Dynamic programming

* This is problem solving technique.

Dynamic programming :

→ using this D.P we can solve

* optimization problem
  (The problem which needs either min result or max result)

Ex: Travel from kadapa to vijayawada with min cost & time.

Sol: walk
bus
train ✓ optimal solution (satisfies constaint & minimal solution (minimization))
car

* The problem which need min result
  → minimization problem.

* The problem which needs maximum result.
  → maximization problem

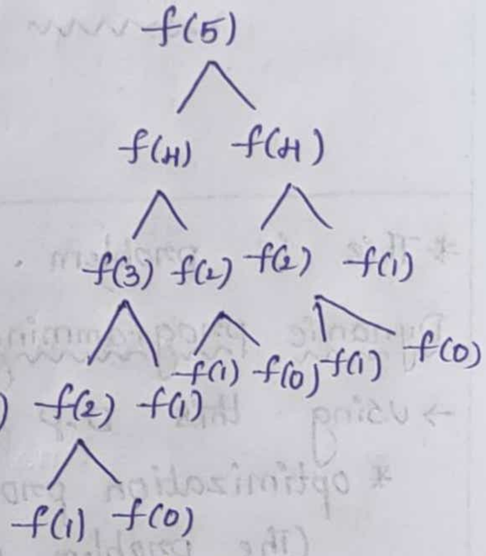
Both are optimization problem.

→ problems solved using DP satisfies two properties
1. overlapping subproblem
2. principal of optimality or optimal substraction.

Overlapping Subproblem :

Ex : Fibonacci series f(5)

```
Alg f(n)
{
  if (n≤1)
    return n
  else
    return as f(n-1) + f(n-2)
}
```



Here f(3) Calls for 2 times   f(1) → 5

f(2) Calls for 3 times

* If some Sub problems Calls for repeatedly, Then we call, Sub problems as Overlapping subproblems
* Some Sub problems occurs repeatedly with Same value ~~age~~ we can say that this fibonacci follow the this Overlapping subproblem.

Drawback of overlapping Subproblems :

Repeatedly occurence of Same Subproblem

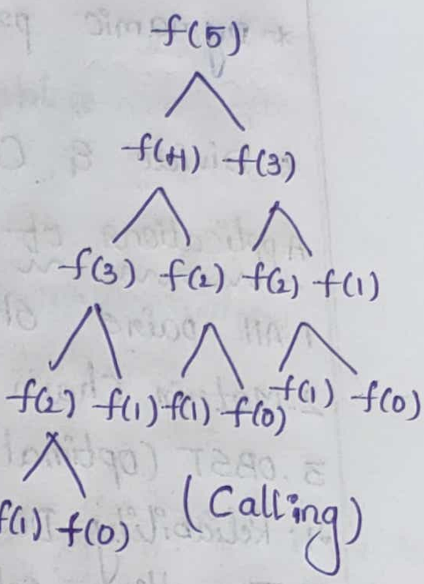
* In order to overcome that drawback, we have 2 techniques.
  1. Memorization
  2. Tabularization.
* we Store the value the in tabular, instead of that recompute.

If we want Compute, again we will search for, table first (which Compute 1st time)

1. In memorization, we will do follow, Top-down approach & Recurssion.
2. In Tabularization, we will follow bottom up approach, & iteration.

1 :- Memorization

Nil	Nil	Nil	Nil	Nil	Nil
0	1	2	3	4	5



0	1	1	2	3	5
0	1	2	3	4	5

* Here we will move from top to bottom & Call for same function (recursion)

2. Tabularization :

```
Alg f(n) ⟶ f(n)
{
  if (n≤1)
     return n;
  else
   {
    f(0)=0;
    f(1)=1;
     for(i=2; i<=n, i++)
      f(i) = f(i-1) + f(i-2);
   }
}
```

0	1	1	2	3	5
0	1	2	3	4	5

$f(2) = f(1) + f(0)$
$f(3) = f(2) + f(1)$
$f(4) = f(3) + f(2)$
$f(5) = f(4) + f(3)$

Here we will call the function from bottom to top & use for loop i.e, iteration.

* In Dynamic programming, we divide the problems into subproblem, but here there is overlapping of sub problem (call function repeatedly with same values whereas in divide & Conquer there is recursion, but there is no overlapping i.e. call same function with different values.

* Dynamic programming → follows Iteration (In this chapter)

* Divide & Conquer → follows Recursion

Applications of DP :

1. All pairs shortest path
2. Matrix Chain multiplication
3. OBST (optimal Binary search tree)
4. Reliability Design.
5. Travelling sales man problem.

Matrix Chain multiplication :

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}_{2\times1}$

* ABC
→ (AB) C
→ A(BC)
→ (AC) B

$A \times B = \begin{bmatrix} 1\times2+2\times6 \\ 3\times2+4\times6 \end{bmatrix} = \begin{bmatrix} 14 \\ 30 \end{bmatrix}$

Here, 4 multiplication operations are required Scalar.

Ex : $A_{1\times2}$ $B_{2\times3}$ $C_{3\times4}$

1. (AB) C

1×2 2×3

$1\times2\times3$ 3×4

$= 6$

$(AB)_{1\times3} C_{3\times4}$ 6+12

$= 1\times3\times4$ → 18

$= 12$

6 multiplication operations are required for AB total 18 required.

2. A(BC)

A (BC)

2×3 3×4

$2\times3\times4$

$= 24$

24

$A_{1\times2}$ $\underset{2\times4}{BC}$

$1\times2\times4$

$= 8$

24
+ 8
32

$A(BC) = A_{1\times2} (B_{2\times3} C_{3\times4})$

$1\times2\times4 + 2\times3\times4$

$= 8 + 24 = 32$

* ~~3~~2 m operators are required
∴ The optimal solution for this problem
$= (AB)C$

Constraint for this problem
→ Requirment of minimum no. of multiplication operators.

<u>problem statement</u> :

Here n matrices will be given. we have to find out the order in which the matrices are to multiplied in order to less no. of scalar multiplications operations are used.

Formula :

$$\boxed{m_{ij} = \min \{ m_{ik} + m_{k+1,j} + P_i P_{k+1} P_{j+1} ,\ i \le k < j \}}$$

Ex: Find the minimum cost of multiplying the matrices with the order.

$A_1 = 1 \times 3$ ($P_1 \times P_2$) $A_2 = 3 \times 2$ ($P_2 \times P_3$) $A_3 = 2 \times 4$ ($P_3 \times P_4$) $A_4 = 4 \times 6$ ($P_4 \times P_5$)

$P_1 = 1,\ P_2 = 3,\ P_3 = 2,\ P_4 = 4,\ P_5 = 6.$

$m_{11} = 0$	$m_{22} = 0$	$m_{33} = 0$	$m_{44} = 0$
$m_{12} = 126$ $k = 1$	$m_{23} = 24$ $k = 2$	$m_{34} = 48$ $k = 3$	
$m_{13} = 14$ $k = 2$	$m_{24} = 84$ $k = 2$		
$m_{14} = 38$ $k = 3$			

$m_{11} = m_{22} = m_{33} = m_{44} = 0$

* m_{12} $\quad i=1, j=2$

$m_{ij} = \min \{ m_{11} + m_{22} + P_1P_2P_3 \}$

$1 \le K < 2$

$K=1$

$= \min \{ 0+0+(1)(3)(2) \}$

$= \min \{6\}$

$= 6$

~~* $m_{23} \Rightarrow i=2 \ j=3$~~

~~$m_{ij} = \min \{ m_{22} + m_{33} + P_2P_3P_4 \}$~~ $\quad \cancel{3 \times 2 \times 4} \ \cancel{= 24}$

~~$2 \le K < 3$~~

~~$K=2$~~

* $m_{13} \Rightarrow i=1 \ j=3$ $\quad 1 \le K < 3$

$m_{ij} = \min \{ m_{11} + m_{23} + P_1P_2P_4, \quad m_{12} + m_{33} + P_1P_3P_4 \}$ $\quad K=1 \quad K=2.$

$= \min \{ 0+24+1\times3\times4, \ 6+0+1\times2\times4 \}$

$= \min \{36, 14\} \Rightarrow 14 \quad 6 \quad K=2$

$\Rightarrow m_{24} = m_{ij} = \min \{ m_{22} + m_{3.4} + P_2P_3P_5 \} m_{33} + m_{4.4} + P_2P_4P_5 \}$

$2 \le K < 4$ $\quad \Rightarrow \min \{ 0+48+3\times2\times6, \ 24+0+3\times4\times6 \}$

$K=2,3$ $\quad \Rightarrow \min \{84, 96\} \Rightarrow 84 \quad 6K=2$

$\Rightarrow m_{14} = m_{ij} = \min \{ m_{11} + m_{24} + P_1P_2P_5,$

$1 \le K < 4$ $\quad m_{12} + m_{3,4} + P_1P_3P_5,$

$K=1,2,3$ $\quad m_{13} + m_{4.4} + P_1P_4P_5 \}$

$= \min \{ 0+84+1\times3\times6,$

$6+48+1\times2\times6,$

$14+0+1\times4\times6 \}$

$= \min \{102, 66, 38\}$

$= 38 \quad 6K=3.$

Hence,

* The minimum no. of Scalar multiplication operation's required to multiply the given matrices is 38

$\Rightarrow$ cost = 38 at K=3

So, we have to find the order, so we use the K-value's for this

Here $K=3 \Rightarrow (A_1 A_2, A_3) A_4 \rightarrow$ because $K=3$ at m_{14}

$\Rightarrow (A_1, A_2) A_3) A_4 \rightarrow K=2$ at m_{13}

$\Rightarrow$ check $\Rightarrow ((A_1, A_2) A_3) A_4$

1×3 3×2 2×4 4×6

$\Rightarrow 1\times3\times2$

$\Rightarrow 1\times2\times4$

$\Rightarrow 1\times4\times6$

$\Rightarrow 1\times3\times2 + 1\times2\times4 + 1\times4\times6 = 38$

So, the Cost is 38.

