Advanced Econometrics I

Assignment # 2

Paul Sun

econsunrq@outlook.com

Instructor Nan Liu

Delivered 2025/10/10

Deadline 2025/10/15

Question List

Question 1

(20 points) Suppose X follows the exponential distribution with parameter $\lambda = 2$.

(a) Calculate $\Pr(X > 2)$ and find the median of X.

(b) Let $Z = 0.5X + 3$. Calculate its PDF and its variance.

Answer to Q - 1.a

$X \sim \text{Exponential}(2)$, its CDF is [1]

$$F_X(x) = 1 - e^{-x/2}, \quad x > 0$$

and PDF is

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Calculate

$$\begin{aligned} \Pr(X > 2) &= 1 - \Pr(X \leq 2) = 1 - F_X(2) \\ &= 1 - \left(1 - e^{-2/2}\right) = e^{-1} \end{aligned}$$

Let m denote the median. By definition, $\Pr(X \leq m) = 0.5$, so

$$F_X(m) = 1 - e^{-m/2} = 0.5$$

Solving for m gives the median $2\ln 2$.

Answer to Q - 1.b

Setting $z = g(x) = 0.5x + 3$, it follows that $g^{-1}(z) = 2z - 6$. The PDF of Z is

$$\begin{aligned} f_Z(z) &= f_X\left[g^{-1}(z)\right] \left| \frac{\mathrm{d}}{\mathrm{d}z} g^{-1}(z) \right| \quad z \in \mathcal{Z} \\ &= \frac{1}{2} e^{-z+3} \cdot 2 = e^{-z+3} \end{aligned}$$

Since $x \in (0, +\infty)$, we have $z \in (-3, +\infty)$, which represents the support $\mathcal{Z}$.

[1] The form of the CDF depends on how the parameter λ is defined. Using $F_X(x) = 1 - e^{-x/2}$ is also acceptable, though it leads to different numerical results.

Question 2

Answer to Q - 2

(10 points) Suppose the density of X is $f(x) = c(x-1)^2$, for $-1 < x < 2$; and 0 otherwise. Let $W = X^2$. Find the PDF of W and calculate $\mathrm{E}(W)$.

The PDF must integrate to 1 over its support, so

$$\int_{-1}^{2} c(x-1)^2 \, \mathrm{d}x = 1$$

Solving for c gives $c = 1/3$.

Since $x \in (-1, 2)$, the variable w lies in the interval $[0, 4)$. The solution to $W = X^2$ are $X_1 = \sqrt{W}$ and $X_2 = -\sqrt{W}$. To derive the PDF of W, we apply the following formula

$$f_W(w) = \sum_i f_X[x_i] \left| \frac{\mathrm{d}x_i}{\mathrm{d}w} \right| = \sum_i \frac{f_X(x_i)}{2\sqrt{w}}$$

For $w \in (0, 1)$, both $\sqrt{w}$ and $-\sqrt{w}$ lie in the support of X, so

$$\begin{aligned}
f_W(w) &= \frac{1}{2\sqrt{w}} \left[f_X\left(\sqrt{w}\right) + f_X\left(-\sqrt{w}\right) \right] \\
&= \frac{1}{6\sqrt{w}} \left[\left(\sqrt{w} - 1\right)^2 + \left(-\sqrt{w} - 1\right)^2 \right] \\
&= \frac{1}{6\sqrt{w}} \left[\left(w - 2\sqrt{w} + 1\right) + \left(w + 2\sqrt{w} + 1\right) \right] \\
&= \frac{1}{3\sqrt{w}} (w + 1)
\end{aligned}$$

For $w \in [1, 4)$, only $\sqrt{w}$ lies in the support of X, so

$$\begin{aligned}
f_W(w) &= \frac{1}{2\sqrt{w}} f_X\left(\sqrt{w}\right) \\
&= \frac{1}{2\sqrt{w}} \left(\sqrt{w} - 1\right)^2
\end{aligned}$$

In summary,

$$f_W(w) = \begin{cases} \dfrac{1}{3\sqrt{w}} (w + 1) & w \in [0, 1) \\ \dfrac{1}{2\sqrt{w}} \left(\sqrt{w} - 1\right)^2 & w \in [1, 4) \\ 0 & \text{otherwise} \end{cases}$$

Given that $W = X^2$, Calculate the expectation

$$\mathrm{E}(W) = \mathrm{E}\left(X^2\right) = \int_{-1}^{2} x^2 \cdot \frac{1}{3}(x-1)^2 \,\mathrm{d}x = \frac{1}{3}\int_{-1}^{2}\left(x^4 - 2x^3 + x^2\right)\mathrm{d}x = \frac{7}{10}$$

Question 3

(15 points) Suppose X has MGF $M(t) = e^{t^2}$. Define $Z = 2X + 5$.

