1.2

Basic Concepts in Probability

Introduction to uncertainty

Every day we have been coming across statements like the ones mentioned below:

1. Probably it will rain tonight.
2. It is quiet likely that there will be a good yield of paddy this year.
3. Probably I will get a first class in the examination.
4. India might win the cricket series against Australia
   and so on.

In all the above statements some element of uncertainty or chance is involved. A numerical measure of uncertainty is provided by a very important branch of statistics known as **Theory of Probability**. In the words of *Prof. Ya-Lin-Chou*: *Statistics is the science of decision making with calculated risks in the face of uncertainty*.

History of Probability

The history of probability suggests that its theory developed with the study of *games of chance*, such as *rolling of dice*, *drawing a card from a pack of cards*, etc. Two French gamblers had once decided that any one person who will first get a 'particular point' will win the game. If the game is stopped before reaching that point, the question is how to share the stake. This and similar other problems were then posed by the great French mathematician *Blaise Pascal*, who after consulting another great French mathematician *Pierre de Fermat*, gave the solution of the problems and then laid down a strong foundation of probability. Later on, another French mathematician, *Laplace*, improved the definition of probability.

Coins, Dice and Playing Cards: The basic concepts in probability are better explained using *coins*, *dice* and *playing cards*. The knowledge of these is very much useful in solving problems in probability.

Coin: A coin is round in shape and it has two sides. One side is known as ***head (H)*** and the other is known as ***tail (T).*** When a coin is tossed, the side on the top is known as the result of the toss.

Die: A die is cube in shape in which length, breadth and height are equal. It has six faces which have same area and numbered from 1 to 6. The plural of die is dice. When a die is thrown, the number on the top face is the result of the throw.

Pack of Cards: A pack of cards 52 cards. It is divided into four suits called *spades*, *clubs*, *hearts* and *diamonds*. Spades and clubs are black; hearts and diamonds are red in colour. Each suit consists of 13 cards, of which *nine* cards are numbered from 2 to 10, an ace, jack, queen and king. We shuffle the cards and then take a card from the top which is the result of selecting a card.

Basic Concepts in Probability

The following basic concepts are very important in understanding the definitions of the probability:

Experiment: The process of making an observation or measurement and observation about a phenomenon is known as an ***experiment***.

Example1: Sitting in the balcony of the house and watching the movement of clouds in the sky is an experiment.

Example2: For given values of pressure (P), measuring the corresponding values of volume (V) of a gas and observing that $P \cdot V = k$(constant) is an experiment. The experiments are of two types:

Deterministic experiment: If an experiment produces the same result when it is conducted several times under identical conditions, then the experiment is known as ***determinant experiment***.
All the experiments in physical and engineering sciences are deterministic.

Random Experiment: If an experiment produces different results even though it is conducted several times under identical conditions, then the experiment is known as ***random experiment***. All the experiments in social sciences are random.

Trial: Conducting a random experiment once is known as a ***trial***.

Outcome: A result of a random experiment in a trial is known as an ***outcome***.
Outcomes are denoted by lowercase letters $a, b, c, d, e, \dots$.

Equally Likely Outcomes: Outcomes of a random experiment are said to be ***equally likely*** if all have the same chance of occurrence. Getting a H and T in a balanced coin are equally likely. The outcomes $1,2,3,4,5$ and 6 are equally likely if the die is a cube.

Sample space: The set of all possible outcomes of a random experiment is known as a ***sample space*** and denoted by **S**.

Event: A subset of the sample space is known as an ***event***.
The events are denoted by uppercase letters $\mathrm{A}, \mathrm{B}, \mathrm{C}$ etc.

Happening of an event: We say that an event happens (or occurs) if any one outcome in it happens (or occurs).

Elementary Event: A singleton set consisting an outcome of a random experiment is known as an ***elementary event***.

Favorable outcomes: The outcomes in an event are known as ***favorable outcomes*** or ***cases*** of that event.

Impossible Event: An event with no outcome in it is known as ***impossible event*** and is denoted by $\boldsymbol{\phi}$.

Certain or Sure Event: An event consisting of all possible outcomes of a random experiment is known as ***certain*** or ***sure event*** and it is same as the sample space.

Exhaustive Events: The events in a sample space are said to be ***exhaustive*** if their union is equal to the sample space. The events $A_1, A_2, \dots, A_n$ in S are said to be exhaustive if

$$\bigcup_{i=1}^{n} A_i = S$$

Mutually Exclusive Events: Two or more events in the sample space are said to be ***mutually exclusive*** if the happening of one of them precludes the happening of the others. Mathematically two events A and B in S are said to be mutually exclusive if $A \cap B = \phi$.

Example3: Consider a random experiment of tossing a coin. The possible outcomes are H and T. Thus, the sample space is given by $S = \{H, T\}$ and $n(S) = 2$ where $n(S)$ is the total number of outcomes in S.

Example 4: Consider a random experiment of tossing two coins (or two tosses of a coin). The sample space is given by $S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$ and $n(S) = 2^2 = 4$.

Example 5: Consider a random experiment of tossing three coins (or three tosses of a coin). The sample space is given by

$$S = \{H, T\} \times \{H, T\} \times \{H, T\} = \{H, T\} \times \{HH, HT, TH, TT\}$$

$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

and $n(S) = 2^3 = 8$.

Let us define some events in the sample space as below:

E_1: Three heads

E_2: Three tails

E_3: Exactly one head

E_4: Exactly two heads

E_5: At least one head

E_6: At least two heads

Then these events are represented by the following subsets of S:

$E_1 = \{HHH\}$;

$E_2 = \{TTT\}$

$E_3 = \{HTT, THT, TTH\}$;

$E_4 = \{HHT, HTH, THH\}$;

$E_5 = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$ and
$E_6 = \{HHH, HHT, HTH, THH\}$.

Note that $E_1 \cup E_2 \cup E_3 \cup E_4 = S$ and hence E_1, E_2 ,E_3 andE_4 are exhaustive events in S. Further, $E_i \cap E_j = \phi$, where $i \neq j$. Hence, E_1, E_2 ,E_3 and E_4 are mutually exclusive events in S.

Note: In general, if a random experiment consists of tossing N coins (or N tosses of a coin), then $n(S) = 2^N$.

Example 6: Let us consider a random experiment of throwing a die. Since we can obtain any one of the six faces 1,2,3,4,5 and 6, the sample space is given by $S = \{1,2,3,4,5,6\}$ and $n(S) = 6$.

Now define $E_1 = \{1,3,5\}, E_2 = \{2,4,6\}$ and $E_3 = \{3,6\}$. We say that E_1 happens or occurs if we get the outcome $1,3$ or 5. In otherwords, we say that E_1 happens if we get an odd number. Similarly, we say that E_2 happens if we get an even number and E_3 happens if we get a multiple of 3.

Since E_1, E_2 and E_3 are subsets of S; E_1, E_2 and E_3 are events in S. Since $E_1 \cup E_2 = S$, E_1 and E_2 are exhaustive events in S. Since $E_1 \cup E_3 = \{1,3,5,6\} \neq S$, E_1 and E_3 are not exhaustive events in S. Since $E_1 \cap E_2 = \phi$, E_1 and E_2 are mutually exclusive events in S. Since $E_1 \cap E_3 = \{3\}$, E_1 and E_3 are not mutually exclusive events in S. Similarly E_2 and E_3 are not mutually exclusive events in S.

Example 7: In a random experiment of throwing two dice (or two throws of a die), the sample space is given by

$$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$

$$\{(1,1),(1,2),\dots,(1,6),$$

$$(2,1),(2,2),\dots,(2,6),$$

$$(3,1),(3,2),\dots,(3,6)$$

$$(4,1),(4,2),\dots,(4,6)$$

$$(5,1),(5,2),\dots,(5,6)$$

$$(6,1),(6,2),\dots,(6,6)\}$$

where in the outcome (a, b), a represents the number obtained on the first die and b represents the number on the second die. Obviously $(a, b) \neq (b, a)$ unless $a = b$. The number of outcomes in S is given by $S = 6^2 = 36$.

Let us define the following events in S.

E_1: Sum of points on two dice is 5

E_2: Sum of points on two dice is 6

E_3: Sum of points on two dice is even

E_4: Sum of points on two dice is odd

E_5: Sum of points on two dice is greater than 12

E_6: Sum of points on two dice is divisible by 3

E_7: Sum is greater than or equal to 2 and is less than or equal to 12

Then the events E_1 to E_7 as subsets of S are given below.

$E_1 = \{(1,4),(2,3),(3,2),(4,1)\}$ and $n(E_1) = 4$

$E_2 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ and $n(E_2) = 5$

The sum of the points on the two dice is even if the points obtained on each die is (i) even or (ii) odd. Thus

$$E_3 = (\{2,4,6\} \times \{2,4,6\}) \cup (\{1,3,5\} \times \{1,3,5\})$$

$$\{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6),(1,1),(1,3),(1,5),$$
$$(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$$

and $n(E_3) = (3 \times 3) + (3 \times 3) = 9 + 9 = 18.$

Similarly,

$$E_4 = (\{2,4,6\} \times \{1,3,5\}) \cup (\{1,3,5\} \times \{2,4,6\})$$

$$\{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5),(1,2),(1,4),(1,6),$$
$$(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$$

and $n(E_4) = (3 \times 3) + (3 \times 3) = 18.$

Further, $E_5 = \phi, i.e.,$ E_5 is an impossible event and $E_7 = S, i.e.,$ E_7 is a certain event. Hence $n(E_5) = 0$ and $n(E_7) = 36$.