Ex 2 : $A_1 = 5\times4$ ($P_1 \times P_2$), $A_2 = 4\times6$ ($P_2 \times P_3$), $A_3 = 6\times2$ ($P_3 \times P_4$), $A_4 = 2\times7$ ($P_4 \times P_5$)

$m_{11}=0$	$m_{22}=0$	$m_{33}=0$	$m_{44}=0$
$m_{12}=120$ $K=1$	$m_{23}=48$ $K=2$	$m_{34}=84$ $K=3$	
$m_{13}=88$ $K=1$	$m_{24}=104$ $K=2$		
$m_{14}=158$ $K=3$.			

m_{12} : $1 \le K < 2 \Rightarrow K=1$

$m_{12} = \min\{m_{11} + m_{22} + P_1 P_2 P_3\}$

$= \min\{0+0+5\times4\times6\} = \min\{120\}$

$m_{12} = 120$.

m_{23} : $2 \le k < 3 \Rightarrow k = 2$

$m_{23} = \min\{m_{22} + m_{33} + P_2 P_3 P_4\}$

$= \min\{0 + 0 + 4 \times 6 \times 2\} = \min\{48\}$

$m_{23} = 48.$

m_{34} : $3 \le k < 4 \Rightarrow k = 3$

$m_{34} = \min\{m_{33} + m_{44} + P_3 P_4 P_5\}$

$= \min\{0 + 0 + 6 \times 2 \times 7\} = \min\{84\}$

$m_{34} = 84.$

m_{13} : $1 \le k < 3 \Rightarrow k = 1, 2.$

$$m_{13} = \min\left\{\frac{k=1}{m_{11} + m_{23} + P_1 P_2 P_4}, \frac{k=2}{m_{12} + m_{33} + P_1 P_3 P_4}\right\}$$

$$= \min\left\{\frac{k=1}{0 + 48 + 5 \times 4 \times 2}, \frac{k=2}{120 + 0 + 5 \times 6 \times 2}\right\}$$

$= \min\{88, 180\}$

$m_{13} = 88$, $k = 1$

m_{24} : $2 \le \cancel{k} < 4 \Rightarrow k = 2, 4$

$$m_{24} = \min\left\{\frac{k=2}{m_{22} + m_{34} + P_2 P_3 P_5}, \frac{k=3}{m_{23} + m_{44} + P_2 P_4 P_5}\right\}$$

$$m_{24} = \min\left\{\frac{k=2}{0 + 84 + 4 \times 6 \times 7}, \frac{k=3}{48 + 0 + 4 \times 2 \times 7}\right\}$$

$m_{24} = \min\{252, 104\}$

$m_{24} = 104$, $k = 3.$

m_{14} : $1 \le k < 4 \Rightarrow k = 1, 2, 3$

$$m_{14} = \min\left\{\frac{k=1}{m_{11} + m_{24} + P_1 P_2 P_5}, \frac{k=2}{m_{12} + m_{34} + P_1 P_3 P_5}, \frac{k=3}{m_{13} + m_{44} + P_1 P_4 P_5}\right\}$$

$= \min \left\{ \frac{k=1}{0+104+5\times4\times7}, \frac{k=2}{120+84+5\times6\times7}, \frac{k=3}{88+0+5\times2\times7} \right\}$

$= \min \{ 244, 414, 158 \}$

$m_{14} = 158$, $k=3$.

$\Rightarrow$ cost $=158$

So, we have to find order

$\Rightarrow ((A_1)A_2\, A_3)\, A_4$

$4\times6 \quad 6\times2 \quad 2\times7$

$\Rightarrow \underbrace{4\times6\times2}$

$\Rightarrow \underbrace{5\times4\times2} \quad = 5\times2\times7$

$\Rightarrow 4\times6\times2 + 5\times4\times2 + 5\times2\times7$

$= 48 + 40 + 70 = 158$

Example : $A_1 = 10\times5$, $A_2 = 5\times20$, $A_3 = 20\times30$, $A_4 = 30\times6$.

$P_1 = 10$, $P_2 = 5$, $P_3 = 20$, $P_4 = 30$, $P_5 = 6$

$m_{11} = 0$, $m_{22} = 0$, $m_{33} = 0$, $m_{44} = 0$

$\Rightarrow m_{ij} = \min \{ m_{ik} + m_{k+1,j} + P_i P_{k+1} P_{j+1} \} \quad \{ i \le k < j \}$

$\Rightarrow m_{12} = \min \{ m_{11} + m_{22} + P_1 P_2 P_3 \}$

$1 \le k < 2$, $k=1$

$= \min \{ 0+0+10\times5\times20 \} = \boxed{\{1000\} \rightarrow k=1}$

$\Rightarrow m_{23} = \min \{ m_{22} + m_{33} + P_2 P_3 P_4 \}$

$= \min \{ 0+0+5\times20\times30 \}$

$\boxed{= \min \{3000\} \text{ at } k=2}$

$\Rightarrow m_{34} = m_{ij} = \min \{ m_{33} + m_{44} + P_3 P_4 P_5 \}$

$3 \le k < 4$

$= \min \{ 0+0+20\times30\times6 \}$

$\boxed{= \min \{3600\} \text{ at } k=3}$

$\Rightarrow m_{13} = m_{ij} = \min \{ m_{11} + m_{23} + P_1 \cdot P_2 \cdot P_4 ; \ m_{12} + m_{33} + P_1 \cdot P_3 \cdot P_4 \}$

$1 \le k < 3$, $k = 1,2$

$= \min \{ 0 + 3000 + 10\times5\times30, \ 1000 + 0 + 10\times20\times30 \}$

$= \min \{ 4500, 7000 \}$

$\boxed{= \min \{4500\} \text{ at } k=1}$

$\Rightarrow m_{24} = m_{ij} = \min \{ m_{22} + m_{34} + P_2 \cdot P_3 \cdot P_5$
$2 \le k < 4$ $\quad m_{23} + m_{44} + P_2 \cdot P_4 \cdot P_5 \}$
$k = 2, 3$

$= \min \{ 0 + 3600 + 5 \times 20 \times 6,$
$3000 + 0 + 5 \times 30 \times 6 \}$

$= \min \{ 4200, 3900 \}$

$\boxed{= \min \{ 3900 \} \text{ at } k = 3}$

$m_{14} = m_{ij} = \min \{ m_{11} + m_{24} + P_1 \times P_2 \times P_5,$
$1 \le k < 4$ $\quad m_{12} + m_{34} + P_1 \times P_3 \times P_5$
$K = 1, 2, 3$ $\quad m_{13} + m_{44} + P_1 \times P_4 \times P_5 \}$

$= m_{ij} = \min \{ 0 + 3900 + 10 \times 5 \times 6,$
$1000 + 3600 + 10 \times 20 \times 6,$
$4500 + 0 + 10 \times 30 \times 6 \}$

$= \min \{ 4200, 5800, 6300 \}$

$\boxed{\Rightarrow \min \{ 4200 \} \text{ at } k = 1}$

Cost = 4200
order = $(A_1, A_2\ A_3\ A_4)$
$\Rightarrow (A_1)(A_2\ A_3) A_4$
$\Rightarrow (A_1 (A_2\ A_3)) A_4$
$\Rightarrow 5 \times 20 \times 30 + 10 \times 5 \times 30 + 10 \times 30 \times 6$

$\rightarrow (A_1((A_2\ A_3))\ A_4))$
$10 \times 5 \quad 5 \times 20 \quad 20 \times 30 \quad 30 \times 6$
$\Rightarrow 5 \times 20 \times 30$
$\Rightarrow 10 \times 5 \times 30$
$\Rightarrow 10 \times 30 \times 6.$

Algorithm for matrix multiplication :

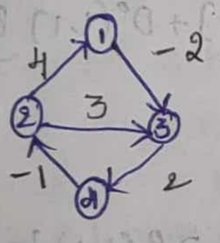
Algorithm matrix – chain – null ( )
```
{
  for i=1 to n do
        m[i,j] = 0
  for len = 2 to n do
  {
     for i=1 to (n-len+1) do
     {
          j = i+len-1;
          m[ij] = 8 ,
          For k=1 to j-1 do
          {
             q = m[i,k] + m[k+1,j] + P[i-1] * P[k] *
                                               P[j]
```

```
        if (q<m[i,j])
        {
            m[i,j]=q;
            S[i,j] =k;
        }
      }
    }
  }
  return m[1,n];
}
```

* Time Complexity for this Algorithm is $O(n^3)$.

All pairs Shortest path problem :

* Here weighted directed graph is given. Now, We have to find short path from every node to Every another node.
* This is problem statement.
* weight → Cost associated edge.



1 → 2, 3, 4   2 → 1, 3, 4   3 → 1, 2   4 → 1, 2, 3

* (a,b) & (b,a) path are not Same in the weighted directed graphs (whereas, directions are given)
* sources – all nodes
  destinations – All nodes
* 2-3
  2 1 3 → via 1
  2 4 3 → via 4

D^1, D^2, D^3, D_4

* D^1 → Cost form every node to every other node via 1.

* D^2 – via node 2
  D^3 – via node 3
  } Cost adjacency matrices

D^4 D^0 → Direct path.

Cost adjacency matrices :

D^0	1	2	3	4
1	0	∞	-2	∞
2	4	0	3	∞
3	∞	∞	0	2
4	∞	-1	∞	0

* if there is no path → ∞ the cost is → ∞

D^1	1	2	3	4
1	0	∞	-2	∞
2	4	0	2	∞
3	∞	∞	0	2
4	∞	-1	∞	0

$1 \xrightarrow{1} 1 \Rightarrow 1-0$
$1 \xrightarrow{1} 2 \Rightarrow 1-2$
$2 \xrightarrow{1} 3 \rightarrow 2-1+1-3$
$4+(-2) = 2$

* $\boxed{D^K(i,j) = \min\{(D^{K-1}(i,j) ; D^{K-1}(i,k) + D^{K-1}(k,j)\}}$

Ex : $D^1(2,3) = \min\{(D^0(2,3) ; D^0(2,1) + D^0(1,3))\}$
$= \min\{3 ; 4 + (-2)\}$
$= \min\{3, 2\}$
$= 2$

$D^1(2,4) = \min\{D^0(2,4) ; D^0(2,1) + D^0(1,1)\}$
$= \min\{\infty, 4 + \infty\}$
$= \infty$

$D^1(3,1) = \min\{D^0(3,1) + D^0(3,1) + D^0(1,1)\}$
$= \min\{\infty, \infty + 0\}$
$= \min\{\infty, \infty\}$
$= \infty$

$D^1(3,2)$

*

D^2	1	2	3	4
1	0	∞	-2	∞
2	4	0	2	∞
3	∞	∞	0	2
4	3	-1	1	0

$D^2(1,2) = \min\{D^1(1,2), D^1(1,2) + D^1(2,2)\}$
$= \min\{\infty, \infty + 0\}$
$= \min\{\infty, \infty\}$
$= \infty$

