Periodic motion:

The motion which repeats at regular intervals of time is called "periodic motion".

Oscillatory motion:

The periodic motion in which a body moves to and fro about a fixed point is called "oscillatory motion".

Time period: (T)

The smallest interval of time after which the motion is repeated is called its "period". It is denoted by T. It's S.I unit is 'second'.

Frequency:

The reciprocal of time period (T) give number of repetitions that occur per unit time. This ~~quantity~~ is called "frequency of periodic motion". It is denoted by ν (or) f. It's S.I unit is sec^{-1} (or) Hertz.

1 Hertz = 1 oscillation per second.

Displacement as a function of time:

The distance from starting point as a function of time is its position. The change in position with respect to time can be represented as displacement.

Restoring force

Consider a uniform, massless spring in its unstretched state. When the spring is stretched (or compressed) the restoring force developed in the spring tries to bring the spring back to its Equilibrium position. The restoring force developed is directly proportional to the elongation (or compression) produced in the spring.

If 'F' is the restoring force on producing an elongation (or) compression 'x' in the spring, then

$$F \propto x$$

$$\boxed{F = -kx.}$$

The negative sign indicates, that the restoring force is in a direction opposite to 'x'. Here 'k' is a constant for a given spring and is called force constant (or) spring constant (or) stiffness constant of the spring. It is given by

$$\boxed{k = \frac{F}{x}}$$

$1N = 10^3 \, dyne/cm = 10^5 \, dyne/m$

S.I unit → N/m

C.G.S unit → $dyne/cm$.

Dimensional formula $= MT^{-2}$.

Simple Harmonic motion:

"A body is said to be in simple Harmonic motion, if it moves to and fro along a straight line, about its mean position such that, at any point its acceleration is directly proportional to its displacement in magnitude but opposite in direction and is directed always towards the mean position."

If 'a' is the acceleration of the body at any given displacement 'x' from the mean position, then for the body is said to be in SHM,

$$a \propto -x$$

$$\boxed{a = -bx}$$

Where 'b' is constant of propostionality and negative sign indicates 'a' and 'x' are always in opposite directions.

There are two types of simple Harmonic motion.

(1) Linear simple harmonic motion

(i) oscillations of a liquid column in a U-tube.

(ii) oscillations of a swing with small amplitude.

(iii) oscillations of a loaded spring.

(2) Angular simple Harmonic motion.

(i) oscillations of a balance wheel in a watch

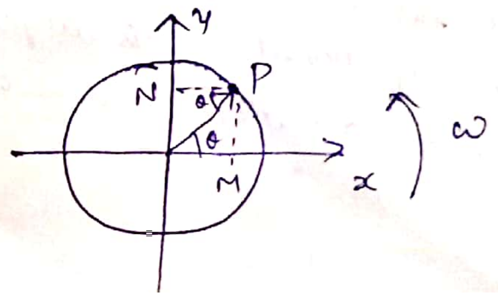
(ii) oscillations of a torsional pendulum.

(iii) oscillations of a freely suspended magnet in a uniform magnetic field.

Characterstics of SHM:

(i) Displacement:

"The shortest distance between ~~to~~ mean position and position of a particle at any instant is called displacement of the particle in SHM."



From the reference circle the displacement of 'N' at any instant of time 't' is given by

$ON = A\sin\theta$. $\left(\sin\theta = \frac{ON}{OP} \text{ when } OP = A\right)$

$\boxed{y = A\sin\omega t}$. $\left(\because \theta = \omega t,\ OP = A\right)$

$\because$ In SHM displacement of particle varies sinusoidally with time. when particle is at mean position $y_{min} = 0$.

when particle is at extreme position $y_{max} = A$.

(ii) <u>Amplitude:</u>

The maximum displacement of the particle executing SHM on either side of the mean position is known as the Amplitude of SHM.

Amplitude $y_{max} = A$.

(iii) <u>Velocity:</u>

"The rate of change of displacement is called velocity."

The velocity of particle in SHM is given by

$$V = \frac{dy}{dt}.$$

$$V = \frac{d}{dt}(A\sin\omega t)$$

$$= A\cos\omega t\,(\omega)$$

$$\boxed{V = A\omega\cos\omega t}$$

$$V = A\omega\sqrt{1-\sin^2\omega t}$$

$$V = A\omega\sqrt{1-\frac{y^2}{A^2}}$$

$$V = \cancel{A}\omega\frac{\sqrt{A^2-y^2}}{\cancel{A}} = \omega\sqrt{A^2-y^2}$$

Maximum Velocity at $y = 0$.

5

$$V_{max} = \omega\sqrt{A^2 - 0^2}$$

$$\boxed{V_{max} = A\omega}$$

Minimum Velocity at $y = A$

$$V_{min} = \omega\sqrt{A^2 - A^2}$$

$$\boxed{V_{min} = 0}$$

(iv) Acceleration:

"The rate of change of velocity of a particle with time is called acceleration"

The acceleration of a particle in SHM is given by

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}(A\omega\cos\omega t)$$

$$a = -A\omega^2 \sin\omega t$$

$$\boxed{a = -\omega^2 y} \qquad (\because y = A\sin\omega t)$$

$$a \propto -y$$

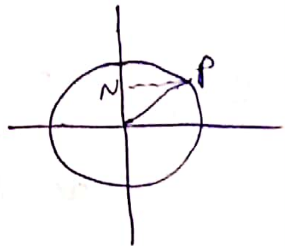
Thus the accn is directly as well as negative direction of the displacement.

(v) Time period:

The time taken by a particle in SHM to complete one oscillation about mean position is called time period of SHM.

In the reference circular motion, the time for one revolution of particle P is

Equal to the period of oscillation of foot of the perpendicular 'N' in SHM



$$\boxed{\text{Time period} = T = \frac{2\pi}{\omega}}$$

where 'ω' is the angular frequency

We know that from acceleration

$$a = -\omega^2 y.$$

$$\omega = \sqrt{\frac{a}{y}} \quad \text{(magnitude only)}$$

$$\frac{2\pi}{T} = \sqrt{\frac{a}{y}}.$$

$$T = 2\pi\sqrt{\frac{y}{a}}.$$

$$\boxed{T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.}$$

(Vi) Frequency:

"The number of oscillations completed by a particle in SHM in one second is called frequency."

$$\text{Frequency} = \frac{1}{T}.$$

$$\boxed{n = \frac{1}{T} \Rightarrow \frac{1}{2\pi}\sqrt{\frac{a}{y}}}$$

Since $\omega = \frac{2\pi}{T}$

$$\omega = \frac{2\pi}{1/n} = 2\pi n.$$

'ω' is also known as angular frequency.

