Advanced Macroeconomics I

Assignment # 1

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Question List

Question 1

Describe how, if at all, each of the following developments affects the break-even and actual investment lines in our basic diagram for the Solow model

(a) The rate of depreciation falls.

(b) The rate of technological progress rises.

(c) The production function is Cobb-Douglas, $f(k) = k^{\alpha}$, and capital's share, α, rises.

(d) Workers exert more effort, so that output per unit of effective labor for a given value of capital per unit effective labor is higher than before.

Answer to Q - 1.a

The decrease in the rate of depreciation δ leads to a decrease in the slope of the break-even investment line, which means the balanced-growth-path level of k increases and the actual investment line stays unchanged.

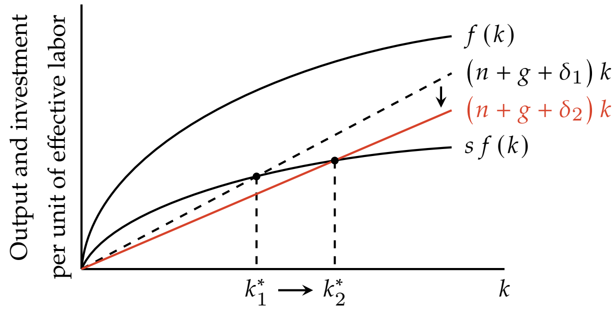


Figure 1: The effects of a decrease in δ

Answer to Q - 1.b

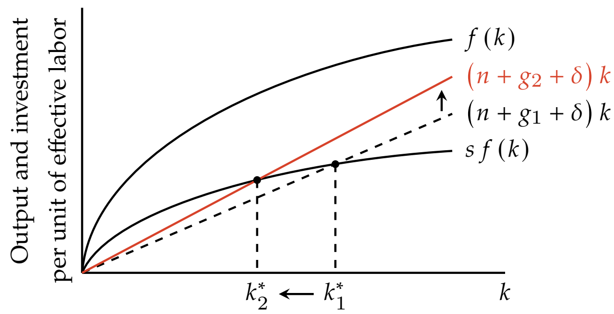


Figure 2: The effects of an increase in g

Answer to Q - 1.c

Consider the production function satisfies Cobb-Douglas form, and the intensive form is $f(k) = k^{\alpha}$ with $0 < \alpha < 1$ (cases that violates the assumptions of the Solow model are excluded). Since $f(k)$ is the power function, and all power functions with different exponents (α) pass through the point $(1, 1)$, we have to process by cases.

Cases 1 - When break-even investment line has a steep slope so that the point $(1,1)$ lies **below** it, the increase in α leads to the **decrease** in BGP level of k.

Cases 2 - When break-even investment line has a steep slope so that the point $(1,1)$ lies **above** it, the increase in α leads to the **increase** in BGP level of k.

The Figure 3 illustrates the case 2.

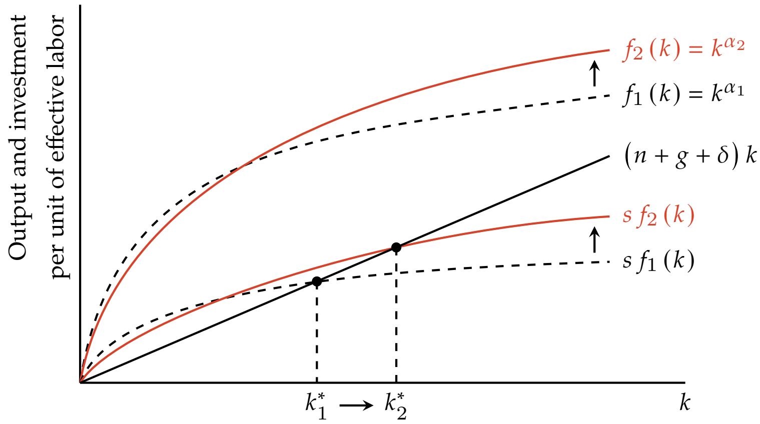


Figure 3: The effects of an increase in α

Answer to Q - 1.d

The increase in worker's efforts leads to an upward shift in the production function, which means both the balanced-growth-path level of k and the actual investment line increase. Since n, g and δ remain unchanged, the break-even investment line is unaffected.

