Mathematical Economics

Assignment # 1

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Question List

AI Usage Statement

This assignment was entirely completed by myself.

AI was used only for English language polishing.

Question 1

Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

(a) What is the rank of A?

(b) Express A as $E_1 E_2 \cdots E_n I$, where each E_i is an elementary matrix.

(c) Calculate $\det A$.

(d) Invert A using Theorem 9.4.

(e) Use Cramer's rule to solve $Ax = b$.

Answer to Q - 1.a

Apply elementary row operations on A

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{r_1 + 3r_2 + 2r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{-r_2,\, -r_3/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reduced row echelon form (RREF) of A is equivalent to the identity matrix. Therefore, the rank of A is 3, indicating A is full rank.

Answer to Q - 1.b

Express the elementary row operations we applied earlier in terms of elementary matrices.

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1/2 \end{bmatrix}}_{E_6'} \underbrace{\begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}}_{E_5'} \underbrace{\begin{bmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E_4'} \underbrace{\begin{bmatrix} 1 & & 2 \\ & 1 & \\ & & 1 \end{bmatrix}}_{E_3'} \underbrace{\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix}}_{E_2'} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}}_{E_1'}$$

Since $E_6' \cdots E_1' A = I$, we can derive $E_1 \cdots E_6$ by taking inverse on $E_6' \cdots E_1' A = I$.

$$E_1 \ \cdots \ E_6 = \left(E_6' \cdots E_1'\right)^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -2 \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$$

Answer to Q - 1.c

From the conclusions we obtained earlier, we know that

$$A \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

which implies

$$\det A = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \cdot (-1) \cdot (-2) = 2$$

Answer to Q - 1.d

Theorem 9.4

Let A be a nonsingular matrix, then

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$$

By definition of the adjoint matrix of A

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}^T$$

C_{ij} is the (i,j)-th cofactor. M_{ij} is the (i,j)-th minor. cofactor and minor satisfy $C_{ij} = (-1)^{i+j} M_{ij}$. Calculate $\operatorname{adj} A$

$$\operatorname{adj} A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -2 & 1 \\ 4 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} -4 & 2 & 4 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

By Theorem 9.4, we have

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A = \frac{1}{2} \begin{bmatrix} -4 & 2 & 4 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$

Answer to Q - 1.e

For applying Cramer's rule $x_i = \dfrac{\det B_i}{\det A}$, calculate

$$\det B_1 = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 5 & 2 & 2 \end{vmatrix} = 18 \quad \det B_2 = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \\ 1 & 5 & 2 \end{vmatrix} = -2 \quad \det B_3 = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{vmatrix} = -2$$

The solution to $Ax = b$ is

$$x = \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}$$

Question 2

Find the equation of the tangent line to the graph of the given function for the specified value of x.

(a) $f(x) = x^2$ at $x_0 = 3$

(b) $f(x) = \dfrac{x}{x^2+2}$ at $x_0 = 1$

Answer to Q - 2.a

Take derivative on $f(x)$ with respect to x

$$f'(x) = 2x$$

At $x_0 = 3$, the slope of the tangent is $f'(x) = 6$. We know that $(3, 9)$ is on the curve, the tangent can be written as

$$y - 9 = 6(x - 3)$$

$$y = 6x - 9$$

Answer to Q - 2.b

Similarly,

$$f'(x) = \frac{-x^2 + 2}{(x^2+2)^2}$$

At $x_0 = 1$, we have $f(1) = 1/3$ and $f'(1) = 1/9$. Hence, the tangent line is

$$y - \frac{1}{3} = \frac{1}{9}(x - 1)$$

$$y = \frac{1}{9}x - \frac{2}{9}$$

Question 3

Use differentials to estimate

(a) $\sqrt{50}$

(b) $\sqrt{9997}$

Key formula

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

Answer to Q - 3.a

Let $f(x) = \sqrt{x}$. The differential is

$$\mathrm{d}f = f'(x)\,\mathrm{d}x = \frac{\mathrm{d}x}{2\sqrt{x}}$$

Set $x_0 = 49$, $\Delta x = 1$. Then, we have

$$\sqrt{50} = \sqrt{49+1} \approx \sqrt{49} + \frac{1}{2\sqrt{49}} \cdot 1 = \frac{99}{14} \approx 7.07$$