$D^2(1,3) = \min\{D^1(1,3), D^1(1,2) + D^1(2,3)\}$
$= \min\{-2, \infty + 2\}$
$= -2$

$D^2(1,4) = \min\{D^1(1,4), D^1(1,2) + D^1(2,4)\}$
$= \min\{\infty, \infty + \infty\}$
$= \infty$

$D^2(3,1) = \min\{D^1(3,1), D^1(3,2) + D^1(2,1)\}$
$= \min\{\infty, \infty + 4\}$
$= \infty$

$D^2(3,4) = \min\{D^1(3,4), D^1(3,2) + D^1(2,4)\}$
$= \min\{2, \infty + \infty\}$
$= 2$

$D^2(4,1) = \min\{D^1(4,1), D^1(4,2) + D^1(2,1)\}$
$= \min\{\infty, -1 + 4\}$
$= 3$

$D^2(4,3) = \min\{D^1(4,3), D^1(4,2) + D^1(2,3)\}$
$= \min\{\infty, -1 + 2\}$
$= -1$

*

D^4	1	2	3	4
1	0	-1	-2	0
2	4	0	2	4
3	5	1	0	2
4	3	-1	1	0

$D^4(2,1) = \min\{D^3(2,1), D^3(2,4) + D^3(4,1)\}$
$= \min\{4, 4+3\}$
$= 4$

$D^4(1,2) = \min\{D^3(1,2), D^3(1,4) + D^3(4,2)\}$
$= \min\{\infty, 0 + -1\}$
$= -1$

$D^4(1,3) = \min\{D^3(1,3), D^3(1,4) + D^3(4,3)\}$
$= \min\{-2, 0+1\}$
$= -2$

$D^4(2,3) = \min\{D^3(2,3), D^3(2,4) + D^3(4,3)\}$

$= \min\{2, 4+1\}$

$= 2.$

$D^4(3,1) = \min\{D^3(3,1), D^3(3,4) + D^3(4,1)$

$= \min\{15, 0+15\}$

$= 15$

$D^4(3,2) = \min\{D^3(3,2), D^3(3,4) + D^3(4,2)\}$

$= \min\{\infty, 2+-1\}$

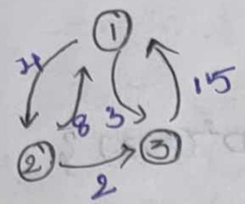
$= 1$

Algorithm :

```
Algorithm Allpair(w, D)
{
  for i=1 to n do          -> n
  {
    for j=1 to n do        -> n.n        n^2      w[i,j]
    {                                            weight from
      D[i,j]=w[i,j];                             i to j
    }
  }
  for k=1 to n do          -> n
  {
    for i=1 to n do        -> n.n                 n^3
    {
      for j=1 to n do      -> n.n.n
      {
        D[i,j] = min(D[i,j], D[i,k]+D[k,j]);
      }
    }
  }
}
```

* Time Complexity $= O(n^3)$

Ex: 2 -



Cost adjacency matrices :

D^0	1	2	3
1	0	4	3
2	8	0	2
3	15	∞	0

D^1	1	2	3
1	0	4	3
2	8	0	2
3	15	19	0

$D^1(1,2) = \min\{D^0(1,2), D^0(1,1) + D^0(1,2)\}$
$= \min\{4, 0+4\}$
$= 4$

$D^1(1,3) = \min\{D^0(1,3), D^0(1,1) + D^0(1,3)\}$
$= \min\{3, 0+3\}$
$= 3.$

$D^1(2,1) = \min\{D^0(2,1), D^0(2,1) + D^0(1,1)\}$
$= \min\{8, 8+0\}$
$= 8$

$D^1(2,3) = \min\{D^0(2,3), D^0(2,1) + D^0(1,3)\}$
$= \min\{2, 8+3\}$
$= 2$

$D^1(3,1) = \min\{D^0(3,1), D^0(3,1) + D^0(1,1)\}$
$= \min\{15, 15+0\}$
$= 15$

$D^1(3,2) = \min\{D^0(3,2), D^0(3,1) + D^0(1,2)\}$
$= \min\{\infty, 15+4\}$
$= 19.$

D^2	1	2	3
1	0	4	3
2	8	0	2
3	15	19	0

$D^2(1,2) = \min\{D(1,2), D^2(1,2) + D^2(2,2)\}$
$= \min\{4, 4+0\}$
$= 4$

$D^2(1,3) = \min\{D^2(1,3), D^2(1,2) + D^2(2,3)\}$
$= \min\{3, 4+2\}$
$= 3$

$D^2(2,1) = \min\{D^1(2,1), D^2(2,2) + D^2(2,1)\}$
$= \min\{2, 0+2\}$
$= 2$

$D^2(2,3) = \min\{D^1(2,3), D^1(2,2) + D^1(2,3)\}$
$= \min\{2, 0+2\}$
$= 2$

$D^2(3,1) = \min\{D^1(3,1), D^1(3,2) + D^1(2,3)\}$
$= \min\{15, 19+2\}$
$= 15$

$D^2(3,2) = \min\{D^1(3,2), D^1(3,2) + D^1(2,2)\}$
$= \min\{19, 19+0\}$
$= 19$

D^3	1	2	3
1	0	4	3
2	8	0	2
3	15	19	0

$D^3(1,2) = \min\{D^2(1,2), D^2(1,3) + D^2(3,2)\}$
$= \min\{4, 3+19\}$
$= 4$

$D^3(1,4) = \min\{D^2(1,3), D^2(1,3) + D^2(3,3)\}$
$= \min\{3, 3+0\}$
$= 3$

$D^3(2,1) = \min\{D^2(2,1), D^2(2,3) + D^2(3,1)\}$
$= \min\{8, 2+15\}$
$= 8.$

$D^3(2,3) = \min\{D^2(2,3), D^2(2,3) + D^2(3,3)\}$
$= \min\{2, 2+0\}$
$= 2$

$D^3(3,1) = \min\{D^2(3,1), D^2(3,3) + D^2(3,1)\}$
$= \min\{15, 0+15\}$
$= 15$

$D^3(3,2) = \min\{D^2(3,2) + D^2(3,3) + D^2(3,2)\}$
$= \min\{19, 0+19\}$
$= 19$

$D^3(3,3) = \min\{D^2(3,3), D^2(3,3) + D^2(3,3)\}$
$= \min\{0,0\}$
$= 0.$

Optimal Binary Search Trees :

BST :-

→ It Contains atmost 2 children.

→ left child < parent < right child.

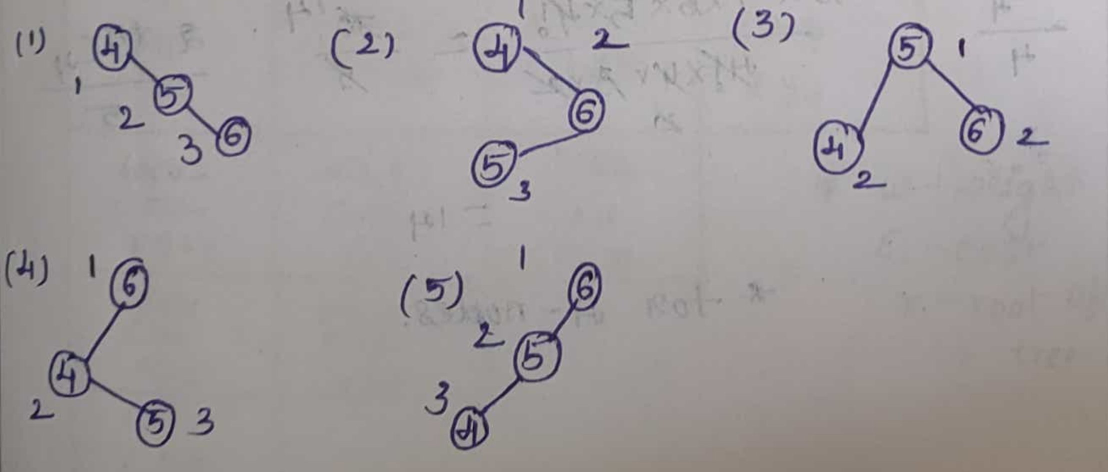
* If n nodes given, then we can construct,

$$\frac{{}^{2n}C_n}{n+1}$$ no. of Binary Search Tree.

Ex: 3 nodes.

$$\frac{{}^{2\times 3}C_3}{4} = \frac{{}^{6}C_3}{4} = \frac{\frac{\cancel{6}\times 5\times \cancel{4}\times \cancel{3!}}{\cancel{3!}\times \cancel{3!}}}{\cancel{4}} = 5$$

Ex: 4, 5, 6.



which give minimal cost → optimal B.S.T

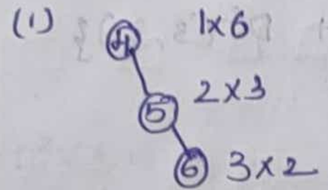
(3) gives minimum Cost → 5

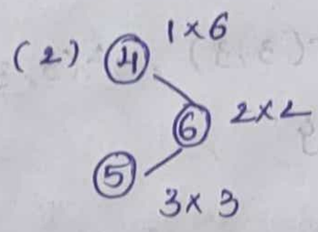
So, (3) is the required one.

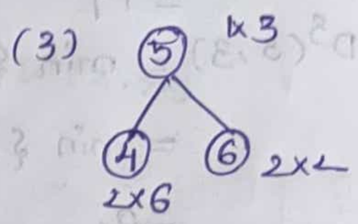
* If frequencies are given, than now we find Cost.

frequencies (6, 3, 2)

Ex : 4, 5, 6

(1)


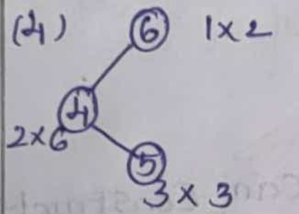
(2)


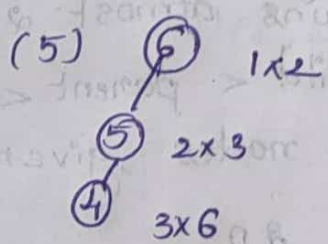
(3)


$= 6+6+6$
$= 18$

$= 6+4+9$
$= 19$

$= 3+12+14$
$= 19.$

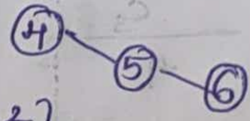
(4)


(5)


$= 2+12+9$
$= 23$

$= 2+6+18$
$= 26$

(1) gives minimum Cost → 18

So, the optimal B.S.T is


for given frequency (6, 3, 2)

$\frac{{}^8C_4}{4}$, $\frac{\cancel{4}\,\cancel{8}\times 7\times\cancel{6}\times 5\times\cancel{4!}}{\cancel{4!}\times\cancel{4}\times\cancel{3}\times\cancel{2}} = \frac{\cancel{70}\,14}{\cancel{5}}$ $\quad \frac{2\times{}^4C_4}{5}$

$= 14$

* for 4 - nodes.