(a) By calculating the MGF of Z, derive the PDF of Z.

(b) Note that the 0.975 quantile of standard normal distribution is 1.96. Calculate the 0.025 quantile of Z.

(c) Find b such that $\Pr(|Z-5| < b) = 0.95$.

Answer to Q - 3.a

Since

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

we obtain

$$M_Z(t) = e^{5t} M_X(2t) = e^{5t} \cdot e^{(2t)^2} = e^{4t^2+5t}$$

which satisfies the form of MGF of normal distribution. Thus, $Z \sim \mathrm{Normal}(5, 8)$ and its PDF is

$$f_Z(z) = \frac{1}{\sqrt{16\pi}} \exp\left\{\frac{-(z-5)^2}{16}\right\}, \quad z \in \mathbb{R}$$

Answer to Q - 3.b

Now we are going to find z such that

$$\begin{aligned} F_Z(z) = 0.25 \quad &\Leftrightarrow \quad \Pr(Z \le z) = 0.025 \\ &\Leftrightarrow \quad \Pr\left(\frac{Z-5}{2\sqrt{2}} \le \frac{z-5}{2\sqrt{2}}\right) = 0.25 \\ &\Leftrightarrow \quad \Pr\left(Y \le \frac{z-5}{2\sqrt{2}}\right) = 0.025 \end{aligned}$$

where $Y \sim \mathrm{Normal}(0,1)$. According to the statement, we have

$$\Pr(Y \le 1.96) = 0.975 \quad \Leftrightarrow \quad \Pr(Y > 1.96) = 0.025$$

By symmetry of normal distribution, $\Pr(Y \le -1.96) = 0.025$. Thus,

$$\frac{z-5}{2\sqrt{2}} = -1.96$$

Solving for z gives $z = 5 - 3.92\sqrt{2}$.

Answer to Q - 3.c

Do some algebraic manipulation

$$\begin{aligned}\Pr(|Z-5| < b) &= \Pr\left[-b < \left(2\sqrt{2}\cdot Y + 5\right) - 5 < b\right] \\ &= \Pr\left(-\frac{b\sqrt{2}}{4} < Y < \frac{b\sqrt{2}}{4}\right) = 0.95\end{aligned}$$

As we discussed earlier, we have another conclusion that

$$\begin{aligned}&\Pr(Y \le 1.96) - \Pr(Y \le -1.96) \\ =\ &\Pr(-1.96 \le Y \le 1.96) \\ =\ &0.975 - 0.025 = 0.95\end{aligned}$$

Thus,

$$\frac{b\sqrt{2}}{4} = 1.96 \quad \Rightarrow \quad b = 3.92\sqrt{2}$$

Question 4

(15 points) Suppose X follows the geometric distribution with parameter p.

(a) Show that it has the Markov property, i.e. $\Pr(X > s \mid X > t) = \Pr(X > s - t)$ for any positive integers $s > t$.

(b) Derive the MGF of X.

(c) Calculate the variance of X.

Answer to Q - 4.a

As the problem states, the PMF of X is

$$f_X(k) = p\,(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

Calculate such a summation of the geometric series

$$\Pr(X > t) = \sum_{k=t+1}^{\infty} p\,(1-p)^{k-1} = \lim_{n\to\infty} p \cdot \frac{(1-p)^{(t+1)-1}\left[1-(1-p)^n\right]}{1-(1-p)} = (1-p)^t$$

Verify the Markov property,

$$
\begin{aligned}
\Pr(X > s \mid X > t) &= \frac{\Pr(X > s, X > t)}{\Pr(X > t)} = \frac{\Pr(X > s)}{\Pr(X > t)} \\
&= \frac{\sum_{k=s+1}^{\infty} p(1-p)^{k-1}}{\sum_{k=t+1}^{\infty} p(1-p)^{k-1}} = \frac{(1-p)^s}{(1-p)^t} \\
&= (1-p)^{s-t} = \Pr(X > s - t)
\end{aligned}
$$

