

Mathematical Economics

Assignment # 1

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Question 1

Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

- What is the rank of A ?
- Express A as $E_1 E_2 \cdots E_n I$, where each E_i is an elementary matrix.
- Calculate $\det A$.
- Invert A using Theorem 9.4.
- Use Cramer's rule to solve $Ax = b$.

Answer to Q - 1.a

Apply elementary row operations on A

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ \xrightarrow{r_1 + 3r_2 + 2r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{-r_2, -r_3/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reduced row echelon form (RREF) of A is equivalent to the identity matrix. Therefore, **the rank of A is 3**, indicating A is **full rank**.

Answer to Q - 1.b

Express the elementary row operations we applied earlier in terms of elementary matrices.

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1/2 \end{bmatrix}}_{E'_6} \underbrace{\begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}}_{E'_5} \underbrace{\begin{bmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E'_4} \underbrace{\begin{bmatrix} 1 & 2 & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E'_3} \underbrace{\begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix}}_{E'_2} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 & 1 \end{bmatrix}}_{E'_1}$$

Since $E'_6 \cdots E'_1 A = I$, we can derive $E_1 \cdots E_6$ by taking inverse on $E'_6 \cdots E'_1 A = I$.

$$E_1 \cdots E_6 = (E'_6 \cdots E'_1)^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$$

Answer to Q - 1.c

From the conclusions we obtained earlier, we know that

$$A \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

which implies

$$\det A = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \cdot (-1) \cdot (-2) = 2$$

Answer to Q - 1.d

Theorem 9.4

Let A be a nonsingular matrix, then

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj}A$$

By definition of the adjoint matrix of A

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}^T$$

C_{ij} is the (i,j) -th cofactor. M_{ij} is the (i,j) -th minor. cofactor and minor satisfy $C_{ij} = (-1)^{i+j} M_{ij}$. Calculate $\text{adj}A$

$$\text{adj}A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -2 & 1 \\ 4 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} -4 & 2 & 4 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

By Theorem 9.4, we have

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj}A = \frac{1}{2} \begin{bmatrix} -4 & 2 & 4 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$

Answer to Q - 1.e

For applying Cramer's rule $x_i = \frac{\det B_i}{\det A}$, calculate

$$\det B_1 = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 5 & 2 & 2 \end{vmatrix} = 18 \quad \det B_2 = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \\ 1 & 5 & 2 \end{vmatrix} = -2 \quad \det B_3 = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{vmatrix} = -2$$

The solution to $Ax = b$ is

$$x = \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}$$

Question 2

Find the equation of the tangent line to the graph of the given function for the specified value of x .

(a) $f(x) = x^2$ at $x_0 = 3$

(b) $f(x) = \frac{x}{x^2 + 2}$ at $x_0 = 1$

Answer to Q - 2.a

Take derivative on $f(x)$ with respect to x

$$f'(x) = 2x$$

At $x_0 = 3$, the slope of the tangent is $f'(x) = 6$. We know that $(3, 9)$ is on the curve, the tangent can be written as

$$y - 9 = 6(x - 3)$$

$$y = 6x - 9$$

Answer to Q - 2.b

Similarly,

$$f'(x) = \frac{-x^2 + 2}{(x^2 + 2)^2}$$

At $x_0 = 1$, we have $f(1) = 1/3$ and $f'(1) = 1/9$. Hence, the tangent line is

$$y - \frac{1}{3} = \frac{1}{9}(x - 1)$$

$$y = \frac{1}{9}x - \frac{2}{9}$$

Question 3

Use differentials to estimate

(a) $\sqrt{50}$

(b) $\sqrt{9997}$

Key formula

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

Answer to Q - 3.a

Let $f(x) = \sqrt{x}$. The differential is

$$df = f'(x) dx = \frac{dx}{2\sqrt{x}}$$

Set $x_0 = 49$, $\Delta x = 1$. Then, we have

$$\sqrt{50} = \sqrt{49 + 1} \approx \sqrt{49} + \frac{1}{2\sqrt{49}} \cdot 1 = \frac{99}{14} \approx 7.07$$