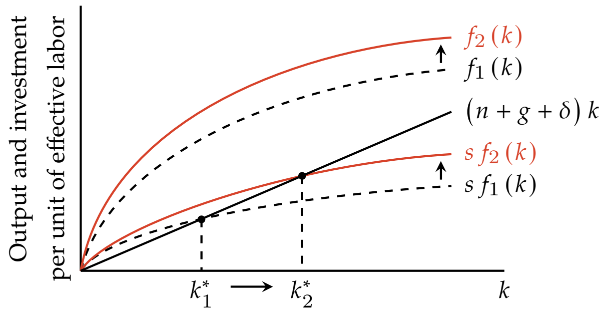


Figure 4: The effects of an increase in worker's efforts

Question 2

Consider a Solow economy that is on its balanced-growth-path. Assume for simplicity that there is no technological progress. Now suppose that the rate of population growth falls.

(a) What happens to the balanced-growth-path values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new balanced-growth-path.

(b) Describe the effect of the fall in population growth on the path of output (that is, total output, not output per worker).

Answer to Q - 2.a

As the figure below shows, the decrease in the rate of population growth (n) leads to a decrease in the slope of the break-even investment line ($n + g$). When the economy moves to its new BGP, it will achieve a higher level of capital per worker (k^*), which leads to a higher level of output per worker (y^*). Since $c = (1 - s)y$, the consumption per worker (c^*) also increases.

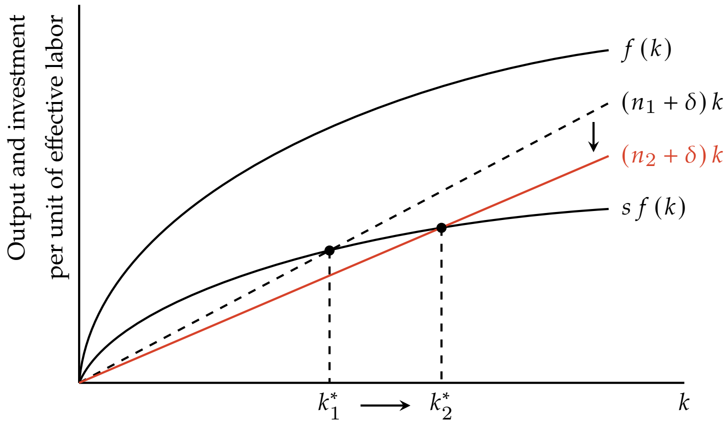


Figure 5: The effects of a decrease in n

Firstly, although there is a sudden change in n at time t_0, this change does not immediately translate into changes in these variables, since none of them are directly determined by n in their expressions.

Secondly, as we discussed in *The Speed of Convergence* (Romer, 2019 [1]), the speed of convergence of k diminishes as k approaches k^*. Therefore, the path of k is concave, and so are the paths of y and c.

We illustrate the dynamics of k, which can also represent the dynamics of y and c.

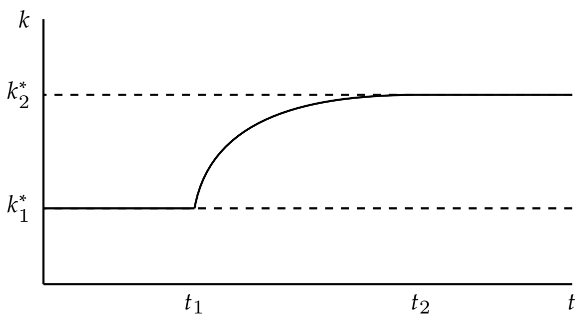


Figure 6: The dynamics of k with a decrease in n

[1] Romer, David. (2019). *Advanced Macroeconomics*. 5th edition. McGraw-Hill Education.

Answer to Q - 2.b

The key effect on total output (Y) comes from the growth rate. Since there is no technological progress, the growth rate of output is

$$\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} + \frac{\dot{y}}{y} = n$$

Since $\dot{Y}/Y$ is the derivative of $\ln Y$ with respect to t, the growth rate of output is represented by the slope of the $\ln Y(t)$ curve in phase diagram. The blue part represents n_1, and the red part represents n_2. Since y_1^* moves to y_2^* at a time-varying speed after the shock, we can see that the slope do not change suddenly but gradually decreases from n_1 to n_2.