The sum of the points on the two dice is divisible by 3 if their sum is 3,6,9 or 12. Thus

$E_6 = \{(1,2),(2,1),(1,5),(2,4),(3,3),(4,2),(5,1),(3,6),(4,5),(5,4),(6,3),(6,6)\}$ and $n(E_6) = 12$.

Note: In general, if the random experiment consists of throwing of N dice (or N throws of a die), the number of outcomes in S is given by $n(S) = 6^N$.

Example 8: Let us consider the random experiment of tossing a coin and a die together. Then the sample space is given by

$S = \{H, T\} \times \{1,2,3,4,5,6\}$

$= \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

and $n(S) = 2 \times 6 = 12$.

Note: In the above examples 3 to 8, if the coins and dice are unbiased, the outcomes in the sample spaces are equally likely. Normally, the coins are balanced and hence are unbiased. If a die is a cube, then all the surfaces have the same area and also it is unbiased.

Example 9: Let us consider the random experiment of selecting two balls simultaneously from and urn containing 4 balls of different colours red(R), blue(B), yellow(Y) and white(W). Then the sample space is given by

$$S = \{RB, RY, RW, BY, BW, YW\} \text{ and } n(S) = 4_{C_2} = 6$$

Example 10: If the random experiment consists of selecting two balls one after the other with replacement in Example 9, the sample space is given by

$S = \{R, B, Y, W\} \times \{R, B, Y, W\} =$
$\{RR, RB, RY, RW, BR, BB, BY, BW, YR, YB, YY, YW, WR, WB, WY, WW\}$ and $n(S) = 4 \times 4 = 16$.

Example 11: If the random experiment consists of selecting two balls one after the other without replacement in Example7, the sample space is given by

$S = \{RB, RY, RW, BR, BY, BW, YR, YB, YW, WR, WB, WY\}$ and
$n(S) = 4 \times 3 = 12.$

Example12: Consider a random experiment of tossing a coin until head appears. Its sample space is given by

$$S = \{H, TH, TTH, TTTH, \dots\}$$

where TTH represents tail in first, tail in second and head in third tosses and so on. Obviously, $n(S)$ is infinite.

Example13: Consider a random experiment of tossing a coin repeatedly until head or tail appears twice in succession. Thus the sample space is given by

$$S = \{HH, TT, THH, HTT, HTHH, THTT, \dots\}$$

and $n(S)$ is infinite.

1.3.

Definitions of Probability

The probability of a given event is an expression of likelihood or chance of occurrence of an event. How the number is assigned would depend on the interpretation of the term 'probability'. There is no general agreement about its interpretation. However, broadly speaking, there are four different schools of thought on the concept of probability.

Mathematical (or classical or A priori) definition of probability

Let S be a sample space associated with a random experiment. Let A be an event in S. We make the following assumptions on S:

(i) It is discrete and finite

(ii) The outcomes in it are equally likely

Then the probability of happening (or occurrence) of the event A is defined by

$$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Number\ of\ outcomes\ in\ S} = \frac{n(A)}{n(S)}$$

Note:

i) The probability of non-happening (or non-occurrence) of A is given by

$$P(\bar{A}) = \frac{Number\ of\ outcomes\ in\ \bar{A}}{Number\ of\ outcomes\ in\ S} = \frac{n(\bar{A})}{n(S)} = \frac{n(S)-n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)} = 1 - P(A)$$

That is $P(\bar{A}) = 1 - P(A)$

ii) If $A = \phi$, then $P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$. That is, probability of an impossible event is zero.

iii) If $A = S$, then $P(S) = \frac{n(S)}{n(S)} = 1$. That is, probability of a certain event is one.

iv) For any event A in S, $0 \leq P(A) \leq 1$.

v) The odds in favour of A are given by $n(A) : n(\bar{A}) = P(A) : P(\bar{A})$.

vi) The odds against of A are given by $n(\bar{A}) : n(A) = P(\bar{A}) : P(A)$.

vii) If the odds in favour of A are $a : b$, then $P(A) = \frac{a}{a+b}$.
viii) If the odds against of A are $c : d$, then $P(A) = \frac{d}{c+d}$.
ix) $n(A)$ and $n(S)$ are counted by using methods of counting discussed in **Module 1.1**.

Limitations: The mathematical definition of probability breaks down in the following cases:

(i) The outcomes in the sample space are not equally likely.
(ii) The number of outcomes in the sample space is infinite.

Statistical (or Empirical or Relative Frequency or Von Mises) Definition of Probability

If a random experiment is performed repeatedly under identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.

Symbolically, if in N trials an event A happens a_N times, then the probability of the happening of A is given by

$$P(A) = \lim_{N \to \infty} \frac{a_N}{N} \quad \text{... (1.3.1)}$$

Note:

i) Since the probability is obtained objectively by repetitive empirical observations, it is known as Empirical Probability.
ii) The empirical probability approaches the classical probability as the number of trials becomes indefinitely large.

Limitations of Empirical Probability

(i) If an experiment is repeated a large number of times, the experimental conditions may not remain identical.
(ii) The limit in (1.3.1) may not attain a unique value, however large N may be.

Subjective definition of probability: In this method, probabilities are assigned to events according to the knowledge, experience and belief about the happening of the events. The main limitation of this definition is, it varies from person to person.

Axiomatic Definition of Probability: Let S be a sample space and let $\mathbb{B}$ be a σ-field associated with S. A probability function (or measure) P is a real valued set function having domain B and which satisfies the following three axioms:

1. $P(A) \geq 0$, for every $A \in \mathbb{B}$ (Non-negativity)
2. $P(S) = 1, i.e, P$ is normed (Normality)
3. If $A_1, A_2, \dots, A_n, \dots$ are mutually exclusive events in S, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ (σ - additive or countably additive)

Thus, the probability function is a normed measure on (the measurable space) $(\mathrm{S}, \mathbb{B}, \mathrm{P})$ is called a **Probability space**. This definition is useful in proving theorems on probability.

Note: The elements of $\mathbb{B}$ are events in S.

Solved Examples using Mathematical Definition of Probability

In this section, we use mathematical definition of probability for computing probabilities. Also we use methods of counting for counting the number of outcomes in an event and sample space.

Example 1: A uniform die is thrown at random. Find the probability that the number on it is (i) even (ii) odd (iii) even or multiple of 3 (iv) even and multiple of 3 (v) greater than 4

Solution:

(i) The number of favourable cases to the event of getting an even number is 3, viz., 2,4,6.

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(ii) The number of favourable cases to the event of getting an odd number is 3, viz., 1, 3, 5.

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(iii) The number of favourable cases to the event of getting even or multiple of 3 is 4, viz., 2, 3, 4, 6.

$\therefore$ Required probability $= \frac{4}{6} = \frac{2}{3}$

(iv) The number of favourable cases to the event of getting even and multiple of 3 is 1, viz., 6.

$\therefore$ Required probability $= \frac{1}{6}$

(v) The number of favourable cases to the event of getting greater than 4 is 2, viz., 5 and 6.

$\therefore$ Required probability $= \frac{2}{6} = \frac{1}{3}$

Example 2: Four cards are drawn at random from a pack of 52 cards. Find the probability that

(i) They are a king, a queen, a jack and an ace.
(ii) Two are kings and two are aces.
(iii) All are diamonds.
(iv) Two are red and two are black.
(v) There is one card of each suit.
(vi) There are two cards of clubs and two cards of diamonds.

Solution: Four cards can be drawn from a well shuffled pack of 52 cards in ${}^{52}C_4$ ways, which gives the exhaustive number of cases.

(i) 1 king can be drawn out of the 4 kings is ${}^{4}C_1 = 4$ ways. Similarly, 1 queen, 1 jack and an ace can each be drawn in ${}^{4}C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are ${}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1$.

Hence, required probability $= \frac{{}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_4} = \frac{256}{{}^{52}C_4}$

(ii) Required probability $=\dfrac{{}^4C_2 \times {}^4C_2}{{}^{52}C_4}$

(iii) Since 4 cards can be drawn out of 13 cards (since there are 13 cards of diamond in a pack of cards) in ${}^{13}C_4$ ways,

Required probability $=\dfrac{{}^{13}C_4}{{}^{52}C_4}$

(iv) Since there are 26 red cards (of diamonds and hearts) and 26 black cards (of spades and clubs) in a pack of cards,

Required probability $=\dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

(v) Since, in a pack of cards there are 13 cards of each suit,

Required probability $=\dfrac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

(vi) Required probability $=\dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$

Example 3: What is the chance that a non-leap year should have fifty-three Sundays?

Solution: A non-leap year consists of 365 days, $i.e.$, 52 full weeks and one over-

day. A non-leap year will consist of 53 Sundays if this over-day is Sunday. This over-day can be anyone of the possible outcomes:

(i) Sunday (ii) Monday (iii) Tuesday (iv) Wednesday (v) Thursday (vi) Friday (vii) Saturday, $i.e.$, 7 outcomes in all. Of these, the number of ways favourable to the required event viz., the over-day being Sunday is 1.

$\therefore$ Required probability $=\frac{1}{7}$

Example 4: Find the probability that in 5 tossings, a perfect coin turns up head at least 3 times in succession.

Solution: In 5 tossings of a coin, the sample space is:

$$S = \{H, T\} \times \{H, T\} \times \{H, T\} \times \{H, T\} \times \{H, T\}, (H : \text{head} ; T : \text{tail})$$

$\therefore$ Exhaustive number of cases $= 2^5 = 32$.

The favourable cases for getting at least three heads in succession are :

Starting with 1st toss: $HHHTH$, $HHHTT$, $HHHHT$, $HHHHH$

Starting with 2nd toss: $THHHT$, $THHHH$

Starting with 3rd toss: $TTHHH$, $HTHHH$

Hence, the total number of favourable cases for getting at least 3 heads in succession are 8.