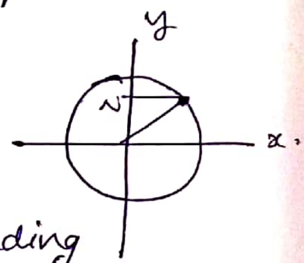
SI unit of frequency 'n' is hertz (Hz)

SI unit of angular frequency (ω) is rad/sec.

(vii) Phase :

The phase of a particle Executing SHM at any instant is defined, as its state or condition as regards its position and direction of motion at that instant.

Let us consider the motion of 'N' on the 'Reference circle'. During the course of one oscillation, Each time when it crosses a point, the condition of its vibration, is given by the displacement, Velocity and acceleration at that point at that instant.



These characteristics of vibration depend on the corresponding angular displacement with respect to mean position, and this angle is known as the phase or phase angle of the particle.

$$\boxed{\text{phase (or) phase angle } \theta = \omega t = 2\pi\left(\frac{t}{T}\right)}$$

The phase of the particle may also be given in terms of time 't', that has Elapsed since the particle last passed its mean position, in the positive direction, measured as a function of its time period "T".

When $t = T$ (or) $\theta = 2\pi$ the End of Every one period T, the particle returns to the same phase, by describing a phase angle '2π'.

(VIII) Epoch (ϕ)

The starting phase of oscillation is called 'Epoch'

(or)

The phase of oscillation of a particle at $t=0$ is called 'Epoch'.

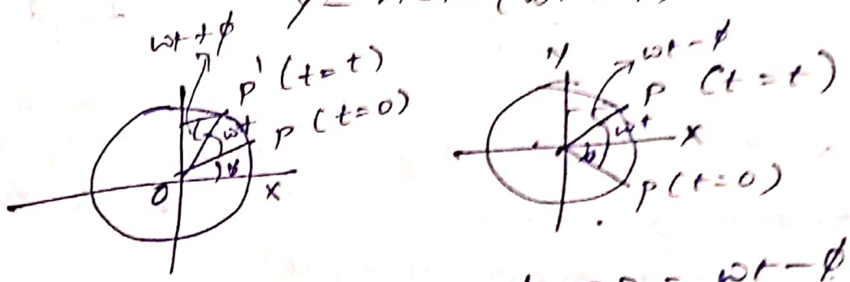
Since all the phases of vibration repeat, Exactly at Equal intervals of time period T, the oscillations can be followed starting from any phase angle.

If a SHM is followed starting from the mean position at $t=0$, the initial phase angle (Epoch) is Equal to zero. The displacement Equation of such a particle can be written as

$$y = A \sin\theta$$
$$y = A \sin\omega t$$

If the oscillations are followed from any initial phase or Epoch ϕ at $t=0$, the Equation of motion of the SHM can be written as

$$y = A \sin(\omega t \pm \phi)$$



$$\angle XOP = \omega t - \phi.$$

$$\angle XOP' = \omega t + \phi$$

If the oscillations are followed from the positive Extreme position at $t=0$, the corresponding Epoch is Equal to $\pi/2$ and the SHM can be represented by the Equation

$$y = A \sin(\omega t + \pi/2) = A \cos\omega t.$$

Similarly if the oscillations are followed from the negative Extreme position at $t=0$, the corresponding Epoch is Equal to $-\pi/2$ and the Equation of motion of the SHM can be written as

$$y = A \sin(\omega t - \pi/2)$$
$$= -A \cos\omega t.$$

Simple Harmonic oscillator:

The most common Example of harmonic oscillation is a mass on a spring. For a simple Harmonic oscillator, Consider a Block of mass 'm' attached to a light spring of force constant (k) as shown in fig(a). The block is kept on a smooth horizontal surface and initially the spring is in its natural length.

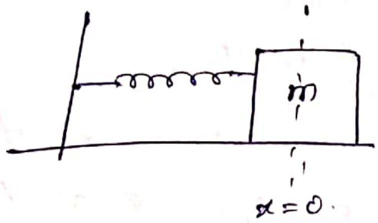


fig (a)

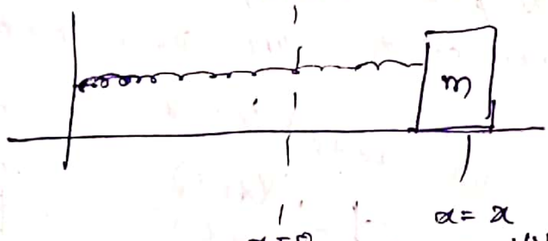


fig (b)

If the block is displaced from its Equilibrium position towards right, the restoring force due to the spring will act as shown in fig (b).

From Newton's IInd law

We know that

$$F_{net} = -kx$$

$$\Rightarrow \quad ma = -kx \quad \text{—— (1)}$$

Where $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

$$\therefore \quad m\frac{d^2x}{dt^2} = -kx.$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} \cdot x = 0.$$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{—— (2)}$$

Where $\omega^2 = \frac{k}{m}$ (or) $\omega = \sqrt{\frac{k}{m}}$.

This is the ~~Eqn~~ Second-order differential Equation of simple harmonic oscillation. (10)

Solution of Simple Harmonic oscillator:

Let us assume that the sine or Cosine functions are the solutions of the simple Harmonic oscillator.

Such that $x = A\cos\omega t$ (or) $A\sin\omega t$
where 'A' is the arbitrary constant sometimes it is also called as Amplitude of the wave function.

$$x = A\cos\omega t \quad \text{—— (3)}$$

$$\frac{dx}{dt} = A\frac{d}{dt}(\cos\omega t)$$

$$= A\omega(-\sin\omega t)$$

$$\frac{d^2x}{dt^2} = -A\omega\frac{d}{dt}(\sin\omega t)$$

$$= -A\omega(\omega\cos\omega t)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos\omega t \quad \text{—— (4)}$$

Substitute (3) & (4) values in Equation (2).

$$\therefore -A\omega^2\cos\omega t + \omega^2(A\cos\omega t)$$

$$= 0$$

$\therefore$ L.H.S and R.H.S ~~are~~ of the Equation (2) are Equal and Satisfied.

$\therefore$ The solution of the differential (second order) Equation is

~~A~~ $\boxed{x = A\cos\omega t}$ (or)

$$\boxed{x = \frac{A}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)}$$

Time period and frequency of the Simple Harmonic oscillator

We know $\omega^2 = \frac{k}{m}$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$ Angular frequency

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$ Time period.

$$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$ frequency.