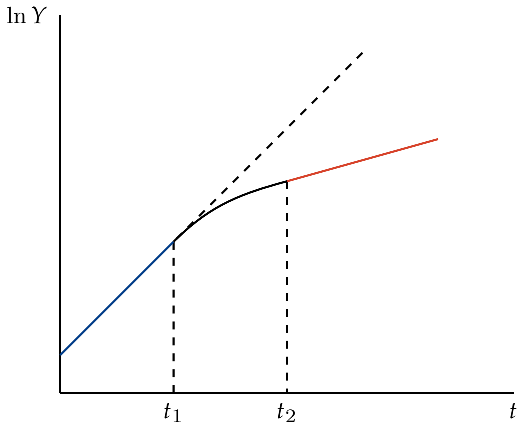


Figure 7: The dynamics of $\ln Y$ with a decrease in n

Question 3

Factor payments in the Solow model. Assume that both labor and capital are paid their marginal products. Let w denote $\partial F(K, AL)/\partial L$ and r denote $[\partial F(K, AL)/\partial K] - \delta$.

(a) Show that the marginal product of labor w is $A\left[f(k) - kf'(k)\right]$.

(b) Show that if both capital and labor are paid their marginal products, constant returns to scale implies that the total amount paid to the factors of production equals total net output. That is, show that under constant returns, $wL + rK = F(K, L) - \delta K$.

(c) The return to capital (r) is roughly constant over time, as are the shares of output going to capital and to labor. Does a Solow economy on a balanced-growth-path exhibits these properties?

(d) Suppose the economy begins with a level of k less than k^*. As k moves toward k^*, is w growing at a rate greater than, less than, or equal to its growth rate on the balanced-growth-path? What about r?

Answer to Q - 3.a

By the expression of w, we have

$$\begin{aligned} w &\equiv \frac{\partial F(K, AL)}{\partial L} = \frac{\partial \left[AL \cdot f(k)\right]}{\partial L} \\ &= Af(k) + AL \cdot \frac{\partial f\left[K/(AL)\right]}{\partial L} \\ &= Af(k) - AL \cdot f'(k) \cdot \frac{K}{AL^2} \\ &= A\left[f(k) - kf'(k)\right] \end{aligned}$$

Answer to Q - 3.b

Substituting the expressions for w and r, we obtain

$$\begin{aligned} wL + rK &= AL\left[f(k) - kf'(k)\right] + \left[f'(k) - \delta\right] K \\ &= ALf(k) - ALkf'(k) + Kf'(k) - \delta K \\ &= F(K, L) - \delta K \end{aligned}$$

Answer to Q - 3.c

Calculate the growth rates of r.

$$\frac{\dot{r}}{r} = \frac{\dot{\left[f'(k) - \delta\right]}}{f'(k) - \delta} = \frac{\dot{k}f''(k)}{f'(k) - \delta}$$

When the economy is on the BGP, we have $\dot{k} = 0$, so $\dot{r}/r = 0$, which means the return to capital is constant over time.

The growth rate of the share assigning to capital on the BGP is

$$\frac{\dot{\left[(r \cdot K)/Y\right]}}{(r \cdot K)/Y} = \frac{\dot{r}}{r} + \frac{\dot{K}}{K} - \frac{\dot{Y}}{Y} = \frac{\dot{k}f''(k)}{f'(k) - \delta} + \left(n + g + \frac{\dot{k}}{k}\right) - \left(n + g + \frac{\dot{y}}{y}\right) = 0$$

Therefore, the share of output going to capital is constant over time. Since the share of output going to labor is $1 - (r \cdot K)/Y$, it is also constant over time.

Answer to Q - 3.d

Calculating the growth rates of w.

$$\begin{aligned} \frac{\dot{w}}{w} &= \frac{\dot{\left[A\left(f(k) - kf'(k)\right)\right]}}{A\left(f(k) - kf'(k)\right)} = \frac{\dot{A}}{A} + \frac{\dot{\left[f(k) - kf'(k)\right]}}{f(k) - kf'(k)} \\ &= g - \frac{k\dot{k}f''(k)}{f(k) - kf'(k)} = g - \frac{k\dot{k}f''(k)}{f(k) - rk} \end{aligned}$$

The Inada conditions requires that $f''(k) < 0$, and the output per worker (y) is necessarily greater than the portion allocated to labor (rK), we conclude that $\dot{w}/w > g$. When the economy is on the BGP, $\dot{k} = 0$, so $\dot{w}/w = g$. Therefore, as k moves toward k^*, which means w is growing at a rate greater than its growth rate on the BGP.

