Advanced Econometrics I

Assignment # 1

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Question List

AI Usage Statement

This assignment was entirely completed by myself.

AI was used only for English language polishing.

Question 1

(10 points) Let S be a sample space. Show that the intersection of two sigma algebra is a sigma algebra.

Answer to Q - 1

Set $\mathcal{B}_1$ and $\mathcal{B}_2$ are two sigma algebras. Define $\mathcal{B}_3 \equiv \mathcal{B}_1 \cap \mathcal{B}_2$.

Verify each details in the definition of a sigma algebra to show that $\mathcal{B}_3$ is a sigma algebra.

First, according to the definition, we have $\varnothing$ in both $\mathcal{B}_1$ and $\mathcal{B}_2$. Thus, $\mathcal{B}_3$ contains $\varnothing$.

Second, consider A is an element of $\mathcal{B}_3$, also an element of $\mathcal{B}_1$ and $\mathcal{B}_2$. It implies that A^c is in both $\mathcal{B}_1$ and $\mathcal{B}_2$. So $\mathcal{B}_3$ is closed under complementation.

Third, consider a sequence of sets $A_1, A_2, \cdots \in \mathcal{B}_3$. Since each A_i is in both $\mathcal{B}_1$ and $\mathcal{B}_2$, and both $\mathcal{B}_1$ and $\mathcal{B}_2$ are sigma algebras, we have

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_1 \quad \text{and} \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_2.$$

Hence, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_3$, showing that $\mathcal{B}_3$ is closed under countable unions. $\square$

Question 2

(10 points) Suppose $\Pr(A) = 1/3$ and $\Pr(B^c) = 1/4$.

(a) Is it possible that A and B are mutually exclusive?

(b) Suppose $\Pr(A \cap B) = 1/8$. Calculate $\Pr(A \cup B)$.

Answer to Q - 2.a

Impossible.

Mutually exclusive means that $\Pr(A \cap B) = 0$. Calculate

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= \frac{1}{3} + \left(1 - \frac{1}{4}\right) - \Pr(A \cap B) \\ &= \frac{13}{12} - \Pr(A \cap B)\end{aligned}$$

$\Pr(A \cup B)$ would be greater than 1 if $\Pr(A \cap B) = 0$. Hence, A and B cannot be mutually exclusive.

Answer to Q - 2.b

Substitute $\Pr(A \cap B) = 1/8$ into the expression for $\Pr(A \cup B)$ obtained above,

$$\Pr(A \cup B) = \frac{13}{12} - \Pr(A \cap B) = \frac{23}{24}$$

Question 3

(10 points) Suppose any conditioning event has positive probability. If $A \subset B$, prove that $\Pr(B|A) = 1$ and $\Pr(A|B) = \Pr(A) / \Pr(B)$.

Answer to Q - 3

Since $A \subset B$, we have $A \cap B = A$, and therefore $\Pr(A \cap B) = \Pr(A)$.

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A)}{\Pr(A)} = 1$$

Also,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \qquad \square$$

Question 4

(10 points) Suppose two events A and B are independent, and $B \subset A$. What could we say about $\Pr(A)$ or $\Pr(B)$?

Answer to Q - 4

Basically, both $\Pr(A)$ and $\Pr(B)$ can take any value between 0 and 1.

A and B are independent, which implies $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$. Since $B \subset A$, we also have $\Pr(A \cap B) = \Pr(B)$. Both of these conditions must hold simultaneously, which gives

$$\Pr(A) \cdot \Pr(B) = \Pr(B)$$

Case 1: $\Pr(B) \neq 0$, then $\Pr(A) = 1$.

Case 2: $\Pr(B) = 0$, then $\Pr(A) \in [0, 1]$.

Question 5

(20 points) Suppose A, B, C are three events such that A and B are disjoint, A and C are independent, and B and C are independent. Suppose also that $4\Pr(A) = 2\Pr(B) = \Pr(C)$, $\Pr(A \cup B \cup C) = 5\Pr(A)$, and $\Pr(A) > 0$. Determine the value of $\Pr(A)$.

Answer to Q - 5

For three events, the probability of their union can be expressed as

$$\begin{aligned}\Pr(A \cup B \cup C) = {} & \Pr(A) + \Pr(B) + \Pr(C) \\ & - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) \\ & - \Pr(A \cap B \cap C)\end{aligned}$$

A and B are disjoint, so $A \cap B = \varnothing$.