Answer to Q - 3.b

Set $x_0 = 10000$, $\Delta x = -3$. Then, we have

$$\sqrt{9997} = \sqrt{10000 - 3} \approx \sqrt{10000} + \frac{1}{2\sqrt{10000}} \cdot (-3) = \frac{99}{14} = 99.985$$

Question 4

Calculate the derivatives of the following composite functions

(a) $\sin(x^4)$

(b) e^{x^2+3x}

(c) $\log(x^2 + 4)$

Answer to Q - 4.a

$$\frac{d \sin(x^4)}{dx} = \frac{d \sin(x^4)}{dx^4} \frac{dx^4}{dx} = 4x^3 \cos(x^4)$$

Answer to Q - 4.b

$$\frac{de^{x^2+3x}}{dx} = \frac{de^{x^2+3x}}{d(x^2+3x)} \frac{d(x^2+3x)}{dx} = (2x+3)e^{x^2+3x}$$

Answer to Q - 4.b

$$\frac{d \log(x^2+4)}{dx} = \frac{d \log(x^2+4)}{d(x^2+4)} \frac{d(x^2+4)}{dx} = \frac{2x}{x^2+4}$$

Question 5

Calculate an expression for the inverse of $f(x) = x^2 + x + 2$, specifying the domain carefully. Then use Theorem 4.3 (from Lecture 1) to compute the derivative of its inverse function at the point $f(1)$. Check your answer by directly taking the derivative of the inverse function you obtained.

Answer to Q - 5

Theorem 4.3

Let f be a C^1 function defined on the interval I in \mathbb{R}^1 . If $f'(x) \neq 0$ for all $x \in I$, then

- f is invertible on I
- its inverse g is a C^1 function on the interval $f(I)$
- for all z in the domain of the inverse function g

$$g'(z) = \frac{1}{f'[g(z)]}$$

Rewrite the expression of $f(x)$

$$f(x) = x^2 + x + 2 = \left(x + \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4}$$

Hence, the domain of the inverse is $[7/4, +\infty)$. Set $z = f(x)$ and calculate the inverse

$$\begin{aligned} z &= x^2 + x + 2 \\ z &= \left(x + \frac{1}{2}\right)^2 + \frac{7}{4} \\ x &= \pm \sqrt{z - \frac{7}{4}} - \frac{1}{2} \end{aligned}$$

Denote the inverse of $f(x)$ by $x = g(z)$.

The inverse of a strictly monotonic function is also strictly monotonic with the same type of monotonicity. $g(z)$ should be a strictly increasing function when $x \geq -1/2$, and a strictly decreasing function when $x < -1/2$. Therefore,

$$g(z) = \begin{cases} \sqrt{z - 7/4} - 1/2 & x \geq -1/2 \\ -\sqrt{z - 7/4} - 1/2 & x < -1/2 \end{cases} \quad z \in [7/4, +\infty)$$

Differentiate $f(x)$

$$f'(x) = 2x + 1$$

At $x = 1 > -1/2$, $z = f(1) = 4$. By Theorem 4.3, we have

$$g'(4) = \frac{1}{f'[g(4)]} = \frac{1}{f'(5/2)} = 2 \cdot \frac{5}{2} + 1 = \frac{1}{6}$$

To check the result by using theorem, differentiate $g(z)$ ($x \geq -1/2$)

$$g'(z) = \frac{1}{2\sqrt{z - 7/4}}$$

Finally, $g'(4) = 1/6$, the result is correct.

Question 6

Let the demand function be $x(p) = 2 - 4p/3$ and cost function be $C(x) = x^3$.

- (a) Derive the profit function $\Pi(x)$.
- (b) Derive the marginal revenue $MR(x)$.
- (c) Derive the marginal cost $MC(x)$.
- (d) Calculate output x^* such that $MR(x^*) = MC(x^*)$.
- (e) Is x^* a local maximum? Is x^* a global maximum? Why?