Energy in SHM

Potential Energy: ~~Consider~~ When a particle ~~of~~ ~~no~~ Executing Simple Harmonic oscillations, the restoring force also increases

i.e $F = ma$

$-kx = ma$

$(\because a \propto -x$ and $\frac{k}{m} = \omega^2$ [proportionality Constant and angular frequency])

When the particle undergoes a displacement then the potential Energy stored in the simple harmonic oscillator is nothing but the work done on the system due to displacement is:

$$W = \int_0^x dW = \int_0^x \underset{\text{force}}{kx}\, \underset{\text{displacement}}{dx}$$ (only magnitude)

$$\boxed{W = \frac{1}{2}kx^2}$$

This work done is associated with the particle as potential Energy.

$$\therefore P.E = \frac{1}{2}kx^2$$

$$\boxed{P.E = \frac{1}{2}m\omega^2 x^2} \quad (1)$$

Kinetic Energy: If the particle has velocity 'v' at any position then

$$v = \omega\sqrt{A^2 - x^2}$$

The kinetic Energy of the particle is given by

$$K.E = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\omega^2 (A^2 - x^2)\right)$$

$$\boxed{K.E = \frac{1}{2}m\omega^2 (A^2 - x^2)} \quad (2)$$

$\therefore$ The total Energy of the system is

$$E = P.E + K.E$$

$$= \cancel{\frac{1}{2}m\omega^2 x^2} + \frac{1}{2}m\omega^2 A^2 - \cancel{\frac{1}{2}m\omega^2 x^2}$$

$$\boxed{E = \frac{1}{2}m\omega^2 A^2}$$

$$E \propto A^2$$

Where 'A' is the amplitude of the oscillator which remains constant for a simple harmonic oscillator.

Damped oscillations: The oscillations of a body whose amplitude goes on decreasing with time are defined as damped oscillations.

Differential Equation of a damped oscillations (13)

If 'v' is the velocity of the oscillator then damping force $F_d = -bv$ where 'b' is damping constant.

The resulting force acting on damping harmonic oscillator is

$$\Sigma F = F_{restoring} + F_{damping}$$

$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\cdot\frac{dx}{dt} + \frac{K}{m}x = 0 \quad \text{--- (1)}$$

Let $\omega_0^2 = \frac{k}{m}$ and $2\gamma = \frac{b}{m}$.

Then the differential equation can be written as.

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0} \quad \text{--- (2)}$$

Solution of the Equation

Let us assume that $x = Ae^{\alpha t}$ is the arbitrary solution of the differential Equation.

$$\frac{dx}{dt} = A(\alpha)e^{\alpha t} \quad \text{--- (3)}.$$

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t} \quad \text{--- (4)}.$$

Substitute (3) & (4) in (1)

$$\Rightarrow A\alpha^2 e^{\alpha t} + 2\gamma A\alpha e^{\alpha t} + \omega_0^2 (Ae^{\alpha t})$$

$$\Rightarrow Ae^{\alpha t}\left[\alpha^2 + 2\gamma\alpha + \omega_0^2\right]$$

$Ae^{\alpha t} = 0$ (or) $\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$.

$\Rightarrow$ since $Ae^{\alpha t} \neq 0$.

$\therefore \alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$

$$\alpha = \frac{-(2\gamma) \pm \sqrt{(2\gamma)^2 - 4(\omega_0^2)(1)}}{2(1)}$$

$$\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\therefore \quad x = Ae^{\left(-\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\right)t}$$

or

$$\boxed{x = A_1 e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t}}$$

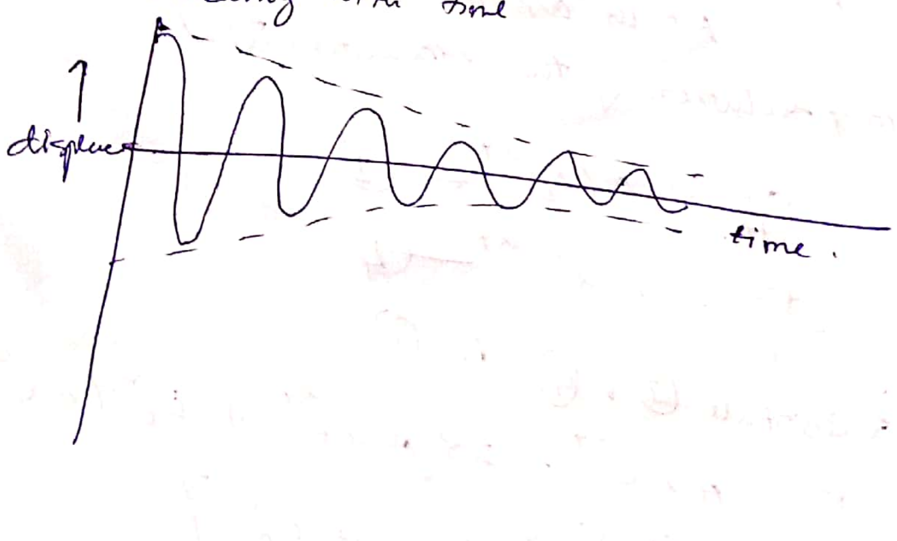
$$\boxed{x = e^{-\gamma t}\left[A_1 e^{\left(\sqrt{\gamma^2 - \omega_0^2}\right)t} + A_2 e^{-\left(\sqrt{\gamma^2 - \omega_0^2}\right)t}\right]}$$

where A_1 & A_2 are the arbitrary constants.

This is the solution of the Damping oscillation.

The Amplitude of the oscillation is

$\boxed{x = x_m \cdot e^{-\gamma t}}$ i.e it is Exponentially decreasing with time



Continuation of Damped oscillations.

$$x = e^{-\gamma t}\left[A_1 e^{(\sqrt{\gamma^2-\omega_0^2})t} + A_2 e^{-(\sqrt{\gamma^2-\omega_0^2})t}\right]$$

$$x = e^{-\gamma t}\left[A_1 e^{ct} + A_2 e^{-ct}\right] \quad \text{when } c = \sqrt{\gamma^2-\omega_0^2}$$

To Evaluate A_1 and A_2.

We have to use Initial conditions.