Answer to Q - 3.b

Set $x_0 = 10000$, $\Delta x = -3$. Then, we have

$$\sqrt{9997} = \sqrt{10000-3} \approx \sqrt{10000} + \frac{1}{2\sqrt{10000}} \cdot (-3) = \frac{99}{14} = 99.985$$

Question 4

Calculate the derivatives of the following composite functions

(a) $\sin\left(x^4\right)$

(b) e^{x^2+3x}

(c) $\log\left(x^2+4\right)$

Answer to Q - 4.a

$$\frac{\mathrm{d}\sin\left(x^4\right)}{\mathrm{d}x} = \frac{\mathrm{d}\sin\left(x^4\right)}{\mathrm{d}x^4}\frac{\mathrm{d}x^4}{\mathrm{d}x} = 4x^3\cos\left(x^4\right)$$

Answer to Q - 4.b

$$\frac{\mathrm{d}e^{x^2+3x}}{\mathrm{d}x} = \frac{\mathrm{d}e^{x^2+3x}}{\mathrm{d}\left(x^2+3x\right)}\frac{\mathrm{d}\left(x^2+3x\right)}{\mathrm{d}x} = (2x+3)\,e^{x^2+3x}$$

Answer to Q - 4.b

$$\frac{\mathrm{d}\log\left(x^2+4\right)}{\mathrm{d}x} = \frac{\mathrm{d}\log\left(x^2+4\right)}{\mathrm{d}\left(x^2+4\right)}\frac{\mathrm{d}\left(x^2+4\right)}{\mathrm{d}x} = \frac{2x}{x^2+4}$$

Question 5

Calculate an expression for the inverse of $f(x) = x^2+x+2$, specifying the domain carefully. Then use Theorem 4.3 (from Lecture 1) to compute the derivative of its inverse function at the point $f(1)$. Check your answer by directly taking the derivative of the inverse function you obtained.

Answer to Q - 5

Theorem 4.3

Let f be a C^1 function defined on the interval I in $\mathbb{R}^1$. If $f'(x) \neq 0$ for all $x \in I$, then

- f is invertible on I
- its inverse g is a C^1 function on the interval $f(I)$
- for all z in the domain of the inverse function g

$$g'(z) = \frac{1}{f'\left[g(z)\right]}$$

Rewrite the expression of $f(x)$

$$f(x) = x^2+x+2 = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4}$$

Hence, the domain of the inverse is $[7/4, +\infty)$. Set $z = f(x)$ and calculate the inverse

$$z = x^2+x+2$$

$$z = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4}$$

$$x = \pm\sqrt{z-\frac{7}{4}} - \frac{1}{2}$$

Denote the inverse of $f(x)$ by $x = g(z)$.

The inverse of a strictly monotonic function is also strictly monotonic with the same type of monotonicity. $g(z)$ should be a strictly increasing function when $x \geq -1/2$, adn a strictly decreasing function when $x < -1/2$. Therefore,

$$g(z) = \begin{cases} \sqrt{z - 7/4} - 1/2 & x \geq -1/2 \\ -\sqrt{z - 7/4} - 1/2 & x < 1/2 \end{cases} \quad z \in [7/4, +\infty)$$

Differentiate $f(x)$

$$f'(x) = 2x + 1$$

At $x = 1 > -1/2$, $z = f(1) = 4$. By Theorem 4.3, we have

$$g'(4) = \frac{1}{f'\left[g(4)\right]} = \frac{1}{f'(5/2)} = 2 \cdot \frac{5}{2} + 1 = \frac{1}{6}$$

To check the result by using theorem, differentiate $g(z)$ $(x \geq -1/2)$

$$g'(z) = \frac{1}{2\sqrt{z - 7/4}}$$

Finally, $g'(4) = 1/6$, the result is correct.

Question 6

Let the demand function be $x(p) = 2 - 4p/3$ and cost function be $C(x) = x^3$.

(a) Derive the profit function $\Pi(x)$.

(b) Derive the marginal revenue $\mathrm{MR}(x)$.

(c) Derive the marginal cost $\mathrm{MC}(x)$.

(d) Calculate output x^* such that $\mathrm{MR}(x^*) = \mathrm{MC}(x^*)$.

(e) Is x^* a local maximum? Is x^* a global maximum? Why?