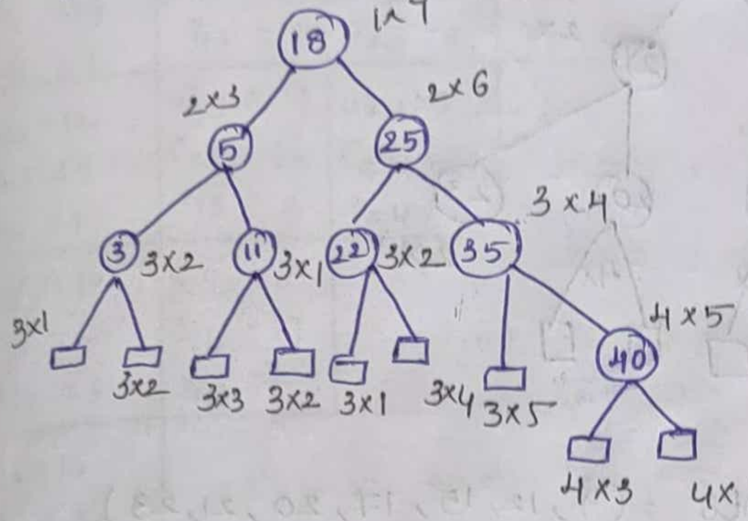
② Elements = (3, 5, 11, 18, 22, 25, 35, 40)

→ 1430.

$p(1:8) = (2, 3, 1, 7, 2, 6, 4, 5)$

$q(0:8) = (1, 2, 3, 2, 1, 4, 5, 3, 1)$

one of the BST



⇒ 7+6+12+6+3+6+12+3+6+9+6+3+12+15+12+4

= 142.

Q: Construct OBST

$(a_1, a_2, a_3, a_4) = (\text{do}, \text{if}, \text{int}, \text{while})$

$P(1:4) = (3, 3, 1, 1)$

$q(0:4) = (2, 3, 1, 1)$

Sol

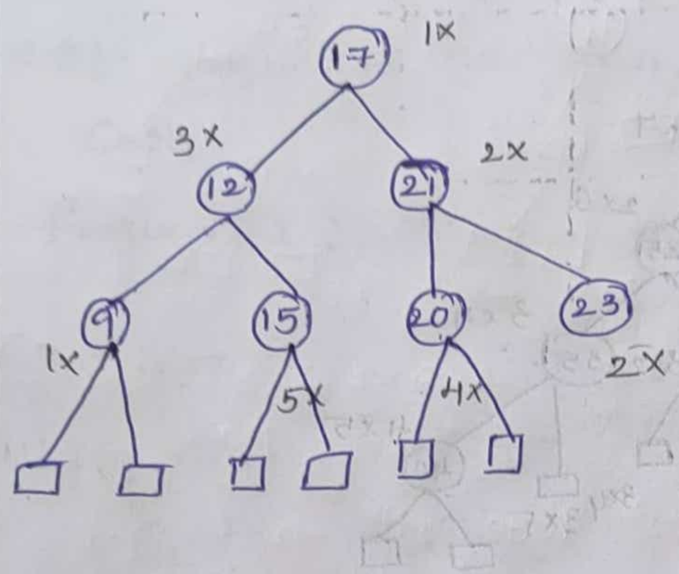
0	1	2	3	4
w_{00} c_{00} r_{00}	w_{11} c_{11} r_{11}	w_{22} c_{22} r_{22}	w_{33} c_{33} r_{33}	w_{44} c_{44} r_{44}
w_{01} c_{01} r_{01}	w_{12} c_{12} r_{12}	w_{23} c_{23} r_{23}	w_{34} c_{34} r_{34}	
w_{02} c_{02} r_{02}	w_{13} c_{13} r_{13}	w_{24} c_{24} r_{24}		
w_{03} c_{03} r_{03}	w_{14} c_{14} r_{14}			
w_{04} c_{04}				

* w - weight
  c - cost
  r - root of tree.

Ex: Elements = (9, 12, 15, 17, 20, 21, 23)

P(1:7) = (1, 3, 5, 6, 4, 2, 2)

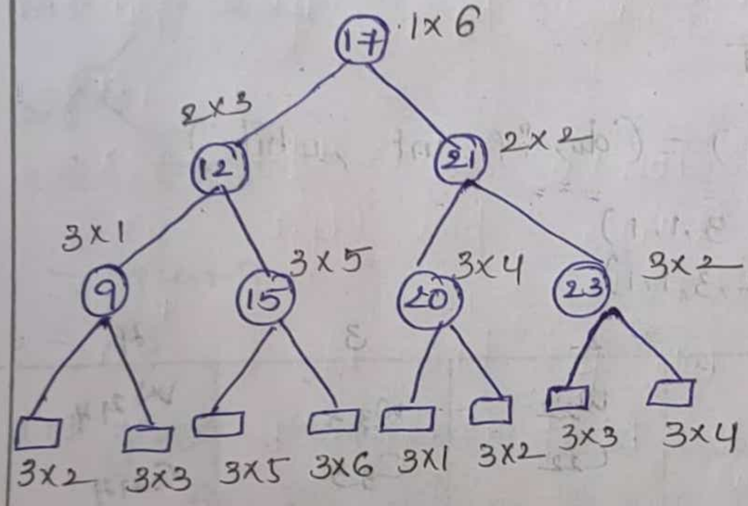
q(0:7) = (2, 3, 5, 6, 1, 2, 3, 4).



Ex: Elements = (9, 12, 15, 17, 20, 21, 23)

P(1:7) = (1, 3, 5, 6, 4, 2, 2)

q(0:7) = (2, 3, 5, 6, 1, 2, 3, 4)



= 6 + 6 + 4 + 3 + 15 + 12 + 6 + 6 + 9 + 15 + 18 + 3 + 6 + 9 + 12

= 130.

Ex: Construct the OBST

(a_1, a_2, a_3, a_4) = (do, if, int, while)

p(1:4) = (3, 3, 1, 1)

q(0:4) = (2, 3, 1, 1, 1)

0	1	2	3	4
$w_{00} = 2$ $c_{00} = 0$ $r_{00} = 0$	$w_{11} = 5$ $C_{11} = 0$ $r_{11} = 0$	$w_{22} = 1$ $C_{22} = 0$ $r_{22} = 0$	$w_{33} = 1$ $C_{33} = 0$ $r_{33} = 0$	$w_{44} = 1$ $C_{44} = 0$ $r_{44} = 0$
$w_{01} = 8$ $C_{01} = 19$ $r_{01} = 1$	$w_{12} = 7$ $C_{12} = 7$ $r_{12} = 2$	$w_{23} = 3$ $C_{23} = 3$ $r_{23} = 3$	$w_{34} = 3$ $C_{34} = 3$ $r_{34} = 4$	
$w_{02} = 12$ $C_{02} = 19$ $r_{02} = 1$	$w_{13} = 9$ $C_{13} = 12$ $r_{13} = 2$	$w_{24} = 5$ $C_{24} = 8$ $r_{24} = 3$		
$w_{03} = 14$ $C_{03} = 25$ $r_{03} = 2$	$w_{14} = 11$ $C_{14} = 19$ $r_{14} = 2$			
$w_{04} = 16$ $C_{04} = 32$ $r_{04} = 2$				

$\Rightarrow$ w - weight
c - cost
r - root.

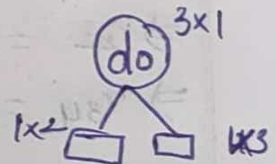
Formulas :

$w_{ij} = q_i$ $C_{ij} = 0$ $r_{ij} = 0$	$w_{i\,i+1} = q_i + q_{i+1} + P_{i+1}$ $C_{i\,i+1} = q_i + q_i + P_i + 1$ $r_{i\,i+1} = i+1$	$w_{ij} = w_{ij} + P_j + q_j$ $C_{ij} = \min\{C_{ik-1} + C_{kj}\} + w_{ij}$ $\boxed{i \le k \le j}$ $r_{ij} = k.$

$w_{01} = w_{i\,i+1} = q_0 + q_1 + P_1 = 2+3+3 = 8$

$C_{01} = C_{i\,i+1} = q_0 + q_{i+1} + P_{i+1} = 2+3+3 = 8 \Rightarrow$

$r_{01} = r_{i\,i+1} = i+1 = 0+1 = 1$

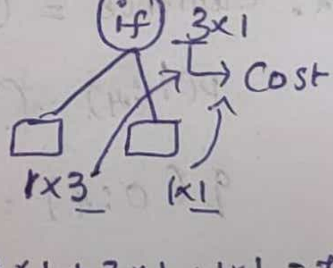


Cost = 2+3+3
= 8.

$w_{12} = q_1 + q_2 + P_2 = 3+1+3 = 7$

$C_{12} = q_1 + q_2 + P_2 = 7$

$r_{12} = 2$



$= 3\times1 + 3\times1 + 1\times1 = 7$

$w_{02} = w_{01} + P_2 + q_2 = 8+3+1 = 12$

$C_{02} = \min\{C_{ik-1} + C_{kj}\} + w_{ij}$

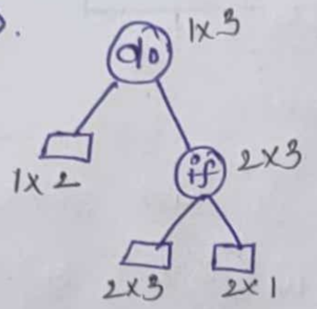
$i \le k \le j$

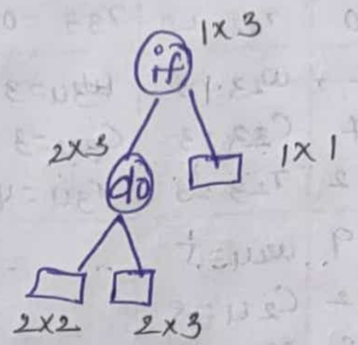
$1 \le k \le 2$, $k = 1, 2$ $= \min\left\{\begin{array}{l} C_{00} + C_{12} \\ C_{01} + C_{22} \end{array}\right\} + w_{02} = \{0+7, 8+0\} + 12$

$= \min\{7\} + 12$

$= 19$ & $k = 1$

(do, if)

⇒ 



Cost $= 3+6+2+6+2$
$= 19$

Cost $= 3+6+4+6+1$
$= 20.$

⇒ The cost of the OBST for the given Example is 32

⇒ The weight of the OBST for the given Example is 16

⇒ $r_{ij} = k$

⇒ r_{ik} → left child $= r_{02} = 1$ (do).