Answer to Q - 6.a

Calculate the inverse of demand function

$$x = 2 - \frac{4p}{3} \Rightarrow p = \frac{3}{2} - \frac{3x}{4}$$

Revenue function

$$R(x) = p \cdot x = \left(\frac{3}{2} - \frac{3x}{4} \right) x = \frac{3x}{2} - \frac{3x^2}{4}$$

Profit function can be written as

$$\Pi(x) = R(x) - C(x) = \frac{3x}{2} - \frac{3x^2}{4} - x^3$$

Answer to Q - 6.b

$$MR(x) = \frac{dR(x)}{dx} = \frac{3}{2} - \frac{3x}{2}$$

Answer to Q - 6.c

$$MC(x) = \frac{dC(x)}{dx} = 3x^2$$

Answer to Q - 6.d

$$MR(x) = MC(x)$$

$$\frac{3}{2} - \frac{3x}{2} = 3x^2$$

The solution to the equation above are $x^* = 1/2$ and $x^* = -1$. Since x represents the number of products, which implies x can not be negative, $x^* = -1$ should be dropped.

Question 7

An *orthogonal basis* of an Euclidean space is a basis whose vectors are pairwise orthogonal. That is, for any v_i and v_j in the basis with $i \neq j$, $v_i \perp v_j$. Let $v_1 = (1, 2, 3)$. Find an orthogonal basis of \mathbb{R}^3 that includes v_1 .

Answer to Q - 7

Let $v_2 = (v_2^1, v_2^2, v_2^3)$ and $v_3 = (v_3^1, v_3^2, v_3^3)$ are two possible vector. For v_1 and v_2 , orthogonality requires

$$v_1 \cdot v_2 = 1 \cdot v_2^1 + 2 \cdot v_2^2 + 3 \cdot v_2^3 = 0$$

The entries of v_2 can be arbitrarily selected as long as they satisfy the expression above. One possible vector is $v_2 = (2, -1, 0)$.

v_3 has to be orthogonal to both v_1 and v_2 simultaneously, we have

$$v_1 \cdot v_3 = 1 \cdot v_3^1 + 2 \cdot v_3^2 + 3 \cdot v_3^3 = 0$$

$$v_2 \cdot v_3 = 2 \cdot v_3^1 - 1 \cdot v_3^2 = 0$$

Solve the equation system, one possible vector is $v_3 = (3, 6, -5)$.

Question 8

Which of the following are basis in \mathbb{R}^3 ?

(1) $(1, 1, 1), (1, 2, 1), (1, 0, 1)$

(2) $(6, 3, 9), (5, 2, 8), (4, 1, 7)$

(3) $(1, 1, 1), (1, 2, 1), (1, 0, 0)$

Answer to Q - 8

If three vectors constitute a basis of \mathbb{R}^3 , they are linear independent. We can determine whether three vectors form a basis by calculating the determinant. Since the determinant of a matrix equals the determinant of its transpose, it does not matter whether we treat the vectors as row vectors or column

vectors.

For Group 1

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} \xrightarrow{c_3 - c_1} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

For Group 2

$$\begin{vmatrix} 6 & 3 & 9 \\ 5 & 2 & 8 \\ 4 & 1 & 7 \end{vmatrix} \xrightarrow{c_3 - c_1 - c_2} \begin{vmatrix} 6 & 3 & 0 \\ 5 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix} \xrightarrow{2c_2 - c_3} \begin{vmatrix} 6 & 3 & 0 \\ 6 & 3 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 0$$

For Group 3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} \xrightarrow{r_2 - r_1} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1$$

Therefore, only $\{(1, 1, 1), (1, 2, 1), (1, 0, 0)\}$ is a basis of \mathbb{R}^3 .

Question 9

The **column space** of a matrix A is the set of all linear combinations of the column vectors of A .
The **null space** of a matrix A is the set of all solutions to $Ax = 0$.

(a) Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Is b in the column space of A ?

(b) Given a basis for the null space of A .

(c) Given a basis for the column space of A .

Answer to Q - 9.a

No.

In this question, we are going to find a vector x which satisfies $Ax = b$. If such x exists, b is in the

column space of A ; otherwise, it is not. Do elementary row operations on augmented matrix.

$$\begin{aligned}
 [A|b] &= \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & 4 \end{array} \right] \xrightarrow{r_2+4r_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ -3 & 7 & 6 & 4 \end{array} \right] \\
 &\xrightarrow{r_3+3r_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 13 \end{array} \right] \xrightarrow{r_2-3r_3} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & 0 & 0 & -24 \\ 0 & -2 & -6 & 13 \end{array} \right]
 \end{aligned}$$

Since there is no x which can establish $0 \cdot x = -24$, b is not in $\text{Col}(A)$.