Suppose at $t=0$, $x = x_0 \rightarrow$ Maximum and Velocity $\frac{dx}{dt} = 0$. min

At $t=0$:

$$x = e^{-0}\left[A_1 e^{0} + A_2 e^{-0}\right]$$

$$x_0 = A_1 + A_2 \quad \text{—(1)}$$

$$\frac{dx}{dt} = v = 0 = -\gamma e^{-\gamma t}\left[A_1 e^{ct} + A_2 e^{-ct}\right] + e^{-\gamma t}\left[A_1 c e^{ct} - A_2 c e^{-ct}\right]$$

$$0 = -\gamma\left[A_1 + A_2\right] + \left[A_1 c - A_2 c\right]$$

$$0 = -\gamma\left[A_1 + A_2\right] + c\left[A_1 - A_2\right]$$

$$\gamma\left[A_1 + A_2\right] = c\left[A_1 - A_2\right]$$

$$\frac{\gamma}{c} x_0 = \left[A_1 - A_2\right] \quad \text{[From (1)]} \quad \text{—(2)}$$

From (1) + (2).

$$2A_1 = x_0 + \frac{\gamma x_0}{c}$$

$$A_1 = \frac{x_0}{2}\left[1 + \frac{\gamma}{c}\right] \quad \text{—(3)}$$

From (1) − (2)

$$2A_2 = x_0 - \frac{\gamma x_0}{c}.$$

$$A_2 = \frac{x_0}{2}\left[1 - \frac{\gamma}{c}\right] \quad \text{--- (4)}.$$

$$\therefore \quad x = e^{-\gamma t}\left[\frac{x_0}{2}\left(1+\frac{\gamma}{c}\right)e^{ct} + \frac{x_0}{2}\left(1-\frac{\gamma}{c}\right)e^{-ct}\right].$$

$$= e^{-\gamma t}\left[\frac{x_0}{2}e^{ct} + \frac{x_0\gamma}{2c}e^{ct} + \frac{x_0}{2}e^{-ct} - \frac{x_0\gamma}{2c}e^{-ct}\right]$$

$$= e^{-\gamma t}\left\{\frac{x_0}{2}\left(e^{ct}+e^{-ct}\right) + \frac{x_0\gamma}{2c}\left(e^{ct}-e^{-ct}\right)\right.$$

$$\boxed{x = x_0 e^{-\gamma t}\left[\left(\frac{e^{ct}+e^{-ct}}{2}\right) + \frac{\gamma}{c}\left(\frac{e^{ct}-e^{-ct}}{2}\right)\right]}$$

This is the general solution of the Damped oscillation differential equation. whose Amplitude $x_0 e^{-\gamma t}$ shows that it is exponentially decreasing with time.

$$\gamma = \frac{b}{2m}, \quad c = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}.$$

Case - i) A periodic (or) dead beat motion:

$b^2 > \omega_0^2$

This is the case for heavy damping.

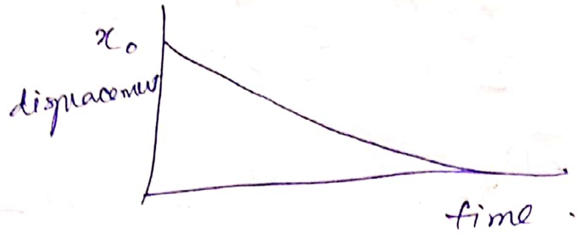
$\sqrt{b^2 - \omega_0^2}$ is real and positive.

As displacement x decreases from its initial value x_0 and it tends to zero when $t \to \infty$. In this case no oscillations occur. Hence such a motion is called aperiodic or overdamped.

Under very heavy damping, the particle passes its

Equilibrium position at most once before returning asymptotically to rest. ~~This is not with a~~

(2)



Case - 2 Critically damped motion:

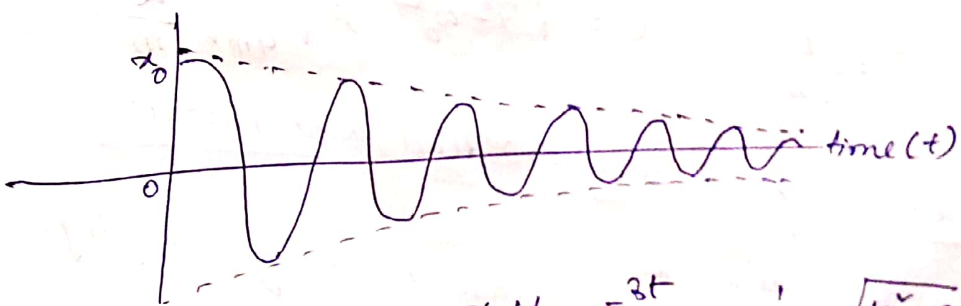
When $\gamma^2 = \omega_0^2$ (or) $C = \sqrt{\gamma^2 - \omega_0^2} = 0$.

In this case, the general solution of the damped oscillatory motion behave like the motion is neither overdamped nor oscillatory and is said to be critically damped. In other words, it is a transition stage between dead beat motion and damped oscillatory motion.

Case - 3 oscillatory damped SHM

When $\gamma^2 < \omega_0^2$ (light damping)

Then $\sqrt{\gamma^2 - \omega_0^2}$ is imaginary (or) negative quantity, Then the motion of the particle is oscillatory, the displacement varies as a sine curve.



$$A = \frac{x_0 \omega_0}{\sqrt{\omega_0^2 - \gamma^2}} e^{-\gamma t} \qquad \omega' = \sqrt{\omega_0^2 - \gamma^2}.$$

The frequency (f) and time period (t) of the damped SHM are.

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

Forced oscillations.

When the harmonic oscillator oscillates with its natural frequency in a medium like air, its oscillations get damped i.e the amplitude decreases Exponentially with time to zero. If the amplitudes of oscillations are to be maintain indefinitely, Energy must be supplied Externally. Such oscillations of the body under the action of External periodic force are known as as forced oscillations and the oscillator is called as forced harmonic oscillator (or) driven harmonic oscillator.

Let $F = F_0 e^{i\omega t}$ be an External periodic force applied to a damped harmonic oscillator. Here F_0 is the maximum value of the applied force and 'ω' its angular frequency, as the oscillating system. Then the differential Equation for motion of driven Oscillator is

$$F_{net} = F_{restoring} + F_{damping} + F_{External}.$$

$$ma = -kx - bv + F_0 e^{i\omega t}$$

$$ma + kx + bv = F_0 e^{i\omega t}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = F_0 e^{i\omega t}$$

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f_0 e^{i\omega t}} \quad \text{--- (1)}$$

where $2\gamma = \frac{b}{m}$, $\omega_0 = \sqrt{\frac{k}{m}}$

(19)

Equation ① represents the motion of forced oscillations.

Solution of the differential Equation:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \quad \text{--- ①}$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}(\cos\omega t + i\sin\omega t)$$

There are two parts in the Equation.

real part $\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$ --- ②

imaginary part and.