Answer to Q - 4.b

Calculate the following expectation,

$$
\begin{aligned}
\mathrm{E}\left(e^{tX}\right) &= \sum_{k=1}^{\infty} e^{tk} \cdot p(1-p)^{k-1} \\
&= p e^t \cdot \sum_{k=0}^{\infty} \left[e^t \cdot (1-p)\right]^k \\
&= p e^t \cdot \lim_{n \to \infty} \frac{1 - \left[(1-p) e^t\right]^n}{1 - (1-p) e^t} \\
&= \frac{p e^t}{1 - (1-p) e^t}
\end{aligned}
$$

By definition, that is the MGF of X. The last step is to find the range of t such that the MGF exists. Note that the limit converges and the denominator is non-zero only when

$$
(1-p) e^t < 1 \quad \Rightarrow \quad t < -\ln(1-p)
$$

Answer to Q - 4.c

Differentiate the MGF with respect to t once

$$
M_X'(t) = \frac{\mathrm{d}M_X(t)}{\mathrm{d}t} = \frac{p e^t}{\left[1 - (1-p) e^t\right]^2}
$$

and differentiate again

$$
M_X''(t) = \frac{\mathrm{d}M_X'(t)}{\mathrm{d}t} = \frac{p e^t \left[1 + (1-p) e^t\right]}{\left[1 - (1-p) e^t\right]^3}
$$

Evaluate the first and second derivatives at $t = 0$

$$
M_X'(0) = \frac{p}{\left[1 - (1-p)\right]^2} = \frac{1}{p}
$$

and

$$
M_X''(0) = \frac{p\left[1 + (1-p)\right]}{\left[1 - (1-p)\right]^3} = \frac{2-p}{p^2}
$$

Thus, the variance is

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = M_X''(0) - \left[M_X'(0)\right]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Question 5

(15 points) Suppose a discrete random variable X has variance 0.5 and its MGF satisfies that $M(t) = a + b\left(e^t + e^{-t}\right)$. Calculate the CDF, the mean and the variance of X.

Answer to Q - 5

Differentiate the MGF with respect to t twice

$$M_X'(t) = b\left(e^t - e^{-t}\right) \quad \text{and} \quad M_X''(t) = b\left(e^t + e^{-t}\right)$$

Since X has variance 0.5, we have

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = M_X''(0) - \left[M_X'(0)\right]^2 = 2b - 0^2 = 0.5$$

Solving for b gives $b = 0.25$.

By definition, $M_X(0) = \mathrm{E}\left(e^{0\cdot X}\right) = 1$, we have

$$M_X(0) = a + b\left(e^0 + e^0\right) = a + 2\cdot 0.25 = 1$$

Thus, $a = 0.5$, which gives the MGF of X

$$M_X(t) = 0.5 + 0.25\left(e^t + e^{-t}\right)$$

We can see that it satisfies the form of MGF of discrete random variable

$$M_X(t) = \sum_x \mathrm{Pr}(X = x)\, e^{tx} = \underbrace{\mathrm{Pr}(X=-1)}_{=0.25} e^{-t} + \underbrace{\mathrm{Pr}(X=0)}_{=0.5} e^0 + \underbrace{\mathrm{Pr}(X=1)}_{=0.25} e^t$$

So, the discrete random variable X takes values in $\{-1, 0, 1\}$, and

$$F_X(x) = \begin{cases} 0 & x < -1 \\ 0.25 & -1 \le x < 0 \\ 0.75 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Finally, calculate the mean and variance of X

$$\mathrm{E}(X) = M_X'(0) = 0.25\left(e^0 - e^0\right) = 0 \quad \text{and} \quad \mathrm{Var}(X) = 0.5$$

Question 6

(7 points) An electronic product will fail a shock arrives. Suppose during the time $[0, t)$, the number of shocks follow Poisson distribution with parameter λt. What is the lifetime of the product?