Answer to Q - 6.a

Calculate the inverse of demand function

$$x = 2 - \frac{4p}{3} \quad \Rightarrow \quad p = \frac{3}{2} - \frac{3x}{4}$$

Revenue function

$$R(x) = p \cdot x = \left(\frac{3}{2} - \frac{3x}{4}\right) = \frac{3x}{2} - \frac{3x^2}{4}$$

Profit function can be written as

$$\Pi(x) = R(x) - C(x) = \frac{3x}{2} - \frac{3x^2}{4} - x^3$$

Answer to Q - 6.b

$$\mathrm{MR}(x) = \frac{\mathrm{d}R(x)}{\mathrm{d}x} = \frac{3}{2} - \frac{3x}{2}$$

Answer to Q - 6.c

$$\mathrm{MC}(x) = \frac{\mathrm{d}C(x)}{\mathrm{d}x} = 3x^2$$

Answer to Q - 6.d

$$\mathrm{MR}(x) = \mathrm{MC}(x)$$
$$\frac{3}{2} - \frac{3x}{2} = 3x^2$$

The solution to the equation above are $x^* = 1/2$ and $x^* = -1$. Since x represents the number of products, which implies x can not be negative, $x^* = -1$ should be dropped.

Question 7

An *orthogonal basis* of an Euclidean space is a basis whose vectors are pairwise orthogonal. That is, for any v_i and v_j in the basis with $i \neq j$, $v_i \perp v_j$. Let $v_1 = (1, 2, 3)$. Find an orthogonal basis of $\mathbb{R}^3$ that includes v_1.

Answer to Q - 7

Let $v_2 = \left(v_2^1, v_2^2, v_2^3\right)$ and $v_3 = \left(v_3^1, v_3^2, v_3^3\right)$ are two possible vector. For v_1 and v_2, orthogonality requires

$$v_1 \cdot v_2 = 1 \cdot v_2^1 + 2 \cdot v_2^2 + 3 \cdot v_2^3 = 0$$

The entries of v_2 can be arbitrarily selected as long as they satisfy the expression above. One possible vector is $v_2 = (2, -1, 0)$.

v_3 has to be orthogonal to both v_1 and v_2 simultaneously, we have

$$v_1 \cdot v_3 = 1 \cdot v_3^1 + 2 \cdot v_3^2 + 3 \cdot v_3^3 = 0$$
$$v_2 \cdot v_3 = 2 \cdot v_3^1 - 1 \cdot v_3^2 = 0$$

Solve the equation system, one possible vector is $v_3 = (3, 6, -5)$.

Question 8

Which of the following are basis in $\mathbb{R}^3$?

1. $(1,1,1)$, $(1,2,1)$, $(1,0,1)$
2. $(6,3,9)$, $(5,2,8)$, $(4,1,7)$
3. $(1,1,1)$, $(1,2,1)$, $(1,0,0)$

Answer to Q - 8

If three vectors constitute a basis of $\mathbb{R}^3$, they are linear independent. We can determine whether three vectors form a basis by calculating the determinant. Since the determinant of a matrix equals the determinant of its transpose, it does not matter whether we treat the vectors as row vectors or column vectors.

For Group 1

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} \xlongequal{c_3 - c_1} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

For Group 2

$$\begin{vmatrix} 6 & 3 & 9 \\ 5 & 2 & 8 \\ 4 & 1 & 7 \end{vmatrix} \xlongequal{c_3 - c_1 - c_2} \begin{vmatrix} 6 & 3 & 0 \\ 5 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix} \xlongequal{2c_2 - c_3} \begin{vmatrix} 6 & 3 & 0 \\ 6 & 3 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 0$$

For Group 3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} \xlongequal{r_2 - r_1} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1$$

Therefore, only $\{(1,1,1),(1,2,1),(1,0,0)\}$ is a basis of $\mathbb{R}^3$.

Question 9

The **column space** of a matrix A is the set of all linear combinations of the column vectors of A. The **null space** of a matrix A is the set of all solutions to $Ax = 0$.

(a) Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Is b in the column space of A?

(b) Given a basis for the null space of A.

(c) Given a basis for the column space of A.

Answer to Q - 9.a

No.

In this question, we are going to find a vector x which satisfies $Ax = b$. If such x exists, b is in the column space of A; otherwise, it is not. Do elementary row operations on augmented matrix.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & 4 \end{array}\right] \xrightarrow{r_2+4r_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ -3 & 7 & 6 & 4 \end{array}\right]$$

$$\xrightarrow{r_3+3r_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 13 \end{array}\right] \xrightarrow{r_2-3r_3} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & 0 & 0 & -24 \\ 0 & -2 & -6 & 13 \end{array}\right]$$

Since there is no x which can establish $0 \cdot x = -24$, b is not in $\text{Col}(A)$.