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f_0 \sin\omega t \quad \text{--- ③}.$$

Let us consider the solution of the Equation ① is

$$x = Ae^{i\omega t}$$

where 'A' is the complex quantity.

$$\frac{dx}{dt} = i\omega A e^{i\omega t} \quad \text{--- ④}$$

$$\frac{d^2x}{dt^2} = i^2\omega^2 A e^{i\omega t} = -\omega^2 A e^{i\omega t} \quad \text{--- ⑤}$$

Substitute ④ & ⑤ in ①.

$$-\omega^2 A e^{i\omega t} + 2\gamma(i\omega A e^{i\omega t}) + \omega_0^2(Ae^{i\omega t}) = f_0 e^{i\omega t}$$

$$\Rightarrow -\omega^2 A + 2\gamma i\omega A + A\omega_0^2 = f_0.$$

$$A(-\omega^2 + \omega_0^2 + i\omega 2\gamma) = f_0.$$

$$\boxed{A = \frac{f_0}{2i\omega\gamma + (\omega_0^2 - \omega^2)}}$$

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$$A = \frac{F_0}{2i\omega\gamma + (\omega_0^2 - \omega^2)}.$$ (20)

$$A = \frac{iF_0}{i\left[2i\omega\gamma + (\omega_0^2 - \omega^2)\right]}.$$

$$A = \frac{iF_0}{-2\omega\gamma + i(\omega_0^2 - \omega^2)}$$

$$= \frac{iF_0}{-\omega\left(\frac{b}{m}\right) + i\left(\frac{k}{m} - \omega^2\right)}$$

$$= \frac{iF_0 . m}{-b\omega + i(k - m\omega^2)}$$

$$A = \frac{iF_0 m}{\omega\left[-b + i\left(\frac{k}{\omega} - m\omega\right)\right]}$$

$$A = \frac{-iF_0 m}{\omega\left[b - i\left(\frac{k}{\omega} - m\omega\right)\right]}$$

$$A = \frac{-iF_0 m}{\omega\left[b + i\left(m\omega - \frac{k}{\omega}\right)\right]}.$$

$$A = \frac{-iF_0 m}{\omega Z_m}.$$

where Z_m is the Mechanical Impedance

$$\boxed{Z_m = b + i\left(m\omega - \frac{k}{\omega}\right)} \quad (6)$$

The Magnitude of the Mechanical impedance is

$$\boxed{Z_m = \sqrt{b^2 + \left(m\omega - \frac{k}{\omega}\right)^2}} \quad \left[\because z = x + iy \quad |z| = \sqrt{x^2 + y^2}\right].$$

The direction of the applied force and displacement 'x' is given by.

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$z = \cos\phi + i\sin\phi$.

$$\tan\phi = \frac{\sin\phi\ (\text{imagr part})}{\cos\phi\ (\text{real part})}$$

$$\boxed{\tan\phi = \frac{\left(m\omega - \frac{k}{\omega}\right)}{b}} \quad \text{— (7)}$$

$\therefore$ The solution of the given equation is.

$$x = Ae^{i\omega t}$$

$$\boxed{x = \left(\frac{-iF_0 m}{\left[\omega\left[b + i\left(m\omega - \frac{k}{\omega}\right)\right]\right]}\right) e^{i\omega t}}$$

Case i) If $F = F_0\cos\omega t$ i.e the real part of the solution.

$$x = \frac{-iF_0 m}{\omega Z_m}\left[\cos\omega t + i\sin\omega t\right].$$

$$x = \frac{F_0 m}{\omega Z_m}\left[-i\cos\omega t + \sin\omega t\right] \quad \text{(8)}$$

$$x = \frac{F_0 m}{\omega Z_m}\left[-i\cos\omega t\right] + \frac{F_0 m}{\omega Z_m}\sin\omega t. \quad \text{— (8)}$$

The real part of the above Equation is .

$$x = \frac{F_0 m}{\omega Z_m} \sin\omega t = \frac{F_0 m}{\omega Z_m} \overset{\cos}{\cancel{\sin}}(\omega t - \pi/2) .$$

(22)

Compare with

$$x = F_0 \cos\omega t$$

we find that total phase difference b/w the displacement and applied force is $-\pi/2$.

and amplitude is

$$\frac{F_0 m}{\omega Z_m} = \frac{F_0 m}{\omega [b + i(m\omega - \frac{k}{\omega})]} .$$

Case-(ii)

If $F = F_0 \sin\omega t$ i.e the imaginary part of the solution.

Compare with the Equation (8) with imaginary part.

$$x = -\frac{F_0 m \cos\omega t}{\omega Z_m}$$

$$x = -\frac{F_0 m}{\omega Z_m} \sin(\pi/2 - \omega t)$$

$$x = \frac{F_0 m}{\omega Z_m} \sin(\omega t - \pi/2)$$

we again find that the total phase difference between the displacement 'x' and applied force 'F' is $-\pi/2$. The amplitude also $\frac{F_0 m}{\omega Z_m}$

Phase difference between driven oscillator and driving force is always ~~behind~~ $-\pi/2$

i.e driven oscillator is always behind the driving force in phase.

Amplitude resonance:

When a periodic force $F = F_0 \cos \omega t$ is applied to a damped oscillator, the displacement is given by

$$\boxed{x = \frac{F_0 m}{\omega Z_m} \sin \omega t}$$

Where Z_m is the Mechanical impedance offered by the oscillator and its magnitude is given by

$$\boxed{Z_m = \sqrt{b^2 + \left(m\omega - \frac{k}{\omega}\right)^2}}$$

and $$\boxed{\tan\phi = \frac{\left(m\omega - \frac{k}{\omega}\right)}{b}}$$

~~b~~ Where 'b' is the damping constant, 'm' is the mass and 'k' is the stiffness (or) spring constant of the oscillator.

~~Given~~ We know that

$$Z_m = \sqrt{\frac{b^2 + (m\omega^2 - k)^2}{\omega^2}}$$

$$= \frac{1}{\omega}\sqrt{b^2\omega^2 + (m\omega^2 - k)^2} \qquad \omega_0^2 = \frac{k}{m}.$$

$$= \frac{1}{\omega}\sqrt{b^2\omega^2 + (m\omega^2 - m\omega_0^2)^2}$$

$$= \frac{1}{\omega}\sqrt{b^2\omega^2 + m^2(\omega^2 - \omega_0^2)^2}$$

$$= \frac{1}{\omega}\sqrt{\frac{m^2 b^2\omega^2}{m^2} + m^2(\omega^2 - \omega_0^2)^2}$$

$$= \frac{m}{\omega}\sqrt{\left(\frac{b}{m}\right)^2\omega^2 + (\omega^2 - \omega_0^2)^2}$$

$$= \frac{m}{\omega}\sqrt{(2\gamma)^2\omega^2 + (\omega^2 - \omega_0^2)^2} \quad \left(\because 2\gamma = \frac{b}{m}\right)$$

$$\boxed{Z_m = \frac{m}{\omega}\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

$$x = \frac{F_0 \cancel{m} \sin\omega t}{\cancel{\omega}\left[\frac{\cancel{m}}{\cancel{\omega}}\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}\right]}$$

$$\boxed{x = \frac{F_0 \sin\omega t}{\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}}$$

$\therefore$ Amplitude is $\dfrac{F_0}{\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}} = A$

$$\boxed{\therefore \quad x = A \sin\omega t}$$

At frequency of displacement resonance ($\omega = \omega_0$) i.e the driving frequency is Equal To the natural frequency of the oscillator. This is the case of amplitude resonance. At this stage the amplitude is given by

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$$A = \frac{F_0}{\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

$$A = \frac{F_0}{\sqrt{4\gamma^2\omega_0^2 + 0^2}} = \frac{F_0}{2\gamma\omega_0}$$

Thus at a driving force Equal To the resonant frequency the maximum amplitude is inversely proportional To damping coefficient 'γ'.

i.e For small damping, amplitude is large and for large damping, amplitude is small.