$\therefore$ Required probability $= \frac{Number\ of\ favourable\ cases}{Exhaustive\ number\ of\ cases} = \frac{8}{32} = \frac{1}{4} = 0.25$

Example 5: A bag contains 20 tickets marked with numbers 1 to 20. One ticket is drawn at random. Find the probability that it will be a multiple of (i)2 or 5, (ii)3 or 5

Solution: One ticket can be drawn out of 20 tickets in ${}^{20}C_1 = 20$ ways, which determine the exhaustive number of cases.

(i) The number of cases favourable to getting the ticket number which is:

(a) a multiple of 2 are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, $i.e.$, 10 cases.

(b) a multiple of 5 are 5, 10, 15, 20 i.e., 4 cases

Of these, two cases viz., 10 and 20 are duplicated.

Hence the number of distinct cases favourable to getting a number which is a multiple of 2 or 5 are: $10 + 4 - 2 = 12$.

$\therefore$ Required probability $= \frac{12}{20} = \frac{3}{5} = 0.6$

(ii) The cases favourable to getting a multiple of 3 are 3, 6, 9, 12, 15, 18 i.e., 6 cases in all and getting a multiple of 5 are 5, 10, 15, 20 i.e., 4 cases in all. Of these, one case viz., 15 is duplicated.

Hence, the number of distinct cases favourable to getting a multiple of 3 or 5 is $6 + 4 - 1 = 9$.

$\therefore$ Required probability $= \frac{9}{20} = 0.45$

Example 6: An urn contains 8 white and 3 red balls. If two balls are drawn at random, find the probability that

(i) both are white, (ii) both are red, (iii) one is of each color.

Solution: Total number of balls in the urn is $8 + 3 = 11$. Since 2 balls can be drawn out of 11 balls in ${}^{11}C_2$ ways,

Exhaustive number of cases $= {}^{11}C_2 = \frac{11 \times 10}{2} = 55$

(i) If both the drawn balls are white, they must be selected out of the 8 white balls and this can be done in ${}^{8}C_2 = \frac{8 \times 7}{2} = 28$ ways.

$\therefore$ Probability that both the balls are white $= \frac{28}{55}$

(ii) If both the drawn balls are red, they must be drawn out of the 3 red balls and this can be done in ${}^{3}C_2 = 3$ ways. Hence, the probability that both the drawn balls are red $= \frac{3}{55}$.

(iii) The number of favourable cases for drawing one white ball and one red ball is ${}^{8}C_1 \times {}^{3}C_1 = 8 \times 3 = 24$

$\therefore$ Probability that one ball is white and other is red $= \frac{24}{55}$

Example 7: The letters of the word 'article' are arranged at random. Find the probability that the vowels may occupy the even places.

Solution: The word 'article' contains 7 distinct letters which can be arranged among themselves in $7!$ ways. Hence exhaustive number of cases is $7!$.

In the word 'article' there are 3 vowels, viz., a, i and e and these are to be placed in, three even places, viz., 2nd, 4th and 6th place. This can be done in $3!$, ways. For each arrangement, the remaining 4 consonants can be arranged in $4!$ ways. Hence, associating these two operations, the number of favourable cases for the vowels to occupy even places is $3! \times 4!$.

$\therefore$ Required probability $= \frac{3!4!}{7!} = \frac{3!}{7\times6\times5} = \frac{1}{35}$

Example 8: Twenty books are placed at random in a shelf. Find the probability that a particular pair of books shall be:

(i) Always together **(ii) Never together**

Solution: Since 20 books can be arranged among themselves in $20!$ ways, the exhaustive number of cases is $20!$.

(i) Let us now regard that the two particular books are tagged together so that we shall regard them as a single book. Thus, now we have $(20 - 1) = 19$ books which can be arranged among themselves in $19!$ ways. But the two books which are fastened together can be arranged among themselves in $2!$ ways.

Hence, associating these two operations, the number of favourable cases for getting a particular pair of books always together is $19! \times 2!$.

$\therefore$ Required probability is $\frac{19!\times2!}{20!} = \frac{2}{20} = \frac{1}{10}$.

(ii) Total number of arrangement of 20 books among themselves is $20!$ and the total number of arrangements that a particular pair of books will always be together is $19!\,2!$, [See part (i)]. Hence, the number of arrangements in which a particular pair of books is never together is

$$20! - 2 \times 19! = (20-2) \times 19! = 18 \times 19!$$

$\therefore$ Required probability $= \frac{18\times 19!}{20!} = \frac{18}{20} = \frac{9}{10}$

Aliter: P [A particular pair of books shall never be together]

$= 1 - P$[A particular pair of books is always together]$= 1 - \frac{1}{10} = \frac{9}{10}$.

Example 9: n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.

Solution: The n persons can be seated in n chairs at a round table in $(n-1)!$ ways, which gives the exhaustive number of cases.

If two specified persons, say, A and B sit together, then regarding A and B fixed together, we get $(n-1)$ persons in all, who can be seated at a round table in $(n-2)!$ ways. Further, since A and B can interchange their positions in 2! ways, total number of favourable cases of getting A and B together is $(n-2)! \times 2!$.

Hence, the required probability is: $\frac{(n-2)!\times 2!}{(n-1)!} = \frac{2}{n-1}$

Aliter: Let us suppose that of the n persons, two persons, say, A and B are to be seated together at a round table. After one of these two persons, say A occupies the chair, the other person B can occupy any one of the remaining $(n-1)$ chairs. Out of these $(n-1)$ seats, the number of seats favourable to making B sit next to A is 2 (since B can sit on either side of A). Hence the required probability is $\frac{2}{n-1}$.

Example 10: In a village of 21 inhabitants, a person tells a rumour to a second person, who in turn repeats it to a third person, etc. at each step the recipient of the rumour is chosen at random from the 20 people available. Find the probability that the rumour will be told 10 times without:

(i) returning to the originator ; **(ii) being repeated to any person**

Solution: Since any person can tell the rumour to any one of the remaining $21 - 1 = 20$ people in 20 ways, the exhaustive number of cases that the rumour will be told 10 times is 20^{10}.

(i) Let us define the event :

E_1 : The rumour will be told 10 times without returning to the originator.

The originator can tell the rumour to any one of the remaining 20 persons in 20 ways, and each of the $10 - 1 = 9$ recipients of the rumour can tell it to any of the remaining $20 - 1 = 19$ persons (without returning it to the originator) in 19 ways. Hence the favourable number of cases for E_1 are 20×19^9. The required probability is given by :

$$P(E_1) = \frac{20 \times 19^9}{20^{10}} = \left(\frac{19}{20}\right)^9$$

(ii) Let us define the event :

E_2 : The rumour is told 10 times without being repeated to any person.

In this case the first person (narrator) can tell the rumour to any one of the available $21 - 1 = 20$ persons; the second person can tell the rumour to any one of the remaining $20 - 1 = 19$ persons; the third person can tell the rumour to anyone of the remaining $20 - 2 = 18$ persons; …; the 10th person can tell the rumour to any one of the remaining $20 - 9 = 11$ persons.

Hence the favourable number of cases for E_2 are $20 \times 19 \times 18 \times \ldots \times 11$.

$\therefore$ Required probability $= P(E_2) = \frac{20 \times 19 \times 18 \times \ldots \times 11}{20^{10}}$

Example 11: If 10 men, among whom are A and B, stand in a row, what is the probability that there will be exactly 3 men between A and B?

Solution: If 10 men stand in a row, then A can occupy any one of the 10 positions and B can occupy any one of the remaining 9 positions. Hence, the exhaustive number of cases for the positions of two men A and B are $10 \times 9 = 90$.

The cases favourable to the event that there are exactly 3 men between A and B are given below:

(i) A is in the 1st position and B is in the 5th position.

(ii) A is in the 2nd position and B is in the 6th position.

. . . .

. . . .

. . . .

(vi) A is in the 6th position and B is in the 10th position.

Further, since A and B can interchange their positions, the total number of favourable cases $= 2 \times 6 = 12$.

$\therefore$ Required probability $= \frac{12}{90} = \frac{2}{15} = 0.1333$

Example12: A five digit number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.

Solution: The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is $5!$. Out of these, the number of arrangements which begin with 0 (and therefore will give only 4-digited numbers) is $4!$.

Hence the total number of five digited numbers that can be formed from digits 0, 1, 2, 3, 4 is $5! - 4! = 120 - 24 = 96$

The number formed will be divisible by 4 if the number formed by the two digits on extreme right (i.e., the digits in the unit and tens places) is divisible by 4. Such numbers are:

$$04, \quad 12, \quad 20, \quad 24, \quad 32 \text{ and } 40$$

If the numbers end in 04, the remaining three digits viz., 1, 2 and 3 can be arranged among themselves in 3! in each case.

If the numbers end with 12, the remaining three digits 0, 2, 3 can be arranged in $3!$ ways. Out of these we shall reject those numbers which start with 0 (i.e., have 0 as the first digit). There are $(3-1)! = 2!$ such cases. Hence, the number of five digited numbers ending with 12 is : $3! - 2! = 6 - 2 = 4$

Similarly the number of 5 digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is: $3 \times 3! + 3 \times 4 = 18 + 12 = 30$

Hence, required probability $= \frac{30}{96} = \frac{5}{16}$

Example13: There are four hotels in a certain town. If 3 men check into hotels in a day, what is the probability that each checks into a different hotel?

Solution: Since each man can check into any one of the four hotels in ${}^4C_1 = 4$ ways, the 3 men can check into 4 hotels in $4 \times 4 \times 4 = 64$ ways, which gives the exhaustive number of cases.