Answer to Q - 9.b

Earlier, we derive $\text{rank}(A) = 2$, so the rank of $\text{Null}(A)$ is $3 - 2 = 1$. Any vector satisfying $Ax = 0$ can be a basis of $\text{Null}(A)$. The solutions of $Ax = 0$ also are the following one's.

$$\tilde{A} \equiv \left[\begin{array}{ccc} 1 & -3 & -4 \\ 0 & 0 & 0 \\ 0 & -2 & -6 \end{array} \right] x = 0$$

We choose the solution $x = (5, 3, -1)$ to be the basis of $\text{Null}(A)$.

Answer to Q - 9.c

A maximal linearly independent subset of the columns of A constitutes a basis for $\text{Col}(A)$. Examining the matrix \tilde{A} (introduced in the previous subsection), we find that columns 1 and 2 constitutes such a set. A basis of $\text{Col}(A)$ can be $\{(1, 0, 0), (-3, 0, -2)\}$.

Question 10

A set V is called a **vector subspace** (or simply subspace) of \mathbb{R}^n if V is a subset of \mathbb{R}^n such that (1) $0 \in V$; (2) $u + v \in V$ for all $u, v \in V$; and (3) $\lambda v \in V$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

- Is the line $y = 2x + 1$ a vector subspace of \mathbb{R}^2 ?
- Given an example of a nonempty subset U of \mathbb{R}^2 such that (1) $u + v \in U$ for all $u, v \in U$; (2) $-u \in U$ for all $u \in U$, but U is not a vector subspace of \mathbb{R}^2 .
- Given an example of a nonempty subset U of \mathbb{R}^2 such that $\lambda u \in U$ for all $u \in U$ but U is not a vector subspace of \mathbb{R}^2 .
- Let v_1, v_2, \dots, v_k be a collection of vectors in \mathbb{R}^n . Prove that $\mathcal{L}[v_1, v_2, \dots, v_k]$ is a vector

subspace of \mathbb{R}^n .

- (e) Prove that $\mathcal{L}[v_1, v_2, \dots, v_k]$ is the smallest vector subspace of \mathbb{R}^n that contains v_1, v_2, \dots, v_k .

Answer to Q - 10.a

No. $(0, 0)$ does not lie on the line.

Answer to Q - 10.b

The subspace U of \mathbb{R}^2 consisting of the first and third quadrants.

Answer to Q - 10.c

The subspace U of \mathbb{R}^2 consisting of the x-axes and y-axes.

Answer to Q - 10.d

By definition,

$$\mathcal{L}[v_1, v_2, \dots, v_k] \equiv \{\mu_1 v_1 + \mu_2 v_2 + \dots + \mu_k v_k \mid \forall \mu_1, \mu_2, \dots, \mu_k \in \mathbb{R}\}$$

Let $\mu_1 = \mu_2 = \dots = \mu_k = 0$, we have zero vector in \mathcal{L} .

Any vector u and v in \mathcal{L} can be represented by v_1, v_2, \dots, v_k , so the linear combination $u + v$ can be represented by v_1, v_2, \dots, v_k also. Hence, $u + v \in \mathcal{L}$ and \mathcal{L} is closed under addition.

Similarly, \mathcal{L} is closed under scalar multiplication.

Therefore, \mathcal{L} is a vector subspace. □

Answer to Q - 10.e

We arbitrarily discard one vector from \mathcal{L} . Suppose the discarded vector is b , and b is not any of v_1, v_2, \dots, v_k . By the definition of \mathcal{L} , there exists $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{R}$ such that

$$b = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_k v_k$$

Let $u = \mu_1 v_1$, $v = \mu_2 v_2 + \dots + \mu_k v_k$. $u + v = b$ is not in \mathcal{L} .

Since b is chosen arbitrarily, it implies that no vector in \mathcal{L} can be discarded. Combine with the previous conclusion, \mathcal{L} is the smallest subspace containing v_1, v_2, \dots, v_k . □