<u>Velocity resonance</u>: We know that the displacement Equation of a driven oscillator is

$$x = \frac{-i F_0 m e^{i\omega t}}{\omega Z_m}$$

where $Z_m = \frac{m}{\omega}\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}$

$$x = \frac{-i F_0 m}{\omega Z_m}(\cos\omega t + i\sin\omega t)$$

case-i) When comparing the real and imaginary parts

i.e $F = F_0 \cos\omega t$ with the displacement part

(26)

$$x = \frac{F_0 m}{\omega Z_m} \sin\omega t .$$

i.e velocity $= \frac{dx}{dt} = \frac{F_0 m}{\omega Z_m} \cdot (\omega . \cos\omega t).$

$$V = \frac{F_0 m}{\omega Z_m} \omega . \cos\omega t .$$

$$= \frac{F_0 \cancel{m} \, \omega \cos\omega t}{\omega \left(\frac{m}{\omega}\right) \sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

$$v = \frac{F_0 \omega \cos\omega t}{\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

At resonance : $\omega = \omega_0$.

$$V = \frac{F_0 \omega_0 \cos\omega_0 t}{2\gamma\omega_0}$$

$$v = \frac{F_0}{2\gamma} \cos\omega_0 t$$

The maximum value of $\cos\omega_0 t$ is '1'.

$$\boxed{\therefore V_{max} = \frac{F_0}{2\gamma}}$$

case-ii) When comparing the $F = F_0 \sin\omega t$ (imaginary part).

$$x = \frac{-F_0 m \cos\omega t}{\omega Z_m}.$$

i.e. velocity $= \frac{dx}{dt} = \frac{-F_0 m}{\omega Z_m}(-\omega \sin\omega t)$

$$V = \frac{F_0 m \sin\omega t}{Z_m}$$

$$V = \frac{F_0 m \sin\omega t}{\left(\frac{m}{\omega}\right)\left(\sqrt{4\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}\right)}$$

At $\omega = \omega_0$ (resonant frequency)

$$V = \frac{F_0 \omega_0 \sin\omega_0 t}{\sqrt{4\gamma^2\omega^2 + 0}}$$

$$V = \frac{F_0 \cancel{\omega_0} \sin\omega_0 t}{2\gamma \cancel{\omega_0}}$$

$$V = \frac{F_0 \sin\omega_0 t}{2\gamma}$$

The maximum value of $\sin\omega_0 t$ is '1'.

$$\boxed{\therefore V = \frac{F_0}{2\gamma}}$$

Thus at frequency of velocity resonance, the velocity amplitude ~~resonance~~ depends upon the driving force F_0 and damping constant 'γ'.

Quality factor:

The quality factor (Q) of a Mechanical oscillator is given by

$$Q = \frac{m\omega_0}{b}$$

where b = damping constant

Resonance absorption band width $\omega_2 - \omega_1 = \frac{b}{m}$

where $\omega_2 > \omega_0$
$\omega_1 < \omega_0$ i.e. $\boxed{\omega_2 > \omega_0 > \omega_1}$

At resonance absorption the average power drops to half maximum value.

Hence $Q = \frac{m\omega_0}{b} = \frac{\omega_0}{\omega_2 - \omega_1}$

$$\boxed{Q = \frac{\text{Frequency at resonance}}{\text{Full band width at half maximum power}}}$$

$\therefore$ Q-value of an oscillator is the ratio of frequency of maximum velocity response to the band width at half maximum power.

Figure of merit:

It is defined as the ratio of the frequency at velocity resonance (maximum velocity response) to the full bandwith at half maximum power

i.e Figure of merit $= \frac{\text{Frequency at resonance}}{\text{Full band width at half maximum power}}$

$$= \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{\left(\frac{b}{m}\right)} = \frac{m\omega_0}{b} = Q.$$

$\therefore$ Figure of merit is also known as quality factor.

Coupled oscillations:

When the motion of one oscillating system influences another one, then we can say that they are 'coupled'.

The two (or) more oscillators linked together in such a way that an exchange of energy transfer takes place between them are called "coupled oscillators."

Normal Co-ordinates:

Normal co-ordinates are those co-ordinates which help us to express the equations of motion of the harmonic oscillators of a coupled ~~oscillators~~ system in the form of a set of linear differential equations with constant coefficients and in which each equation contains only one variable.

Normal mode of vibration:

The manner in which a coupled system oscillates is called a 'mode'.

The mode of a coupled system may be harmonic (or) non-harmonic.

The harmonic modes of a coupled system are called Normal modes.

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Characteristics of Normal mode:

(1) A Normal mode has its own frequency known as Normal frequency

(2) In Each Normal mode all the components of the system vibrate with same normal frequency.

(3) The normal modes of vibration are Entirely independent of each other.

(4) The total Energy of the oscillator is Equal to the sum of the energies of all the normal modes.

(5) If at any time only one mode is Excited and vibrates the other modes will always be at rest and unexcited and these will acquire no energy from the vibrating mode.

Degrees of freedom:

A degree of freedom of a system is independent way by which the system may acquire Energy.

A degree of freedom is assigned its own particular normal co-ordinates.

The number of degrees of freedom and the number of normal co-ordinates of a system is the number of different ways in which the system can acquire energy.

Each harmonic oscillator has two degrees of freedom as it may have both kinetic as well as potential Energy.

The kinetic energy of a simple harmonic oscillator of mass 'm' and having displacement co-ordinate (x) is given by

$$\Rightarrow \frac{1}{2} m \dot{x}^2$$

The potential Energy is given by

$$\Rightarrow \frac{1}{2} K x^2$$ where $k \rightarrow$ stiffness or spring constant.