⇒ r_{kj} → right child $= r_{24} = 3$ (int)

⇒ $r_{01} = 1$

⇒ r_{06} = left child = 0 (NO)

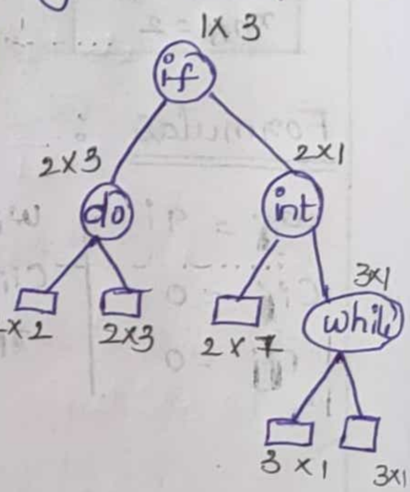
⇒ r_{11} = right child = 0 (NO)

⇒ $r_{24} = 3$

⇒ $r_{22} = 0$ left child = 4 (NO)

⇒ $r_{34} = 4$ (right child) = 4

if



$= 3+6+2+4+6+2+3+3+3$
$= 32$ → Cost

Cost $= 3+3+2+3+1+1+1+1+1$
$= 16$

Ex: Elements $= (a_1, a_2, a_3, a_4)$

$p(1:4) = \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\right)$

$q(0:4) = \left(\frac{1}{4}, \frac{3}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\right)$

0	1	2	3	4
$w_{00} = \frac{1}{4}$ $c_{00} = 0$ $r_{00} = 0$	$w_{11} = \frac{3}{16}$ $c_{11} = 0$ $r_{11} = 0$	$w_{22} = \frac{1}{16}$ $c_{22} = 0$ $r_{22} = 0$	$w_{33} = \frac{1}{16}$ $c_{33} = 0$ $r_{33} = 0$	$w_{44} = \frac{1}{16}$ $c_{44} = 0$ $r_{44} = 0$
$w_{01} = 11/16$ $c_{01} = 11/16$ $r_{01} = 1$	$w_{12} = 3/8$ $c_{12} = 3/8$ $r_{12} = 2$	$w_{23} = 3/16$ $c_{23} = 3/16$ $r_{23} = 3$	$w_{24} = 3/16$ $c_{34} = 3/16$ $r_{34} = 4$	
$w_{02} = 7/8$ $c_{02} = 5/4$ $r_{02} = 1$	$w_{13} = 1/2$ $c_{13} = 11/16$ $r_{13} = 2$	$w_{24} = 5/16$ $c_{24} = 1/2$ $r_{24} = 3$		
$w_{03} = 1$ $c_{03} = 27/16$ $r_{03} = 1$	$w_{14} = 5/8$ $c_{14} = 9/8$ $r_{14} = 2$			
$w_{04} = 9/8$ $c_{04} = 9/4$ $r_{04} = 1$				

$\Rightarrow w_{01} = q_0 + q_{0+1} + p_{0+1} = q_0 + p_1 + q_1$

$= \frac{1}{4} + \frac{3}{16} + \frac{1}{4} = \frac{11}{16}$

$\Rightarrow c_{01} = q_0 + q_1 + p_1 = \frac{11}{16}$

$\Rightarrow r_{01} = 0 + 1 = 1$

$\Rightarrow w_{12} = q_1 + q_2 + p_2 = \frac{3}{16} + \frac{1}{16} + \frac{1}{8} = \frac{3}{8}$

$\Rightarrow c_{12} = 3/8$

$\Rightarrow r_{12} = 2.$

$\Rightarrow w_{23} = q_2 + q_3 + p_3 = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$

$\Rightarrow c_{23} = 3/16$

$\Rightarrow r_{23} = 2 + 1 = 3$

$\Rightarrow w_{24} = q_3 + q_4 + p_4 = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$

$\Rightarrow c_{34} = 3/16$

$\Rightarrow r_{34} = 3 + 1 = 4$

$\Rightarrow w_{02} = w_{01} + P_2 + q_2 = {}^{11}/_{16} + {}^{1}/_{8} + {}^{1}/_{16} = {}^{7}/_{8}$

$\Rightarrow C_{02} = \min\{C_{00}+C_{12}, C_{01}+C_{22}\} + w_{02}$

$0 \le K < 2$

$K=0,1,2 = \min\{0+\frac{3}{8}, \frac{11}{16}+0\} + \frac{7}{8}$

$= \min\{\frac{5}{4}, \frac{25}{10}\} = \frac{5}{4}$ & $16 = 1$

$\Rightarrow w_{13} = w_{12} + P_3 + q_3 = \frac{3}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$

$\Rightarrow C_{13} = \min\{C_{11}+C_{23}, C_{12}+C_{33}\} + w_{13}$

$1 < k \le 3$

$K=2,3 = \min\{0+\frac{3}{16}, \frac{3}{8}+0\} + \frac{1}{2}$

$= \min\{\frac{11}{16}, \frac{7}{8}\} = \frac{11}{16}$ at $K=2$

$\Rightarrow w_{24} = w_{23} + P_4 + q_4 = \frac{3}{16} + \frac{1}{16} + \frac{1}{16} = \frac{5}{16}$

$\Rightarrow C_{24} = \min\{C_{22}+C_{34}, C_{23}+C_{44}\} + w_{24}$

$2 < K \le 4$

$K=3,4 = \min\{0+\frac{3}{16}, \frac{3}{16}+0\} + \frac{5}{16}$

$= \min\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$ at $K=3$.

$\Rightarrow w_{03} = w_{02} + P_3 + q_3 = \frac{7}{8} + \frac{1}{16} + \frac{1}{16} = 1$

$\Rightarrow C_{03} = \min\{C_{00}+C_{12}, C_{01}+C_{23}, C_{02}+C_{33}\} + w_{03}$

$0 < K \le 3$

$K=1,2,3 = \min\{0+\frac{11}{16}, {}^{11}/_{16}+{}^{3}/_{16}, {}^{5}/_{4}+0\} + 1$

$= \min\{\frac{27}{16}, \frac{15}{8}, \frac{9}{4}\}$

$= \min\{\frac{27}{16}\}$ at $K=1$

$\Rightarrow w_{14} = w_{13} + P_4 + q_4 = \frac{1}{2} + \frac{1}{16} + \frac{1}{16} = \frac{5}{8}$

$\Rightarrow C_{14} = \min\{C_{11}+C_{24}+C_{12}+C_{34}, C_{13}+C_{44}\} + w_{14}$

$1 < K \le 4$

$K=2,3,4 = \min\{0+\frac{1}{2}, \frac{3}{8}+\frac{3}{16}, \frac{11}{16}+0\} + \frac{5}{8}$

$= \min\{\frac{9}{8}, \frac{19}{16}, \frac{21}{16}\}$

$= \min\{\frac{9}{8}\}$ at $K=2$

$\Rightarrow w_{04} = w_{03} + P_4 + q_4 = 1 + \frac{1}{16} + \frac{1}{16} = \frac{9}{8}$

$\Rightarrow C_{04} = \min\{C_{00}+C_{14}, C_{01}+C_{24}, C_{02}+C_{34}, C_{03}+C_{44}\}$

$+ w_{04}$

$0 < K \le 4$

$K = 1,2,3,4$ $= \min \{ 0 + \frac{9}{8}, \frac{11}{16} + \frac{1}{2}, \frac{5}{4} + \frac{3}{16}, \frac{27}{16} + 0 \} + \ldots$

$= \min \{ \frac{9}{4}, \frac{37}{16}, \frac{41}{16}, \frac{45}{16} \}$

$= \min \{ 9/4 \}$ at $k=1$

$\Rightarrow r_{ij} = K$

$\Rightarrow r_{00}$ = left child (no)

$\Rightarrow r_{14} = 2 = (a_2)$

$\Rightarrow r_{14} = 2$

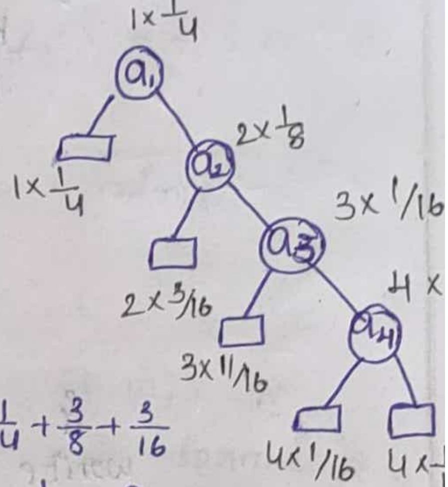
$\Rightarrow r_{11} = 0$

$\Rightarrow r_{24} = 3 \rightarrow (a_3)$

$\Rightarrow a_{24} = 3$

$\Rightarrow r_{22} = 0$

$\Rightarrow r_{34} = 4$



$\Rightarrow \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{16} + \frac{1}{4} + \frac{1}{4} = 9/4$ = Cost

Weight = 9/8.

Algorithm :

```
Algorithm OBST(p, q, n)
{
  for i=0 to n-1 do
  {
     w[i, i] = q;;
     C[i, i] = 0-0;
     r[i, i] = 0;
     w[i, i+1] = q[i] + q[i+1] + p[i+1];
     r[i, i+1] = i+1;
     C[i, i+1] = q[i] + q[i+1] + p[i+1];
  }
     w[n, n] = q[n]
     C[n, n] = 0.0;
     r[n, n] = 0;
  for m=2 to n do
  {
     for i=0 to n-m do
     {
```

```
j = i+m;
w[i,j] = w[i,j-1] + p[j] + q[j];
c[i,j] = 8;
for k in i+1 to j
{
    k = c[i,k-1] + c[k,j] + w[i,j];
    if (k < c[i,j])
    {
        e[i,j] = k;
        r[i,j] = k;
    }
}
write [c[0,n], w[0,n], r[0,n]])
}
}
}
```

* The time complexity is for these algorithm, is $O(n^3)$ time. Since Here nested for loops are used. Each of these loops takes an at most n-values.