If three men are to check into different hotels, then first man can check into any one of the 4 hotels in ${}^4C_1 = 4$ ways; the second man can check into any one of the remaining 3 hotels in ${}^3C_1 = 3$ ways; and the third man can check into any one of the remaining two hotels in ${}^2C_1 = 2$ ways. Hence, favourable number of cases for each man checking into a different hotel is: ${}^4C_1 \times {}^3C_1 \times {}^2C_1 = 4 \times 3 \times 2 = 24$

$\therefore$ Required probability $= \frac{24}{64} = \frac{3}{8} = 0.375$

P1:

In a single throw with two uniform dice find the probability of throwing (i) Five, (ii) Eight

Solution:

Exhaustive number of cases in a single throw with two dice is $6^2 = 36$.

(i) Sum of '5' can be obtained on the two dice in the following mutually exclusive ways:
$(1,4), (2,3), (3,2), (4,1)$ $i.e.$, 4 cases in all, where the first and second number in the bracket ( ) refer to the numbers on the 1st and 2nd dice respectively.
$\therefore$ Required probability $= \frac{4}{36} = \frac{1}{9}$

(ii) The cases favourable to the event of getting sum of 8 on two dice are:
$(2,6), (3,5), (4,4), (5,3), (6,2)$ $i.e.$, 5 distinct cases in all.
$\therefore$ Required probability $= \frac{5}{36}$

P2:

If six dice are rolled, then find the probability that all show different faces

Solution:

In a random roll of six dice, the exhaustive number of cases is $n(S) = 6^6$.

Define the event E : All the six dice show different faces.

We can get any one of the six faces 1,2,3,4,5,6, on the first dice. For the happening of E, the second die must show any one of the remaining 5 faces, the third die must show any one of the remaining 4 faces, and so on, the 6^{th} die must show the remaining last face.

Hence, by the product rule, the number of cases favourable to the happening of E are $n(E) = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{6!}{6^6}$$

P3:

The letters of the word ‘failure’ are arranged at random. Find the probability that the consonants may occupy only odd positions.

Solution:

There are 7 distinct letters in the word ‘failure’ and they can be arranged among themselves in 7! ways, which gives the exhaustive number of cases.

In the word ‘failure’ there are 4 vowels viz., a, i, u and e, and 3 consonants viz., f, l, r. These 3 consonants are to be placed in the 4 odd places viz., 1^{st}, 3^{rd}, 5^{th} and 7^{th} and this can be done in ${}^4C_3$ ways. Further these 3 consonants can be arranged among themselves in $3!$ ways and the remaining 4 vowels can be arranged among themselves in $4!$ ways. Associating all these operations, total number of favourable cases for the consonants to occupy only odd positions is ${}^4C_3 \times 3! \times 4!$.

$\therefore$ Required probability $= \dfrac{{}^4C_3 \times 3! \times 4!}{7!} = \dfrac{4 \times 3!}{7 \times 6 \times 5} = \dfrac{4}{35}$.

P4:

Find the probability that in a random arrangement of the letters of the word ASSASSINATION, the four S's, come consecutively.

Solution:

The word 'ASSASSINATION' contains 13 letters in which A occurs 3 times, S 4 times, N 2 times and I 2 times, and T and O once each.

Hence, the total number of permutations (exhaustive number of cases) is: $\frac{13!}{3!4!2!2!}$

The four S's can come consecutively if they occupy the 10 positions as given below.

(i) First four positions,
(ii) Second to 5th positions,
(iii) Third to 6th positions.
… … … …
(x) Last four positions i.e., 10th to 13th positions.

In each of the above 10 arrangements, the remaining nine letters viz., 'AAINATION' of which 3 are A's, 2 are N's and 2 are I's, and the rest all different which can be arranged among themselves in $\frac{9!}{3!2!2!}$ ways.

Hence, total number of favourable cases is $\frac{10\times9!}{3!2!2!}$

$\therefore$ Required probability $= \frac{10\times9!}{3!2!2!} \div \frac{13!}{3!4!2!2!} = \frac{10\times9!4!}{13!} = \frac{10\times4!}{13\times12\times11\times10} = \frac{2}{143}$

P5:

In a random arrangement of the letters of the word 'MATHEMATICS', find the probability that all the vowels come together.

Solution:

The total number of permutations of the letters of the word 'MATHEMATICS' are $\frac{11!}{2!2!2!}$, because it contains 11 letters, of which 2 are A's, 2 M's, 2 T's, and remaining are all different.

The word MATHEMATICS contains 4 vowels viz., AEAI, (2 A's being identical). To obtain the total number of arrangements in which these 4 vowels come together, we regard them as tied together, forming only one letter so that, the total number of letters in MATHEMATICS may be taken as $11 - 3 = 8$, out of which 2 are M's, 2 are T's and rest distinct and therefore, their number of arrangements is given by $\frac{8!}{2!2!}$

Further, the four vowels AEAI, two of which are identical and rest distinct can be arranged among themselves in $\frac{4!}{2!}$ ways. Hence, the total number of arrangements favourable to getting all vowels together is: $\frac{8!}{2!2!} \times \frac{4!}{2!}$

$\therefore$ Required probability $= \frac{8!4!}{2!2!2!} \div \frac{11!}{2!2!2!} = \frac{8!4!}{11!} = \frac{4!}{11\times10\times9} = \frac{4}{165}$

1.3. Definitions of Probability

Exercise:

1. What is the probability that a leap year selected at random will contain (a) 53 Tuesdays and (b) 53 Sundays or 53 Mondays?

2. In a single throw of two dice, what is the probability of getting (a) a total of 8 ; and (b) total different from 8 :

3. Prove that in a single throw with a pair of dice, the probability of getting the sum of 7 is equal to $\frac{1}{6}$ and the probability of getting the sum of 10 is equal to $\frac{1}{12}$

4. In the play of two dice, the thrower loses if his first throw is 2,4, or 12. He wins if his first throw is a 5 or 11. Find the ratio between his probability of losing and probability of winning in the first throw.
   Hint: Number of favourable cases for getting
   (a) 2,4 or 12 is $1 + 3 + 1 = 5$; (b) 5 or 11 is $4 + 2 = 6$

5. If a pair of dice is thrown, find the probability that the sum of the digits on them is neither 7 nor 11.

6. Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has :
   (a) an even number? (b) a number 5 or a multiple of 5?
   (c) a number which is greater than 75? (d) a number which is a square?

7. There are 17 balls, numbered from 1 to 17 in a bag. If a person selects one ball at random, what is the probability that the number printed on the ball will be an even number greater than 9?

8. An integer is chosen at random from the first 200 positive integers. What is the probability that integer chosen is divisible by 6 or 8?

9. One ticket is drawn at random from a bag containing 30 tickets numbered from 1 to 30. Find the probability that

   (a) It is multiple of 5 or 7; (b) It is multiple of 3 or 5

10. A number is chosen from each of the two sets :

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad ; \quad B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

   If P_1 is the probability that the sum of the two numbers be 10 and P_2 the probability that their sum be 8, find $P_1 + P_2$.

11. A bag contains 7 white and 9 black balls. Two balls are drawn in succession at random. What is the probability that one of them is white and the other is black?

12. A bag contains eight balls, five being red and three white. If a man selects two balls at random from the bag. What is the probability that he will get one ball of each colour?

13. A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that balls drawn are black.

14. A bag contains 4 white, 5 red and 6 green balls. Three balls are drawn at random. What is the probability that a white, a red and a green ball are drawn?

15. A bag contains 8 black, 3 red and 9 white balls. If 3 balls are drawn at random, find the probability that
(a) all are black, (b) 2 are black and 1 is white, (c) 1 is of each colour,
(d) the balls are drawn in the order black, red and white, (e) None is red.

16. The Federal Match Company has forty female employees and sixty male employees. If two employees are selected at random, what is the probability that (i) both will be males, (ii) both will be females, (iii) there will be one of each sex?

17. If a single draw is made from a pack of 52 cards, what is the probability of securing either an ace of spades or a jack of clubs.

18. Four cards are drawn from a full pack of cards. Find the probability that two are spades and two are hearts?

19. What is the probability of getting 9 cards of the same suit in one hand at a game of bridge?

20. The letters of the word **Triangle** are arranged at random. Find the probability that the word so formed (a) starts with T, (b) ends with E, (c) starts with T and ends with E.

21. In a random arrangement of the letters of the word **VIOLENT**, find the chance that the vowels I,O,E occupy odd positions only.

22. In a random arrangement of the letters of the word **Allahabad**, find the chance that the vowels occupy the even places.

23.The letters of the word **ARRANGE** are arranged at random. Find the chance that : (a) The two R's come together (b) The two R's does not come together (c) The two R's and the two A's come together

24. (a) If the letters of the word REGUALTIONS be arrange at random, what is the chance that there will be exactly 4 letters between the R and the E?

(b) What is the probability that four S's come consecutively in the word

MISSISSIPPI?

25. A and B stand in a ring with 10 others persons. If the arrangement of the persons is at random, find the chance that (a) there are exactly three persons between A and B (b) A and B stand together

26.The first 12 letters of the English alphabet are written at random. What is the probability that (a) there are 4 letters between A and B (b) A and B are written down side by side.

27.Seven persons sit in a row at random. Find the chance that (a) three persons A, B, C sit together in a particular order (b) A, B, C sit together in any order (c) B and C occupy the end seats (iv) C always occupies the middle seat.

28.A six figure number is formed by the digits 4, 5, 6, 7, 8, 9 ; no digit being repeated. Find the probability that the number formed is (a) divisible by 5 (b) not divisible by 5.

29.Five digited numbers are formed from the digits 1, 2, 3, 4, 5. Find the chance that the number formed is greater than 2300.