Similarly, as we derived above,

$$\frac{\dot{r}}{r} = \frac{\dot{k} f''(k)}{f'(k) - \delta} < 0$$

it is clear that as k moves toward k^*, the growth rate of r is less than its rate on the balanced-growth-path (which is zero), even though the value of r itself changes but the one on BGP not.

Question 4

This question asks you to use a Solow-style model to investigate some ideas that have been discussed in the context of Thomas Piketty's recent work (see Piketty, 2014 [1]; Piketty and Zucman, 2014 [2]; Rognlie, 2015 [3]). Consider an economy described by the assumptions of the Solow model, except that factors are paid their marginal products (as in Question 3), and all labor incomes consumed, and all other income is saved. Thus, $C(t) = L(t)[\partial F(K, AL)/\partial L]$.

(a) Show that the properties of the production function and our assumptions about the behavior of L and A imply that the capital-output ratio, K/Y, is rising if and only if the growth rate of K is greater than $n + g$ - that is, if and only if k is rising.

(b) Assume that the initial conditions are such that $\partial F(K, AL)/\partial K$ at $t = 0$ is strictly greater than $n + g + \delta$. Describe the qualitative behavior of the capital-output ratio over time. (For example, does it grow or fall without bound? Gradually approach some constant level from above or below? Something else?) Explain your reasoning.

(c) Piketty found that: Since the return to capital exceeds the growth rate of the economy, the capital-output ratio tend to grow without bound. If you found in (b) that K/Y grows without bound, explain intuitively whether the driving force of this unbounded growth is that the return to capital exceeds the economy's growth rate. If you found in (b) that K/Y doesn't grow without bound, explain intuitively what is wrong with Piketty's statement.

(d) Suppose $F(\cdot)$ is Cobb-Douglas and that the initial situation is as in part (b). Describe the qualitative behavior over time of the share of *net* capital income (that is, $K[\partial F(K, AL)/\partial K - \delta]$) in *net* output (that is, $Y(t) - \delta K(t)$). Explain your reasoning. Is the common statement that an excess of the return to capital over the economy's growth rate cause capital's share to rise over time correct in this case?

Go to Question 5, Question 6.

Answer to Q - 4.a

Differentiate K/Y with respect to k, we have

$$\begin{aligned}\frac{\partial}{\partial k}\frac{K}{Y} &= \frac{\partial}{\partial k}\frac{k\cdot AL}{f(k)\cdot AL} = \frac{\partial}{\partial k}\frac{k}{f(k)} \\ &= \frac{f(k)-kf'(k)}{\left[f(k)\right]^2} > 0\end{aligned}$$

which means K/Y rises if and only if k rises.

Next, we will show that k rises if and only if the growth rate of K is greater than $n+g$. By the property of the growth rate, we have

$$\frac{\dot{K}}{K} = \frac{(k\cdot \dot{AL})}{k\cdot AL} = \frac{\dot{k}}{k} + \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = \frac{\dot{k}}{k} + g + n$$

Therefore, $\dot{K}/K > n+g$ if and only if $\dot{k}/k > 0$, which means k is rising.

Answer to Q - 4.b

In (a), we have concluded that the behavior of K/Y is equivalent to that of k, so we will focus on k in following analysis.

Since there is nothing about whether factors are paid by their marginal products when we talk about the "equilibrium" or "golden-rule" things, we can still use the conclusions we derived in the Solow model.

MPK at $t=0$ is strictly greater than $n+g+\delta$ means that $k(0)$ is smaller than the golden-rule level, which requires that MPK $= n+g+\delta$. Another interesting fact is that at the moment when k reaches the BGP, MPK $< n+g+\delta$ holds, which means k must increase to its BGP level (since we assume that MPK is diminishing). All the analysis has been completed, and we now illustrate the dynamics of k in the figure below.

We can see that k will gradually approach a constant level from below, instead of growing without bound. So do K/Y.