0/1 Knapsack problem.

* Knapsack is nothing but bag.
* Here n-objects are given which have some weight & profit and bag is given, the bag have same max capacity say "m" we have to select those objects that gives the max profit for inserting into the knapsack. But we have to check the condition. The weight of the inserted object exceed the max capacity (or) not. it should not exceed the max capacity.

Ex:

n	w	P
1	2	1
2	3	2
3		3

$\Rightarrow x = $ | 1 | 0 | 1 |

N = 6

Objects - 1, 3

max profit = 4

* mathematically $\sum_{i=1}^{n} w_i x_i \le M$

$\sum_{i=1}^{n} n_i x_i$ is maximise

$\Rightarrow$ we have to find S^0, S^1, S^0, S^3

$\Rightarrow S \Rightarrow$ means Set which contains (p, w)

$\Rightarrow$ To find 's' values. we have two formula's :

$$\boxed{\begin{aligned} S^{i+1} &= S^i + S^i_1 \\ S^i_1 &= S^i + \{P_{i+1} + w_{i+1}\} \end{aligned}}$$

$\rightarrow$ Merge

$\rightarrow$ Addition.

find optimal solution

n	w	P
1	2	1
2	3	2
3	4	5

m = 6

$S^0 = \{P, w\} \Rightarrow S^0 = \{(0,0)\}$

<u>i=0</u>

$S^1 = S^{0+1} = S^0 + S^0_1$

$S^0_1 = S^0 + \{P_1, w_1\} = \{(0,0)\} + \{(1,2)\}$

$= \{(1,2)\}$

$S^1 = \{(0,0)(1,2)\}$

<u>i=1</u>

$S^{1+1} = S^1 + S^1_1$

$\Rightarrow S^1_1 = S^1 + \{P_2, w_2\} = \{(0,0)(1,2)\} + (2,3)$

$= \{(2,3)(3,5)\}$

$S^2 \Rightarrow \{ \underset{0}{\underline{(0,0)}} \underset{1}{\underline{(1,2)}} \underset{2}{\underline{(2,3)}} \underset{1+2}{\underline{(3,5)}} \}$

$\underline{\underline{i=2}}$

$S^{2+1} = S^2 + S_1^2$

$\Rightarrow S_1^2 = S^2 + (p_3, w_3) = \{(0,0)(1,2)(2,3)(3,5)\} + (5,4)$

$= \{(5,4)(6,6)\underset{\times}{(7,7)}\underset{\times}{(8,9)}\}$

$S^2 \Rightarrow \{\underset{0}{(0,0)}\ \underset{1}{(1,2)}\ \underset{2}{(2,3)}\ \underset{1+2}{(3,5)}\ \underset{3}{(5,4)}\ \underset{1+3}{(6,6)}\}$

$\Rightarrow x = \boxed{1 \mid 0 \mid 1}$ $\Rightarrow$ max profit $= 6$

$x_1 = 1$

$x_2 = 0$

$x_3 = 1$

* find optimal Solution

n	w	p
1	2	1
2	3	2
3	4	5

$m = 6.$

$S^0 = \{p, w\} \Rightarrow S^0 = \{(0,0)\}$

$\underline{\underline{i=0}}$

$S^1 = S^{0+1} = S^0 + S_1^0$

$S_1^0 = S^0 + \{P_1, w_1\} = \{(0,0)\} + \{(1,2)\}$

$= \{(1,2)\}$

$S^1 = \{(0,0)(1,2)\}$

$\underline{\underline{i=1}}$

$S^{1+1} = S^1 + S_1^1$

$S_1^1 = S^1 + \{P_2, w_2\} = \{(0,0)(1,2)\} + (2,3)$

$= \{(2,3)(3,5)\}$

$S^2 \Rightarrow \{(0,0)(1,2)(2,3)(3,5)\}$

$\underline{\underline{i=2}}$

$S^{2+1} = S^2 + S_1^2$

$\Rightarrow S_1^2 = S^2 + (p_3, w_3) = \{(0,0)(1,2)(2,3)(3,5)\} + (5,4)$

$= \{(5,4)(6,6)\underset{\times}{(7,7)}\underset{\times}{(8,9)}\}$

$S^3 \Rightarrow \{ \underset{0}{(0,0)}\ \underset{1}{(1,2)}\ \underset{2}{(2,3)}\ \underset{1+3}{(3,5)}\ \underset{3}{(5,4)}\ \underset{1+3}{(6,6)} \}$

$\Rightarrow$ max profit = 6

$\Rightarrow x = \boxed{1 \mid 0 \mid 1}$

$x_1 = 1$

$x_2 = 0$

$x_3 = 1$

Purging rule (or) Dominance rule :

Suppose $(p_j, w_j)\ (p_k, w_k)$

if $p_j \le p_k$ and $w_j \ge w_k$ then (p_j, w_j) can be eliminated.

Ex:- find the optimal solution.

n	P	W
1	1	2
2	2	3
3	5	4
4	6	5

$m = 8$.

Formula : $S^{i+1} = S^i + S_1^i$

$S_1^i = S^i + \{p_{i+1}, w_{i+1}\}$

$\Rightarrow S^0 = \{p, w\} = S^0 = \{(0,0)\}$.

$\underline{i=0}$

$S^{1+1} = S^0 + S_1^0 \Rightarrow S_1^0 = S^0 + \{p_1, w_1\}$

$= S^0 + (1,2) = \{(0,0) + (1,2)\}$

$\Rightarrow S^1 = \{(0,0), (1,2)\} = \{(1,2)\}$

Purging rule

$= \{(0,0)\ (1,2)\}$

$p_j w_j \quad p_k w_k$

$= \underset{T}{0 \le 1}$ & $\underset{F}{0 \ge 2} \rightarrow F$

$\underline{i=1}$

$S^{1+1} = S^1 + S_1^1$

$S_1^1 = S^1 + \{p_2, w_2\} = \{(0,0)\ (1,2)\} + (2,3)$

$= \{(2,3), (3,5)\}$

$S^2 \Rightarrow \{(0,0)\ (1,2)\ (2,3)\ (3,5)\}$.

Purging rule

$= (1,2)\ (2,3)$

$p_j w_j \quad p_k w_k$

$\underset{T}{1 \le 2}$ & $\underset{F}{2 \ge 3} \rightarrow F$

$= (1,2)\ (3,5)$, $(2,3)\ (3,5)$

$p_j w_j\ p_k w_k \quad p_j w_j\ p_k w_k$

$\underset{T}{1 \le 3}$ $\underset{F}{2 \ge 5} \rightarrow F$ $\quad \underset{T}{2 \le 3}$ & $\underset{F}{3 \ge 5} \rightarrow F$

$\underline{i=2}$

$S^{2+1} = S^2 + S_1^2$

$\Rightarrow S_1^2 = S^2 + \{P_3, w_3\} = \{(0,0)(1,2)(2,3)(3,5)\} + (5,4)$

$= \{(5,4)(6,6)(7,7)(8,9)\}$

$S^2 = \{(0,0)(1,2)(2,3)(3,5)(5,4)(6,6)(7,7)(8,9)\}$

$\underline{i=3}$

$S^{3+1} = S^3 + S_1^3$

$\Rightarrow S_1^3 = S^3 + \{P_4, w_4\}$

$= \{(0,0)(1,2)(2,3)(3,5)(5,4)(6,6)(7,7)(8,9)\} + (6,5)$

$\Rightarrow \{(6,5),(7,7)(8,8)(9,10)(11,11)(12,11)(13,12)(14,14)\}$

$S^4 = \{(0,0)(1,2)(2,3)(3,5)(5,4)(6,6)(7,7)(6,5)(7,7)(8,8)\}$

$\Rightarrow$ max profit = 8 $[8,8]$

$\Rightarrow x = \boxed{0\,|\,1\,|\,0\,|\,1}$

$x_1 = 0, \; x_2 = 1, \; x_3 = 0, \; x_4 = 1.$ ← vectors.

purging rule : $\underline{i=2}$

$= (3,5)(5,4) \rightarrow (2,3)(5,4)$

$P_j w_j \quad P_K w_K$

$3 \le 5$ & $5 \ge 4 \rightarrow T$ $\quad 2 \le 5$ & $3 \ge 4 \rightarrow F$

T T T T

$\rightarrow (1,2)(5,4)$ $\quad \rightarrow (5,4)(6,6)$

$P_j w_j \quad P_K w_K$ $\quad P_j w_j \quad P_K w_K$

$1 \le 5$ & $2 \ge 4 \rightarrow F$ $\quad 5 \le 6$ & $4 \ge 6 \rightarrow F$

T F T F

$\rightarrow (6,6)(7,7)$ $\quad \rightarrow (8,9)$ is the Exceed weight.

$P_j w_j \quad P_K w_K$ $\quad$ so $(8,9)$ will be eliminated

$6 \le 7$ & $6 \ge 7 \rightarrow F$

T F

→ (3,5) element can be eliminated.

$S^3 = \{(0,0)(1,2)(2,3)(5,4)(6,6)(7,7)\}$
$\quad\quad 0 \quad 1 \quad 2 \quad 3 \quad 1+3 \quad 2+3.$

purging rule : $i=3$

→ (6,6) & (6,5) → (5,4) (6,5)
$p_j\, w_j \quad p_K\, w_K \qquad p_j\, w_j \quad p_K\, w_K$

$6 \le 6$ & $6 \ge 5 \to T$ $\qquad 5 \le 6$ & $4 \ge 5 \to F$
T T $\qquad$ T F.