Answers:

1. (a) $\frac{2}{7}$ (b) $\frac{3}{7}$
2. (a) $\frac{5}{36}$ (b) $\frac{31}{36}$
3.
4.
5. $\frac{7}{9} = 0.78$

6. (a) 0.5 (b) 0.2 (c) 0.25 (d) 0.10

7. $\frac{4}{17}$

8. $\frac{1}{4}$

9. (a) $\frac{1}{3}$ (b) $\frac{7}{15}$

10. $\frac{16}{81}$

11. $\frac{21}{40}$

12. $\frac{{}^5C_1 \times {}^3C_1}{{}^8C_2} = \frac{15}{28}$
13. $\frac{3}{28}$
14. $\frac{24}{91}$

15. (a) $\frac{14}{285}$ (b) $\frac{21}{95}$ (c) $\frac{18}{95}$ (d) $\frac{3}{95}$ (e) $\frac{34}{57}$

16. (a) 0.357, (b) 0.157, (c) 0.4848

17. $\frac{1}{26}$

18. $\frac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4} = \frac{468}{20825}$

19. $4 \times {}^{13}C_9 \times {}^{39}C_4 / {}^{52}C_4$

20. (a) $\frac{1}{8}$ (b) $\frac{1}{8}$ (c) $\frac{1}{56}$

21. $\frac{{}^{4}C_3 \times 3!4!}{7!} = \frac{4}{35}$

22. $\frac{1\times 5!}{2!} \div \frac{9!}{4!2!} = \frac{1}{126}$

23. (a) $\frac{6!}{2!} \div \frac{7!}{2!2!} = \frac{2}{7}$ (b) $(1260 - 360) \div 1260 = \frac{5}{7}$ (c) $\frac{5!}{1260} = \frac{2}{21}$

24. (a) $\frac{6}{55}$ (b) $\frac{4}{165}$

25. (a) $\frac{2}{11}$ (b) $\frac{2}{11}$

26. (a) $\frac{7}{66}$ (b) $\frac{1}{6}$

27. (a) $\frac{5!}{7!} = \frac{1}{42}$ (b) $\frac{5!\,3!}{7!} = \frac{1}{7}$ (c) $\frac{5\times 2!}{7!} = \frac{1}{21}$ (d) $\frac{6!}{7!} = \frac{1}{7}$

28. (a) $\frac{1}{6}$ (b) $\frac{5}{6}$

29. $\frac{3!+3\times 4!}{5!} = \frac{13}{20}$

1.4

Theorems in Probability

In this module, we shall prove some theorems which help us to evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach based on the three axioms given in axiomatic definition of probability in module 1.3 on definitions of probability.

In a problem on probability, we are required to evaluate probability of certain statements. These statements can be expressed in terms of set notation and whose probabilities can be evaluated using theorems in probability. Let A and B be two events in S. Certain statements in set notation are given in the following table.

S. No.	Statement	Set notation
1.	At least one of the events A or B occurs	$A \cup B$
2.	Both the events A and B occur	$A \cap B$
3.	Neither A nor B occurs	$\bar{A} \cap \bar{B}$
4.	Event A occurs and B does not occur	$A \cap \bar{B}$
5.	Exactly one of the events A or B occurs	$(\bar{A} \cap B) \cup (A \cap \bar{B}) = A \Delta B$
6.	Not more than one of the events A or B occurs	$(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})$
7.	If event A occurs, so does B	$A \subset B$
8.	Events A and B are mutually exclusive	$A \cap B = \phi$
9.	Complement of event A	$\bar{A}$
10.	Sample space	S

Example 1: Let A, B and C are three events in S. Find expression for the events in set notation.

(i) only A occurs

(ii) both A and B, but not C, occur

(iii) all three events occur

(iv) at least one occurs

(v) at least two occur

(vi) one and no more occurs

(Vii) two and no more occur

(viii) none occurs

Solution:

(i) $A \cap \bar{B} \cap \bar{C}$

(ii) $A \cap B \cap \bar{C}$

(iii) $A \cap B \cap C$

(iv) $A \cup B \cup C$

(v) $(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \overline{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

(vii) $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

(viii) $(\bar{A} \cap \bar{B} \cap \bar{C}) = (\overline{A \cup B \cup C})$

Theorems on Probability

Theorem 1: Probability of the impossible event is zero, i.e., $P(\phi) = 0$.

Proof: We know that $S \cup \phi = S \Rightarrow P(S) = P(S \cup \phi)$

$$\Rightarrow P(S) = P(S) + P(\phi) \text{ (Axiom 3)}$$

$$\Rightarrow P(\phi) = 0$$

Theorem 2: Probability of the complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$.

Proof: Since A and $\bar{A}$ are mutually exclusive events in S,

$$A \cup \bar{A} = S \Rightarrow P(A \cup \bar{A}) = P(S) \Rightarrow P(A) + P(\bar{A}) = 1 \text{ (Axioms 2 and 3)}$$

$$\Rightarrow P(\bar{A}) = 1 - P(A)$$

Corollary 1: $\mathbf{0 \leq P(A) \leq 1}$

Proof: We have $P(A) = 1 - P(\bar{A}) \leq 1$ ($\because$ $P(\bar{A}) \geq 0$, by Axiom 1)

Further, $P(A) \geq 0$ (by Axiom 1). Therefore, $0 \leq P(A) \leq 1$

Corollary 2: $\boldsymbol{P(\phi) = 0}$

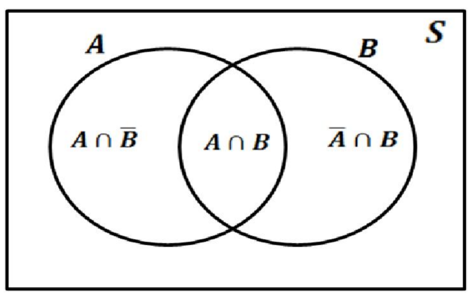
Proof: Since $\phi = \bar{S}$, $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$ (by Axiom 2)

$\Rightarrow P(\phi) = 0$

Theorem 3: For any two events $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

(i) $\boldsymbol{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$ **(ii)** $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$

Proof:



(i) From the Venn diagram, we have,

$$B = (A \cap B) \cup (\bar{A} \cap B),$$

where $(\bar{A} \cap B)$ and $(A \cap B)$ are mutually exclusive events. Hence by Axiom 3,

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$
$$\Longrightarrow \boldsymbol{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$$

(ii) Similarly, we have,

$$A = (A \cap B) \cup (A \cap \bar{B}),$$

where $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive events. Hence by Axiom 3

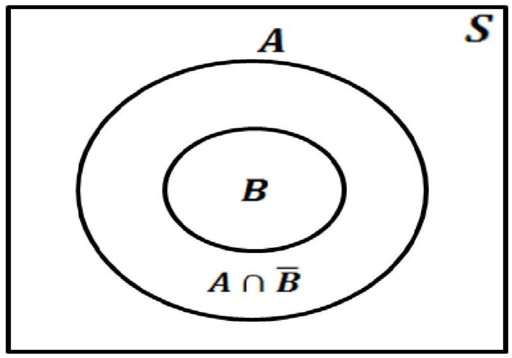
$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$\Longrightarrow \boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$$

Theorem 4: If $\boldsymbol{B \subset A}$, then

(i) $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(B)}$ **(ii)** $\boldsymbol{P(B) \leq P(A)}$

Proof:



(i) If $B \subset A$, then B and $A \cap \bar{B}$ are mutually exclusive events and

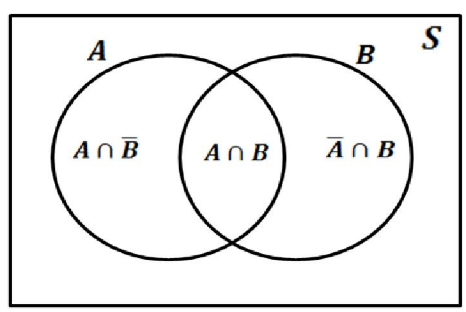
$$A = B \cup (A \cap \bar{B})$$

$$\Longrightarrow P(A) = P(B) + P(A \cap \bar{B}) \text{ (Axiom 3)}$$

$$\Longrightarrow P(A \cap \bar{B}) = P(A) - P(B)$$

(ii) We have $P(A \cap \bar{B}) \geq 0$ (Axiom 1). Hence $P(A) - P(B) \geq 0 \Rightarrow P(B) \leq P(A)$.
Thus, $B \subset A \Rightarrow P(B) \leq P(A)$.

Theorem 5: Addition Theorem of Probability for Two Events:
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two events in $\boldsymbol{S}$. Then
$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$



Proof: From Venn diagram, we have

$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually exclusive events in S.

$\therefore\ P(A \cup B) = P(A) + P(\bar{A} \cap B)$ (Axiom 3)

$= P(A) + P(B) - P(A \cap B)$ (From Theorem 3)

Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note:

1. If A and B are mutually exclusive events then $A \cap B = \phi$ and hence $P(A \cap B) = P(\phi) = 0$. Thus, if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
2. The addition theorem of probability for three events is given by

$$\boldsymbol{P(A \cup B \cup C) =}$$

$$\boldsymbol{P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

This can be proved first by taking $A \cup B$ as one event and C as second event and repeated application of Theorem 5

$$P(A \cup B \cup C) = P\big((A \cup B) \cup C\big) = P(A \cup B) + P(C) - P\big((A \cup B) \cap C\big)$$

$$= P(A \cup B) + P(C) - P\big((A \cap C) \cup (B \cap C)\big)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

3. Addition Theorem of Probability for $\boldsymbol{n}$-Events

Let $\boldsymbol{A_1, A_2, .., A_n}$ be $\boldsymbol{n}$ events in $\boldsymbol{S}$. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} P(A_i \cap A_j) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j<k}}^{n} \sum_{k=1}^{n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Example 2: If two dice are thrown, what is the probability that the sum is (i) greater than 8, (ii) neither 7 nor 11 (iii) an even number on the first die or a total of 8?