Answer to Q - 6

Let X denote the number of shocks during the time interval $[0, t)$. By definition, $X \sim \text{Poisson}(\lambda t)$, and its PMF is

$$f_X(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

Certainly, the product will not break more than once, so we now regard the time until the first shock arrives as the lifetime of the product. The length of lifetime is random because no one can predict when the product will break, so we set T as a random variable representing the lifetime. The CDF of T is

$$\begin{aligned}
F_T(t) &= \Pr(T \leq t) = 1 - \Pr(T > t) \\
&= 1 - \Pr\left(\{\text{no shock during } [0, t)\}\right) \\
&= 1 - \Pr(X = 0) = 1 - f_X(0) \\
&= 1 - e^{-\lambda t}
\end{aligned}$$

which is exponential distribution exactly. And the PDF of T is

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) = \lambda e^{-\lambda t}, \quad t > 0$$

Question 7

(8 points) Suppose X is a continuous random variable. Prove the following

(a) $\min_a \mathrm{E}|X - a| = \mathrm{E}|X - m|$, where m is the median of X.

(b) $\min_a \mathrm{E}|X - a|^2 = \mathrm{Var}(X)$

Answer to Q - 7.a

By the definition of expectation,

$$\mathrm{E}|X - a| = \int_{-\infty}^{+\infty} |X - a|(x)\,\mathrm{d}x = \int_{-\infty}^{a} (x - a) f(x)\,\mathrm{d}x - \int_{a}^{+\infty} (x - a) f(x)\,\mathrm{d}x$$

Differentiate with respect to a

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{E}|X-a| &= \frac{\mathrm{d}}{\mathrm{d}a}\int_{-\infty}^{a}(x-a)f(x)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}a}\int_{a}^{+\infty}(x-a)f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a}\frac{\mathrm{d}}{\mathrm{d}a}(x-a)f(x)\,\mathrm{d}x - \int_{a}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}a}(x-a)f(x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{a}f(x)\,\mathrm{d}x + \int_{a}^{+\infty}f(x)\,\mathrm{d}x \\
&= -\Pr(X\le a)+\Pr(X>a)
\end{aligned}$$

Setting the first-order condition to zero gives $\Pr(X\le a)=\Pr(X>a)$, which means $a=m$.

Without being fully rigorous, we have to assume that the functions are differentiable and the moments exists. So do the next problem subsection.

In the second step of the derivation, we use the Leibniz integral rule [2] to exchange the order of differentiation and integration.

Answer to Q - 7.b

Notice that

$$g(a)\equiv \mathrm{E}|X-a|^2=\mathrm{E}\left(X^2-2aX+a^2\right)=\mathrm{E}\left(X^2\right)-2a\,\mathrm{E}(X)+a^2$$

Take the first derivative with respect to a to derive the FOC

$$g'(a)=-2\mathrm{E}(X)+2a=0$$

Solving a gives $a=\mathrm{E}(X)$. In conclusion,

$$\begin{aligned}
\min_a \mathrm{E}|X-a|^2 &= g[\mathrm{E}(X)]=\mathrm{E}\left(X^2\right)-2[\mathrm{E}(X)]^2+[\mathrm{E}(X)]^2 \\
&= \mathrm{E}\left(X^2\right)-[\mathrm{E}(X)]^2=\mathrm{Var}(X)
\end{aligned}$$

Question 8

(10 points) Let X be a non-negative random variable with cumulative distribution function $F(\cdot)$. Define the indicator function $I(t)$ as 1 if $X>t$ and 0 otherwise.

(a) Show that $\displaystyle\int_0^{+\infty} I(t)\,\mathrm{d}t = X$.

(b) Show that $\mathrm{E}(X)=\displaystyle\int_0^{+\infty}[1-F(t)]\,\mathrm{d}t$.

[2] G. Casella and R. L. Berger. (2024). *Statistical Inference*. 2nd edition. CRC Press. See **Theorem 2.4.3** in Section 2.4

Answer to Q - 8.a

In the integral, $0 < t < +\infty$, but we notice that $I(t) = 1$ only when $0 < t < X$. Thus,

$$\int_0^{+\infty} I(t)\,\mathrm{d}t = \int_{X>t} 1\,\mathrm{d}t =_X^0 1\,\mathrm{d}t = X$$

Answer to Q - 8.b

Starting from the definition of expectation,

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^{+\infty} x f(x)\,\mathrm{d}x = \int_0^{+\infty} \left(\int_0^{x} 1\,\mathrm{d}t\right) f(x)\,\mathrm{d}x \\
&= \int_0^{+\infty} I(t) f(x)\,\mathrm{d}x\,\mathrm{d}t =_\infty^0 \Pr(X > t)\,\mathrm{d}t \\
&=_\infty^0 [1 - F(t)]\,\mathrm{d}t
\end{aligned}$$