→ (2,3) (6,5) → (1,2) (6,5)
$2 \le 6$ & $3 \ge 5 \to F$ $\qquad 1 \le 6$ & $2 \ge 5 \to F$
T F $\qquad$ T F

→ (6,5) (7,7) → (5,4) (7,7)
$6 \le 7$ & $5 \ge 7 \to F$ $\qquad 5 \le 7$ & $4 \ge 7 \to F$
T F $\qquad$ T F

→ (2,3) (7,7) → (1,2) (7,7)
$2 \le 7$ & $3 \ge 7 \to F$ $\qquad 1 \le 7$ & $2 \ge 7 \to F$
T F $\qquad$ T F

→ (7,7) (8,8) → (6,5) (8,8)
$7 \le 8$ & $7 \ge 8 \to F$ $\qquad 6 \le 8$ & $5 \ge 8 \to F$
T F $\qquad$ T F.

→ {(11,9) (12,11) (13,12)} These elements are Exceeded weight so these elements will be eliminated.

→ (6,6) element can be eliminated

$S^4 = \{(0,0)(1,2)(2,3)(5,4)(6,5)(7,7)(8,8)\}$
$\quad\quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 2+3 \quad 2+4$

→ The maximum profit 8 by (2,4)

$x_1 = 0, \; x_2 = 1, \; x_3 = 0, \; x_4 = 1$ ← vectors.

Reliability design :

Ex:-

D	C	r
D_1	30	0.9
D_2	15	0.8
D_3	20	0.5

$C = 100$

→ find Reliability of the system
→ and no.of copies that are maintained for Each device.

Sol :

D	C	r	U
D_1	30	0.9	2
D_2	15	0.8	3
D_3	20	0.5	2

⇒ we have to find S^0 —— S^3

$S^0 = \{(r, c)\} = \{(1, 0)\}$

S^i = Device number
i = no.of Copies.

$S^1 = S_1^1 \cup S_2^1$

$\Rightarrow S_1^1 = \{(0.9, 30)\}$

$S_2^1 = \{(0.99, 60)\}$ $[\because 1-[1-r_i)^2]$

$S^1 = \{(0.9; 30); (0.99, 60)\}$
$= \{\underline{(0.9, 30)}, (0.99, 60)\}$

$S^2 = S_1^2 \cup S_2^2 \cup S_3^2$

$S_1^2 = \{(0.8, 15)\} = \{(0.72, 45), (0.792, 75)\}$

$S_2^2 = \{(0.96, 30)\} = \{\underline{(0.864, 60)}, (0.9504, 90)\} \rightarrow \times$

$S_3^2 = \{(0.992, 45)\} = \{(0.8928, 75), (0.98208, 105)\} \rightarrow \times$

$S^2 = \{(0.72, 45), (0.792, 75), (0.864, 60), (0.89, 75)\}$
×

$S^2 = S_1^3 \cup S_2^3$

$S_1^3 = \{(0.5, 20)\} = \{(0.36, 65), (0.432, 80), (0.445, 95)\}$

$S_2^3 = \{0.75, 40)\} = \{(0.54, 85), \underline{(0.648, 100)}, (0.6675, 115)\}$
×

$S^3 = \{(0.36, 65), (0.432, 80), \underset{\times}{(0.445, 95)}\ (0.54, 85)$
$(0.648, 100)\}$

optimal pair $= (0.648, 100)$

The reliability of the system is 0.648

Cost is 100.

$\Rightarrow$ check :-

	D_1	D_2	D_3
Copy $\leftarrow$	1	2	3

$C \leftarrow 30 + 30 + 40 = 100$

$r \leftarrow 0.9 \times 0.96 \times 0.75 = 0.648.$

Ex-2 : $C = 105$

D	C	r	v
D_1	30	0.9	2
D_2	20	0.8	3
D_3	15	0.5	3

Here $S^0 = \{(1,0)\}$

$S^1 = S_1^1 \cup S_2^1$ where

$S_1^1 = \{(0.9, 30)\}$, $S_2^1 = \{(0.99, 60)\}$

$\therefore S^1 = \{(0.9; 30), (0.99, 60)\}$

$S^2 = S_1^2 \cup S_2^2 \cup S_3^2$, where

$S_1^2 = \{(0.8, 20)\} = \{(0.72, 50), (0.792, 80)\}$

$S_2^2 = \{(0.96, 40)\} = \{(0.864, 70), (0.9504, 100)\}$

$S_3^2 = \{(0.992, 60)\} = \{0.89, 90), (0.98, 120)\}$

Here $(0.9504, 100), (0.98, 120)$ Can be eliminated

$S^2 = \{(0.72, 50), (\boxed{0.792, 80}), (0.864, 70), (0.89, 90)\}$

Here $(0.792, 80)$ Can be eliminated.

$S^2 = \{(0.72, 50), (0.864, 70), (0.89, 90)\}$

$S^3 = S_1^3 \cup S_2^3 \cup S_3^3$, where.

$S_1^3 = \{(0.5, 15)\} = \{(0.36, 65), (0.43, 85), (0.445, 105)\}$

$S_2^3 = \{(0.75, 30)\} = \{(0.54, 80), (0.648, 100), (0.66, \cancel{120})\}$

$S_3^3 = \{(0.875, 45)\} = \{(0.63, 95), (0.\cancel{756}, 115), (\cancel{0.778}, 135)\}$

Here $(0.66, 120)$, $(0.756, 115)$, $(0.778, 135)$ Can be eliminated.

$\therefore S^3 = \{(0.36, 65), (0.43, 85), (0.445, 105), (0.54, 80)$
$(0.648, 100), (0.63, 95)\}$.

Here $(0.43, 85)$ $(0.44, 105)$ Can be eliminated.

$\therefore S^3 = \{(0.36, 65), (0.54, 80)$ $(0.648, 100), (0.63, 95)\}$

$\therefore$ Reliability of the system is 0.648

Cost is 100

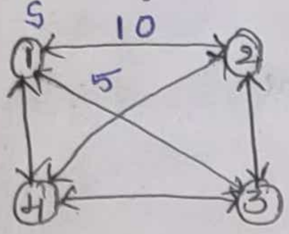
optimal pair = $(0.648, 100)$

Algorithm for 0/1 Knapsack : [Recursine]

```
int knapsack (int W, int wt[], int val[], int n)
{
    if (n==0 || W==0)
    return 0;
  if (wt[n-1] > W)
        return knapsack (W, wt, val, n-1);
  else
        return max (
            val [n-1]
                + knapsack [W - wt[n-1],
                    wt, val, n-1),
          knapsack (W, wt, val, n-1);
}
```

* Time Complexity for 0/1 knapsack algorithm is $O(2^n)$

Travelling Salesman problem:



1 → 2 → 3 → 4 → 1

1 2 3 4 1   1 3 4 2 1

1 2 4 3 1   1 3 2 4 1

1 4 3 2 1

1 4 2 3 1

- → No city is visited more than once Source node Twice
- → every city should be Covered
- → After visiting every city, he should return to his native place.

Aim: To suggest the Best path which will take minimum Cost

total possibilities are 6:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 10 & 15 & 20 \\ 2 & 5 & 0 & 9 & 10 \\ 3 & 6 & 13 & 0 & 12 \\ 4 & 8 & 8 & 9 & 0 \end{array}$$

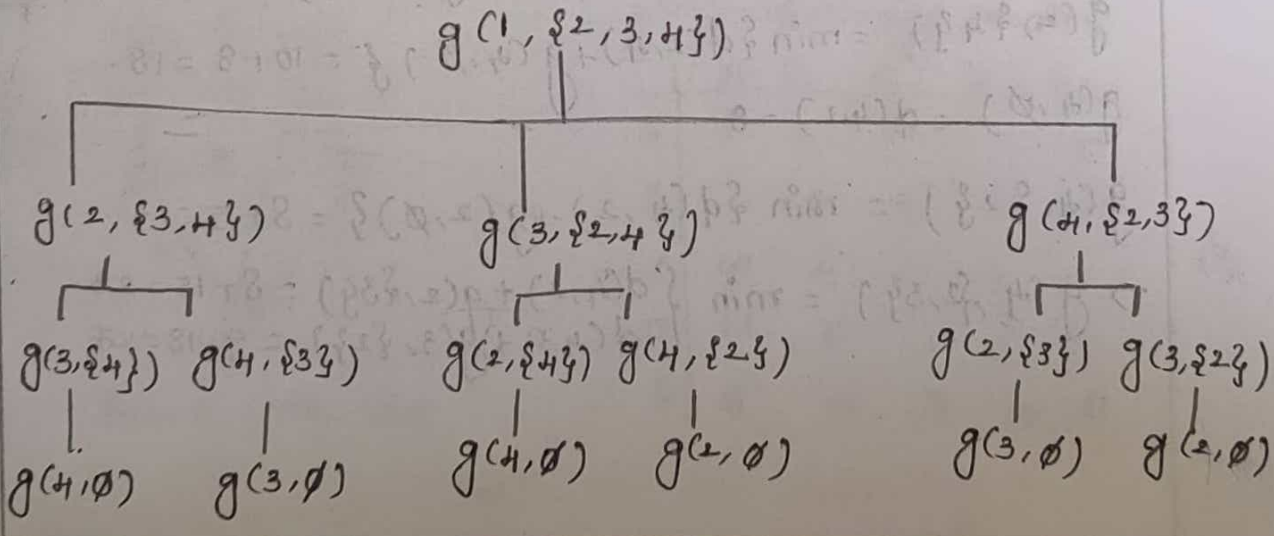
$1 \xrightarrow{10} 2 \xrightarrow{9} 3 \xrightarrow{12} 4 \xrightarrow{8} 1 \rightarrow 39$

$1 \xrightarrow{10} 2 \xrightarrow{10} 4 \xrightarrow{9} 3 \xrightarrow{6} 1 \rightarrow 35$

n – nodes   1 – – – 1

Time complexity : (n-1)

$(n-1)! = O(n!)$



total 15 Subproblems :

$\rightarrow$ 3 are repeated

$\therefore$ There are 12 Unique Subproblems

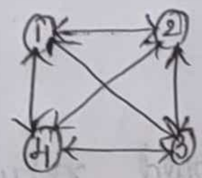
n - nodes

$(n-1)\, 2^{n-2}$ =

$n \cdot (n-1)\, 2^{n-2} = O(n^2 \cdot 2^n)$: time Complexity

Np-hard problem.