Solution:

(i) If two dice are thrown, then $n(S) = 6^2 = 36$. Let T be the event getting the sum of the numbers greater than 8 on the two dice. Then $T = A \cup B \cup C \cup D$, where A, B, C and D repectively the events of getting the sum of 9,10,11 and 12. Note that A, B, C and D are pair wise mutually exclusive events. Therefore

$$P(T) = P(A) + P(B) + P(C) + P(D)$$

Note that $A = \{(3,6), (4,5), (5,4), (6,3)\}$ and $P(A) = \frac{4}{36}$

$B = \{(4,6), (5,5), (6,4)\}$ and $P(B) = \frac{3}{36}$

$C = \{(5,6), (6,5)\}$ and $P(C) = \frac{2}{36}$

$D = \{(6,6)\}$ and $P(D) = \frac{1}{36}$

$\therefore P(T) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$

(ii) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11.Then

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ and $P(A) = \frac{6}{36}$

$B = \{(5,6), (6,5)\}$ and $P(B) = \frac{2}{36}$

$\therefore$ Required probability $= P$ (neither 7 nor 11)

$= P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B)]$ ($\because$ A and B are mutually exclusive events)

$= 1 - \left[\frac{6}{36} + \frac{2}{36}\right] = 1 - \frac{8}{36} = 1 - \frac{2}{9} = \frac{7}{9}$

(iii) Let A be the event of getting an even number on the first die and B be the event of getting the sum of 8. Therefore,

$A = \{2,4,6\} \times \{1,2,3,4,5,6\} \Rightarrow n(A) = 3 \times 6 = 18,$

$B = \{(2,6), (3,5), (4,4), (5,3), (6,2)\} \Rightarrow n(B) = 5,$

$A \cap B = \{(2,6),(4,4),(6,2)\} \Rightarrow n(A \cap B) = 3$ and
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}$

Example 3: A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution: Let us define the following events:

A : The card drawn is a king

B : The card drawn is a heart

C : The card drawn is a red card

Then, A, B and C are not mutually exclusive.

$n(A) = 4$, $n(B) = 13$, $n(C) = 26$, $n(A \cap B) = 1$, $n(A \cap C) = 2$,
$n(B \cap C) = 13$, $n(A \cap B \cap C) = 1$.

$$P(A \cup B \cup C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{2}{52} - \frac{13}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

Compound event: The simultaneous occurrence of two or more events is termed as compound event.

Compound probability: The probability of a compound event is known as compound probability.

Conditional probability: The probability of an event A occuring when it is known that some event B has occurred, is called a conditional probability of the event A, given that B has occurred and denoted by $\boldsymbol{P(A|\ B)}$.

Definition: The conditional probability of the event A, given that B has occurred, denoted by $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) > 0$$

If $P(B) = 0$, $P(A|B)$ is not defined.

Example 4: Consider a family with two children. Assume that each child is likely to be a boy as it is to be a girl. What is the conditional probability that both children are boys, given that (i) the older child is a boy (ii) at least one of the child is a boy?

Solution: We have the sample space $S = \{(b, b), (b, g), (gg\ ), (gg\ )\}$.Define the events:

A : Older child is a boy

B : Younger child is a boy

Therefore, $A = \{(b, b), (b, g)\}$, $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$, $B = \{(b, b), (gb\ )\}$

Then $A \cap B$: both are boys, $A \cap B = \{(b, b)\}$ and $P(A \cap B) = \frac{n(A \cap B)}{n(s)} = \frac{1}{4}$

$A \cup B$: At least one is a boy

and $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(i) $P\big((A \cap B)|A\big) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

(ii) $P\big((A \cap B)|(A \cup B)\big) = \frac{P[(A \cap B) \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Independent events: Two events A and B are said to be independent if the happening or non-happening of A is not affected by the happening or non-happening of B. Thus, A and B are independent if and only if the conditional probability of the event A given that B has happened is equal to the probability of A. That is,

$$P(A|B) = P(A) \text{ if } P(B) > 0$$

Similarly $P(B|A) = P(B)$ if $P(A) > 0$

By the definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Thus, A and B are independent events, if and only if

$$P(A \cap B) = P(A|B) \cdot P(B) = P(A) \cdot P(B)$$

In general, $A_1, A_2, \dots, A_n$ are independent events, if and only if

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n)$$

Pair wise Independent Events: A set of events $A_1, A_2, \dots, A_n$ are said to be pairwise independent if every pair of different events are independent.

That is, $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ for all i and j, $i \neq j$.

Mutual Independent Events: A set of events $A_1, A_2, \dots, A_n$ are said to be mutually independent, if $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$ for every subset $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ of $\{A_1, A_2, \dots, A_n\}$.

Note: Pair wise independence does not imply mutual independence.

Theorem 6: Multiplication Theorem for Two events

Let A and B be any two events, then

$$P(A \cap B) = \begin{cases} P(A).P(B|A) & \text{if } P(A) > 0 \\ P(B).P(A|B) & \text{if } P(B) > 0 \\ P(A).P(B) & \text{if A and B are independent} \end{cases}$$

The proof follows from definition of conditional probability.

Note: Multiplication Theorem for n-Events $A_1, A_2, A_3, \ldots, A_n$

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = \begin{cases} P(A_1).P(A_2 \mid A_1).P(A_3 \mid (A_1 \cap A_2))\ldots P(A_n \mid (A_1 \cap A_2 \cap \ldots \cap A_{n-1})) \\ P(A_1).P(A_2).P(A_3)\ldots P(A_n), \text{if } A_1, A_2, \ldots, A_n \text{ are independent} \end{cases}$$

Theorem 7: If A_1 and A_2 are independent events, then A_1 and $\overline{A_2}$ are also independent.

Proof: **(See P3)**

Theorem 8: If A_1 and A_2 are independent events, then $\overline{A_1}$ and $\overline{A_2}$ are also independent.

Proof: **(See P4)**

Example 5: A fair dice is thrown twice. Let A, B and C denote the following events:

A : First toss is odd; B : Second toss is even; C Sum of numbers is 7

- **(i) Find $P(A), P(B)$ and $P(C)$.**
- **(ii) Show that A, B and C are pair wise independent**
- **(iii) Show that A, B and C are not independent**

Solution:

(i) The number of outcomes in the sample space S is given by $n(S) = 6^2 = 36$. We have,

$$A = \{1,3,5\} \times \{1,2,3,4,5,6\} \text{ and } n(A) = 3 \times 6 = 18$$

$$B = \{1,2,3,4,5,6\} \times \{2,4,6\} \text{ and } n(B) = 6 \times 3 = 18$$

$$C = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \text{ and } n(C) = 6$$

Therefore, $P(A) = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{18}{36} = \frac{1}{2}$ and $P(C) = \frac{6}{36} = \frac{1}{6}$.

(ii) $A \cap B = \{(1,2), (1,4), (1,6), (3,2), (3,4), (3,6), (5,2), (5,4), (5,6)\}$

$\therefore n(A \cap B) = 9$ and $P(A \cap B) = \frac{9}{36} = \frac{1}{4}$

But $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Thus, $P(A \cap B) = P(A) \cdot P(B) \Rightarrow A$ and B are independent.

Next consider $A \cap C = \{(1,6), (3,4), (5,2)\}$

$\therefore n(A \cap C) = 3$ and $P(A \cap C) = \frac{3}{36} = \frac{1}{12}$.

But $P(B) \cdot P(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

Thus, $P(B \cap C) = P(B) \cdot P(C) \Rightarrow B$ and C are independent

(iii) Consider $A \cap B \cap C = \{(1,6), (3,4), (5,2)\}$

$\therefore n(A \cap B \cap C) = 3$ and $P(A \cap B \cap C) = \frac{3}{36} = \frac{1}{12}$

But $P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24}$

Thus, $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

$\Rightarrow A, B$ and C are not independent.

Theorem 9: If A_1 and A_2 are independent events, then

$$\boldsymbol{P(A_1 \cup A_2) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})}$$

Proof: Consider $RHS = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})$

$$= 1 - \left[\left(1 - P(A_1)\right) \cdot \left(1 - P(A_2)\right) \right]$$

$$= 1 - \left(1 - P(A_1) - P(A_2) + P(A_1) \cdot P(A_2)\right)$$

$$= 1 - 1 + P(A_1) + P(A_2) - P(A_1) \cdot P(A_2)$$

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= P(A_1 \cup A_2)$$

Thus, $P(A_1 \cup A_2) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})$.

Generalization: If $A_1, A_2, \dots, A_n$ are n independent events, then

$$\boldsymbol{P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2}) \cdot \dots \cdot P(\overline{A_n})}$$

Example 6: A problem in probability is given to three students A, B and C whose chances of solving it are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that the problem will be solved if they all try independently.

Solution: Let E_1, E_2 and E_3 denote the events that the problem is solved by A, B and C respectively. Then, we have

$$P(E_1) = \frac{1}{3} \Rightarrow P(\overline{E_1}) = \frac{2}{3}$$

$$P(E_2) = \frac{1}{4} \Rightarrow P(\overline{E_2}) = \frac{3}{4}$$

$$P(E_3) = \frac{1}{5} \Rightarrow P(\overline{E_3}) = \frac{4}{5}$$

The problem is solved if atleast one of them is able to solve it.