A and C are independent, so $\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$.

B and C are independent, so $\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$.

Now, substituting these result and the relation $4\Pr(A) = 2\Pr(B) = \Pr(C)$ into the formula for the union, we can rewrite the expression entirely in terms of $\Pr(A)$.

$$\begin{aligned}\Pr(A \cup B \cup C) & = \Pr(A) + 2\Pr(A) + 4\Pr(A) \\ & \quad - \Pr(\varnothing) - \Pr(A) \cdot \Pr(C) - \Pr(B) \cdot \Pr(C) \\ & \quad - \Pr(\varnothing \cap C) \\ & = 7\Pr(A) - 0 - 4[\Pr(A)]^2 - 8[\Pr(A)]^2 - 0 \\ & = 7\Pr(A) - 12[\Pr(A)]^2\end{aligned}$$

Given $\Pr(A \cup B \cup C) = 5\Pr(A)$,then

$$6[\Pr(A)]^2 - \Pr(A) = 0$$

$$\Pr(A)[6\Pr(A) - 1] = 0$$

The solution to the equation above are $\Pr(A) = 1/6$ and $\Pr(A) = 0$ (dropped).

Question 6

(20 points) Amy and Bob are rolling a fair dice in turn. Amy starts first, and she keeps rolling until she get "one". And then it's Bob's turn. So does Bob. Try to give the probability of the event $A_n = \{\text{the n-th roll is made by Amy}\}$. This two cases are mutually exclusive.

Answer to Q - 6

The event "the n-th roll is made by Amy" can be partitioned into the following two mutually exclusive events:

(1) The (n-1)-th roll was made by Amy and the outcome was not a 1;

(2) The (n-1)-th roll was made by Bob and the outcome was a 1.

Since the die is fair, let B denote the event "rolling a 1" (regardless of who rolls). Then we obtain the following formula

$$\begin{aligned}
\Pr(A_n) &= \underbrace{\Pr(A_{n-1}) \cdot \Pr(B^c)}_{\text{case 1}} + \underbrace{\Pr\left(A_{n-1}^c\right) \cdot \Pr(B)}_{\text{case 2}} \\
&= \Pr(A_{n-1}) \cdot \frac{5}{6} + [1 - \Pr(A_{n-1})] \cdot \frac{1}{6} \\
&= \frac{2}{3} \cdot \Pr(A_{n-1}) + \frac{1}{6}
\end{aligned}$$

Rewrite the result

$$x_n - \frac{2}{3} \cdot x_{n-1} = \frac{1}{6}$$

Solve the difference equation by applying the general solution formula [1].

$$x_n = \left(x_0 - \frac{c}{1-a}\right) a^n + \frac{c}{1-a}$$

Here, we have $c = 1/6$ and $a = 2/3$, so

$$\begin{aligned}
\Pr(A_n) \equiv x_n &= \left(1 - \frac{1/6}{1-2/3}\right)\left(\frac{2}{3}\right)^n + \frac{1/6}{1-2/3} \\
&= \frac{1}{2}\left(\frac{2}{3}\right)^n + \frac{1}{2}
\end{aligned}$$

[1] Reference: https://lvjr.bitbucket.io/course/wjf/wjf09-i.pdf, page 169

Question 7

(20 points) Suppose that 5% of men and 0.25% of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is a male? (Assume that males and females are in equal numbers.)

Answer to Q - 7

By the given conditions, we have

$$\begin{aligned}
&\Pr(\text{male}) = \Pr(\text{female}) = 0.5\\
&\Pr(\text{c-b} \mid \text{male}) = 0.05\\
&\Pr(\text{c-b} \mid \text{female}) = 0.0025
\end{aligned}$$

Using Bayes' formula

$$\begin{aligned}
\Pr(\text{male} \mid \text{c-b}) &= \frac{\Pr(\text{c-b} \mid \text{male}) \cdot \Pr(\text{male})}{\Pr(\text{c-b} \mid \text{male}) \cdot \Pr(\text{male}) + \Pr(\text{c-b} \mid \text{female}) \cdot \Pr(\text{female})}\\
&= \frac{0.05 \cdot 0.5}{0.05 \cdot 0.5 + 0.0025 \cdot 0.5} = \frac{20}{21} \approx 95.24\%
\end{aligned}$$