Mathematical Economics

Assignment # 2

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Question List

Question 1

For any set X, we can define the *discrete metric* $d\colon X \times X \to \mathbb{R}$ by

$$d\left(x, y\right) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Show that the discrete metric satisfies the 3 requirements of a metric: (1) $d\left(x, y\right) \geq 0$ and $d\left(x, y\right) = 0$ if and only if $x = y$; (2) $d\left(x, y\right) = d\left(y, x\right)$; (3) $d\left(x, y\right) \leq d\left(x, z\right) + d\left(z, y\right)$.

(b) Show that the sequence $x_n = 1/n$ does not converge in the discrete metric.

Answer to Q - 1.a

Non-negativity

Since the metric d only takes the values 0 or 1, it clearly satisfies non-negativity. By definition, $d\left(x, y\right) = 0$ holds if and only if $x = y$.

Symmetry

The value of $d\left(x, y\right)$ depends on whether x equals y or not, and since equality and inequality are symmetric relations, it follows that $d\left(x, y\right) = d\left(y, x\right)$.

Triangle Inequality

We consider two cases. If $x = y$, then $d\left(x, y\right) = 0$. By the non-negativity property,

$$0 = d\left(x, y\right) \leq d\left(x, z\right) + d\left(z, y\right).$$

If $x \neq y$, then $d\left(x, y\right) = 1$. At least one of $d\left(x, z\right)$ or $d\left(z, y\right)$ must be 1. Hence,

$$1 = d\left(x, y\right) \leq d\left(x, z\right) + d\left(z, y\right).$$

Therefore, the triangle inequality holds. □

Answer to Q - 1.b

Prove by contradiction. Suppose, for the sake of contradiction, that the sequence $\{x_n\}$ converges to some limit a. According to the definition, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d\left(x_n, a\right) < \varepsilon.$$

Now, let's $\varepsilon = 0.5$. The inequality requires $1/n = a$ for $n = N, N+1, \ldots$, which is clearly impossible.

In conclusion, the sequence $x_n = 1/n$ does not converge in the discrete metric d. □

Question 2

We say a sequence $\{x_n\}$ in a metric space (X, d) is bounded if there exists $a \in X$ and $r > 0$ such that $d(x_n, a) < r$ for all n. Prove that any convergent sequence is bounded.

Answer to Q - 2

Without any additional assumptions, consider a convergent sequence $\{x_n\}$ with limit a. Let ε take value 1. By definition, there exists $N \in \mathbb{N}$ such that

$$d(x_n, a) \leq 1, \quad \forall n > N$$

This shows that $\{x_n\}_{n>N}$ is bounded. Introduce a new notation

$$r = \max\{x_1, \dots, x_n, 1\}$$

Then $d(x_n, a) < r$ for all $N \in \mathbb{N}$. Therefore, every convergent sequence is bounded. □

Question 3

Suppose that $\{x_n\}$ is a sequence of real numbers that converges to x and that all x_n and x are nonzero.

(a) Prove that there is a positive number B such that $|x_n| \geq B$ for all n.

(b) Using (a), prove that $\{1/x_n\}$ converges to $1/x$.

(c) Prove that if $x_n \to x$, $y_n \to y$, and $x_n, x \neq 0$, then $y_n / x_n \to y/x$.

Answer to Q - 3.a

Let $\varepsilon = |x|/2 > 0$. By definition, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|x_n - x| < \frac{|x|}{2}$$

Hence, for all $n > N$,

$$|x_n| = |(x_n - x) + x| \geq |x| - |x_n - x| > |x| - \frac{|x|}{2} > 0$$

For $n \leq N$, it is clear that $|x_n| > 0$ (given $x_n \neq 0$). Finally, let

$$B = \min\left\{|x_1|, \dots, |x_n|, \frac{|x|}{2}\right\} > 0.$$

Then for all $n \in \mathbb{N}$, we have $|x_n| \geq B$. $\square$

Answer to Q - 3.b

Consider

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n| \cdot |x|} \leq \frac{|x_n - x|}{B \cdot |x|}$$

Since $\{x_n\}$ converges to x, for any positive number ε, there exists $N \in \mathbb{N}$

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| < \frac{\varepsilon'}{B \cdot |x|} \equiv \varepsilon, \quad n > N$$

Thus, the limit of the inverse equals the inverse of the limit. $\square$

Answer to Q - 3.c

Note that

$$\left|\frac{y_n}{x_n} - \frac{y}{x}\right| = \frac{|y_n \cdot x - x_n \cdot y|}{|x_n| \cdot |x|}$$

Consider the numerator separately

$$\begin{aligned} |y_n \cdot x - x_n \cdot y| &= |y_n \cdot x - xy + xy - x_n \cdot y| \\ &= |(y_n - y) \cdot x + (x - x_n) \cdot y| \\ &\leq |y_n - y| \cdot |x| + |x_n - x| \cdot |y| \end{aligned}$$

Assign ε_2 to the y-term, ε_1 and B_1 to the x-term. By definition,

$$|y_n \cdot x - x_n \cdot y| < \frac{\varepsilon_2 |x| + \varepsilon_1 |y|}{B_1 \cdot |x|} \equiv \varepsilon, \quad n > N$$

Thus, the division is preserved in the limit. $\square$

Question 4

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $x_n \to x$ and $y_n \to y$.