Thus, $P(E_1 \cup E_2 \cup A_3) = 1 - P(\overline{E_1}) \cdot P(\overline{E_2}) \cdot P(\overline{E_3}) = 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = 1 - \frac{2}{5} = \frac{3}{5}$

P1:

If one card is selected at random from a pack of 52 cards, find the probability that it is (i) a diamond or a spade (ii) an ace or a spade.

Solution:

(i) Let us denote the event of getting a diamond by A and a Spade by B. Then there are no common cards to diamonds and spades and hence $A \cap B = \phi$. Thus, A and B are mutually exclusive events. Further, $P(A) = \frac{13}{52} = \frac{1}{4}$ and $(B) = \frac{13}{52} = \frac{1}{4}$. Hence, by addition theorem, we have

P (a diamond or a spade) $= P(A \cup B) = P(A) + P(B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(ii) Let us denote the event of getting an ace by A and a Spade by B. Then there is one card common for aces and spades and hence $A \cap B \neq \phi$. Thus, A and B are not mutually exclusive events. Further, $P(A) = \frac{4}{52}$, $P(B) = \frac{13}{52}$ and $P(A \cap B) = \frac{1}{52}$. Hence, by addition theorem, we have

P (an ace or a spade) $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

P2:

Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population: 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers?

Solution:

Let E, F and G denote the events that the adult reads newspapers A, B and C respectively. Then we are given:

$P(E) = 0.20$, $P(F) = 0.16$, $P(G) = 0.14$, $P(E \cap F) = 0.08$, $P(E \cap G) = 0.05$, $P(F \cap G) = 0.04$ and $P(E \cap F \cap G) = 0.02$. Thus,

$$P(E \cup F \cup G)$$

$$= P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$

$$= 0.2 + 0.16 + 0.14 - 0.08 - 0.05 - 0.04 + 0.02 = 0.35$$

P3:

Theorem: If A_1 and A_2 are independent events, then A_1 and $\overline{A_2}$ are also independent.

Proof: Given that A_1 and A_2 are independent events.

$P(A_1 \cap \overline{A}_2) = P(A_1) - P(A_1 \cap A_2)$ (By Theorem 3(ii))

$= P(A_1) - P(A_1) \cdot P(A_2)$ ($\because$ A_1 and A_2 are independent)

$= P(A_1)\big(1 - P(A_2)\big)$

$= P(A_1) \cdot P(\overline{A_2})$

$\Rightarrow P(A_1 \cap \overline{A_2}) = P(A_1) \cdot P(\overline{A_2})$

Thus, A_1 and $\overline{A_2}$ are independent.

P4:

Theorem: If A_1 and A_2 are independent events, then $\overline{A_1}$ and $\overline{A_2}$ are also independent.

Proof: Given that A_1 and A_2 are independent events.

Consider $P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1 \cup A_2}) = 1 - P(A_1 \cup A_2)$

$$= 1 - \big(P(A_1) + P(A_2) - P(A_1 \cap A_2)\big)$$

$$= 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2)$$

$$= 1 - P(A_1) - P(A_2) + P(A_1) \cdot P(A_2) \quad (\because A_1 \& A_2 \text{ are independent})$$

$$= P(\overline{A_1}) - P(A_2)\big(1 - P(A_1)\big)$$

$$= P(\overline{A_1}) - P(A_2) \cdot P(\overline{A_1})$$

$$= P(\overline{A_1})\big(1 - P(A_2)\big)$$

$$= P(\overline{A_1}) \cdot P(\overline{A_2})$$

$\Rightarrow P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1}) \cdot P(\overline{A_2})$

Thus, $\overline{A_1}$ and $\overline{A_2}$ are also independent.

1.5

Bayes'Theorem and Its Applications

One of the important applications of the conditional probability is in the computation of unknown probabilities on the basis of the information supplied by the experiment or past records. For example, suppose an event has occurred through one of the various mutually exclusive events or reasons. Then the conditional probability that it has occurred due to a particular event or reason is called it as **inverse** or **posteriori probability**. These probabilities are computed by Bayes' theorem, named so after the British mathematician **Thomas Bayes** who propounded it in 1763. The revision of old (given) probabilities in the list of the additional information supplied by the experiment or past records is of extreme help in arriving at valid decisions in the face of uncertainty.

Bayes' Theorem (Rule for the Inverse Probability)

Let $E_1, E_2, \dots, E_n$ be n be mutually exclusive and exhaustive events in the sample space S with $P(E_i) \neq 0$ for $i = 1, 2, \dots, n$. Let A be an arbitrary event which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$. Then

$$P(E_i \mid A) = \frac{P(E_i).P(A|E_i)}{\sum_{i=1}^{n} P(E_i).P(A|E_i)} = \frac{P(E_i).P(A|E_i)}{P(E_i)} \qquad \text{for } i = 1, 2, \dots, n$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$, we have $A = A \cap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (A \cap E_i)$.

Since $(A \cap E_i) \subset E_i (i = 1, 2, \dots, n)$ are mutually exclusive events, we have by addition theorem of probability

$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap E_i) \right)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$\Rightarrow P(A)=\sum_{i=1}^{n} P(E_i).P(A|E_i)$$ (By multiplication theorem of probability)

Also we have

$$P(A \cap E_i) = P(E_i|A).P(A)$$

$$\Rightarrow P(E_i|A) = \frac{P(A \cap E_i)}{P(A)}$$

$$\Rightarrow P(E_i|A)=\frac{P(E_i).P(A|E_i)}{\sum_{i=1}^{n} P(E_i).P(A|E_i)} \quad \text{for } i = 1, 2, \dots, n$$

which is the Bayes' rule.

Note:

1. The probabilities $P(E_1), P(E_2), \dots, P(E_n)$ are known as the 'a priori probabilities', because they exist before we gain any information from the experiment itself.
2. The probabilities $P(A|\, E_i)$ $i = 1, 2, \dots n$ are called 'likelihoods' because they indicate how likely the event A under consideration is to occur, given each and every a priori probability.
3. The probabilities $P(E_i|A), i = 1, 2, .., n$ are called 'posteriori probabilities' because they are determined after the results of the experiment are known.
4. $P(A)=\sum_{i=1}^{n} P(E_i).P(A|E_i)$ is known as **total probability**.
5. Bayes' theorem is extensively used by *business, management* and *engineering* executives in arriving at valid decisions in the face of uncertainty.

Example 1: In a bolt factory machines A, B, C manufacutre respectively 25%, 35% and 40% of the total. Of their output $5, 4, 2$ percent are known to be defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by

- **(i) Machine A.**
- **(ii) Machine B or C**

Solution: Let E_1, E_2 and E_3 denote respectively the events that the bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event that it is defective. Then we have:

E_i	E_1	E_2	E_3	Total
$P(E_i)$	0.25	0.35	0.40	1
$P(E\|E_i)$	0.05	0.04	0.02	
$P(E \cap E_i) = P(E_i).P(E\|E_i)$	0.0125	0.0140	0.0080	$P(E) = 0.0345$

$$P(E) = \sum_{i=1}^{3} P(E_i).P(E|E_i) = 0.0345$$

(i) Hence, the probability that a defective bolt chosen at random is manufactured by factory A is given by Bayes' rule as:

$$P(E_1|E) = \frac{P(E_1)P(E|E_1)}{\sum P(E_i)P(E|E_i)} = \frac{0.0125}{0.0345} = 0.36$$

(ii) Similarly,

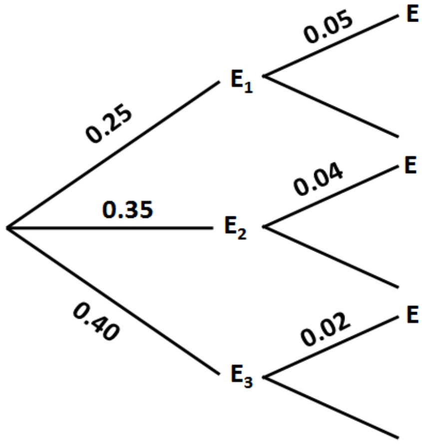
$$P(E_2|E) = \frac{0.0140}{0.0345} = \frac{28}{69} = 0.41\,; \qquad P(E_3|E) = \frac{0.0080}{0.0345} = \frac{16}{69} = 0.23$$

Hence, the probability that a defective bolt chosen at random is manufactured by machine B or C is:

$$P(E_2|E) + P(E_3|E) = 0.41 + 0.23 = 0.64$$

(OR Required probability is equal to $1 - P(E_1|E) = 1 - 0.36 = 0.64$)

Aliter: **TREE DIAGRAM**



Events	Probability
$E_1 \cap E$	$0.25 \times 0.05 = 0.0125$
$E_2 \cap E$	$0.35 \times 0.04 = 0.0140$
$E_3 \cap E$	$0.40 \times 0.02 = 0.0080$
Total	**0.0345**

From the above diagram the probability that a defective bolt is manufactured by factory A is

$$P(E_1|E) = \frac{0.0125}{0.0345} = 0.36$$

Similarly, $P(E_2|E) = \frac{0.0140}{0.0345} = 0.41$ and $P(E_3|E) = \frac{0.0080}{0.0345} = 0.23$

Hence, the probability that a defective bolt chosen at random is manufactured by machine B or C is:

$$P(E_2|E) + P(E_3|E) = 0.41 + 0.23 = 0.64$$

(OR Required probability is equal to $1 - P(E_1|E) = 1 - 0.36 = 0.64$)

Remark: Since $P(E_3)$ is greatest, on the basis of 'a *priori*' probabilities alone, we are likely to conclude that a defective bolt drawn at random from the product is manufactured by machine C. After using the additional information we obtained the *posterior* probabilities which give $P(E_2|E)$ as maximum. Thus, we shall now say that it is probable that the defective bolt has been manufactured by machine B, a result which is different from the earlier conclusion. However, latter conclusion is a much valid conclusion as it is based on the entire information at our disposal. Thus, Bayes's rule provides a very powerful tool in improving the quality of probability and this helps the management executives in arriving at valid decisions in the face of uncertainty. Thus, the additional information reduces the importance of the prior probabilities. The only requirement for the use of *Bayesian rule* is that all the hypotheses under consideration must be valid and that none is assigned ‘a prior’ probability 0 or 1.