Answer to Q - 9.b

Earlier, we derive $\text{rank}(A) = 2$, so the rank of $\text{Null}(A)$ is $3 - 2 = 1$. Any vector satisfying $Ax = 0$ can be a basis of $\text{Null}(A)$. The solutions of $Ax = 0$ also are the following one's.

$$\tilde{A} \equiv \begin{bmatrix} 1 & -3 & -4 \\ 0 & 0 & 0 \\ 0 & -2 & -6 \end{bmatrix} x = 0$$

We choose the solution $x = (5, 3, -1)$ to be the basis of $\text{Null}(A)$.

Answer to Q - 9.c

A maximal linearly independent subset of the columns of A constitutes a basis for $\text{Col}(A)$. Examining the matrix $\tilde{A}$ (introduced in the previous subsection), we find that columns 1 and 2 constitutes such a set. A basis of $\text{Col}(A)$ can be $\{(1, 0, 0), (-3, 0, -2)\}$.

Question 10

A set V is called a **vector subspace** (or simply subspace) of $\mathbb{R}^n$ if V is a subset of $\mathbb{R}^n$ such that (1) $0 \in V$; (2) $u + v \in V$ for all $u, v \in V$; and (3) $\lambda v \in V$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

(a) Is the line $y = 2x + 1$ a vector subspace of $\mathbb{R}^2$?

(b) Given an example of a nonempty subset U of $\mathbb{R}^2$ such that (1) $u+v \in U$ for all $u, v \in U$; (2) $-u \in U$ for all $u \in U$, but U is not a vector subspace of $\mathbb{R}^2$.

(c) Given an example of a nonempty subset U of $\mathbb{R}^2$ such that $\lambda u \in U$ for all $u \in U$ but U is not a vector subspace of $\mathbb{R}^2$.

(d) Let $v_1, v_2, \ldots, v_k$ be a collection of vectors in $\mathbb{R}^n$. Prove that $\mathcal{L}[v_1, v_2, \ldots, v_k]$ is a vector subspace of $\mathbb{R}^n$.

(e) Prove that $\mathcal{L}[v_1, v_2, \ldots, v_k]$ is the smallest vector subspace of $\mathbb{R}^n$ that contains $v_1, v_2, \ldots, v_k$.

Answer to Q - 10.a

No. $(0,0)$ does not lie on the line.

Answer to Q - 10.b

The subspace U of $\mathbb{R}^2$ consisting of the first and third quadrants.

Answer to Q - 10.c

The subspace U of $\mathbb{R}^2$ consisting of the the x-axes and y-axes.

Answer to Q - 10.d

By definition,

$$\mathcal{L}[v_1, v_2, \ldots, v_k] \equiv \left\{ \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_k v_k \,|\, \forall\, \mu_1, \mu_2, \ldots, \mu_k \in \mathbb{R} \right\}$$

Let $\mu_1 = \mu_2 = \cdots = \mu_k = 0$, we have zero vector in $\mathcal{L}$.

Any vector u and v in $\mathcal{L}$ can be represented by $v_1, v_2, \ldots, v_k$, so the linear combination $u+v$ can be represented by $v_1, v_2, \ldots, v_k$ also. Hence, $u+v \in \mathcal{L}$ and $\mathcal{L}$ is closed under addition.

Similarly, $\mathcal{L}$ is closed under scalar multiplication.

Therefore, $\mathcal{L}$ is a vector subspace. □

Answer to Q - 10.e

We arbitrarily discard one vector from $\mathcal{L}$. Suppose the discarded vector is b, and b is not any of $v_1, v_2, \ldots, v_k$. By the definition of $\mathcal{L}$, there exists $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{R}$ such that

$$b = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_k v_k$$

Let $u = \mu_1 v_1$, $v = \mu_2 v_2 + \cdots \mu_k v_k$. $u + v = b$ is not in $\mathcal{L}$.

Since b is chosen arbitrarily, it implies that no vector in $\mathcal{L}$ can be discarded. Combine with the previous conclusion, $\mathcal{L}$ is the smallest subspace containing $v_1, v_2, \ldots, v_k$. $\square$