(a) Prove that $|x_n - y_n| \to |x - y|$

Hint: You can choose to prove the more general version: If $x_n \to x$ and $y_n \to y$ in a metric space (X, d), then $d(x_n, y_n) \to d(x, y)$.

(b) If $x_n \leq y_n$ for all n, prove that $x \leq y$.

Answer to Q - 4.a

Applying the triangle inequality, we have

$$\begin{aligned} d\left(x_n, y_n\right) &\leq d\left(x_n, x\right) + d\left(x, y_n\right) \\ &\leq d\left(x_n, x\right) + d\left(y_n, y\right) + d\left(x, y\right) \end{aligned}$$

Similarly,

$$\begin{aligned} d\left(x, y\right) &\leq d\left(x, y_n\right) + d\left(y_n, y\right) \\ &\leq d\left(x_n, x\right) + d\left(y_n, y\right) + d\left(x_n, y_n\right) \end{aligned}$$

which gives

$$\underbrace{d\left(x, y\right) - d\left(x_n, x\right) - d\left(y_n, y\right)}_{\text{converge to } d(x,y)} \leq d\left(x_n, y_n\right) \leq \underbrace{d\left(x_n, x\right) + d\left(y_n, y\right) + d\left(x, y\right)}_{\text{converge to } d(x,y)}$$

Using the Squeeze Theorem, $d\left(x_n, y_n\right) \to d\left(x, y\right)$. □

Answer to Q - 4.b

This problem is equivalent to showing that the limit of the non-negative sequence $z_n \equiv y_n - x_n \geq 0$ is non-negative. We will prove it by contradiction.

Firstly,

$$\begin{aligned} \lim_{n\to\infty} z_n &= \lim_{n\to\infty} \left(y_n - x_n\right) \\ &= \lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n \\ &= y - x \end{aligned}$$

Suppose, for the sake of contradiction, that $y - x < 0$. We pick $\varepsilon = \left(x - y\right)/2 > 0$. By definition, there exists $N \in \mathbb{N}$ such that

$$\left|z_n - \left(y - x\right)\right| < \frac{x - y}{2}, \quad n > N$$

Since $z_n \geq 0$ and $y - x < 0$, we have

$$\left|z_n - \left(y - x\right)\right| = z_n - \left(y - x\right) < \frac{x - y}{2}$$

which implies

$$z_n < \frac{y - x}{2} < 0$$

This is a contradiction. □

Question 5

Show that any open set is the union of open balls. Conclude that any open set is its own interior.

Answer to Q - 5

By definition, a set S is open if all the elements in S are interior points. That is, for any $x \in S$, there exists $B_\varepsilon (\subset) S$. We can express S as follows:

$$S = \bigcup_{x \in S} B_\varepsilon (x)$$

Thus, any open set is the union of open balls. $\square$

Interior is the union of all open sets contained in S. Of course, S contains itself, a open set. So any open set is its own interior. $\square$

Question 6

Prove that a set S is closed if and only if $S = \mathrm{Cl}(S)$.

Answer to Q - 6

At the beginning, we state that $\mathrm{Cl}(S)$ is the smallest closed set which containing S. The reason is that $\mathrm{Cl}(S)$ is defined as the intersection of all closed sets containing S, which means that any closed set containing S must also contain $\mathrm{Cl}(S)$.

Sufficiency. If the set S is a closed set, then $\mathrm{Cl}(S)$, the smallest closed set containing S, is itself.

Necessity. Trivially, $\mathrm{Cl}(S)$ is a closed set by definition, so S is a closed set when $S = \mathrm{Cl}(S)$. $\square$

Question 7

Show that a subset A of a metric space (X, d) is open if and only if for any sequence $\{x_n\}$ in X converging to some $x \in A$, we have $x_n \in A$ for all n sufficiently large.

Answer to Q - 7

Sufficiency. A is open means that for any $x \in A$, there always exists a ball $B_\varepsilon(x) \subset A$. Consider a sequence $\{x_n\}$ converging to x. By the definition of convergence, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \quad n > N$$

which means that $x_n \in B_\varepsilon(x) \subset A$ for all $n > N$.

Necessity. Prove by contradiction. Suppose A is not open for sake of contradiction, which means that A contains at least one point x that is not an interior point. By definition, for any $\varepsilon > 0$, the ball $B_\varepsilon(x)$ is not a subset of A. This implies that there exists a sequence $\{x_n\}$ such that $x_n \notin A$ for all n and $x_n \to x$. This contradicts the assumption that for any sequence $\{x_n\}$ in X converging to some $x \in A$, we have $x_n \in A$ for all n sufficiently large. □

Question 8

A set A in a metric space (X, d) is said to be **dense** in X if $\mathrm{Cl}(A) = X$. Prove that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. The following theorem from the Math Camp might be useful.

Theorem

For any $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists a rational number $r \in \mathbb{Q}$ such that $0 < |x - r| < \varepsilon$.