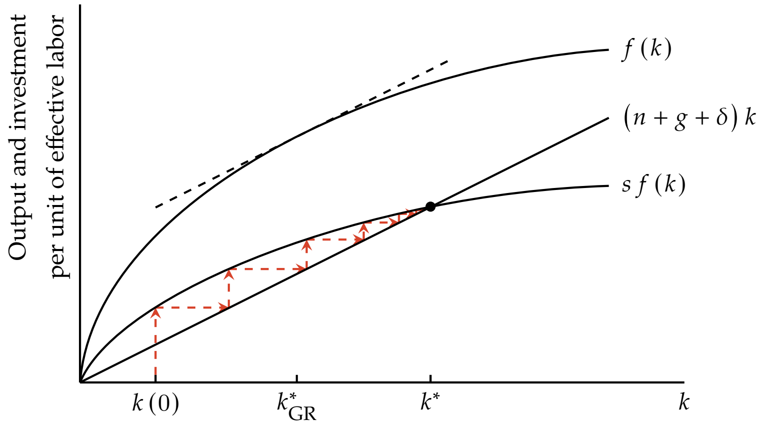
Answer to Q - 4.c

In Piketty's opinion, if the return to capital exceeds the growth rate of the economy, which means $r > n+g$, the economy would grow over time without bound. It sounds quite reasonable, since the

[1] **Piketty, Thomas. 2014.** *Capital in the Twenty-First Century.* Translated by Arthur Goldhammer. Cambridge: Harvard University Press.

[2] **Piketty, Thomas, and Gabriel Zucman. 2014.** "Capital Is Back: Wealth-Income Ratios in Rich Countries 1700-2010." *Quarterly Journal of Economics* 129 (August): 1255-1310.

[3] **Rognlie, Matthew. 2015.** "Deciphering the Fall and Rise in the Net Capital Share: Accumulation or Scarcity?" *Brookings Papers on Economic Activity*, no. 1, 1-54.



capital always earns more than the economy consumes in every period.

The difference between Piketty's statement and our analysis is whether the capital is paid by its marginal product or not. Piketty relaxed some assumptions of the Solow model, leading him to a different result.

Under assumptions, the Solow model owns only one steady state (the original point is excluded), which leads to K/Y cannot growth without bound. This conclusion relies on the fact that MPK is diminishing. If the capital is paid by MPK, r would be diminishing as well, which means $r > n+g$ might not hold over time.

In summary, if the capital is paid by MPK (Piketty did not specify this point), K/Y would grow **with** bound; if the capital is not paid by MPK (what Piketty assumed), K/Y have chances growing **without** bound. I do not think there is anything wrong in Piketty's study, it depends on whether the reality goes with his assumptions or not.

Answer to Q - 4.d

Let the production function be

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1$$

The share of net capital income in net output is

$$\frac{r \cdot K}{Y} = \frac{(\mathrm{MPK} - \delta)\,K}{Y} = \frac{\left[\alpha K^{\alpha-1}(AL)^{1-\alpha} - \delta\right] K}{K^{\alpha}(AL)^{1-\alpha} - \delta K} = \alpha - \frac{\delta}{\alpha}$$

The ratio K/Y remains unchanged whatever happened to the economy. Hence, the statement is wrong.

Question 5

Consider Question 4. Suppose there is a marginal increase in K. Derive an expression (in terms of K/Y, δ, the marginal product of capital F_K, and the elasticity of substitution between capital and effective labor in the gross production function $F(\cdot)$) that determines whether a marginal increase in K increases, reduces, or has no effect on the share of net capital income in net output.

Answer to Q - 5

The share of net capital income in net output (denoted as $\mathcal{N}$) is

$$\mathcal{N} \equiv \frac{r \cdot K}{Y - \delta K} = \frac{(\mathrm{MPK} - \delta) K}{Y - \delta K} = \frac{(F_K - \delta) K}{Y - \delta K}$$

Differentiate $\mathcal{N}$ with respect to K to check how $\mathcal{N}$ changes when K increases marginally.

$$\frac{\partial \mathcal{N}}{\partial K} = \frac{(K F_{KK} + F_K - \delta)(Y - \delta K) - (F_K - \delta)^2 K}{(Y - \delta K)^2} \tag{Q5.1}$$

As the problem requires, we will express the above equation adding the terms of K/Y and the elasticity of substitution

$$\sigma \equiv -\left[\frac{\partial \log F_K / F_{AL}}{\partial \log K/(AL)}\right]^{-1} = -\frac{F_K / F_{AL}}{K/(AL)} \left\{\frac{\partial\, [F_K / F_{AL}]}{\partial\, [K/(AL)]}\right\}^{-1}$$