If the Normal modes of a harmonic oscillator are represented by normal co-ordinates 'x' and 'y' then the total energy corresponding to the two modes will be

$$E_x = a\dot{x}^2 + b x^2$$

$$E_y = c\dot{y}^2 + d y^2$$

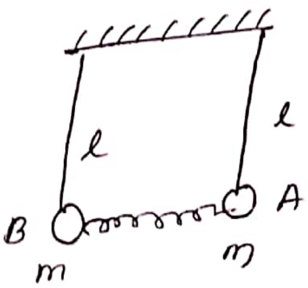
where a, b, c and d are constants,

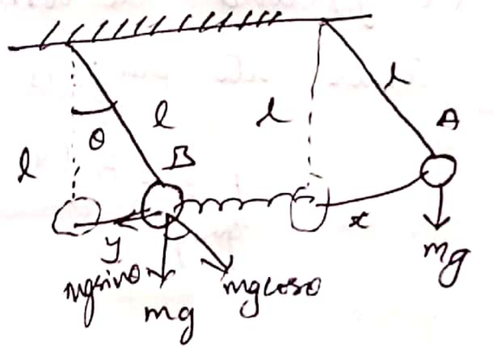
$$\left. \begin{matrix} a\dot{x}^2 \\ c\dot{y}^2 \end{matrix} \right\} \longrightarrow \text{kinetic energy}$$

$$\left. \begin{matrix} bx^2 \\ dy^2 \end{matrix} \right\} \longrightarrow \text{potential Energy.}$$

Equation of motion of stiffness coupled system of two pendulums

Consider a coupled system of two identical pendulums each having a pendulum bob of mass 'm' suspended by a light weightless, rigid rod of length l. The bobs are connected by a light spring of stiffness 'k', whose normal length is equal to the distance b/w the bobs when none of the two pendulums is displaced from its equilibrium position. Such pendulums are known as stiffness coupled.





The component of force due to gravity tending to bring the bobs of the pendulums back to its mean position.

$$= -mg\sin\theta$$

$$= -mg\,y/l \quad \text{for pendulum B}$$

$$(\because \sin\theta = y/l)$$

negative sign indicates the direction of force component

$$= -mg\sin\theta$$

$$= -mg\frac{x}{l} \quad (\text{for pendulum 'A'})$$

The Equations of Motion for the pendulum 'A' and 'B' respectively are

$\Rightarrow \quad \Sigma F = F_g + F_{spring}$.

for pendulum 'A'

$$\Rightarrow \quad m\ddot{x} = -mg\frac{x}{l} - k(x-y) \quad \text{(For pendulum 'A')} \quad \text{—— (1)}$$

[Since the pendulums 'A' and 'B' set into oscillations with a small amplitude in the plane of the paper and let 'x' and 'y' be the displacements. then the elongation of length $(x-y)$ and the corresponding force which comes into play is

$$F_{spring} = -k(x-y) \;]$$

$$m\ddot{y} = -mg\frac{y}{l} + k(x-y) \quad \text{—— (2)}.$$

Divide by 'm' on both sides.

$$\ddot{x} = -g\frac{x}{l} - \frac{k}{m}(x-y) \quad \text{—— (3)}$$

$$\ddot{y} = -g\frac{y}{l} + \frac{k}{m}(x-y) \quad \text{—— (4)}$$

$$\ddot{x} + \frac{g}{l}x = -\frac{k}{m}(x-y) \quad \text{—— (5)}$$

$$\ddot{y} + \frac{g}{l}(y) = \frac{k}{m}(x-y) \quad \text{—— (6)}.$$

(5) + (6) $\Rightarrow$

$$\ddot{x} + \left(\frac{g}{l}\right)x + \ddot{y} + \left(\frac{g}{l}\right)y = 0$$

$$\ddot{x} + \omega_0^2 x + \ddot{y} + \omega_0^2 y = 0 \quad \left(\because \omega_0^2 = \frac{g}{l}\right)$$

$$\ddot{x} + \ddot{y} + \omega_0^2(x+y) = 0 \longleftarrow (7)$$

(6) − (5)

$$\ddot{x} - \ddot{y} + \left(\frac{g}{l}\right)x - \left(\frac{g}{l}\right)y = \frac{-2k}{m}(x-y)$$

$$\ddot{x} - \ddot{y} + \left(\omega_0^2(x-y)\right) + \frac{2k}{m}(x-y) = 0.$$

$$\ddot{x} - \ddot{y} + \left(\omega_0^2 + \frac{2k}{m}\right)(x-y) = 0 \longleftarrow (8)$$

Let us choose new co-ordinates 'X' and 'Y' so that

$$X = x+y \qquad Y = x-y$$

$$\dot{X} = \dot{x} + \dot{y} \qquad \dot{Y} = \dot{x} - \dot{y}$$

$$\ddot{X} = \ddot{x} + \ddot{y} \qquad \ddot{Y} = \ddot{x} - \ddot{y}$$

Then Eq (7)

$$\ddot{X} + \omega_0^2 X = 0 \longleftarrow (9)$$

Eq (8)

$$\ddot{Y} + \left(\omega_0^2 + \frac{2k}{m}\right) Y = 0. \longleftarrow (10)$$

The Equation of Motion described in (9) & (10) have only one variable. The motion of the coupled System is thus described in terms of two co-ordinates 'X' and 'Y'. Each Equation of motion is a linear differential Equation of a simple Harmonic oscillator with constant coefficients with only one Variable.

The co-ordinates 'X' and 'Y' are therefore normal co-ordinates of the coupled system.

Thus the coupled system of two simple pendulums has two normal modes, one described by normal co-ordinate 'X' and other by Y.

In phase mode $Y = 0$.

i.e $x - y = 0$

$x = y$ for all times. The completely

described by

$$\ddot{X} + \frac{g}{l} X$$

$$\ddot{X} + \omega_0^2 X = 0$$

As the relative displacement of the two pendulums always remain same ($\because x = y$), the stiffness of the coupling has no effect. The spring always remains at its normal length. The frequency of oscillations is same as that of either pendulum in isolation

i.e $\omega_0^2 = \frac{g}{l}$

Both pendulums are always swinging in same phase. Such vibrations are called in phase vibrations and the corresponding mode is known as in-phase mode.

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out of phase mode: $X = 0$

$$X = 0, \quad x + y = 0$$
$$x = -y.$$

$$\ddot{Y} + \left(\omega_0^2 + \frac{2k}{m}\right) Y = 0.$$

'x' of the pendulum 'A' is positive that of B(y) is negative but Equal to that of A ($x = -y$). At zero displacement the two pendulums will be moving in opposite directions. Thus the periodic motion of the two pendulums will be 180° out of phase. Such vibrations are called out of phase vibrations and the corresponding mode is known as out of phase mode.