Ex :



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 10 & 15 & 20 \\ 2 & 5 & 0 & 9 & 10 \\ 3 & 6 & 13 & 0 & 12 \\ 4 & 8 & 8 & 9 & 0 \end{array}$$

Formula :

$g(i, S) = \min \{ d(i,j) + g(j, \{S - \{j\}\}) \}, \; j \in S.$

$$g(1, \{2,3,4\}) = \min \begin{cases} d(1,2) + g(2, \{3,4\}) = 10+25 = 35 \\ d(1,3) + g(3, \{2,4\}) = 15+25 = 40 \\ d(1,4) + g(4, \{2,3\}) = 20+23 = 43 \end{cases}$$

$$\rightarrow g(2, \{3,4\}) = \min \begin{cases} d(2,3) + g(3, \{4\}) = 9+20 = 29 \\ d(2,4) + g(4, \{3\}) = 10+15 = 25 \end{cases}$$

$g(3, \{4\}) = d(3,4) + g(4, \emptyset) = 12+8 = 20$

$g(4, \emptyset) = d(4,1) = 8$

$g(4, \{3\}) = d(4,3) + g(3, \emptyset) = 9+6 = 15$

$$\rightarrow g(3, \{2,4\}) = \min \begin{cases} d(3,2) + g(2, \{4\}) = 13+18 = 31 \\ d(3,4) + g(4, \{2\}) = 12+13 = 25 \end{cases}$$

$g(2, \{4\}) = \min \{ d(2,4) + g(4, \emptyset) \} = 10+8 = 18.$

$g(4, \emptyset) = d(4,1) = 8$

$g(4, \{2\}) = \min \{ d(4,2) + g(2, \emptyset) \} = 8+5 = 13.$

$$\rightarrow g(4, \{2,3\}) = \min \begin{cases} d(4,2) + g(2, \{3\}) = 8+15 = 23 \\ d(4,3) + g(3, \{2\}) = 9+18 = 27 \end{cases}$$

$g(2, \{3\}) = d(2,3) + g(3, \emptyset) = 9 + 6 = 15$

$g(3, \emptyset) = d(3,1) = 6$

$g(3, \{2\}) = d(3,2) + g(2, \emptyset) = 13 + 5 = 18$

$g(2, \emptyset) = d(2,1) = 5.$

Algorithm for Travelling Salesman problem :

if $|S| = 2$ then

{

S must be $\{1, i\}$

$C(S, i) = d(1, i)$

}

else

{

If $|S| > 2$

$C(S, i) = \min \{ C(S - \{i\}, j) + d(j, 1) \}$

}

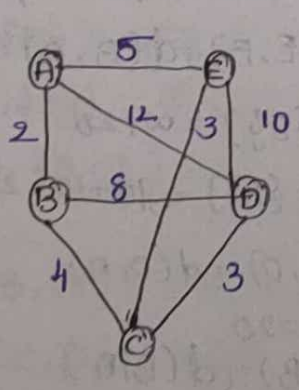
Where $j \in S$, $j != i$ and $j != 1$

Time Complexity :

Time Complexity for Travelling Salesman problem

$$O(n^2 - 2^2)$$

Ex :



Source node is A

Cost adjacency matrix

	A	B	C	D	E
A	0	2	∞	12	5
B	2	0	4	8	∞
C	∞	4	0	3	3
D	12	8	3	0	10
E	5	∞	3	10	0

Sol: Formula $= g(i, S) = \min \{ d(i,j) + g(j, \{S - \{j\}\}) \}, j \in S$

$\rightarrow g(A, \{B,C,D,E\}) = \min \begin{cases} d(A,B) + g(B, \{C,D,E\}) = 2+19 = 21 \\ d(A,C) + g(C, \{B,D,E\}) = \cancel{09}\,23 \\ d(A,D) + g(D, \{B,C,E\}) = 12+19 = 31 \\ d(A,E) + g(E, \{B,C,D\}) = 5+16 = 21 \end{cases}$

$\rightarrow g(B, \{C,D,E\}) = \min \begin{cases} d(B,C) + g(C, \{D,E\}) = 4+18 = 22 \\ d(B,D) + g(D, \{C,E\}) = 8+11 = 19 \\ d(B,E) + g(E, \{C,D\}) = \infty + 18 \end{cases}$

$\rightarrow g(C, \{D,E\}) = \min \begin{cases} d(C,D) + g(D, \{E\}) = 3+15 = 18 \\ d(C,E) + g(E, \{D\}) = 3+22 = 25 \end{cases}$

$g(D, \{E\}) = d(D,E) + g(E, \emptyset) = d(D,E) + d(E,A) = 10+5 = 15.$

$\rightarrow g(C, \{B,D,E\}) = \min \begin{cases} d(C,B) + g(B, \{D,E\}) = 4+23 = 27 \\ d(C,D) + g(D, \{B,E\}) = 3+12+\infty = 15+\infty \\ d(C,E) + g(E, \{B,D\}) = 3+20 = 23. \end{cases}$

$\rightarrow g(B, \{D,E\}) = \min \begin{cases} d(B,D) + g(D, \{E\}) = 8+15 = 23 \\ d(B,E) + g(E, \{D\}) = \infty + 22 \end{cases}$

$g(D, \{E\}) = d(D,E) + g(E, \emptyset) = d(D,E) + d(E,A)$
$= 10+5 = 15$

$g(E, \{D\}) = d(E,D) + g(D, \emptyset) = d(E,D) + d(D,A)$
$= 10+12 = 22.$

$\rightarrow g(D, \{B,E\}) = \min \begin{cases} d(D,B) + g(B, \{E\}) = 8+5+\infty = 13+\infty \\ d(D,E) + g(E, \{B\}) = 10+\infty+2 = 12+\infty \end{cases}$

$g(B, \{E\}) = d(B,E) + g(E, \emptyset) = d(B,E) + d(E,A) = \infty + 5$

$g(E, \{B\}) = d(E,B) + g(B, \emptyset) = d(E,B) + d(B,A) = \infty + 2$

$\rightarrow g(E, \{B,D\}) = \min \begin{cases} d(E,B) + g(B, \{D\}) = \infty + 20 \\ d(E,D) + g(D, \{B\}) = 10+10 = 20. \end{cases}$

$g(B, \{D\}) = d(B,D) + g(D, \emptyset) = d(B,D) + d(D,A)$
$= 8+12 = 20$

$g(D, \{B\}) = d(D,B) + g(B, \emptyset) = d(D,B) + d(B,A)$
$= 8+2 = 10.$

$\therefore g(C, \{B,D,E\}) = 23 \rightarrow (2)$

$\rightarrow g(D,\{B,C,E\}) = \min \begin{cases} d(D,B) + g(B,\{C,E\}) = 8+12 = 20 \\ d(D,C) + g(C,\{B,E\}) = 3+5+\infty = 8+\infty \\ d(D,E) + g(E,\{B,C\}) = 10+9 = 19 \end{cases}$

$\rightarrow g(B,\{C,E\}) = \min \begin{cases} d(B,C) + g(C,\{E\}) = 4+8 = 12 \\ d(B,E) + g(E,\{C\}) = \infty + 3 + \infty \end{cases}$

$g(C,\{E\}) = d(C,E) + g(E,\emptyset) = d(C,E) + d(E,A)$
$= 3+5 = 8.$

$g(E,\{C\}) = d(C,E) + g(C,\emptyset) = d(E,C) + d(C,A)$
$= 3+\infty.$

$\rightarrow g(C,\{B,E\}) = \min \begin{cases} d(C,B) + g(B,\{E\}) = 4+5+\infty = 9+\infty \\ d(C,E) + g(E,\{B\}) = 3+2+\infty = 5+\infty \end{cases}$

$g(B,\{E\}) = d(B,E) + g(E,\emptyset) = d(B,E) + d(E,A) = \infty + 5$

$g(E,\{B\}) = d(E,B) + g(B,\emptyset) = d(E,B) + d(B,A) = \infty + 2$

$\rightarrow g(E,\{B,C\}) = \min \begin{cases} d(E,B) + g(B,\{C\}) = \infty + 4 + \infty \\ d(E,C) + g(C,\{B\}) = 3+6 = 9 \end{cases}$

$g(B,\{C\}) = d(B,C) + g(C,\emptyset) = d(B,C) + d(C,A) = 4+\infty$

$g(C,\{B\}) = d(C,B) + g(B,\emptyset) = d(C,B) + d(B,A) = 4+2 = 6.$

$\therefore g(D,\{B,C,E\}) = 19 \rightarrow$ (3)

$\rightarrow g(E,\{B,C,D\}) = \min \begin{cases} d(E,B) + g(B,\{C,D\}) = \infty + 19. \\ d(E,C) + g(C,\{B,D\}) = 3+13 = 16. \\ d(E,D) + g(D,\{B,C\}) = 10+9 = 19. \end{cases}$

$\rightarrow g(B,\{C,D\}) = \min \begin{cases} d(B,C) + g(C,\{D\}) = 4+15 = 19 \\ d(B,D) + g(D,\{C\}) = 8+3+\infty = 11+\infty \end{cases}$

$g(C,\{D\}) = d(C,D) + g(D,\emptyset) = d(C,D) + d(D,A) = 3+12 = 15$

$g(D,\{C\}) = d(D,C) + g(C,\emptyset) = d(D,C) + d(C,A) = 3+\infty$

$\rightarrow g(C,\{B,D\}) = \min \begin{cases} d(C,B) + g(B,\{D\}) = 4+20 = 24 \\ d(C,D) + g(D,\{B\}) = 3+10 = 13. \end{cases}$

$g(B,\{D\}) = d(B,D) + g(D,\emptyset) = d(B,D) + d(D,A) = 8+12 = 20.$

$g(D,\{B\}) = d(D,B) + g(B,\emptyset) = d(D,B) + d(B,A) = 8+2 = 10.$

$\rightarrow g(D,\{B,C\}) = \min \begin{cases} d(D,B) + g(B,\{C\}) = 8+4+\infty = 12+\infty \\ d(D,C) + g(C,\{B\}) = 3+6 = 9. \end{cases}$

$g(B,\{C\}) = d(B,C) + g(C,\emptyset) = d(B,C) + d(C,A) = 4+\infty$

$g(C,\{B\}) = d(C,B) + g(B,\emptyset) = d(C,B) + d(B,A) = 4+2 = 6.$

$\therefore g(E,\{B,C,D\}) = 16 \longrightarrow$ (4)

$\therefore g(A,\{B,C,D,E\}) = \min(21, \infty, 31, 21) = 21$

Cost=21 and

paths : 1) A-B-D-C-E-A = 2+8+3+3+5 = 21

2) A-E-C-D-B-A = 5+3+3+8+2 = 21