Example 2: In a railway reservation office, two clerks are engaged in checking reservation forms. On an average, the first clerk checks 55% of the forms, while the second does the remaining. The first clerk has an error rate of 0.03 and second has an error rate of 0.02 , A reservation form is selected at random from the total number of forms checked during a day, and is found to have an error. Find the probability that it was checked (i) by the first (ii) by the second clerk.

Solution: Let us define the following events:

E_1: The selected form is checked by clerk 1.

E_2: The selected form is checked by clerk 2.

E: The selected form has an error.

Then we are given:

$$P(E_1) = 55\% = 0.55 \;; \quad P(E_2) = 45\% = 0.45 \;;$$

$$P(E|E_1) = 0.03 \quad ; \quad P(E|E_2) = 0.02$$

Required to find $P(E_1|E)$ and $P(E_2|E)$. By Bayes’ Rule the probability that the form containing the error was checked by clerk 1, is given by;

$$P(E_1|E) = \frac{P(E_1)\,P(E_1|E)}{P(E_1)\,P(E|E_1)+P(E_2)P(E|E_2)} = \frac{0.55\times 0.03}{0.55\times 0.03+0.45\times 0.02}$$

$$= \frac{0.0165}{0.0165+0.0090} = \frac{0.0165}{0.0255} = 0.647$$

Similarly, the probability that the form containing the error was checked by clerk 2, is given by

$$P(E_2|E) = \frac{P(E_2)\,P(E|E_2)}{P(E_1)P(E|E_1)+P(E_2)P(E|E_2)} = \frac{0.45\times 0.02}{0.55\times 0.03+0.45\times 0.02} = \frac{0.0090}{0.0255} = 0.353$$

(OR $P(E_2|E) = 1 - P(E_1|E) = 1 - 0.647 = 0.353$)

Example 3: The results of an investigating by an expert on a fire accident in a skyscraper are summarized below:

(i) Prob. (there could have been short circuit) $= \mathbf{0.8}$

(ii) Prob. (LPG cylinder explosion) $= \mathbf{0.2}$

(iii) Chance of fire accident is 30% given a short circuit and 95% given an LPG explosion.

Based on these, what do you think is the most probable cause of fire?

Solution: Let us define the following events:

E_1: Short circuit ; E_2: LPG explosion ; E: Fire accident

Then, we are given:

$$P(E_1) = 0.8 \ ; \quad P(E_2) = 0.2 \ ;$$

$$P(E|E_1) = 0.30 \ ; \quad P(E|E_2) = 0.95$$

By Bayes' Rule:

$$P(E_1|E) = \frac{P(E_1).P(E|E_1)}{P(E_1)P(E|E_1)+P(E_2)P(E|E_2)} = \frac{0.80\times 0.30}{0.8\times 0.30+0.2\times 0.95} = \frac{0.240}{0.240+0.190} = \frac{24}{43}$$

$$P(E_2|E) = \frac{P(E_2)P(E|E_2)}{P(E_1)P(E|E_1)+P(E_2)P(E|E_2)} = \frac{0.190}{0.430} = \frac{19}{43}$$

(OR $P(E_2|E) = 1 - P(E_1|E) = 1 - \frac{24}{43} = \frac{19}{43}$)

Since $P(E_1|E) > P(E_2|E)$, short circuit is the most probable cause of fire.

Example 4: The contents of urns I, II and III are respectively as follows:

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they came from urns I, II, III?

Solution:

Let E_1, E_2 and E_3 denote the events of choosing 1st , 2nd and 3rd urn respectively and let E be the event that the two balls drawn from the selected urn are white and red. Then we have:

	E_1	E_2	E_3
$P(E_i)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$P(E\|E_i)$	$\frac{1\times 3}{6_{C_2}} = \frac{1}{5}$	$\frac{2\times 1}{4_{C_2}} = \frac{1}{3}$	$\frac{4\times 3}{12_{C_2}} = \frac{2}{11}$
$P(E\cap E_i) = P(E_i)\times P(E\|E_i)$	$\frac{1}{3}\times\frac{1}{5} = \frac{1}{15}$	$\frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$	$\frac{1}{3}\times\frac{2}{11} = \frac{2}{33}$

We have:

$$\sum P(E_i)\,P(E|E_i) = \frac{1}{15} + \frac{1}{9} + \frac{2}{33} = \frac{33+55+30}{495} = \frac{118}{495}$$

Hence by Bayes's rule, the probability that the two white and red balls drawn are from 1st urn is:

$$P(E_1|E) = \frac{P(E_1)P(E|E_1)}{\sum P(E_i)P(E|E_i)} = \frac{\frac{1}{15}}{\frac{118}{495}} = \frac{33}{118}$$

Similarly, we have

$$P(E_2|E) = \frac{P(E_2)P(E|E_2)}{\sum P(E_i)P(E|E_i)} = \frac{\frac{1}{9}}{\frac{118}{495}} = \frac{55}{118}$$

and $\quad P(E_3|E) = \frac{\frac{2}{33}}{\frac{118}{495}} = \frac{30}{118}$ $\qquad$ (Or $\quad P(E_3|E) = 1 - \frac{33}{118} - \frac{55}{118} = \frac{30}{118}$)

P1:

A company has two plants to manufacture scooters. Plant I manufactures 80 percent of the scooters and plant II manufactures 20 percent. At plant I, 85 out of 100 scooters are rated standard quality or better. At plant II, only 65 out of 100 scooters are rated standard quality or better.

(i) What is the probability that scooter selected at random came from plant, I if it is known that the scooter is of standard quality?

(ii) What is the probability that the scooter came from plant II, if it is known that the scooter is of standard quality?

Solution:

Let us define the following events:

E_1: Scooter is manufactured by plant I

E_2: Scooter is manufactured by plant II

E: Scooter is rated as standard quality.

Then we are given:

$P(E_1) = 0.80,\qquad P(E_2) = 0.20;\qquad P(E|E_1) = 0.85\qquad P(E|E_2) = 0.65$

(i) Required probability is: (By Bayes' Rule)

$$P(E_1|E) = \frac{P(E_1)P(E|E_1)}{P(E_1)P(E|E_1)+P(E_2)\,P(E|E_2)}$$

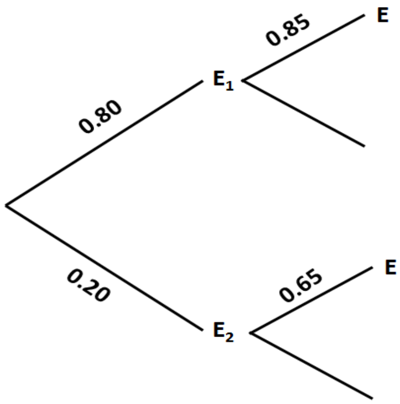
$$= \frac{0.80\times0.85}{0.80\times0.85\_0.20\times0.65} = \frac{0.68}{0.68+0.13} = \frac{0.68}{0.81} = 0.84$$

(ii) Required probability is given by:

$$P(E_2|E) = \frac{P(E_2)P(E|E_2)}{P(E_1)P(E|E_1)+P(E_2)P(E|E_2)} = \frac{0.20\times0.65}{0.80\times0.85+0.20\times0.65} = \frac{0.13}{0.81} = 0.16$$

(OR required probability is equal to$1 - 0.84 = 0.16$)

Aliter: TREE DIAGRAM



Events	Probability
$E \cap E_1$	$0.80 \times 0.85 = 0.68$
$E \cap E_2$	$0.20 \times 0.65 = 0.13$
Total	**0.81**

(i) $P(E_1|E) = \frac{0.68}{0.81} = 0.84$; (ii) $P(E_2|E) = \frac{0.13}{0.81} = 0.16$

P2:

In a class of 75 students, 15 were considered to be very intelligent, 45 as medium and the rest of them are below average. The probability that a very intelligent student fails in a viva – voice examination is 0.005; the medium student failing has a probability 0.05 ; and the corresponding probability for a below average student is 0.15. If a student is known to have passed the viva – voice examination, what is the probability that he is below average?

Solution:

Let us define the following events

E_1: The student of the class is very intelligent

E_2: The student is medium

E_3: The student is below average

A: The student passes in the viva- voice examination.

Then, we are given:

$$P(E_1) = \frac{15}{75} = 0.2 \quad ; \quad P(E_2) = \frac{45}{75} = 0.6 \quad ; \quad P(E_3) = \frac{15}{75} = 0.2$$

$$P(A|E_1) = 1 - 0.005 = 0.995 \quad ; P(A|E_2) = 1 - 0.05 = 0.95$$

$$P(A|E_3) = 1 - 0.15 = 0.85$$

Required to find $P(E_3|A)$.

If a student is known to have passed the viva – examination, the probability that he is below average, is given by Bayes' rule as follows.

$$P(E_3|A) = \frac{P(E_3)\, P(A|E_3)}{P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + P(E_3)\, P(A|E_3)}$$

$$= \frac{0.2 \times 0.850}{0.2 \times 0.995 + 0.6 \times 0.950 + 0.2 \times 0.850}$$

$$= \frac{0.170}{0.199 + 0.570 + 0.170} = \frac{0.170}{0.939} = 0.181$$