Answer to Q - 8

In this question, we are going to show that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.

We that that the real number set $\mathbb{R}$ is consisted of rational numbers and irrational numbers, so $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ when the irrational number are all the limit points of $\mathbb{Q}$.

Suppose $x \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number. For any $\varepsilon > 0$, by the given theorem, there exists a rational number $r \in \mathbb{Q}$ such that $0 < |x - r| < \varepsilon$. This implies that any open ball $B_\varepsilon(x)$ contains $r \neq x$. Hence, every irrational number is a limit point of $\mathbb{Q}$, which means $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. □

Question 9

For each of the following sets, state whether it is open, closed, or neither. Justify your answer in one or two sentences. Then find its interior, closure and boundary.

(a) $A = \{1/n : n = 1, 2, \ldots\}$

(b) $A = \{1, 2, \ldots\}$

(c) $\mathbb{R}^n$

(d) $\varnothing$ in any metric space

(e) $A = \mathbb{Q} \times \mathbb{R}$

(f) $A \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \cdot \boldsymbol{z} < c\}$ where $\boldsymbol{z} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

Answer to Q - 9

(a) **A is not open.** For example, any open ball centered at 1 will include points not in A, so 1 is not an interior point. **A is not closed.** The limit point 0 of A is not contained in A.

The interior is $\varnothing$.

The closure is $A \cup \{0\}$.

The boundary is $A \cup \{0\}$.

(b) **A is neither open nor closed.** The reason is similar to (a).

The interior is $\varnothing$.

The closure is A.

The boundary is A.

(c) **$\mathbb{R}^n$ is open.** All open balls centered at any point in $\mathbb{R}^n$ are subsets of $\mathbb{R}^n$. **$\mathbb{R}^n$ is closed.** It contains all its limit points.

The interior is $\mathbb{R}^n$.

The closure is $\mathbb{R}^n$.

The boundary is $\varnothing$.

(d) **$\varnothing$ is open.** There are none in $\varnothing$ that are not interior points. **$\varnothing$ is closed.** The boundary of $\varnothing$ is $\varnothing$, which is contained in $\varnothing$.

The interior, closure, boundary are all $\varnothing$.

(e) **A is not open.** Any open ball centered at the point of A must contain points not in A (the first coordinate is irrational). **A is not closed.** The limit point (π, π) of A is not contained in A.

The interior is $\varnothing$.

The closure is $\mathbb{R}^2 \setminus A$.

The boundary is A.

(f) **A is open.** For any point $x \in A$, we can always find a small enough open ball centered at x that is a subset of A. **A is not closed.** The limit point x such that $x \cdot z = c$ is not contained in A.

The interior is $A = \{x : x \cdot z < c\}$.

The closure is $\{x : x \cdot z \leq c\}$.

The boundary is $\{x : x \cdot z = c\}$.

Question 10

Prove that if $x_n \to x$ in a metric space (X, d), then every subsequence $x_{n_j} \to x$.

Answer to Q - 10

Suppose $\{x_{n_j}\}_{n_j}$ is an arbitrary subsequence of $\{x_n\}_n$ that indices n_j is a subsequence of $\mathbb{N}$.

By definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \quad n > N$$

We can find J such that

$$d(x_{n_j}, x) < \varepsilon, \quad n_j > N, \quad \forall j \geq J$$

This shows $x_{n_j} \to x$. □

Question 11

Suppose $A \subset X$ and $x \in X$ where (X, d) is a metric space. The *distance* from x to A is defined by

$$d(x, A) := \inf_{y \in A} d(x, y)$$

(a) Given an example where $d(x, A) < d(x, y)$ for all $y \in A$.

(b) Does $d(x, A) = 0$ imply $x \in A$?

(c) Prove that for any closed $S \subset \mathbb{R}^n$, there exists $y \in S$ such that $d(x, y) = d(x, S)$.

Hint: Try to construct a sequence $y_n \in S$ such that $y_n \to y$ and $d(x, y_n) \to d(x, S)$. You might want to use Theorem 12.14.

Theorem 12.14

Let C be a compact subset of $\mathbb{R}^n$ and let $\{\boldsymbol{x}_n\}$ be any sequence in C. Then, $\{\boldsymbol{x}_n\}$ has a convergent subsequence whose limit lies in C.

The following proposition about infimum might be useful.

Proposition

Suppose S is a nonempty set of real numbers. Then b is the infimum of S if and only if (a) $x \geq b$ for all $x \in S$, and (b) for any $\varepsilon > 0$, there exists an $x \in S$ such that $x < b + \varepsilon$.

Answer to Q - 11.a

$d(x, A) < d(x, y)$ holds when x is a boundary point of A but $x \notin A$. For example, let $A = (0, 1)$ and $x = 1$. Then $d(x, A) = 0$ but $d(x, y) > 0$ for all $y \in A$, since $d(x, y) = 0$ only when $x = y$.

Answer to Q - 11.b

No. The same example in (a) works here. Note that

$$d[1, (0, 1)] = \inf_{y \in (0,1)} d(1, y) = \lim_{y \to 1^-} d(1, y) = 0$$

y arbitrarily close to 1, but can not be 1 forever.