Since $F(\cdot)$ satisfies constant returns to scale, we have

$$F_K = f'(k) \quad \text{and} \quad F_{AL} = f'(k) - k f(k)$$

Using these two equations, we can calculate the differential of F_K / F_{AL} further.

$$\frac{\partial\, [F_K / F_{AL}]}{\partial\, [K/(AL)]} = \frac{\partial}{\partial k} \frac{f(k)}{f(k) - k f'(k)} = \frac{f(k) f''(k)}{\left[f'(k) - k f(k)\right]^2} = \frac{Y F_{KK}}{F_{AL}^2}$$

Replacing it back to the expression of σ, we have

$$\sigma = -\frac{F_K}{F_{AL}} \frac{AL}{K} \frac{F_{AL}^2}{Y F_{KK}} = -\frac{F_K (AL \cdot F_{AL})}{K Y F_{KK}} = \frac{F_K (K F_K - Y)}{K Y F_{KK}}$$

We used Euler's theorem in the last step.

Rearranging equation (Q5.1) and substituting the expression of σ yields

$$\begin{aligned} \frac{\partial \mathcal{N}}{\partial K} &= \frac{Y}{(Y - \delta K)^2} \left\{\left[\frac{F_K}{\sigma}\left(1 - \frac{K}{Y} F_K\right) + F_K - \delta\right]\left(1 - \delta \frac{K}{Y}\right) - (F_K - \delta)^2 \frac{K}{Y}\right\} \\ &= \underbrace{\frac{Y - K F_K}{(Y - \delta K)^2}}_{>0} \left[\frac{F_K}{\sigma}\left(\sigma - 1 + \delta \frac{K}{Y}\right) - \delta\right] \end{aligned}$$

How $\mathcal{N}$ behaves depends on the sign of the term in the brace.

Question 6

Consider the same setup at the start of Question 4. Show that the economy converges to a balanced-growth-path, and that the balanced-growth-path level of k equals the golden-rule level of k. What is the intuition for the result?

Answer to Q - 6

In this part, based on the assumptions of the Solow model, we additionally assume that

(1) Factors are not paid by their MP, which gives $r \neq \text{MPK}$ (or plus $-\delta$ on right-hand side).

(2) All labor incomes are consumed, and all other income is saved.

Now, though we put forward two assumptions, the economy do have a BGP as well.

Consider $k \equiv K/(AL)$ and apply the property of growth rate to it, we obtain

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - g - n$$

Next, substituting the motion equation of K yields

$$\begin{aligned} \frac{\dot{k}}{k} &= \frac{sF(K, AL) - \delta K}{K} - g - n \\ &= \frac{sF(K, AL)}{K} - \delta - g - n \\ &= \frac{sf(k)}{k} - (n + g + \delta) \end{aligned}$$

That is the motion equation of k

$$\dot{k} = sf(k) - (n + g + \delta)k$$

The economy reaches its BGP when $\dot{k} = 0$. Similar to the treatments applied in the Solow model, we can prove that the BGP exists and is unique, steady.

For intuition, we know that the steady condition and the golden-rule condition respectively are

$$sf(k) = (n + g + \delta)k \quad \text{and} \quad f'(k) = n + g + \delta$$

That is

$$\underbrace{sf(k)}_{\text{saving}} = \underbrace{kf'(k)}_{\text{return}}$$

which means the golden rule level would be reached when the economy consumes all labor incomes.

For example, let production function be Cobb-Douglas form, and the intensive form is $f(k) = k^{\alpha}$.

The steady condition is

$$s\,f\,(k) = \big(n + g + \delta\big)\,k \quad \Rightarrow \quad s\,k^{\alpha-1} = n + g + \delta$$

and the golden-rule condition is

$$f'\,(k) = n + g + \delta \quad \Rightarrow \quad \alpha\,k^{\alpha-1} = n + g + \delta$$

which means the economy realizes golden-rule level when $s = \alpha$ holds. α is the share of capital income in total output, and s is the saving rate. When $s = \alpha$, the economy consumes $(1 - \alpha)$ portion of its output, which is exactly the full amount of labor income, and saves all the rest. Therefore, the economy reaches its golden-rule level when all labor incomes are consumed, and all other income is saved.