Total Energy

we have

$$\ddot{X} + \omega_0^2 X = 0$$
$$\ddot{Y} + \left(\omega_0^2 + \frac{2k}{m}\right) Y = 0.$$

where the solutions of the Equations are.

$$X = X_0 \cos \omega_1 t = x + y.$$
$$Y = Y_0 \cos \omega_2 t = x - y.$$

X_0, Y_0 are the Normal amplitudes.

$$\omega_1 = \omega_0 = \sqrt{\frac{g}{l}}.$$

where

$$\omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}}$$

Let us consider

$$X_0 = Y_0 = 2a.$$

Then $X = x+y = 2a \cos\omega_1 t$

$Y = x - y = 2a \cos\omega_2 t.$

$$X+Y = 2x = 2a[\cos\omega_1 t + \cos\omega_2 t]$$

$$\Rightarrow \cancel{2}x = \cancel{2}a[\cos\omega_1 t + \cos\omega_2 t]. \quad (1)$$

$$\dot{x} = a[-\omega_1 \sin\omega_1 t - \omega_2 \sin\omega_2 t]. \quad (2)$$

$$X - Y = 2y = 2a[\cos\omega_1 t - \cos\omega_2 t]$$

$$y = a[\cos\omega_1 t - \cos\omega_2 t] \quad (3)$$

$$\dot{y} = a[-\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t] \quad (4).$$

Eq (1) $\Rightarrow$ $x = a[\cos\omega_1 t + \cos\omega_2 t]$

$$x = 2a\left[\cos\frac{(\omega_2-\omega_1)}{2}t \cos\left(\frac{\omega_2+\omega_1}{2}\right)t\right]$$

Eq (3) $\Rightarrow$ $y = a[\cos\omega_1 t - \cos\omega_2 t].$

$$= -2a \sin\left(\frac{\omega_2-\omega_1}{2}\right)t \sin\left(\frac{\omega_1+\omega_2}{2}\right)t$$

Let us consider $\frac{\omega_2-\omega_1}{2} = \omega_m$ modulated frequency (or) beat frequency

and $\frac{\omega_1+\omega_2}{2} = \omega_{av}$ average frequency.

Hence the modulated amplitude is.

$A = 2a \cos\omega_m t$ for pendulum 'A' | $A = -2a \sin\omega_m t$ Pendulum 'B'

$$x = 2a \cos\left(\frac{\omega_2-\omega_1}{2}\right)t \cos\left(\frac{\omega_1+\omega_2}{2}\right)t$$

$$\boxed{x = 2a \cos\omega_m t \cos\omega_{av} t}$$

where $\omega_m = \frac{\omega_2-\omega_1}{2}$

$\omega_{av} = \frac{\omega_1+\omega_2}{2}$

$$y = -2a \sin\left(\frac{\omega_2-\omega_1}{2}\right)t \sin\left(\frac{\omega_1+\omega_2}{2}\right)t$$

$$\boxed{y = -2a \sin\omega_m t \sin\omega_{av} t}$$

The modulated amplitude of pendulums

A

$A = 2a \cos\omega_m t$

Amplitude for A

$\therefore \quad x = A \cos\omega_a t$

$\dot{x} = A\omega_a(-\sin\omega_a t)$

$|V_{max}| = A\omega_a$

$\therefore \quad E_A = \frac{1}{2} m v^2$

$= \frac{1}{2} m A^2 \omega_a^2$

$= \frac{1}{2} m\, 4a^2 \cos^2\omega_m t \cdot \omega_a^2$

$= 2ma^2\omega_a^2 \cos^2\omega_m t$

B

$B = 2a\sin\omega_m t$.

for B.

$y = -B\sin\omega_a t$

$\dot{y} = -B\omega_a \cos\omega_a t$

$|V_{max}| = B\omega_a$.

$E_B = \frac{1}{2} m (B^2\omega_a^2)$

$= \frac{1}{2} m\omega_a^2 (4a^2 \sin^2\omega_m t)$

$E_B = 2ma^2\omega_a^2 \sin^2\omega_m t$

$$E_A + E_B = 2ma^2\omega_a^2 = E_0$$

$$E_A - E_B = 2ma^2\omega_a^2\left[\cos^2\omega_m t - \sin^2\omega_m t\right]$$

$$= E_0\left[\cos 2\omega_m t\right]$$

$$E_A + E_B = E_0$$

$$E_A - E_B = E_0 \cos 2\omega_m t$$

$$2E_A = E_0 \left[1 + \cos 2\omega_m t\right].$$

$$E_A = \frac{E_0}{2} \left[1 + \cos\left(\omega_2 - \omega_1\right) t\right].$$

$$E_B = \frac{E_0}{2} \left[1 - \cos\left(\omega_2 - \omega_1\right) t\right].$$

This shows that total energy is constant but it flows back and forth b/w the pendulums at the Modulated (or) beat frequency.

N-coupled oscillators:

$$u_n = u_0 e^{i(\omega t - kna)}$$

$$m\frac{d^2u_n}{dt^2} = \beta(u_{n+1} + u_{n-1} - 2u_n)$$

$$m(i^2\omega^2)u_0 e^{i(\omega t - kna)} = \beta\left(u_0 e^{i(\omega t - k(n+1)a)} + u_0 e^{i(\omega t - (k(n-1))a)}\right.$$
$$\left. - 2u_0 e^{i(\omega t - kna)}\right]$$

$$-m\omega^2 u_n = \beta\left(u_0 e^{i(\omega t - kna)} e^{-ika} + u_0 e^{i(\omega t - kna)} e^{+ika}\right.$$
$$\left. - 2u_n\right)$$

$$= \beta\left(e^{-ika}u_n + u_n e^{+ika} - 2u_n\right)$$

$$= \beta u_n\left(e^{\left(\frac{ika}{2}\right)^2} + e^{\left(\frac{ika}{2}\right)^2} - 2e^{\frac{ika}{2}}e^{-\frac{ika}{2}}\right)$$

$$-m\omega^2 u_n = \beta u_n\left(\left(e^{\frac{ika}{2}} - e^{-\frac{ika}{2}}\right)^2\right.$$

$$-m\omega^2 = \beta\left(e^{\frac{ika}{2}} - e^{-ika/2}\right)^2$$

$$\left[\sin x = \frac{e^{ix} - e^{-ix}}{2i}\right]$$

$$= \beta\left(2i\sin\left(\frac{ka}{2}\right)\right)^2$$

$$m\omega^2 = 4\beta\sin^2\frac{ka}{2}$$

$$\omega = \pm\sqrt{\frac{4\beta}{m}}\ \sin\left(\frac{ka}{2}\right)$$

Let $c \rightarrow$ logitudinal stiffness

$\rho \rightarrow m/l$.

$\Rightarrow$ $c = \beta a$, $\rho = \frac{m}{a}$.

$$\omega = \pm\sqrt{\frac{4(c/a)}{\rho a}}\ \sin\left(\frac{ka}{2}\right)$$

$$\omega = \pm\frac{2}{a}\sqrt{\frac{c}{\rho}}\ \sin\left(\frac{ka}{2}\right)$$

$$\omega = \pm\frac{2}{a}(v_s)\sin\left(\frac{ka}{2}\right).$$

$\downarrow$

Velocity of the sound wave.