$$

(b) First of all, we need to assume that growth rate for $T(t)$. Say, $T(t)$ growths at rate t.

Now, we need to get the product per unit of effective labor:

$$\begin{aligned} \frac{Y}{T(t)L} &= AK^{\alpha}H^{\eta}\frac{[T(t)L]^{1-\alpha-\eta}}{T(t)L} \\ \hat{y} &= AK^{\alpha}H^{\eta}\frac{1}{(T(t)L)^{\alpha+\eta}} \\ \hat{y} &= A\hat{k}^{\alpha}\hat{h}^{\eta} \end{aligned}$$

Output can be used on a one-to-one basis for consumption or investment in either type of capital. So, assuming a constant saving rate s and common depreciation rate δ, we can derive that:

$$\dot{\hat{h}} + \dot{\hat{k}} = sA\hat{k}^{\alpha}\hat{h}^{\eta} - (\delta + n + x)(\hat{h} + \hat{k})$$

This equation brings an important assumption, that marginal contribution of both type of capital has to be equal. Otherwise, all investment will end up at only one type of capital. That is, $\frac{\partial \hat{y}}{\partial \hat{k}} = \frac{\partial \hat{y}}{\partial \hat{h}}$, that is:

$$\alpha\frac{\hat{y}}{\hat{k}} = \eta\frac{\hat{y}}{\hat{h}} \rightarrow \hat{h} = \frac{\eta}{\alpha}\hat{k}$$

So, we can re-write the fundamental Slow-Swan equation:

$$\begin{aligned} \tfrac{\eta}{\alpha}\hat{k} + \dot{\hat{k}} &= sA\hat{k}^{\alpha}\tfrac{\eta}{\alpha}\hat{k}^{\eta} - (\delta + n + x)(\hat{h} + \tfrac{\eta}{\alpha}\hat{k}) \\ \dot{\hat{k}} &= s\hat{A}\hat{k}^{\alpha+\eta} - (\delta + n + x)\hat{k} \end{aligned}$$

Where $\hat{A} = \frac{\eta^{\eta}\alpha^{(1-\eta)}}{\alpha+\eta}$ Therefore, the growth rate of the physical capital is given by=

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{A}\hat{k}^{\alpha+\eta-1} - (\delta + n + x)$$

(c) The main steady-state condition is given by:

$$s\hat{A}\hat{k}^{\alpha+\eta-1} = (\delta + n + x)$$

(d) We know that *speed of convergence* β measures how much the growth rate declines as the capital stock increases in a proportional sense, that is,

$$\beta \equiv -\frac{\partial\frac{\dot{\hat{k}}}{\hat{k}}}{\partial log\hat{k}}$$

. So we put $\frac{\dot{\hat{k}}}{\hat{k}}$ in function of $log(k)$ That is;

$$\begin{aligned} \frac{\dot{\hat{k}}}{\hat{k}} &= s\hat{A}\hat{k}^{\alpha+\eta-1} - (\delta + n + x) \\ &= s\hat{A}e^{(\alpha+\eta-1)\ln\hat{k}} - (\delta + n + x) \end{aligned}$$

Then,

$$\frac{\partial\frac{\dot{\hat{k}}}{\hat{k}}}{\partial log\hat{k}} = (\alpha + \eta - 1)s\hat{A}\hat{k}^{\alpha+\eta-1}$$

Therefore,

$$\beta = (1 - \alpha - \eta)s\hat{A}\hat{k}^{\alpha+\eta-1}$$

We know that at the steady-state, $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ we have that the in the neighborhood of the steady state, the speed of convergence equals to:

$$\beta = (1 - \alpha - \eta)(\delta + n + x)$$

Question4 (20 points)

Let us consider the standard Solow model introducing technology:

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$$

Where A is a technology variable.

(a) Why this model differs from the Solow–Swan Model? Do we need any extra assumption on how A is growing?

(b) Solve the steady state (make a graph)

(c) Draw the solow diagram with technological progress

Question5 (10 points)
Suppose the U.S. Congress enacts legislation that discourages saving and investment, such as the elimination of the investment tax credit that occurred in 1990. As a result, suppose the investment rate falls permanently from s' to s''. (i.e. $s' > s''$). Examine this policy change in the Solow model with technological progress, assuming that the economy begins in steady state. Sketch a graph of how (the natural log of) output per worker evolves over time with and without the policy change. Make a similar graph for the growth rate of output per worker. Does the policy change permanently reduce the level or the growth rate of output per worker?